# Lecture 1

| Intuitive notion | Mathematical notion |
|------------------|---------------------|
| Experiment | Probability space |
| Possible outcomes | Sample space |
| Single outcome | Member of the sample space |
| Set of outcomes | Event |
| Random quantity | Random variable |
| Average | Expectation |
| Dispersion index | Standard deviation (Variance) |

**Table 1:** correspondence between intuitive notions and mathematical notions

אות של פונקצית פרוב׳

## 1 Discrete Probability Space

**Definition 1.1** (Discrete Probability space)**.** *A discrete probability space is a pair* $(\Omega, \mathbb{P})$*, where* $\Omega$ *is a finite or countably infinite set, called the* sample space*, and* $\mathbb{P} : \Omega \to [0, 1]$ *is a function, called the* probability function*, which satisfies*

$\Omega$ -: או של המאורע
"קבוצה סופית או בת מניה"

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

### Example 1:

1. *Rolling a fair die:*

הסתם קביל זהה בכולם

$$\Omega = \{1, 2, 3, 4, 5, 6\},$$
$$\forall i \in \Omega \ \ \mathbb{P}(i) = \frac{1}{6}.$$

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2. *Tossing an unfair coin:* the probability of its result being heads is $\frac{1}{3}$.

$$\Omega = \{h, t\},$$
$$\mathbb{P}(h) = \frac{1}{3}, \ \mathbb{P}(t) = \frac{2}{3}.$$

3. *Tossing two unfair coins:* probability of $\frac{1}{3}$ for heads in each.

$$\Omega = \{(h, h), (h, t), (t, h), (t, t)\},$$
$$\mathbb{P}((h, h)) = \frac{1}{9}, \ \mathbb{P}((h, t)) = \frac{2}{9}, \ \mathbb{P}((t, h)) = \frac{2}{9}, \ \mathbb{P}((t, t)) = \frac{4}{9}.$$

Observe that $\frac{1}{9} + \frac{2}{9} + \frac{2}{9} + \frac{4}{9} = 1$.

4. *Tossing an unfair coin until the first time it comes up heads:* probability of $\frac{1}{3}$ for heads per toss.

$$\Omega = \{1, 2, 3, \ldots\},$$
$$\mathbb{P}(k) = \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3}.$$

Observe that

$$\sum_{k \in \Omega} \mathbb{P}(k) = \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1} \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1.$$

5. *Sampling a point uniformly at random from a segment of real numbers:* we will consider the segment $[0, 1]$.

$$\Omega = [0, 1],$$
$$\forall \omega \in \Omega \ \mathbb{P}(\omega) = 0.$$

This is problematic with our current definition requiring $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$. We will revisit this problem towards the end of this course. Note that the same problem occurs if we wish to sample a rational number uniformly, even though the set of all rational numbers is countable.

אפשר לגרש

## 1.1 Events :- מס-קבוצה של מרחב הדגימה

**Definition 1.2** (Event)**.** *An* event *is a subset of the sample space* $\Omega$. *We will extend the definition of* $\mathbb{P}$ *for events in the following way: For an event* $A \subseteq \Omega$ *define*

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\omega).$$

## Example 2:

1. *Tossing a fair die:* Let $A$ be the event that the outcome of the die is even. Using the modeling of the probability space from the previous example, we get that

$$A = \{2, 4, 6\},$$

$$\mathbb{P}(A) = \mathbb{P}(2) + \mathbb{P}(4) + \mathbb{P}(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

2. *Tossing two unfair coins:* probability of $\frac{1}{3}$ for heads in each. Let $A$ be the event that the result of at least one of the coin tosses is heads. Using the modeling of the probability space from the previous example, we get that

$$\Omega = \{(h, h), (h, t), (t, h), (t, t)\},$$

$$\mathbb{P}((h, h)) = \frac{1}{9}, \ \mathbb{P}((h, t)) = \frac{2}{9}, \ \mathbb{P}((t, h)) = \frac{2}{9}, \ \mathbb{P}((t, t)) = \frac{4}{9},$$

$$A = \{(h, h), (h, t), (t, h)\},$$

$$\mathbb{P}(A) = \mathbb{P}((h, h)) + \mathbb{P}((h, t)) + \mathbb{P}((t, h)) = \frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{5}{9}.$$

Now, let $B$ be the event that the result of the first coin toss is heads. Then

$$B = \{(h, h), (h, t)\},$$

$$\mathbb{P}(B) = \mathbb{P}((h, h)) + \mathbb{P}((h, t)) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3},$$

which matches our intuition – this event is similar to tossing one coin which comes up heads. In the following lectures we will explain why this intuition is correct.

בהרצאה בכיתה

## 1.2   Basic Properties Of Probability Spaces

שאלה 1

**Lemma 1.3.** *Let $(\Omega, \mathbb{P})$ be a probability space. Then*

$$\mathbb{P}(\emptyset) = 0,$$
$$\mathbb{P}(\Omega) = 1.$$

*Proof.* Follows immediately from the definition of an event, and from the fact that

$$\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1.$$

□

עבור שיטול אוכ לאיזבבה אי סמך של הסתברויות כולן 1

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**Definition 1.4.** *For an event $A$ define its complementary event* $\boxed{A^c \text{ by } A^c = \Omega \setminus A}$

**Lemma 1.5.** *Let $(\Omega, \mathbb{P})$ be a probability space. Then for all events $A$*

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$

*Proof.*

$$\mathbb{P}(A^c) = \sum_{\omega \in A^c} \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) - \sum_{\omega \in A} \mathbb{P}(\omega) = 1 - \mathbb{P}(A).$$

$$\boxed{P(f) = \sum_{\omega \in A^c} P(\omega) = \sum_{\omega \in \Omega} P(\omega) - \sum_{\omega \in A} P(\omega) = 1 - P(\omega)}$$ □

**Lemma 1.6** (Monotonicity). *Let $(\Omega, \mathbb{P})$ be a probability space.* *If $A \subseteq B$ are events, then*

*אם A מוכל ב B*

$$\mathbb{P}(A) \leq \mathbb{P}(B).$$

*Proof.*

$$\mathbb{P}(B) = \sum_{\omega \in B} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in B \setminus A} \mathbb{P}(\omega) \geq \sum_{\omega \in A} \mathbb{P}(\omega) = \mathbb{P}(A),$$

*where the inequality is due to the fact that $\mathbb{P}$ is a non-negative function, i.e., $\mathbb{P}(\omega) \geq 0$ for every $\omega \in \Omega$.*

*אי שליליות ( כל אלמנט אי שלילי )* □

**Lemma 1.7.** *Let $(\Omega, \mathbb{P})$ be a probability space, and let $A$ and $B$ be disjoint events. Then*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B).$$

*More generally, if $A_1, A_2, \ldots$ is an infinite sequence of pairwise disjoint events (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$), then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

*Proof.* We start with proving the statement for two events. It holds that

$$\mathbb{P}(A \cup B) = \sum_{\omega \in A \cup B} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in B} \mathbb{P}(\omega) = \mathbb{P}(A) + \mathbb{P}(B),$$

*where the second equality is due to the fact that $A$ and $B$ are disjoint. The proof of the general statement follows similar lines:*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{\omega \in \bigcup_{n=1}^{\infty} A_n} \mathbb{P}(\omega) = \sum_{n=1}^{\infty} \sum_{\omega \in A_n} \mathbb{P}(\omega) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

□

*איחוד זרים = סכום ה הסתברויות*

**Lemma 1.8.** *Let $(\Omega, \mathbb{P})$ be a probability space, and let $A$ and $B$ be events. Then*

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

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*first proof of Lemma 1.8.* We will prove the equivalent statement $\mathbb{P}(A)+\mathbb{P}(B) = \mathbb{P}(A \cup B)+\mathbb{P}(A \cap B)$. Indeed

$$\begin{aligned}
\mathbb{P}(A) + \mathbb{P}(B) &= \mathbb{P}((A \setminus B) \cup (A \cap B)) + \mathbb{P}((B \setminus A) \cup (A \cap B)) \\
&= \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) + \mathbb{P}(A \cap B) \\
&= \mathbb{P}(A \cup B) + \mathbb{P}(A \cap B),
\end{aligned}$$

where the second and third equalities hold by Lemma 1.7. □

*second proof of Lemma 1.8.*

$$\mathbb{P}(A \cup B) = \sum_{\omega \in A \cup B} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in B} \mathbb{P}(\omega) - \sum_{\omega \in A \cap B} \mathbb{P}(\omega) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

□

$\text{אולי נחמד?}$

**Lemma 1.9** (Union Bound). *Let $(\Omega, \mathbb{P})$ be a probability space, and let $A$ and $B$ be events. Then*

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B).$$

*More generally, if $A_1, A_2, \ldots$ is an infinite sequence of events, then*

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

*Proof.* We will prove the general case. It holds that

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{\omega \in \bigcup_{n=1}^{\infty} A_n} \mathbb{P}(\omega) \leq \sum_{n=1}^{\infty} \sum_{\omega \in A_n} \mathbb{P}(\omega) = \sum_{n=1}^{\infty} \mathbb{P}(A_n),$$

where the inequality is due to the fact that every term $\mathbb{P}(\omega)$ in the sum on the left-hand side appears at least once in the sum on the right-hand side, and $\mathbb{P}$ is a non-negative function. □

$$P(A \cup B) = \sum_{\omega \in A \cup B} P(\omega) = \sum_{\omega \in A} P(\omega) + \sum_{\omega \in B} P(\omega) \leq P(A) + P(B)$$

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