Problem 3

Define $S_n = \sum_{k=1}^{n} \left(3k^2 - 3k + 1\right)$.

We are tasked to compute S_1, S_2, S_3, S_4 and conjecture a formula for S_n.

Step 1: Expand the sum

$$S_n = \sum_{k=1}^{n} \left(3k^2 - 3k + 1\right) = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

Using known summation formulas:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$
$$\sum_{k=1}^{n} 1 = n.$$

Substitute into the equation:

$$S_n = 3 \cdot \frac{n(n+1)(2n+1)}{6} - 3 \cdot \frac{n(n+1)}{2} + n.$$

Simplify:

$$S_n = \frac{n(n+1)(2n+1)}{2} - \frac{3n(n+1)}{2} + n.$$

$$S_n = \frac{n(n+1)}{2}(2n+1-3) + n.$$

$$S_n = \frac{n(n+1)(2n-2)}{2} + n = n(n+1)(n-1) + n.$$

Step 2: Evaluate S_1, S_2, S_3, S_4

1. S_1:

$$S_1 = 1(1+1)(1-1) + 1 = 0 + 1 = 1.$$

2. S_2:

$$S_2 = 2(2+1)(2-1) + 2 = 2(3)(1) + 2 = 6 + 2 = 8.$$

3. S_3:

$$S_3 = 3(3+1)(3-1) + 3 = 3(4)(2) + 3 = 24 + 3 = 27.$$

4. S_4:

$$S_4 = 4(4+1)(4-1) + 4 = 4(5)(3) + 4 = 60 + 4 = 64.$$

Step 3: Conjecture Formula for S_n

From the results:

$$S_n = 1, 8, 27, 64.$$

These correspond to the cubes of integers:

$$S_n = n^3.$$

Thus, the formula is:

$$S_n = n^3.$$