Course Name
Fall 2025

Assignment 00

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Due: Date Here

Q1

Solution.
a) First we show that if u is a solution to $u_t + uu_x = 0$, then $w = u^2$ is a smooth solution of (2).

$$\begin{aligned} u_t + uu_x &= 0 \\ (2u)u_t + (2u)u_x &= 0 \\ 2uu_t + 2uu_x &= 0 \\ \left(u^2\right)_t + \left(\frac{2}{3}(u^2)^{\frac{3}{2}}\right)_x &= 0 \ \text{ Since } u \text{ is smooth.} \\ w_t + \left(\frac{2}{3}w^{\frac{3}{2}}\right)_x &= 0 \end{aligned}$$

In the other direction, we start with $w = u^2$ as a solution to $w_t + \left(\frac{2}{3}w^{\frac{3}{2}}\right)_x = 0$.

$$\begin{aligned} w_t + \left(\frac{2}{3}w^{\frac{3}{2}}\right)_x &= 0 \\ (u^2)_t + \left(\frac{2}{3}u^3\right)_x &= 0 \\ 2uu_t + 2u^2u_x &= 0 \ \text{ Since } u \text{ is smooth.} \\ 2u(u_t + uu_x) &= 0 \end{aligned}$$

Thus we have two cases. If $u = 0$, then trivially, (1) holds. Otherwise if $u > 0$, then we can divide by $2u$ giving us that,

$$u_t + uu_x = 0$$

as required. This concludes the proof in both directions.

b) Consider the initial conditions given by,

$$u(x,0) = \begin{cases} 2, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$w(x,0) = \begin{cases} 4, & x < 0 \\ 1, & x > 0 \end{cases}$$

This initial condition defines a shock starting at $t = 0$, we can find the shock speed for u and $w = u^2$. We can compute the shock speed for u and w with the Rankine-Hugoniot condition,

$$
\begin{aligned}
s_u = \frac{f(u_r) - f(u_l)}{u_r - u_l} &= \frac{1}{2}\frac{u_r^2 - u_l^2}{u_r - u_l} \\
\text{sub} \quad u_r = 1,\, u_l = 2 \\
&= \frac{1}{2}\frac{1 - 4}{1 - 2} \\
&= \frac{3}{2}
\end{aligned}
$$

$$
\begin{aligned}
s_w &= \frac{f(w_r) - f(w_l)}{w_r - w_l} \\
&= \frac{2}{3} \cdot \frac{w_r^{\frac{3}{2}} - w_l^{\frac{3}{2}}}{w_r - w_l} \\
&= \frac{2}{3} \cdot \frac{1 - 4^{\frac{3}{2}}}{1 - 4} \\
&= \frac{14}{9}
\end{aligned}
$$

Where s_u is the shock speed for u and s_w is the shock speed for w. Now we can show that there is a time t such that $u^2 \neq w$. Define x_u as the position at time t of shockwave u and x_w as the position at time t of shockwave w. Then they can be expressed as a function of t by,

$$
\begin{aligned}
x_u &= \frac{3}{2}t \\
x_w &= \frac{14}{9}t
\end{aligned}
$$

Taking any $t > 0$ shows us that immediately after our initial condition our shocks will no longer align and $u^2 = w$ will not hold. For example, taking $t = 1$ gives us that,

$$
x_u = \frac{3}{2} \neq \frac{14}{9} = x_w
$$

At $t = 1$, we will have that $u = 2$ and $w = 1$, however, $w^2 \neq u = 2$. This is because the u shockwave moves slower than the w shockwave and so it will lag behind, no longer preserving the initial condition $u^2 = w$.