TER Master 1: Amenable groups

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Abstract

We give a quick introduction to the amenable group, which includes: some preparation work(definition of unitary representation and their main proprieties), a definition of the amenable group, main theorems about it, and small research on the amenability of some classical groups. It must be mentioned, that in this work there are no new theoretical results, that why it has to be considered as a literature review of already existing papers about amenable groups. The main of them is Kazhdan's T-property, more precisely Appendix A about the unitary representation of topological groups in Hilbert's spaces and Appendix G about amenability. The main theorems with their proof that are reviewed in this TER are the following: Theorem about a fixed point, Markov-Kakutani theorem about amenability of all abilene topological groups, and Hulanicki-Reiter theorem.

Some words about this paper structure. My TER is divided into 3 parts: Introduction, Theoretical part, and Practical use. The first part says about the main definition that will be used in this work and will contain all the necessary preparation to give these definitions. The second part is going to prove all main theorems about amenable groups and some of their harder proprieties which were not included in the first part. The last part gives some examples of amenable and not amenable groups, as well as some geometrical proprieties of amenable groups.

In this work, it is supposed that the theorems that were given during bachelor and mater 1 courses are well-known, so they are used with no proof

1 Introduction

1.1 Unitary group representation

In this section, we will define and study unitary group representation and their main proprieties.

Definition 1. *Let $\mathcal{H}$ be a Hilbert space. We say $U \in \mathcal{L}(\mathcal{H})$ is unitary if it is invertible and the inverse is its adjoint operator U^*:*

$$UU^* = U^*U = 1$$

The group of all unitary operators we note $\mathcal{U}(\mathcal{H})$

Now we can pass to the main definition of this part:

Definition 2. *A unitary representation of a topological group G in a Hilbert space $\mathcal{H}$ is a group homomorphism: $\pi : G \longrightarrow \mathcal{U}(\mathcal{H})$ which is strongly continuous in sense that: for all $v \in \mathcal{H}$ the mapping $g \longrightarrow \pi(g)v$ is continuous*
We will often use notation (π, H) instead of $\pi :\longrightarrow \mathcal{U}(\mathcal{H})$

When we talk about representation, it's useful not only to know representation by itself but also to know how it acts on a smaller part of $\mathcal{H}$.

Definition 3. *We say that $\mathcal{K} \subset \mathcal{H}$ is a G-invariant if and only if for all $g \in G$ and all $v \in K$ we have: $\pi(g)v \in K$*

Indeed we now can obtain a new representation of G by restricting π to $\pi^{\mathcal{K}} : \mathcal{K} \longrightarrow \mathcal{K}$. Such representation is called a subrepresentation of $(\pi, \mathcal{H})$. One of the important things about unitary representation is that it is completely reducible, it means that every closed invariant subspace has a closed invariant complement. In more detail:

Proposition 1. *Let $(\pi, \mathcal{H})$ be a unitary representation of G, and let $\mathcal{K}$ be a G-invariant subspace, Then $\mathcal{K}^{\perp}$, the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}$ is also G-invariant*

Proof. Since $\pi(g) \in \mathcal{H}$, we have for every $g \in G$, $v \in \mathcal{K}^{\perp}$ and $w \in \mathcal{K}$:

$$\langle \pi(g)v, w \rangle = \langle v, \pi(g)^* w \rangle = \langle v, \pi(g^{-1})w \rangle$$

and since $\mathcal{K}$ is G-invariant we have that $\pi(g^{-1})w \in \mathcal{K}$, that is why $\langle v, \pi(g^{-1})w \rangle = 0$, and as result $\pi(g)v \in \mathcal{K}^{\perp}$ □

From this proposition, we can easily see that if the representation has a subrepresentation we can simplify the study of it just by dividing the representation into smaller parts. So it is a reason why it is interesting to learn about the representation that can not be simplified in this way. That is where we have the next definition.

Definition 4. *A unitary representation $(\pi, \mathcal{H})$ is said to be irreducible if the only G-invariant closed subspaces of $\mathcal{H}$ are the trivial ones, that is 0 and $\mathcal{H}$*

The other thing that we often use to study things is the morphisms of structure. In the case of representation, the role of such morphisms plays an intertwining operator:

Definition 5. *An intertwining operator between two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of G is a continuous linear operator T from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$*

Now we can easily define the notation of equivalency:

Definition 6. *We say that representations π_1 and π_2 are equivalent, if there exists an intertwining operator $T : (\pi_1, \mathcal{H}_1) \longrightarrow (\pi_2, \mathcal{H}_2)$ which is isometric and one to one. We denote it by $\pi_1 \simeq \pi_2$*

Proposition 2 will show that T only needs to be invertible. To formulate that proposition we need some specific dictionary, that we are now defining

Definition 7. *An operator U from a Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$ is a partial isometry if there exists a closed subspace $\mathcal{M}$ of $\mathcal{H}_1$ such that restriction of U to $\mathcal{M}$ is an isometry and such that $U = 0$ on $\mathcal{M}^{\perp}$. The subspace $\mathcal{M} = (KerU)^{\perp}$ is the initial space and $U(\mathcal{M})$ the final space of the partial isometry U*

Remark 1. *An operator U is a partial isometry between two Hilbert spaces if and only if U^*U is a projection*

Now we can use this definition to generalize the polar decomposition of an operator to the operators between two Hilbert spaces. Let T be a continuous operator from Hilbert space $\mathcal{H}_1$ to another Hilbert space $\mathcal{H}_2$. We set $|T| = (T^*T)^{\frac{1}{2}}$ and define $U : |T|(\mathcal{H}_1) \longrightarrow T(\mathcal{H}_1)$. U is well defined because:

$$|T|v = 0 \Longleftrightarrow (T^*T)^{\frac{1}{2}}v = 0 \Longleftrightarrow T^*Tv = 0 \Longleftrightarrow Tv = 0$$

Moreover, U is an isometry:

$$\||T|v\|^2 = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2$$

Extend now linearly U to $\mathcal{H}_1$ by setting $U = 0$ on $|T|(\mathcal{H}_1)^{\perp} = KerT$. Then U is a partial isometry with initial space $(KerT)^{\perp}$ and the final space $T(\mathcal{H}_1)$. Then we have the polar decomposition of T:

$$T = U|T|.$$

Proposition 2. *Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two unitary representation of G. Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be an intertwining operator between π_1 and π_2; set $\mathcal{M}_1 = (KerT)^{\perp}$ and let $\mathcal{M}_2$ denote the closure of the image T.*
Then $\mathcal{M}_1$ and $\mathcal{M}_2$ are closed invariant subspaces of $\mathcal{H}_1$ and $\pi_1^{\mathcal{M}_1} \simeq \pi_2^{\mathcal{M}_2}$

Proof. We first check that $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ intertwines π_2 and π_1. For all $g \in G$ we have:

$$T^*\pi_2(g) = (\pi_2(g^{-1})T)^* = (T\pi_1(g^{-1}))^* = \pi_1 g T^*$$

It follows that $T^*T \in \mathcal{L}(\mathcal{H}_1)$ intertwines π_1 with itself:

$$T^*T\pi_1 = T^*\pi_2 T_1 = \pi_1 T^*T$$

Since $|T| = (T^*T)^{\frac{1}{2}}$ is a limit of polynomials in T^*T, $|T|$ also intertwines π_1 with itself. Let $T = U|T|$ be the polar decomposition of T; because $KerU = KerT$ and the restriction of U to $\mathcal{M}_1 = (KerT)^{\perp}$ is an isometry onto $\mathcal{M}_2$. It remains to check that U intertwines π_1 with π_2. Let g in G. Then:

$$\pi_2(g)U|T|v = \pi_2(g)Tv = T\pi_1(g)v = U|T|\pi_1(g)v = U\pi_1(g)|T|v$$

for all v in $\mathcal{H}_1$. This shows that $\pi_2(g)U = U\pi_1(g)$ on the image of $|T|$, and therefore on its closure $\mathcal{M}_1$. And because $U = 0$ on $KerT$ and $KerT$ is $\pi_1(G)$-invariant $\pi_2(g)U = U\pi_1(g)$ on $\mathcal{H}_1$. So as a result, because U is an isometry, onto and intertwining operator, we have that $\pi_1 \simeq \pi_2$ □

Corollary. *The two representations $(\pi_1,\mathcal{H}_1)$ and $(\pi_2,\mathcal{H}_2)$ of G are equivalent if and only if it exists one to one intertwining operator between them*

Proof. By definition, it is clear that if $\pi_1 \simeq \pi_2$, then it exists a one-to-one intertwining operator. The interesting part is to prove the sufficiency:
Let's suppose that it exists onto operator T that intertwines π_1 and π_2. But then $KerT = 0$, so $(KerT)^{\perp} = \mathcal{H}_1$ and also $ImT = \mathcal{H}_2$. Applying the previous proposition we obtain that $\pi_1 \simeq \pi_2$ □

Another important thing that we can easily define with the use of the notation of intertwining operator and equivalent representation is a notation of partial order.

Definition 8. *A unitary representation ρ of G is contained in a representation π of G if ρ is equivalent to a subrepresentation of π. This is denoted by $\rho \leq \pi$*

We will end the first part of the introduction by giving some examples of unitary representation

Examlples:

1. A unitary character of G is a continuous homomorphism $\mathcal{X} : G \longrightarrow S^1$, where S^1 is the multiplicative group of all complex numbers of modulus 1. We can identify a one-dimensional representation π of G with its character by $g \longrightarrow Trace(\pi(g))$. Such representation is called unit representation or unit character of G

   Remark 2. *A 1-dimensional representation is obviously irreducible.*

2. The direct sum of representation:
   Let $(\pi_i, \mathcal{H}_i)_{i\in I}$ be a family of unitary representation. Let $\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_i$ be a Hilbert sum of $\mathcal{H}_i$. The direct sum of the representations π_i is the unitary representation π of G on $\mathcal{H}$, that we define for all $g \in G$ and $\oplus_i v_i \in \mathcal{H}$ by:

$$\pi(g)(\oplus_i v_i) = \oplus_i \pi_i(g) v_i$$

If all the representations π_i are equivalent to the same representation σ we can note sometime $\pi = n\sigma$, where $n = |I|$

3. The tensor products of representation:
First, we will recall what the tensor product of Hilbert spaces is. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and $\mathcal{H} \otimes \mathcal{K}$ their algebraic tensor product. The completion of $\mathcal{H} \otimes \mathcal{K}$, with respect to the unique inner product for which

$$\langle v_1 \otimes h_1, v_2 \otimes h_2 \rangle = \langle v_1, v_2 \rangle \langle h_1, h_2 \rangle, \text{ for all } v_1, v_2 \in \mathcal{H},\ h_1, h_2 \in \mathcal{K}$$

is called the Hilbert tensor product of $\mathcal{H}$ and $\mathcal{K}$ and is again denoted by $\mathcal{H} \otimes \mathcal{K}$
Now we define the tensor product of two unitary representation $(\pi, \mathcal{H})$, $(\rho, \mathcal{K})$ - $(\pi \otimes \rho, \mathcal{H} \otimes \mathcal{K})$ by

$$(\pi \otimes \rho)(g)(v \otimes h) = \pi(g)v \otimes \rho(g)h, \text{ for every } v \in \mathcal{H},\ h \in \mathcal{K},\ g \in G$$

4. Translations:
Let $G = \mathbb{R}$ and $\mathcal{H} = L_2(\mathbb{R})$. Then $\tau : \mathbb{R} \longrightarrow \mathcal{U}(L_2)$ given by $\tau : \alpha \longrightarrow \tau_\alpha$ where $\tau_\alpha(f)(x) = f(x - \alpha)$ is a unitary representation.

1.2 Schur's lemma

As it was seen in the previous section the irreducible presentation can be interesting to learn about. That is why in this section we will talk about how we can detect irreducible representation. The main instrument for that is Schur's lemma but before we can talk about it, we have to do some preparation work:

Definition 9. *For a unitary representation $(\pi, \mathcal{H})$ of G, the set of all operators $T \in \mathcal{L}(\mathcal{H})$ which intertwine π with itself is called the commutant of $\pi(G)$ and is denoted by $\pi(G)'$*

Remark 3. *It is obvious that $\pi(G)'$ is a subalgebra of $\mathcal{L}(\mathcal{H})$. Moreover, $\pi(G)'$ is closed under taking the adjoint, which means that if $T \in \pi(G)'$ then $T^* \in \pi(G)'$, see Proposition 1*

Proposition 3. *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$ and let $P \in \mathcal{L}(\mathcal{H})$ be the orthogonal projection onto $\mathcal{K}$. Then $\mathcal{K}$ is G-invariant if and only if $P \in \pi(G)'$*

Proof. Let $Q = 1 - P$ be the orthogonal projection on $K^{\perp}$. Then for all v in $\mathcal{H}$ and g in G we have

$$(*)\ \pi(g)v = \pi(g)Pv + \pi(g)QV$$

Assume now that $\mathcal{K}$ is G-invariant. Then $K^{\perp}$ is also G-invariant, see Proposition 1.Hence, $\pi(g)Pv \in \mathcal{K}$ and $\pi(g)Qv \in \mathcal{K}^{\perp}$. This shows that relation $(*)$ is the orthogonal decomposition of $\pi(g)v$ with respect to $\mathcal{K}$. In particular, $\pi(g)Pv = P\pi(g)v$.
Conversely, assume that $\pi(g)P = P\pi(g)$ for all g in G. Then for all v in $\mathcal{K}$ we have:

$$\pi(g)v = \pi(g)Pv = P\pi(g)v$$

Since $\pi(g)v = P\pi(g)v$, we obtain that $\pi(g)v \in \mathcal{K}$. So $\mathcal{K}$ is G-invariant □

Now when all preparation work is done. We can pass to the main theorem of this part which is a Schur's lemma:

Theorem 1 (Schur's lemma)**.** *A unitary representation $(\pi, \mathcal{H})$ of G is irreducible if and only if $\pi(G)'$ consists of the scalar multiples of the identity operator* 1

Proof. Assume that $\pi(G)'$ consists only of the multiples of I. Let $\mathcal{K}$ be a closed G-invariant subspace of $\mathcal{H}$ with corresponding orthogonal projection P. By the Proposition 3, $P \in \pi(G)'$. So, by hypothesis, we know that $P = \lambda$. But of course $P^2 = P$, that is why we have $\lambda = 0$ or $\lambda = 1$, that is correspond to $\mathcal{K} = 0$ or $\mathcal{K} = \mathcal{H}$ respectively. Thus, π is irreducible.
Now, assume that π is irreducible and let $T \in \pi(G)'$. Set $T_1 = (T + T^*)/2$ and $T_2 = (T - T^*)/2i$. Then $T = T_1 + iT_2$ and $T_1, T_2 \in \pi(G)'$(by remark to Definition 9). Since T_1 and T_2 are selfadjoint, we can assume that T is selfadjoint.
We claim that the spectrum of $\sigma(T)$ of T consists of a single real number λ. Once proved, we obtain that $T = \lambda$
Assume by contradiction that there exist two different numbers $\lambda_1, \lambda_2 \in \sigma(T)$. Let

U_1 and U_2 be disjoint neighbourhoods in $\sigma(T)$ of λ_1 and λ_2 respectively. Then we can find real-valued functions $f_1, f_2 \in C(\sigma(T))$ with:

$$0 \leq f_i \leq 1_{U_i} \text{ and } f_i \neq 0, \text{ for } i = 1, 2$$

Let $f_i(T), 1_{U_i}(T) \in \mathcal{L}(\mathcal{H})$ be defined by functional calculus. We have

$$0 \leq f_i(T) \leq 1_{U_i}(T) \text{ and } f_i(T) \neq 0, \text{ for } i = 1, 2$$

Also, we obtain that

$$1_{U_i}(T)^2 = 1_{U_i}$$

as a result $1_{U_i}(T)$ is an orthogonal projection. Now $1_{U_i}(T) \in \pi(G)'$, since $T \in \pi(g)'$. Hence, $1_{U_i}(T) = 0$ or $1_{U_i}(T) = 1$ by irreducibility of π. As we have shown $1_{U_i} \neq 0$, it follows that $1_{U_i}(T) = 1$. This is a contradiction since $1_{U_1} 1_{U_2} = 0$ □

Corollary. *If G is an abelian topological group. Then any irreducible unitary representation of G is one-dimensional*

Proof. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. Since G is commutative, $\pi(G)$ is contained in $\pi(G)'$. Thus by Schur's Lemma, for every $g \in G$ the operator $\pi(g)$ is of the form $\mathcal{X}(g)1$. And then it is clear that $dim\mathcal{H} = 1$ and $\mathcal{X}$ is a unitary character of G, see example 1 p.6 □

1.3 The Haar measure of a locally compact group

Definition 10 (Left Haar measure)**.** *Let G be topological group and $\mathcal{B}$ Borel σ-algebra. Then regular Borel measure μ is called left Haar measure on G if and only if for all $B \in \mathcal{B}$ and $g \in G$*

$$\mu(gB) = \mu(B) \textit{ or equivalently } \int_G f(g^{-1}x)d\mu = \int_G f(x)d\mu$$

Remark 4. *We can define in the same way right Haar measure. But in general left Haar measure has no reason to be the right one*

Remark 5. *If the left(or right) Haar measure exists then it is unique up to a multiplicative constant*

In general, there exist groups that have no Haar measure. But all groups that we will study in this project have it. This is a result of the following theorem(that we will admit)

Theorem 2. *If G is a locally compact group, then it admits left Haar measure*

Examples

1. The Lebesgue measure is a Haar measure on $\mathbb{R}^n$ by construction
2. If G is discrete, the counting measure μ defined by $\mu(B) = \#B$ for any subset B of G is a Haar measure. We will use this example later to discuss the amenability of such groups
3. Let G be (ax+b)-group (over $\mathbb{R}$). Thus,

$$\mathrm{G} = \{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \; a \in \mathbb{R}^*, b \in \mathbb{R} \; \}$$

   Then $|a|^{-2}dadb$ is a left Haar measure while $|a|^{-1}dadb$ is a right one

1.4 The representations of compact group

The last chapter will be deduced to some important type of unitary group under the hypothesis that our group is compact. Some of them we will use when the amenability will be discussed.
In this section, G will always be a locally compact group, with a fixed left Haar measure, that will be denoted dg

Definition-proposition 11. *Let by $L^2(G)$ be denoted the Hilbert space of square-integrable functions with respect to the Haar measure. Then $(\lambda_G, L^2(G))$ is a unitary representation, called left regular representation, where λ_G is defined by:*

$$\lambda_G(g)f(x) = f(g^{-1}x), \textit{ for all } f \in L_2(G), x \in G$$

Proof. It is clear that λ_G is a group homomorphism from G to $\mathcal{L}(L_2(G))$. We have to verify that λ_G is a continuous and $Im(\lambda_G) \subset \mathcal{U}(L_2(G))$:

1. The operator λ_G is continuous:
   Let K be the compact subset of G and fix $f \in C^0(K)$. Then for all $g_1, g_2 \in G$. We obtain:

   $$\int_K |f(g_1x) - f(g_2x)|^2 dg = \int_K |f(g_1g_2^{-1}x) - f(x)|^2 dg \longrightarrow 0, \text{ if } g_1 \longrightarrow g_2, \text{ by dominated convergence theorem}$$

   Since G is a locally compact and $C^\infty(K)$ is dense in $L_2(K)$, we obtain the same result for all function in $Ł_2(G)$

2. We have the following relation $Im\lambda_G \subset \mathcal{U}(L_2(G)))$ We obtain that λ_G is unitary, by left invariance of Haar measure:

   $$||\lambda_G(g)f||_2^2 = \int_G |f(g^{-1}x)|^2 dg = \int_G |f(x)|^2 dg = ||f||_2^2, \text{ for all } g \in G$$

   And $\lambda_G(g)$ is inversible and its inverse is a $\lambda_G(g^{-1)}$.

□

Remark 6. *We have already seen left regular representation with translation, see p.8 example 3.*

With the use of the notation of regular representation, we can obtain some characteristic properties of compact group:

Proposition 4. *For a locally compact group G, the following proprieties are equivalent:*

1. *The group G is compact.*
2. *Unit character 1_G is contained in λ_G.*
3. *The Haar measure on G is finite.*

Proof. 1$\Longrightarrow$2: If G is a compact group, then since Haar measure is a regular measure, it is finite. As a result, constants are square integrable, which implies that $1_G \leq \lambda_G$
2$\Longrightarrow$3: If $1_G \leq \lambda_G$ then there exists $f \in L_2(G)$, such that: $f(g^{-1}x) = f(x)$, for all

$g \in G$, it means that f is constant. Since constant is integrable, Haar measure is finite.
3$\Longrightarrow$1: We denote by μ Haar measure on G. Let U be a compact neighbourhood of e. Assume that G is not compact. Then it exists an infinite sequence $g_1, g_2, ...$ in G with

$$g_{n+1} \notin \bigcup_{i=1}^{n} g_i U, \text{ for all } n \in \mathbb{N}$$

Choose a neighbourhood V of e with $V = V^{-1}$ and $V^2 \subset U$. Then

$$g_n V \cap g_m V = \emptyset, \text{ for all } n \neq m$$

Hence,

$$\mu(G) \geq \mu(\bigcup_{n\in\mathbb{N}} g_n V) = \sum_{n\in\mathbb{N}} \mu(g_n V) = \sum_{n\in\mathbb{N}} \mu(V) = \infty$$

because μ is invariant and since $\mu(V) > 0$ □

Another strong theorem about compact groups is the following. We will admit it

Theorem 3 (Peter-Weyl)**.** *Let G be a compact group.*

1. *Every unitary representation of G is the direct sum of irreducible subrepresentations.*

2. *Every irreducible unitary representation of G is finite-dimensional*

1.5 Weak containment

This section will be deduced to weak containment, which is a much less strict notation of the partial order on the representations. More precisely

Definition 12. *Let $(\pi, \mathcal{H})$ and $\rho, \mathcal{K}$ be unitary representations of the topological group G. We say that π is weakly contained in ρ, if for every $v \in \mathcal{H}$, every compact subset Q of G and every $\varepsilon > 0$, there exist $v_1, v_2, ..., v_n$ in $\mathcal{K}$ such that, for all $x \in Q$*

$$|\langle \pi(x), v, v\rangle - \sum_{i=1}^{n} \langle \rho(x) v_i, v_i\rangle| < \varepsilon$$

We write for this $\pi \prec \rho$.
If $\pi \prec \rho$ *and* $\rho \prec \pi$, *we say that* π *and* ρ *are weakly equivalent and denote this by* $\pi \sim \rho$

Now we will announce two important propositions that we will admit.

Proposition 5. *Let* $(\pi, \mathcal{H})$ *and* $(\rho, \mathcal{K})$ *be unitary representations of the locally compact group* G *such that* $\pi \prec \rho$. *Assume that* π *is irreducible. Let* v *be a unit vector of* $\mathcal{H}$. *Then* $\langle \pi(\cdot)v, v\rangle$ *can be approximate, uniformly by* $\langle \rho(\cdot)h_n, h_n\rangle$ *with* $(h_n)_n \in \mathcal{K}$ *i.e*

$$\langle \pi(x), v, v\rangle = \lim_n \langle \rho(x)h_n, h_n\rangle$$

Corollary. *Let* $(\pi, \mathcal{H})$ *be a unitary representation of* G. *Then* $1_G \prec \pi$ *if and only if, for every compact subset* Q *of* G *and every* $\varepsilon > 0$, *there exists a unit vector* v *in* $\mathcal{H}$ *such that*

$$\sup_{x\in Q} ||\pi(x)v - v|| < \varepsilon$$

Proof. For $x \in G$ and unit vector $v \in H$ we have

1. $||\pi(x)v - v||^2 = 2(1 - Re\langle \pi(x)v, v\rangle) \leq 2|1 - \langle \pi(x)v, v\rangle|$
2. $|1 - \langle \pi(x)v, v\rangle \leq 2(1 - Re\langle \pi(x)v, v\rangle) = ||\pi(x)v - v||^2$

Assume that $1_G \prec \pi$. Let Q and ε be as above. By Proposition 5 there exists a unit vector $v \in \mathcal{H}$ such that $\sup_{x\in Q} |1 - \langle \pi(x)v, v\rangle| \leq \varepsilon^2/2$. Hence $\sup_{x\in Q} ||\pi(x)v - v|| \leq \varepsilon$ by 1.
Conversely, if $\sup_{x\in Q} ||\pi(x)v - v|| \leq \varepsilon$, then $1_G \prec \pi$ by 2 □

Proposition 6. *For a locally compact group* G, *the following properties are equivalent:*

1. $1_G \prec \lambda_G$
2. $\pi \prec \lambda_G$, *for every unitary representation* π *of* G

2 Main theory about amenable groups

2.1 Invariant means and definition of amenable group

Definition 13. *A mean m on a ring $\mathcal{R}$ of subsets of X is a finitely additive probability measure on $\mathcal{R}$, that is, m is a function from $\mathcal{R}$ ti $\mathbb{R}$ with the following properties:*

1. $m(A) \geq 0$ *for all* $A \in \mathcal{R}$
2. $m(X) = 1$
3. $m(A_1 \cup A_2 \cup ... \cup A_n) = m(A_1) + m(A_2) + ... + m(A_n)$ *if* $A_1, .., A_n \in \mathcal{R}$ *are pairwise disjoint*

If a group G acts on X leaving $\mathcal{R}$ invariant, then m is said to be a G-invariant mean if

4. $m(gA) = m(A)$, *for all* $g \in G$ *and* $A \in \mathcal{R}$

For a ring $\mathcal{R}$ of subsets of X, denote by E the vector space of complex-valued function on X generated by the characteristic functions $\mathcal{X}_A$ of subsets A in $\mathcal{R}$. There is a natural bijective correspondence between means m on $\mathcal{R}$ and linear functionals M on E such that

$$m(A) = M(\mathcal{X}_a), \text{ for all } A \in \mathcal{R}$$

Now if X is equipped with a structure of measure space, we can extend this bijection to $L^\infty(X)$. More precisely:
Let $(X, \mathcal{B}, \mu)$ be a measure space. We fix by E the smallest closed subspace such that:

1. contains all simple function
2. is closed under complex conjugation
3. is closed under group action

Definition 14. *A mean on E is a linear functional $M : E \longrightarrow \mathbb{C}$ with the following properties:*

1. *$M(1_X) = 1$*

2. *$M(\varphi) \geq 0$ for all $\varphi \in E$ with $\varphi \geq 0$*

Let G be a group acting on E. We say that M is G-invariant if, moreover,

3. *$M(g\varphi) = M(\varphi)$ for all $g \in G$ and $\varphi \in E$*

Remark 7. *A mean M on E is automatically continuous. Indeed,*

$$-||\varphi||_\infty 1_X \leq \varphi \leq ||\varphi||_\infty 1_X$$

Hence, $|M(\varphi)| \leq ||\varphi||_\infty$ by 1 and 2

Remark 8. *By the Hahn-Banach theorem, we can find an extension of M to all subspace of $L^\infty(X)$ that contains E. Such extension we also call mean*

Proposition 7. *A means on $L^\infty(X, \mathcal{B}, \mu)$ is in one-to-one correspondence to absolutely continuous with respect to μ means on $\mathcal{B}$*

Proof. A mean M on $L^\infty(X, \mathcal{B}, \mu)$ defines a mean m on the σ-algebra $\mathcal{B}$ by $m(A) = M(\mathcal{X}_a)$, for all $A \in \mathcal{B}$
Conversely, if m is a mean on $\mathcal{B}$ which is absolutely continuous with respect to μ, then there exists a unique mean M on $L^\infty(X, \mathcal{B}, \mu)$ such that $m(A) = M(\mathcal{X}_A)$ for all $A \in \mathcal{B}$. Indeed, define

$$M(\varphi) = \sum_{i=1}^{m} \alpha_i m(A_i)$$

if $\varphi = \sum\limits_{i=1}^{m} \alpha_i \mathcal{X}_{A_i}$ is a measurable simple function on X. By finite additivity of m, this definition does not depend on the given representation of φ as a linear combination of characteristic functions of measurable subsets. Let now φ be a measurable bounded function on X. There exists a sequence $(\varphi_n)_n$ of measurable simple functions on X converging uniformly on X to φ. It is clear that $(M(\varphi_n))_n$ is a Cauchy sequence in $\mathbb{C}$ and that its limit does not depend on the particular choice of $(\varphi_n)_n$. Define then

$$M(\varphi) = \lim_n M(\varphi_n)$$

Since m is absolutely continuous with respect to μ, the number $M(\varphi)$ depends only on the equivalence class $[\varphi]$ of φ in $L(X, \mathcal{B}, \mu)$ and we can define $M([\varphi]) = M(\varphi)$ □

From now on, we will identify mean M on a space E in the sense of definition 13 and the corresponding mean m on $\mathcal{B}$, and we will use the same notation m in both cases.
Let G be a topological group. Let l^∞ be the Banach space of bounded functions on G. We say that the group acts on l^∞ by left translation if the action is defined by:

$$\varphi \longrightarrow_g \varphi,\ \varphi \in l^\infty(G),\ g \in G$$

Where ${}_g\varphi$ is defined by ${}_g\varphi(x) = \varphi(gx)$
We denote by $UCB(G)$ the closed subspace of l^∞, consisting of all left uniformly continuous functions on G.

Remark 9. *Function $\varphi \in l^\infty(G)$ is in $UCB(G)$ if and only if the mapping $x \longrightarrow_x \varphi$ from G to $l^\infty(G)$ is continuous.*

Remark 10. *The space $UCB(G)$ is invariant under left translation by elements from G*

Definition 15. *We consider group G acting on itself by left translations. The topological group G is said to be amenable if there exists an invariant mean on $UCB(G)$*

Example: Since by Proposition 4 all compact groups have finite Haar measure, all of them are amenable with normalized Haar measure as unique invariant mean

Remark 11. *For a non-compact group G, invariant means on $UCB(G)$ - when they exist - are in general not unique*

2.2 Proprieties of set of all means

This section is deduced to study of proprieties of the subset of all means on $UCB(G)$ in $UCB(G)^*$, topological duo to $UCB(G)$, endowed with the *weak** topology, where G locally compact group. We denote this subset by $\mathcal{M}$. We also fix all introduced notation for this subsection

Proposition 8. *$\mathcal{M}$ is contained in the unit ball of $UCB(G)^*$*

Proof. Let $\varphi \in UCB(G)$ with $||\varphi||_\infty = 1$ and $m \in \mathcal{M}$. Since $|m(\varphi)| \leq ||\varphi||_\infty$ by Remark 7, we obtain that $||m|| \leq 1$. And $||m||$ is exactly 1 because $m(1_G) = 1$ □

Proposition 9. *$\mathcal{M}$ is weak* closed subset in $UCB(G)^*$*

Proof. Let $(m_n)_n$ be a sequence of mean, that converge in the weak sense. We denote by m their limit. We have to verify that m is also mean. For that:

$$m(1_G) = \lim_n m_n(1_G) = \lim_n 1 = 1$$

and we also obtain that, for all $\varphi \in UCB(G)$ such that $\varphi \geq 0$:

$$m(\varphi) = \lim_n m_n(\varphi) \geq 0$$

□

Remark 12. *Since by Propositions 8, 9 $\mathcal{M}$ is a closed subset of the unit ball, with the use of Banach–Alaoglu theorem we obtain that $\mathcal{M}$ is a compact subset of $UCB(G)^*$*

Another important result that we will admit is that every mean m can be closed by point evaluations or in some cases by integration via $L^1(G)$ functions, more precisely:

Proposition 10. *Let $\mathcal{M}_0$ be the convex hull of all point evaluations, then $\mathcal{M}_0$ is weak* dense in $\mathcal{M}$*

Proposition 11. *We suppose that G is also equipped with Haar measure, and the spaces $L^p(G)$ are taken with respect to this measure. We denote by $L_1(G)_{1,+}$ the convex set of all $f \in L^1(G)$ with $f \geq 0$ and $||f||_1$. Let denote by $\mathcal{M}_1$ the set of all operators on $UCB(G)$ defined via integration against $f \in L^1(G)_{1,+}$. Then $\mathcal{M}_1$ is dense subset of $\mathcal{M}$*

2.3 Proprieties of amenable groups

Before we formulate the first theorem we will recall some definitions:

Definition 16. *Let X be a topological vector space, then we say that X is locally convex if it admits a local base at the origin consisting of convex, balanced sets,*

Definition 17. *Let X be a convex subset of a locally convex topological vector space. A mapping $\alpha : X \longrightarrow X$ is affine if for all $x, y \in X$ and $t \in [0,1]$*

$$\alpha(tx + (1-t)y) = t\alpha(x) + (1-t)\alpha(y)$$

Definition 18. *Let X be a convex subset of a locally convex topological vector space. A continuous auction $G \times X \longrightarrow X$ of G on X is said to be an affine action if*

$$X \longrightarrow X,\, x \longrightarrow gx$$

is an affine mapping for all g in G

Now we can use the notation of affine action to characterise the first propriety of the amenable group

Theorem 4. *For a topological group G the following proprieties are equivalent:*

1. *G is amenable*
2. ***Fixed point property:** any continuous affine of G on a non-empty compact convex subset X of a locally convex topological vector space has a fixed point*

Proof. 2$\Longrightarrow$1: The $UCB(G)^*$ space is a locally convex vector space. The set $\mathcal{M}$ of all means on $UCB(G)$ is *weak** compact convex subset of the unit ball of $UCB(G)^*$ by Proposition 8 and Remark 12. Observe that $\mathcal{M}$ is non-empty: for instance, the point evaluation $f \longrightarrow f(e)$ is a mean on $UCB(G)$. There is a continuous affine action of G on $\mathcal{M}$ given by $gm(\varphi) = m({}_{g^{-1}}\varphi)$ for all $g \in G$. Since by 2 it has a fixed point, it exists $m \in \mathcal{M}$, such that $m(\varphi) = m({}_{g^{-1}}\varphi)$ for all $g \in G$ by that we obtain that G is amenable.
$1 \Longrightarrow 2$: Assume now that G is amenable, and that a continuous affine action of G of non-empty compact convex set X in a locally convex vector space V is given. Fix an element x_0 in X. Let $t : G \longrightarrow X$, $g \longrightarrow gx_0$ be the corresponding

orbital mapping. For every function $f \in C(X)$, the function $f \circ t$ is in $UCB(G)$. Indeed, as X is compact, $f \circ t$ is bounded. Let m be a mean on $UCB(G)$. Then a probability measure μ_m on X is defined by

$$\mu_m(f) = m(f \circ t), \text{ for all } f \in C(X)$$

We define b_m as the X-valued integral $b_m = \int\limits_X x d\mu_m$. The element b_m is the unique element in X with the property that $\varphi(b_m)$=$\mu_m(\varphi)$ for every $\varphi \in V^*$. In particular, the mapping $\Phi : \mathcal{M} \longrightarrow X, m \longrightarrow b_m$ is continuous in weak sense, more precisely: let $(m_n)_{n\in\mathbb{N}}, m \in \mathcal{M}$ and $(m_n)_n \longrightarrow m$ in *weak** topology. Then for every $\varphi \in V^*$ we have:

$$\varphi(b_{m_n}) = \mu_n(\varphi) = m_n(\varphi \circ t) \longrightarrow m(\varphi \circ t) = \mu_m(\varphi) = \varphi(b_m)$$

Observe that $b_m = gx_0$ if $m = \delta_g$ is the evaluation at $g \in G$.
We claim that $b_g m = gb_m$ for every $g \in G$. Indeed, this is clearly true if m is a convex combination of point evaluations. And by Proposition 10 the set of all convex combinations of point evaluations is dense in $\mathcal{M}$. Since G acts by continuous affine mapping on X and since Φ is continuous, the claim follows. If now m is an invariant mean on $UCB(G)$, then $b_m \in X$ is a fixed point for the action of G $\square$

Corollary (Markov-Kakutani theorem)**.** *Every abelian topological group G is amenable*

Proof. Given a continuous affine action of G on a non-empty compact convex subset X in a locally convex vector space V, define for every integer $n \geq 0$ and every g in G, a continuous affine transformation $A_n(g) : X \longrightarrow X$ by:

$$A_n(g)x = \tfrac{1}{n+1}\sum_{i=0}^{n} g^i x, \ x \in X$$

Let $\mathcal{G}$ be the semigroup of continuous affine transformations of X generated by the set $\{A_n(g) : n \geq 0, g \in G\}$. Since X is compact, $\gamma(X)$ is a closed subset of X for every γ in $\mathcal{G}$.
We claim that $\bigcap\limits_{\gamma\in\mathcal{G}} \gamma(X)$ is not empty. Indeed, since X is compact, it suffices to show that $\gamma_1(X) \cap ... \cap \gamma_n(X)$ is non empty for all $\gamma_1, ...\gamma_n$ in $\mathcal{G}$. Let $\gamma = \gamma_1\gamma_2...\gamma_n \in \mathcal{G}$. Then, since $\mathcal{G}$ is abelian, $\gamma(X)$ is contained in $\gamma_i(X)$ for all $i =$

$1, ..., n$. Thus, $\gamma_1(X) \cap ...\gamma_n(X)$ contains $\gamma(X)$, and this proves the claim
Let $x_0 \in \bigcap_{\gamma \in \mathcal{G}} \gamma(X)$. We claim that x_0 is a fixed point for G. Indeed, for every g in G and every $n \in \mathbb{N}$, there exists some x in X such that $x_0 = A_n(g)x$. Hence for every φ in V^*

$$|\varphi(x_0 - gx_0)| = |\varphi(A_n(g)x) - \varphi(gA_n(g)x| = \frac{1}{n+1}|\varphi(x) - \varphi(g^{n+1}x)| \leq \frac{2C}{n+1}$$

where C $= ||\varphi||_\infty$, which is finite, since X is compact. As this holds for all n, it follows that $\varphi(x_0) = \varphi(gx_0)$ for all $\varphi \in V^*$. Hence, $x_0 = gx_0$ for all g in G, by Theorem 4 we obtain that G is amenable ☐

Now we will discuss the behavior of amenability under exact sequences and directed unions.

Proposition 12. *Let G be a topological group and N be a closed normal subgroup.*

1. *If G is amenable, then G/N is amenable.*
2. *If N and G/N are amenable, then G is amenable*

Proof. 1. follows from the fact that $UCB(G/N)$ can be viewed as subspace of $UCB(G))$ of function that is stable under action of N
2. Assume that N and G/N are amenable, and that a continuous affine action of G on a non-empty convex set X is given. Let X^N be the closed subspace of all fixed points of N in X. It is clear that X^N is a compact convex subset of X, that X^N is invariant under G, and that the action of G on X^N factorizes to an action of G/N. Since N is amenable, X^N is not empty by Theorem 4. Hence, G/N has fixed point x_0 in X^N, by the amenability of G/N. Clearly, x_0 is a fixed point for G. The claim follows now from the Theorem 4 ☐

Corollary. *Every compact extension of a soluble topological group is amenable*

Proposition 13. *Let $(G_i)_{i \in I}$ be a directed family of closed subgroups of G such that $\bigcup_{i \in I} G_i$ is dense in G. If G_i is amenable for every $i \in I$, then G is amenable*

Proof. Let G' denote the union $\bigcup_{i \in I} G_i$, with the inductive limit topology. The inclusion $G' \longrightarrow G$ is uniformly continuous with the dense image so that the restriction mapping $UCB(G) \longrightarrow UCB(G')$ is an isomorphism. We can therefore assume without loss of generality that $G = \bigcup_{i \in I} G_i$

For every $i \in I$ let $\mathcal{M}_i$ be the set of all G_i-invariant means on $UCB(G)$. We claim that $\bigcap_{i \in I} \mathcal{M}_i$ is non empty. This will prove proposition, since any $m \in \bigcap_{i \in I} \mathcal{M}_i$ is a G-invariant mean on $UCB(G)$.

Observe that every $\mathcal{M}_i$ is a closed subset of the compact set $\mathcal{M}$ of all means on $UCB(G)$. Hence, it suffices to show that family $(M_i)_{i \in I}$ has the finite intersection propriety.

Let F be a finite subset of I. Let $i \in I$ be such that $G_j \subset G_i$ for all $j \in F$. Choose a G_i-invariant mean m_i on $UCB(G_i)$. Define a mean $\tilde{m}_i$ on $UCB(G)$ by

$$\tilde{m}_i(f) = m_i(f|_{G_i}), \text{ for all } f \in UCB(G)$$

Then $\tilde{m}_i$ is G_i-invariant and therefore $\tilde{m}_i \in \mathcal{M}_j$ for all $j \in F$. Thus $\bigcap_{j \in F} M_j \neq \emptyset$, as claimed $\square$

Corollary. *Every locally finite group is amenable*

Remark 13. *We will see later, that under certain not restrictive conditions, it is also true that subgroups of the amenable group are amenable*

Now we will pass to the more analytical characterization of amenable groups. Let G be a locally compact group, so by Theorem 2 it exists Haar measure on it. We will always assume that G is equipped with a fixed left Haar measure; the space $L^p(G)$ are taken with respect to this measure.

Theorem 5. *Let G be a locally compact group. The following proprieties are equivalent:*

1. *G is amenable*
2. *there exists a topological invariant mean on $L^\infty(G)$, that is, a mean m on $L^\infty(G)$ such that $(m * \varphi) = m(\varphi)$ for all f in $L^1(G)_{1,+}$ and φ in $L^\infty(G)$*
3. *Reiter's Property (P_1): for every compact subset of Q of G and every $\varepsilon > 0$, there exists f in $L^1(G)_{1,+}$ such that*

$$\sup_{x\in Q} ||_{x^{-1}}f - f||_1 \leq \varepsilon$$

4. *Reiter's Property* (P_1^*)*: for every finite subset of* Q *of* G *and every* $\varepsilon > 0$*, there exists* f *in* $L^1(G)_{1,+}$ *such that*

$$\sup_{x\in Q} ||_{x^{-1}}f - f||_1 \leq \varepsilon$$

5. *there exists an invariant mean on* $L^\infty(G)$

Proof. $1 \Longrightarrow 2$.We will first show an important lemma

Lemma 6. *If* m *an invariant mean on* $UCB(G)$*, then for every* φ *in* $UCB(G)$ *and every* f *in* $L^1(G)_{1,+}$*,* $f * \varphi$ *is in* $UCB(G)$ *and the next equality holds:*

$$m(f * \varphi) = m(\varphi)$$

Proof. First we will show that $f * \varphi$ is in UCB(G):

1. $f * \varphi$ is almost everywhere borne by Young's inequality:

$$||f * \varphi||_\infty \leq ||f||_1 ||\varphi||_\infty$$

2. $f * \varphi$ is uniformly continuous because:

$$_gf * \varphi(x) = \int_G f(y)\varphi(g^{-1}xy^{-1})dy \longrightarrow \int_G f(y)\varphi(xy^{-1})dy,$$

uniformly when $g \longrightarrow e$ by uniformly continuity of φ and dominate convergence theorem

Now let show that $m(\varphi * f) = m(\varphi)$, by invariance and uniform continuity of m:

$$m(f * \varphi) = m(\int_G f(y)_{y^{-1}}\varphi dg) = \int_G m(f(y)_{y^{-1}}\varphi)dg = \int_G f(y)m(_{y^{-1}}\varphi)dg = m(\varphi)\int_G f(y)dg = m(\varphi)$$

□

Now we can continue proof of the theorem. Let $(f_i)_i$ be a net in $L^1(G)_{1,+}$ with $supp(f_i) \longrightarrow e$. Then, for each $\varphi \in L^\infty(G)$ and $f \in L^1(G)_{1,+}$, we obtain that $f * \varphi \in UCB(G)$ by dominant convergence theorem and

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$$\lim_i ||f * f_i * \varphi - f * \varphi|| = 0$$

and, hence, $m(f * \varphi) = \lim_i m(f * f_i * \varphi) = \lim_i m(f_i * \varphi)$ by previous lemme. This shows that $m(f * \varphi) = m(f' * \varphi)$ for all $f, f' \in L^1(G)_{1,+}$ and all $\varphi \in L^\infty(G)$. Fix any $f_0 \in L^1(G)_{1,+}$ and define a mean $\tilde{m}$ on $L^\infty(G)$ by

$$\tilde{m}(\varphi) = m(f_0 * \varphi), \;\; \varphi \in L^\infty(G)$$

Then $\tilde{m}$ is a topological invariant mean. Indeed,

$$\tilde{m}(f * \varphi) = m(f_0 * f * \varphi) = m(f_0 * \varphi) = \tilde{m}(\varphi)$$

for all $\varphi \in L^\infty(G)$ and $f \in L^1(G)_{1,+}$ $2 \implies 3$. Let m be a topological invariant mean on $L^\infty(G)$. Since $L^1(G)_{1,+}$ is *weak** dense in the set of all means on $L^\infty(G)$, there exists a net $(f_i)_i$ in $L^1(G)_{1,+}$ converging to m in the *weak** topology. The topological invariance of m implies that, for every $f \in L^1(G)_{1,+}$

$$(*) \lim_i (f * f_i - f_i) = 0$$

in the weak topology on $L^1(G)$
$2 \implies 3$ Let m be a topological invariant mean on $L^\infty(G)$. Since $L^1(G)_{1,+}$ is weak dense in the set of all means on $L^\infty(G)$, see Proposition 11, there exists a net $(f_i)_i$ in $L^1(G)_{1,+}$ converging to m in the *weak** topology. The topological invariance of m implies that, for every $f \in L^1(G)_{1,+}$

$$(*) \lim_i (f * f_i - f_i) = 0$$

Now we consider the product space

$$E = \prod_{f \in L^1(G)_{1,+}} L^1(G)$$

with the product of the norm topology. Then E is a locally convex space, and the weak topology on E is the product of the weak topologies. The set

$$\Sigma = \{f * g - g : f, g \in L^1(G)_{1,+}\}$$

is convex and, by $(*)$, its closure in the weak topology contain 0. Since E is locally convex, the closures of Σ in both senses: weak and norm coincide. Hence, there exists a net $(g_j)_j$ in $L^1(G)_{1,+}$ such that, for every f in $L^1(G)_{1,+}$.

$$(**)\ \lim_j ||f * g_j - g_j||_1 = 0$$

Since the g_j have bounded L^1-norm, $(**)$ holds uniformly for all f in any norm compact subset K of $L^1(G)$.
Let now Q be a compact subset of G containing e, and let $\varepsilon > 0$. Fix any f in $L^1(G)_{1,+}$. Since the mapping

$$G \longrightarrow L^1(G), x \longrightarrow_{x^{-1}} f$$

is continuous, $\{_{x^{-1}}f : x \in Q\}$ is a compact subset of $L^1(G)$. Hence, there exists j such that

$$|||_{x^{-1}}f * g_j - g_j||_1 \leq \varepsilon$$

for all $x \in Q$. Set $g = f * g_j$. Then $g \in L^1(G)_{1,+}$ and, for all $x \in Q$, we have

$$||_{x^{-1}}g - g||_1 \leq ||_{x^{-1}}f * g_j - g_j||_1 + ||f * g_j - g_j||_1 \leq 2\varepsilon$$

$3 \implies 4$ is obvious
$4 \implies 5$ There exists a net $(f_i)_i$ in $L^1(G)_{1,+}$ such that

$$(***)\ \lim_i ||_{x^{-1}}f_i - f_i||_1 = 0$$

for all $x \in G$. Let m be a *weak** limit of $(f_i)_i$ in the set of all means on $L^\infty(G)$. It follows from $(***)$ that m is invariant. This shows that $4 \implies 5$.
$5 \implies 1$ is obvious, because $UCB(G)$ can be viewed as subspace of $L^\infty(G)$ □

Now we can pass to the link between amenability and weak containment

Theorem 7 (Hulanicki-Reiter). *Let G be a locally compact group. The following properties are equivalent:*

1. *G is amenable*
2. *The following include hold $1_G \prec \lambda_G$*
3. *for every unitary representation π of G $\pi \prec \lambda_G$*

Proof. The equivalence of 2 and 3 was already proved in Proposition 6. In view of the previous theorem, it suffices to show that 2 is equivalent to Reiter's Property(P_1) Assume that 2 holds. Then, given a compact subset Q of G and $\varepsilon > 0$ by Proposition 5 there exists $f_{Q,\varepsilon} \in L^2(G)$ with $||f_{Q,\varepsilon}|| = 1$ such that

$$\sup_{x\in Q} ||\lambda_G(x)f_{Q,\varepsilon} - f_{Q,\varepsilon}||_2 < \varepsilon$$

Set $g_{Q,\varepsilon} = |f_{Q,\varepsilon}|^2$. Then $g_{Q,\varepsilon} \in L^1(G)_{1,+}$ and by the Cauchy-Schwartz inequality,

$$||_{x^{-1}}g_{Q,\varepsilon} - g_{Q,\varepsilon}||_1 \leq ||\lambda_G(x)f_{Q,\varepsilon} + f_{Q,\varepsilon}||_2||\lambda_G(x)f_{Q,\varepsilon} - f_{Q,\varepsilon}||_2 \leq$$
$$2||\lambda_G(x)f_{Q,\varepsilon} - f_{Q,\varepsilon}|| \leq 2\varepsilon$$

for all $x \in Q$. Hence, G has Reiter's Property.
Conversely, assume that G has Reiter's Property(P_1). For a compact subset Q of G and $\varepsilon > 0$, let $f \in L^1(G)_{1,+}$ be such that

$$\sup_{x\in Q} ||_{x^{-1}}f - f||_1 < \varepsilon$$

Let $g = \sqrt{f}$. Then $g \in L^2(G)$ and $||g||_2 = 1$. Moreover, using the inequality $|a-b|^2 \leq |a^2 - b^2|$ for all real numbers a and b, we have

$$||\lambda_G(x)g - g||_2^2 \leq \int_G |g(x^{-1}y)^2 - g(y)^2|dy = ||_{x^{-1}}f - f||_1 < \varepsilon$$

for all $x \in Q$. This shows that $1_G \prec \lambda_G$ by Proposition 5 $\square$

Corollary. *Closed subgroups of amenable locally compact groups are amenable*

Proof. Since the restriction of λ_G to H is weakly contained in λ_H,the claim follows from the previous theorem $\square$

3 Use of theory

3.1 Research of amenability of some groups

In this section, we will look at examples of amenable and not amenable groups. In general, it is much easier to give the example of an amenable group. So we will start with it

Examples of amenable group:

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, all of these groups are amenable because they are abelian, see Theorem 4.

2. Every finite group or even direct sum of a finite group is amenable

3. All compact Lie's groups are amenable, for example, Lie group of torus, $O(N), SO(N)$.
4. Every solvable and hence nilpotent group is amenable

Now we can pass to examples of not amenable groups. Our main example is a free group of 2 elements, more precisely:

Proposition 14. *Let F_2 be the free group on two generators a and b, with the discrete topology. Then F_2 is not amenable*

Proof. Assume, by contradiction, that there exists an invariant mean m on $UCB(F_2)$ because F_2 is a discrete group we obtain that $UCB(F_2) = l^\infty(F_2)$. For each subset B of F_2, write $m(B)$ for the value of m on the characteristic function of B. Every element in F_2 can be written as a reduced word in a, a^{-1}, b, b^{-1}. Let A be the subset of all words in F_2 beginning with an a. Then $F_2 = a^{-1}A \cup A$. As $m(aA) = m(A)$ and $m(F_2) = 1$, it follows that $m(A) \geq 1/2$. On the other hand A, bA, b^2A are mutually disjoint subsets of F_2. Hence

$$1 = m(F_2) \geq m(A) + m(bA) + m(b^2A) = 3m(A) \geq \frac{3}{2}$$

This is a contradiction ∎

So now with use of Proposition 14 and corollary to Theorem 7, we can conclude that group $SL(\mathbb{A}, n)$, $GL(\mathbb{A}, n), SO(\mathbb{A}, n)$ with discrete topology are **example of non-amenable group**, where A is any ring extension of $\mathbb{Z}$ and $n \geq 3$.

3.2 Banach-Tarski paradox

The Banach-Tarski paradox says that is, it is possible to partition a unit ball and apply isometries to the pieces of the partition to yield two solid spheres of the same radius as the original. In this subsection, we will formulate the Banach-Tarski paradox and see the link between such type of phenomena and amenability.

Definition 19. *Let G acts on X and $Y \subset X$. We say that Y is a G-paradoxical, when for positive integers m, n there exist pairwise disjoint subsets $A_1, A_2, ..A_m$,*

$B_1, B_2, ..., B_n$ partition: $Y = \bigcup_{i \leq m} A_i \cup \bigcup_{j \leq n} B_j$ and $g_1, g_2, .., g_m, h_1, ..., h_n \in H$, such that

$$Y = \bigcup_{i \leq m} g_i(A_i) = \bigcup_{j \leq n} h_j(B_j)$$

Now, we introduce the next proposition how we can detect these sets.

Proposition 15. *Let G be G-paradoxical, and G acts on X freely. We obtain that X is also a G-paradoxical.*

Proof. Let for all $i \in \{1, 2, ..., m\}$, $j \in \{1, 2, ..., n\}$ g_i, h_j be in G and A_i B_j be the subset of G witness that G is paradoxical. Let M be the choice set containing exactly one element of each orbit of G in X. We obtain the following partitions of $X = \bigcup_{g \in G} gM$. Also it is easy to notice that if for $g, h \in G$ we have $gM = hM$ it implies that $g = h$, so that sets are pairwise disjoint
Now for each i, j let $A'_i = \bigcup_{g \in A_i} gM$ and $B'_j = \bigcup_{h \in B_j} h(M)$. Then $X = \bigcup_{i \leq m} g_i(A'_i) = \bigcup_{j \leq n} h_j(B'_j)$. That claims the proposition □

We can see that this proposition is the opposite of the fixed point property for the amenable group, see theorem 4. From it, we can guess that amenable groups can not be used to generate paradoxical sets. This fact was proven by Tarski in the next theorem that we will admit:

Theorem 8. *Group G is amenable if and only if it is not $G-$paradoxical*

Now we can formulate and give a short proof of the main theorem

Theorem 9 (Banach-Tarski)**.** *Any unit ball in $\mathbb{R}^n$ with $n \geq 3$is $SO(3)$- paradoxical.*

Proof. Since $SO(3)$ with a discrete topology is non-amenable, it is $SO(3)-$paradoxical, by Proposition 19 since $SO(3)$ acts freely on a unit ball it is also $SO(3)$ paradoxical. □