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dirigée par Georges Skandalis

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To My Parents

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Résumé

Cette thèse est consacrée à l'étude de trois questions différentes concernant les groupoïdes de Lie et leurs applications.

Le premier chapitre présente quelques préliminaires sur les groupoïdes de Lie.

Dans le chapitre 2, on exprime la déformation de Witten à l'aide d'une déformation au cone normal et la théorie de C^*-modules ce qui nous permet de retrouver les inégalités de Morse. Notre méthode se généralise au cas des feuilletages.

Dans le chapitre 3, on donne une construction simple du groupoïde de déformation construit par Choi-Pönge et Van Erp-Yuncken. Rappelons que celui-ci décrit le calcule pseudo-différentiel inhomogéne grâce au travail de Debord-Skandalis et Van Erp-Yuncken. Notre construction montre que le groupoïde de déformation est en fait une déformation au cone normal classique itérée.

Dans le chapitre 4, suivant le travail de Antonini, Azzali et Skandalis, on construit un élément en KK-théorie équivariante qui permet d'exprimer directement les invariants de Chern-Simons en K-théorie.

Dans l'appendice on donne quelques rappels sur la KK-théorie équivariante et la KK-théorie réelle introduite par Antonini, Azzali et Skandalis.

Mots-clés

Groupoïdes de Lie, déformation au cone normal, déformation de Witten, fonctions de Morse, calcul pseudo-différentiel inhomogène, KK-théorie, invariants de Chern-Simons.

Deformation groupoids and applications

Abstract

This thesis is devoted to the study of three different questions concerning Lie groupoids and their applications.

The first chapter presents some preliminaries on Lie groupoids.

In Chapter 2, Witten's deformation is expressed using deformation to the normal cone construction and the theory of C^*-modules, which allows us to reprove the Morse inequalities. Our method is generalised to the case of foliations.

In Chapter 3, we give a simple construction of the deformation groupoid built by Choi-Pönge and Van Erp-Yuncken. Recall that this groupoid describes the inhomogeneous pseudo-differential calculus thanks to the work of Debord-Skandalis and Van Erp-Yuncken. Our construction shows that the deformation groupoid is actually an iterated classical deformation to the normal cone.

In Chapter 4, following the work of Antonini, Azzali and Skandalis, we construct an element in equivariant KK-theory that allows us to express the Chern-Simons invariants directly in K-theory.

In the appendix we give some reminders about the equivariant KK-theory and the real KK-theory introduced by Antonini, Azzali and Skandalis.

Keywords

Lie groupoids, deformation to normal cone, Witten deformation, Morse functions, inhomogeneous pseudo-differential calculus, KK-theory, Chern-Simons invariants.

Contents

Introduction

Thanks to a theorem of Gelfand, noncommutative C^*-algebras can be regarded as noncommutative locally compact spaces. Noncommutative geometry was introduced by Connes out of attempts to generalise tools and results from algebraic topology, differential geometry, Riemannian geometry and global analysis to some noncommutative C^*-algebras that can be thought of as noncommutative manifolds.

Lie groupoids. Examples of noncommutative manifolds arise naturally from Lie groupoids, which were introduced by Ehresmann [41]. Associated to Lie groupoids, Connes defines a, usually noncommutative, C^*-algebra (see also Renault [84] for the general case of locally compact groupoids). The C^*-algebra is the completion of the algebra of smooth functions on the Lie groupoid with convolution as the product law. This construction generalises various previous classical constructions like the C^*-algebra of Lie groups.

In [79, 80, 78], Pradines defined a Lie groupoid associated to a foliated manifold. Its C^*-algebra should be regarded as the algebra of continuous functions on the quotient space, an ill-defined space in general.

K-theory and index theory. Among (co)-homology theories, topological K-theory was easily extended to the noncommutative setting.

In the '70s, it was clear to Atiyah and his collaborators that the celeberated Atiyah-Singer index theorem [7] is a Poincaré duality statement for K-theory. In [9], Atiyah defines the K-homology of a compact manifold M using Hilbert spaces, Fredholm operators and representations of the commutative C^*-algebra $C(M)$ of continuous functions on M, and using this definition he proves that an elliptic (pseudo-)differential operator defines naturally an element in the K-homology of M. Atiyah's definition immediately extends to the noncommutative world and led Kasparov [59, 60] to define KK-theory, a far reaching bi-variant generalisation of both K-theory and K-homology.

In [24, 28], Connes generalised the Atiyah-Singer index theorem to foliated manifolds. He discovered that Lie groupoids played a fundamental role in the comprehension of the index theorem in this setting. More precisely, a pseudo-differential operator has a Schwartz kernel. The Schwartz kernel of the composition of two operators is then the convolution of the respective Shcwartz kernel of each one. The Lie groupoid of Pradines captures the Schwartz kernel of pseudo-differential operators which act longitudinally along the leaves of a foliation. Using this Lie groupoid, Connes (and later with Skandalis [27]) defined the analytic index of a longitudinal pseudo-differential operator which is longitudinally elliptic as an element of the K-theory of the C^*-algebra of the Lie groupoid of the foliation. Later on with Skandalis, they defined the topological index and proved its equality with the analytic one, generalising Atiyah-Singer index theorem for families [8].

Deformation groupoids. In the late '80s, Connes [30, chapter 2] defined a new Lie groupoid called the tangent groupoid, which combines pseudo-differential operators with their symbols, and using it, he gave a simple conceptual proof of Atiyah-Singer index theorem in the case of closed manifolds. His construction and ideas were later extended and used by

1. Hilsum and Skandalis [49] to define the shriek maps in KK-theory for maps between spaces of leaves.

2. Monthubert and Pierrot [69], and Nistor, Weinstein and Xu [72] when they generalised the tangent groupoid construction [1] and proved its relation to the analytic index of pseudo differential operators following A. Connes.

3. Debord and Skandalis [37] in a very general setting in which they show by the functoriality of the deformation to the normal cone (DNC) construction that the deformation to the normal cone of a Lie groupoid along a Lie subgroupoid is naturally a Lie groupoid.

This thesis deals with Lie groupoids and their applications. The first three chapters use deformation groupoids. In chapter 2, an application of deformation groupoids towards Witten deformation is given. In chapter 3, the Heisenberg deformation defined to capture the nonhomogeneous pseudo-differential calculus is shown to be a special case of the deformation to the normal cone. Chapter 4 uses Le Gall's

[1] in the notation used here, they defined the groupoid $\mathrm{DNC}(G, G^0)$

[63] equivariant KK-theory, generalising Kasparov's [60] equivariant KK-theory. A primitive description, as an element in the equivariant KK-theory, of the α-invariant defined by Atiyah-Patodi-Singer [6] is given.

This thesis is divided into 4 chapters and 2 appendices.

Chapter 1. In this chapter, we recall the definition of a Lie groupoid (see [67] for more details), the C^* algebra of a Lie groupoid (see [84] for more details), the pseudo-differential calculus associated to a Lie groupoid (see [36, 65, 66, 101, 100] for more details), De Rham and Laplace operators on a Lie groupoid, the deformation to the normal cone construction following [37]. Iterated deformation to the normal cone construction is also introduced. Finally, we extend a result of Chernoff [19] to Lie groupoids. This result was previously known in some particular cases by Hilsum [48] and Vassout [101]. See also the work of Roe [87, 86, 85].

Chapter 2. In this chapter, we give an application of deformation groupoids to Witten's deformation of a Morse function. A Morse function f is a real valued smooth function on a compact manifold M with nondegenerate critical points. This is a generic condition by results of Morse. In [70], Morse proved the so called Morse inequalities highlighting a relation between the number of critical points of f and the Betti numbers $\dim(H^\cdot(M))$. He did so by studying the level sets $f^{-1}(]-\infty,a])$ and seeing how they change as a passes by a critical value. In [104], Witten proposed an analytic way to prove Morse inequalities. His method consists of deforming the De Rham operator d to become $d_t = e^{-\frac{f}{t}} d e^{\frac{f}{t}}$, and then studying the associated Laplacian $\Delta_t = (d_t + d_t^*)^2$. Since the operator d_t is conjugate to d, it follows that $\ker(\Delta_t)$ is isomorphic to $\ker(\Delta)$. Hence by Hodge theory, $\dim(\ker(\Delta_t^i)) = \dim(H^i(M,\mathbb{R}))$ for all $t > 0$, where Δ_t^i denotes the Laplacian acting on forms of degree i. He then proves that, as $t \to 0^+$, the spectrum $sp(\Delta_t^i)$ gets separated into two parts, the first part is finite and consists of an eigenvalue for each critical point of f of index i, and the second part consists of eigenvalues which converge to $+\infty$. Morse inequalities are then corollaries of this decomposition.

We apply the deformation to the normal cone construction to obtain a smooth manifold whose underlying set is equal to

$$\mathrm{DNC}(M, \mathrm{Crit}(f)) = M \times]0, +\infty[\coprod_{a \in \mathrm{Crit}(f)} T_a M \times \{0\}.$$

The natural projection $\pi_{\mathbb{R}} : \mathrm{DNC}(M, \mathrm{Crit}(f)) \to \mathbb{R}$ is a submersion, hence the fibers define a (rather trivial) foliation. Using Connes [24] approach to indices of elliptic operators along the leaves, Kasparov's KK-theory [60], more precisely the Baaj-Julg [10] formalism, we deduce Witten's theorem on the decomposition of the spectrum of the Laplacian Δ_t^i as a corollary of the construction of a regular operator on $\mathrm{DNC}(M, \mathrm{Crit}(f))$ with compact resolvent (as an operator on a C^*-module). In fact we prove the following

Theorem 0.1. *Let*

$$\lambda_1^p(t) \leq \lambda_2^p(t) \cdots$$

denote the spectrum of Δ_t^p, then for every $i \in \mathbb{N}$,

$$\lim_{t \to 0^+} t\lambda_i^p(t) = \lambda_i^p(0),$$

where $\lambda_i^p(0)$ is the i'th eigenvalue of harmonic oscillator

$$\Delta_0^p := \bigoplus_{a \in \mathrm{Crit}(f)} \big(d + d^* + c(d_a^2(f))\big)^2 : \bigoplus_{a \in \mathrm{Crit}(f)} L^2(\Lambda_{\mathbb{C}}^p T_a M) \to \bigoplus_{a \in \mathrm{Crit}(f)} L^2(\Lambda_{\mathbb{C}}^p T_a M),$$

where $L^2(\Lambda_{\mathbb{C}}^p T_a M)$ is the set of all L^2 functions from $T_a M$ to $\Lambda_{\mathbb{C}}^p T_a M$, $d_a^2 f$ is the 1*-differential form on $T_a M$, and c is the Clifford multiplication.*

The small eigenvalues of Δ_t^p correspond to the 0 eigenvalue of Δ_0^p which correspond to critical points of f of index p.

Our methods rely only on C^*-algebraic methods, so called soft analysis. In particular, without any extra difficulty we extend the previous theorem to the case of foliations.

Chapter 3. In order to construct a parametrix for Hörmander's [51] subelliptic operators on a contact manifold, Folland and Stein [43, 42] defined a non-commutative pseudo-differential calculus where the principal cosymbol is a function on a bundle of Heisenberg groups. A fundamental characteristic of this pseudo-differential calculus is that a vector field defines a differential operator of order 1 if it is everywhere tangent to the contact subbundle and of order 2 if not. Later on, this was generalised to an arbitrary subbundle of the tangent bundle, and even further to a filtration of the tangent bundle under conditions on the Lie bracket (see [17, 16, 39, 33, 23, 94, 40, 13, 47, 88]) To such a structure one associates a bundle of graded nilpotent Lie groups over which the cosymbols are functions. Let us remark that the general situation is more involved because the bundle of graded nilpotent Lie groups doesn't need to be locally trivial and hence the analogue of the theorem of Darboux doesn't hold in general.

This calculus was later used by many authors, for instance by Connes and Moscovici [26, 25] to define a transversal signature operator on foliated manifolds and do computations in cyclic cohomology, following a construction of Hilsum and Skandalis [49], by Julg and Kasparov [55] to compute the $SU(n,1)$ equivariant KK-theory following the work of Rumin [89].

In [36], Debord and Skandalis showed how to recover the classical pseudo-differential calculus thanks to the tangent groupoid. In [76, 95], Ponge and van-Erp independentaly define a deformation groupoid for a contact manifold. van Erp and Yuncken [98] used this groupoid to give an alternate presentation of the pseudo-differential calculus mentioned above. This groupoid was also used by van Erp [96, 95] and later (with Baum [12]) to formulate and prove an index formula in the same spirit as that of Atiyah-Singer. Their index theorem is for differential operators whose cosymbol is invertible in the above calculus associated to a contact structure. These operators are necessarily hypoelliptic, hence their analytic index is well defined but they are rarely elliptic.

The groupoid defined by Ponge and van-Erp was later extended by Choi and Ponge [20, 22, 21], and independently by van Erp and Yuncken [99] following work by Julg and van Erp [56]. This extension was also used by van Erp [97] to formulate and prove an index theorem for hypoelliptic operators on foliated manifolds.

In this chapter, we prove that the deformation groupoids defined in [76, 95, 77, 20, 22, 21, 99] are special cases of the deformation to the normal cone construction. Let $H \subseteq TM$ be a vector bundle. Recall the tangent groupoid

$$\mathrm{DNC}(M \times M, M) = M \times M \times \mathbb{R}^* \coprod TM \times \{0\} \rightrightarrows M \times \mathbb{R}$$

defined by Connes. The space $H \times \{0\} \subseteq TM \times \{0\} \subseteq \mathrm{DNC}(M \times M, M)$ is a Lie subgroupoid. Hence by the naturality of the DNC construction, the space

$$\mathrm{DNC}(\mathrm{DNC}(M \times M, M), H \times \{0\}) \rightrightarrows \mathrm{DNC}(M \times \mathbb{R}, M \times \{0\}) = M \times \mathbb{R}^2$$

is a Lie groupoid. We prove that the fiber over $M \times \{1\} \times \mathbb{R}$ is the Heisenberg Lie groupoid. Furthermore the groupoid $\mathrm{DNC}(\mathrm{DNC}(M\times M, M), H\times\{0\}) \rightrightarrows M\times\mathbb{R}^2$ is a quite natural object to study because it contains 'the deformations in all the directions'. In Section 3.4, we show that the general case (replacing H by a filtartion of TM) is just an iterated deformation to the normal cone construction. Our approach gives rise to noncommutative Lie groupoids/symbols precisely because we deform Lie groupoids with respect to subgroupoids and not with respect to spaces, and contrary to the methods used in [76, 95, 77, 20, 22, 21, 99] no analysis on local coordinates is needed to construct the Lie groupoid, only functoriality of the DNC construction.

The methods developed here can be used to give a variety of examples of Lie groupoids. In particular we extend the Heisenberg Lie groupoid to cover the case of transverse (to a foliation) hypoelliptic pseudo-differential calculus without any difficulty (examples 3.10 and 3.14).

Chapter 4. The fourth chapter is independent of the other three chapters. It is on the Chern-Simons invariants in KK-theory. In [18], Chern and Simons defined invariants associated to a flat vector bundle over a compact connected smooth manifold. Their invariants were originally defined as differential forms and hence as elements in the De Rham cohomology.

Atiyah, Patodi, and Singer [4, 5, 6] in their celebrated articles highlighted the connection between the Chern-Simons invariants and index theory. They transported the Chern-Simons invariants to K-theory. To this end, they defined the K-theory with coefficients in $\mathbb{C}/\mathbb{Z}$, and then using Atiyah-Hirzebruch theorem on the bijectivity of the Chern character they transported the Chern-Simons invariants to K-theory. The resulting element is the so-called α-invariant of a flat vector bundle or equivalently of the holonomy representation of the fundamental group. The pairing (Kasparov product) of the α-invariant with the class of a Dirac operator $[D] \in KK^1(M,\mathbb{C})$ gives the η-invariant as proved in Atiyah, Patodi, and Singer [4, 5, 6].

The α-invariant lives in the K-theory of the underlying manifold with coefficients in $\mathbb{C}/\mathbb{Z}$. If V is a flat vector bundle associated to a representation of the fundamental group of a compact manifold M, then the Atiyah-Hirzebruch theorem implies that the element $[V]-[\mathbb{C}^{\dim(V)}]$ in $K^0(M)$ is torsion. A property of the α-invariant is that its boundary under Bockstein homomorphism is equal to $[V]-[\mathbb{C}^{\dim(V)}]$.

Closely related, and in a sense more primitive invariants are the relative Chern-Simons invariants and the relative α invariant which are defined respectively in the De Rham cohomology with coefficients in $\mathbb{C}$ and in the K-theory with coefficients in $\mathbb{C}$. These invariants are defined for flat vector bundles which are equipped with a trivialisation. The relation between the two is that when one takes the relative invariant modulo $\mathbb{Z}$, then the choice of a trivialisation disappears, and the relative invariant becomes the usual invariant.

When the holonomy representation is unitary, all the different invariants stated above become either in $\mathbb{R}$ or $\mathbb{R}/\mathbb{Z}$. In chapter 4, we restrict ourselves to the relative α-invariant of trivialised unitary flat vector bundles.

It was suggested in [6] that the α-invariant should have an intrinsic K-theoretical definition that uses the theory of Von Neuman algebras of type II. This motivated research in this direction by many authors, see for example [2, 11, 38, 57], etc ...

We continue this line of research by constructing a universal classifying element in the KK-theory of the classifying space of trivialised unitary flat vector bundles. An element directly defined in KK-theory without passing through De Rham cohomology might shed some light on the interaction between Chern-Simons invariants and KK-theory.

We follow Antonini, Azzali, Skandalis [2] definition of KK-theory with real coefficients. By using their work on the α-inavraints [2], we construct an element in $KK^1_{U_n\rtimes U_n^\delta,\mathbb{R}}(C(U_n),C(U_n))$ which when pulled back by the classifying map (seen as a generalised homomorphism in the sense of Hilsum-Skandalis [49]) of a trivialised unitary flat vector bundle $f : M \to U_n \rtimes U_n^\delta$ gives the relative α-invariant.

Appendix A Some basic facts on regular operators on C^*-modules are recalled. Some of the results are stated without proof; references to Lance's book [62] are then given, some others were given in a master course by Skandalis, and their proofs are written for the sake of completeness. Propositions A.12 and A.11 are used in chapter 1 and 2 respectively.

Appendix B In this appendix we recall the definition of real KK-theory given by Antonini, Azzali, and Skandalis [2]. Some results on KK-theory are also stated that are used in chapter 4. We refer the reader to [63, 62, 93, 14] for more details on KK-theory.

Chapter 1

Groupoids; a short introduction

The connection between Lie groupoids and pseudo-differential operators and index theory was exploited by Connes [28, 30, 24, 29] who defined a pseudo-differential calculus associated to a Lie groupoid, then showed how the analytic index of an elliptic operator in this calculus is naturally an element in the K theory of the C^*-algebra of the Lie groupoid (see Connes [24] and Renault [84]).

In this chapter we recall some results in the theory of Lie groupoids and recent developments from the point of view of noncommutative geometry.

In Section 1.1, we recall the notion of Lie groupoids, Lie algebroids, $\mathcal{VB}$ groupoids, Morita equivalence of Lie groupoids

In Section 1.2, we recall the construction of C^*-algebras of Lie groupoids.

In Section 1.4, we recall the deformation to the normal cone construction following [37]. This is the main construction that will be used in Chapter 2 and 3.

In Section 1.5, the deformation to normal cone construction is iterated.

In Section 1.6, the definition of pseudo-differential operators on Lie groupoids is recalled. A result of Chernoff [19] on the self adjointness of first order differential operators is stated. This result was extended to Lie groupoids of foliations by Hislum [48], and to Lie groupoids whose base is compact by Vassout [101]. See also the work of Roe [87, 86, 85]. We extend it to arbitrary Lie groupoids.

In Section 1.7, the construction of the De Rham operator and the Laplacian on Lie groupoids is recalled.

In this thesis, the following conventions will be used:

- If $V \subseteq M$ is a smooth submanifold, then N_V^M denotes the normal bundle.
- If $E \to M$ is a vector bundle on a smooth manifold, then $\Gamma(E)$, $\Gamma_c(E)$, $\Gamma^\infty(E)$, $\Gamma_c^\infty(E)$ denote respectively the space of C^0 sections, C^0 sections with compact

support, C^∞ sections, C^∞ sections with compact support.

1.1 Lie groupoids and Lie algebroids

Definition 1.1. A *groupoid* is a small category whose morphisms are invertible. To fix notation, we will denote by

- G the set of morphisms;
- G^0 the set of objects. We will always identify G^0 with the subset of G consisting of identity morphisms.
- $s : G \to G^0$ the source map;
- $r : G \to G^0$ the range map;
- G^x will denote the set $r^{-1}(x)$ for $x \in G^0$.
- G_x will denote the set $s^{-1}(x)$ for $x \in G^0$.
- $R_\gamma : G_{r(\gamma)} \to G_{s(\gamma)}$ the right multiplication by $\gamma \in G$.
- $L_\gamma : G^{s(\gamma)} \to G^{r(\gamma)}$ the left multiplication by $\gamma \in G$.

A *Lie groupoid* is a groupoid G such that its set of morphisms is endowed with a smooth structure such that

- G^0 is an embedded smooth submanifold of G;
- the map s is a submersion;
- the inverse map $\gamma \to \gamma^{-1}$ is smooth;
- the product map $G \times_{s,r} G \to G$ is smooth.

We will by abuse of language call $G \rightrightarrows G^0$ the Lie groupoid. If G, H are Lie groupoids, then a groupoid morphism is a smooth function $f : G \to H$ which is a functor.

The definition of a Lie groupoid is originally due to Ehresmann [41]. The definition used here, which is also the most used definition of a Lie groupoid, is due to Pradines [81, 80, 79, 78]. Other definitions exist as well. For a more detailed discussion about the different definitions of a Lie groupoid and the relation between them see [67, section III.1].

Remark 1.2. The manifold G^0 is a always assumed Hausdorff second countable smooth manifold. On the other hand, examples of Lie groupoids where G is only a locally Hausdorff smooth manifold are considered. See [24, section 6] for more details.

Definition 1.3. A *Lie algebroid* on a smooth manifold M is a vector bundle $E \to M$ together with a smooth bundle of linear maps $\natural : E \to TM$ and a Lie algebra structure on the vector space of smooth sections $\Gamma^\infty(E)$ such that if $X, Y \in \Gamma^\infty(E)$ and $f \in C^\infty(M)$, then

$$[X, fY] = f[X, Y] + \natural(X)(f)Y.$$

By regarding $[[X, Y], fZ]$, it follows that if $X, Y \in \Gamma^\infty(E)$, then $\natural([X, Y]) = [\natural(X), \natural(Y)]$.

If G is a Lie groupoid, then the normal bundle $N_{G^0}^G \to G^0$ is naturally endowed with the structure of a Lie algebroid on G^0, and is called the Lie algebroid of G. See [67, section III.3] for more details.

Examples 1.4. Let M be a smooth manifold. The following are examples of Lie groupoids:

1. The trivial Lie groupoid where $G = G^0 = M$;

2. The pair groupoid $M \times M \rightrightarrows M$, where $s(y, x) = x$, $r(y, x) = y$. Its Lie algebroid can be identified with TM such that the anchor $\natural$ is the identity;

3. If G^0 is a point, then a Lie groupoid is simply a Lie group whose Lie algebroid is its Lie algebra;

4. Let G and H be Lie groupoids. The product $G \times H \rightrightarrows G^0 \times H^0$ is naturally a Lie groupoid.

5. Let H be a Lie group acting on M by the right. The crossed product groupoid

$$M \rtimes H = \{(y, h, x) \in M \times H \times M : yh = x\} \rightrightarrows M$$

is a Lie groupoid, where $s(y, h, x) = x$, $r(y, h, x) = y$ and $(z, h, y) \cdot (y, h', x) = (z, hh', x)$.

6. A smooth vector bundle $V \xrightarrow{\pi} M$ can be regarded as a Lie groupoid where $G = V$, $G^0 = M$, $s = r = \pi$ and $v \cdot v' = v + v'$. Its Lie algebroid is equal to V. The anchor map is 0, and the Lie bracket is zero.

7. Let $F \subseteq TM$ be an involutive subundle (by Frobenius theorem, a regular foliation). The monodromy groupoid

$$\mathrm{Mond}(M, F) = \{(y, [\gamma]_m, x) : x, y \in M\} \rightrightarrows M$$

   where γ is a smooth leafwise path from x to y and $[\gamma]_m$ is the leafwise homotopy class of γ. The *holonomy groupoid* $\mathcal{G}(M, F) = \{(y, [\gamma], x)\} \rightrightarrows M$ where $[\gamma]$ is the class of γ up to holonomy. The Lie algebroid of the two Lie groupoids is equal to F and $\natural$ is the inclusion.

   One can prove that the monodromy groupoid and the holonomy groupoid are the 'largest' and the 'smallest' groupoid respectively whose Lie algebroid is equal to F (See [32, proposition 1]). See also [24, section 5].

8. A Thom-Mather stratified manifold gives rise to a Lie groupoid [35].

A fundamental difference between Lie groups and Lie groupoids is the failure of Lie's third theorem. A Lie algebroid E is called integrable if there exists a Lie groupoid whose Lie algebraoid is isomorphic to E. There exist Lie algebroids which aren't integrable. Crainic and Fernandes found the necessary and sufficient conditions for the integrability of Lie algebroids [31]. The following integrability result due to Debord is often useful in applications.

Theorem 1.5 ([34])**.** *If the anchor map is injective on a dense subset, then the Lie algebroid is integrable.*

Definition 1.6. A Lie subgroupoid of a Lie groupoid G is a Lie groupoid H such that

1. H (respectively H^0) is a submanifold of G (respectively G^0),

2. The source map, range map and multiplication map of H are the restriction of those of G. In other words, H is a subcategory of G.

Let us recall the notion of a $\mathcal{VB}$-groupoid from [82, 67].

Definition 1.7. Let H be a Lie groupoid. A $\mathcal{VB}$-groupoid over H is given by

- a vector bundle G over H
- a vector bundle G^0 over H^0

- a Lie groupoid structure on G whose space of objects is G^0, such that the map range map $r : G \to G^0$, the inverse map $i : G \to G$, the multiplication map $m : G \times_{s,r} G \to G$ are respectively bundle maps over the range map $r : H \to H^0$, the inverse map $i : H \to H$ and the multiplication map $m : H \times_{s,r} H \to H$.

By abuse of notation we will call $G \rightrightarrows G^0$ a $\mathcal{VB}$-groupoid over H.

A $\mathcal{VB}$-subgroupoid of G is a Lie subgroupoid $K \rightrightarrows K^0$ of G such that K is a subbundle of G and K^0 is a subbundle of G^0.

Example 1.8. Let $E \to M$ be a vector bundle, $F \subseteq E$ a subbundle. The groupoid

$$E \rtimes F = \{(e, e') \in E \oplus E : e - e' \in F\} \rightrightarrows E$$

is a $\mathcal{VB}$-groupoid over M. Furthermore the only $\mathcal{VB}$-subgroupoids of $E \rtimes F$ are $E' \rtimes F'$ where $E' \subseteq E$, $F' \subseteq F \cap E'$ are vector subbundles.

We recall the notion of Morita equivalence for Lie groupoids.

Definition 1.9. A smooth Morita equivalence between two Lie groupoids G and H is a smooth manifold X, and two smooth submersions $p : X \to G^0$, $q : X \to H^0$, and two smooth maps

$$X \times_{p,r} G \to X, \quad H \times_{s,q} X \to X$$
$$(x, g) \to xg, \quad (h, x) \to hx$$

such that

1. If g (respectively h) is an identity, then $xg = x$ (respectively $hx = x$).
2. If $(x, g) \in X \times_{p,r} G$ (respectively $(h, x) \in H \times_{s,q} X$), then $p(xg) = s(g)$ (respectively $q(hx) = r(h)$). Furthermore if $g' \in G^{s(g)}$ (respectively $h' \in H_{r(h)}$), then $(xg)g' = x(gg')$ (respectively $h'(hx) = (h'h)x$).
3. The two actions commute, that is if $x \in X$, $g \in G^{p(x)}$, $h \in H_{q(x)}$, then $q(xg) = q(x)$ and $p(hx) = p(x)$ and $h(xg) = (hx)g$.
4. The map
$$H \times_{s,q} X \to X \times_p X, \quad (h, x) \to (hx, x)$$
is a diffeomorphism, and similarly
$$X \times_{p,r} G \to X \times_q X, \quad (x, g) \to (xg, x)$$

is a diffeomorphism.

Notice that the previous definition could be equivalently formulated as the existence of a Lie groupoid structure on $G \sqcup X \sqcup X^{-1} \sqcup H \rightrightarrows G^0 \sqcup H^0$ such that

1. G and H are Lie subgroupoids
2. Every element in X has a source in G^0 and a range in H^0
3. G^0 and H^0 meet all the orbits.

A classical example of a Morita equivalence is the following: if G is a Lie group acting freely and properly on a smooth manifold M, then the manifold M defines a Morita equivalence between the trivial Lie groupoid M/G and the crossed product Lie groupoid $M \rtimes G$.

Quotient of Lie groupoids. Let $G \rightrightarrows G^0$ be a Lie groupoid, $H \subseteq G$ a Lie subgroupoid. The Lie groupoid H acts on the smooth manifold G_{H^0} by right translation. This action is clearly free and proper. Hence by [15, section 5.9.5], the quotient space G_{H^0}/H is a smooth manifold, that will be denoted by G/H.

1.2 C^*-algebra of Lie groupoid

Definition 1.10. Let V be a real finite dimensional vector space, $\alpha \in]0, +\infty[$. An α-density on V is a map $f : \Lambda^{\dim(V)}V \to \mathbb{C}$ such that for every $\lambda \in \mathbb{R}$, $v \in \Lambda^{\dim(V)}V$ one has $f(\lambda v) = |\lambda|^\alpha f(v)$. The space of α-densities is a 1-dimensional complex vector space denoted by $|\Lambda|^\alpha V^*$.

If E is a real vector bundle, then the bundle of vector spaces $x \to |\Lambda|^\alpha E_x^*$ is naturally endowed with the structure of a vector bundle that will be denoted by $|\Lambda|^\alpha E^*$.

Let M be a smooth manifold. One defines the Hilbert space $L^2 M$ without choosing a measure by defining $L^2 M$ as the completion of $\Gamma_c(|\Lambda|^{\frac{1}{2}} T^*M)$ with respect to the inner product $\langle f, g \rangle = \int \bar{f} g$.

Definition 1.11. The bundle of half-densities on a Lie groupoid $G \rightrightarrows G^0$ is the line bundle $\mathcal{D}G_\gamma = |\Lambda|^{\frac{1}{2}} \mathfrak{A}G^*_{s(\gamma)} \otimes |\Lambda|^{\frac{1}{2}} \mathfrak{A}G^*_{r(\gamma)}$, where $\gamma \in G$.

Let $f, g \in \Gamma_c(\mathcal{D}G)$. The following formulas have a natural meaning and turn $\Gamma_c(\mathcal{D}G)$ into a $*$-algebra.

$$f \star g(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2),\ f^*(\gamma) = \overline{f(\gamma^{-1})}$$

Definition 1.12. A representation of a Lie groupoid $G \rightrightarrows G^0$ on a complex Hilbert space H is a $*$-algebra homomorphism $\phi : \Gamma_c(\mathcal{D}G) \to B(H)$ which is continuous with respect to the Frechet topology on $\Gamma_c(\mathcal{D}G)$ and the strong topology on $B(H)$.

Example 1.13. Let $G \rightrightarrows G^0$ be a Lie groupoid, $x_0 \in G^0$. The regular representation of G (at x_0) is the Hilbert space $L^2(G_{x_0})$ with the action given by

$$f \cdot \xi(\gamma) = \int_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\xi(\gamma_2),$$

where $f \in \Gamma_c(\mathcal{D}G)$, $\xi \in L^2(G_{x_0})$, $\gamma \in G_{x_0}$. This action is easily seen to be continuous.

A theorem by Renault [84] (see also [1]) says that representations of G could be equivalently defined as $*$-algebra homomorphism $\phi : \Gamma_c(\mathcal{D}G) \to B(H)$ which are bounded with respect to an L^1-norm of the following type

$$\|f\|_1 := \sup_{x\in G^0} \max\{\int_{G_x} |f|, \int_{G^x} |f|\},$$

after a suitable choice of measure on G_x and G^x. See [84, 103] for more details.

Definition 1.14.

- *The maximal norm* $\|f\|_{max}$ of an element $f \in \Gamma_c(\mathcal{D}G)$ is the supremum of the operator norm of the action of f over all continuous representations of G. The completion of $\Gamma_c(\mathcal{D}G)$ with respect to $\|\cdot\|_{max}$ is called *the maximal C^*-algebra* of G and is denoted by $C^*_{max}G$.
- *The reduced norm* $\|\cdot\|_r$ an element $f \in \Gamma_c(\mathcal{D}G)$ is the supremum of the operator norm of the action of f over all regular representation of G obtained in Example 1.13. The completion of $\Gamma_c(\mathcal{D}G)$ with respect to $\|\cdot\|_r$ is called *the reduced C^*-algebra* of G and is denoted by C^*_rG.

Obviously one has

$$\|\cdot\|_r \leq \|\cdot\|_{max}.$$

Let us remark that this construction can be performed in the general case of a locally compact groupoid [84].

Remark 1.15. In what follows, we will write $C^*(G)$ when the choice between maximal or the reduced C^*-algebra is superfluous.

Examples 1.16.

1. If $G = G^0 = M$, then $C^*(G) = C_0(M)$ the C^*-algebra of continuous functions on M vanishing at infinity.

2. If $G = M \times M \rightrightarrows M$, then $C^*(G) = \mathcal{K}(L^2M)$ the C^*-algebra of compact operators on L^2M. This is a consequence of Hilbert-Schmidt theorem.

3. The case of a Lie group is a classical construction. An example where this C^*-algebra is computable is the case of an abelian group. Pontryagin duality (Fourier transform) says that $C^*(G) = C_0(\hat{G})$, where $\hat{G}$ is the Pontryagin dual of G. More generally if G is a Lie group acting on a smooth manifold M, then $C^*_{max}(M \rtimes G)$ (respectively $C^*_r(M \rtimes G)$) is equal to the classical cross product $C_0(M) \rtimes_{max} G$ (respectively $C_0(M) \rtimes_r G$). See [103] for more details.

4. If $V \rightrightarrows M$ is a vector bundle, then the Fourier transform on each fiber gives an isomorphism between $C^*(V)$ and $C_0(V^*)$.

The C^*-algebra $C_0(G^0)$ embeds inside the multiplier algebra $M(C^*G)$.

Proposition 1.17. *Let G be a Lie groupoid. The action of $C_c(G^0)$ on $\Gamma_c(\mathcal{D}G)$ given by*

$$(f{\cdot}g)(\gamma) = f(r(\gamma))g(\gamma), \quad (g{\cdot}f)(\gamma) = g(\gamma)f(s(\gamma)), \quad f \in C_c(G^0), g \in \Gamma_c(\mathcal{D}G), \gamma \in G$$

extends to give a $*$-homomorphism $C_0(G^0) \to M(C^*G)$.

Let $E \to G^0$ be a Hermitian vector bundle. We define the C^*G-C^*-module

$$C^*E := \Gamma_0(E) \otimes_{C_0(G^0)} C^*G,$$

where $\Gamma_0(E)$ denotes the $C_0(G^0)$-C^*-module of all continuous sections of E vanishing at infinity.

Theorem 1.18 ([71])**.** *The C^*-algebras (either reduced or maximal) of Morita equivalent groupoids are Morita equivalent (see [62] for the definition of Morita equivalent C^*-algebras).*

1.3 Transversal measure on groupoids

If $f \in \Gamma_c(\mathcal{D}G)$, then the restriction of f to G^0 is a section of $|\Lambda|\mathfrak{A}G$ that will be denoted by $f_{|G^0}$.

Definition 1.19. A linear map $\mu : \Gamma_c(|\Lambda|\mathfrak{A}G) \to \mathbb{C}$ is called

1. positive if for any positive 1-density f in $\Gamma_c(|\Lambda|\mathfrak{A}G)$, $\mu(f) \geq 0$.
2. G-invariant if for any $f, g \in \Gamma_c(\mathcal{D}G)$, then

$$\mu((f \star g)_{|G^0}) = \mu((g \star f)_{|G^0}).$$

A transversal measure on G is a positive G-invariant $\mu : \Gamma_c(|\Lambda|\mathfrak{A}G) \to \mathbb{C}$.

Definition 1.20. Let A be a C^*-algebra. A trace on A is a map $\tau : A^+ \to [0, +\infty]$ such that

1. for all $a, b \in A^+$, $\lambda, \mu \in \mathbb{R}^+$, one has $\tau(\lambda a + \mu b) = \lambda\tau(a) + \mu\tau(b)$ with the convention $0 \cdot +\infty = 0$.
2. for all $a \in A$, $\tau(a^*a) = \tau(aa^*)$.

The trace τ is called semi-finite, if for any $a \in A^+$,

$$\tau(a) = \sup_{b<a, \tau(b)<+\infty} \tau(b).$$

Proposition 1.21 ([?])**.** *Let τ be a trace on a C^*-algebra A, then*

1. *the set $A_\tau := \mathrm{Vect}\{a \in A^+ : \tau(a) < +\infty\}$ is a two sided of A.*
2. *There exists a unique linear map on A_τ extending τ.*

Theorem 1.22 (Connes)**.** *If μ is a transversal measure on G, then the map*

$$\Gamma_c(\mathcal{D}G) \to \mathbb{C}$$

extends to a lower semi-continuous semi-finite trace on C^*G that will be denoted by Tr_μ.

Proof. □

Let μ_t be a continuous family of G-transversal measures, that is

$$\mu : \Gamma_c(|\lambda|\mathfrak{A}G) \to C[0,1]$$

is a linear map such that $\mu_t = \mathrm{ev}_t \circ \mu$ is positive and G-invariant. We denote by C^*G_μ the set of all $a \in C^*G$ such that there exists an sequence $a_n \in \Gamma_c(|\Lambda|\mathfrak{A}G)$ such that $a_n \to a$ in C^*G and

$$\sup_{t\in[0,1]} \mathrm{Tr}_{\mu_t}\left((a-a_n)^*(a-a_n)\right) \to 0.$$

Proposition 1.23. *The set C^*G_μ us a dense two sided $*$-ideal.*

1.4 Deformation to the normal cone

In this section, we recall the deformation to the normal cone construction following [37]. *The deformation to the normal cone* of a manifold M along an immersed submanifold V is a manifold whose underlying set is

$$\mathrm{DNC}(M,V) := M \times \mathbb{R}^* \sqcup N_V^M \times \{0\},$$

where N_V^M is the normal bundle of V inside M. The smooth structure is defined by covering the manifold with two sets; the first is $M \times \mathbb{R}^*$ and the second is $\phi(N_V^M) \times \mathbb{R}^* \sqcup N_V^M \times \{0\}$ where $\phi : N_V^M \to M$ is a tubular embedding[1]. The smooth structure on $\phi(N_V^M) \times \mathbb{R}^* \sqcup N_V^M \times \{0\}$ is given by declaring the following map a diffeomorphism

$$\begin{aligned}
&\tilde{\phi} : N_V^M \times \mathbb{R} \to \phi(N_V^M) \times \mathbb{R}^* \sqcup N_V^M \times \{0\}\\
&\tilde{\phi}(x,X,t) = (\phi(x,tX),t) \in M \times \mathbb{R}^*, \quad t \neq 0\\
&\tilde{\phi}(x,X,0) = (x,X,0) \in N_V^M \times \{0\}.
\end{aligned}$$

Proposition 1.24 (cf. chapter IV from [58])**.** *The above charts are compatible and the smooth structure is independent of ϕ.*

[1]To simplify the exposition, we will always assume that tubular neighbourhoods are diffeomorphisms on N_V^M. In the case of immersed manifolds, the tubular neighbourhoods are only local in V.

Compatibility is clear. Independence of ϕ follows by noticing that the following functions are smooth functions that generate the smooth structure:

1. the function

$$\begin{aligned}(\pi_M, \pi_{\mathbb{R}}) : \mathrm{DNC}(M, V) &\to M \times \mathbb{R} \\ (x, t) &\to (x, t), \quad t \neq 0 \\ (x, X, 0) &\to (x, 0)\end{aligned}$$

2. Let $f \in C^\infty(M)$ be a smooth function which vanishes on V. Therefore $df : N_V^M \to \mathbb{R}$ is well defined. The following function is smooth

$$\begin{aligned}\mathrm{DNC}(f) : \mathrm{DNC}(M, V) &\to \mathbb{R} \\ (x, t) &\to \frac{f(x)}{t}, \quad t \neq 0 \\ (x, X, 0) &\to df_x(X)\end{aligned}$$

The group $\mathbb{R}^*$ acts smoothly on $\mathrm{DNC}(M, V)$. The action is given by $\lambda_u(x, t) = (x, ut)$ and $\lambda_u(x, X, 0) = (x, \frac{X}{u}, 0)$ for $u \in \mathbb{R}^*$.

Proposition 1.25 (Functoriality of DNC)**.** *Let M, M' be smooth manifolds, $V \subseteq M$, $V' \subseteq M'$ submanifolds, $f : M \to M'$ a smooth map such that $f(V) \subseteq V'$. Then the map defined by*

$$\begin{aligned}\mathrm{DNC}(M, V) &\to \mathrm{DNC}(M', V') \\ (x, t) &\to (f(x), t), \quad t \neq 0 \\ (x, X, 0) &\to (f(x), df_x(X), 0)\end{aligned}$$

is a smooth map[2] *that will be denoted by* $\mathrm{DNC}(f)$*. Furthermore the map* $\mathrm{DNC}(f)$ *is*

- *a submersion if and only if f is a submersion and $f_{|V} : V \to V'$ is also a submersion.*

- *an immersion if and only if f is an immersion and for every $v \in V$, $T_vV = df_v^{-1}(TV')$.*

[2]In the case where V' is an immersed submanifold, one must also suppose that $f_{|V} : V \to V'$ is continuous.

Proof. Smoothness of $\mathrm{DNC}(f)$ follows from the description of smooth maps given above. For statements concerning submersions and immersions. Let $U \subseteq \mathrm{DNC}(M,V)$ be the set where the differential of $\mathrm{DNC}(f)$ is onto (respectively injective). It is clear that U is an open set that is invariant under the $\mathbb{R}^*$ action and contains $M \times \mathbb{R}^*$. To prove that $U = \mathrm{DNC}(M,V)$, it suffices to prove that $V \times \{0\} \subseteq U$. If $v \in V$, then one sees directly that

$$T_{(v,0)} \mathrm{DNC}(M,V) = \mathbb{R} \oplus T_v V \oplus T_v M/T_v V$$

The differential of $\mathrm{DNC}(f)$ is then $d\,\mathrm{DNC}(f)_{(v,0)}(t,X,Y) = (t, df_v(X), df_v(Y))$. The proposition is then clear. $\square$

The map

$$N_V^M \to N_{V'}^{M'}, \quad (x,X) \to (f(x), df_x(X))$$

will be denoted by Nf.

Remark 1.26. It follows from Proposition 1.25, that if G is a Lie group acting smoothly on a manifold M that leaves a submanifold V invariant, then G acts smoothly on $\mathrm{DNC}(M,V)$. This action commutes with the $\mathbb{R}^*$ action λ. In particular the group $G \times \mathbb{R}^*$ acts on $\mathrm{DNC}(M,V)$.

Proposition 1.27. *Let M_1, M_2, M be manifolds, $V_i \subseteq M_i, V \subseteq M$ submanifolds, $f_i : M_i \to M$ smooth maps such that*

1. *$f_i(V_i) \subseteq V$ for $i \in \{1,2\}$*
2. *the maps f_i are transverse*
3. *the maps $f_i|V_i : V_i \to V$ are transverse*

Then

1. (a) *the maps $Nf_i : N_{V_i}^{M_i} \to N_V^M$ are transverse.*

   (b) *the natural map*
   $$N_{V_1 \times_V V_2}^{M_1 \times_M M_2} \to N_{V_1}^{M_1} \times_{N_V^M} N_{V_2}^{M_1}$$
   is a diffeomorphism.

   Similarly for DNC, *we have*

2. (a) *the maps* $\mathrm{DNC}(f_i) : \mathrm{DNC}(M_i, V_i) \to \mathrm{DNC}(M, V)$ *are transverse.*

   (b) *the natural map*

$$\mathrm{DNC}(M_1 \times_M M_2, V_1 \times_V V_2) \to \mathrm{DNC}(M_1, V_1) \times_{\mathrm{DNC}(M,V)} \mathrm{DNC}(M_2, V_2)$$

   is a diffeomorphism.

Proof. Statements 1. (a) and 1. (b) are clear. Bijectivity of the natural map $\mathrm{DNC}(M_1 \times_M M_2, V_1 \times_V V_2) \to \mathrm{DNC}(M_1, V_1) \times_{\mathrm{DNC}(M,V)} \mathrm{DNC}(M_2, V_2)$ is clear as well. To prove that it is a diffeomorphism and that the maps $\mathrm{DNC}(f_i)$ are transverse, we use the same argument as in Proposition 1.25. The two conditions are open conditions which are $\mathbb{R}^*$-invariant. Hence it suffices to check that them at $V_1 \times_V V_2$ which follows directly from 1. (a) and 1. (b). $\square$

Theorem 1.28. *Let G be a Lie groupoid, H a Lie subgroupoid. Then*

1. *the space $N_H^G \rightrightarrows N_{H^0}^{G^0}$ is a Lie groupoid whose structure maps are Ns, Nr and whose Lie algebroid is equal to $N_{\mathfrak{A}H}^{\mathfrak{A}G}$. Furthermore, N_H^G is a $\mathcal{VB}$-groupoid over H.*

2. *the manifold* $\mathrm{DNC}(G, H) \rightrightarrows \mathrm{DNC}(G^0, H^0)$ *is a Lie groupoid whose structure maps are* $\mathrm{DNC}(s)$, $\mathrm{DNC}(r)$ *and Lie algebroid is equal to* $\mathrm{DNC}(\mathfrak{A}G, \mathfrak{A}H)$.

3. *if $K \subseteq H$ is a Lie subgroupoid, then the restriction of the normal bundle $N_H^G\big|_K \rightrightarrows N_{H^0}^{G^0}\big|_{K^0}$ is a Lie subgroupoid of $N_H^G \rightrightarrows N_{H^0}^{G^0}$ whose Lie algebroid is $N_{\mathfrak{A}H}^{\mathfrak{A}G}\big|_{\mathfrak{A}K}$. Furthermore $N_H^G\big|_K$ is a $\mathcal{VB}$-groupoid over K.*

Proof. Statements 1 and 2 are direct consequences of propositions 1.24, 1.25 and 1.27. The third statement follows from the first and because the projection map onto the base

$$\begin{tikzcd} N_H^G \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & H \arrow[d, shift left] \arrow[d, shift right] \\ N_{H^0}^{G^0} \arrow[r] & H^0 \end{tikzcd}$$

is a submersive morphism of groupoids, hence the inverse image of the Lie subgroupoid K is a Lie groupoid. $\square$

From now on, for a Lie groupoid G and a Lie subgroupoid H, we will use $\mathcal{N}_H^G$ to denote the space N_H^G equipped with the structure of a Lie groupoid given by Theorem 1.28.

Remarks 1.29. 1. Let $E \to M$ be a vector bundle, $V \subseteq M$ a submanifold, $F \to V$ a subbundle of the restriction of E to V. By Theorem 1.28, the space $\mathrm{DNC}(E, F)$ is a vector bundle over $\mathrm{DNC}(M, V)$. Since a section of $\mathrm{DNC}(E, F)$ is determined by its values on the dense set $M \times \mathbb{R}^*$. It follows that

$$\Gamma(\mathrm{DNC}(E, F)) = \{X \in \Gamma(E \times \mathbb{R}) : X_{|V \times \{0\}} \in \Gamma(F)\},$$

where Γ denotes the set of global sections (continuous or smooth).

In the particular case where F is the zero bundle, it is clear that by dividing by t, we have an isomorphism from $\mathrm{DNC}(E, V)$ to $\pi_M^* E$ where $\pi_M : \mathrm{DNC}(M, V) \to M$ is the projection map. It follows that to a Euclidean metric on E, one associates canonically a Euclidean metric on $\mathrm{DNC}(E, V)$.

Moreover, the vector bundles $\mathrm{DNC}(E, V)^*$ and $\mathrm{DNC}(E^*, V)$ are canonically isomorphic by the isomorphism

$$\begin{aligned} \mathrm{DNC}(E^*, V) &\to \mathrm{DNC}(E, V)^* \\ \alpha &\to \left(e \to \frac{1}{t^2}\alpha(e)\right) \text{ for } t \neq 0. \end{aligned}$$

2. Let $V = V_0 + a \subseteq \mathbb{R}^n$ be an affine subspace where V_0 is the underlying vector space, $a \in \mathbb{R}^n$. Let L be the orthogonal of V_0, π_{V_0}, π_L the orthogonal projections. The space $\mathrm{DNC}(\mathbb{R}^n, V)$ will be identified with $\mathbb{R}^{n+1}$ by the following map

$$\begin{aligned} &\mathrm{DNC}(\mathbb{R}^n, V) \to \mathbb{R}^{n+1} \\ &(x, t) \to (a + \pi_{V_0}(x - a) + \frac{\pi_L(x - a)}{t}, t), \quad t \neq 0 \\ &(x, X, 0) \to (x + X, 0), \end{aligned}$$

where in the last identity we identified $N_V^{\mathbb{R}^n}$ with L.

Examples 1.30. 1. If M is a smooth manifold, then

$$\mathrm{DNC}(M \times M, M) = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}$$

is the first deformation groupoid constructed by A. Connes. He used it to give a short elegant proof of Atiyah Singer index theorem [30]. The product law is given by

$$(x, y, t) \cdot (y, z, t) = (x, z, t),\ (x, X, 0) \cdot (x, Y, 0) = (x, X + Y, 0).$$

If V is a submanifold of M, then $\mathrm{DNC}(V \times V, V)$ is a Lie subgroupoid of $\mathrm{DNC}(M \times M, M)$. It is clear that the quotient space is equal to

$$\mathrm{DNC}(M \times M, M)/\,\mathrm{DNC}(V \times V, V) = \mathrm{DNC}(M, V).$$

2. Let $L \subseteq G^0$ be a submanifold. Since N_L^G is equal to $N_L^{G^0} \oplus \ker(ds)_{|L}$. It follows that the groupoid $\mathcal{N}_L^G \rightrightarrows N_L^{G^0}$ is equal to

$$\{(X, Y, Z) : X, Z \in N_L^{G^0}, Y \in \mathfrak{A}G, Z = X + \natural(Y)\},$$

with the obvious structural maps.

Remark 1.31 (Convergence to infinity)**.** Let M be a closed manifold, V a closed embedded submanifold. A sequence in $\mathrm{DNC}(M, V)$ converging to infinity has a subsequence z_n converging to infinity of one of the following forms

1. $z_n = (x_n, t_n)$, where $x_n \to x$ in M and $t_n \to +\infty$.

2. $z_n = (x_n, t_n)$, where $x_n \to x \notin V$ and $t_n \to 0$.

3. $z_n = (x_n, t_n)$, where $x_n \to x \in V$, $t_n \to 0$ and $\left\| \frac{x_n - x}{t_n} \right\| \to \infty$.

4. $z_n = (x_n, \xi_n, 0)$, where $x_n \to x \in V$, $\|\xi_n\| \to \infty$.

Remarks 1.32. 1. We could have replaced $\mathbb{R}^*$ by $]0, 1]$ or $]0, \infty[$ or $\mathbb{R}P^1 \backslash \{0\}$. The version with $]0, 1]$ will be used in Chapter 2. We will sometimes use $\mathrm{DNC}_{\mathbb{R}}(M, V)$, $\mathrm{DNC}_{[0,1]}(M, V)$, $\mathrm{DNC}_{[0,\infty[}(M, V)$, $\mathrm{DNC}_{\mathbb{R}P^1}(M, V)$ to denote each one respectively.

2. If V is a not injectively immersed submanifold, then $\mathrm{DNC}(M, V)$ can still be defined. It is then a non Hausdorff manifold [49].

1.5 DNC iterated

Let M be a smooth manifold, $V_0 \subseteq M$ a submanifold, $V_1 \subseteq \mathrm{DNC}(M, V_0)$ a submanifold. One defines

$$\mathrm{DNC}^2(M, V_0, V_1) := \mathrm{DNC}(\mathrm{DNC}(M, V_0), V_1).$$

This space being a deformation space admits an $\mathbb{R}^*$-action that will be denoted by $\lambda^{(1)}$, and a projection map $\pi_{\mathbb{R}}^{(1)} : \mathrm{DNC}^2(M, V_0, V_1) \to \mathbb{R}$.

If V_1 is $\mathbb{R}^*$-invariant, then by Remark 1.26, the group $\mathbb{R}^*$ acts on $\mathrm{DNC}^2(M, V_0, V_1)$. This action will be denoted by $\lambda^{(0)}$, furthermore the group $(\mathbb{R}^*)^2$ acts on $\mathrm{DNC}^2(M, V_0, V_1)$ by $\lambda^{(0)} \times \lambda^{(1)}$.

Let $\pi_{\mathbb{R}} : \mathrm{DNC}(M, V) \to \mathbb{R}$ be the projection constructed in Section 1.4. If $\pi_{\mathbb{R}}(V_1)$ is an affine subspace of $\mathbb{R}$ and $\pi_{\mathbb{R}}|V_1 : V_1 \to \pi_{\mathbb{R}}(V_1)$ is a submersion, then the map

$$\pi_{\mathbb{R}}^{(0,1)} := \mathrm{DNC}(\pi_{\mathbb{R}}) : \mathrm{DNC}^2(M, V_0, V_1) \to \mathrm{DNC}(\mathbb{R}, \pi_{\mathbb{R}}(V_1)) = \mathbb{R}^2$$

is a smooth submersion, where we identified $\mathrm{DNC}(\mathbb{R}, \pi_{\mathbb{R}}(V_1))$ with $\mathbb{R}^2$ using Remarks 1.29.

If V_1 is furthermore $\mathbb{R}^*$-invariant, then one has for all $u, t \in \mathbb{R}^*$

$$\pi_{\mathbb{R}}^{(0,1)} \lambda_u^{(1)} = \left(\frac{\pi_{\mathbb{R}}^{(0)}}{u}, u\pi_{\mathbb{R}}^{(1)}\right),\ \pi_{\mathbb{R}}^{(0,1)} \lambda_u^{(0)} = (u\pi_{\mathbb{R}}^{(0)}, \pi_{\mathbb{R}}^{(1)}), \tag{1.1}$$

where $\pi_{\mathbb{R}}^{(0,1)} = (\pi_{\mathbb{R}}^{(0)}, \pi_{\mathbb{R}}^{(1)})$.

By induction, given a sequence of submanifolds

$$V_0 \subseteq M,\ V_1 \subseteq \mathrm{DNC}(M, V_0),\ V_2 \subseteq \mathrm{DNC}^2(M, V_0, V_1), \cdots,\ V_k \subseteq \mathrm{DNC}^k(M, V_0, \dots, V_{k-1}).$$

We define the space

$$\mathrm{DNC}^{k+1}(M, V_0, \cdots, V_k) := \mathrm{DNC}(\mathrm{DNC}^k(M, V_0, \cdots, V_{k-1}), V_k).$$

If for each $1 \leq i \leq k$, $\pi_{\mathbb{R}}^{(0,\dots,i-1)}(V_i)$ is an affine subspace of $\mathbb{R}^i$ and $\pi_{\mathbb{R}}^{(0,\dots,i-1)} : V_i \to \pi_{\mathbb{R}}^{(0,\dots,i-1)}(V_i)$ is a submersion, then by Proposition 1.25, the map

$$\pi_{\mathbb{R}}^{(0,\cdots,k)} := \mathrm{DNC}(\pi_{\mathbb{R}}^{(0,\cdots,k-1)}) : \mathrm{DNC}^{k+1}(M, V_0, \cdots, V_k) \to \mathrm{DNC}(\mathbb{R}^k, \pi_{\mathbb{R}}^{(0,\dots,k-1)}(V_k)) = \mathbb{R}^{k+1}$$

is a smooth submersion, where we identified $\mathrm{DNC}(\mathbb{R}^k, \pi_{\mathbb{R}}^{(0,\dots,k-1)}(V_k))$ with $\mathbb{R}^{k+1}$ using Remarks 1.29.

If each V_i is $(\mathbb{R}^*)^i$ invariant, then the space $\mathrm{DNC}^{k+1}(M, V_0, \cdots, V_k)$ admits $k+1$ pairwise commuting actions of $\mathbb{R}^*$-denoted $\lambda^{(k)}, \dots, \lambda^{(0)}$.

Propositions 1.25, 1.27 and Theorem 1.28 have obvious extensions to DNC^k. In particular we have by induction if $G \rightrightarrows G^0$ is a Lie groupoid, $H_0 \subseteq G$, $H_1 \subseteq \mathrm{DNC}(G, H_0)$, $\dots$, $H_k \subseteq \mathrm{DNC}^k(G, H_0, \dots, H_{k-1})$. are Lie subgroupoids, then

$$\mathrm{DNC}^{k+1}(G, H_0, H_1, \cdots, H_k) \rightrightarrows \mathrm{DNC}^{k+1}(G^0, H_0^0, \cdots, H_k^0)$$

is a Lie groupoid.

1.6 Pseudo-differential operators on groupoids

We start by recalling a few classical definitions due to A. Connes [28] in the case of foliation groupoid and in the general case due to Monthubert and Pierrot [69], and independantely Nistor, Weinstein and Xu [72]. We follow closely the presentation given in [101].

Definition 1.33. Let $P : C_c^\infty(G) \to C^\infty(G)$ be a continuous (with respect to the Fréchet topology) $\mathbb{C}$-linear map. The map P is called G-equivariant if

1. for every function $f \in C_c^\infty(G), \gamma \in G$, $P(f)(\gamma)$ only depend on $f_{|G_{s(\gamma)}}$. In other words P consists of a family of maps $P_x : C_c^\infty(G_x) \to C^\infty(G_x)$ for $x \in G^0$.
2. For every $\gamma \in G$, $R_{\gamma^{-1}} \circ P_{r(\gamma)} \circ R_\gamma = P_{s(\gamma)}$, where R_γ is right multiplication by γ.

By equivariance, the Schwartz kernel of P is equal to $k_P(\gamma\gamma'^{-1})$, where k_P is a distribution on G. More precisely we have

$$Pf(\gamma) = \int_{G_{s(\gamma)}} k_P(\gamma\gamma'^{-1}) f(\gamma') d\gamma'.$$

The operator P is said to be compactly supported if k_P is compactly supported, and smoothing with compact support if $k_P \in C_c^\infty(G)$.

The operator P is called a G-pseudo differential operator if P is a G-equivariant and in addition the operator P is a bundle (with respect to the fibration s) of pseudo differential operators.

Remark 1.34. Another point of view is to see pseudo-differential operators as conormal distributions in $I(G, G^0)$ (using Hormander notation [52]). See [66, 65] for a detailed development of this point of view.

If P is a G-invariant pseudo-differential operator, then the principal symbol of P, denoted $\sigma(P) : \mathfrak{A}G \to \mathbb{R}$, where $\sigma(P)(x, v)$ is the principal symbol of P_x at (x, v), where we use $\mathfrak{A}_x G = T_x G_x$.

All the above extends straightforwardly to operators acting on sections of r^*E, where E is a vector bundle on G^0. To avoid the choice of measures as was done in Section 1.2, pseudo-differential operators will be maps

$$P : \Gamma_c^\infty \left(r^* \left(E \otimes |\Lambda|^{\frac{1}{2}} \mathfrak{A}G \right) \right) \to \Gamma^\infty \left(r^* \left(E \otimes |\Lambda|^{\frac{1}{2}} \mathfrak{A}G \right) \right).$$

Let g be a Euclidean metric on $\mathfrak{A}G$. For every $\gamma \in G$, one has the isomorphism

$$T_\gamma G_{s(\gamma)} \xrightarrow{d_\gamma R_{\gamma^{-1}}} T_{r(\gamma)} G_{r(\gamma)} = \mathfrak{A}_{r(\gamma)} G.$$

It follows that g defines a Riemannian metric on G_x for every $x \in G^0$.

Proposition 1.35. *There exists a Euclidean metric g on $\mathfrak{A}G$ such that for every $x \in G^0$, the induced Riemannian metric on G_x is complete. Such metrics are called complete Euclidean metrics.*

Proof. Let g be any Euclidean metric on $\mathfrak{A}G$, and let $h : G^0 \to]0, +\infty[$ be a smooth function such that if $x \in G^0$, then the ball in G_x of radius $h(x)$ with center x is relatively compact. It is straightforward to verify that the euclidean metric $\frac{1}{h^2} g$ is complete. See [73] for more details. □

We recall a theorem of Chernoff [19], and then extend it to Lie groupoids.

Theorem 1.36 (Chernoff)**.** *Let M be a complete Riemannian manifold, D a symmetric first-order differential operator acting on a Hermitian vector bundle E,*

$$c(x) = \sup_{v \in T_x^* M : \|v\| = 1} \|\sigma(D)(x, v)\|.$$

If c is bounded above, then the differential equation

$$\frac{d}{dt} u(x, t) = iDu, \quad u(x, 0) = f(x), \quad (x, t) \in M \times \mathbb{R}$$

admits a unique global solution for any $f \in C_c^\infty(M)$. Furthermore the distribution defined by $f \mapsto u$ has support inside

$$\{(x, x', t) \in M \times M \times \mathbb{R} : d(x, x') \leq t \sup_{x \in M} c(x)\}.$$

This support is proper for each fixed $t \in \mathbb{R}$.

Corollary 1.37. *With the notation of Theorem 1.36 the operator D is essentially self adjoint.*

We extend Chernoff's theorem to Lie groupoids.

Proposition 1.38. *Let $G \rightrightarrows G^0$ be a Lie groupoid, g a complete Euclidean metric on $\mathfrak{A}G$, $E \to G^0$ a Hermitian vector bundle, D a symmetric first order G-invariant differential operator on G acting on r^*E, $c : G^0 \to \mathbb{R}$ the function*

$$c(x) = \sup_{v \in \mathfrak{A}G_x^* : \|v\| = 1} \|\sigma(D)(x, v)\| .$$

If c is bounded above, then the closure of D is a regular self adjoint operator acting on C^*E.

Proof. Let $f \in \Gamma_c^\infty\left(r^*\left(E \otimes |\Lambda|^{\frac{1}{2}}\mathfrak{A}G\right)\right)$, and consider the differential equation

$$\partial_t u(\gamma, t) = iDu(\gamma, t),\ u(\gamma, 0) = f(\gamma), \quad (\gamma, t) \in G \times \mathbb{R}.$$

By the classical theory of linear differential equations a unique C^∞ solution to this equation exists locally. By Theorem 1.36 and our assumptions, we deduce that a solution exists globally on G_x for each x. In particular solutions to this equation exist globally on G. Furthermore the distribution kernel associated to this equation is proper for each fixed t. Let

$$V_t : \Gamma_c^\infty\left(r^*\left(E \otimes |\Lambda|^{\frac{1}{2}}\mathfrak{A}G\right)\right) \to \Gamma_c^\infty\left(r^*\left(E \otimes |\Lambda|^{\frac{1}{2}}\mathfrak{A}G\right)\right), \quad f \to u(\cdot, t)$$

be the convolution to the left by the distribution kernel. If $f, g \in \Gamma_c^\infty\left(r^*\left(E \otimes |\Lambda|^{\frac{1}{2}}\mathfrak{A}G\right)\right)$, then

$$\frac{d}{dt}\langle V_t f, V_t g\rangle = \langle iDV_t f, V_t g\rangle + \langle V_t f, iDV_t g\rangle = \langle i(D - D^*)V_t f, V_t g\rangle = 0.$$

Hence the operators V_t extend to an isometry acting on the C^*G-module C^*E. This operator is adjointable (and therefore C^*G-linear) because of the equation

$$\langle \xi, V_t \eta \rangle = \langle V_{-t}\xi, \eta \rangle,$$

which proves as well that V_t is a unitary in $\mathcal{L}(C^*E)$. The proposition follows then from Proposition A.12. $\square$

Proposition 1.39 ([101])**.** *Under the same hypothesis as Proposition 1.38, if furthermore D is an elliptic operator, then for every $f \in C_0(G^0)$[3] and $g \in C_0(\mathbb{R})$, the operator $g(D)f$ is compact.*

Proof. By a denisty argument it is enough to prove the proposition for $f \in C_c^\infty(G)$, and $g \in C_c(\mathbb{R})$. Let Q be a parametrix for D^2, that is $D^2 Q = 1 + R$ with R a G-pseudo differential operator of order ≤ -1. The support of Q can be chosen to be a subset of an arbitrary open neighbourhood of G^0. Since D^2 is a differential operator, its Schwartz kernel is supported in a subset of G^0. In particular the support of R can be choosen as well to be a subset of an arbitrary neighbourhood of G^0. We choose the supports of R and Q so that Qf and Rf are G-invariant pseudo differential operators with compact support. It follows from [101, theorem 18], that Qf and Rf extend to compact operators on C^*E. It follows from the identity

$$(1 + D^2)^{-1} f = Qf - (1 + D^2)^{-1} Rf + (1 + D^2)^{-1} Qf,$$

that $(1 + D^2)^{-1} f$ is compact. Since $g(x)(1 + x^2)$ is bounded, it follows that $g(D)f$ is compact as well. $\square$

1.7 De Rham operator

Let G be a Lie groupoid. The De Rham exterior derivative along the leaves[4] of $s : G \to G^0$ is a G-differential operator denoted by

$$d : \Gamma_c^\infty \left(r^* \left(\Lambda_{\mathbb{C}} \mathfrak{A}G^* \right) \right) \to \Gamma_c^\infty \left(r^* \left(\Lambda_{\mathbb{C}} \mathfrak{A}G^* \right) \right).$$

When restricted to G-invariant sections of $r^* \left(\Lambda_{\mathbb{C}} \mathfrak{A}G^* \right)$, the operator d becomes

$$d : \Gamma_c^\infty(\Lambda_{\mathbb{C}} \mathfrak{A}G^*) \to \Gamma_c^\infty(\Lambda_{\mathbb{C}} \mathfrak{A}G^*)$$

[3] Recall Proposition 1.17
[4] Recall that $\ker(ds)$ is canonically identified with $r^*(\mathfrak{A}G)$

which can be defined intrinsically using the algebroid structure on $\mathfrak{A}G$ and Cartan's formula

$$\begin{aligned} d\alpha(X_0,\ldots,X_k) &= \sum_i (-1)^i \natural(X_i)\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k), \end{aligned}$$

where $\alpha \in \Gamma_c^\infty(\Lambda_{\mathbb{C}}\mathfrak{A}G^*)$ and $X_i \in \Gamma^\infty(\mathfrak{A}G)$ and $\natural : \mathfrak{A}G \to TG^0$ is the anchor map.

Propositions 1.40 and 1.41 will be used in Chapter 2.

Proposition 1.40. *Let W be a Riemannian manifold, X a vector field. The operator $\mathcal{L}_X + \mathcal{L}_X^*$ is a $C^\infty(W)$-linear operator (i.e. a 0-order differential operator), where $\mathcal{L}$ denotes Cartan Lie derivative.*

Proof. The operator $\mathcal{L}$ is an ungraded derivation. Therefore,

$$\mathcal{L}_X(f\alpha) = (\mathcal{L}_X f)\alpha + f\mathcal{L}_X\alpha,$$

where $f : W \to \mathbb{R}$ is a real valued smooth function and $\alpha \in \Gamma_c^\infty(\Lambda_{\mathbb{C}}T^*W)$. Taking the dual one deduces that

$$f\mathcal{L}_X^*(\alpha) = (\mathcal{L}_X f)\alpha + \mathcal{L}_X^*(f\alpha).$$

Therefore

$$\mathcal{L}_X(f\alpha) + \mathcal{L}_X^*(f\alpha) = f\left(\mathcal{L}_X(\alpha) + \mathcal{L}_X^*(\alpha)\right).$$ □

If $G \rightrightarrows G^0$ is Lie groupoid, $X \in \Gamma_c^\infty(\mathfrak{A}G)$, then the operator $\mathcal{L}_X : \Gamma_c^\infty(\Lambda_{\mathbb{C}}\mathfrak{A}G^*) \to \Gamma_c^\infty(\Lambda_{\mathbb{C}}\mathfrak{A}G^*)$ is defined by Cartan's formula

$$\mathcal{L}_X = di_X + i_X d.$$

The operator $\mathcal{L}_X$ is a differential operator of order 1 on G^0. We then have

Proposition 1.41. *The operator $\mathcal{L}_X + \mathcal{L}_X^*$ is $C^\infty(G^0)$-linear (i.e. a 0-order differential operator).*

Proof. Same as that of Proposition 1.40. □

Chapter 2

Witten deformations

In this chapter, an application of the deformation to the normal cone construction towards Witten deformation is given.

The organisation of this chapter is as follows: in Section 2.1, we state and prove Proposition 2.2, a simple proposition for Lie groupoids using which we will deduce all the results of this chapter;

In Section 2.2, we deduce Theorem 0.1 as a corollary of Proposition 2.2;

In Section 2.3, we extend the results of Section 2.2 to foliations.

2.1 Preliminary proposition

We will deduce the properties of Witten's deformation ultimately using the following simple proposition.

Proposition 2.1. *Let W be a complete Riemannian manifold, $\# : T^*W \to TW$ the musical isomorphism given by the Riemannian metric, α a 1-form on W such that*

1. *the form $d\alpha$ is bounded*
2. *the section of $\mathrm{End}(\Lambda_{\mathbb{C}} T^*W)$ given by $\mathcal{L}_{\alpha^\#} + \mathcal{L}^*_{\alpha^\#}$ (cf. Proposition 1.40) is bounded*
3. *$\|\alpha\|$ is a proper function,*

then the operator $d + d^* + c(\alpha)$ acting on $L^2(\Lambda_{\mathbb{C}} T^*W)$ is a self-adjoint elliptic operator with compact resolvent, where $L^2(\Lambda_{\mathbb{C}} T^*W)$ is the Hilbert space of L^2 sections of $\Lambda_{\mathbb{C}} T^*W$.

Here $\|\alpha\|$ is the function on W which sends a point x to $\|\alpha_x\|$. Similarly for $d\alpha$ and $d^*\alpha$.

In fact we will need the Lie groupoid version of Proposition 2.1.

Proposition 2.2. *Let G be a Lie groupoid, g a complete euclidean metric (see Proposition 1.35) on $\mathfrak{A}G$, $\# : \mathfrak{A}G^* \to \mathfrak{A}G$ the musical isomorphism given by g, $\alpha \in \Gamma^\infty(\mathfrak{A}G^*)$. If*

1. *the form $d\alpha$ is bounded*
2. *the section of $\mathrm{End}(\Lambda_{\mathbb{C}}\mathfrak{A}G^*)$ given by $\mathcal{L}_{\alpha\#} + \mathcal{L}^*_{\alpha\#}$ (cf. Proposition 1.41) is bounded*
3. *$\|\alpha\| : G^0 \to \mathbb{R}$ is a proper function,*

then the closure of the operator $d + d^* + c(\alpha)$ acting on the C^*G Hilbert module $C^*\left(\Lambda_{\mathbb{C}}\mathfrak{A}G^*\right)$ is a regular self adjoint elliptic operator with compact resolvent.

Remark 2.3. Thanks to [61] (see also [10, 64]), Proposition 2.2 implies that the Kasparov product of $d + d^*$ and $c(\alpha)$ is the operator $d + d^* + c(\alpha)$.

Proof. Since

$$\|\sigma\left(d + d^* + c(\alpha)\right)(x, v)\| = \|\sigma\left(d + d^*\right)(x, v)\| = \|v\|_{g_x},$$

it follows that $d + d^* + c(\alpha)$ is elliptic and from Proposition 1.38 that the closure of $d + d^* + c(\alpha)$ is a regular self adjoint operator.

It follows from Cartan's formula that the graded commutator is equal to

$$[d, i_{\alpha\#}] = \mathcal{L}_{\alpha\#}.$$

Since

$$[d, c(\alpha)] = [d, \alpha\wedge] + [d, i_{\alpha\#}] = d\alpha \wedge \cdot + \mathcal{L}_{\alpha\#}.$$

Hence by the hypothesis of proposition 2.2

$$[d + d^*, c(\alpha)] = [d, c(\alpha)] + [d, c(\alpha)]^* = d\alpha \wedge \cdot + i_{(d\alpha)\#} + \mathcal{L}_{\alpha\#} + \mathcal{L}^*_{\alpha\#}$$

is bounded, where $i_{(d\alpha)\#}(\cdot)$ is the adjoint of $d\alpha \wedge \cdot$. Therefore the closure of $(d + d^*)^2 + c(\alpha)^2 = (d + d^* + c(\alpha))^2 - [d + d^*, c(\alpha)]$ is a regular self adjoint operator.

By Proposition A.11, one has

$$(1 + (d + d^*)^2 + c(\alpha)^2)^{-1} \leq (1 + (d + d^*)^2)^{-1}$$
$$(1 + (d + d^*)^2 + c(\alpha)^2)^{-1} \leq (1 + (c(\alpha))^2)^{-1} = (1 + \|\alpha\|^2)^{-1}$$

It follows from [74, proposition 1.4.5] that there exists $a, b \in \mathcal{L}(C^*\Lambda_{\mathbb{C}}\mathfrak{A}G^*)$ such that

$$(1+(d+d^*)^2+c(\alpha)^2)^{-\frac{1}{2}} = a(1+(d+d^*)^2)^{-\frac{1}{4}}, \quad (1+(d+d^*)^2+c(\alpha)^2)^{-\frac{1}{2}} = (1+\|\alpha\|^2)^{-\frac{1}{4}}b.$$

Hence

$$(1 + (d + d^*)^2 + c(\alpha)^2)^{-1} = a(1 + (d + d^*)^2)^{-\frac{1}{4}}(1 + \|\alpha\|^2)^{-\frac{1}{4}}b.$$

Since by our assumptions $(1 + \|\alpha\|^2)^{-\frac{1}{4}} \in C_0(G^0)$. It follows from Proposition 1.39 that

$$(1 + (d + d^*)^2)^{-\frac{1}{4}}(1 + \|\alpha\|^2)^{-\frac{1}{4}} \in \mathcal{K}(C^*\Lambda_{\mathbb{C}}\mathfrak{A}G^*)$$

Hence $(1 + (d + d^*)^2 + c(\alpha)^2)^{-1}$ is compact as well.

Since$[d + d^*, c(\alpha)]$ is bounded, and

$$\begin{aligned}&\left(1 + (d + d^* + c\,(\alpha))^2\right)^{-1}\\ &= \left(1 - \left(1 + (d + d^* + c(\alpha))^2\right)^{-1}[d + d^*, c(\alpha)]\right)\left(1 + (d + d^*)^2 + c(\alpha)^2\right)^{-1},\end{aligned}$$

it follows that $(1 + (d + d^* + c(\alpha))^2)^{-1}$ is compact. □

2.2 Classical Witten deformation

Let M be a closed manifold, $f : M \to \mathbb{R}$ a Morse function (a smooth function whose critical points are nondegenerate), $\mathrm{Crit}(f)$ its critical points (a finite set), $\pi_{\mathbb{R}} : \mathrm{DNC}_{[0,1]}(M, \mathrm{Crit}(f)) \to [0, 1]$, $\pi_M : \mathrm{DNC}_{[0,1]}(M, \mathrm{Crit}(f)) \to M$ the natural projections. By Theorem 1.28, the following is naturally a Lie groupoid

$$\begin{aligned}G &= \mathrm{DNC}_{[0,1]}(M \times M, \mathrm{Crit}(f) \times \mathrm{Crit}(f))\\ &= M \times M \times]0, 1] \sqcup_{a,b\in\mathrm{Crit}(f)} T_aM \times T_bM \times \{0\} \rightrightarrows \mathrm{DNC}_{[0,1]}(M, \mathrm{Crit}(f)),\end{aligned}$$

whose algebroid is equal to $\mathrm{DNC}(TM, \mathrm{Crit}(f))$.

Remark 2.4. In this section, all deformations will be on $[0, 1]$. We could equally well work on $\mathbb{R}$ but then the conclusion of Corollary 2.5 would have to be changed to that the resolvent is locally compact in the $\mathbb{R}$ direction. Notice that the Lie groupoid G has boundary. Since G is the restriction to $[0,1]$ of the Lie groupoid $\mathrm{DNC}_{\mathbb{R}}(M\times M,\mathrm{Crit}(f))$. It follows that all results of chapter 1 still hold for G.

Let g be a Riemannian metric on M. In Remarks 1.29, on $\mathfrak{A}G=\mathrm{DNC}(TM,\mathrm{Crit}(f))$ a Euclidean metric is defined which is equal to $\frac{g}{t^2}$ on $M\times\{t\}$ for $t\neq 0$ and the constant Riemannian metric g_a on $T_aM\times\{0\}$. This metric is complete by the completeness of the metric g on M.

Let α be the 1-form given by Proposition 1.25

$$\alpha=\mathrm{DNC}(df):\mathrm{DNC}(M,\mathrm{Crit}(f))\to\mathrm{DNC}(T^*M,\mathrm{Crit}(f)).$$

After identifying $\mathrm{DNC}(T^*M,\mathrm{Crit}(f))$ with $\mathrm{DNC}(TM,\mathrm{Crit}(f))^*=\mathfrak{A}G^*$ (see Remarks 1.29), the form α is equal to $\frac{df}{t^2}$ on $M\times\{t\}$ for $t\neq 0$ and to d^2_af on $T_aM\times\{0\}$ for $a\in\mathrm{Crit}(f)$.

Let us verify the condition of Proposition 2.2.

1. The form α is clearly closed.

2. On $M\times\{t\}$, one has
$$\alpha^{\#_{\frac{g}{t^2}}}=df^{\#_g},$$
where $\#$ is the musical isomorphism. Hence $\mathcal{L}_{\alpha^\#}$ is independent of t. Since the Riemannian metric is multiplied by a scalar and $\mathcal{L}_{\alpha^\#}$ is an operator of degree 0. It follows that $\mathcal{L}^*_{\alpha^\#}$ doesn't depend on t as well for $t\neq 0$. Hence the norm of the section $\mathcal{L}_{\alpha^\#}+\mathcal{L}^*_{\alpha^\#}$ is independent of t for $t\neq 0$. Therefore, it is bounded by its boundness on M.

3. On $M\times\{t\}$, one has
$$\|\alpha\|_{\frac{g}{t^2}}=\left\|\frac{df}{t^2}\right\|_{\frac{g}{t^2}}=\frac{\|df\|_g}{t}$$
and on $T_aM\times\{0\}$,
$$\|\alpha\|_{g_a}=\left\|d^2_af\right\|_{g_a}.$$
It follows from Remark 1.31 that $\|\alpha\|$ is a proper function on the space $\mathrm{DNC}_{[0,1]}(M,\mathrm{Crit}(f))$.

Corollary 2.5. *The operator $d+d^*+c(\alpha)$ acting on the $C([0,1])$ module $C^*\Lambda_{\mathbb{C}}\ker(d\pi_{\mathbb{R}})^*$ is a regular self adjoint operator with compact resolvent.*

Proof. The manifold $\mathrm{DNC}_{[0,1]}(M, \mathrm{Crit}(f))$ gives naturally to a Morita equivalence between the Lie groupoid $\mathrm{DNC}_{[0,1]}(M \times M, \mathrm{Crit}(f) \times \mathrm{Crit}(f)) \rightrightarrows \mathrm{DNC}_{[0,1]}(M, \mathrm{Crit}(f))$ and the trivial Lie groupoid $[0,1] \rightrightarrows [0,1]$. The corollary then follows from Proposition 2.2 and Theorem 1.18. $\square$

Corollary 2.6. *Let $d_t = e^{-\frac{f}{t}} d e^{\frac{f}{t}}$, $\Delta_t = (d_t + d_t^*)^2$ be the Witten Laplacian acting on $L^2(\Lambda_{\mathbb{C}} T^* M)$. If*

$$\lambda_1^p(t) \leq \lambda_2^p(t) \cdots$$

denotes the spectrum of Δ_t acting on p-forms, then the function

$$t \to \begin{cases} t\lambda_i^p(t) & \textit{if}\, t \neq 0 \\ \lambda_i^p(0) & \textit{if}\, t = 0 \end{cases}$$

is continuous, where $\lambda_i^p(0)$ is the i'th eigenvalue of Harmonic osccilator

$$\bigoplus_{a \in \mathrm{Crit}(f)} \left(d + d^* + c(d_a^2(f))\right)^2 : \bigoplus_{a \in \mathrm{Crit}(f)} L^2(T_a M, \Lambda_{\mathbb{C}}^p T_a M) \to \bigoplus_{a \in \mathrm{Crit}(f)} L^2(T_a M, \Lambda_{\mathbb{C}}^p T_a M),$$

where $L^2(T_a M, \Lambda_{\mathbb{C}}^p T_a M)$ is the set of all L^2 functions from $T_a M$ to $\Lambda_{\mathbb{C}}^p T_a M$, $d_a^2 f$ is the 1-differential form on $T_a M$, and c is the Clifford multiplication.

Proof. After normalizing the metric $\frac{g}{t^2}$, the operator $(d + d^* + c(\alpha))^2$ on $M \times \{t\}$ is equal to $t^2 \Delta_{t^2}$. The corollary then follows from Lemma 2.7. $\square$

Lemma 2.7. *Let E be a $C[0,1]$ module, $L \in \mathcal{K}(E)$ a compact operator. If the spectrum of $(L^*L)^{\frac{1}{2}}$ acting on the fiber E_t of E at $t \in [0,1]$ is denoted by $\mu_1(t) \geq \mu_2(t) \cdots$, then for every i, the function $t \to \mu_i(t)$ is continuous.*

Proof. If $L = \sum_i e_i \langle f_i, \cdot \rangle$ is a finite rank operator, then the lemma is clear. By [45, theorem 2.1], one has if T_1, T_2 are compact operators, then for every $i \in \mathbb{N}$,

$$|\mu_i(T_1) - \mu_i(T_2)| \leq \|T_1 - T_2\|.$$

It follows that for each $i \in \mathbb{N}$, the map

$$\mathcal{K}(E) \to L^\infty([0,1]), \quad T \to \mu_i(T)$$

is continuous. Since the image of the dense subspace of finite rank operators is inside the closed subspace of continuous functions, the lemma follows. $\square$

Remark 2.8. Lemma 2.7 is false if L is only supposed to be in $\mathcal{L}(E)$, and L_t is compact for every t. A trivial example is the $C([0,1])$ module $E = C_0(]0,1])$ and L the identity.

The calculation of the spectrum of the harmonic oscillator in Corollary 2.6 is a classical calculation. In particular we have

Proposition 2.9 ([83, section V.3])**.** *If $a \in \mathrm{Crit}(f)$ and*

$$\xi_1(a) \leq \cdots \leq \xi_{\mathrm{Ind}(a)} \leq 0 \leq \xi_{\mathrm{Ind}(a)+1} \leq \cdots \leq \xi_{\dim(M)},$$

denote the eigenvalues of $d_a^2 f$ with respect to g_a, then the spectrum of

$$\left(d + d^* + c(d_a^2(f))\right)^2 : L^2(T_aM, \Lambda^p_{\mathbb{C}} T_aM) \to L^2(T_aM, \Lambda^p_{\mathbb{C}} T_aM)$$

is the weighted set[1]

$$\coprod_{\substack{J \subseteq \{1,\dots,\dim(M)\} : |J| = p \\ \alpha \in \mathbb{N}^{\dim(M)}}} \{ \sum_{j \in J \cap \{\mathrm{Ind}(a)+1,\dots,\dim(M)\}} \xi_j(a) + \sum_{j \in J^c \cap \{1,\dots,\mathrm{Ind}(a)\}} \xi_j(a) + \sum_{j=1}^{\dim(M)} \alpha_j \xi_j(a) \}.$$

Corollary 2.10 (Morse inequalities)**.** *If C_i denotes the number of critical points of f, then for every k,*

$$\sum_{i=0}^{k} (-1)^{k-i} C_i \geq \sum_{i=0}^{k} (-1)^{k-i} \dim H^i(M, \mathbb{R})$$

Proof. Multiplication by e^f is an isomorphism between the complex $(\Omega^*(M), d + df)$ and $(\Omega^*(M), d)$. The corollary follows from Hodge theory and Corollary 2.6. $\square$

2.3 Foliated case

Let $F \subseteq TM$ be a subbundle (not necessarily integrable) of the tangent bundle of a closed manifold M, $f : M \to \mathbb{R}$ a smooth function. We are interested in the set

$$\mathrm{Crit}_F(f) := \{x \in M : df_x(F_x) = 0\}.$$

[1] the union is with multiplicity

Let $x_0 \in \mathrm{Crit}_F(f)$, $X \in \Gamma^\infty(TM)$, $Y \in \Gamma^\infty(F)$. One defines

$$d^2_{x_0} f(X,Y) := (XYf)(x_0).$$

Proposition 2.11. *The number $d^2_{x_0} f(X,Y)$ only depends on $X(x_0)$ and $Y(x_0)$. In other words $d^2_{x_0} f : T_{x_0}M \times F_{x_0} \to \mathbb{R}$ is a well defined bilinear form.*

Proof. This is clear for X. Let $Y' \in \Gamma^\infty(F)$ be another section such that $Y'(x_0) = Y(x_0)$. It follows that $Y' - Y$ could be written as the sum of elements of the form gZ, where $g : M \to \mathbb{R}$ is a function that vanishes at x_0, and $Z \in \Gamma^\infty(F)$. Hence

$$X(Y + gZ)f(x_0) = XYf(x_0) + X(g)(x_0)Zf(x_0) + g(x_0)XZf(x_0) = XYf(x_0),$$

where the second term vanishes because $x_0 \in \mathrm{Crit}_F(f)$ and the third because $g(x_0) = 0$. ☐

Proposition 2.12. *Let $Z = F^\perp \subseteq T^*M$. The section $df : M \to T^*M$ is transversal to Z if and only if for every $x \in \mathrm{Crit}_F(f)$, the bilinear map $d^2_x f$ is of maximal rank, that is the induced linear map $d^2_x f : T_xM \to F^*_x$ is surjective. Furthermore if this is the case, then $\mathrm{Crit}_F(f)$ is a smooth manifold whose tangent bundle is $T\,\mathrm{Crit}_F(f) = \ker(d^2_x f)$.*

Proof. It is clear that df is transverse to Z if and only if $df_{|F} : M \to F^*$ is transverse to the zero section. This is clearly equivalent to $d^2_x f$ being of maximal rank for every $x \in \mathrm{Crit}_F(f)$. ☐

By Thom's multijet transversality theorem (see [46, theorem 4.9]), the transversality condition of Proposition 2.12 is satisfied generically. We now fix such a function f, and suppose that F is integrable.

Remark 2.13. The manifold $\mathrm{Crit}_F(f)$ is of complementary dimension to F. It is transverse to F at a point $x \in \mathrm{Crit}_F(f)$ if and only if the critical point x of the function $f|l_x$ is non degenerate, where l_x is the leaf containing x. In particular, if the foliation doesn't admit a closed transversal, then there exist no smooth function which is Morse on each leaf.

Let

$$G = \mathrm{DNC}_{[0,1]}(\mathcal{G}(M,F), \mathrm{Crit}_F(f)) \rightrightarrows \mathrm{DNC}_{]0,1]}(M, \mathrm{Crit}_F(f))$$

be the deformation of foliation groupoid $\mathcal{G}(M,F)$ along the submanifold of its units $\mathrm{Crit}_F(f)$. By Theorem 1.28, this is a Lie groupoid whose Lie algebroid is equal to $\mathrm{DNC}(F, \mathrm{Crit}_F(f))$. Recall that by Examples 1.30,

$$\mathcal{N}^{\mathcal{G}(M,F)}_{\mathrm{Crit}_F(f)} = \{(X,Y,Z) : X, Z \in \mathcal{N}^M_{\mathrm{Crit}_F(f)}, Y \in F,\ X = Y + Z\} \rightrightarrows N^M_{\mathrm{Crit}_F(f)}.$$

Let g be a Euclidean metric on F. The Lie algebroid $\mathrm{DNC}(F, \mathrm{Crit}_F(f))$ admits then a Euclidean metric by Remarks 1.29. On $M \times \{t\}$, it is equal to $\frac{g}{t^2}$, and on $\mathcal{N}^M_{\mathrm{Crit}_F(f)}$ it is equal to $g_{|\mathrm{Crit}_F(f)}$. This metric is complete because for $t \neq 0$, the metric g is complete on each leaf, and for $t = 0$, it is complete by the description of $\mathcal{N}^{\mathcal{G}(M,F)}_{\mathrm{Crit}_F(f)}$ given above.

Let

$$\alpha = \mathrm{DNC}(df_{|F}) : \mathrm{DNC}(M, \mathrm{Crit}_F(f)) \to \mathrm{DNC}(F^*, \mathrm{Crit}_F(f)).$$

The map α is regarded as a section of

$$\mathfrak{A}G^* = \mathrm{DNC}(F, \mathrm{Crit}_F(f))^* = \mathrm{DNC}(F^*, \mathrm{Crit}_F(f)).$$

See Remarks 1.29 for the last equality. On $M \times \{t\}$, $\alpha = \frac{df_{|F}}{t^2}$, and on $\mathcal{N}^M_{\mathrm{Crit}_F(f)} \times \{0\}$ it is equal to $d^2 f$.

Let us show that hypothesis Proposition 2.2 hold.

1. It is clear that the form α is closed.

2. On $M \times \{t\}$, one has
$$\alpha^{\#_{\frac{g}{t^2}}} = (df_{|F})^{\#_g},$$
where $\#$ is the musical isomorphism. Hence $\mathcal{L}_{\alpha^\#}$ is independent of t for $t \neq 0$. Since the Riemannian metric is multiplied by a scalar and $\mathcal{L}_{\alpha^\#}$ is an operator of degree 0. It follows that $\mathcal{L}^*_{\alpha^\#}$ doesn't depend on t as well. Hence the norm of the section $\mathcal{L}_{\alpha^\#} + \mathcal{L}^*_{\alpha^\#}$ is independent of t for $t \neq 0$. Therefore, it is bounded by its boundness on M.

3. On $M \times \{t\}$, one has
$$\|\alpha\|_{\frac{g}{t^2}} = \left\| \frac{df_{|F}}{t^2} \right\|_{\frac{g}{t^2}} = \frac{\|df_{|F}\|_g}{t}$$

and on $N^M_{\mathrm{Crit}_F(f)} \times \{0\}$,

$$\|\alpha\|_g = \left\|d^2 f\right\|_{g_a}.$$

It follows from Remark 1.31 that $\|\alpha\|$ is proper as a function on $\mathrm{DNC}_{[0,1]}(M, \mathrm{Crit}_F(f))$.

Corollary 2.14. *The closure of the operator $d + d^* + c(\alpha)$ acting on $C^*\Lambda_{\mathbb{C}}\mathfrak{A}G^*$ is a regular self adjoint operator with a compact resolvent.*

By Corollary 2.14, the operator $d + d^* + c(\alpha)$ defines an element in $KK^0(\mathbb{C}, C^*G)$. By regarding the evaluation at 0 and at 1 of the previous element, one deduces:

Corollary 2.15. *The Euler characteristic $e(F)$ of $\mathcal{F}$ as an element in $KK(\mathbb{C}, C^*(\mathcal{G}(M, F)))$ can be represented by the element $d+d^*+c(\alpha)$ in $KK^0(\mathbb{C}, C^*\mathcal{N}^{\mathcal{G}(M,F)}_{\mathrm{Crit}_F(f)})$. More precisely,*

$$e(F) = \mathrm{Ind}^{\mathcal{G}(M,F)}_{\mathrm{Crit}_F(f)}([d + d^* + c(\alpha)]),$$

where Ind *is defined in [37].*

Chapter 3

Deformation to the normal cone with weight

In this chapter, we give an elementary construction of the deformation groupoids associated to the inhomogeneous pseudo-differential calculus. These groupoids were defined by Ponge [76] and van-Erp [95] independentaly and were later generalised by Choi and Ponge [76, 95, 77, 20, 22, 21, 99] and independently by van Erp and Yuncken [99]. Our construction is elementary is the sense that no analysis on local coordinates is required only the naturality of the deformation to the normal cone construction is needed.

This chapter is organised as follows; In Section 3.1, Proposition 3.1 is proved. This proposition will be used in Section 3.2, to prove that our construction gives the Lie groupoid defined in [77].

Let $H \subseteq TM$ be a subbundle. Recall the tangent groupoid defined by Connes

$$\mathrm{DNC}(M \times M, M) = M \times M \times \mathbb{R}^* \sqcup TM \times \{0\} \rightrightarrows M \times \mathbb{R}.$$

In Section 3.2, we prove that the the fiber over $M \times \{1\} \times \mathbb{R}$ of the Lie groupoid

$$\mathrm{DNC}(\mathrm{DNC}(M \times M, M), H \times \{0\}) \rightrightarrows M \times \mathbb{R}^2$$

have the same features as the Heisenberg groupoid.

In Section 3.3, an alternate description (in a more general situation) of the horizontal fiber regarded in Section 3.2 is given, also local charts of our space are given which shows that our space coincides with the space in [20, 22, 21] as a Lie groupoid.

In Section 3.4, we generalize the construction given in Section 3.2 but for a filtration of the tangent bundle proving that iterated deformation to the normal cone gives rise the Heisenberg groupoid in the general case. This section is independent of Section 3.2 and provides another proof of Theorem 3.2. Finally in Example 3.13, we show that the deformation constructed in [90] is just the quotient (see Section 1.1) of Heisenberg Lie groupoids.

3.1 Preliminary proposition

In this section, we prove Proposition 3.1, which will be used in Section 3.2 to prove that the fiber over $M \times \{1\} \times \mathbb{R}$ of $\mathrm{DNC}(\mathrm{DNC}(M \times M, M), H \times \{0\})$ have the same features as the Heisenberg groupoid.

Proposition 3.1. *Let $G \rightrightarrows G^0$ be a Lie groupoid, $H \subseteq G$ a Lie subgroupoid which is a bundle of connected Lie groups such that*

$$(dr - ds)(T_h G) \subseteq T_{s(h)} H^0, \quad \forall h \in H.$$

Then

1. *the Lie groupoid $\mathcal{N}_H^G \rightrightarrows N_{H^0}^{G^0}$ is a bundle of Lie groups.*
2. *the Lie groupoid $\mathcal{N}_H^G\big|_{H^0} \rightrightarrows N_{H^0}^{G^0}$ is a bundle of abelian Lie groups which is isomorphic (as a bundle of Lie groups) to $\mathfrak{A}G/\mathfrak{A}H \times_{H^0} N_{H^0}^{G^0}$.*
3. *In this way the Lie groupoid $\mathcal{N}_H^G \rightrightarrows N_{H^0}^{G^0}$ sits in an exact sequence of a bundle of Lie groups over $N_{H^0}^{G^0}$ whose fiber at $(x_0, X_0) \in N_{H^0}^{G^0}$ is*

$$1 \to \mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0} \to \left(\mathcal{N}_H^G\right)_{(x_0,X_0)} \to H_{x_0} \to 1.$$

Furthermore the action associated to this exact sequence of the Lie algebra $\mathfrak{A}H_{x_0}$ on the abelian group $\mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0}$ is as follows; if $X, Y \in \Gamma^\infty(\mathfrak{A}G)$ such that $X_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$, then by our assumption,

$$[X,Y](x_0) \mod \mathfrak{A}H_{x_0}$$

only depends on $X(x_0) \in \mathfrak{A}H_{x_0}$ and $Y(x_0) \mod \mathfrak{A}H_{x_0} \in \mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0}$. In particular the above exact sequence is central if and only if this action is trivial.

Proof. 1. The condition $(dr - ds)(T_h G) \subseteq T_{s(h)} H^0$ can be restated as the equality of the maps $Ns, Nr : T_h G/T_h H \to T_{s(h)} G^0/T_{s(h)} H^0$. Those two maps are the source and the target maps of the Lie groupoid $\mathcal{N}_H^G = \sqcup_{h\in H} T_hG/T_hH \rightrightarrows N_{H^0}^{G^0}$. By assumption, they coincide which means that $\mathcal{N}_H^G \rightrightarrows N_{H^0}^{G^0}$ is a bundle of Lie groups.

2. If $X \subseteq Y \subseteq Z$ are manifolds, then $N_Y^Z\big|_X = N_X^Z/N_X^Y$. It follows that $\mathcal{N}_H^G\big|_{H^0}$ is the surjective image by a groupoid morphism of the Lie groupoid $\mathcal{N}_{H^0}^G$. One has

$$\mathcal{N}_{H^0}^G = \{(X,Y,Z) : X, Z \in N_{H^0}^{G^0}, Y \in \mathfrak{A}G/\mathfrak{A}H, Z = X + \natural(Y)\}.$$

By assumption, the map $\natural : \mathfrak{A}G/\mathfrak{A}H \to N_{H^0}^{G^0}$ is the zero map. Hence $\mathcal{N}_{H^0}^G \rightrightarrows N_{H^0}^{G^0}$ is a bundle of abelian Lie groups, hence $\mathcal{N}_H^G\big|_{H^0} \rightrightarrows N_{H^0}^{G^0}$ as well.

3. (a) exactness at $\mathfrak{A}G_{x_0}/\mathfrak{A}H_{x_0}$ is clear, because $\mathcal{N}_H^G\big|_{H^0}$ is a subgroupoid of $\mathcal{N}_H^G$

   (b) exactness at $\left(\mathcal{N}_H^G\right)_{(x_0,X_0)}$ follows directly from the definitions.

   (c) the map $s : G \to G^0$ is a submersion, hence exactness at H_{x_0}.

Let us prove that $[X, Y]$ only depends on $X(x_0)$ and $Y(x_0)$, where $X, Y \in \Gamma^\infty(\mathfrak{A}G)$ such that $X_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$.

- If Y vanishes at x_0, then locally it can be written as the sum of sections of the form fZ, where $f : M \to \mathbb{R}$ vanishes at x_0 and $Z \in \Gamma^\infty(\mathfrak{A}G)$. One has
$$[X, fZ] = f(x_0)[X,Z](x_0) + df_{x_0}(\natural(X(x_0)))Z(x_0) = 0,$$
because $X(x_0) \in \mathfrak{A}H_{x_0}$ and H is a bundle of Lie groups, hence $\natural(X(x_0)) = 0$.

- If $Y_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$, then $[X,Y](x_0) \in \mathfrak{A}H_{x_0}$ because the Lie bracket computation could be carried out inside $\mathfrak{A}H$.

- If X vanishes at x_0, then $dX_{x_0} : T_{x_0}G^0 \to \mathfrak{A}_{x_0}G$ is well defined. It is well known that $[X,Y](x_0) = -dX_{x_0}(\natural(Y(x_0)))$. This formula can be proved locally by writing X as sum of fZ. The condition $X_{|H^0} \in \Gamma^\infty(\mathfrak{A}H)$ implies that $dX_{x_0}(T_{x_0}H^0) \subseteq \mathfrak{A}H^0$. The assumption on $dr - ds$ implies that $\natural(Y(x_0)) \in T_{x_0}H^0$, hence $[X,Y](x_0) \in \mathfrak{A}H_{x_0}$.

That this is the action associated to the abelian extension of $\left(\mathcal{N}_H^G\right)_{(x_0,X_0)}$ is then clear. $\square$

3.2 Computations in the case of a single subbundle

Let M be a smooth manifold, $H \subseteq \mathcal{N}_M^{M\times M} = TM$ a subbundle. In this section we prove Theorem 3.2, which proves the claim made in the introduction (at least on the algebraic level) that the fiber of the groupoid $\mathrm{DNC}^2(M\times M, M, H\times\{0\}) \rightrightarrows \mathrm{DNC}^2(M, M, M\times\{0\}) = M\times\mathbb{R}^2$ over $M\times\{1\}\times\mathbb{R}$ is equal to the groupoid constructed in [20, 22, 21, 99]. In Section 3.3, we will write local charts which will prove that in fact the fiber is equal as a smooth manifold to the one constructed in [20, 22, 21, 99].

Before stating the theorem, let us recall the constuction of the Levi form $\mathcal{L}$: the map

$$\Gamma^\infty(H)\times\Gamma^\infty(H)\to\Gamma^\infty(TM/H), \quad (X,Y)\to[X,Y] \mod H$$

is $C^\infty(M)$-linear because

$$[fX,Y] = f[X,Y] - XfY = f[X,Y] \mod H.$$

Hence it comes from an anti symmetric bilinear bundle map $\mathcal{L} : H\times H\to TM/H$.

Theorem 3.2. *The groupoid $\mathcal{N}_{H\times\{0\}}^{\mathrm{DNC}(M\times M,M)} \rightrightarrows N_{M\times\{0\}}^{M\times\mathbb{R}} = M\times\mathbb{R}$ is isomorphic (by an isomorphism which is equal to the identity on the objects) to the bundle of Lie groups $H\oplus TM/H\times\mathbb{R} \rightrightarrows M\times\mathbb{R}$ equipped with the group law*

$$(h,n,t)\cdot(h',n',t) = \left(h+h', n+n'+\frac{t}{2}\mathcal{L}(h,h')\right).$$

Proof. First we apply Proposition 3.1 to $\mathrm{DNC}(M\times M, M) \rightrightarrows M\times\mathbb{R}$ and $H\times\{0\} \rightrightarrows M\times\{0\}$. Let us check the condition of Proposition 3.1 and the triviality of the action.

- Since $\pi_{\mathbb{R}}\circ r = \pi_{\mathbb{R}}\circ s$, the condition of Proposition 3.1 is satisfied.
- the triviality of the action is immediate to check. If X is a section of TM over $M\times\mathbb{R}$ which vanishes on $M\times\{0\}$, Y is a section of TM over $M\times\mathbb{R}$ which vanishes on $M\times\{0\}$ and whose ∂_t-derivative on $M\times\{0\}$ is in H, then the vector field $[X,Y]$ vanishes over $M\times\{0\}$.

The central exact sequence of bundles of Lie groups over $(x_0,t_0)\in N_{M\times\{0\}}^{M\times\mathbb{R}} =$

$M \times \mathbb{R}$ given by Proposition 3.1 is then equal to

$$1 \to T_{x_0}M/H_{x_0} \to \left(\mathcal{N}_{H\times\{0\}}^{\mathrm{DNC}(M\times M,M)}\right)_{(x_0,t_0)} \to H_{x_0} \to 1.$$

There exists a quite natural section of this exact sequence: let $h \in H_{x_0}$, $f : \mathbb{R} \to M$ any smooth function such that $f'(0) = h$ and $f'(t) \in H_{f(t)}\ \forall t$,

$$\begin{aligned}
&\sigma_{x_0,t_0}(h,\cdot) : \mathbb{R} \to \mathrm{DNC}(M \times M, M) \\
&\sigma_{x_0,t_0}(h,u) = (f(tu), f(0), tu) && \text{if } tu \neq 0 \\
&\sigma_{x_0,t_0}(h,0) = (x_0, h, 0) && \text{if } tu = 0
\end{aligned}$$

One then sees immediately that the map

$$\begin{aligned}
\mathfrak{S}_{x_0,t_0} : H_{x_0} &\to \left(\mathcal{N}_{H\times\{0\}}^{\mathrm{DNC}(M\times M,M)}\right)_{(x_0,t_0)} \\
h &\to \left(\frac{\partial}{\partial u}\Big|_{u=0}\sigma_{x,t}(h,u) \mod T_{(x_0,h,0)}(H \times \{0\})\right)
\end{aligned}$$

is well defined (i.e, doesn't depend on the choice of f) and is a section of the above exact sequence.

The map $\mathfrak{S}_{x_0,t_0}$ is not a group homomorphism. For $h_1, h_2 \in H_{x_0}$, we have

$$\mathfrak{S}_{x_0,t_0}(h_1)\mathfrak{S}_{x_0,t_0}(h_2)\mathfrak{S}_{x_0,t_0}(-h_1-h_2) = \frac{t_0}{2}\mathcal{L}(h_1,h_2) \in T_{x_0}M/H_{x_0}.$$

This follows from the definition of $\mathcal{L}$. □

Corollary 3.3. *The fiber of the groupoid*

$$\mathrm{DNC}^2(M \times M, M, H \times \{0\}) \rightrightarrows M \times \mathbb{R}^2$$

over $M \times \{1\} \times \mathbb{R}$ is equal to (as an algebraic groupoid) to

$$M \times M \times \mathbb{R}^* \sqcup H \oplus TM/H \times \{0\} \rightrightarrows M \times \mathbb{R},$$

where the groupoid structure on $M \times M \times \mathbb{R}^*$ is the pair groupoid, and on $H \oplus TM/H$ is the bundle of nilpotent Lie groups

$$(h,n)\cdot(h',n') = \left(h + h', n + n' + \frac{1}{2}\mathcal{L}(h,h')\right).$$

Since H is $\mathbb{R}^*$ invariant, by Section 1.5 we have two group actions λ^1, λ^0 of $\mathbb{R}^*$. Under the above identification the two actions λ^1 and λ^0 become

$$\lambda^1_s(h,n,t) = (\frac{h}{s}, \frac{n}{s}, \frac{t}{s}), \quad \lambda^0_s(h,n,t) = (h, \frac{n}{s}, ts).$$

3.3 Another description of $N_{H\times\{0\}}^{\mathrm{DNC}(M,V)}$

Let M be a smooth manifold, V a submanifold, $H \subseteq N_V^M$ a smooth subbundle, $\mathcal{H}$ the lift of H to TM. In other words $\mathcal{H}$ is a subbundle of the restriction of TM to V such that $TV \subseteq \mathcal{H}$ and $H = \mathcal{H}/TV$. In this section we give an alternate description of the fiber $\left(\pi^{(0,1)}\right)^{-1}(\{1\}\times\mathbb{R})$ of the space $\mathrm{DNC}^2(M, V, H\times\{0\})$.

Definition 3.4. Let $\tilde{N}^M_{V,H}$ the set of smooth functions $f : \mathbb{R} \to M$ such that $f(0) \in V$ and $f'(0) \in \mathcal{H}_{f(0)}$.

Let $N^M_{V,H}$ be the quotient of $\tilde{N}^M_{V,H}$ by the equivalence relation where $f, g \in \tilde{N}^M_{V,H}$ are equivalent if and only if

1. $f(0) = g(0)$
2. $f'(0) - g'(0) \in T_{f(0)}V$.
3. for every smooth function $l : M \to \mathbb{R}$ which vanishes on V and whose derivative dl vanishes on $\mathcal{H}$, one has $(l\circ f)''(0) = (l \circ g)''(0)$.

Let $\pi_{\mathbb{R}} : \mathrm{DNC}(M, V) \to \mathbb{R}$ be the projection. Since $\pi_{\mathbb{R}}(H) = 0$, the map $N\pi_{\mathbb{R}} : N_H^{\mathrm{DNC}(M,V)} \to N_0^{\mathbb{R}} = \mathbb{R}$ is well defined. We claim that the set $N^M_{V,H}$ is in a natural bijection with $(N\pi_{\mathbb{R}})^{-1}(1)$. To see this let $f \in \tilde{N}^M_{V,H}$. Since $f(0) \in V$, the function

$$\mathrm{DNC}(f) : \mathbb{R} \to \mathrm{DNC}(M, V), \quad t \to (f(t), t), \neq 0,\ 0 \to (f'(0), 0)$$

is smooth. And since $f'(0) \in H$ it follows that $\mathrm{DNC}^2(f) : \mathbb{R} \to \mathrm{DNC}^2(M, V, H)$ is a welll defined smooth map. Its value at zero is an element in $N_H^{\mathrm{DNC}(M,V)}$ which is clearly in $(N\pi_{\mathbb{R}})^{-1}(1)$.

Proposition 3.5. *the map*

$$\beta : N^M_{V,H} \to (N\pi_{\mathbb{R}})^{-1}(1), \quad [f] \to [\mathrm{DNC}(f)]$$

is a well defined bijection

Let us remark that the map β is not a linear map and in fact the space $N_{V,H}^M$ is not a vector bundle.

Proof. In Section 1.4, two types of functions on $\mathrm{DNC}(M,V)$ were described which generate the ring of smooth functions on $\mathrm{DNC}(M,V)$. By regarding each type we see that for two functions $f, g \in \tilde{N}_{V,H}^M$, the classes in $N_H^{\mathrm{DNC}(M,V)}$ of $\mathrm{DNC}(f)$ and $\mathrm{DNC}(g)$ are equal if and only if the classes of f and g are equal in $N_{V,H}^M$. Hence β is well defined and injective. Surjectivity follows by looking at a local chart as described below. $\square$

Let $\psi : N_V^M \to M$ be a tubular neighbourhood embedding, $L : H \oplus N_V^M/H \to N_V^M$ a linear isomorphism given by the choice of a complementary subbundle of H inside N_V^M, $\phi = \psi \circ L$.

By the local charts descriped in Section 1.4, the following is a local chart for $\mathrm{DNC}(M,V)$:

$$\begin{aligned}
&\tilde{\phi} : H \oplus N_V^M/H \times \mathbb{R} \to \mathrm{DNC}(M,V)\\
&(h,n,t) \to (\phi(th,tn),t), \quad t \neq 0\\
&(h,n,0) \to (L(h,n,0),0).
\end{aligned}$$

Therefore the following is a local chart for $\mathrm{DNC}^2(M,V,H\times\{0\})$

$$\begin{aligned}
&H \oplus N_V^M/H \times \mathbb{R} \times \mathbb{R} \to \mathrm{DNC}^2(M,V,H\times\{0\})\\
&(h,n,t,u) \to (\phi(uth,u^2tn),ut,u) \in M \times \mathbb{R}^* \times \mathbb{R}^* \quad t \neq 0, u \neq 0\\
&(h,n,0,u) \to (L(h,un),0,u) \in N_V^M \times \{0\} \times \{u\}, \quad u \neq 0\\
&(h,n,t,0) \to (h,n,t,0) \in N_H^{\mathrm{DNC}(M,V)} \times \{0\},
\end{aligned}$$

where in the last identity we identified $N_H^{\mathrm{DNC}(M,V)}$ with $H \oplus N_V^M \oplus \mathbb{R}$ using $\tilde{\phi}$. In this local picture, $\pi^{(0,1)}$ is the projection $(h,n,t,u) \to (t,u)$.

Let

$$\mathrm{DNC}_H(M,V) := M \times \mathbb{R}^* \sqcup N_{V,H}^M \times \{0\}.$$

We equip $\mathrm{DNC}_H(M,V)$ with a smooth structure by identifying it with $\left(\pi_{\mathbb{R}}^{(0,1)}\right)^{-1}(\{1\}\times$

$\mathbb{R}$) using the map β. Its local charts are hence given by

$$\begin{aligned} H \oplus N_V^M/H \times \mathbb{R} &\to \mathrm{DNC}_H(M,V) \\ (h,n,u) &\to (\phi(uh,u^2n),u), \quad u \neq 0 \\ (h,n,0) &\to ([t \mapsto \phi(L(th,t^2n))],0). \end{aligned}$$

The space $\mathrm{DNC}_H(M,V)$ is called *the deformation to the normal cone of M along V with weight H.*

Remark 3.6. All the other fibers $\left(\pi_{\mathbb{R}}^{(0,1)}\right)^{-1}(\{t\}\times\mathbb{R})$ for $t \neq 0$ are isomorphic to $\left(\pi_{\mathbb{R}}^{(0,1)}\right)^{-1}(\{1\}\times\mathbb{R})$ by a rescaling in the u-variable. The fiber $\left(\pi_{\mathbb{R}}^{(0,1)}\right)^{-1}(\{0\}\times\mathbb{R})$ is equal to $\mathrm{DNC}(N_V^M,H)$. In particular the space $\mathrm{DNC}^2(M,V,H)$ should be seen as a deformation of the space $\mathrm{DNC}_H(M,V)$ to the simpler space $\mathrm{DNC}(N_V^M,H)$.

Since H is $\mathbb{R}^*$-invariant, by Section 1.5 it follows that there is an $(\mathbb{R}^*)^2$ action on $\mathrm{DNC}^2(M,V,H\times\{0\})$. It follows from Equation (1.1) in Section 1.5 that $\left(\pi_{\mathbb{R}}^{(0,1)}\right)^{-1}(\{1\}\times\mathbb{R})$ is invariant under the diagonal $\lambda_u^{(1)}\lambda_u^{(0)}$. This action is described by $u\cdot(x,t)=(x,tu)$ and $u\cdot([f],0)=([f(\frac{\cdot}{u})],0)$ for $f\in \tilde{N}_{V,H}^{M}$.

Corollary 3.7. *Let $(M,V),(M',V')$ be smooth manifold pairs, $H\subseteq N_V^M, H'\subseteq N_{V'}^{M'}$ subbundles, $g: M\to M'$ a smooth map such that $g(V)\subseteq V'$ and $dg(H)\subseteq H'$. Then the maps*

- $Ng : N_{V,H}^M \to N_{V',H'}^{M'} \quad [f]\to[g\circ f]$
- $\mathrm{DNC}(g): \mathrm{DNC}_H(M,V)\to \mathrm{DNC}_{H'}(M',V')$
$$\begin{aligned}(x,t)&\to(g(x),t)\\ ([f],0)&\to([g\circ f],0)\end{aligned}$$

are well defined and smooth.

Proof. This is a corollary of Proposition 1.25 applied twice and the identification of $\mathrm{DNC}_H(M,V)$ with $\left(\pi_{\mathbb{R}}^{(0,1)}\right)^{-1}(\{1\}\times\mathbb{R})\subseteq \mathrm{DNC}^2(M,V,H\times\{0\})$. □

Proposition 3.8. *Let M_1,M_2,M be manifolds, $V_i\subseteq M_i, V\subseteq M$ submanifolds, $H_i\subseteq N_{V_i}^{M_i}$, $H\subseteq N_V^M$ vector subbundles, $f_i: M_i\to M$ smooth maps such that*

1. *$f_i(V_i)\subseteq V$*
2. *the maps $f_i: M_i\to M$ are transverse*

3. *the maps* $f_i|V : V_i \to V$ *are transverse*

4. $H = df_1(H_1) + df_2(H_2)$,

then

1. *the maps* $\mathrm{DNC}(f_i) : \mathrm{DNC}_{H_i}(M_i, V_i) \to \mathrm{DNC}_H(M, V)$ *are transverse.*

2. *the natural map*

$$\mathrm{DNC}_{H_1\times_H H_2}(M_1\times_M M_2, V_1\times_V V_2) \to \mathrm{DNC}_{H_1}(M_1, V_1)\times_{\mathrm{DNC}_H(M,V)}\mathrm{DNC}_{H_2}(M_2, V_2)$$

is a diffeomorphism.

Proof. This is a corollary of Proposition 1.27 applied twice and the identification of $\mathrm{DNC}_H(M, V)$ with $\left(\pi_{\mathbb{R}}^{(0,1)}\right)^{-1}(\{1\} \times \mathbb{R}) \subseteq \mathrm{DNC}^2(M, V, H \times \{0\})$. □

Theorem 3.9. *Let* $G \rightrightarrows G^0$ *be a groupoid,* $G' \rightrightarrows G'^0$ *a subgroupoid,* $H \subseteq \mathcal{N}_{G'}^{G}$ *a* $\mathcal{VB}$*-subgroupoid [82, 67]. Then*

1. *the space* $\mathcal{N}_{G',H}^{G} \rightrightarrows N_{G'^0,H^0}^{G^0}$ *is a Lie groupoid whose algebroid is equal to* $N_{\mathfrak{A}G',\mathfrak{A}H}^{\mathfrak{A}G}$.

2. *the space* $\mathrm{DNC}_H(G, G') \rightrightarrows \mathrm{DNC}_{H^0}(G^0, G'^0)$ *is a Lie groupoid whose Lie algebroid is equal to* $\mathrm{DNC}_{\mathfrak{A}H}(\mathfrak{A}G, \mathfrak{A}G')$.

Proof. This is a corollary of Corollary 3.7 and Proposition 3.8. □

Example 3.10. Let $F \subseteq TM$ be an integrable subundle. We regard the foliation groupoid $\mathcal{G}(M, F) \rightrightarrows M$ as an immersed subgroupoid of $M \times M \rightrightarrows M$ by the map

$$(x, [\gamma], y) \to (x, y).$$

This map is not injective but the Lie groupoid $\mathrm{DNC}(M \times M, \mathcal{G}(M, F)) \rightrightarrows M \times \mathbb{R}$ is still well defined by Remarks 1.32. Its underlying manifold is a second countable locally Hausdorff manifold.

The vector bundle TM/F will be denoted by $\nu(F)$. If $\gamma : [0, 1] \to M$ is path tangent to the leaves, then its holonomy defines a map $d\gamma : \nu(F)_{\gamma(0)} \to \nu(F)_{\gamma(1)}$. One then sees that the groupoid

$$\mathcal{N}_{\mathcal{G}(M,F)}^{M\times M} = \{(x, [\gamma], y, X) : (x, [\gamma], y) \in \mathcal{G}(M, F), X \in \nu(F)_y\} \rightrightarrows M.$$

The product is then given by

$$(x, [\gamma], y, X) \cdot (y, [\gamma'], z, Y) = (x, [\gamma\gamma'], z, d\gamma'(X) + Z).$$

Let $H \subseteq \nu(F)$ be a holonomy invariant subbundle, i.e such that for any leafwise path $\gamma : [0, 1] \to M$, one has $d\gamma(H_{\gamma(0)}) = H_{\gamma(1)}$. It follows that

$$L := \{(x, [\gamma], y, X) \in \mathcal{N}^{M\times M}_{\mathcal{G}(M,F)} : X \in H_y\} \subseteq \mathcal{N}^{M\times M}_{\mathcal{G}(M,F)}$$

is a Lie subgroupoid. The groupoid

$$\mathcal{N}^{M\times M}_{\mathcal{G}(M,F),L} = \{(x, [\gamma], y, X, Y) : X \in H_y, Y \in \nu(F)_y\} \rightrightarrows M$$

has then the groupoid law

$$(x, [\gamma], y, X, Y)\cdot(y, [\gamma'], z, X', Y') = (x, [\gamma\gamma'], z, d\gamma'(X)+X', d\gamma'(Y)+Y'+\frac{1}{2}\mathcal{L}(d\gamma'(X), X')),$$

where $\mathcal{L} : H \times H \to v(F)/H$ is a Levi form defined similarly to the one defined in Section 3.2.

3.4 Carnot Groupoid

A more general groupoid will be constructed starting from the following data: Let M be a smooth manifold, $0 = H^0 \subseteq H^1 \subseteq \cdots \subseteq H^{k+1} = TM$ be vector bundles such that

$$[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^{i+j}),$$

where $H^i = TM$ for $i > k$. We will calculate the Lie algebroid of this groupoid and hence show that it is equal to the groupoid constructed in [77, 20, 22, 21, 99].[1]

Since $[\Gamma^\infty(H^i), \Gamma^\infty(H^i)] \subseteq \Gamma^\infty(H^{i+j})$, it follows that the map

$$\begin{aligned}\Gamma^\infty(H^i/H^{i-1}) \times \Gamma^\infty(H^j/H^{j-1}) &\to \Gamma^\infty(H^{i+j}/H^{i+j-1})\\ (X, Y) &\to [X, Y] \mod \Gamma^\infty(H^{i+j-1})\end{aligned}$$

[1] Following Ponge's recommendation, the deformation groupoid in the case of a filtration of TM should be called the Carnot groupoid.

is a $C^\infty(M)$-bilinear map, hence it comes from an antisymmetric bilinear map

$$\mathcal{L} : H^i/H^{i-1} \times H^j/H^{j-1} \to H^{i+j}/H^{i+j-1}.$$

For each $a \in M$, the map $\mathcal{L}$ defines the structure of a Lie algebra on $\mathcal{G}(H^\cdot)_a := \oplus_i H^i_a/H^{i+1}_a$ by

$$[X, Y] = \mathcal{L}(X, Y), \quad \text{for } X \in H^i_a/H^{i+1}_a, Y \in H^j_a/H^{j+1}_a.$$

By Baker–Campbell–Hausdorff formula, the vector space $\mathcal{G}(H^\cdot)_a$ admits the structure of a nilpotent Lie group. It is clear that the structure of group is C^∞ in a, hence $\mathcal{G}(H^\cdot)$ is a bundle of nilpotent Lie groups. We will define a Lie groupoid denoted by $\mathrm{DNC}_{H^\cdot}(M \times M, M)$ by induction on k whose underlying set is equal to

$$M \times M \times \mathbb{R}^* \sqcup \mathcal{G}(H^\cdot) \times \{0\}.$$

and whose Lie algebroid is equal to

$$\Gamma^\infty(\mathfrak{A}_{H^\cdot}) = \{X \in \Gamma^\infty(TM \times \mathbb{R}) : \partial^i_t X_{|t=0} \in \Gamma^\infty(H^i) \forall i \geq 0\}$$

For $k = 1$, this is just $\mathrm{DNC}_{H^1}(M \times M, M) \rightrightarrows M \times \mathbb{R}$ defined in Section 3.3. By induction assuming it is defined for $k - 1$, that is the Lie groupoid

$$\begin{aligned}\mathrm{DNC}_{H^1,\dots,H^{k-1}}(M \times M, M) &= M \times M \times \mathbb{R}^* \sqcup \mathcal{G}(H^1, \dots, H^{k-1}) \times \{0\} \\ &= M \times M \times \mathbb{R}^* \sqcup H^1 \oplus H^2/H^1 \oplus \cdots \oplus TM/H^{k-1} \times \{0\}\end{aligned}$$

is well defined. The subset $H^1 \oplus H^2/H^1 \oplus \cdots \oplus H^k/H^{k-1}$ is a Lie subgroupoid of $\mathcal{G}(H^1, \dots, H^{k-1})$ precisely because

$$[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^k), \quad i + j = k.$$

Therefore the space

$$\begin{aligned}&\mathrm{DNC}(\mathrm{DNC}_{H^1,\dots,H^{k-1}}(M \times M, M), H^1 \oplus H^2/H^1 \oplus \cdots \oplus H^k/H^{k-1} \times \{0\}) \\ &\qquad \rightrightarrows \mathrm{DNC}(M \times \mathbb{R}, M \times \{0\}) = M \times \mathbb{R}^2\end{aligned}$$

is a Lie groupoid, where we used Remarks 1.29. The Lie algebroid of this groupoid

is then
$$\mathrm{DNC}(\mathfrak{A}_{H^1,\dots,H^{k-1}}, H^1 \oplus \cdots \oplus H^k/H^{k-1})$$

Using Remarks 1.29, we get that the space of sections of this algebroid is then equal to

$$\begin{aligned}\Gamma^\infty(&\mathrm{DNC}(\mathfrak{A}_{H^1,\dots,H^{k-1}}, H^1 \oplus \cdots \oplus H^k/H^{k-1})\\ &= \{X \in \Gamma^\infty(TM \times \mathbb{R} \times \mathbb{R}) : \partial_t^i(X)(0,u) \in \Gamma^\infty(H^i)\ \forall 0 \leq i \leq k-1, u \in \mathbb{R}\\ &\& \ \partial_t^k(X)(0,0) \in \Gamma^\infty(H^k)\}.\end{aligned}$$

We define $\mathrm{DNC}_{H^1,\cdots,H^k}(M \times M, M)$ as the fiber of $\mathrm{DNC}(\mathrm{DNC}_{H^1,\dots,H^{k-1}}(M \times M, M), H^1 \oplus H^2/H^1 \oplus \cdots \oplus H^k/H^{k-1} \times \{0\})$ over $M \times \{1\} \times \mathbb{R}$. This is clearly a Lie groupoid.

It follows from the above description of $\Gamma^\infty(\mathrm{DNC}(\mathfrak{A}_{H^1,\dots,H^{k-1}}, H^1 \oplus \cdots \oplus H^k/H^{k-1}))$ by restricting to the diagonal we get that if

$$X \in \Gamma^\infty(\mathrm{DNC}(\mathfrak{A}_{H^1,\dots,H^{k-1}}, H^1 \oplus \cdots \oplus H^k/H^{k-1}))$$

then $\partial_t^i X(0,0) \in \Gamma^\infty(H^i)$ for all $0 \leq i \leq k$, where we used that $X(0,u) = 0$. This fnishes the induction, and proves that Lie algebroid of $\mathrm{DNC}_{H^\cdot}(M \times M, M)$ is equal to $\mathfrak{A}_{H^\cdot}$. Hence we proved the following

Theorem 3.11. *The groupoid* $\mathrm{DNC}_{H^\cdot}(M \times M, M)$ *is the same as the groupoid constructed in [20, 22, 21, 99].*

Remarks 3.12.

1. In [99], a more general case is reagrded where starting from a groupoid G, subbundles $H^1 \subseteq \cdots \subseteq H^r = \mathfrak{A}G$ such that $[\Gamma^\infty(H^i), \Gamma^\infty(H^j)] \subseteq \Gamma^\infty(H^{i+j})$ they construct a groupoid $\mathrm{DNC}_{H^\cdot}(G, G^0)$. It is clear that the above construction works equally well for this case with only notational changes. The advantage of our approach is that we can do the more general case of a groupoid inside another without any extra difficulty.

2. The groupoid

$$\begin{aligned}\mathrm{DNC}^{k+1}(M\times M, M, H^1 \times \{0\}, \dots, H^1 \oplus \cdots \oplus H^k/H^{k-1} \times \mathbb{R}^{k-1} \times \{0\})\\ \rightrightarrows \mathrm{DNC}^{k+1}(M, M \times \{0\}, \dots, M \times \mathbb{R}^{k-1} \times \{0\}) = M \times \mathbb{R}^{k+1}.\end{aligned}$$

is a Lie groupoid which contains the 'deformations in all the directions'.

This groupoid admits an $(\mathbb{R}^*)^{k+1}$ action as in Section 1.5. The fiber over $(1,\dots,1,0)$ is then equal to $\mathrm{DNC}_{H^1,\dots,H^k}(M\times M,M)$. The action $\mathbb{R}^*$ defined on $\mathrm{DNC}_{H^1,\dots,H^k}(M\times M,M)$ defined in [99] is then just the diagonal action of $\mathbb{R}^{k+1}$ which by induction is easily seen to preserve the fiber $(1,\dots,1,0)$.

For example, in the case $k=2$, this gives

$$\begin{aligned}\mathrm{DNC}^3(M\times M, H^1\times\{0\}, H^1\oplus H^2/H^1\times\mathbb{R}) &= M\times M\times\mathbb{R}^*\times\mathbb{R}^*\times\mathbb{R}^*\\ &\sqcup TM\times\{0\}\times\mathbb{R}^*\times\mathbb{R}^*\sqcup H^1\oplus TM/H^1\times\mathbb{R}\times\{0\}\times\mathbb{R}^*\\ &\sqcup H^1\oplus H^2/H^1\oplus TM/H^2\times\mathbb{R}\times\mathbb{R}\times\{0\}\end{aligned}$$

Let us remark that the subgroupoid $H^1\oplus H^2/H^1\oplus TM/H^2\times\mathbb{R}\times\mathbb{R}\times\{0\}$ is not trivial as a groupoid, it has a structure

$$\begin{aligned}(h_1,h_2,h_3,t,u,0)\cdot(k_1,k_2,k_3,t,u,0) = \Big(h_1+k_1, h_2+k_2+\frac{t}{2}[h_1,k_1],\\ h_3+k_3+\frac{tu}{2}\left([h_1,k_2]+[h_2,k_1]\right)+\frac{t^2u}{12}\left([h_1,[h_1,k_1]]+[k_1,[k_1,h_1]]\right), t, u, 0\Big)\end{aligned}$$

Similarly for $M\times\{0\}\times\mathbb{R}^*\times\mathbb{R}^*$ and $H^1\oplus TM/H^1\times\mathbb{R}\times\{0\}\times\mathbb{R}^*$.

3. The existence of the Lie groupoid $\mathrm{DNC}_{H^\cdot}(M\times M,M)$ follows from Theorem 1.5.

Example 3.13. Let $V\subseteq M$ a smooth submanifold such that $H^i\cap TV$ is of locally of finite rank. It is then clear that $[\Gamma^\infty(H^i\cap TV),\Gamma^\infty(H^j\cap TV)]\subseteq\Gamma^\infty(H^{i+j}\cap TV)$. Let $G(H^\cdot)$ the bundle of nilpotent Lie groups $\oplus_i H/H^{i-1}$, $G(H^\cdot\cap TV)$ be the bundle of nilpotent Lie groups $\oplus(H^i\cap TV)/(H^{i-1}\cap TV)$. In [90], the authors define a smooth manifold whose underlying set is equal to $M\times\mathbb{R}^*\sqcup G(H^\cdot)_{|V}/G(H^\cdot\cap TV)$, where $G(H^\cdot)_{|V}$ is the restriction of $G(H^\cdot)$ to V. Similarly to the description in Examples 1.30 of the classical deformation to the normal as a quotient space (see Section 1.1), the space defined in [90] can also be written as $\mathrm{DNC}_{H^\cdot}(M\times M,M)/\mathrm{DNC}_{H^\cdot\cap TV}(V\times V,V)$.

Example 3.14. Following the notation of Example 3.10. Let F be a foliation, $H^1\subseteq\cdots\subseteq H^{k+1}=\nu(F)$ subbundles such that if $X\in\Gamma^\infty(H^i)$ and $Y\in\Gamma^\infty(H^j)$, then

$$[X,Y]\in\Gamma^\infty(H^{i+j}).$$

with the convention $H^s=\nu(F)$ for $s>k$ and such that if $i\in\{1,\dots,k\}$, $\gamma:[0,1]\to M$ a path tangent to the leaves, then $d\gamma H^i_{\gamma(0)}=H^i_{\gamma(1)}$. In Example 3.10, we defined the groupoid $\mathrm{DNC}_{L^1}(M \times M, \mathcal{G}(M, F))$. We can by an induction, similar to the above, construct the groupoid $\mathrm{DNC}_{H^\cdot}(M \times M, \mathcal{G}(M, F))$.

Chapter 4

KK-theory and Chern-Simons invariants

Introduction

In this chapter, we will define a primitive element in equivariant KK-theory of the classifying space of trivialised unitary flat vector bundles.

The organisation of this chapter is as follows;

1. In section 4.1, we recall Chern-Weil theory for tracial C^*-algebras. This section follows closely the presentation in the article by Fomenko and Mishchenko [68].

2. In section 4.2, the definition of the Chern Simons invariants in KK-theory is given.

3. In section 4.3, we construct a more primitive element in equivariant KK-theory, that is done for any compact group.

4.1 Chern-Weil theory

In this section, we recall Chern-Weil theory for tracial C^*-algebras. This section follows closely the article by Fomenko and Mishchenko [68], and article by Simons and Sullivan [68].

Let M be a connected smooth manifold, A a unital C^*-algebra, P a finitely generated (f.g) projective right A-module. A smooth A-vector bundle V with fiber P is a smooth 1-cocycle on M with coefficients in the group of A-linear automorphisms

$GL(P)$. We will denote by $\Gamma^\infty(V)$ the right A-module of smooth sections of V, by $\Omega^\cdot(M,V) := \Omega(M)\hat{\otimes}_{C^\infty(M)}\Gamma^\infty(V)$ the space of differential forms on M with values in V. This is a graded module over $\Omega^\cdot(M,A)$.

Definition 4.1. An A-connection on V is a $\mathbb{C}$-linear map $\nabla : \Gamma^\infty(V) \to \Omega^1(M,V)$ which satisfies Leibniz rule

$$\nabla(sf) = \nabla(s)f + s \otimes df, \quad \forall s \in \Gamma^\infty(V), f \in C^\infty(M,A).$$

Like in the classical theory, a connection ∇ extends to a $\mathbb{C}$-linear map ∇ : $\Omega^\cdot(M,V) \to \Omega^\cdot(M,V)$ satisfying Leibniz rule. Locally, if U is an open set such that $V \simeq U \times P$, then an A-connection is locally a map $\nabla = d + L$, where $L \in \Omega^1(U, \mathrm{End}_A(P))$ It follows that ∇^2 is the left action by $dL + L^2$.

Let $\tau : A \to \mathbb{C}$ be a finite trace such that $\tau(1) = 1$. The trace τ extends to $M_n(A)$, by the formula $\tau(M) = \sum \tau(M_{i,i})$. The trace extends as well to $\mathrm{End}_A(P)$ for an A-projective module P by using a complementary module Q, as follows

$$\mathrm{End}_A(P) \subseteq \mathrm{End}_A(P \oplus Q) \simeq M_n(A) \xrightarrow{\tau} \mathbb{C}.$$

It is straightforward to verify that this extension doesn't depend on the choice of Q. It follows that if ∇ is an A-connection, then for $k \geq 1$, the forms $\tau((dL + L^2)^k)$ glue together to form a complex valued $2k$-form on M, that will be denoted by $\tau(\nabla^{2k})$.

Let V be an A-vector bundle, and ∇ an A-connection on V. The Chern character is defined by the formula

$$\mathrm{Ch}_\tau(V,\nabla) := \exp(\frac{1}{2\pi i}\nabla^2) = \sum_{k=0}^{\infty} \frac{1}{k!(2\pi i)^k}\tau(\nabla^{2k}) \in \Omega^{\mathrm{even}}(M).$$

Let ∇_t be a C^1-path of A-connections. Locally if $\nabla_t = d + L_t$, then $\dot{\nabla}_t = \dot{L}_t$ is hence well defined. The forms $\tau(\dot{L}_t \wedge (dL + L^2)^k)$ glue together to form a complex valued $2k+1$-form on M, that will be denoted by $\tau(\dot{\nabla} \wedge \nabla^{2k})$.

The so-called Chern-Simons forms are defined by the formula

$$\mathrm{CS}_\tau(V,\nabla_t) := \int_0^1 \sum_{k=0}^{\infty} \frac{1}{k!(2\pi i)^k}\tau(\dot{\nabla}_t \wedge \nabla_t^{2k}) \in \Omega^{\mathrm{odd}}(M). \tag{4.1}$$

By a direct local computation, one deduces that

$$d\,\mathrm{CS}_\tau(V,\nabla_t)=\mathrm{Ch}_\tau(V,\nabla_1)-\mathrm{Ch}_\tau(V,\nabla_0). \tag{4.2}$$

Hence the following holds

Proposition 4.2. *The Chern character is a closed differential form whose class is independent of the choice of the connection ∇.*

Proof. To see that the form $\mathrm{Ch}(V,\nabla)$ one compares it locally to the trivial connection using Equation (4.2). □

If $\tilde\nabla_t$ is another C^1-path of connections with same endpoint as ∇_t, then in [91], complex valued differential forms $\mathrm{CS}_\tau(V,\tilde\nabla_t,\nabla_t)$ are defined such that

$$d\,\mathrm{CS}_\tau(V,\tilde\nabla_t,\nabla_t)=\mathrm{CS}_\tau(V,\nabla_t)-\mathrm{CS}_\tau(V,\tilde\nabla_t).$$

It follows that modulo $d\Omega^{\mathrm{even}}(M)$, the forms $\mathrm{CS}_\tau(V,\nabla_t)$ only depend on the endpoint ∇_0 and ∇_1. Hence the notation $\mathrm{CS}_\tau(V,\nabla_1,\nabla_0)\in\Omega^{\mathrm{odd}}(M)/d\Omega^{\mathrm{even}}(M)$ is justified. Given two connections ∇_0,∇_1, then there is a preferred path $t\nabla_0+(1-t)\nabla_1$. One sees that in this case Equation (4.1) becomes

$$\mathrm{CS}_\tau(\nabla_1,\nabla_0)=\sum_{k=0}^{\infty}\frac{(-1)^k k!}{(2\pi i)^{k+1}(2k+1)!}\tau((\nabla_1-\nabla_0)^{2k+1}) \tag{4.3}$$

Let us recall the tensor product of bundles: if A and B are C^*-algebras, P a f.g projective A-module, Q a f.g projective B-module, then $P\otimes_{max}Q$ is a f.g projective $A\otimes_{\max}B$-module. It follows that if V is a A-bundle and W is a B-module then the maximal tensor product $V\otimes_{max}W$ is a well defined smooth $A\otimes_{max}B$-vector bundle. Same holds for minimal tensor product.

Proposition 4.3 ([91])**.** *Let V,W be A-vector bundles and $\nabla_V^0,\nabla_V^1,\nabla_V^2,\nabla_W^0,\nabla_W^1$ be A-connections on the indicated bundles. Then, we have*

1. $\mathrm{Ch}_\tau(V\oplus W,\nabla_V^0\oplus\nabla_W^0)=\mathrm{Ch}_\tau(V,\nabla_V^0)+\mathrm{Ch}_\tau(W,\nabla_W^0)$
2. $\mathrm{Ch}_\tau(V\otimes W,\nabla_V^0\otimes\nabla_W^0)=\mathrm{Ch}_\tau(V,\nabla_V^0)\wedge\mathrm{Ch}_\tau(W,\nabla_W^0)$
3. $\mathrm{CS}_\tau(\nabla_V^0,\nabla_V^1)+\mathrm{CS}_\tau(\nabla_V^1,\nabla_V^2)=\mathrm{CS}_\tau(\nabla_V^0,\nabla_V^2)$
4. $\mathrm{CS}_\tau(\nabla_V^0\oplus\nabla_W^0,\nabla_V^1\oplus\nabla_W^1)=\mathrm{CS}_\tau(\nabla_V^0,\nabla_V^1)+\mathrm{CS}_\tau(\nabla_W^0,\nabla_W^1)$

5.
$$\mathrm{CS}_\tau(\nabla_V^0 \otimes \nabla_W, \nabla_V^1 \otimes \nabla_W) = \mathrm{Ch}_\tau(\nabla_W)\,\mathrm{CS}_\tau(\nabla_V^0, \nabla_V^1),$$

where the product $\mathrm{Ch}_\tau(\nabla_W)\,\mathrm{CS}_\tau(\nabla_V^0, \nabla_V^1)$ *is well defined module exact forms because* $\mathrm{Ch}_\tau(\nabla_W)$ *is closed.*

6.
$$\begin{aligned}\mathrm{CS}_\tau(\nabla_V^0 \otimes \nabla_W^0, \nabla_V^1 \otimes \nabla_W^1) = {} & \mathrm{Ch}_\tau(\nabla_V^0)\,\mathrm{CS}_\tau(\nabla_W^0, \nabla_W^1) \\ & + \mathrm{Ch}_\tau(\nabla_W^1)\,\mathrm{CS}_\tau(\nabla_V^0, \nabla_V^1)\end{aligned}$$

The odd Chern-character (cf. [44]) is defined as follows. Let u be an A-linear automorphism of an A-vector bundle V, and ∇ an A-connection on V, then the odd Chern character is defined by the formula

$$\mathrm{Ch}_\tau(u, \nabla) = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{(2k+1)!(2\pi i)^{k+1}} \tau((u^{-1}\nabla u - u)^{2k+1}), \tag{4.4}$$

where $u^{-1}\nabla u$ is the A-connection on V defined on V by the formula

$$\Omega^*(M, V) \to \Omega^{*+1}(M, V), \quad s \to u^{-1}\nabla(us)$$

Proposition 4.4 ([91])**.** *The form* $\mathrm{Ch}_\tau(u)$ *is closed, and its class depends only on the homotopy class of* u.

Corollary 4.5. *Let* ∇ *be an* A*-connection on an* A*-vector bundle* V *and* $T : V \to V$ *a* A*-linear automorphism, then we have*

$$\mathrm{CS}_\tau(T^{-1}\nabla T, \nabla) = \mathrm{Ch}_\tau(T, \nabla) \tag{4.5}$$

Proof. This follows from Equation (4.3) and Equation (4.4). □

Theorem 4.6 (Atiyah-Hirzebruch, see [53])**.** *Let* M *be a compact smooth manifold, then the Chern character* $\mathrm{Ch} : K^*(M) \otimes \mathbb{C} \to H^*(M, \mathbb{C})$ *is a ring isomorphism.*

Remark 4.7. The normalisation constants in the definition of Chern character and the Chern-Simons forms are not uniform across the literature. Some authors don't divide by powers of $\frac{1}{2\pi i}$. Authors divide by powers of $\frac{1}{2\pi i}$ for the Chern character to be a rational map.

A C^*-metric on an A-bundle V is a smooth family of C^*-metrics on each fiber. If g is a C^*-metric, ∇ an A-connection on V, then ∇^* is an A-connection defined by the equation for X a vector field, $s, s' \in \Gamma^\infty(V)$,

$$Xg(s,s') = g(\nabla_X s, s') + g(s, \nabla^*_X s').$$

Let Γ be the fundamental group of M, $\phi : \Gamma \to GL(P)$ a representation. One can define an A-vector bundle by $\tilde{M} \times_\Gamma P$.

Definition 4.8. A flat structure on an A-vector bundle on M is the choice of an A-vector bundle isomorphism to $\tilde{M} \times_\Gamma P$ for some representation $\phi : \Gamma \to GL(P)$, and for some finitely generated projective A-module P. Furthermore if P is endowed with the structure of a C^*-module such that ϕ is a unitary representation, then we say that V is unitary flat,

A flat vector bundle is a vector bundle equipped with a flat structure.

Proposition 4.9. *A flat structure on an A-vector bundle can be equivalently given by the choice of a flat A-connection, that is an A-connection ∇ such that $\nabla^2 = 0$. Furthermore, the bundle is unitary flat if and only if the connection is unitary with respect to some C^*-metric on the vector bundle. The representation associated to ∇ is called the holonomy representation of ∇.*

Definition 4.10. A trivial A-connection is a flat A-connection ∇ whose holonomy is trivial.

Remark 4.11. It is clear from the definition that giving a trivial connection on a bundle is the same as giving a trivialization of the bundle.

Let ∇_0, ∇_1 be flat A-connections on an A-vector bundle V. It follows immediately from the definition of the Chern character that $\mathrm{Ch}_\tau(\nabla_0) = \mathrm{Ch}_\tau(\nabla_1) = \tau(Id_P)$, where $Id_P \in \mathrm{End}_A(P)$ is the identity morphism. It follows from Equation (4.2) that $\mathrm{CS}_\tau(\nabla_1, \nabla_0)$ gives a cohomology class in $H^{\mathrm{odd}}(M, \mathbb{C})$.

Definition 4.12. The α-invariant of (V, ∇_1, ∇_0) is defined as

$$\alpha_{V,\nabla_1,\nabla_0} = \mathrm{Ch}^{-1}(\mathrm{CS}_\tau(\nabla_1, \nabla_0)) \in K^1(M, \mathbb{C}).$$

From now on we restrict our selves to the case of unitary representations. In this case the imaginary part of the α invariant is zero as can be immediately seen from Equation (4.3). In general this holds for connections that are autoadjoint with respect to a nondegenerate sesquilinear forms introduced in Skandalis and Hilsum [50].

A nondegenerate sesquilinear form is a $\mathbb{C}$-bilinear form $Q : P \times P \to A$ such that $Q(p,q) = Q(q,p)^*$, $Q(p,qa) = Q(p,q)a$, and that there exists a bijective A-linear operator $T : P \to P$ such that $Q(\cdot, T\cdot)$ is a C^*-metric. It is proved in [50] that T can be chosen so that $T^2 = 1$.

Proposition 4.13. *Let V be an A vector bundle with fiber P, a flat connection ∇ whose holonomy is ϕ and ∇_{triv} a trivial connection on V. If there exists a non degenerate sesquilinear form Q on V such that $\phi(\Gamma) \subseteq U(Q)$, then the imaginary part of $\alpha_{\nabla,,\nabla_{triv}}$. Here $U(Q)$ denotes the group of isometries of Q.*

We follow the definition of

Proof. Let T an operator, and g a C^*metric such that $Q(\cdot,\cdot) = g(\cdot, T\cdot)$ and $T^2 = 1$. We will denote by ∇^* the adjoint of ∇ with respect to g. Let $s, s' \in \Gamma(V)$ be two sections and $X \in \Gamma(TM)$ a vector field. Then

$$\begin{aligned} Q(\nabla_X s, s') = g(\nabla_X, Ts') &= X \cdot g(s, Ts') - g(s, \nabla_X^* s') \\ &= X \cdot Q(s, s') - Q(s, T\nabla_X^* Ts') \end{aligned}$$

It follows that $\nabla = T\nabla^* T$. It follows that the form $\omega = \nabla - \nabla^*$ anticommutes with T. Hence

$$\tau(\omega^k) = \tau(\omega^k T^2) = \tau((-1)^k T\omega^k T) = (-1)^k \tau(\omega^k). \tag{4.6}$$

Since one has

$$\overline{\mathrm{CS}(\nabla, \nabla_{triv})} = \mathrm{CS}(\nabla^*, \nabla_{triv}^*) = \mathrm{CS}(\nabla^*, \nabla_{triv}).$$

It follows that the imaginary part of Chern-Simons forms is equal to $\frac{1}{2}\,\mathrm{CS}(\nabla, \nabla^*)$. The result then follows from Equation (4.3) and Equation (4.6). □

4.2 *KK*-theory definition of α-invariants

In this section, we prove Theorem 4.14, which is the main theorem that provides a description of Chern-Simons in *KK*-theory, which is done in Theorem 4.16. This theorem is due to Antonini, Azzali and Skandalis. We extend it in Proposition 4.18.

Let V be a $\mathbb{C}$-vector bundle, ∇ a unitary flat connection, and ∇_{triv}, a trivial connection.

Theorem 4.14 ([2]). *There exists a unital C^*-algebra A equipped with a trace τ such that $\tau(1) = 1$, and a unitary flat A-bundle (W, ∇_W) whose fiber is equal to A, and an isomorphism $T : V \otimes W \to V \otimes W$ such that*

$$T^{-1}(\nabla_{triv} \otimes \nabla_W)T = \nabla_V \otimes \nabla_W. \tag{4.7}$$

Proof.

Lemma 4.15. *Let A a C^*-algebra, τ a tracial state on A, V a flat A-bundle whose holonomy is $\phi : \Gamma \to \mathrm{GL}(P)$.*

We can take $A = C(U_n) \rtimes \Gamma$, where Γ acts on U_n on the right by multiplication by $\phi(\gamma)$, where $\phi : \Gamma \to U_n$ is the holonomy representation of ∇. The algebra A is equipped with the trace

$$\tau(f\gamma) = \delta_e(\gamma) \int_{U_n} f d\mu,$$

where μ is the normalised Haar measure. We take $\psi : \Gamma \to GL(A)$ the inclusion map, W the associated unitary flat A-bundle, and ∇_W the associated flat connection. Let $u \in M_n(C(U_n)) \subseteq M_n(A)$ be the unitary defined as the 'inclusion function' $u : U_n \to M_n(\mathbb{C})$. Notice that both $V \otimes W$, and $\mathbb{C}^n \otimes W$ are flat with holonomy representation $\gamma \to \phi(\gamma)\gamma \in M_n(A)$, and $\gamma \to \gamma \in M_n(A)$, respectively. The unitary u satisfies

$$u\phi(\gamma)\gamma u^{-1} = \gamma.$$

Therefore u defines a map $T_1 : V \otimes W \to \mathbb{C}^n \otimes W$ such that

$$T_1^{-1}(d \otimes \nabla_W)T_1 = \nabla_V \otimes \nabla_W,$$

where d is the trivial connection on $\mathbb{C}^n$. Let $T_2 : \mathbb{C}^n \to V$ be the trivialisation given by ∇_{triv}. This means that $d = T_2^{-1}\nabla_{triv}T_2$. The map $T = (T_2 \otimes \mathrm{Id}_W) \circ T_1$ satisfy Equation (4.7). ∎

The following is the key theorem of this chapter.

Theorem 4.16. *Let $[T] \in KK^1(\mathbb{C}, C(M) \otimes A)$ be the element defined in 4.14, and $[\tau] \in KK^0_{\mathbb{R}}(A, \mathbb{C})$ the element in real KK theory defined by the trace τ. The Kasparov product $[T] \otimes_A [\tau] \in KK^1_{\mathbb{R}}(\mathbb{C}, C(M)) = K^1(M, \mathbb{R})$ is equal to the α invariant $\alpha_{\nabla,\nabla_{triv}}$. In particular it is independent of the choice of A and W.*

Proof. By Corollary 4.5, we have

$$\begin{aligned}\mathrm{Ch}_\tau([T]) =&[\mathrm{CS}_\tau(T^{-1}(\nabla_{triv}\otimes\nabla_W)T, \nabla_{triv}\otimes\nabla_W]\\ =&[\mathrm{CS}_\tau(\nabla_V\otimes\nabla_W, \nabla_{triv}\otimes\nabla_W)]\\ =&[\mathrm{CS}_\tau(\nabla_V, \nabla_{triv})][\mathrm{Ch}_\tau(\nabla_W)]\\ =&\,\mathrm{Ch}(\alpha_{V,\nabla,\nabla_{triv}})\tau(1) = \mathrm{Ch}(\alpha_{V,\nabla,\nabla_{triv}})\end{aligned}$$

It follows that the class $[T]$ in $K^1(M,\mathbb{R})$ is equal to $\alpha_{\nabla,\nabla_{triv}}$ □

Remarks 4.17.
1. In general, it is impossible to find a commutative algebra A satisfying Theorem 4.14. Because if such an algebra exists, the Chern-Simons invariants become rational by the rationality of the Chern-character on locally compact spaces which doesn't hold in general.

2. The proposition is in general false for non unitary flat connections, because in the case were it holds, the imaginary part of Chern-Simons invariant is equal to 0.

In the next proposition, we show that 4.14 admits a sort of generalisation to arbitrary noncommutative C^*-algebras other than $M_n(\mathbb{C})$.

Proposition 4.18. *Let A be a unital C^*-algebra, V a unitary A-flat vector bundle. There exists a unital C^*-algebra B, a $*$-morphism $i : A\otimes C^*_r\Gamma \to B$ such that if W denotes Mishchenko's universal $C^*_r\Gamma$-bundle, then there exists an isomorphism preserving the flat structure*

$$T : i_*((M\times A)\otimes W) \to i_*(V\otimes W),$$

where $M\times A$ is the trivial A-bundle over M with fiber A. Furthermore if τ_A is a tracial state on A, then a tracial state τ_B on B is naturally defined such that $\tau_B(i(a\otimes\gamma)) = \tau_A(a)\delta_e(\gamma)$.

Proof. The free product $A\star_{\mathbb{C}} C(S^1)$ can be described as the universal unital C^*-algebra that is equipped with a $*$-morphism $i : A \to A\star_{\mathbb{C}} C(S^1)$ and a unitary $z\in A\star_{\mathbb{C}} C(S^1)$. In other words, if B is a C^*-algebra with a $*$-morphism $j : A\to B$ and a unitary $w\in B$, then there exists a unique $*$-morphism $\phi : A\star_{\mathbb{C}} C(S^1)\to B$ such that $\phi\circ i = j$ and $\phi(z) = w$. See [102] for more details on free products.

Let $u \in A$ be a unitary, then $u^{-1}z$ is a unitary in $A \star_{\mathbb{C}} C(S^1)$, hence by the universality of $A \star_{\mathbb{C}} C(S^1)$, there exists a unique $*$-morphism

$$\phi_u : A \star_{\mathbb{C}} C(S^1) \to A \star_{\mathbb{C}} C(S^1)$$

such that $\phi_u(a) = a$, and $\phi_u(z) = u^{-1}z$. By uniqueness, one has $\phi_u \circ \phi_v = \phi_{uv}$. Since $\phi_1 = \mathrm{Id}$, it follows that ϕ_u is an automorphism for every u.

Let $\psi : \Gamma \to U(A)$ be the holonomy representation of V. The group Γ acts on $A \star_{\mathbb{C}} C(S^1)$ by the morphisms $\phi_{\psi(\gamma)}$ for $\gamma \in \Gamma$.

Let $B = (A \star_{\mathbb{C}} C(S^1)) \rtimes_r \Gamma$, i the natural $*$-morphism given by

$$i : A \otimes C_r^*\Gamma \to B, \quad i(a \otimes \gamma) = a\gamma.$$

The map i is a $*$-morphism because ϕ_u fixes A for any unitary u. The unitary $z \in A \star_{\mathbb{C}} C(S^1) \subseteq B$ satisfies the following equation

$$i(1 \otimes \gamma) z i(1 \otimes \gamma^{-1}) = \gamma z \gamma^{-1} = \phi_{\psi(\gamma)}(z) = \psi(\gamma)^{-1} z,$$

which means that z defines an isomorphism from $i_*((M \times A) \otimes W) \to i_*(V \otimes W)$ which preserves the flat structure.

Let $\tau_A : A \to \mathbb{C}$ be a tracial state. We will denote by $\ker(\tau_A) \subseteq A$ the kernel of τ_A. By [102, section 1], the algebra $A \star_{\mathbb{C}} C(S^1)$ admits a finite trace $\tau = \tau_A \star \int_{S^1}$. This trace is defined as the unique trace satisfying the following properties

1. If $a \in A$, then $\tau(a) = \tau_A(a)$
2. If $k \in \mathbb{Z}^*$, then $\tau(z^k) = 0$
3. If $k \geq 1$, $a_1, \dots a_k \in \ker(\tau_A) \subseteq A$ and $l_1, \dots l_k \in \mathbb{Z}^*$, then

$$\tau(a_1 z^{l_1} a_2 z^{l_2} \cdots a_k z^{l_k}) = 0.$$

Lemma 4.19. *For every unitary $u \in A$, the automorphism ϕ_u preserves the trace τ.*

Proof. Let $B_+ \subseteq A \star_{\mathbb{C}} C(S^1)$ be the linear span of elements of the form $z^{l_0} a_1 z^{l^1} a_2 \dots a_k z^{l_k}$ for $k \geq 0$, $l_i > 0$, $a_i \in A$. Every element in B_+ can be written as a finite sum of elements of the form $z^{l_0} a_1 z^{l^1} a_2 \dots a_k z^{l_k}$ with $l_i > 0$ and $a_i \in \ker(\tau_A) \subseteq A$. Hence

$\tau_{|B_+} = 0$, furthermore for $k \geq 1$, $b_i \in B_+ \coprod B_+^*$ and $a_i \in \ker(\tau_A) \subseteq A$, one has

$$\tau(a_1 b_1 a_2 \dots a_k b_k) = 0.$$

Furthermore it follows from the the identity $\tau(ab) = \tau((a - \tau(a))b) + \tau(a)\tau(b)$ that $\tau(ab) = 0$ for $a \in A$, $b \in B_+$.

We verify that the properties of τ, are also verified by $\tau \circ \phi_u$. The result then follows from uniqueness of the trace.

1. If $a \in A$, then $\tau(\phi_u(a)) = \tau(a) = \tau_A(a)$.

2. If $k > 0$, then $\phi_u(z^k) = (u^{-1}z)^k$. Hence $\phi_u(z^k) = u^{-1}x \in AB_+$. Therefore $\tau(\phi_u(z^k)) = 0$. By taking the adjoint it follows that $\tau(\phi_u(z^{-k})) = 0$.

3. Any element $a_1 z^{l_1} a_2 z^{l_2} \cdots a_k z^{l_k}$ for $l_i \in \mathbb{Z}^*$ and $a_i \in \ker(\tau_A) \subseteq A$ can be written as $\alpha_1 x_1 \alpha_2 x_2 \dots \alpha_s x_s$ for some $s \geq 1$, $\alpha_i \in \ker(\tau_A) \subseteq A$ and $x_i \in B_+ \coprod B_+^*$ such that x_i alternatively belong to B_+ and B_+^*.

   Suppose that $x_1 \in B_+$. Since $\phi_u(B_+) = u^{-1}B_+$ and $\phi_u(B_+^*) = B_+^* u$. By writing $\phi_u(x_i) = u^{-1}y_i$ or $y_i u$, it follows that

$$\phi_u(\alpha_1 x_1 \alpha_2 x_2 \dots \alpha_s x_s) = \begin{cases} \alpha_1 u^{-1} y_1 \alpha_2 y_2 (u \alpha_3 u^{-1}) y_3 \dots y_s u, & \text{if, } u_s \in B_+^* \\ \alpha_1 u^{-1} y_1 \alpha_2 y_2 (u \alpha_3 u^{-1}) y_3 \dots y_s, & \text{if, } u_s \in B_+ \end{cases}$$

   It then follows that $\tau(\phi_u(\alpha_1 x_1 \alpha_2 x_2 \dots \alpha_s x_s)$ is equal to;

$$\tau(\alpha_1 u^{-1} y_1 \alpha_2 y_2 (u \alpha_3 u^{-1}) y_3 \dots y_s u) = \tau((u \alpha_1 u^{-1}) y_1 \alpha_2 y_2 (u \alpha_3 u^{-1}) y_3 \dots y_s) = 0$$

   in the first case and in the second we have

$$\tau(\alpha_1 u^{-1} y_1 \alpha_2 y_2 (u \alpha_3 u^{-1}) y_3 \dots y_s) = \tau(\alpha_2 y_2 (u \alpha_3 u^{-1}) y_3 \dots \big(y_s \alpha_1 u^{-1} y_1\big)) = 0,$$

   where we used that $y_s \alpha_1 u^{-1} y_1 \in B_+$.

   Hence $\tau(a_1 z^{l_1} a_2 z^{l_2} \cdots a_k z^{l_k}) = 0$ in the case where $x_1 \in B_+$. The case where $x_1 \in B_+^*$ is handled similarly. □

It follows from Lemma 4.19 that $(A \star C(S^1)) \rtimes \Gamma$, admits a tracial state defined by $\tau \rtimes \delta_e(\sum b_i \gamma_i) = \tau(b_e)$, where $b_i \in A \star C(S^1)$. □

4.3 A morphism in KK-theory with real coefficients

In this section, we give the definition of a primitive element in the equivariant KK-theory of the classifying space of trivialised unitary flat vector bundled.

Let G be a compact Lie group, to avoid confusion, we will denote by G the Lie group seen as a group, G^δ the group G with the discrete topology, $\mathbb{G}$ the space G seen as a compact space.

Definition 4.20. The following morphism defined below is denoted Ψ_G

$$\Psi_G : KK^*(\mathbb{C}, C(\mathbb{G})) \to KK^*_{G^\delta,\mathbb{R}}(\mathbb{C}, C(\mathbb{G})) = KK^*_{\mathbb{G}\rtimes G^\delta,\mathbb{R}}(C(\mathbb{G}), C(\mathbb{G})), \tag{4.8}$$

where G^δ is acts by right translation on $\mathbb{G}$. The morphism is the successive composition of the following morphisms

1. Let G act on $\mathbb{G} \times \mathbb{G}$ by the right diagonal action. The space $(\mathbb{G} \times \mathbb{G})/G$ is identified with $\mathbb{G}$ by using the map $(x, y) \to yx^{-1}$. It follows that we have a Morita equivalence from C^*-algebra $C(\mathbb{G})$ to the C^*-algebra $C(\mathbb{G} \times \mathbb{G}) \rtimes G$. By Green-Julg theorem [54], we obtain an isomorphism
$$KK^*(\mathbb{C}, C(\mathbb{G})) \to KK^*_G(\mathbb{C}, C(\mathbb{G}) \otimes C(\mathbb{G})).$$

2. The forgetful map and then the inclusion map of KK-theory inside $KK_{\mathbb{R}}$-theory
$$KK^*_G(\mathbb{C}, C(\mathbb{G} \times \mathbb{G})) \to KK^*_{G^\delta,\mathbb{R}}(\mathbb{C}, C(\mathbb{G}) \otimes C(\mathbb{G})).$$

3. By Corollary B.6, the Haar measure defines an element in $KK^0_{G^\delta,\mathbb{R}}(C(\mathbb{G}), \mathbb{C})$. One takes the Kasparov product with this element (on the second copy on $C(\mathbb{G})$) to obtain a morphism
$$KK^*_{G^\delta}(\mathbb{C}, C(\mathbb{G}) \otimes C(\mathbb{G})) \to KK^*_{G^\delta,\mathbb{R}}(\mathbb{C}, C(\mathbb{G}))$$

Remark 4.21. The composition of the morphism (4.8) with the forgetful morphism $KK^*_{G^\delta,\mathbb{R}}(\mathbb{C}, C(\mathbb{G})) \to KK^*_{\mathbb{R}}(\mathbb{C}, C(\mathbb{G}))$ is the inclusion morphism of $\mathbb{Z}$ in $\mathbb{R}$.

Following our convention, U_n denotes the Lie group, $\mathbb{U}_n$ the space, U_n^δ the discrete group. Let M be a compact manifold, $\Gamma = \pi_1(M)$. The groupoids M and $\tilde{M} \rtimes \Gamma$ are

Morita equivalent (see Definition 1.9). Since a flat vector bundle V whose holonomy representation is $\phi : \Gamma \to U_n^\delta$ defines a functor of groupoids $\tilde{M} \rtimes \Gamma \to U_n$ by sending (x, γ) to $\phi(\gamma)$. It follows that when this morphism is composed with the Morita equivalence of M and $\tilde{M} \rtimes \Gamma$, a flat vector bundle defines a generalised morphism[1] denoted by $f_\phi : M \to U_n^\delta$.

One sees easily that giving a trivialisation of a $\tilde{M} \times_\phi \mathbb{C}^n$ is the same thing as a map $\beta : \tilde{M} \to U_n$ such that $\beta(x\gamma) = \beta(x)\phi(\gamma)$ for every $x \in \tilde{M}$ and $\gamma \in \Gamma$. In particular if V is trivialised flat vector bundle, then the following morphism

$$\tilde{M} \rtimes \Gamma \to U_n \rtimes U_n^\delta, \quad (x, \gamma) \to (\beta(x), \phi(\gamma)).$$

is a morphism of Lie groupoids. By composing with the Morita equivalence of M and $\tilde{M} \rtimes \Gamma$, one obtains a generalised morphism denoted by $f_V : M \to U_n \rtimes U^\delta$.

Theorem 4.22. *Let $G = U_n$, and $[Id] \in K^1(\mathbb{U}_n)$ be classical identity element. The image of $[Id]$ by the morphism defined in 4.20 is a classifying element for Chern-Simons invariants in KK-theory. By this we mean that if V is a trivialised unitary flat vector bundle, then the pull back of $\Psi_{U_n}([Id])$ by f_V which is an element in $KK^1_{C(M),\mathbb{R}}(C(M), C(M)) = K^1(M) \otimes \mathbb{R}$ is equal to the α-invariant of V.*

Proof. We will first describe $\Psi_{U_n}([Id])$. We will check that the final element obtained is the map T constructed in 4.14. The result then follows from Theorem 4.16. The following enumeration follows each successive composition starting with step 0 to denote the element $[Id] \in K^1(\mathbb{U}_n)$.

0. The identity element $[Id] \in K^1(\mathbb{U}_n)$ will be seen as a unitary automorphism of total space of the bundle $U_n \times \mathbb{C}^n \to U_n$ given by

$$\mathbb{U}_n \times \mathbb{C}^n \to \mathbb{U}_n \times \mathbb{C}^n, \quad (x, v) \to (x, xv).$$

1. The first morphism changes this element to become a a unitary isomorphism from $\mathbb{U}_n \times \mathbb{U}_n \times \mathbb{C}^n \to \mathbb{U}_n \times \mathbb{U}_n \times \mathbb{C}^n$ sending $L(x, y, u) = (x, y, yx^{-1}v)$. This will be regarded as the composition of two isomorphisms $L = L_2 L_1^{-1}$

$$L_i : \mathbb{U}_n \times \mathbb{U}_n \times \mathbb{C}^n \to \mathbb{U}_n \times \mathbb{U}_n \times \mathbb{C}^n$$
$$L_i(x_1, x_2, v) = (x_1, x_2, x_i v).$$

[1]See [49] for the definition of a generalised morphism.

Notice that the group U_n acts trivially on $\mathbb{C}^n$ and L is U_n-equivariant. The group U_n doesn't act trivially on the $\mathbb{C}^n$ appearing in the domain of the maps L_1, and L_2. It acts by $z \cdot (x_1, x_2, v) = (x_1 z^{-1}, x_2 z^{-1}, zv)$. Both the maps L_1 and L_2 become equivariant for this action. We have

$$\begin{aligned} L_i(z \cdot (x_1, x_2, v)) = L_i(x_1 z^{-1}, x_2 z^{-1}, zv) &= (x_1 z^{-1}, x_2 z^{-1}, x_i z^{-1} zv) \\ &= (x_1 z^{-1}, x_2 z^{-1}, x_i v) \\ &= z \cdot L_i(x_1, x_2, v). \end{aligned}$$

2. This is the forgetful map, only changing the topology in the last picture of the group U_n to U_n^δ.

3. One views the bundles $\mathbb{U}_n \times \mathbb{U}_n \times \mathbb{C}^n$ as a bundle over the first copy of $\mathbb{U}_n$ with coefficients in $C(\mathbb{U}_n)$. Applying Corollary B.6 amounts to extending the coefficient algebra to $C(\mathbb{U}_n) \rtimes \mathbb{U}_n^\delta$.

4. We will use the notation of Theorem 4.14, and let $P = \tilde{M} \times_\Gamma U_n$.

   The pull back of the element obtained in step 3 by ϕ, becomes the vector bundle $\mathbb{C}^n \otimes W$ over M. The middle vector bundle in step 1, becomes $V \otimes W$, L_1^{-1} and L_2 become respectively $T_2 \otimes Id_W$, and T_1. Applying the Morita equivalence between M and $\tilde{M} \rtimes \Gamma$ finishes the proof. □

Remark 4.23. In most of this chapter compactness is not needed. Most notably, let $\phi : \pi_1(M) \to U_n$ be the holonomy representation of a trivialised unitary flat vector bundle on a not necessarily compact manifold, and $f_\phi : M \to U_n \rtimes U_n^\delta$ the corresponding generalised morphism. The element $f^*\Phi_{U_n}(Id_n)$ is an element in $KK^1_{M,\mathbb{R}}(C_0(M), C_0(M))$. This later group is isomorphic to the K-theory with real coefficients without compact support as proved in [60].

Appendix A

Regular operators

This appendix is on regular operators on C^*-modules. We refer the reader to [62] for more details on C^*-modules and regular operators. Propositions A.7, A.8, A.9, A.10, A.11, A.12 are results that first appeared in Skandalis master course.

Definition A.1. Let A be a C^*-algebra, E and F be A-C^*-modules, $t : \mathrm{Dom}(t) \subseteq E \to F$ a densely defined A-linear operator. The adjoint of t is the operator defined by its graph

$$\mathrm{graph}(t^*) = \{(-tx, x) : x \in \mathrm{Dom}(t)\}^{\perp}.$$

This is the graph of a well defined A-linear operator by the density of $\mathrm{Dom}(t)$.

The operator t is called regular if the following conditions are satisfied

1. The domain of t^* is dense

2. One has

$$\mathrm{graph}(t^*) \oplus \{(-tx, x) : x \in \mathrm{Dom}(t)\} = F \oplus E.$$

   In particular the graph of t is closed.

Example A.2. Let X be a locally compact toplogical space, $f : X \to \mathbb{C}$ a continuous function. The operator

$$\begin{aligned} M_f : \mathrm{Dom}(M_f) \subseteq C_0(X) &\to C_0(X) \\ g &\to fg, \end{aligned}$$

is regular, where $\mathrm{Dom}(M_f) = \{g \in C_0(X) : gf \in C_0(X)\}$.

Proposition A.3 ([62, chapter 9]). *A densely defined closed A-linear operator is regular if and only if dom(t^*) is dense and the operator* $(1+t^*t) : \mathrm{Dom}(t^*t) \to E$ *has dense image.*

Proposition A.4 ([62, chapter 9]). *Let E, F be C^*-modules, $t : \mathrm{Dom}(t) \subseteq E \to F$ a regular operator.*

1. *The operators t^* and t^*t are regular.*
2. *The operator $(1+t^*t) : \mathrm{Dom}(t^*t) \to E$ is a bijection whose inverse is an element in $\mathcal{L}(E)^+$.*
3. *The operator $(1+t^*t)^{-\frac{1}{2}} := \left((1+t^*t)^{-1}\right)^{\frac{1}{2}} \in \mathcal{L}(E)$ is a bijection onto* $\mathrm{Dom}(t)$.
4. *The operator $t(1+t^*t)^{-\frac{1}{2}}$ is an element of $\mathcal{L}(E,F)$ whose adjoint is equal to $t^*(1+tt^*)^{-\frac{1}{2}}$.*

Proposition A.5 ([50]). *Let t be a regular operator acting on a C^*-module E.*

1. *If $L \in \mathcal{L}(E)$ is a bounded operator such that* $\mathrm{Im}(L) \subseteq \mathrm{Dom}(t)$ *then $tL \in \mathcal{L}(E)$.*
2. *Let s be a bijective regular operator. If* $\mathrm{Dom}(s) \subseteq \mathrm{Dom}(t)$, *then $ts^{-1} \in \mathcal{L}(E)$.*

Proposition A.6 ([62, theorem 3.2]). *Let E, F be A-C^*-modules, $L \in \mathcal{L}(E,F)$ an operator with closed image. Then*

1. *L^* has closed*
2. *$E = \ker(L) \oplus \mathrm{Im}(L^*)$*
3. *$F = \ker(L^*) \oplus \mathrm{Im}(L)$*

Proposition A.7. *Let E, F, G be C^*-modules, $t : \mathrm{Dom}(t) \subseteq E \to F$, $s : \mathrm{Dom}(s) \subseteq F \to G$ regular operators. If*

$$\mathrm{Rang}(t) + \mathrm{Dom}(s) = F = \mathrm{Dom}(t^*) + \mathrm{Rang}(s^*)$$

then the operator st is regular and $(st)^* = t^*s^*$.

Proof. Let L be the operator

$$L : \mathrm{graph}(t) \oplus \mathrm{graph}(s) \to F, \quad (x, tx, y, sy) \to tx - y.$$

The operator L is clearly bounded and surjective by the assumptions. Furthermore, since graph(s) and graph(t) are orthocomplemented, it follows that $L \in \mathcal{L}(\text{graph}(t) \oplus \text{graph}(s), F)$. Hence by the open mapping theorem $L^{-1}(\text{Dom}(s))$ is dense. If we denote by $\pi_1 : \text{graph}(t) \oplus \text{graph}(s) \to E$, the projection onto the first coordinate, then it is clear that $\text{Dom}(st) = \pi_1(L^{-1}(\text{Dom}(s)))$ is dense in $\pi_1(\text{graph}(t) \oplus \text{graph}(s)) = \text{Dom}(t)$ which is dense in E. Hence st is densely defined, and $\text{Dom}(st)$ is an essential domain of t.

Let us prove that $(st)^* = t^*s^*$. It is clear that $t^*s^* \subseteq (st)^*$. Let

$$Q : E \oplus F \to E, \quad Q(x, y) = t(1 + t^*t)^{-\frac{1}{2}}x + (1 + s^*s)^{-\frac{1}{2}}y.$$

By Proposition A.4, it follows that Q is in $\mathcal{L}(E \oplus F, F)$ and is onto. Hence by Proposition A.6, there exists $(C, D) \in \mathcal{L}(F, E \oplus F)$ such that $Q(C, D) = Id_F$. Let $x \in \text{Dom}((st)^*)$, $y \in \text{Dom}(s)$. It follows that

$$\begin{aligned} \langle x, sy\rangle = \langle x, sQ(C, D)y\rangle &= \langle x, st(1 + t^*t)^{-\frac{1}{2}}Cy + s(1 + s^*s)^{-\frac{1}{2}}Dy\rangle \\ &= \langle C^*(1 + t^*t)^{-\frac{1}{2}}(st)^*x + D^*s^*(1 + ss^*)^{-\frac{1}{2}}x, y\rangle \end{aligned}$$

Hence $x \in \text{Dom}(s^*)$ and $s^*x = C^*(1+t^*t)^{-\frac{1}{2}}(st)^*x+D^*s^*(1+ss^*)^{-\frac{1}{2}}x$. If $z \in \text{Dom}(st)$, then

$$\langle s^*x, tz\rangle = \langle x, stz\rangle = \langle (st)^*x, z\rangle.$$

Since $\text{Dom}(st)$ is an essential domain of t, it follows that $s^*x \in \text{Dom}(t^*)$, and $t^*s^*x = (st)^*x$.

By the symmetry of of the hypothesis of Proposition A.7, it follows that $st = (t^*s^*)^*$. Hence the graph of st is closed.

Since L is surjective, it follows from Proposition A.6 that $Ker(L)$ is orthocomplemented in graph$(t) \oplus$ graph(s). Hence the operator $\pi(x, tx, y, sy) = (x, sy)$ from $\ker(L)$ to $E \oplus G$ is in $\mathcal{L}(\ker(L), E \oplus G)$. The image of π is graph(st). Since it is closed, it follows from Proposition A.6 that graph(st) is orthocomplemented, and st is regular. □

Corollary A.8. *Let E, F be C^*-modules, $t : \text{Dom}(t) \subseteq E \to F$, $s : \text{Dom}(s) \subseteq E \to$*

F regular operators. If

$$E = \mathrm{Dom}(s) + \mathrm{Dom}(t), \quad F = \mathrm{Dom}(s^*) + \mathrm{Dom}(t^*),$$

then $s + t$ is a regular operator and $(s + t)^* = s^* + t^*$.

Proof. Let t' and s' be the operators

$$\begin{aligned} t' &: \mathrm{Dom}(t) \to E \oplus F, & s' &: \mathrm{Dom}(s) \oplus F \to F \\ x &\to (x, tx), & (x, y) &\to s(x) + y. \end{aligned}$$

It is straightforward to verify that s' and t' are regular operators that satisfy the conditions of Proposition A.7. Therefore $s't' = t + s$ is regular and $(s't')^* = t'^* s'^* = t^* + s^*$. □

Corollary A.9. *let E be a C^*-module, $t : \mathrm{Dom}(t) \subseteq E \to E$ a regular operator. If $\mathrm{Im}(t) \subseteq \mathrm{Dom}(t)$, then the operator $t + t^*$ is a self adjoint regular operator.*

Proof. The operator t is regular, hence

$$\mathrm{graph}(t^*) \oplus \{(-tx, x) : x \in \mathrm{Dom}(t)\} = E \oplus E.$$

It follows that $\mathrm{Im}(t) + \mathrm{Dom}(t^*) = E$. Hence by Corollary A.8, the operator $t + t^*$ is a self adjoint regular operator. □

Proposition A.10 ([75]). *Let $t : \mathrm{Dom}(t) \subseteq E \to F$ be a densely defined closed operator whose adjoint t^* has a dense domain. The following are equivalent;*

1. *The operator t is regular;*

2. *For every representation π of A, one has $\pi(t)^* = \pi(t^*)$;*

3. *For every irreducible representation π of A, one has $\pi(t)^* = \pi(t^*)$.*

Proposition A.11. *Let s and t be regular operators acting on a C^*-module E. If $\mathrm{Dom}(t) \subseteq \mathrm{Dom}(s)$ and for every $\xi \in \mathrm{Dom}(t)$, $\|s\xi\| \leq \|t\xi\|$, then*

$$(1 + t^*t)^{-1} \leq (1 + s^*s)^{-1}$$

Proof. First we prove the following claim: if $\xi \in \operatorname{Dom}(t)$, then $\langle s\xi, s\xi\rangle \le \langle t\xi, t\xi\rangle$. To this end let $\epsilon > 0$. By assumption, we have

$$\begin{aligned}\left\| s\xi\big(\langle t\xi, t\xi\rangle + \epsilon\big)^{-\frac{1}{2}}\right\| &\le \left\| t\xi\big(\langle t\xi, t\xi\rangle + \epsilon\big)^{-\frac{1}{2}}\right\| \\ &= \left\| \big(\langle t\xi, t\xi\rangle + \epsilon\big)^{-\frac{1}{2}}\langle t\xi, t\xi\rangle\big(\langle t\xi, t\xi\rangle + \epsilon\big)^{-\frac{1}{2}}\right\|^{\frac{1}{2}} \\ &\le 1\end{aligned}$$

It follows that

$$\big(\langle t\xi, t\xi\rangle + \epsilon\big)^{-\frac{1}{2}}\langle s\xi, s\xi\rangle\big(\langle t\xi, t\xi\rangle + \epsilon\big)^{-\frac{1}{2}} \le 1$$

Hence

$$\langle s\xi, s\xi\rangle \le \langle t\xi, t\xi\rangle + \epsilon.$$

This proves the claim.

Let $s_1 = (1 + s^*s)^{\frac{1}{2}}$ and $t_1 = (1 + t^*t)^{\frac{1}{2}}$. Both t_1 and s_1 are regular by Proposition A.4, furthermore $\operatorname{Dom}(t_1) = \operatorname{Dom}(t) \subseteq \operatorname{Dom}(s) = \operatorname{Dom}(s_1)$. It follows from the above claim that if $\xi \in \operatorname{Dom}(t)$, then

$$\|s_1\xi\|^2 = \|\langle s_1\xi, s_1\xi\rangle\| = \|\langle \xi + s^*s\xi, \xi\rangle\| = \|\langle \xi, \xi\rangle + \langle s\xi, s\xi\rangle\| \le \|\langle \xi, \xi\rangle + \langle t\xi, t\xi\rangle\| = \left\|t_1\xi^2\right\|.$$

By Proposition A.5, it follows that $L = s_1 t_1^{-1}$ is a bounded operator in $\mathcal{L}(E)$. Since t_1 is surjective and

$$\|Lt_1\xi\| = \|s_1\xi\| \le \|t_1\xi\|,$$

it follows that $\|L\| \le 1$. Hence $LL^* = s_1 t_1^{-1} t_1^{-1} s_1 \le 1$. Hence $t_1^{-2} \le s_1^{-2}$. □

Proposition A.12. *Let S be a regular self adjoint operator acting on a C^*-module E, $V_t = \exp(itS)$, $T : \operatorname{Dom}(T) \subseteq E \to E$ a $\mathbb{C}$-linear map with a dense domain. If*

1. $V_t \operatorname{Dom}(T) = \operatorname{Dom}(T)$
2. $T \subseteq S$.

Then the closure of $\operatorname{graph}(T)$ *is equal to* $\operatorname{graph}(S)$.

Proof. By taking the closure of T, we can suppose that T is closed. Let $f \in \mathcal{S}(\mathbb{R})$

be a Schwartz function. Since

$$f(S) = \int_{-\infty}^{\infty} \hat{f}(t) V_{2\pi t} dt,$$

it follows that $f(S)\,\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T)$ and $Tf(S) = f(S)T$. Since $f(S)$ and $Sf(S)$ are bounded operators and $\mathrm{Dom}(T)$ is dense, it follows that $\{(f(S)x, Sf(S)x) : x \in E\} \subseteq \mathrm{graph}(T)$.

Let $x \in \mathrm{Dom}(S)$, and $0 \leq f_n \leq 1$ Schwartz functions such that $f_n \to 1$ uniformly on every compact. It follows that $f_n(S)x \to x$, and $Sf_n(S) \to Sx$. Hence $(x, Sx) \in \mathrm{graph}(T)$, which implies that $S = T$. $\square$

Appendix B

KK-theory with real coefficients

In this section, we recall the definition of real KK-theory given by [2]. We refer the reader to [62, 93, 14] for more details on C^*-modules and KK-theory.

Let G be a Lie groupoid. The $KK^*_G(A, B)$ group is usually defined only for separable C^*-algebras only. We follow the remarks given by Skandalis[92] in order to define $KK^*_G(A, B)$ for arbitrary C^*-algebras A and B by

$$KK^*_G(A, B) := \varprojlim_{D} KK^*_G(D, B)$$

where the projective limit and injective limit are over all separable C^*-algebra with morphisms $\phi : C \to B$ and $\psi : B \to D$.

When the groupoid G is not second countable (but G^0 is always assumed second countable) then

$$KK^*_G(A, B) := \varprojlim_{H} KK^*_H(A, B)$$

where the projective limit is over all second countable Lie subgroupoids.

Definition B.1. Let $\mathcal{C}$ be the category whose objects are unital C^*-algebras endowed with a tracial state and whose morphisms are $*$-morphisms preserving the trace.

Definition B.2 ([3])**.** Let G be a Lie groupoid, A and B be two G-C^*-algebras. Equivariant KK-theory with real coefficients is defined by

$$KK^*_{G,\mathbb{R}}(A, B) := \varinjlim_{C \in \mathcal{C}} KK^*_G(A, B \otimes C).$$

Here the groupoid G acts on C trivially.

Similarly the equivariant KK-theory with $\mathbb{R}/\mathbb{Z}$ coefficients is defined by

$$KK^*_{G,\mathbb{R}/\mathbb{Z}}(A,B) = \varinjlim_{C\in\mathcal{C}} KK^*_G\left(A, B\otimes \mathrm{Cone}(\mathbb{C}\to C)\right).$$

This definition is justified by the Kunneth formula

Theorem B.3 (Künneth formula, see for example [14])**.** *Let A be a separable C^* algebra in the bootstrap category and B any C^*-algebra then the following sequences are exact*

$$0 \to K_*(A)\otimes K_*(B) \to K_*(A\otimes B) \to \mathrm{Tor}^1_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0$$

Where the first map is degree 0 and the second if of degree 1

Proposition B.4 ([2])**.** *Let M be a compact smooth manifold, then*

$$KK^*_{\mathbb{R}}(\mathbb{C}, C(M)) = K^*(M,\mathbb{R}).$$

Theorems and propositions in [63] pass through the direct limit to $KK_{G,\mathbb{R}}$. In particular Kasparov product exists

$$KK^i_{G,\mathbb{R}}(A,B)\times KK^j_{G,\mathbb{R}}(B,C) \to KK^{i+j}_{G,\mathbb{R}}(A,C).$$

Functoriality and Morita equivalence remains true that is if $f : G \to G'$ is a generalised morphism, in the sense of [49], of groupoids then

$$f^* : KK^*_{G',\mathbb{R}}(A,B) \to KK^*_{G,\mathbb{R}}(f^*A, f^*B)$$

is well defined, moreover if f is a Morita equivalence, then f^* is an isomorphism.

If $\tau : A \to \mathbb{C}$ is a trace on a C^*-algebra, then τ defines naturally an element in $[\tau] \in KK^0_{\mathbb{R}}(A,\mathbb{C})$.

Proposition B.5. *Let A be a unital C^*-algebra, and Γ a countable discrete group, and $c : \Gamma \to U(A)$ a group homomorphism, where $U(A)$ is the group of unitaries of A. The Γ-C^*-algebra A with the trivial Γ action is Γ-Morita equivalent to the Γ-C^*-algebra A with the inner action given by c.*

Proof. The module A with the action $\gamma\cdot a = c(\gamma)a$ is the Morita equivalence. $\square$

Corollary B.6. *Let A be a C^*-algebra, $\tau : A \to \mathbb{C}$ a tracial state, Γ a discrete group acting on A which preserves τ. The trace τ defines an element in $KK^0_{\Gamma,\mathbb{R}}(A,\mathbb{C})$.*

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