Groupoïdes de déformations et applications

Thèse dirigée sous la direction de Georges Skandalis

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Introduction

In noncommutative geometry, ill-defined spaces in classical differential geometry (for example the quotient space by a group action, or the quotient space of a foliation) are studied using methods from functional analysis.

The starting point is a theorem of Gelfand that allows us to regard noncommutative C^*-algebras as noncommutative topological spaces.

Therefore one associates to an ill-defined topological space a noncommutative C^*-algebra whose elements could be regarded as functions on the space.

Introduction

A great success of this strategy is when it goes in the other direction as well. Sometimes one can define noncommutative C^*-algebras that allow us to solve problems in classical differential geometry.

Lie groupoids

A key tool in noncommutative geometry is that of a Lie groupoid.

Definition

A Groupoid is a small category whose morphisms are all invertible.

A Lie groupoid is a groupoid whose set of morphisms G is endowed with the structure of a smooth manifold such that

1. the space of objects G^0 is an embedded submanifold of G.
2. the source map $s : G \to G^0$ is a smooth submersion.
3. the inverse map $\gamma \in G \to \gamma^{-1} \in G$ is smooth
4. the composition map $G \times_{s,r} G \to G$ is smooth.

The notation $G \rightrightarrows G^0$ is used to denote a Lie groupoid.

Examples

1. If G is a Lie group, then $G \rightrightarrows \{e\}$ is naturally a Lie groupoid.
2. If M is a smooth manifold, then $M \rightrightarrows M$ is a trivial Lie groupoid.
3. If $V \to M$ is vector bundle, then $V \rightrightarrows M$ is a Lie groupoid.
4. If M is a smooth manifold, then $M \times M \rightrightarrows M$ is the pair Lie groupoid.

C^*-algebra of a Lie groupoid

Let $f, g \in C_c^\infty(G)$. The product structure on G defines the convolution

$$f \star g(\gamma) = \int_{\gamma'\gamma''=\gamma} f(\gamma')g(\gamma'').$$

This product makes $C_c^\infty(G)$ an associative algebra, furthermore an involutive algebra

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

The completion with respect to some norm is the C^*-algebra of G defined by A. Connes and J. Renault around the 80's, denoted C^*G

Holonomy groupoid

5. Let $F \subseteq TM$ be a foliation, R the equivalence relation

$$xRy \text{ if } x \text{ and } y \text{ are in the same leave.}$$

The foliation groupoid is the graph of R

$$G = \{(x, y) \in M \times M : xRy\} \subseteq M \times M.$$

This is not a smooth manifold in general, but the following is

$$\mathcal{G}(M, F) := \{(x, [\gamma], y) : xRy\},$$

where γ is a leafwise path from x to y and $[\gamma]$ is its class up to holonomy.

6. Let
$$G = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}.$$
The groupoid structure is just the union of examples 2 and 4.
A smooth structure on G is defined such that $G \rightrightarrows M \times \mathbb{R}$ becomes a Lie groupoid.

Smooth structure and topology are defined by declaring the following functions smooth: for every $f \in C^\infty(M)$, the functions

$$\begin{aligned} G &\to M \times M \times \mathbb{R} \\ (x, y, t) &\to (x, y, t) \\ (x, X, 0) &\to (x, x, 0) \end{aligned} \qquad\qquad \begin{aligned} G &\to \mathbb{R} \\ (x, y, t) &\to \tfrac{f(x)-f(y)}{t} \\ (x, X, 0) &\to df_x(X) \end{aligned}$$

are smooth.

Deformation to the normal cone

Recall the deformation to the normal cone construction (DNC). Let $V \subseteq M$ a submanifold. The set

$$\mathrm{DNC}(M,V) = M \times \mathbb{R}^* \sqcup \mathcal{N}_V^M \times \{0\},$$

where $\mathcal{N}_V^M$ is the normal bundle, is equipped with a smooth structure by declaring the following functions smooth: if $f \in C^\infty(M)$ a smooth function vanishing on V, then the functions

$$\begin{aligned} \mathrm{DNC}(M,V) &\to M \times \mathbb{R} \\ (x,t) &\to (x,t) \\ (x,X,0) &\to (x,0) \end{aligned} \qquad\qquad \begin{aligned} \mathrm{DNC}(M,V) &\to \mathbb{R} \\ (x,t) &\to \tfrac{f(x)}{t} \\ (x,X,0) &\to df_x(X) \end{aligned}$$

are smooth. (The space G in previous slide is then $\mathrm{DNC}(M \times M, M)$)

Deformation to the normal cone

This construction is natural: if $f : M \to M'$ is smooth and $f(V) \subseteq V'$, then

$$\mathrm{DNC}(f) : \mathrm{DNC}(M, V) \to \mathrm{DNC}(M', V')$$

is well defined and smooth.

Debord and Skandalis remark (2015) : if G a Lie groupoid and $H \subseteq G$ a Lie subgroupoid, then

$$\mathrm{DNC}(G, H) \rightrightarrows \mathrm{DNC}(G^0, H^0)$$

is naturally a Lie groupoid.

Witten deformation

Let M be a compact smooth manifold, f a Morse function. Witten gave an analytic proof of Morse's inequalities using the following deformation of the Laplacian

$$\Delta_t = (e^{-\frac{f}{t}} d e^{\frac{f}{t}} + e^{\frac{f}{t}} d^* e^{-\frac{f}{t}})^2.$$

Theorem (Witten 1981)

If $\lambda_t^1 \leq \lambda_t^2 \ldots$ denote the eigenvalues of Δ_t, then

$$\lim_{t \to 0^+} \lambda_t^i = \begin{cases} 0, & \text{if } \quad i \leq \# \mathrm{Crit}(f) \\ +\infty & \text{if not} \end{cases}$$

Witten deformation in terms of DNC

Witten deformation is realised as an operator on the space

$$\mathrm{DNC}(M, \mathrm{Crit}(f)) = M \times \mathbb{R}^* \sqcup_{a \in \mathrm{Crit}(f)} T_a M$$

equal to $t^2 \Delta_{t^2}$ on $M \times \{t\}$ for $t \neq 0$ and to the union of the harmonic oscillators on the tangent spaces at $t = 0$.

We prove that this global operator has compact resolvent (in C^*-module sense). Witten's theorem is then an immediate corollary.

Witten deformation in terms of DNC

Proposition

Let W be a complete Riemannian manifold, α a 1-form on W such that

1. $\|\alpha\|$ *is a proper function*
2. *the graded commutator* $[d + d^*, c(\alpha)]$ *is bounded*

then the resolvent of $d + d^* + c(\alpha)$ *is compact.*

Proof.

$d + d^*$ is an elliptic differential operator, hence its resolvent is locally compact.The first condition ensures that $\frac{1}{1+\|\alpha\|^2} \in C_0(W)$. Hence

$$\left(1 + (d + d^*)^2\right)^{-1} \left(1 + \|\alpha\|^2\right)^{-1}$$

is compact. The second condition finishes the proof. □

This proposition is just a calculation of a Kasparov product.

Witten deformation in terms of DNC

We think of a Lie groupoid as a fiber bundle $s : G \to G^0$. One obtains

Proposition

Let G be a Lie groupoid equipped with a complete metric, $\alpha \in \Gamma(\mathfrak{A}G^*)$ such that

1. $\|\alpha\| : G^0 \to \mathbb{R}$ *is a proper function,*
2. *the operator* $[d + d^*, c(\alpha)]$ *is bounded.*

Then the operator $d + d^* + c(\alpha)$ is has compact resolvent (in the C^*-module sense).

The proof is the same

Witten deformation in terms of DNC

Witten's theorem is a corollary as follows : we apply proposition to the groupoid

$$\begin{aligned}\mathrm{DNC}(M\times M,\mathrm{Crit}(f)) \rightrightarrows \mathrm{DNC}(M,\mathrm{Crit}(f))\\ = M\times M\times\mathbb{R}^*\sqcup_{a\in\mathrm{Crit}(f)} T_aM\times T_aM\times\{0\}.\end{aligned}$$

and the 1-form equal to $\frac{df}{t^2}$ when $t\neq 0$ and to d^2f at 0, and Riemannian metric on the s-fibers equal to $\frac{g}{t^2}$ for $t\neq 0$ and the constant metric on the tangent spaces.

After renormalizing the metric from $\frac{g}{t^2}$ to g, the operator $(d+d^*+c(\alpha))^2$ becomes $t^2\Delta_{t^2}$.

Morse inequalities for foliations

Let $F \subseteq TM$ be a foliation, $f : M \to \mathbb{R}$ a smooth function such that

$$\mathrm{Crit}_F(f) = \{x \in M : d_F f(x) = 0\}$$

is smooth (with a Thom transversality condition).

We apply proposition to Lie groupoid $\mathrm{DNC}\left(\mathcal{G}(M,F), \mathrm{Crit}_F(f)\right)$ to obtain $d + d^* + c(\alpha)$ has compact resolvent.

Given a holonomy invariant transverse measure (Ruelle-Sullivan current), we recover Connes and Fack Morse inequalities (2000).

Inhomogeneous pseudo-differential calculus

The Lie algebra of the Heisenberg group

$$\mathbb{H} = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \},$$

is generated by $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, $Z = \frac{\partial}{\partial z}$ with the relation

$$[X, Y] = Z, \ [X, Z] = [Y, Z] = 0.$$

The operator $X^2 + Y^2$ is hypoelliptic.

Parametrix leads to inhomogeneous pseudo-differential calculus (70s).

Inhomogeneous pseudo-differential calculus

Let $H \subseteq TM$ be a subbundle. The map

$$\begin{aligned} \Gamma(H) \times \Gamma(H) &\to \Gamma(TM/H) \\ (X, Y) &\to [X, Y] \mod H \end{aligned}$$

is $C^\infty(M)$-bilinear. Using the bilinear map, for every $x \in M$, the space $H_x \oplus T_xM/H_x$ is equipped with the structure of a nilpotent Lie group.

The algebra of differential operators is filtered by declaring a vector field X to be of order 1 if $X \in \Gamma(H)$ and 2 otherwise. The principal symbol is a 'function' on $H \oplus TM/H$.

Parametrices lead to an associated pseudo-differential calculus.

Groupoid description of pseudo-differential calculus

Debord and Skandalis (2013) gave an elementary description of pseudo-differential operators using the tangent groupoid

$$G = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

More precisely they defined a class of Schwartz functions on G such that if f is Schwartz and $k \in \mathbb{C}$, then

$$\int_0^{+\infty} f(x, y, t) t^{-k} \frac{dt}{t}$$

is a classical pseudo-differential operator of order k. Moreover all pseudo-differential operators arise this way.

Groupoid description of pseudo-differential calculus

Van-Erp and Yuncken generalised this approach to the inhomogeneous pseudo-differential calculus using the following Lie groupoid (defined independently by Van-Erp and Ponge 2006)

$$M \times M \times \mathbb{R}^* \sqcup H \oplus TM/H \times \{0\}.$$

The groupoid structure on $M \times M \times \mathbb{R}^*$ is that of pair groupoid, and on $H \oplus TM/H \times \{0\}$ is that of the fiber bundle of nilpotent Lie groups.

Construction of inhomogeneous deformation groupoid

One has

$$\begin{aligned} H \times \{0\} \subseteq TM \times \{0\} \subseteq & \, \mathrm{DNC}(M \times M, M) \\ & = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \end{aligned}$$

is a subgroupoid. Hence

$$\mathrm{DNC}(\mathrm{DNC}(M \times M, M), H \times \{0\}) \rightrightarrows M \times \mathbb{R}^2$$

is a Lie groupoid. Its restriction to $M \times \{1\} \times \mathbb{R}$ is the required groupoid.

More general case of a filtration

The inhomogeneous pseudo-differential calculus as well as the associated Lie groupoid are defined more generally starting from a filtration

$$H^1 \subseteq H^2 \subseteq \cdots \subseteq H^r \subseteq H^{r+1} = TM$$

such that

$$[X, Y] \in \Gamma(H^{i+j}), \quad \forall X \in \Gamma(H^i), Y \in \Gamma(H^j).$$

See the work of Choi and Ponge (2015) and Van-Erp and Yunken (2015)

More general case of a filtration

In the case where we have two bundles $H^1 \subseteq H^2 \subseteq TM$, we regard

$$\begin{aligned} H^1 \oplus H^2/H^1 \times \{0\} &\subseteq H^1 \oplus TM/H^1 \times \{0\} \\ &\subseteq M \times M \times \mathbb{R}^* \sqcup H^1 \oplus TM/H^1 \times \{0\}. \end{aligned}$$

This is a subgroupoid precisely because $[H^1, H^1] \subseteq H^2$. Hence

$$\mathrm{DNC}(M \times M \times \mathbb{R}^* \sqcup H^1 \oplus TM/H^1 \times \{0\}, H^1 \oplus H^2/H^1 \times \{0\}) \rightrightarrows M \times \mathbb{R}^2$$

is a Lie groupoid. Its restriction to $M \times \{1\} \times \mathbb{R}$ is the required groupoid.

Chern-Simons invariants

On $GL_n(\mathbb{C})$, there exists $GL_n(\mathbb{C})$-invariant closed forms of degree ($2i-1$ for $1 \leq i \leq n$)

$$\Phi_{2i-1}(M_1, \ldots, M_{2i-1}) = \sum_{\sigma \in \mathfrak{S}_{2i-1}} \epsilon(\sigma) \operatorname{Tr}(M_{\sigma(1)} \ldots M_{\sigma(2i-1)}).$$

Let M be a smooth manifold, $\phi : \pi_1(M) \to GL_n(\mathbb{C})$ representation of the fundamental group, $f : \tilde{M} \to GL_n(\mathbb{C})$ smooth map such that

$$f(x\gamma) = \phi(\gamma)^{-1} f(x).$$

The Chern-Simons invariants (1974) is then the De Rham class of $f^*\Phi_{2i-1} \in H^{2i-1}(M, \mathbb{C})$.

The α-invariant

Atiyah, Patodi and Singer (1976) transported this invariant to K-theory:

$$\alpha_{\phi,f} = \mathrm{Ch}^{-1}\left(\sum_i f^*\Phi_{2i-1}\right) \in K^1(M,\mathbb{C}).$$

The goal is to find an instrinsic definition of $\alpha_{\phi,f}$.

We restrict ourselves to representations of U_n.

K-theory with coefficients in $\mathbb{R}$

Definition (Antonini, Azzali, Skandalis 2013)

Let A be a C^*-algebra. Then

$$K_*^{\mathbb{R}}(A) = \varinjlim K_*(A \otimes D)$$

where D is a unital C^*-algebra equipped with a tracial state.

$K_*^{\mathbb{R}}(C(X)) = K^*(X) \otimes \mathbb{R}$ for a compact space X.

Intrinsic definition of α-invariant.

Let $\phi : \pi_1(M) \to U_n$ be a representation, $A = C(U_n) \rtimes_\phi \pi_1(M)$, $W = \tilde{M} \times_{\pi_1(M)} A$.

Let V be the vector bundle $\tilde{M} \times_\phi \mathbb{C}^n$. There exists a *flat* vector bundle isomorphism

$$V \otimes W \xrightarrow{T} \mathbb{C}^n \otimes W.$$

Theorem (Antonini, Azzali, Skandalis 2013)

The cycle

$$[V \otimes W \xrightarrow{T} \mathbb{C}^n \otimes W \xrightarrow{f \otimes Id} V \otimes W] \in K_1(C(M) \otimes A)$$

in $K_1^{\mathbb{R}}(C(M))$ *is equal to* $\alpha_{\phi,f}$.

Intrinsic definition of α-invariant.

Using their construction, we construct an element $\tilde{\alpha} \in KK^{1,\mathbb{R}}_{U_n \rtimes U_n^\delta}$-theory which satisfies the following: when the map $f : \tilde{M} \to U_n$ is seen as a cocycle from the trivial Lie groupoid M to $U_n \rtimes U_n^\delta$, then

$$f^* \tilde{\alpha} = \alpha_{\phi, f}$$

Thanks for your attention