Theoretical Computer Science

Course Presentation

Objectives and motivations

Why Theoretical Computer Science in an Engineering Curriculum?

Theory is stimulated by practice,

practice is helped by theory:

generality, rigor, insight, “control”

- Engineer is a person who:
  - Applies theory to practice
  - Extracts theory from practice

- Deeply and critically understand the principles of Computer Science (careful re-view of basic CS notions)
- Build solid ground to understand and master innovation (e.g.: multimedia, modeling of concurrent and wireless computation)
- Theory as an antidote to overspecialization and as a support to interdisciplinary studies

- Throw a bridge between basic, applicative courses and more advanced ones (SW engineering, Hardware and computer architecture, distributed systems…)
- Direct application of some notions of TCS to practical cases: in follow-up courses such as Formal Languages and Compilers, Formal Methods, and thesis work

The program (1/2)

- Modeling in Computer Science
  (How to describe a problem and its solution):
  Do not go deep into specific models, rather
  provide ability to understand models and invent new ones
- Theory of Computation:
  what can I do by means of a computer
  (which problems can I solve)?

The program (2/2)

- Complexity theory:
  how much does it cost to solve a problem through a computer?
- Only the basics: further developments in follow-up courses

Organization (1/3)

- Requirements :
  - Basics of Computer science (CS 1)
  - Elements of discrete mathematics (Algebra and Logic)
- Lesson and practice classes (all rather classical style…)
  - Student-teacher interaction is quite appreciated:
    - In classroom
    - In my office (when possible)
    - By Email (administrative matters and to fix face-to-face meetings)
      - angelo.morzenti@polimi.it

Organization (2/3)

- Exam tests *ability to apply, not to repeat*:
  mainly written
  open book when in presence (you can consult textbooks and notes)
  not repetitive, challenging (hopefully)
- Standard written (+ possibly oral) exams on the whole program

Organization (3/3)

- Teaching material:
  - Official text :
    - Ghezzi/Mandrioli: Theoretical Foundations of Computer Science [Wiley]
      - Not re-printed by the publisher: you get it, possibly, in a legal way, see the WeBeep course website
    - In Italian: Dino Mandrioli, Paola Spoletini, Informatica Teorica, 2nd Ed. Città Studi, 2011.
  - Dino Mandrioli, Paola Spoletini, Mathematical logic for computer science: an Introduction, Esculapio Società Editrice, 2010.
  - D.Mandrioli, D.Martinenghi, A.Morzenti, M.Pradella, M.Rossi: Lecture Notes on Monadic First- and Second-Order Logic on Strings, lecture notes, 2019, on the WeBeep course website
  - English translation of the Italian Exercise book: Mandrioli, Lavazza, Morzenti, San Pietro, Spoletini, Esercizi di informatica teorica, 3rd edition, Esculapio, on the WeBeep course website
  - Previous exam texts with solutions, on the WeBeep course website
  - Lesson slides (NOT to use as a substitute of the text book!), on the WeBeep course website

Models in Computer Science

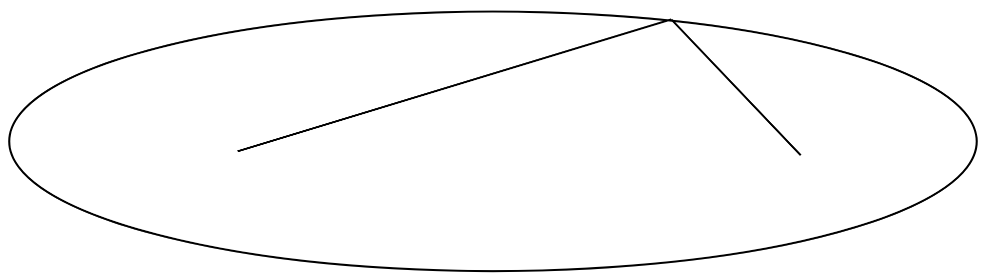
- Not only discrete vs. continuous
  (bits & bytes vs. real numbers and continuous functions)

- Operational models
  (abstract machines, dynamic systems, …)
  based on the concept of a **state** and of means (operations) to represent its evolution (chronologically, i.e., w.r.t. "time")

- Descriptive models
  aimed at expressing properties (desired or feared) of the modeled system, rather than its functioning as an evolution in time through states

Examples

- Operational model of an ellipsis (how you can draw it):



- descriptive model of it (***property*** of the coordinates of its points):

$$\boldsymbol{a \cdot x^{2} + b \cdot y^{2} = c}$$

- Operational definition of sorting:
  - Find minimum element and put it in first position;
  - Find minimum element among the remaining ones and put it in second position;
  - …
- Descriptive definition of sorting:
  - A permutation of the sequence such that

$$\forall i,\ a[i] \le a[i+1]$$

- In fact, differences between operational and descriptive models are not so sharp
  - (grammars are somewhere in between)
- It is however a very useful classification of models
  - Corresponds closely to different “attitudes” that people adopt when reasoning on systems

A first, fundamental, “meta”model: the notion of *language*

- Italian, French, English, …
- C, Pascal, Ada, …
  but also:
- Graphics
- Music
- Multimedia, ...

Elements of a language

- Alphabet or vocabulary
  (from a mathematical viewpoint these are synonyms):
  Finite set of basic symbols
  {a,b,c, …z}
  {0,1}
  {Do, Re, Mi, …}
  ASCII char set
  ...

- String (over an alphabet A):
  ordered finite sequence of elements of A, possibly with repetitions
  a, b, aa, alpha, john, leaves of grass, …
- Length of a string:
  $|a| = 1$, $|ab| = 2$
- Null string, or empty string, ε: $|\varepsilon| = 0$
- A^* = set of all strings, including ε, over A
  $A = \{0,1\}$, $A^* = \{\varepsilon, 0, 1, 00, 01, 10, \ldots\}$

- Operations on strings:
  concatenation (product): x · y

  x = abb, y = baba, x·y = abbbaba
  x = Quel ramo, y = del lago di Como,
  x · y = Quel ramo del lago di Como
  "·" is an associative, non-commutative operation

- A* is called
  free monoid over A built through "·"
- ε is the unit w.r.t. "·" : for any x, $\varepsilon \cdot x = x \cdot \varepsilon = x$

Language

- L is any set of strings, i.e. subset of A*: ($L \subseteq A^*$)
  Italian, C, Pascal, … but also:
  sequences of 0 and 1 with an even number of 1
  the set of musical scores in F minor
  symmetric square matrices
  …
- It is a very broad notion, somehow *universal*

Operations on languages

- Set theoretic operations:
  $\cup$, $\cap$, L_1 - L_2, $\neg L = A^* - L$, $(\overline{L} = \neg L)$

- Concatenation (of languages):
  $\boldsymbol{L_1 \cdot L_2 = \{\ x \cdot y \mid x \in L_1,\ y \in L_2\ \}}$

  $\boldsymbol{L_1 = \{0, 1\}^*,\ \ L_2 = \{a, b\}^*}$ **;**
  $\boldsymbol{L_1 \cdot L_2 = \{\varepsilon, 0, 1, 0a, 11b, abb, 10ba, \ \ldots.\}}$ **NB: ab1 not included!**

Language power, inductive definition

- $L^0 = \{\varepsilon\}; \quad \forall i > 0 \;\; L^i = L^{i\text{-}1} \cdot L$
- $L^* = \bigcup_{n=0}^{\infty} L^n$

  NB: $\{\varepsilon\} \neq \varnothing$!
  $\{\varepsilon\} \cdot L = L$;
  $\varnothing \cdot L = \varnothing$

- $L^+ = \bigcup_{n=1}^{\infty} L^n = (\text{if } \varepsilon \notin L) = L^* - \{\varepsilon\}$

Some practical examples

- let
  - L_1 : set of "MS Office" documents, and
  - L_2 : set of "LibreOffice" documents; then
  - $L_1 \cap L_2$: set of documents that are "compatible MSOffice - LibreOffice"
- Composition of a message over the net:
  - $x \cdot y \cdot z$:
  - x = header (address, …)
  - y = text
  - z = "tail"
-
  - L_1 : set of e-mail messages
  - L_2 : set of SPAM messages
  - Filter: $L_1 - L_2$

- A language can be a means of expression …Of a ***problem***
- $x \in L$?
  - Is a message correct? Is a program correct?
  - $y = x^2$?
  - $z = Det(A)$? (is z the determinant of matrix A?)
- Problems of the kind «computing a function» can always be recast (by means of a suitable encoding) into problems of the kind «language string inclusion»
  - E.g., compute square of a number: computation of function $\boldsymbol{y=x^2}$
  - Language $L_{sq} = \{(x, y) \mid y = x^2\}$
    - To compute a^2 try repeatedly to solve $(a, b) \in L_{sq}$ until you find the right b
- Every string can be encoded in binary
  - every binary string can be interpreted as a natural number ...
  - … hence every problem can be recast (by suitable encoding) into an inclusion problem for a set of natural numbers

Translation

- $y = \tau(x)$
  τ : translation is a function from L_1 to L_2
  - τ_1 : double the "1" ($1 \Rightarrow 11$):
    $\tau_1(0010110) = 0011011110, \ldots$
  - τ_2 : change a with b and viceversa ($a \Leftrightarrow b$):
    $\tau_2(abbbaa) = baaabb, \ldots$
  - Other examples:
    - File compression
    - Self-correcting protocols
    - Computer language compilation
    - translation Italian $\Rightarrow$ English

Conclusion

- The notion of language and the basic associated operations provide a very general and expressive means to describe any type of systems, their properties and related problems:
  - Compute the determinant of a matrix;
  - Determine if a bridge will collapse under a certain load;
  - ….
- Notice that, after all, in a computer any information is a string of bits (hence a string of some language…)

Operational Models

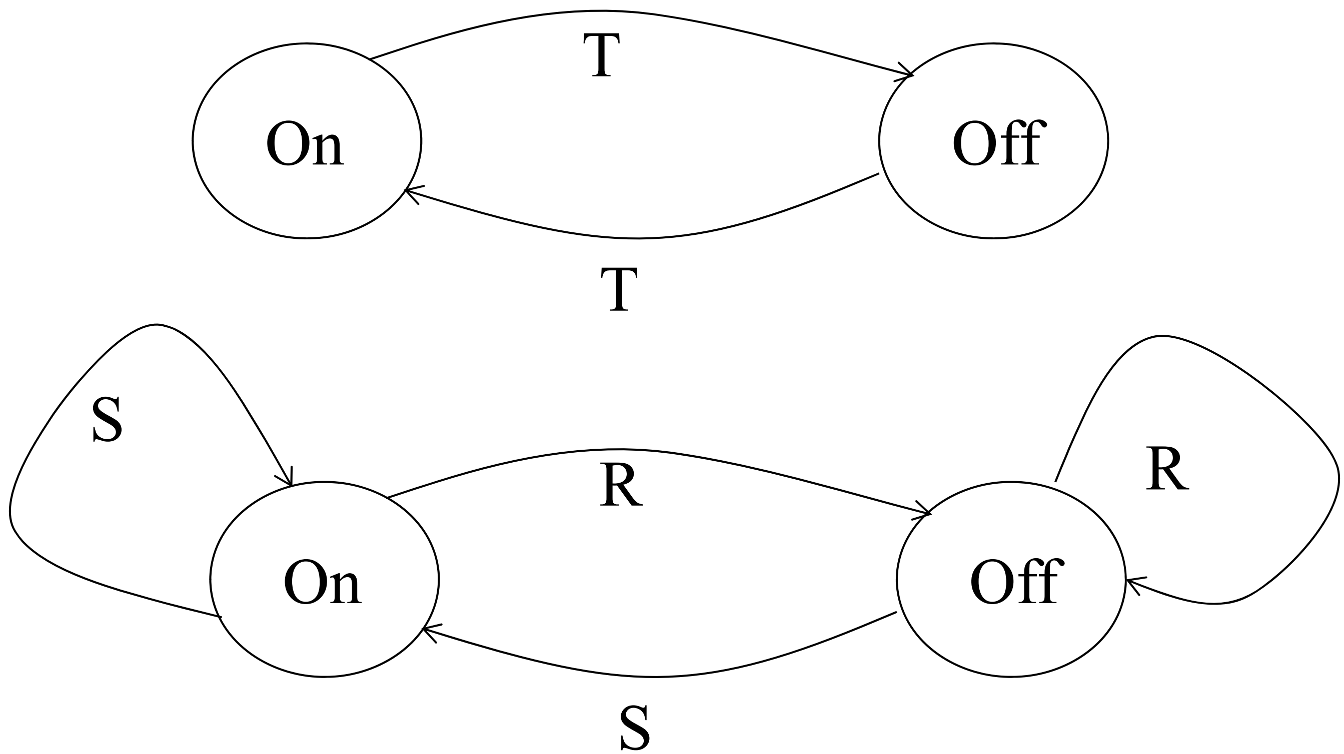
(state machines, dynamical systems)

- Finite State Machines (or *automata*) (*FSA*, or *FA*):
  - A finite state set:
    { on, off }, ….
    {1, 2, 3, 4, …k}, {TV channels}, …

    Graphic representation:

Commands (input) and state transitions

- Two very simple flip-flops:



Turning a light on and off, ...

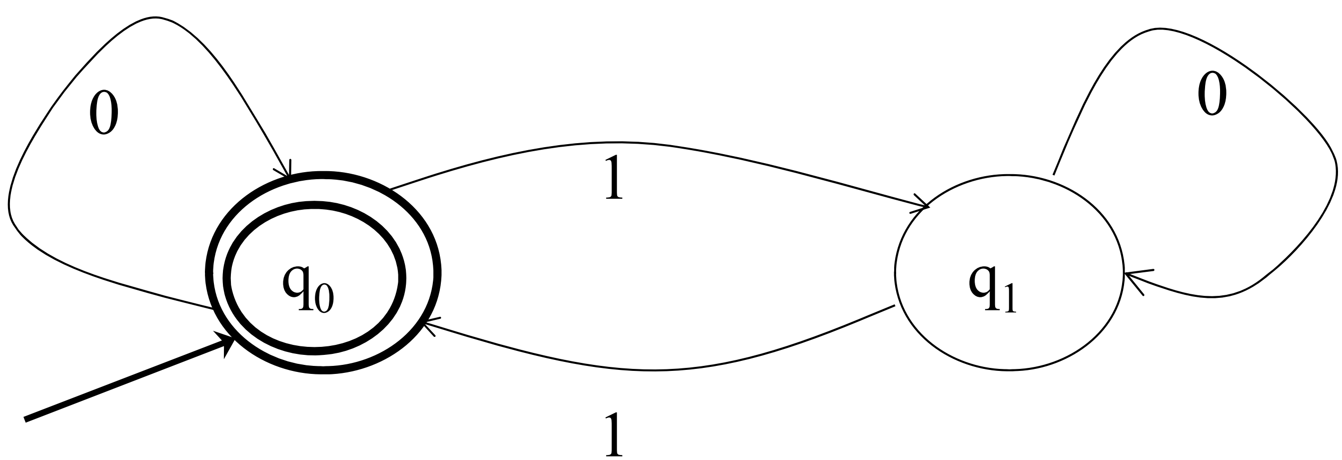
A first formalization

- A finite state automaton is (made of):
  - A finite state set: Q
  - A finite input alphabet: I
  - A transition function (***partial***, in general):
    $\delta: Q \times I \to Q$

Automata as language recognizers (or acceptors) ($\boldsymbol{x} \in \boldsymbol{L}$?)

- A *move sequence* starts from an ***initial state*** and it is *accepting* if it reaches a ***final*** (or *accepting*) *state*.

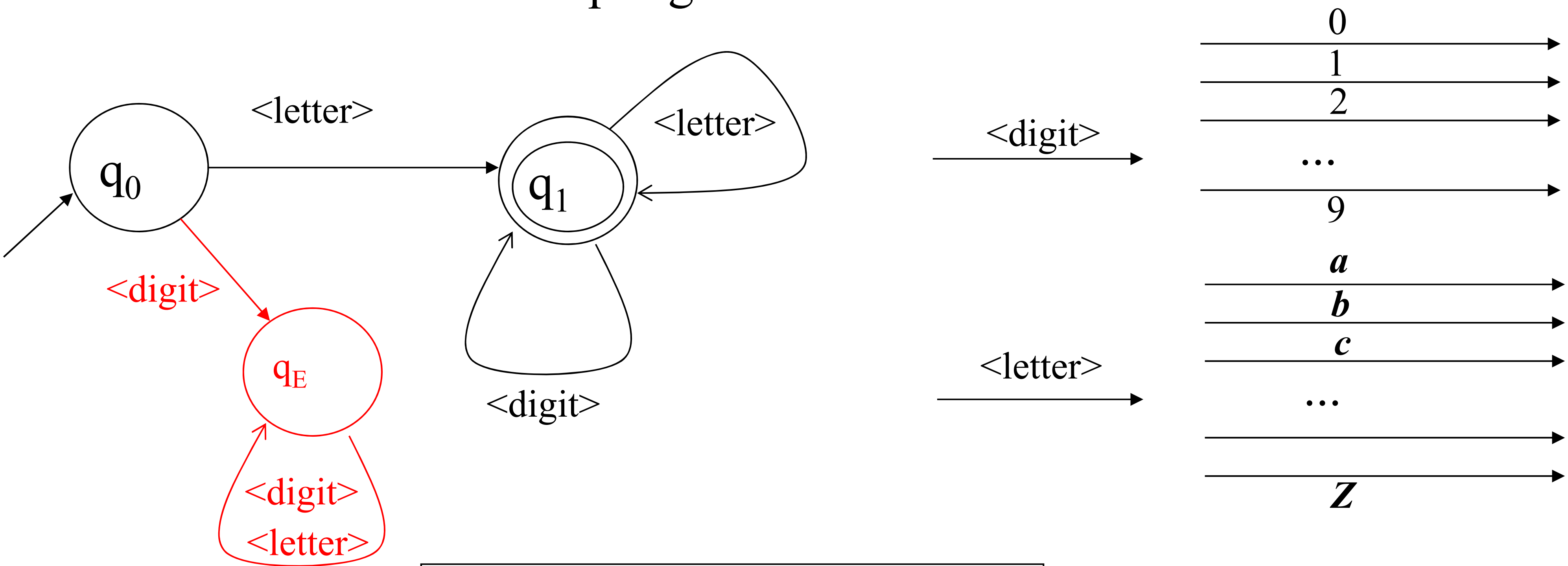
$\boldsymbol{L}$ = {strings with an even number of "1" any number of "0"}



Formalization of the notion of acceptance of language *L*

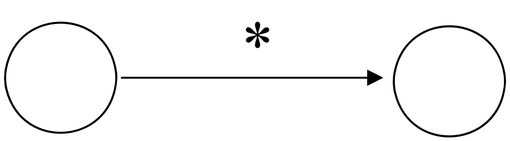
- Move sequence for an input ***string*** (not just one symbol):
  - $\delta^*: Q \times I^* \rightarrow Q$
    δ^* defined inductively from δ, by induction on the string length
    $\delta^* (q, \varepsilon) = q$
    $\delta^* (q, y \cdot i) = \delta(\delta^* (q,y), i)$
- Initial state: $q_0 \in Q$
- Set of final, or accepting states: $F \subseteq Q$
- Acceptance: $x \in L \leftrightarrow \delta^* (q_0, x) \in F$

Accepting Pascal identifiers

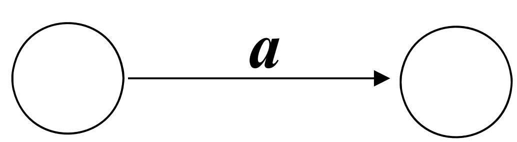


NB: the red part might be omitted

a useful notation/shorthand that is occasionally adopted



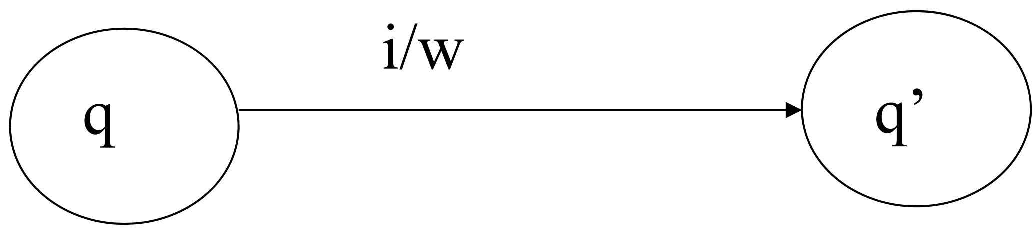
means



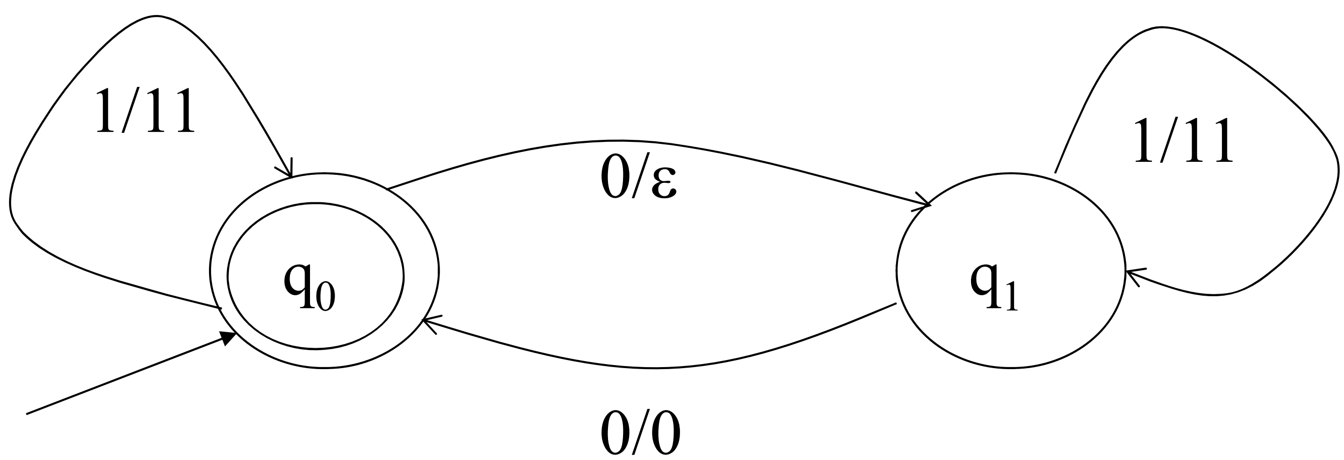
for all $a \in$ **alphabet** I

Automata as language translators
$y = \tau(x)$

Transitions with output: notice that w is a ***string*** (possibly ε)



τ: doubles the number of “1” and halves the “0” (the input must have an even number of “0”)



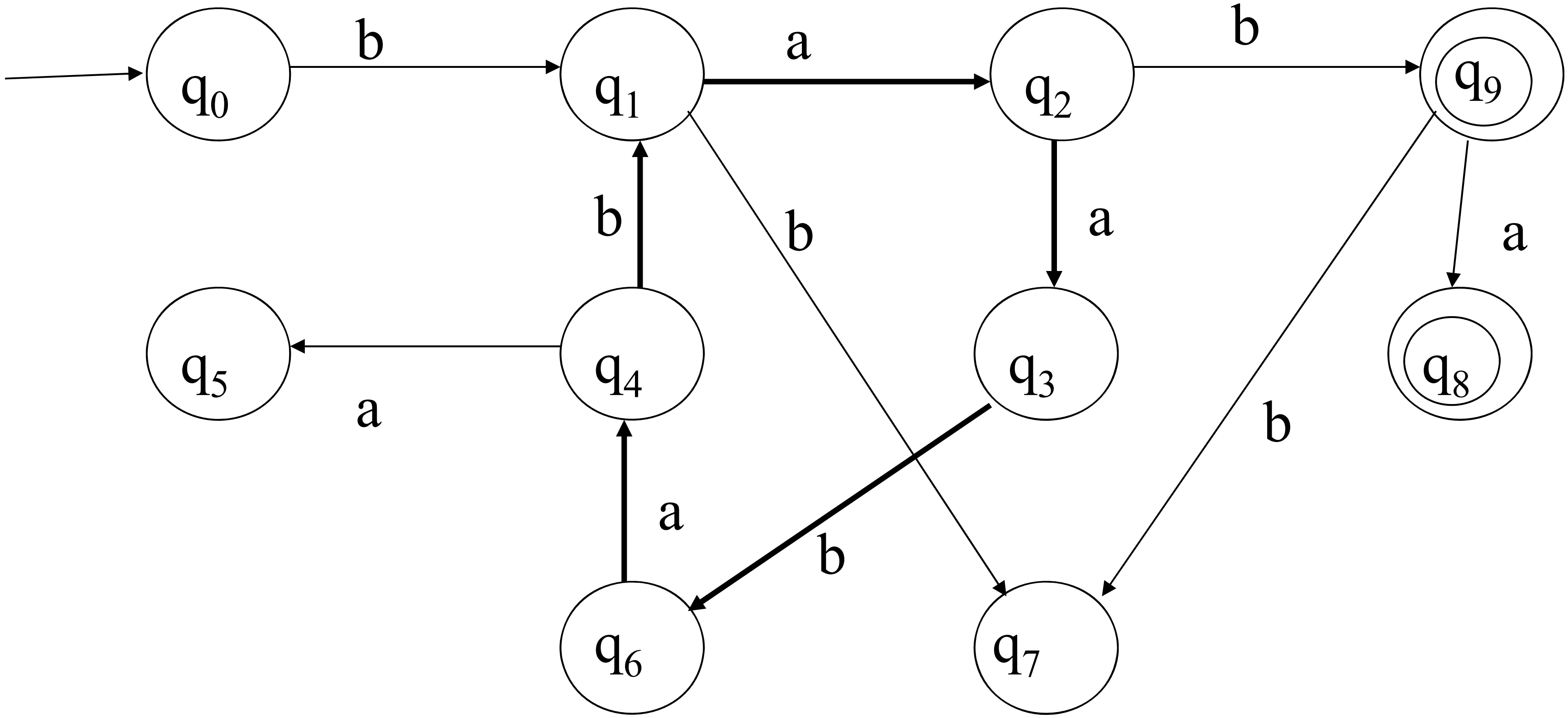
Formalization of translating automata (transducers)

- $T = <Q, I, \delta, q_0, F, O, \eta>$
  - $<Q, I, \delta, q_0, F>$: just like for acceptors
  - O: output alphabet
  - $\eta : Q \times I \rightarrow O^*$ (NB: the output is a string)
- $\eta^* : Q \times I^* \rightarrow O^*$, η^* defined inductively as usual
  $\eta^*(q, \varepsilon) = \varepsilon$
  $\eta^*(q, y \cdot i) = \eta^*(q, y) \cdot \eta(\delta^* (q,y), i)$
- definition of the translation [provided that…]

  $\tau(x)\ _{[x \in L]} = \eta^*(q_0, x)\ [\delta^* (q_0, x) \in F]$

Analysis of the finite state model (for the synthesis refer to other courses - e.g. on digital circuits)

- Very simple, intuitive model, applied in various applicative domains, also outside computer science
- Is there a price for this simplicity?
- …
- A first, fundamental property: the ***cyclic behavior*** of finite automata



There is a cycle q_1 ----***aabab***---> q_1

If, when reading a string, one goes through the cycle once, then one can also go through it 2, 3, …, n, … times

Formally:

- If $x \in L$ and $|x| > |Q|$ then there exists a $\boldsymbol{q} \in \boldsymbol{Q}$ and a $\boldsymbol{w} \in \boldsymbol{I}^+$ such that:

$$\boldsymbol{x = ywz}$$

$$\delta^* (\boldsymbol{q}, \boldsymbol{w}) = \boldsymbol{q}$$

Then $\boldsymbol{y\, w^n\, z} \in \boldsymbol{L}, \forall\, \boldsymbol{n} \geq \mathbf{0}$:

- This is known as the **Pumping Lemma**

Several properties of FA –both good and bad ones– follow from the pumping lemma and other properties of the graph of δ

- Let A be an FA, and L=L(A) the language accepted by A
- $|L| = \infty$? $|L| = \infty$ iff there exists a cycle on the graph of δ with a node on a path from the initial state q_0 to a final state $q \in F$
- $L = \emptyset$? $\exists\ x \in L \leftrightarrow \exists\ y \in L, |y| < |Q|$:
  proof: Eliminate all cycles from A, then look for a path from initial state q_0 to a final state $q \in F$
- …
- Notice that being able to answer the question "$\boldsymbol{x \in L}$?" for every possible given string ***x*** (i.e., for a ***generic*** string) does not enable us to answer other questions, such as the emptiness of the accepted language

A “negative” consequence of the Pumping Lemma (PL)

- Is the language $L = \{a^n b^n \mid n > 0\}$ recognized by any FA?
  - Put another way: can we count using a FA?
- Let us assume so (by contradiction):
- Consider $x = a^m b^m$, with $m > |Q|$ and apply the PL: then $x = ywz$ and $yw^n z \in L\ \forall n$
- There are 3 possible cases:
  - $x = ywz$, $w = a^k$, $k > 0$ ====> $a^{m+r.k} b^m \in L$, $\forall r$: this cannot be (it contradicts the assumption)
  - $x = ywz$, $w = b^k$, $k > 0$ ====> idem
  - $x = ywz$, $w = a^k b^s$, $k,s > 0$ ====> $a^{m-k} a^k b^s a^k b^s b^{m-s} \in L$: this cannot be (it contradicts the assumption)

- Intuitive conclusion: to "count" any ***n*** one needs an infinite memory!
- NB: strictly speaking any computer is a FA: so a computer cannot count?
  - But this is a wrong abstraction of (way of looking at) a computer!
  - It is important to consider a different, less immediate notion of infinity!
  - We consider the memory of a computer infinite as long as all the strings that it manipulates (confortably) fit into its (finite) memory.
- Going from the toy example $\{a^n b^n\}$ to more concrete ones:
  - Recognizing parenthetical structures like those of the programming languages cannot be done with a finite memory
- Therefore we need "more powerful" models

Closure properties of FA

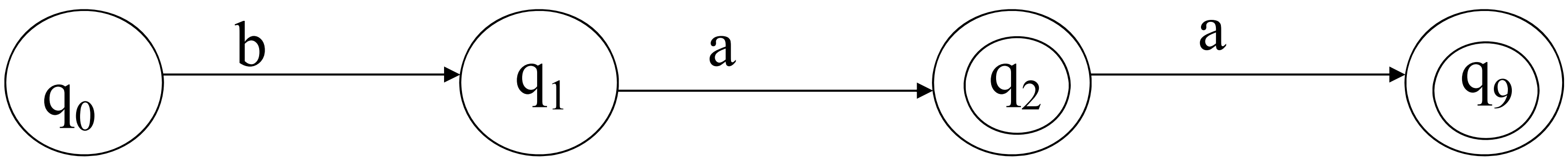
- The mathematical notion of closure:
  - Natural numbers are closed under the sum operation
  - But not under subtraction
  - Integer numbers are closed under sum, subtraction, multiplication, but not …
  - Rational numbers …
  - Real numbers …
  - Closure (under operations and relations) is a very important notion

In the case of Languages:

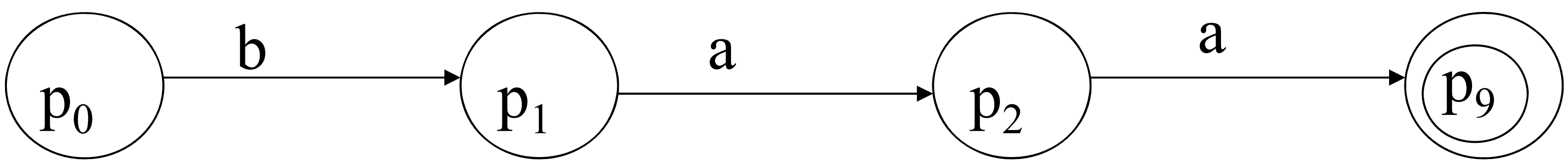
- $\mathcal{L} = \{L_i\}$: a *family* of languages
- $\mathcal{L}$ is closed under OP if and only if (iff)
  for every $L_1, L_2 \in \mathcal{L}$, L_1 OP $L_2 \in \mathcal{L}$
- $\mathcal{R}$: *regular languages*, those accepted by an FA
- $\mathcal{R}$ is closed under set theoretic operations, concatenation, "*", … and virtually "all others"

Intersection

A

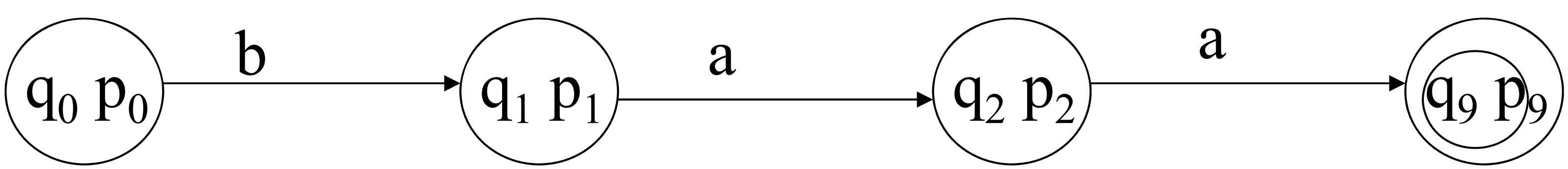


B



One can simulate the “parallel run” of A and B by simply “coupling them”:

<A,B>



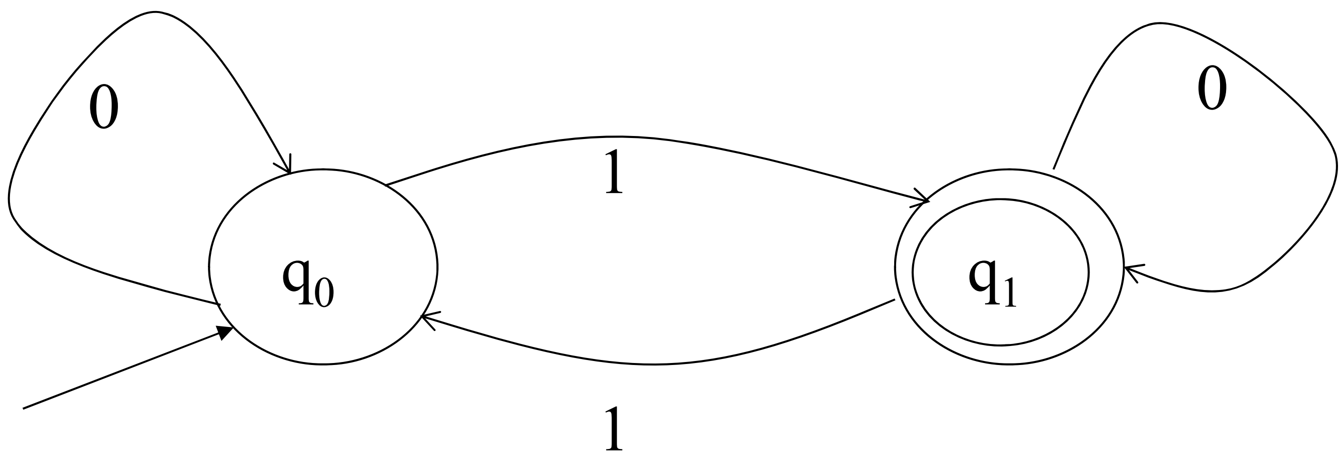
Formally:

- Given $A^1 = <Q^1, I, \delta^1, q_0{}^1, F^1>$ and
  $A^2 = <Q^2, I, \delta^2, q_0{}^2, F^2>$
- The automaton $< A^1, A^2 >$ is defined as:
  $<Q^1 \times Q^2, I, \delta, <q_0{}^1 , q_0{}^2 >, F^1 \times F^2 >$
  $\delta(<q^1 , q^2 > , i) = <\delta^1(q^1, i), \delta^2(q^2,i)>$
- One can show by a simple induction that
  $L(< A^1, A^2 >) = L(A^1) \cap L(A^2)$

Union

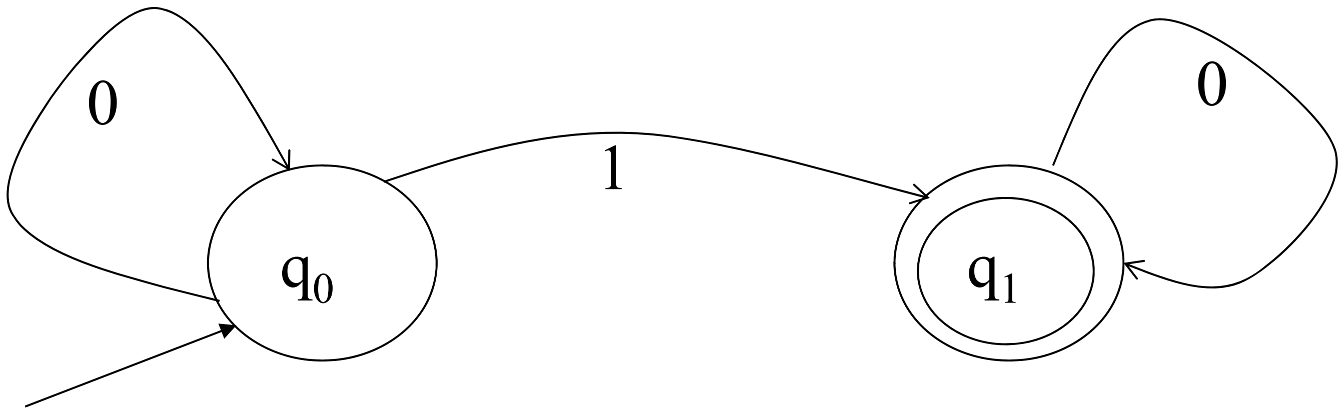
- A similar construction …
  otherwise ... exploit identity $A \cup B = \neg(\neg A \cap \neg B)$
  $\Rightarrow$ need a FA for the complement language

Complement: example automaton accepting
binary strings x with odd number of 1's



An idea: F^ = Q - F:

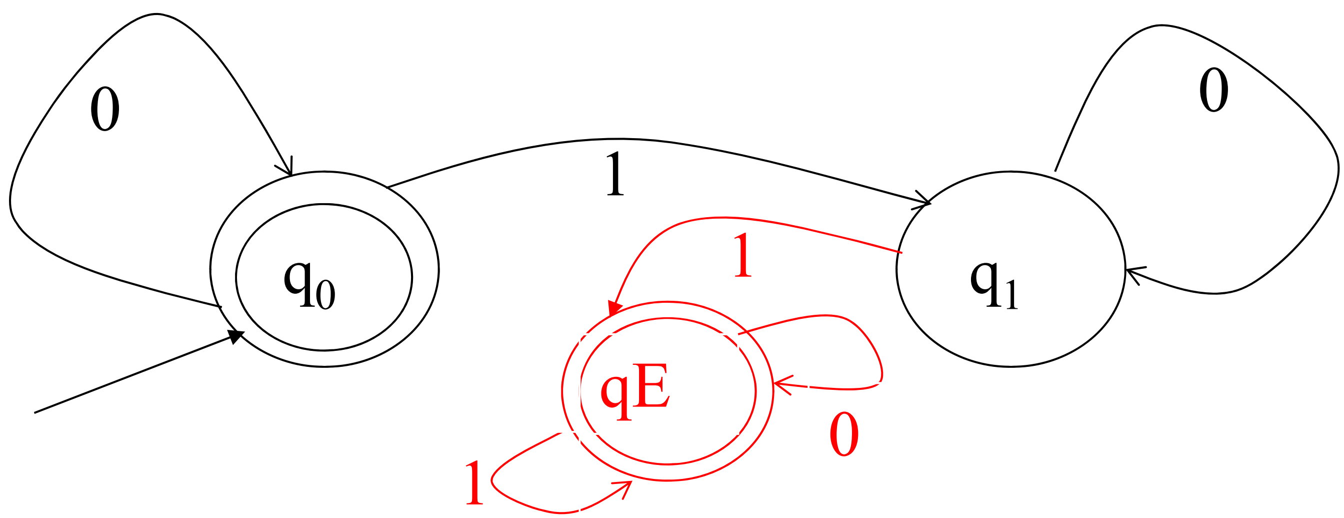
Yes, it works for the automaton above, but ….



L: strings with exactly one '1'

The "complement F construction", i.e., turning F into Q – F, doesn't work

Why? because δ is partial



¬L: strings with no '1' or with more than one '1'

Some general remarks on the complement

- If the analysis of the string terminates for all possible strings then it suffices the "turn a yes into a no" (F into Q-F)
- If, for some string, the analysis of the string does not terminate (it gets blocked or continues forever) then turning F into Q-F does not work
- With FA the problem is easily solved …
- In general if we are unable to provide a positive answer to a problem (e.g., $\boldsymbol{x} \in \boldsymbol{L}$), this does not necessarily mean we can provide a positive answer for the complement problem (e.g., $\boldsymbol{x} \in \neg \boldsymbol{L}$)

Myhill – Nerode Theorem

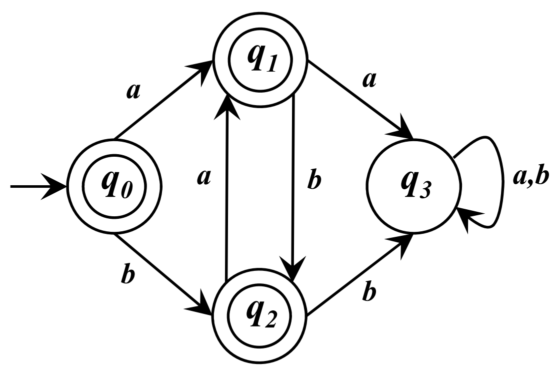
- What do the states a of a DFA $A = < Q, I, \delta, q_0, F >$ represent?
  - every state represents the **set of strings** that, input to A, lead to it
  - these strings "share a common destiny":

    notice that, for every $\boldsymbol{x}, \boldsymbol{y} \in I^*$, if $\delta(q_0, x) = \delta(q_0, y)$ then

    $\forall \boldsymbol{z} \in \mathrm{I}^* \quad \delta(q_0, x \cdot z) = \delta(q_0, y \cdot z)$ hence $x \cdot z \in L(A)$ iff $y \cdot z \in L(A)$

    i.e., either both or none of $\boldsymbol{x} \cdot \boldsymbol{z}$ and $\boldsymbol{y} \cdot \boldsymbol{z} \in L(A)$

- Example: automaton accepting
  - strings $\in \{$a, b$\}^*$ s.t. no two adjacent symbols are equal



  - q_0 associated with $\{\varepsilon\}$, q_1 associated with {strings ending with a},
  - q_2 with {strings ending with b}, q_3 with {strings with two equal adjacent symbols}

- Apply this idea to a language $L \subseteq I^*$: define a relation $\approx_L$ among strings
  - $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{L}$ are ***indistinguishable*** $\boldsymbol{x} \approx_L \boldsymbol{y}$ iff

    $$\forall\, \boldsymbol{z} \in \boldsymbol{I^*} \qquad \boldsymbol{x \cdot z} \in \boldsymbol{L} \;\Leftrightarrow\; \boldsymbol{y \cdot z} \in \boldsymbol{L}$$

  - note that $\boldsymbol{\approx_L}$ is an equivalence relation (‘$\Leftrightarrow$’ is a sort of equality)
  - hence $\boldsymbol{I^*}$ is *partitioned* by $\approx_L$ into equivalence classes
- the cardinality of the equiv. classes of $\approx_L$ may be
  - **finite**. Example: $L = \{ x \in \{a, b\}^* \mid$ no two adjacent symbols of $\boldsymbol{x}$ are equal $\}$
    - $\{\varepsilon\}$, {strings ending with a}, {strings ending with b}, {strings with two equal adjacent symbols}
  - **infinite**. Example: $L = \{ a^n b^n \mid n \geq 0 \}$
    - $\{\varepsilon\}, \{a\}, \{aa\}, \{aaa\}, \{aaaa\}, \ldots \{a^n\}, \ldots$ and others for strings containing b’s

- Myhill-Nerode theorem:

  a language L is accepted by a DFA

  iff $\approx_L$ has a finite number of equivalence classes

- Proof (sketch) of Myhill-Nerode theorem:

  L accepted by a DFA iff $\approx_L$ has a finite number of equivalence classes

- (1): $L=L(A)$ for a DFA $A \Rightarrow \approx_L$ has a finite number of equiv. classes

  immediate: make $\boldsymbol{A}$ complete if necessary (by adding an error sink state)

  then every $x \in I^*$ is in the equiv. class of some state of $\boldsymbol{A}$

- (2): $\approx_L$ finite equiv. classes $\Rightarrow L$ accepted by the DFA $A = (Q, I, \delta, q_0, F)$:
  - $q_0 = [\varepsilon]$ (NB: $[x]$ denotes the equiv. class that includes x)
  - $Q = \{ [x] \mid x \in I^* \}$
  - for every equiv. class $[x]$, $\forall a \in I$ $\delta([x], a) = [x \cdot a]$
    - NB: δ is *well defined*: $\delta([x], a)$ does not depend on the x chosen as a representative of the class, by the definition of $\approx_L$
  - $F = \{ [x] \mid x \in L \}$

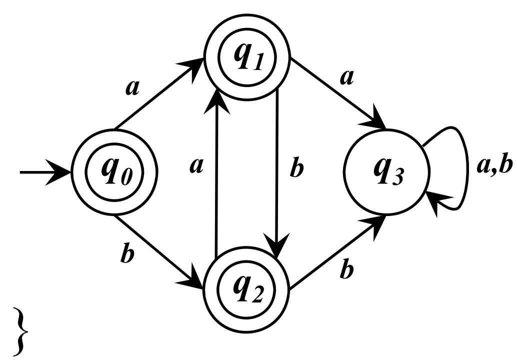
- Another (simplest) example: $L = \{ x \in \{a\}^* \mid |x| \text{ MOD } 3 = 1 \}$
  - Exercise: Identify the equiv. classes of $\approx_L$; Design the accepting automaton

"Applications" of the Myhill-Nerode theorem

- Prove (Disprove) that a language is accepted by a DFA
  - Based on the fact the $\approx_L$ has (does not have) a finite number of equiv. classes
- DFA minimization (find the minimal equivalent DFA)
  - …introducing the notion of ***indistinguishable states***

- "Annotate" the states of a DFA by the associated $\approx_L$ equiv. classes

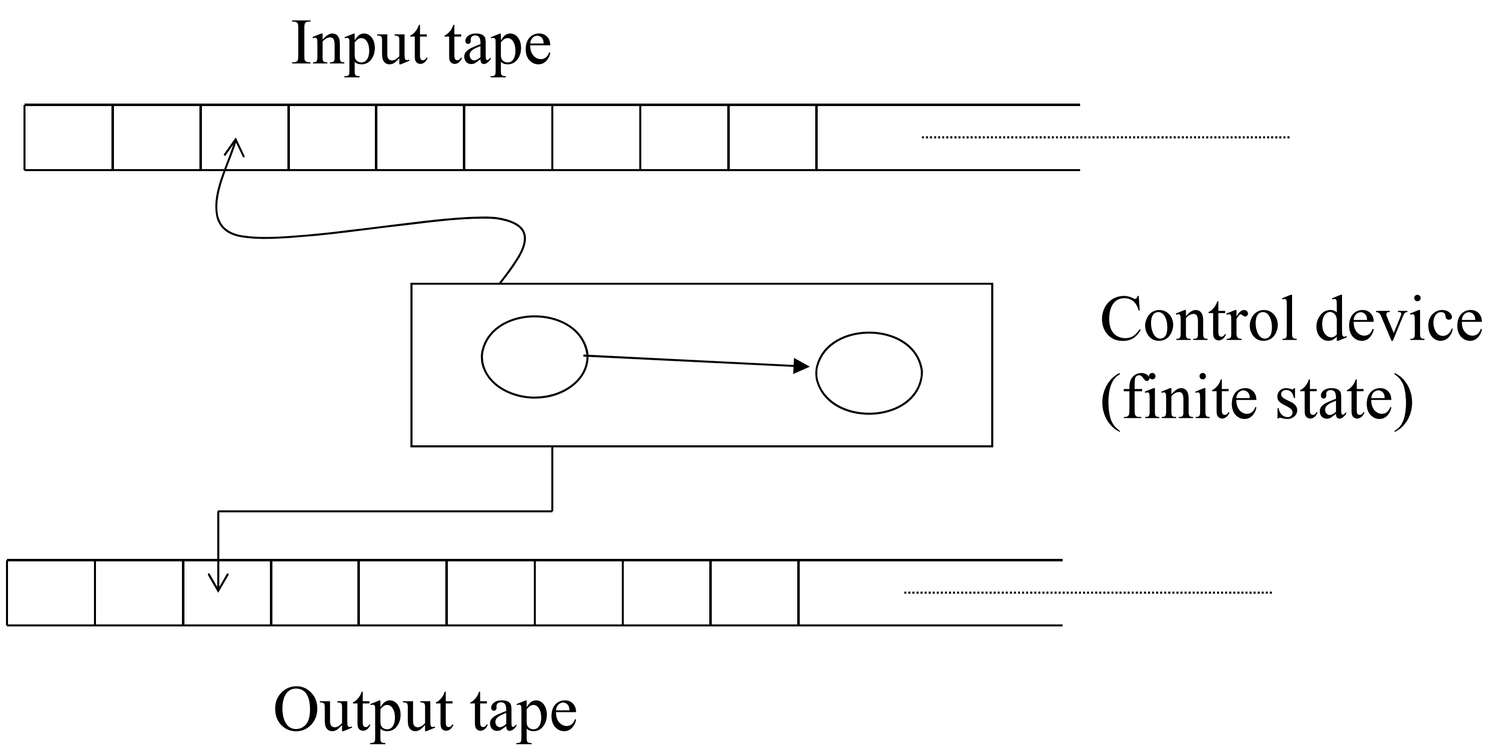
- Example $L = \{ x \in \{a, b\}^* \mid$ no two adjacent symbols of $\boldsymbol{x}$ are equal $\}$



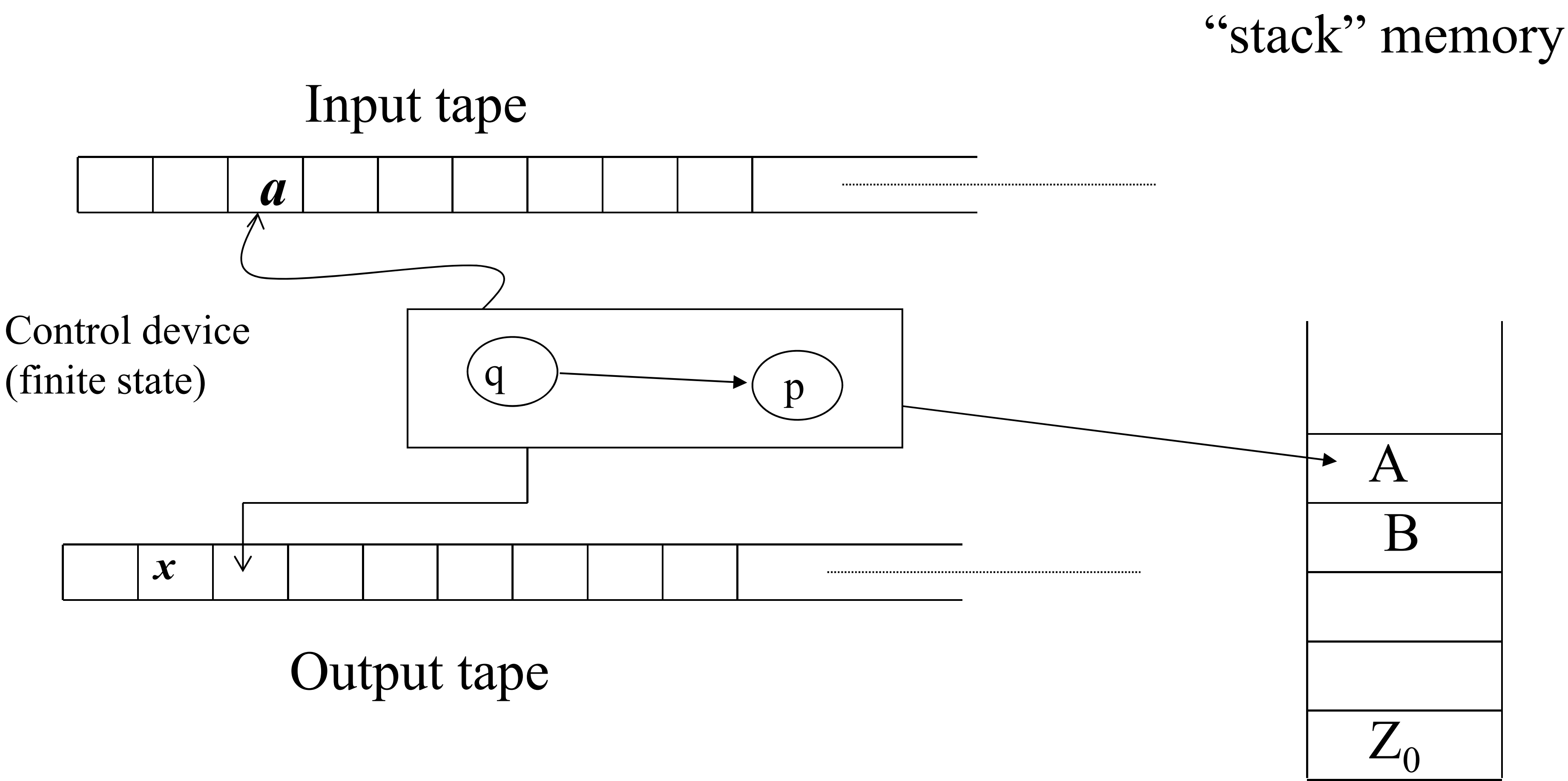
  - $q_0 \leftrightarrow [\varepsilon] = \{\varepsilon\}$,
  - $q_1 \leftrightarrow [a] = \{ x \mid \exists y\ \ x = y{\cdot}a \wedge$ "no aa nor bb in x" $\}$,
  - $q_2 \leftrightarrow [b] = \{ x \mid \exists y\ \ (x = y{\cdot}b) \wedge$ "no aa nor bb in x" $\}$,
  - $q_3 \leftrightarrow [aa] = \{ x \mid \exists y, z \in I^* (x = y{\cdot}aa{\cdot}z \ \vee\ x = y{\cdot}bb{\cdot}z) \}$

Let us increase the power of the FA by increasing its memory

- Consider a more "mechanical" view of the FA:



- Now let us “enrich it” :



The move of the pushdown automaton (with a **stack** memory):

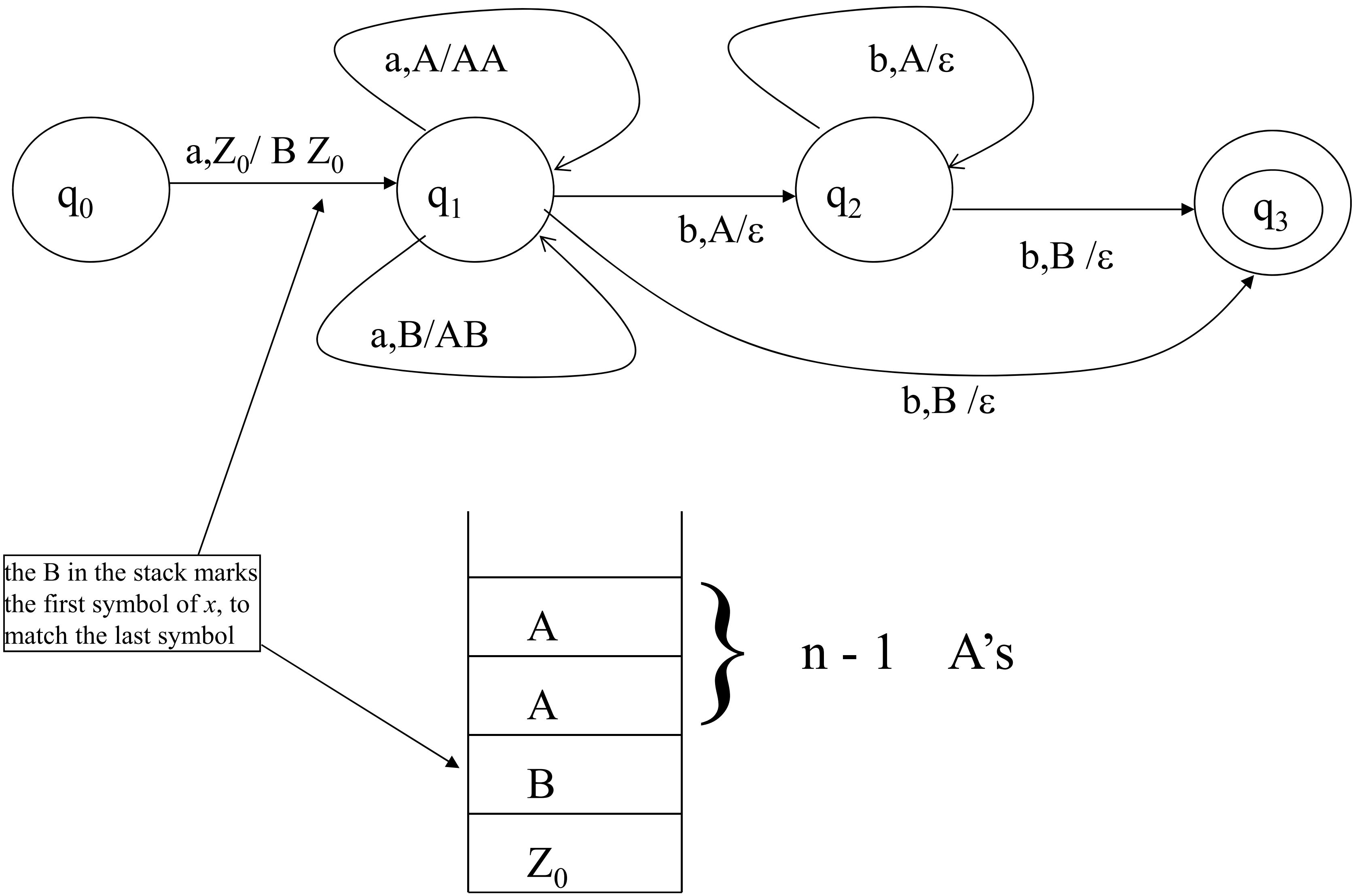
- Depending on :
  - the symbol read on the input tape (but it could also ignore the input …)
  - the symbol read on top of the stack
  - the state of the control device:
- the pushdown automaton
  - changes its state
  - moves ahead the scanning head (or it does not if input was ignored)
  - changes the symbol A read on top of the stack with a ***string*** α of symbols (α may be empty: this amounts to a ***pop*** of A)
  - (if translator) it writes a string (possibly empty) on the output tape (advancing the writing head consequently)

- The input string $\boldsymbol{x}$ is recognized (accepted) if
  - The automaton scans it completely (the scanning head reaches the end of $\boldsymbol{x}$), and
  - Upon reaching the end of $\boldsymbol{x}$ it is in an acceptance state (just like for the FA)
- If the automaton is also a translator, then $\boldsymbol{\tau(x)}$ is the string on the output tape after $\boldsymbol{x}$ has been completely scanned (if x is accepted, otherwise $\tau(x)$ is undefined: $\boldsymbol{\tau(x) = \perp}$)

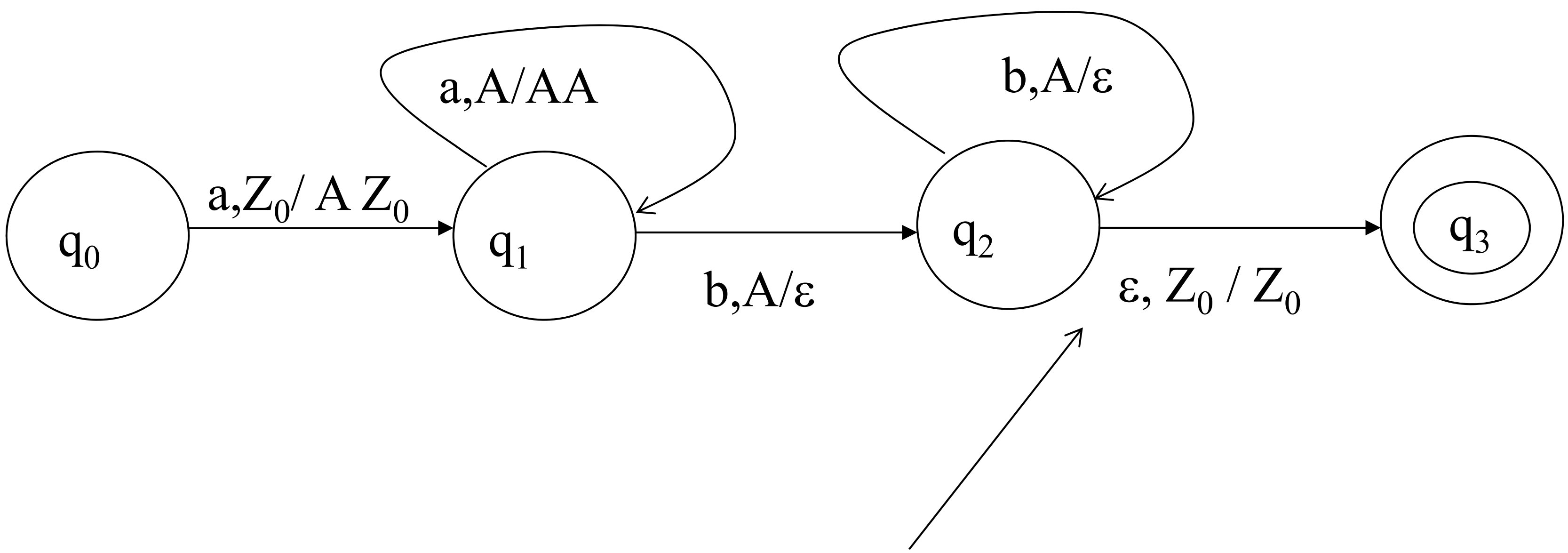
NB: $\perp$ is the "undefined" symbol:

$\boldsymbol{\tau(x)} = \perp$ is just a shorthand for $\neg\exists \boldsymbol{y}\ (\boldsymbol{\tau(x) = y})$

A first example: accepting $\{a^n b^n \mid n > 0\}$



Another one:

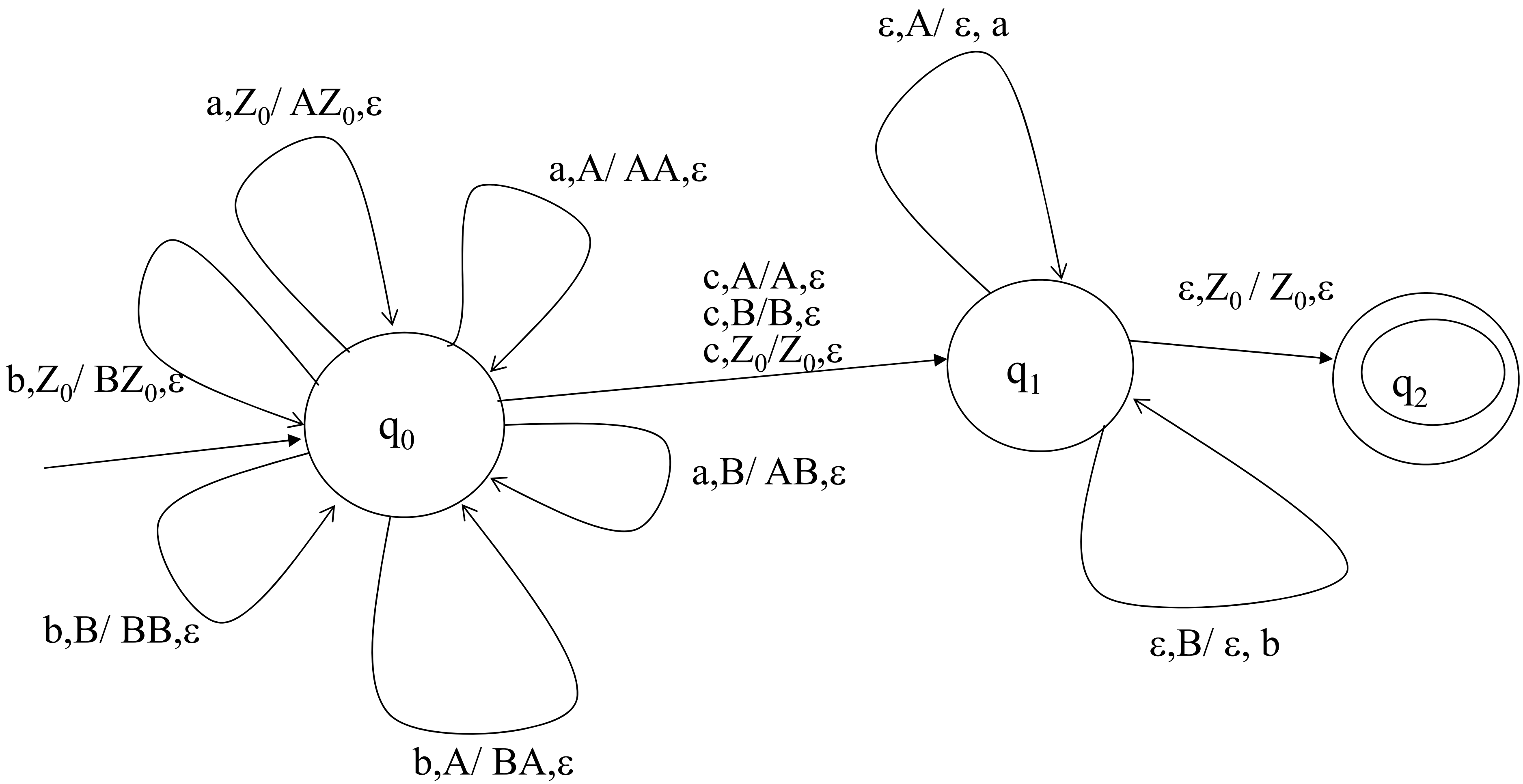


ε-move (AKA ***spontaneous*** move)

the ‘ε’ symbol is «overloaded»: used as an ***additional input symbol***

A (classical) pushdown automaton-translator

It reverses a string: $\tau(wc)=w^R$, $\forall w \in \{a,b\}^+$

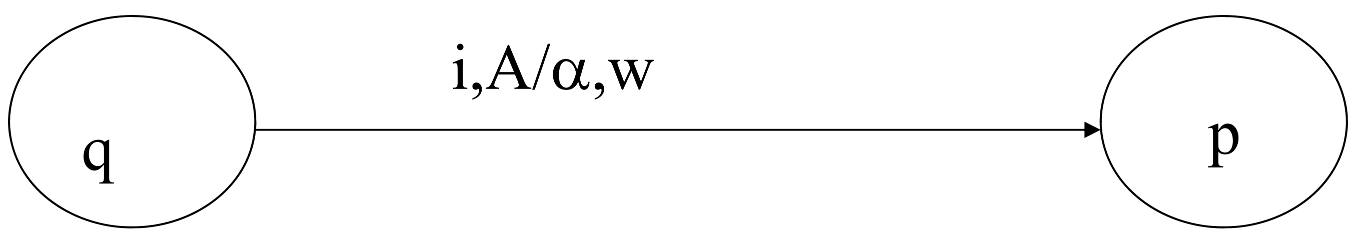


Now we formalize ...

- Pushdown automaton [transducer]: $<Q, I, \Gamma, \delta, q_0, Z_0, F [, O, \eta]>$
- Q, I, q_0, F [O] just like FA [FST]
- Γ stack alphabet (disjointed from other ones for ease of definition)
- Z_0 : initial stack symbol (not essential, useful to simplify definitions)
- $\delta: Q \times (I \cup \{\varepsilon\}) \times \Gamma \rightarrow Q \times \Gamma^*$ δ is partial just as in FSA
- $\eta: Q \times (I \cup \{\varepsilon\}) \times \Gamma \rightarrow O^*$ (η defined where δ is)

Graphical notation:

$$<p,\alpha> = \delta(q,i, A)$$
$$w \quad = \eta(q,i, A)$$



- Configuration (a generalization of the notion of state): $c = \langle q, x, \gamma, [z] \rangle$:
  - q: state of the control device
  - x: ***unread*** portion of the input string (the head is positioned on the first symbol of x)
  - γ : string of symbols in the stack (convention: <top-left, bottom-right>)
  - z: string written (up to now) on the output tape

- ***Transition relation* ‘|--’** among configurations:
  $c = \langle q, x, \gamma, [z] \rangle \;|\!-\!-\; c' = \langle q', x', \gamma', [z'] \rangle$
  - $\gamma = A\beta$

  - **Case 1**: $x = i.y$ and $\delta(q, i, A) = \langle q', \alpha \rangle$ (**ordinary, “input-consuming” move**)
    $[\eta(q,i, A) = w]$
  - $x' = y$
  - $\gamma' = \alpha\beta$
  - $[z' = z.w]$

  - **Case 2**: $\delta(q, \varepsilon, A) = \langle q', \alpha \rangle$ (**spontaneous, ε move**)
    $[\eta(q, \varepsilon, A) = w]$
  - $x' = x$
  - $\gamma' = \alpha\beta$
  - $[z' = z.w]$
- NB: $\forall q,A,i \;\; (\delta(q, i, A) \neq \perp \Rightarrow \delta(q, \varepsilon, A) = \perp)$ (hence $\delta(q, \varepsilon, A) \neq \perp \Rightarrow \delta(q, i, A) = \perp$)
  - i.e., ε-moves are alternative to all input-consuming ones
- Otherwise … nondeterminism!

- Acceptance [and translation] of a string
- |-*- : reflexive, transitive closure of the relation |--
  - i.e., |-*- denotes a number ≥0 of "steps" of the relation |--
- $x \in L$ [$z = \tau(x)$] $\leftrightarrow$

$$c_0 = \langle q_0, x, Z_0, [\varepsilon] \rangle \;|\!-^{*}\!-\; c_F = \langle q, \varepsilon, \gamma, [z] \rangle,\ q \in F$$

input string completely scanned

stack content immaterial

last state final

Pay attention to ε-moves, especially at the end of the string!

Pushdown automata in practice

- They are the heart of compilers
- Stack memory (LIFO) suitable to analyze nested syntactic structures (arithmetical expressions, compound instructions, …)
- Abstract run-time machine for programming languages with recursion
- ….

  Occur very frequently in the course of Languages and translators

Properties of pushdown automata (especially as acceptors)

- $\{a^n b^n \mid n > 0\}$ is accepted by a pushdown automaton (not by a FA)
  - However $\{a^n b^n c^n \mid n > 0\}$ ….
  - NOT: after counting –using the stack- *n a*'s and "de-counting" *n b*'s how can we remember *n* to count the *c*'s?
    The stack is ***not*** a read-only memory: to read it, one must destroy it!
    This limitation of the pushdown automaton can be proved formally through a generalization of the pumping lemma.
- $\{a^n b^n \mid n > 0\}$ accepted by a pushdown automaton;
  $\{a^n b^{2n} \mid n > 0\}$ accepted by a pushdown automaton
- However $\{a^n b^n \mid n > 0\} \cup \{a^n b^{2n} \mid n > 0\}$ …
  - Reasoning intuitively similar to the previous one:
  - If I empty all the stack with *n b*'s then I am unable to count other *b*'s
  - If I empty only half the stack and I do not find any more *b* I cannot know if I am halfway in the stack
  - The formalization of this reasoning is however not trivial ….

Some consequences

- $\mathcal{LP}$ = class languages accepted by pushdown automata
- $\mathcal{LP}$ is not closed under union nor intersection
- Why?
  - [consider the languages $\{a^n b^n c^n \mid n > 0\}$ and $\{a^n b^n \mid n > 0\} \cup \{a^n b^{2n} \mid n > 0\}$]
- Considering the complement …
  The same principle as with FA: change the accepting states into non accepting states.
  There are however new difficulties

- Function δ must be made complete (as with FA) with an error state. Pay attention to the nondeterminism caused by ε-moves!
- The ε-moves can cause cycles $\Rightarrow$ never reach the end of the string $\Rightarrow$ the string is not accepted, but it is not accepted either by the automaton with F^ = Q-F.
- There exists however a construction that associates to every automaton an equivalent loop-free automaton
- Not finished yet: what if there is a sequence of ε-moves at the end of the scanning with some states in F and other ones not in F?

- $<q_1, \varepsilon, \gamma_1>$ |-- $<q_2, \varepsilon, \gamma_2>$ |-- $<q_3, \varepsilon, \gamma_3>$ |-- …
  $q_1 \in F$, $\quad q_2 \notin F$, …. ?
- Then we must "force" the automaton to accept only at the end of a (necessarily finite, as the automaton is loop-free) sequence of ε-moves.
- This is also possible through a suitable construction.

Once more, rather than the technicalities of the construction/proof we are interested in the general mechanism to accept the complement of a language: sometimes the same machine that solves the "positive instance" of the problem can be adapted to solve the "negative instance": this can be trivial or difficult: we must be sure to be able to complete the construction

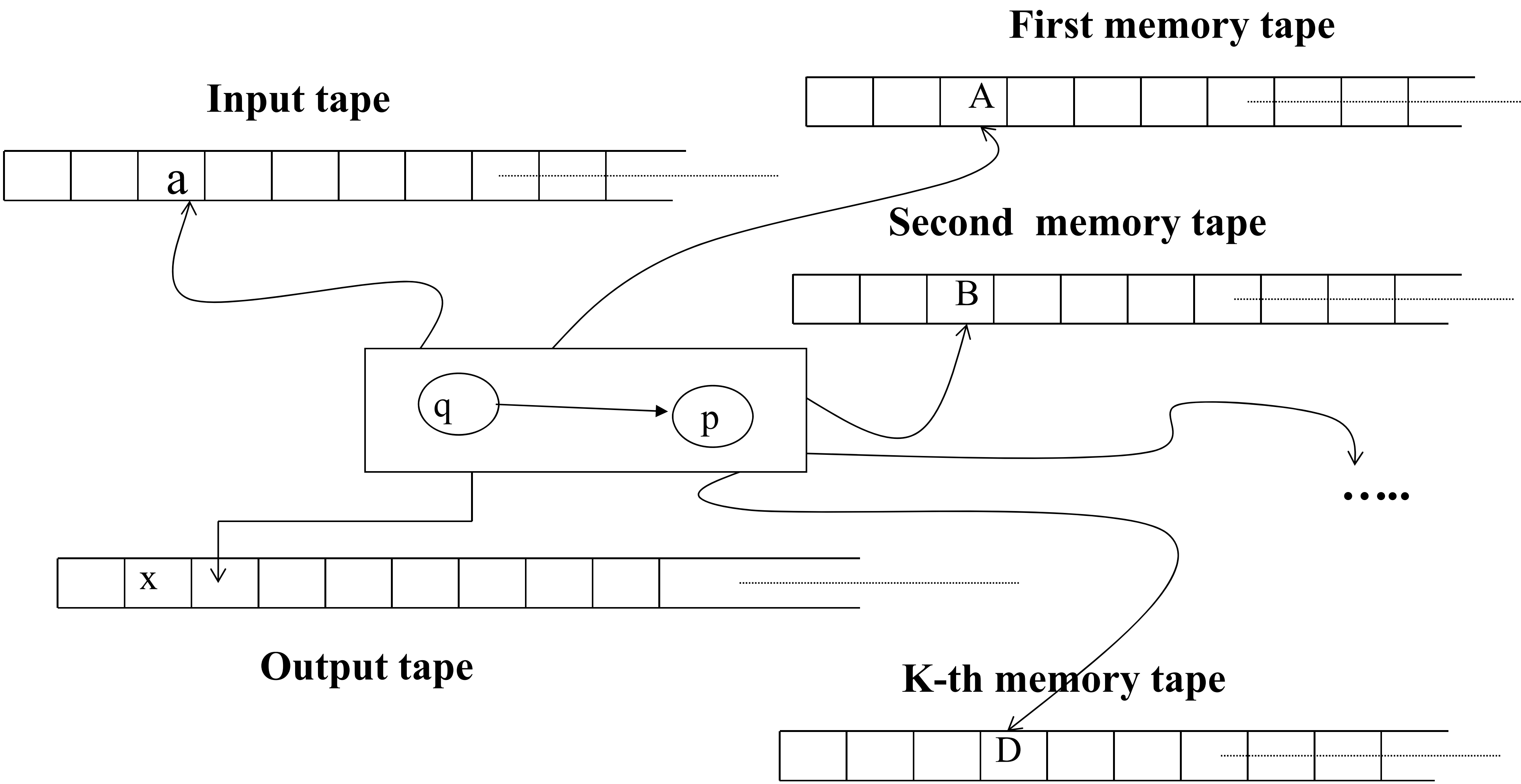
Pushdown automata [as acceptors (PDA) or translators (PDT)] are more powerful than finite state ones (a FA is a trivial special case of a PDA; more, PDA have an unlimited counting ability that FA lack)
However also PDA/PDT have their limitations …
… a new and "last" (for us) automaton:

the ***Turing Machine*** (TM)
Historical model of a "computer", simple and conceptually important under many aspects.

We consider it as an automaton; then we will derive from it some important, universal properties of automatic computation.

For now we consider the "K-tape" version, slightly different from the (even simpler) original model. This choice will be explained later.

k-tape TM


First memory tape
Input tape
A
a
Second memory tape
B
q
p
…..
x
Output tape
K-th memory tape
D

Informal Description and Partial Formalization of the TM

- States and alphabets as with other automata (input, output, control device, memory alphabet)
- For historical reasons and due to some “mathematical technicalities” the tapes are represented as *infinite* cell sequences [0,1,2, …] rather than finite strings. There exists however a special symbol “blank” (“ ”, or “barred b” or “_”) and it is assumed that every tape contains only a finite number of non-blank cells.
  - The equivalence of the two ways of representing the tape content is obvious.
- Scanning and output heads are also as in previous models

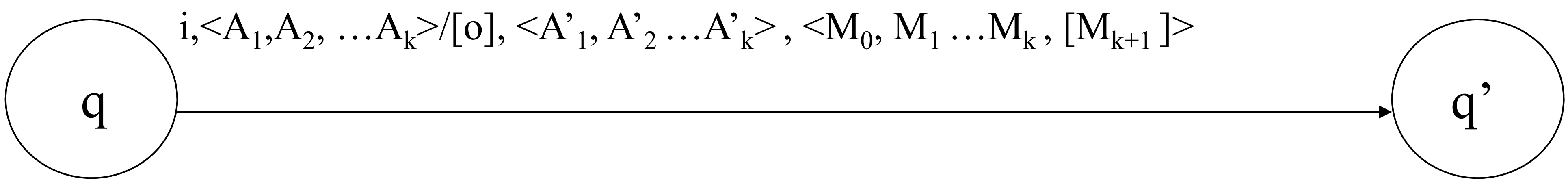
- The move of the TM:
- Reading:
  - one symbol on the input tape
  - k symbols on the k memory tapes (one for each tape)
  - state of the control device
- Action:
  - State change: q ----> q’
  - Write a symbol in place of the one read on each of the k memory tapes: A_i ----> A_i’, $1 <= i <= k$
  - [Write a symbol on the output tape]
  - Move of the k + 2 heads:
    - memory and scanning heads can move one position right (R) or left (L) or stand still (S)
    - The output head can move one position right (R) or stand still (S)

As a consequence:

$$< \delta, [\eta] >: Q \times I \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{R, L, S\}^{k+1} [\times O \times \{R, S\}]$$

(partial!)

Graphical notation:



$\forall i \in [0..k]\ \ M_i \in \{R,L,S\}$ $\qquad$ $M_{k+1} \in \{R,S\}$

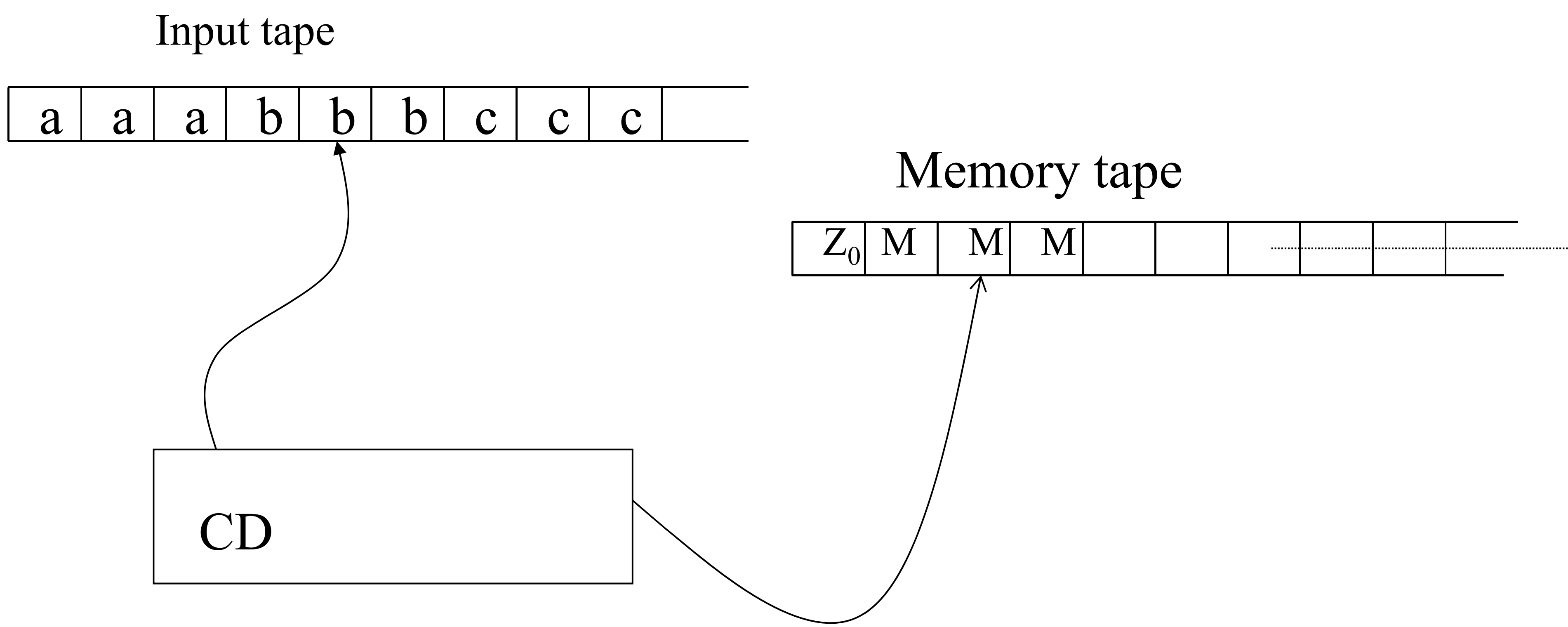
Why do we not loose generality having O rather than O^* for the output?

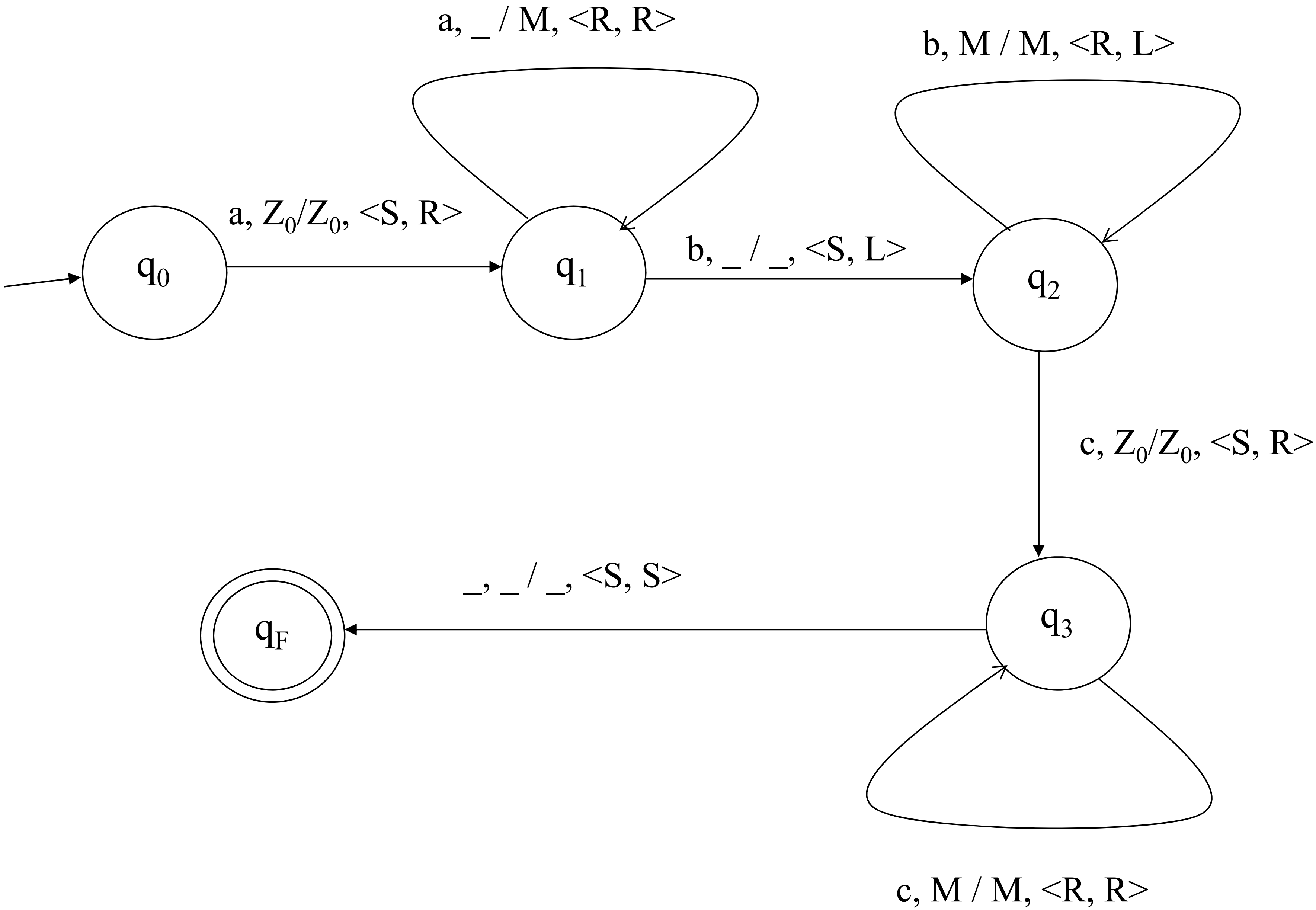
- Initial configuration:
  - •Z_0 followed by all blanks in the memory tapes
  - •[output tape all blank]
  - •Heads in the 0-th position on every tape
  - •Initial state of the control device q_0
  - •Input string x starting from the 0-th cell of the input tape, followed by all blanks

- Final configurations:
  - Accepting states $F \subseteq Q$
  - For ease of notation, convention:
    $<\delta,[\eta]> (q, \ldots) = \perp \ \forall \ q \in F$: no further action from a final state
  - The TM stops when $<\delta,[\eta]> (q, \ldots) = \perp$
  - Input string x is accepted iff:
    - After a ***finite*** number of moves the TM stops (hence it is in a configuration where $<\delta,[\eta]> (q, \ldots) = \perp$)
    - When it stops, its state $q \in F$
- NB: as a consequence
  - x is *not* accepted if:
    - The TM stops in a state $q \notin F$; or
    - The TM never stops (NB: this case is very important)
  - There is a similarity with the PDA (a non-loop-free PDA might also not accept because of a “non stopping run”), however … does there exist a loop-free TM?

Some examples

- A TM accepting $\{a^n b^n c^n \mid n > 0\}$



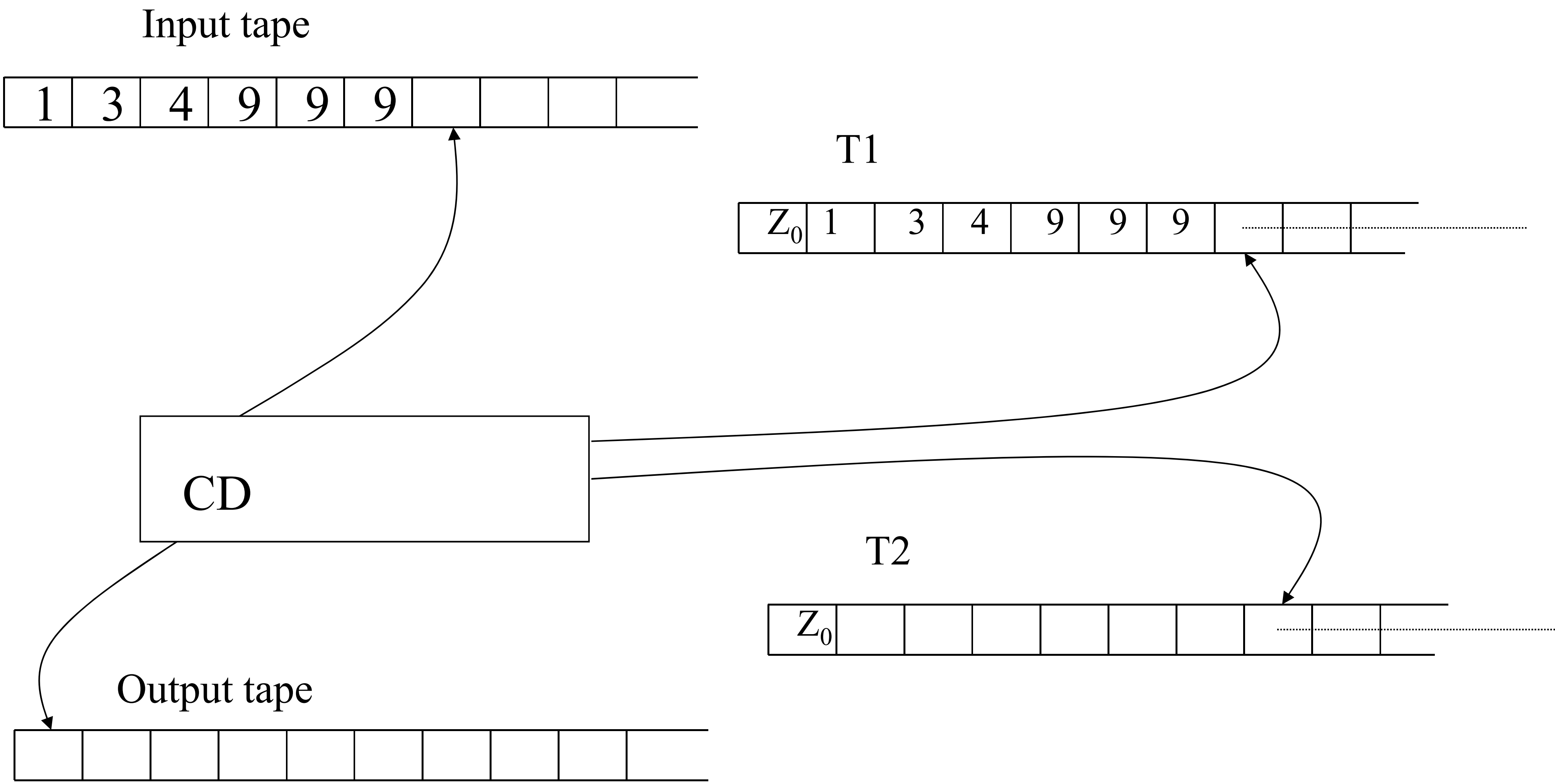

a, _ / M, <R, R>
b, M / M, <R, L>
a, Z_0/Z_0, <S, R>
b, _ / _, <S, L>
q_0
q_1
q_2
c, Z_0/Z_0, <S, R>
_, _ / _, <S, S>
q_F
q_3
c, M / M, <R, R>

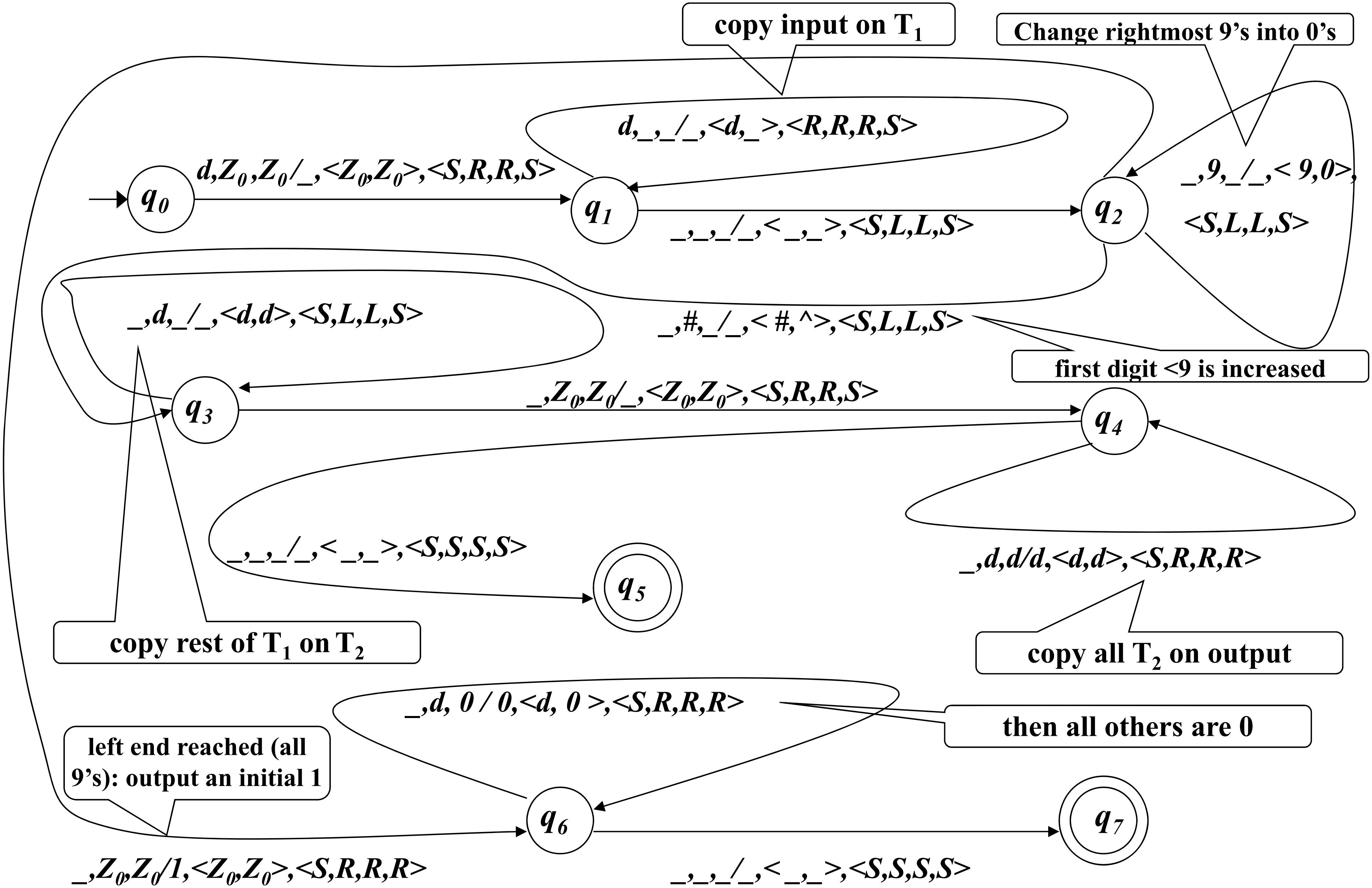
Computing the successor of a number n coded with decimal digits two memory tapes T_1 and T_2

- M copies all digits of n on T_1, to the right of Z_0, while it moves head T_2 by the same number of positions.
- M scans the digits of T_1 from right to left. It writes on T_2 from right to left changing the digits as needed (9's become 0's, first digit≠9 becomes the successive digit, then all other ones are unchanged, …)
- M copies T_2 on the output tape.

- Notation: ***d** : any decimal digit*
- ***_ : blank***
- ***# : any digit*** $\neq 9$
- ***^ : successor of a digit denoted as # (in the same transition)***

Purpose of the example: to show that the TM can compute ***any function***


Input tape
1 3 4 9 9 9
T1
Z_0 1 3 4 9 9 9
CD
T2
Z_0
Output tape


copy input on T_1
Change rightmost 9’s into 0’s
d,_,_/_,<d,_>,<R,R,R,S>
q_0
d,Z_0,Z_0/_,<Z_0,Z_0>,<S,R,R,S>
q_1
_,_,_/_,< _,_>,<S,L,L,S>
q_2
_,9,_/_,< 9,0>,
<S,L,L,S>
_,d,_/_,<d,d>,<S,L,L,S>
_,#,_/_,< #,^>,<S,L,L,S>
first digit <9 is increased
q_3
_,Z_0,Z_0/_,<Z_0,Z_0>,<S,R,R,S>
q_4
_,_,_/_,< _,_>,<S,S,S,S>
q_5
_,d,d/d,<d,d>,<S,R,R,R>
copy rest of T_1 on T_2
copy all T_2 on output
_,d, 0 / 0,<d, 0 >,<S,R,R,R>
then all others are 0
left end reached (all 9’s): output an initial 1
q_6
q_7
_,Z_0,Z_0/1,<Z_0,Z_0>,<S,R,R,R>
_,_,_/_,< _,_>,<S,S,S,S>

Closure properties of TM

- $\cap$: OK (a TM can easily simulate two other ones, both "in series" and "in parallel")
- $\cup$: OK (idem)
- Idem for other operations (concatenation, *, ….)

- What about the complement?

  Negative answer! (Proof later on)
  If there existed loop-free TM's, it would be easy: it would suffice to define the set of halting states (it is easy to make it disjoint from the set of non-halting states) and partition it into accepting and non-accepting states.

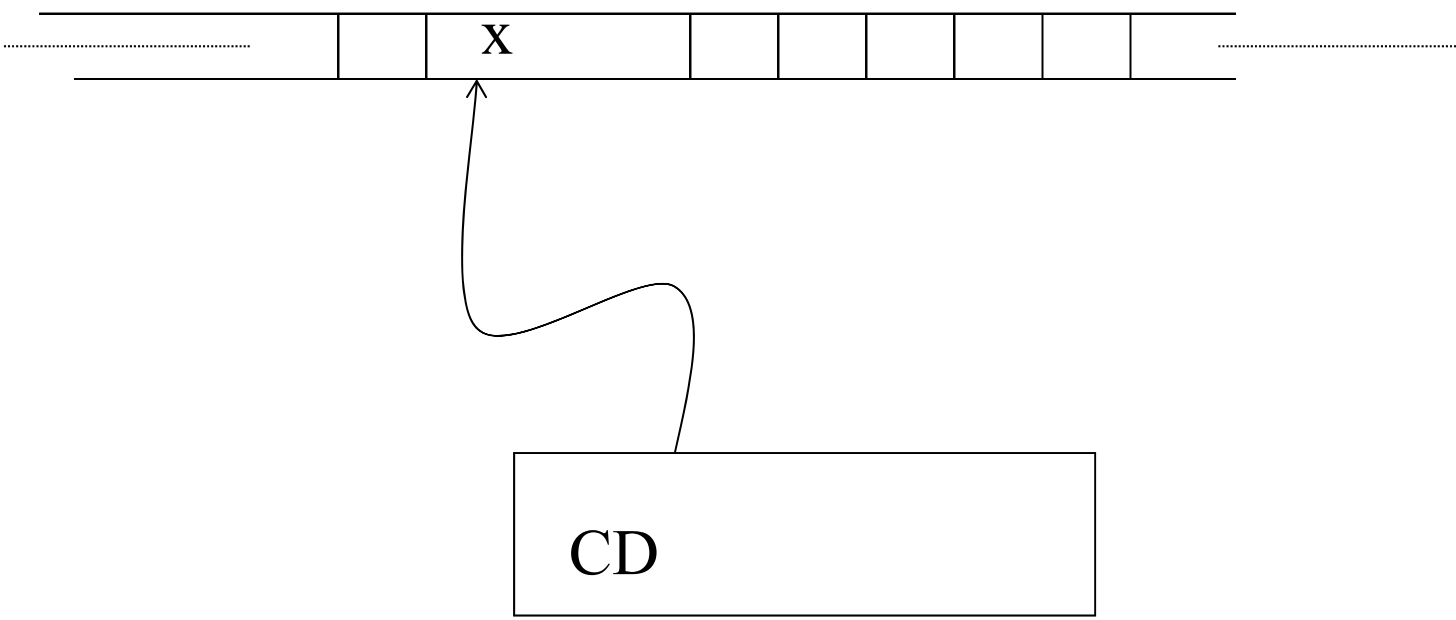
  ==========>

  It is therefore apparent that the problem arises from ***nonterminating computations***

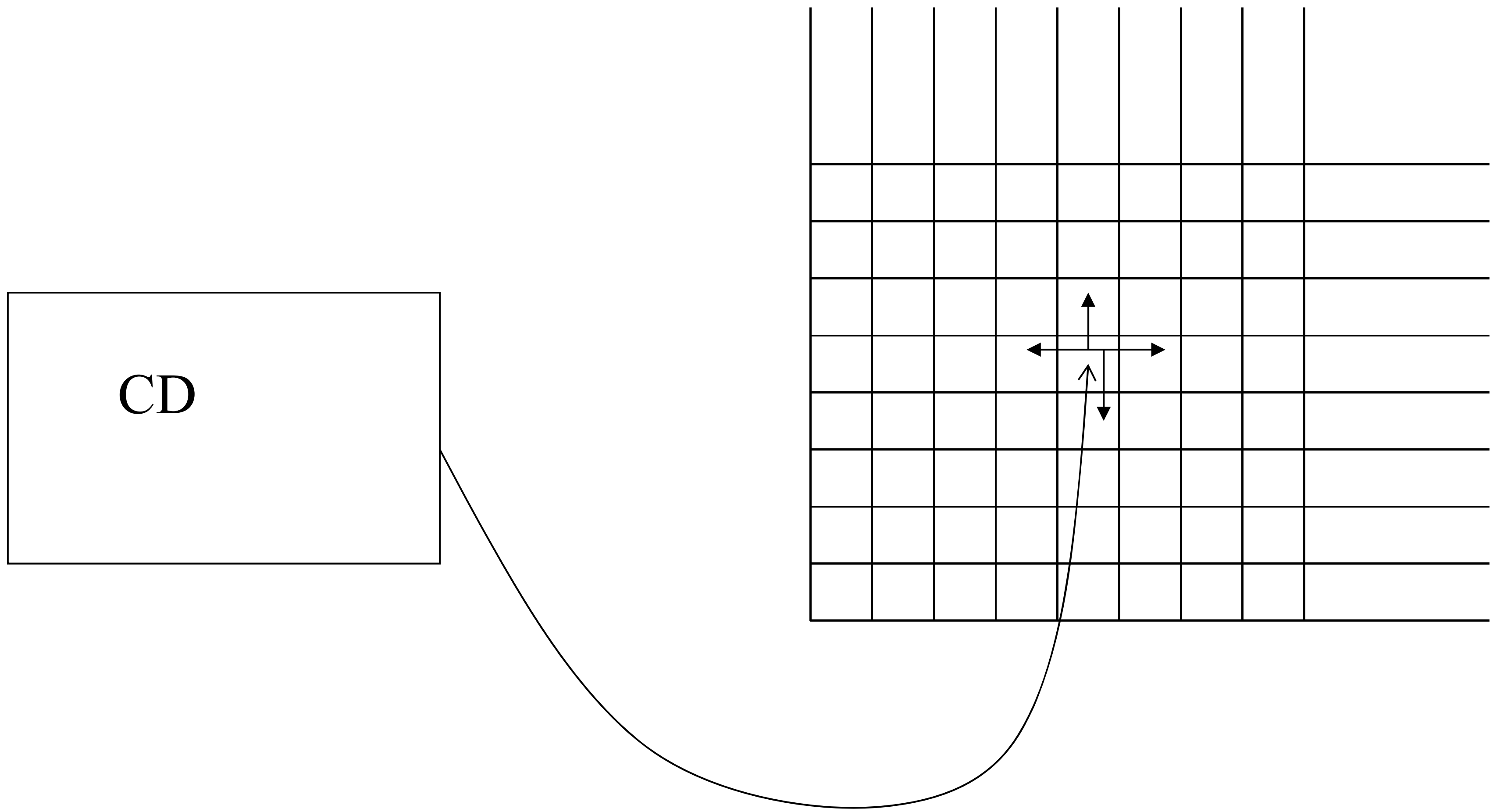
Equivalent TM models

- Single tape TM (NB: ≠ TM with 1 memory tape)

A single tape (usually unlimited in both directions):
serves as input, memory, and output

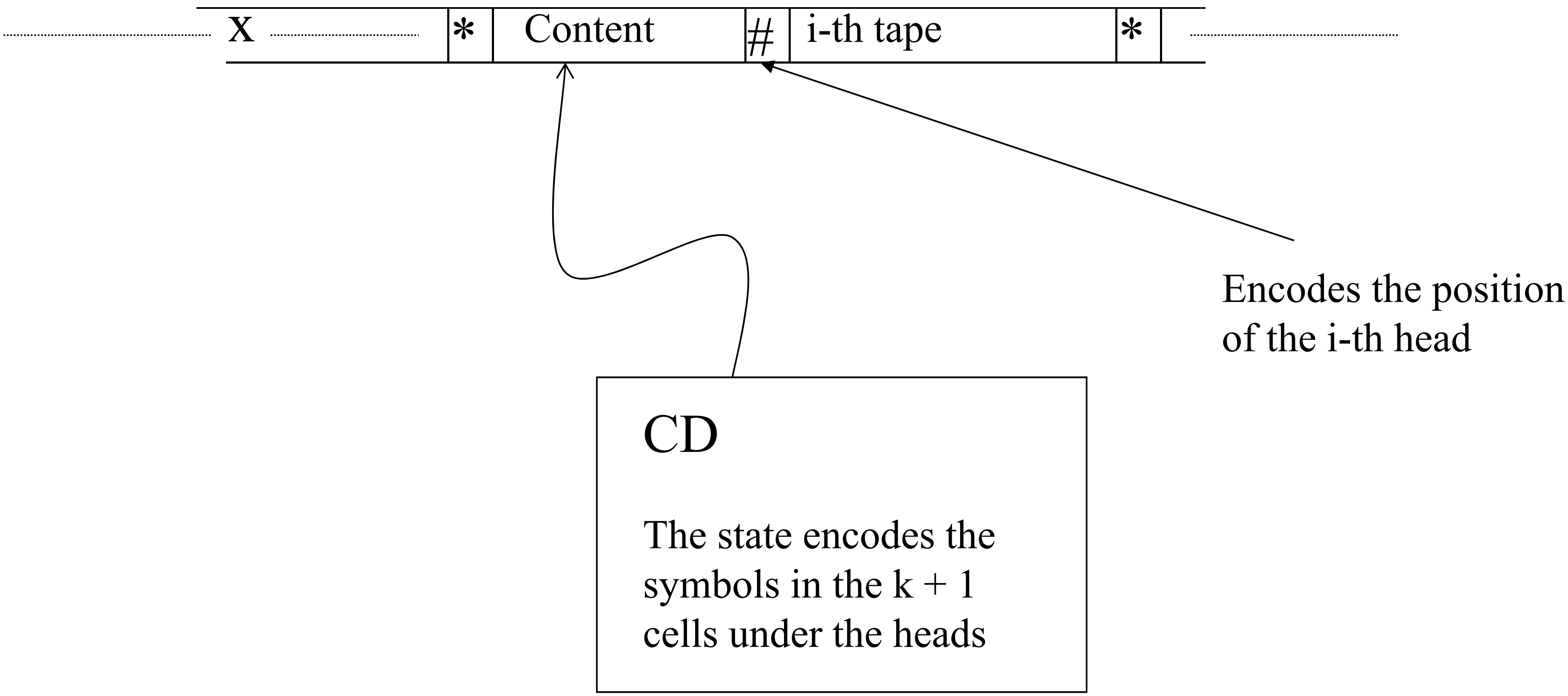


- Bidimensional tape TM



- TM with *k* heads for each tape
- …..

All versions of the TM are equivalent, w.r.t. their accepting or translating ability, for instance:



What relations exist among the various automata (TM in particular) and more traditional/realistic computing models?

- The TM can simulate a Von Neumann machine (which is also "abstract")
- The main difference is in the way the memory is accessed: sequential rather than "random" (direct)
- This does not influence the machine for what concerns computational power (i.e., the class of problems it can solve)
- There can be a (profound) impact for what concerns the *complexity* of the computations, i.e., the time and memory needed
- We will consider the implications in both cases

Nondeterministic (operational) models

- Usually one thinks of an algorithm as a *uniquely determined* sequence of operations: in a certain configuration there is no doubt on what the next "step" will be
- Are we sure that this is desirable?

Let us compare

if x > y **then** max := x **else** max := y

with

if x >= y **then** max := x
y >= x **then** max := y
fi

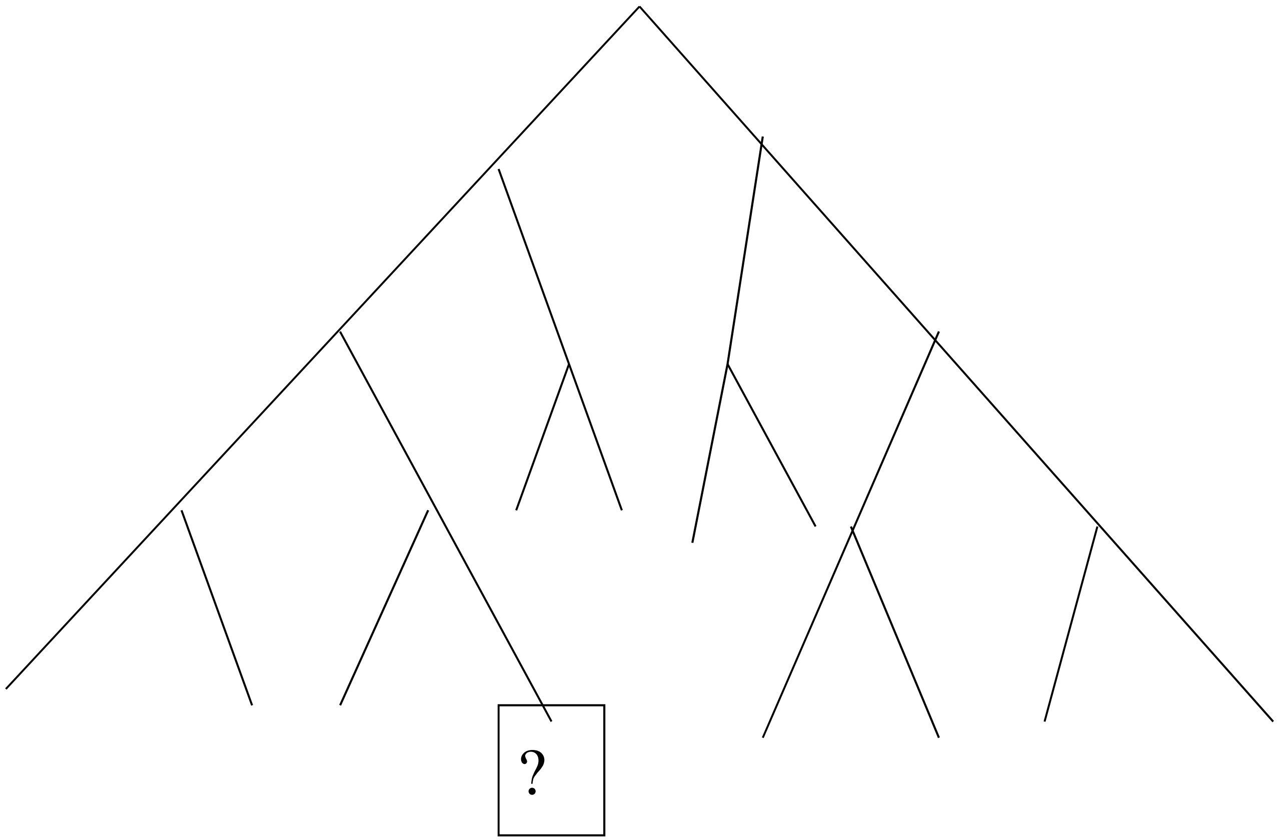
- Is it only a matter of elegance?

- Let us consider the **case** construct of Pascal & others:
  why not having something like the following?

  case
  - x = y **then** S1
  - z > y +3 **then** S2
  - …. **then** …

  endcase

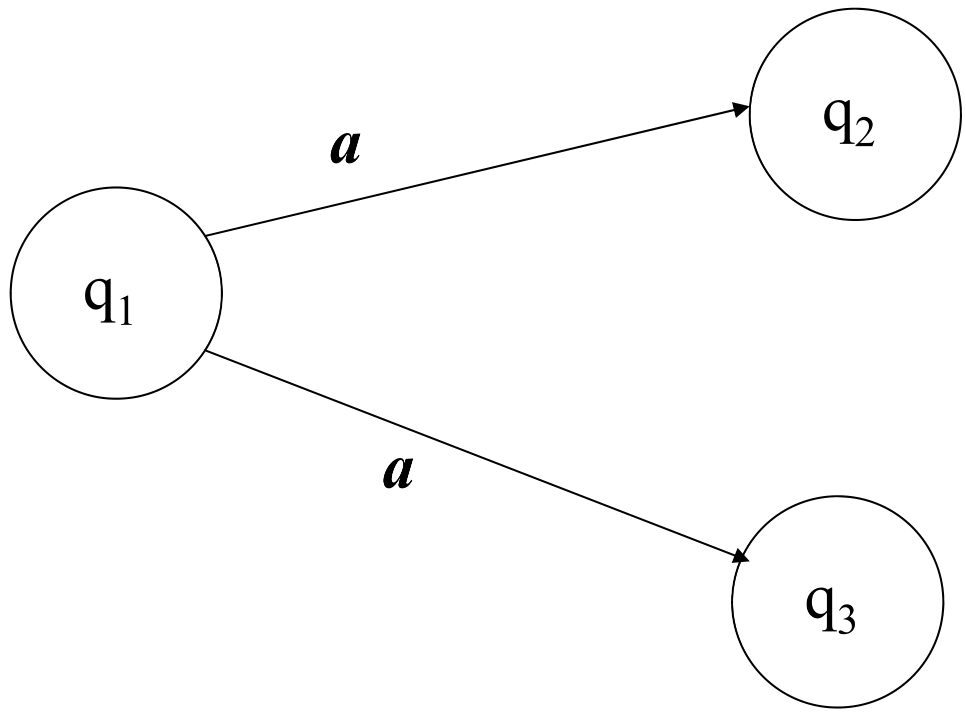
Another form of nondeterminism which is usually “hidden”: blind search



- In fact, the search algorithms are a “simulation” of “basically nondeterministic” algorithms:
- Is the searched element in the root of the tree?
- If yes, OK. Otherwise
  - Search the left subtree
    or
  - Search the right subtree
- Choice of priority among various paths is often arbitrary
- If we were able to assign two tasks in parallel to two distinct machines ---->
- Nondeterminism as a model of computation or at least a model of design of parallel computing
  (For instance Ada and other concurrent languages exploit nondeterminism)

Among the numerous nondeterministic (ND) models: ND version of known models

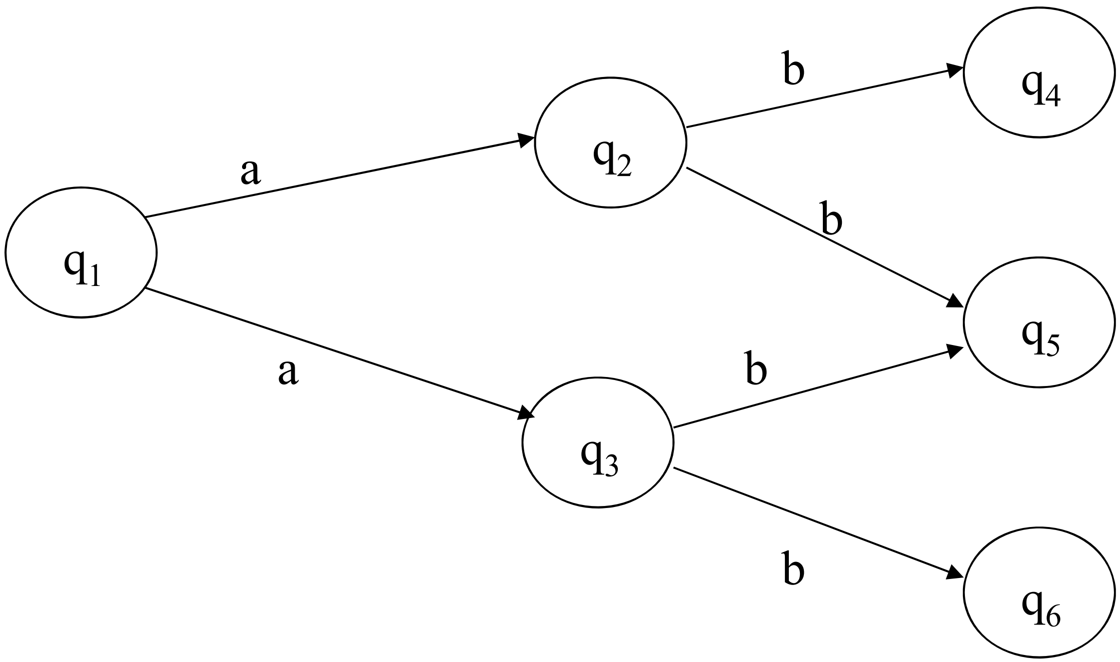
- ND FA (we will soon see how handy it is)



Formally: $\boldsymbol{\delta(q_1,a) = \{q_2, q_3\}}$

$\delta : Q \times I \rightarrow \mathcal{P}(Q)$

δ^* : formalization of a move sequence



$\delta(q_1,a) = \{q_2, q_3\}$, $\delta(q_2,b) = \{q_4, q_5\}$, $\delta(q_3,b) = \{q_6, q_5\}$

$\delta^*(q_1,ab) = \{q_4, q_5, q_6\}$

$$\delta^*(q,\varepsilon) = \{q\}$$

$$\delta^*(q, y.i) = \bigcup_{q' \in \delta^*(q,y)} \delta(q', i)$$

How does a ND FA accept?

$$x \in L \quad \leftrightarrow \quad \delta^*(q_0, x) \cap F \neq \varnothing$$

Among the various possible runs (with the same input) of the ND FA ***it suffices that one of them*** (that is, ***there exists one that***) succeeds and accepts the input string

Another, alternative interpretation of nondeterminism:

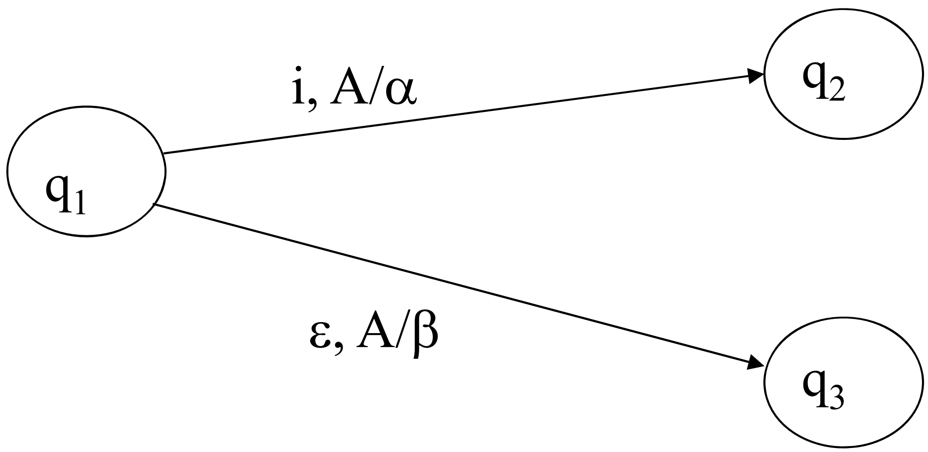
universal nondeterminism (the previous one is ***existential***):

all runs of the automaton accept

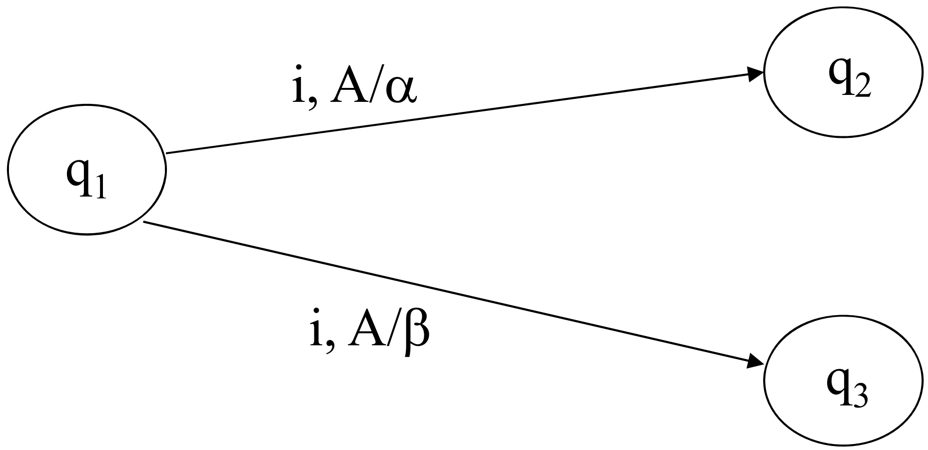
$$(\delta^*(q_0, x) \subseteq F)$$

nondeterministic PDA (NPDA)

- In fact PDA are “natural born” ND :



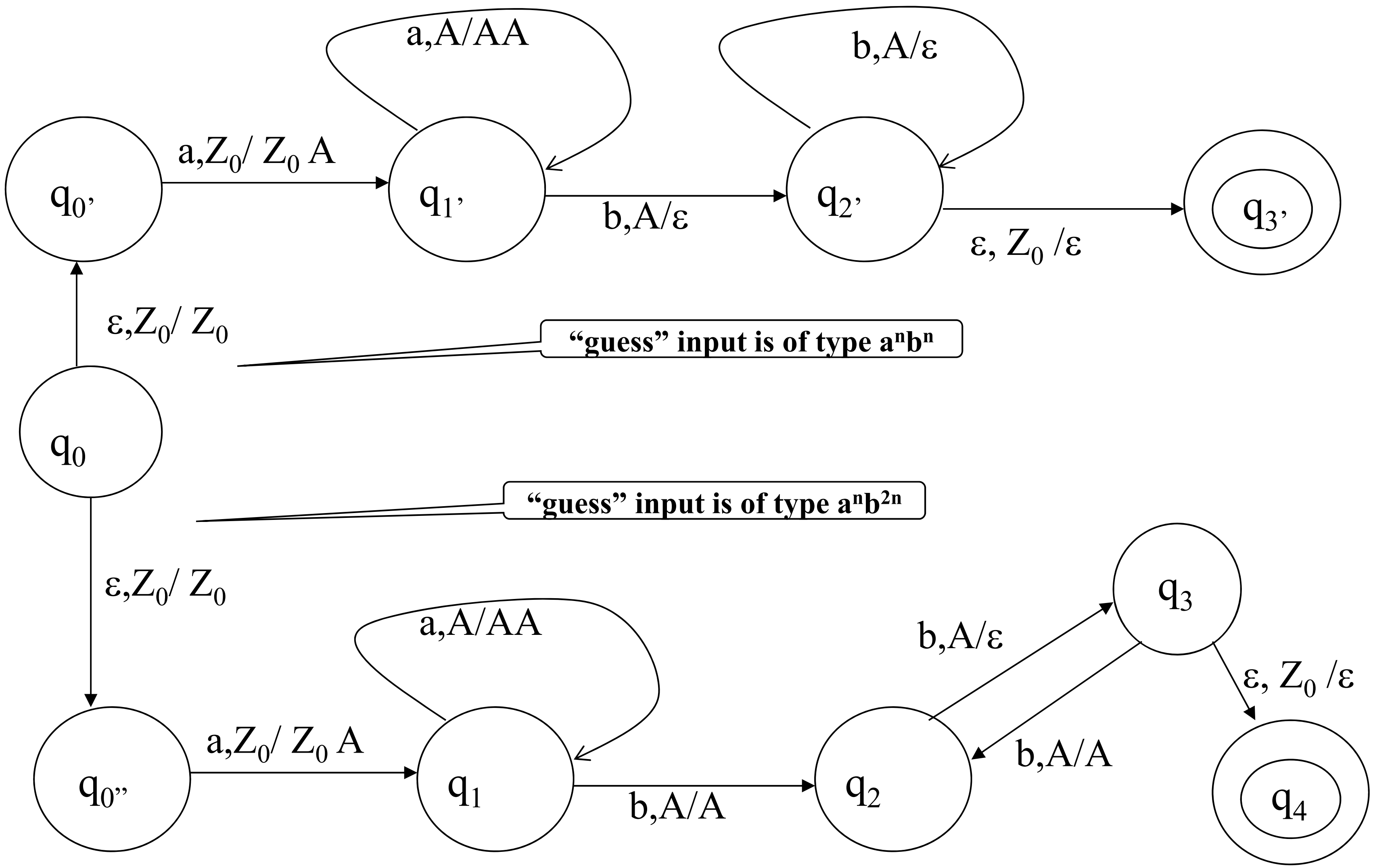
- We might as well remove the deterministic constraint and generalize:



$$\delta : Q \times (I \cup \{\varepsilon\}) \times \Gamma \rightarrow \wp_F(Q \times \Gamma^*)$$

- Why index F? (finite subsets, we do not want infinite ones)
- As usual, the NPDA accepts ***x*** if *there exists a sequence*
- c_0 |-*- <q, ε, γ>, q ∈ F
- in case of nondeterminism, the relation '|--' is not unique (i.e., ***functional***) any more

A "trivial" example: accepting $\{a^n b^n \mid n>0\} \cup \{a^n b^{2n} \mid n>0\}$



Some immediate significant consequences

- NPDA can accept a language that is not accepted by deterministic PDA ----> they are more powerful
- The previous construction can be easily generalized to obtain a *constructive* proof of closure under union of the NPDA
  -a property that deterministic PDA do not enjoy
- The closure under intersection still does not hold ($\{a^nb^nc^n\} = \{a^nb^nc^*\} \cap \{a^*b^nc^n\}$ cannot be accepted by a PDA, not even ND)
  -the two cited examples, $\{a^nb^nc^n\}$ and $\{a^nb^n\} \cup \{a^nb^{2n}\}$, are in fact not so similar…

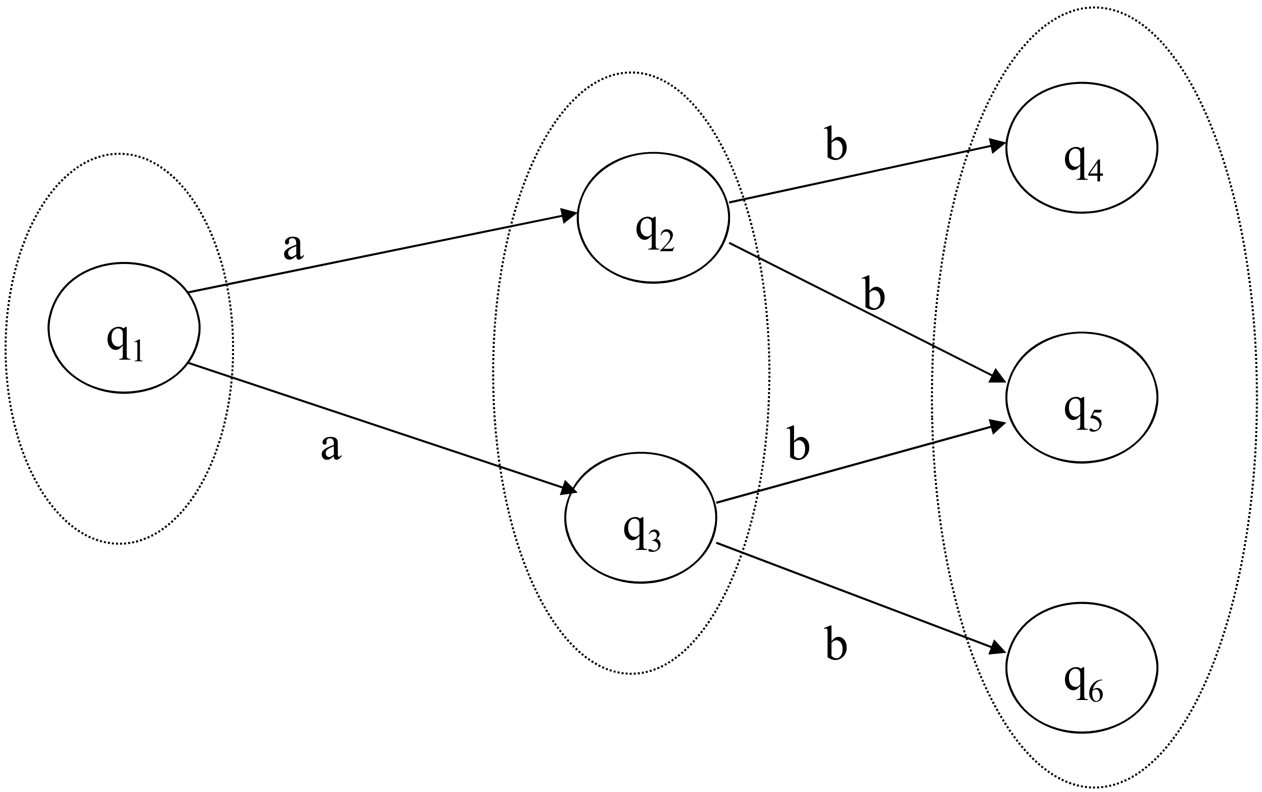
- If a language family is closed under union and not w.r.t. intersection it cannot be closed under complement (why?)
- Hence the family of lang. accepted by NPDA is ***not*** closed under complement
- This highlights a deep change caused by nondeterminism concerning the complement of a problem -in general-:
  if the way of operating of a machine is deterministic and its computation finishes it suffices to change the positive answer into a negative one to obtain the solution of the “complement problem” (for instance, *presence* rather than *absence* of errors in a program)

- In the case of NPDA, though it is possible, like for PDA, to make a computation always finish, there can be two computations
  - $c_o \mid\!-^*\!- \langle q_1, \varepsilon, \gamma_1 \rangle$
  - $c_o \mid\!-^*\!- \langle q_2, \varepsilon, \gamma_2 \rangle$
  - $q_1 \in F, q_2 \notin F$
- In this case ***x*** is accepted
- However, if F turned into Q-F, ***x*** is still accepted:
  with nondeterminism changing a yes into a no does not work!

- …and for other kinds of automata?

  does nondeterminism increase the power of the model?

Nondeterministic Finite-state Automata (NFA)



Starting from q_1 and reading ***ab*** the automaton reaches a state that belongs to the set $\{q_4,q_5,q_6\}$

Let us call again “state” the set of possible states in which the NFA can be during a run.

Formally ...

- Given a NFA an equivalent deterministic one can be *automatically* computed $\Rightarrow$
- NFA are ***not*** more powerful than their deterministic relatives (this is different than with PDA)
  (so what is their use?)
- Let A_{ND} = <Q_N,I, δ_N, q_{0N}, F_N> the NFA from which we build a FA
- Let A_D = <Q_D,I, δ_D, q_{0D}, F_D> the FA we intend to build
  - $\mathbf{Q_D = \wp(Q_N)}$
  - $$\delta_D(q_D, i) = \bigcup_{q_N \in q_D} \delta_N(q_N, i)$$
  - $q_{0D} = \{q_{0N}\}$
  - $$F_D = \{\overline{Q} \subseteq Q_N \mid \overline{Q} \cap F_N \neq \emptyset\}$$

- Though it is true that for all NFA one can find (and *build*) an equivalent deterministic one
- This does not mean that using NFA is useless:
  - It can be easier to "design" a NFA and then obtain from it automatically an equivalent deterministic one, just to skip the (painful) job of build it ourselves deterministic from the beginning (we will soon see an application of this idea)
  - For instance, from a NFA with 5 states one can obtain, in the worst case, one with 2^5 states!
- Consider NFA and FA for languages $L_1=(a,b)^*a(a,b)$ (i.e., strings over $\{a,b\}$ with 'a' as the symbol before the last one) and $L_2=(a,b)^*a(a,b)^4$ (i.e., 'a' as the fourth symbol before the last...)

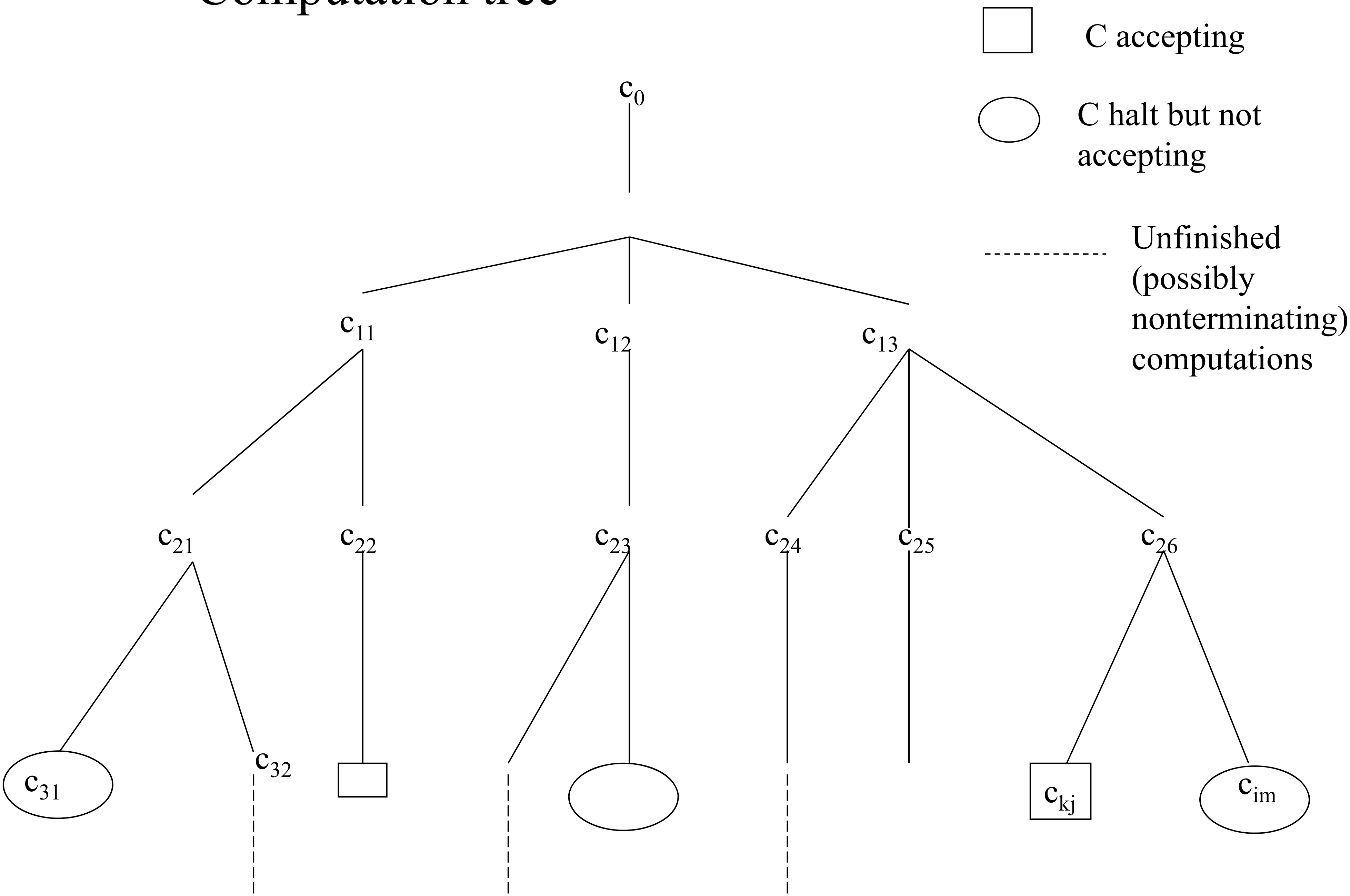
- We still have to consider the TM ...

Nondeterministic TM

$$<\delta, [\eta]>: Q \times I \times \Gamma^k \to \wp(Q \times \Gamma^k \times \{R, L, S\}^{k+1} [\times O \times \{R, S\}])$$

- Configurations, transitions, transition sequences and acceptance are defined as usual
- Does nondeterminism increment the power of TM's?

Computation tree



- ***x*** is accepted by a ND TM iff ***there exists*** a computation that terminates in an accepting state
- Can a deterministic TM establish whether a “sister” ND TM accepts ***x***, that is, accept ***x*** if and only if the ND TM accepts?
- This amounts to “visit” the computation tree of the NDTM to establish whether it contains a path that finishes in an accepting state
- This is a (***almost***) trivial, well known problem of tree visit, for which there are classical algorithms
- The problem is therefore reduced to implementing an algorithm for visiting trees through TM’s: a boring, but certainly feasible exercise … but beware the above “almost” ...

- Everything is easy if the computation tree is finite
- But it could be that some paths of the tree are infinite (they describe nonterminating –hence non-accepting– computations)
- In this case a depth-first visit algorithm (for instance leftmost preorder) might "get stuck in an infinite path" without ever discovering that another branch is finite and leads to acceptance.
- The problem can however be easily overcome by adopting a breadth-first visit algorithm (it uses a queue data structure rather than a stack to manage the nodes still to be visited).
- Hence nondeterminism does ***not*** increase the power of the TM

Conclusions

- Nondeterminism: a useful abstraction to describe search problems and algorithms; or situations where there are no elements of choice or they are equivalent; or parallel computations
- In general it does not increase the computing power, at least in the case of TM's (which are the most powerful automaton seen so far) but it can provide more compact descriptions
- It increases the power of pushdown automata
- It can be applied to various computational models (to every one, in practice); in some cases "intrinsically nondeterministic" models were invented to describe nondeterministic phenomena
- For simplicity we focused only on (D and ND) acceptors but the notion applies also to translator automata
- NB: the notion of ND must not be confused with that of *stochastic* (there exist stochastic models -e.g. Markov chains- that are completely different from the nondeterministic ones)

Grammars

- Automata are a model suitable to recognize/accept, translate, compute (languages): they "receive" an input string and process it in various ways
- Let us now consider a ***generative model***:
  a grammar produces, or *generates*, strings (of a language)
- General notion of a *grammar* or *syntax* (alphabet and vocabulary, and grammar and syntax are synonymous):
  set of ***rules*** to build phrases of a language (strings): it applies to any notion of a language in the widest possible sense.
- In a way similar to normal linguistic mechanisms, a formal grammar generates strings of a language through a process of ***rewriting***:

- “A phrase is made of a subject followed by a predicate”
  “A subject can be a noun or a pronoun, or …”
  “A predicate can be a verb followed by a complement…”
- A program consists of a declarative part and an executable part
  The declarative part …
  The executable part consists of a statement sequence
  A statement can be simple or compound
  ….

- An email message consists of a header and a body
  The header contains an address, ….
- …
- In general this kind of linguistic rules describes a "main object" (a book, a program, a message, a protocol, ….) as a sequence of "composing objects" (subject, header, declarative part, …).
  Each of these is then "refined" by replacing it with more detailed objects and so on, until a sequence of base elements is obtained (bits, characters, …)
  The various rewriting operations can be alternative: a subject can be a noun or a pronoun or something else; a statement can be an assignment, or I/O, ...

Formal definition of a grammar

- $G = \langle V_N, V_T, P, S \rangle$
  - V_N : ***nonterminal*** alphabet or vocabulary
  - V_T : ***terminal*** alphabet or vocabulary
  - $V = V_N \cup V_T$
  - $S \in V_N$: a particular element of V_N called ***axiom*** or ***initial*** **(Start)** ***symbol***
  - $P \subseteq V_N^+ \times V^*$: set of ***rewriting rules***, or ***productions***

    $P = \{\langle\alpha, \beta\rangle \mid \alpha \in V_N^+ \wedge \beta \in V^*\}$

    for ease of notation, write $\langle\alpha, \beta\rangle$ as $\alpha \rightarrow \beta$

    to emphasize the action of ***rewriting***

Example

- $V_N = \{S, A, B, C, D\}$
- $V_T = \{a,b,c\}$
- S
- $P = \{S \rightarrow AB, BA \rightarrow cCD, CBS \rightarrow ab, A \rightarrow \varepsilon\}$

Relation of Immediate Derivation "$\Rightarrow$"

$\alpha \Rightarrow \beta, \alpha \in V^{+}, \;\; \beta \in V^{*}$

if and only if

$\alpha = \alpha_1\alpha_2\alpha_3 \quad , \quad \beta = \alpha_1\beta_2\alpha_3 \quad \wedge \quad \alpha_2 \rightarrow \beta_2 \in P$

α_2 is rewritten as β_2 in the context of α_1 and α_3

With reference to the previous grammar: applying rule **BA → cCD**

aa<u>BA</u>S $\Rightarrow$ aa<u>cCD</u>S

As usual, define the reflexive and transitive closure of $\Rightarrow$

$$\overset{*}{\Rightarrow}$$

it means: "zero or more rewriting steps"

Language generated by a grammar

$$L(G) = \{x \mid x \in V_T^* \wedge S \overset{*}{\Rightarrow} x\}$$

It consists of all strings, containing ***only terminal*** symbols, that can be derived (in any number of steps) from ***S***

NB: not necessarily all derivations lead to a string of terminal symbols

some may "get stuck" (string is not terminal but no rule can be applied)

some may be "never ending"

A first example

$$G_1 = < \{S, A, B\}, \{a, b, 0\}, P, S >$$

$$P = \{S \to aA, A \to aS, S \to bB, B \to bS, S \to 0\}$$

Some derivations

$$S \Rightarrow 0$$

$$S \Rightarrow aA \Rightarrow aaS \Rightarrow aa0$$

$$S \Rightarrow bB \Rightarrow bbS \Rightarrow bb0$$

$$S \Rightarrow aA \Rightarrow aaS \Rightarrow aabB \Rightarrow aabbS \Rightarrow aabb0$$

Through an easy generalization :

$$L(G_1) = \{aa, bb\}^{*}.0$$

Second example

$G_2 =< \{S\},\{a,b\},P,S >$

$P = \{S \to aSb \mid ab\}$ (abbreviation for $S \to aSb$, $S \to ab$)

Some derivations

$S \Rightarrow ab$

$S \Rightarrow aSb \Rightarrow aabb$

$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaabbb$

Through an easy generalization :

$L(G_2) = \{a^n b^n \mid n \geq 1\}$

By substituting production $S \to ab$ with $S \to \varepsilon$ we obtain

$L(G_2) = \{a^n b^n \mid n \geq 0\}$

Third example: G_3

$$\{S \to aACD, A \to aAC, A \to \varepsilon, B \to b, \underline{CD \to BDc}, \underline{CB \to BC}, D \to \varepsilon\}$$

$$S \Rightarrow aACD \Rightarrow a\mathbf{CD} \Rightarrow a\mathbf{BD}c \stackrel{*}{\Rightarrow} abc$$

$$S \Rightarrow aACD \Rightarrow aaACCD \Rightarrow aaCC\mathbf{D} \Rightarrow aaCC \downarrow$$

$$S \Rightarrow aACD \Rightarrow aaACCD \Rightarrow aaC\mathbf{CD} \Rightarrow aa\mathbf{CB}Dc \Rightarrow$$

$$aa\mathbf{B}CDc \Rightarrow aab\mathbf{CD}c \Rightarrow aab\mathbf{B}Dcc \Rightarrow aabb\mathbf{D}cc \Rightarrow aabbcc$$

$$S \stackrel{*}{\Rightarrow} aaaACCCD \Rightarrow aaaCC\mathbf{CD} \Rightarrow aaaCC\mathbf{B}Dc \Rightarrow aaaCCb\mathbf{D}c \Rightarrow aaaCCbc \downarrow$$

...

1. ***S→aACD*, *A→aAC* and *A→ε* generate as many *a*'s as *C*'s, and a final *D***
2. **any $x \in L$ includes only terminal symbols, hence nonterminal symbols must disappear**
3. ***C* disappears only when it "hits" the *D* and then it generates a '*B*' and a '*c*'**
4. ***C*'s and *B*'s must switch to permit all the *C*'s to reach the *D***
5. ***Hence* $C^n D \Rightarrow^* b^n c^n$**
6. **Hence $L=\{a^n b^n c^n \mid n > 0\}$**

Some “natural” questions

- What is the practical use of grammars (beyond funny “tricks” like $\{a^n b^n\}$?)
- What languages can be obtained through grammars?
- What relations exist among grammars and automata (better: among languages *generated* by grammars and languages *accepted* by automata?

Some answers

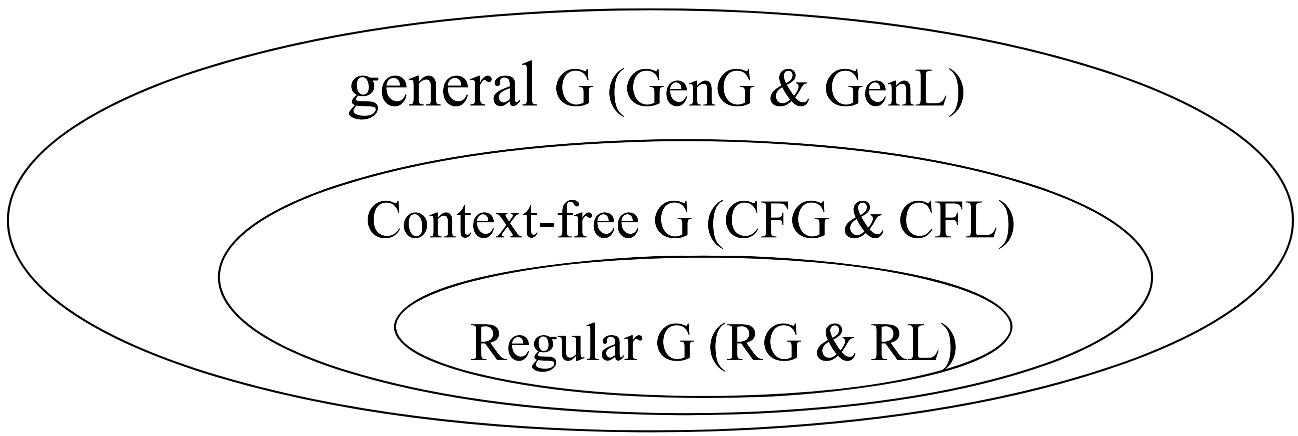
- Definition of the syntax of the programming languages
- Applications are “dual” w.r.t. automata: grammars ***generate*** languages, whereas automata ***recognize*** them
- Simplest example: language *compilation*: the grammar ***defines*** the language, the automaton ***accepts*** and ***translates*** it

Classes of grammars

- Context-free grammars:
  - $\forall$ rule $(\alpha \rightarrow \beta) \in P$, $|\alpha| = 1$, i.e., α is an element $\boldsymbol{A}$ of $\boldsymbol{V_N}$.
  - Context free because the rewriting of α (i.e., of $\boldsymbol{A} \in \boldsymbol{V_N}$) does not depend on its context (the string parts surrounding it do not appear in the left-hand side of the rule)
  - These are in fact the same as the BNF used for defining the syntax of programming languages (so they are well fit to define typical features of programming and natural languages, … but not all)
  - G_1 and G_2 above are context-free not so for G_3

- Regular Grammars:
  - $\forall$ rule $(\alpha \rightarrow \beta) \in P$, $|\alpha| = 1$, $\beta \in ((V_T . V_N) \cup V_T \cup \{\varepsilon\})$
  - Regular grammars are also context free, but not vice versa
  - G_1 above is regular, not so G_2.

Inclusion relations among grammars and corresponding languages



NB: we call Regular Languages those generated by a regular grammar

this appears to be an abuse (a name clash with languages accepted by FA's), but it is not ...

It immediately follows that :

$$\mathcal{RL} \subseteq \mathcal{CFL} \subseteq \mathcal{GenL}$$

But, are these inclusions strict?

The answer comes from the comparison with automata

Relations between grammars and automata (with few surprises)

- Define "equivalence" between RG and FA

  (i.e., the FA accepts same language that the RG generates)

  - **From FA to RG**: given a FA $A=(Q, I, q_0, F)$, let $V_N = Q$, $V_T = I$, $S = \langle q_0 \rangle$, and, for each $\delta(q, i) = q'$ let $\langle q \rangle \rightarrow i\langle q' \rangle$ and, if $q' \in F$, add $\langle q \rangle \rightarrow i$
  - It is an easy intuition (proved by induction) that $\delta^*(q, x) = q'$ iff $\langle q \rangle \Rightarrow^* x\langle q' \rangle$, and hence, if $q' \in F$, $\langle q \rangle \Rightarrow^* x$

- Vice versa, **from RG to FA**:
  - Given a RG, let $Q = V_N \cup \{q_F\}$, $I = V_T$, $\langle q_0 \rangle = S$, $q_F \in F$ and,

    for each $A \rightarrow bC$ let $\delta(A,b) = C$

    for each $A \rightarrow b$ let $\delta(A,b) = q_F$
  - for each $A \rightarrow \varepsilon$ let $A \in F$

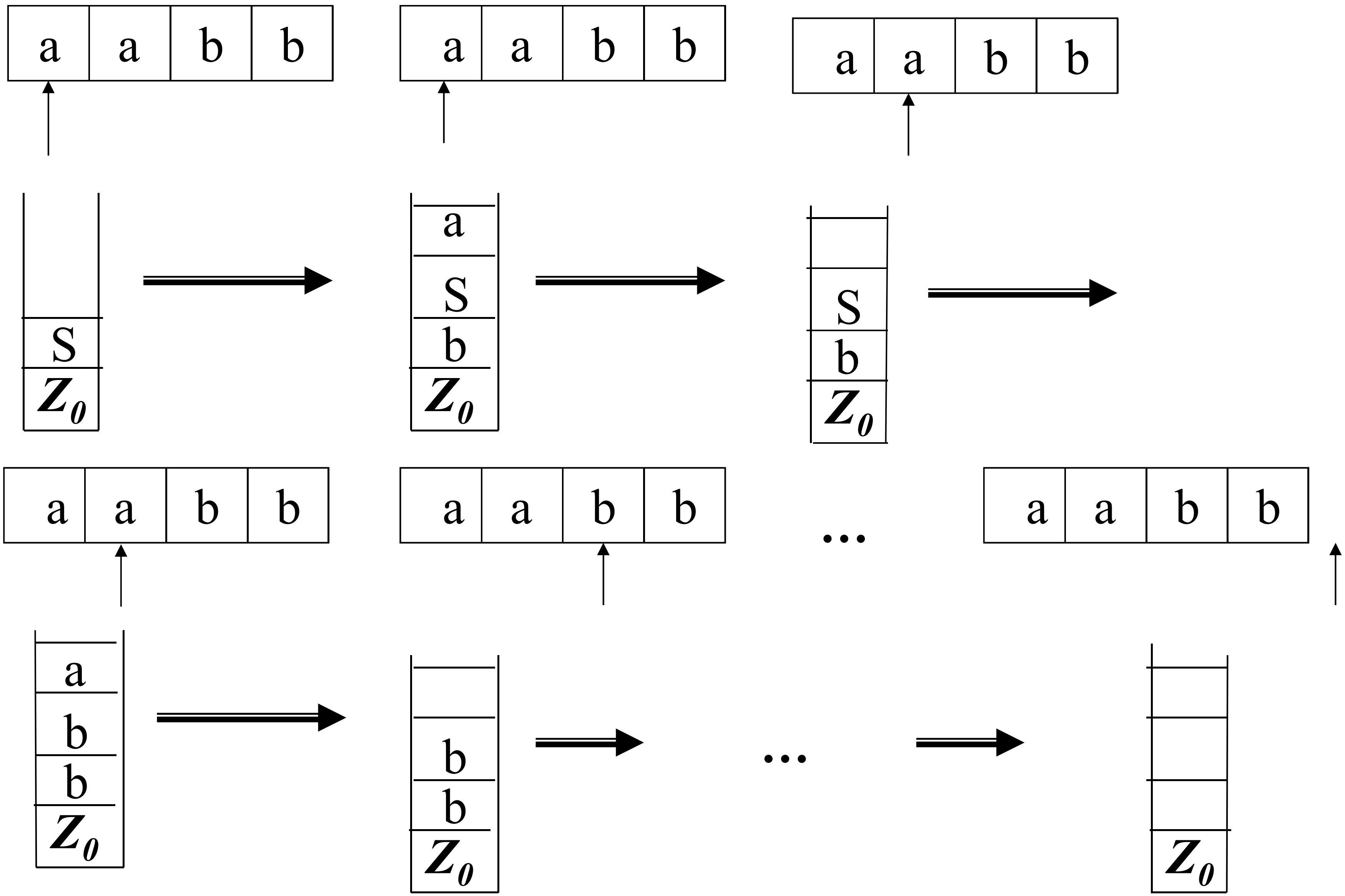
NB: The FA thus obtained is nondeterministic (why?): much easier!

- CFG equivalent to PDA (ND!)

intuitive justification (no proof:
the proof is the “hart” of compiler construction)

S → aSb | ab

$$S \Rightarrow aSb \Rightarrow aabb$$



genG equivalent to TM

- Given G let us construct (in broad lines) a ***ND*** TM, M, accepting L(G):
  - The input string *x* is on the input tape
  - M has one memory tape: it tries in all possible ways to derive *x* on it, and accepts *x* iff it finds a derivation for it
  - The memory tape is initialized with S (better: Z_0S)
  - The memory tape (which, in general, will contain a string α, $\alpha \in V^*$) is scanned searching the left part of some production of *P*
  - When one is found *-not necessarily the first one, M operates a ND choice*- it is substituted by the corresponding right part (if there are many right parts again *M* operates nondeterministically)

– This way:

$$\alpha \Rightarrow \beta \quad \leftrightarrow \quad c = < x, q_s, Z_0\alpha > |-*- < x, q_s, Z_0\beta >$$

If and when the tape holds a string $y \in V_T^*$, it is compared with x. If they coincide, x is accepted, otherwise this particular computation of the nondeterministic TM does not lead to acceptance.

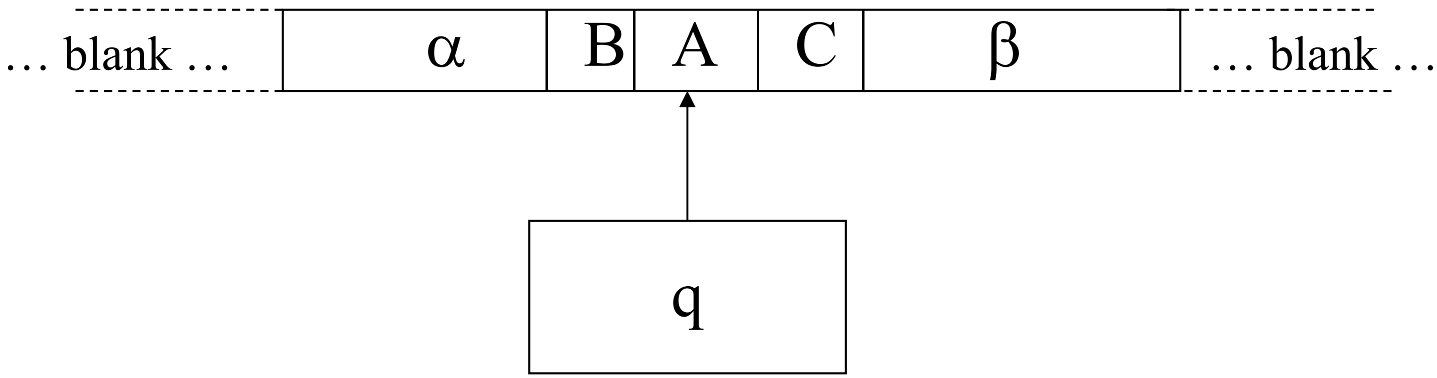
Notice that:

•Using a ND TM facilitates the construction but is immaterial (not necessary)

•It is instead necessary -and, we will see, unavoidable- that, if $x \notin L(G)$, M might "try an infinite number of ways", some of which might never terminate, without being able (rightly) to conclude that $x \in L(G)$, but *not even the opposite*.
This is consistent with the definition of acceptance, which requires M to reach an accepting configuration if and only if $x \in L$, but does not requires M to terminate its computation without accepting (i.e., in a "rejecting state") if $x \notin L$

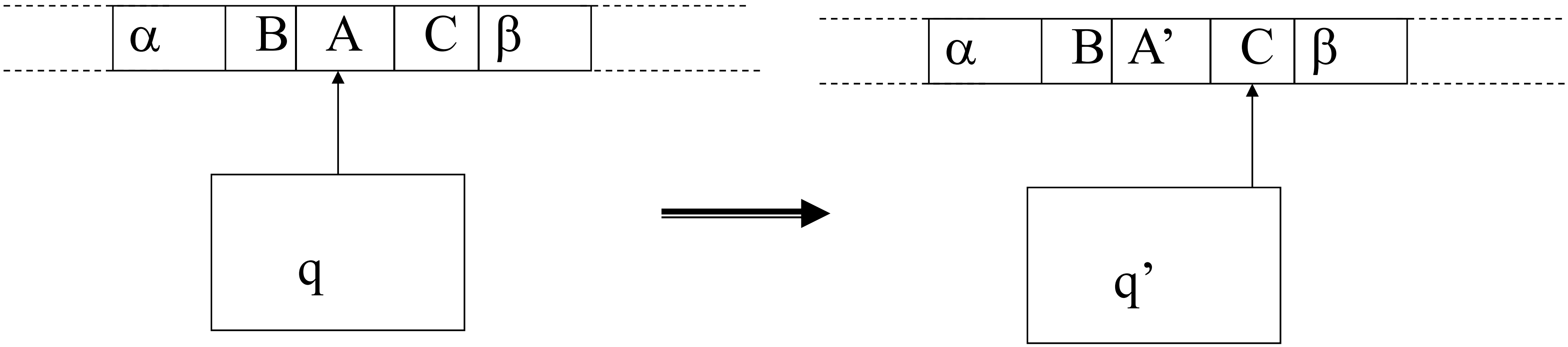
- Given M (***single tape***, for ease of reasoning and without loss of generality) we define (in broad lines) a G generating L(M):
  - First, G generates *all* strings of the type x\$X, $x \in V_T^*$, X being a "copy of x" composed of nonterminal symbols (e.g., for x = aba, x\$X = aba\$ABA)
  - G simulates the successive configurations of M using the string on the right of \$
  - G is defined in a way such that it has a derivation $x\$X \Rightarrow^* x$, with $x \in V_T^*$, if and only if x is accepted by M.
  - The idea: simulate each move of M by an immediate derivation of G

– We represent the configuration



- through the string (special cases are left as an exercise): $\$\alpha BqAC\beta$
- to start, G has therefore a set of derivations of the kind $x\$X \Rightarrow x\$q_oX$ (where q_oX encodes the initial configuration of M)
- for each value of the transition function δ of M, a rule of G is defined :
  - $\delta(q,A) = \langle q', A', R\rangle$ G includes the production $qA \rightarrow A'q'$
  - $\delta(q,A) = \langle q', A', S\rangle$ G includes the production $qA \rightarrow q'A'$
  - $\delta(q,A) = \langle q', A', L\rangle$ G includes the productions $BqA \rightarrow q'BA'$ $\forall$ B in the alphabet of M (recall that M is single tape, hence it has a unique alphabet for input, memory, and output)

– This way, for instance:



– If and only if: $x\$\alpha BqAC\beta \Rightarrow x\$\alpha BA'q'C\beta$,

– etc. …

– We finally add productions allowing G to derive from $x\$\alpha Bq_FAC\beta$ a unique x if –and only if– M reaches an accepting configuration ($\alpha Bq_FAC\beta$), by deleting whatever is to the right of \$, and also \$

The use of mathematical logic as a descriptive formalism / 1

- Logic is a “universal” formalism (very close to natural language)
- It can be applied to a wide variety of contexts, not only to computer science
- Mathematical logic has several connections with areas of (theoretical) computer science
  - many notations and languages based on Logic (e.g., in programming languages, specification languages)
  - relations with automata, formal languages, grammars
  - there are many important results concerning (various types of) logics:
    - expressive power, decidability of satisfiability/validity problem, complexity of decision procedures
- We assume a (moderate) knowledge from previous courses, otherwise
  - Read very carefully Ch.0 of textbook, §0.2 Basic Elements of Mathematical Logic
  - Read very carefully the slide file: MATHEMATICAL LOGIC–INTRODUCTION in the Lessons document folder on WeBeeP
  - for a more in-depth approach:"Introduction to Mathematical Logic" by Elliott Mendelson (for mathematically oriented students very fond of logics)

The use of mathematical logic as a descriptive formalism / 2

- We provide a partly informal treatment of Logic, used mainly as a descriptive notation to define
  - Languages (with a part on the relation between regular languages and a specific class of mathematical logic formulae)
  - Program properties
  - System timing properties
- Many important features (expressive power, decidability, complexity) of various types of logics depend on
  - assumed Interpretation domain (Universe ***U***): the set of possible values of terms (constants, variables, expressions)
  - assumed alphabet of predicate and function symbols
  - possibility for (quantified) variables to denote only first-order objects (elements of the universe) or second-order objects (sets of such elements and relations among them)
- by including suitable predicate and function symbols a «maximal» generality is soon achieved (examples from arithmetics and the famous Gödel incompleteness theorem)
- Often we «play» the following «game»: define some entity in logic using a given, suitably constrained alphabet

1. Mathematical logic to define languages

language $L = \{a^n b^n \mid n \geq 1\}$ specified by the first-order formula

$$\forall x \left(x \in L \leftrightarrow \exists n \, (n \geq 1 \wedge x = a^n \cdot b^n) \right)$$

Notice the alphabet:

variable $\boldsymbol{x}$ denotes a string, $\boldsymbol{n}$ a natural number

constant $\boldsymbol{L}$ denotes a language (set of strings), (constant $\boldsymbol{\varepsilon}$ the empty string)

set-theoretic predicate '$\in$', equality predicate '$=$'

operations '$\cdot$' (string concatenation) and x^n (power on strings)

power operation on strings, x^n, is defined by the formula

$$\forall n \left(\left(n = 0 \rightarrow x^n = \varepsilon \right) \wedge \left(n > 0 \rightarrow x^n = x^{n-1} \cdot x \right) \right)$$

based on the elementary operation for concatenation "$\cdot$" and the integer '-' operation

Notice: for brevity, outermost universal quantifiers are often left implicit, for instance
$\boldsymbol{x \in L \leftrightarrow \exists n (n \geq 1 \wedge x = a^n \cdot b^n)}$ to be intepreted as $\forall \boldsymbol{x}(\boldsymbol{x \in L \leftrightarrow \exists n (n \geq 1 \wedge x = a^n \cdot b^n)})$

- Define language $\boldsymbol{L_1 = a^* b^*}$ without using power operation $\boldsymbol{x^n}$
  (otherwise, simply, $\boldsymbol{x \in L_1 \leftrightarrow \exists m \exists n (x = a^m b^n \wedge n \geq 0 \wedge m \geq 0)}$)

$$x \in L_1 \leftrightarrow \Big((x = \varepsilon) \vee \exists y \big(y \in L_1 \wedge \big(x = ay \vee x = yb \big) \big) \Big)$$

- Define $\boldsymbol{L_2 = b^* c^*}$ similarly
- $\boldsymbol{L_3 = a^* b^* c^*}$ $\boldsymbol{(= L_1 \cdot L_2)}$ can also be defined (without using $\boldsymbol{x^n}$) using the same sort of inductive style:

$$x \in L_3 \leftrightarrow (x \in L_1) \vee (x \in L_2) \vee \exists y \, (y \in L_3 \wedge (x = ay \vee x = yc))$$

- Definition of language $\boldsymbol{L_4 = \{x \mid \#_a x = \#_b x\}}$ based on definition of function $\boldsymbol{\#_a x}$ ($\boldsymbol{\#_b x}$ defined similarly) by:

$$(x = \varepsilon \rightarrow \#_a \, x = 0) \wedge (x = ay \rightarrow \#_a \, x = \#_a \, y + 1) \wedge (x = by \rightarrow \#_a \, x = \#_a \, y)$$

- Then Def. of $\boldsymbol{L_4}$ in the customary way: $\boldsymbol{x \in L_4 \leftrightarrow (\#_a x = \#_b x)}$

A very constrained logic to define languages

- Monadic First Order (MFO) logic (monadic ≡ predicates with one argument) with (in addition to monadic predicates) the only ***binary*** order predicate ‘<’
- boolean connectives ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, etc. ...) and quantifiers ($\exists$, $\forall$) defined as usual
- w.r.t. alphabet I consider string $w \in I^*$, with $|w|=n$, i.e., $w=w_0\ w_1\ \dots\ w_{n\text{-}1}$
- Universe $U = \{0, \dots, n\text{-}1\}$ of string ***positions*** (NB: if $x = \varepsilon$ then $U = \varnothing$)
- for every symbol $a \in I$ monadic predicate $a(x)$, true for string w iff $w_x = a$, i.e., iff symbol a occurs in string w at position x
- binary predicate ‘<’ among string positions has the usual meaning

- Derived definitions
  - $\boldsymbol{x \geq y}$ defined as $\neg(\boldsymbol{x < y})$
  - $\boldsymbol{x \leq y}$ as $\boldsymbol{y \geq x}$
  - $\boldsymbol{x = y}$ as $\boldsymbol{x \leq y \wedge y \leq x}$
  - $\boldsymbol{x \neq y}$ as $\neg(\boldsymbol{x = y})$
  - $\boldsymbol{x > y}$ as $\boldsymbol{y < x}$
  - immediate successor: $\boldsymbol{succ(x, y)}$ ($\boldsymbol{S(x, y)}$ for short) as $\boldsymbol{x < y \wedge \neg\exists z\ (x < z \wedge z < y)}$

  - constant **0**: $\boldsymbol{x = 0}$ as $\boldsymbol{\forall y \neg (y < x)}$
  - all natural integer constants **1**, **2**, **3**, ... : defined as successors of **0**, **1**, **2**, etc.
  - $\boldsymbol{y = x + 1}$ defined as $\boldsymbol{succ(x, y)}$
  - $\boldsymbol{y = x{+}k}$, for every $\boldsymbol{k > 1}$, as $\boldsymbol{\exists z_1 \ldots \exists z_{k\text{-}1}\ (z_1 = x{+}1 \wedge \ldots \wedge y = z_{k\text{-}1}{+}1)}$
  - $\boldsymbol{y = x - 1}$ as $\boldsymbol{succ(y, x)}$
  - $\boldsymbol{y = x - k}$, for every $\boldsymbol{k > 1}$, as $\boldsymbol{x = y + k}$
  - first position in the string : $\boldsymbol{first(x)}$ as $\boldsymbol{\neg\exists y\ (y < x)}$ (equiv. to $\boldsymbol{x = 0}$)
  - last position in the string : $\boldsymbol{last(x)}$ as $\boldsymbol{\neg\exists y\ (y > x)}$

  - **NB**: terms like $\boldsymbol{x + y}$, with $\boldsymbol{x}$ and $\boldsymbol{y}$ both variables are ***not admitted***

Interpretation of a MFO Logic formula w.r.t. a string

- every string $w \in I^*$ corresponds to (defines) an interpretation structure with alphabet I and universe $U = \{ 0, \ldots, |w|\text{-}1 \}$
- ex: for string $w = acbaa$, with $n = |w| = 5$, we have
  - alphabet $I = \{a, b, c\}$
  - f.o. variables denote positions in w; they are interpreted over the Universe $U = [0 .. \; n\text{-}1] = [0 .. 4]$
  - for every alphabet element $i \in I$, predicate $i(x)$ denotes the set of positions x at which $w_x = i$, (NB: $\forall i \neq j \; \forall x \;\; \neg(i(x) \wedge j(x))$)
    - for $w = acbaa$, $a(0)$, $c(1)$, $b(2)$, $a(3)$, $a(4)$ are ***true***; $b(0)$, $c(0)$, $a(1)$, $b(1)$, $a(2)$, $c(2)$, $b(3)$, $c(3)$, $b(4)$, $c(4)$ are ***false***
    - NB: in the atomic formula $a(x)$, a is a predicate ***constant***: it is not quantified
  - less-than relation for $w = acbaa$ is a set of pairs of positions:

    '<' = {(0,1), (0,2), ... (1,2), (1,3), … (3,4)}

Notation: (a ***sentence*** is a formula with all variables quantified)

string $\boldsymbol{w}$ satisfies sentence $\boldsymbol{\varphi}$: $\boldsymbol{w} \vDash \boldsymbol{\varphi}$

string $\boldsymbol{w}$ does not satisfy sentence $\boldsymbol{\varphi}$: $\boldsymbol{w} \nvDash \boldsymbol{\varphi}$

Any sentence $\boldsymbol{\varphi}$ defines the language $\boldsymbol{L}(\boldsymbol{\varphi})$ that includes exactly (all and only) the strings satisfying $\boldsymbol{\varphi}$:

$$\boldsymbol{L}(\boldsymbol{\varphi}) = \{ \boldsymbol{w} \mid \boldsymbol{w} \vDash \boldsymbol{\varphi} \}$$

Examples of MFO sentences defining strings and languages

- φ **:** $\exists x(\, x = 0 \wedge a(x)\,)$ $\quad L(\varphi) = aI^*$ strings starting with an ***a***
- strings where every ***a*** is immediately followed by a ***b*** :

  $$\varphi : \forall x\ (\ a(x) \rightarrow \exists y\ (\ S(x, y) \wedge b(y)\ )\ \ )$$

  $$aaba \not\models \varphi, \quad babbab \models \varphi, \quad bba \not\models \varphi$$

- (non-empty) strings ending with an '***a***': $\exists x(\ last(x) \wedge a(x)\ )$
- strings (of length $\geq$3) where second symbol before the last is ***a***

  $$\exists x(\ \exists y\ (\ y = x{+}2 \wedge\ last(y)\ )\ \wedge\ \ a(x)\ \ )$$

- Every ***singletone*** (one-element) language easily defined; example $L_{abc} = \{\ abc\ \}$

$$\exists x \exists y \exists z\ (x{=}0 \wedge S(x,y) \wedge S(y,z) \wedge last(z) \wedge a(x) \wedge b(y) \wedge c(z)\ )$$

Empty string ε often requires some special care

- for string $\boldsymbol{\varepsilon}$, the set of positions is empty
- convention: when $\boldsymbol{U} = \emptyset$, $\exists \boldsymbol{x\varphi}$ is false and $\forall \boldsymbol{x\varphi}$ true, for any φ

  (see lecture notes, where semantics is precisely defined)

  Lecture Notes on Monadic First- and Second-Order Logic on Strings - Study material on WeBeeP

- a sentence for $\{\varepsilon\}$ must be true for ε but false $\forall \boldsymbol{w} \neq \varepsilon$

$$\neg\exists \boldsymbol{x} \, (\, \boldsymbol{a}(\boldsymbol{x}) \vee \neg \boldsymbol{a}(\boldsymbol{x}) \,) \quad (\textbf{i.e., } \neg\exists x \, (\textbf{\textit{true}}), \textbf{ or } \neg\exists x \textbf{ or } U{=}\emptyset)$$

or, equivalently, $\forall \boldsymbol{x} \, (\, \boldsymbol{a}(\boldsymbol{x}) \wedge \neg \boldsymbol{a}(\boldsymbol{x}) \,)$ $(\textbf{i.e., } \forall x \, (\, \textbf{\textit{false}} \,))$

Properties of MFO (1/2)

- Family of languages expressible in MFO is closed under set-theoretic operations (union, intersection, complement etc.)
  - trivial: use disjunction '$\vee$', conjunction '$\wedge$', negation '$\neg$'
- Every ***finite*** or ***co-finite*** (its complement is finite) language is expressible in MFO
  - from above closure properties under propositional operators and MFO ability to express singletone languages
- Every language over a one-letter alphabet (e.g., $\Sigma=\{a\}$) expressible in MFO is either ***finite*** or ***co-finite***
  - Proof is not trivial, interested students find it in the lecture notes

- Therefore in MFO one cannot express the language $\boldsymbol{L_e = (aa)^*}$ that includes ***exactly*** the even-length strings over $\boldsymbol{I=\{a\}}$
  - because $\boldsymbol{L_e = (aa)^*}$ is neither finite nor co-finite

Properties of MFO (2/2)

- MFO is strictly less powerful than FA (Finite state Automata)
  - from a MFO formula easy to obtain an equivalent FA (we do not show how...)
  - $L_e = (aa)^*$ is recognized by a simple FA (but not expressible in MFO)

- Languages defined by MFO are ***not*** closed under Kleene star '*' :
  - MFO formula $\exists x \exists y\ (x=0 \wedge y=1 \wedge a(x) \wedge a(y) \wedge last(y))$ defines language $L_{e2} = \{aa\}$, and we have $L_e = L_{e2}^*$

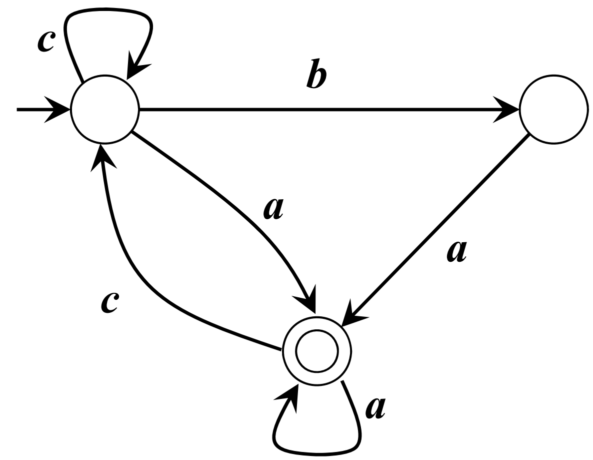
- MFO defines the family of (so-called) ***star-free languages***, that can be obtained starting from any finite language by means of a finite number of union, intersection, complement, and concatenations operations (but not Kleen star '*')

Example of relation between FA and MFO logic formula

$$\varphi = \left(\begin{array}{c} \textbf{\textit{no 'a' is followed by a 'b'}} \\ \wedge \\ \textbf{\textit{every 'b' is followed by an 'a'}} \\ \wedge \\ \textbf{\textit{the string ends with an 'a'}} \end{array} \right)$$

$$\Downarrow$$

$$\varphi = \left(\begin{array}{c} \neg\exists \boldsymbol{x} \exists \boldsymbol{y} (\boldsymbol{S}(x, y) \wedge \boldsymbol{a}(\boldsymbol{x}) \wedge \boldsymbol{b}(\boldsymbol{y})) \\ \wedge \\ \forall \boldsymbol{x} \left(\boldsymbol{b}(\boldsymbol{x}) \rightarrow \exists \boldsymbol{y} \big(\boldsymbol{S}(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{a}(\boldsymbol{y}) \big) \right) \\ \wedge \\ \exists \boldsymbol{x} \, (\boldsymbol{last}(\boldsymbol{x}) \wedge \boldsymbol{a}(\boldsymbol{x})) \end{array} \right)$$



- Are the two models equivalent?
- Is it always possible to
  - find an FA equiv. to a MFO?
  - find an MFO equiv. to a FA?

No, because MFO logic is strictly less powerful than Finite state Automata

Monadic Second Order (MSO) Logic

- Expressive power of MFO can be increased by introducing variables denoting ***monadic predicates*** i.e., ***sets of numbers*** (remember: for every string $\boldsymbol{w}$, the universe $\boldsymbol{U}$ includes the numbers representing positions in $\boldsymbol{w}$)
  - notation: if predicate $\boldsymbol{X}$ represents a set, then $\boldsymbol{X(3)}$ is the same as $\mathbf{3} \in \boldsymbol{X}$
- For clarity we use uppercase identifiers for second order variables denoting predicates: e.g., a possible formula is of the type

$$\exists X \, \varphi(X)$$

  - lowercase identifiers still used for first-order variables, denoting positions in the strings
- Example: the language $\boldsymbol{L_e = (aa)^*}$ is defined by the following MSO formula, where predicate $\boldsymbol{E}$ identifies the even ***positions*** (odd indices)

$$\begin{array}{ll}\exists E \, \forall x \, (& a(x) \wedge \\ & (\, x = 0 \rightarrow \neg E(x) \,) \wedge \\ & \forall y \, (\, y = x+1 \rightarrow (\neg E(x) \leftrightarrow E(y)) \,) \wedge \\ & (\, last(x) \rightarrow E(x) \,) \qquad)\end{array}$$

Another example of MSO language specification

Words over $I = \{a, b\}$ where every two occurrences of b, having no other b in between, are separated by an odd number of a's

$$\varphi = \forall x \forall y \left(\begin{array}{c} b(x) \land x < y \land b(y) \land \forall z\big(x < z \land z < y \rightarrow \neg b(z)\big) \\ \rightarrow \\ \exists X \Big(X(x) \land X(y) \land \forall u \forall v \ \big(S(u,v) \rightarrow \big(X(u) \leftrightarrow \neg X(v)\big)\big)\Big) \end{array} \right)$$

second order quantification **:**
X is a predicate that alternates true and false values

$$a\ b\ a\ a\ a\ b\ a\ b \ \vDash \ \varphi$$

¬X X ¬X X ¬X X ¬X X

OK OK

$$a\ b\ a\ a\ b\ a\ b \ \nvDash \ \varphi$$

¬X X ???? X ¬X X

KO OK

Important result: the family of languages defined by MSO sentences is equal to RL (regular languages)

Büchi's theorem (1960)

- A language (of finite-length words) is recognizable by a finite automaton iff it is definable by a **MSO** sentence
- The two ***conversions*** (automaton $\Leftrightarrow$ formula) are both ***effective***

We provide a (sketchy) proof of the «automaton $\Rightarrow$ formula» part

We only exemplify (provide an intuition for) the «formula $\Rightarrow$ automaton» part

Büchi's theorem - «automaton ⇒ formula» part

- automaton $\boldsymbol{A} = (\boldsymbol{Q}, \boldsymbol{I}, \boldsymbol{q_0}, \delta, \boldsymbol{F})$, $\boldsymbol{Q} = \{\boldsymbol{q_0}, \ldots \boldsymbol{q_k}\}$, reads a string $\boldsymbol{w}$
- use $|\boldsymbol{Q}|$ distinct predicates $\boldsymbol{X_i}$ = {positions of $\boldsymbol{w}$ where $\boldsymbol{A}$ reaches state $\boldsymbol{q_i}$}
  = { $\boldsymbol{x}$ | $\boldsymbol{A}$ is in state $\boldsymbol{q_i}$ when it reads $\boldsymbol{w_x}$ }
- formula $\boldsymbol{\varphi}$ equivalent to $\boldsymbol{A}$:

state positions pairwise disjointed

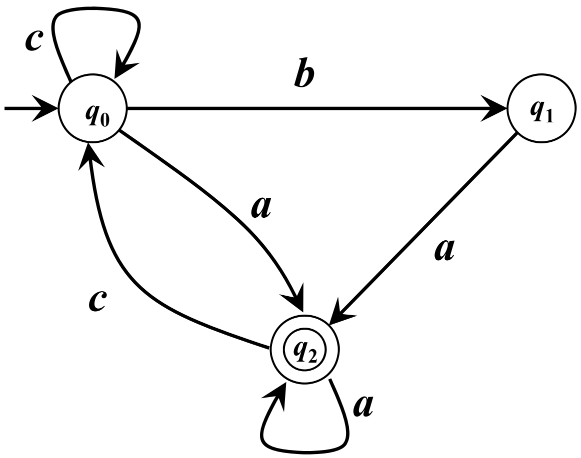
A in the **initial state** right before reading the first symbol

$$\varphi = \exists \boldsymbol{X_0} \ldots \exists \boldsymbol{X_k} \left(\begin{array}{c} \wedge_{i \neq j} \forall x\, \neg\left(X_i(x) \wedge X_j(x)\right) \\ \wedge\, \forall x\, (first(x) \rightarrow X_0(x)) \\ \wedge\, \forall x \forall y \left(S(x,y) \rightarrow \bigvee_{q_j \in \delta(q_i, a)} \left(X_i(x) \wedge a(x) \wedge X_j(y)\right) \right) \\ \wedge\, \forall x \left(last(x) \rightarrow \bigvee_{\exists q_j \in F:\, q_j \in \delta(q_i, a)} \left(X_i(x) \wedge a(x)\right) \right) \end{array} \right)$$

final state: reached after reading the last symbol of the string

transitions: reading $\boldsymbol{a}$ at position $\boldsymbol{x}$, *A* goes from state $\boldsymbol{q_i}$ to state $\boldsymbol{q_j}$

Example of Büchi's theorem - «automaton ⇒ formula»



$$\exists X_0 \exists X_1 \exists X_2 \left(\begin{array}{c} \forall x \big(\neg(X_0(x) \wedge X_1(x)) \wedge \neg(X_0(x) \wedge X_2(x)) \wedge \neg(X_1(x) \wedge X_2(x)) \big) \\ \wedge\, \forall x\, (first(x) \rightarrow X_0(x)) \\ \wedge\, \forall x \forall y \left(S(x,y) \rightarrow \left(\begin{array}{c} X_0(x) \wedge c(x) \wedge X_0(y) \ \vee\ X_0(x) \wedge b(x) \wedge X_1(y) \\ \vee\, X_0(x) \wedge a(x) \wedge X_2(y) \ \vee\ X_1(x) \wedge a(x) \wedge X_2(y) \\ \vee\, X_2(x) \wedge a(x) \wedge X_2(y) \ \vee\ X_2(x) \wedge c(x) \wedge X_0(y) \end{array} \right) \right) \\ \wedge\, \forall x \big(last(x) \rightarrow (X_0(x) \wedge a(x) \vee X_1(x) \wedge a(x) \vee X_2(x) \wedge a(x)) \big) \end{array} \right)$$

Büchi's theorem - «formula $\Rightarrow$ automaton», part ¼
A VERY SKETCHY ILLUSTRATION
(Interested students refer to lecture notes)

- an illustrative example: (regular) language L of strings $w \in \{a, b\}^*$ that include 2 consecutive a's (i.e., $L = \{a, b\}^* \, aa \, \{a, b\}^*$)
  - costruction of FA recognizing L in [Thomas96] and [Wehr07]: a (technically complex) proof by induction on the structure of φ
- *MSO* formula: $\varphi = \exists x \, \exists y \, S(x, y) \wedge a(x) \wedge a(y)$ **procedure in 4 steps**

1. f.o. variables x, y turned into second order: $X, Y \subseteq U = [\, 0 \,..\, |w|\text{-}1 \,]$
   - X, Y represent variables: we introduce in the s.o. logic the $\boldsymbol{Sing}$ predicate (singleton, a derived predicate) to state that they are true for only one value
   - s.o. logic includes (as a derived predicate) '$\subseteq$' with the usual meaning
   - formula φ becomes (*NB*: in the formula below a denotes a predicate!)

$$\varphi' = \exists X \, \exists Y \, S(X, Y) \wedge Sing(X) \wedge Sing(Y) \wedge X \subseteq a \wedge Y \subseteq a$$

Büchi's theorem - «formula $\Rightarrow$ automaton», part 2/4

2. string $w \in \{a, b\}^*$ enriched with 2 components for ***X*** and ***Y***, obtain an «enriched string» ***w'*** that includes elements for evaluating ***X*** and ***Y***, hence it can be used to evaluate ***φ'***

$w' \vDash \varphi'$

$w' \nvDash \varphi'$ $(X \not\subseteq a)$

$w' \nvDash \varphi'$ $(w' \nvDash Sing(Y))$

$z' \nvDash \varphi'$ $(\nexists z' : z \vDash \varphi')$

w'	$w = b\ a\ a$ $X = 0\ 1\ 0$ $Y = 0\ 0\ 1$

w'	$w = b\ a\ a$ $X = 1\ 0\ 0$ $Y = 0\ 1\ 0$

w'	$w = b\ a\ a$ $X = 0\ 1\ 0$ $Y = 0\ 1\ 1$

z'	$z = b\ b\ a\ b$ $X = .\ .\ .\ .\ .$ $Y = .\ .\ .\ .\ .$

$$w' \in \{ \{a, b\} \times \{0, 1\} \times \{0, 1\} \}^*$$

for *X*

for *Y*

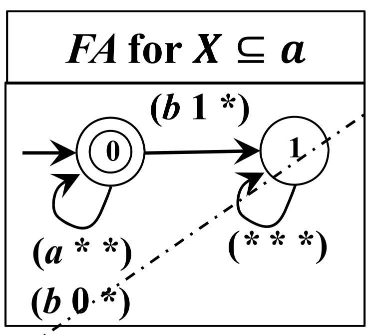
e.g., ***Y* = 0 0 1 encodes «2∈*Y*» *and* it means: «the value of *y* is 2»**

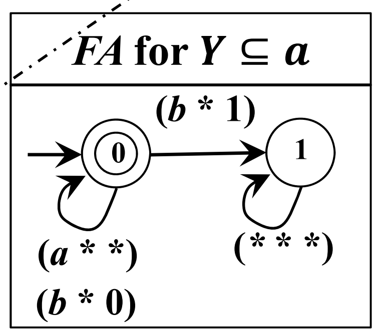
Büchi's theorem - «formula $\Rightarrow$ automaton», part 3/4

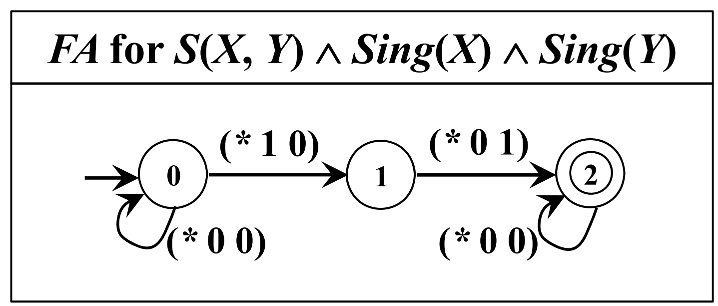
3. build automaton ***A'***
   - with input alphabet $\boldsymbol{I' = \{a, b\} \times \underbrace{\{0, 1\}}_{\text{for } X} \times \underbrace{\{0, 1\}}_{\text{for } Y}}$
   - ***A'*** accepts ***w'*** iff $\boldsymbol{w' \vDash \varphi'}$
   - ***A'*** built from automata for subformulas

$$\boldsymbol{X \subseteq a, \quad Y \subseteq a, \quad S(X, Y) \wedge Sing(X) \wedge Sing(Y)}$$

exploiting closure properties of FA under set-theoretic operations

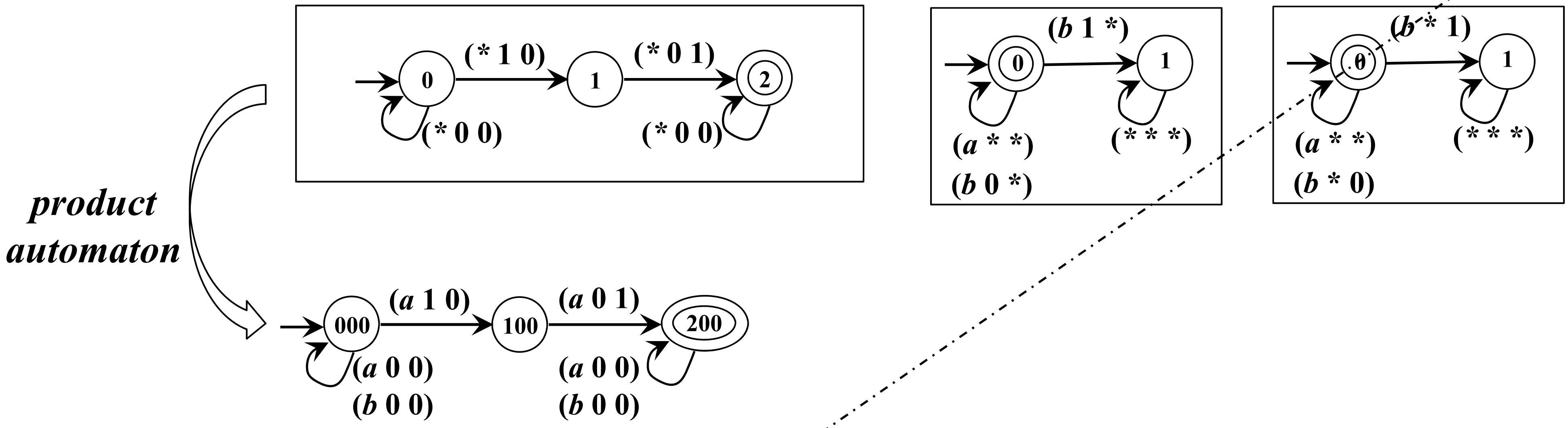




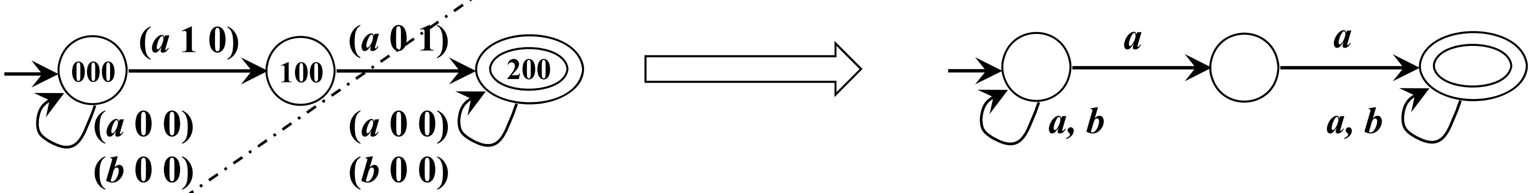


NB: in the above FA's, '*' means «any value», e.g. «(b 1 *)» stands for «(b 1 0) or (b 1 1)»

Büchi's theorem - «formula ⇒ automaton», part 4/4



4. obtain, by «projection», an automaton (in general, nondeterministic) ***A*** that accepts ***w*** if and only if ***A'*** accepts ***w'***



2. Logic to define program properties

- The alphabet includes sets, relations and operators that are in the basic repertoire of programming languages
- Specification of a search algorithm:
  The logical variable ***found*** must be true if and only if there exists an element of the array ***a***, having ***n*** elements, equal to the searched element ***x***:

$$found \leftrightarrow \exists i(1 \leq i \leq n \wedge a[i] = x)$$

- Specification of an algorithm to reverse an array
  - (output ***b*** contains the same elements as input ***a***, in the reverse order):

$$\forall i\big(1 \leq i \leq n \;\rightarrow\; b[i] = a[n - i + 1]\big)$$

In more general terms

{Precondition: *Pre*}
Program - or program fragment - *P*
{Postcondition: *Post*}

P must be such that: ***if*** *Pre* holds before the execution of *P* ***then*** *Post* holds after its execution:

- Search in an ordered array:

$$\{\forall i(1 \le i < n \rightarrow a[i] \le a[i+1])\}$$

$$P$$

$$\{found \leftrightarrow \exists i(1 \le i \le n \wedge a[i] = x)\}$$

NB: this does not at all mean that *P* must be a binary search algorithm (or any other one exploiting the ordering in the array). It only means that the ***implementer*** of *P* ***may*** exploit the fact that before the execution of *P* the array is ordered. A sequential search algorithm would be correct w.r.t. this specification. Instead, a binary search algorithm would not be correct w.r.t. a specification having as a precondition simply ***true*** (i.e. with no assumption on the input array).

- Sorting an array of ***n*** elements with no repetitions:

  $\{\neg \exists i, j (1 \le i \le n \wedge 1 \le j \le n \wedge i \ne j \ \wedge a[i] = a[j])\}$

  $SORT$

  $\{\forall i \left(1 \le i < n \rightarrow a[i] \le a[i+1]\right)\}$

  Is this an adequate ***specification***?
  (Let us think of the analogy: “specification = contract”)
- One should not give anything for granted: what about the following ***implementation***:

```
for (k=1; k<=n; k++) a[k]=k;
```

  does it “***satisfy the contract***”?

- An alternative specification
- NB: use variable ***b*** to denote the array before the sorting operation

$$\{\neg \exists i, j(1 \le i \le n \ \wedge \ 1 \le j \le n \ \wedge \ i \ne j \ \wedge \ a[i] = a[j]) \qquad \wedge \qquad \%\ no\ duplicates\ in\ a$$
$$\forall i \left(1 \le i \le n \to a[i] = b[i]\right)\}$$
$$SORT$$
$$\{\forall i \left(1 \le i < n \ \to \ a[i] \le a[i+1]\right) \qquad \wedge$$
$$\forall i \left(1 \le i \le n \ \to \ \exists j \left(1 \le j \le n \ \wedge \ a[i] = b[j]\right)\right) \qquad \wedge \ \%\ all\ a[i]\ are\ in\ some\ b[j]$$
$$\forall j \left(1 \le j \le n \ \to \ \exists i \left(1 \le i \le n \ \wedge \ b[j] = a[i]\right)\right)\} \qquad \%\ all\ b[j]\ are\ in\ some\ a[i]$$

- If we eliminate the first line in the precondition, is the specification still satisfactory?
  - (consider, in case of duplicates, their number in the array before and after the sorting)
- In fact, even a well-known, intuitive notion like sorting may be subject to misunderstandings
- In case of critical applications (having important implications on safety, or economic issues) the precision of formal notations is essential for writing good specifications

3. Mathematical logic for specifying system (timing) properties

- distinguished (integer- or real-valued) variables ***t***, ***t'***, t_1, etc. denote time points
- predicates with a time variable as argument denote «facts» that may or may not hold at given time instants
- "If I push the button the light goes on within Δ time units (t.u.)":
  - ***P_B***(***t***): a predicate denoting Push Button at time ***t***
  - ***L_On***(***t***): a predicate denoting that the Light is On at time ***t***

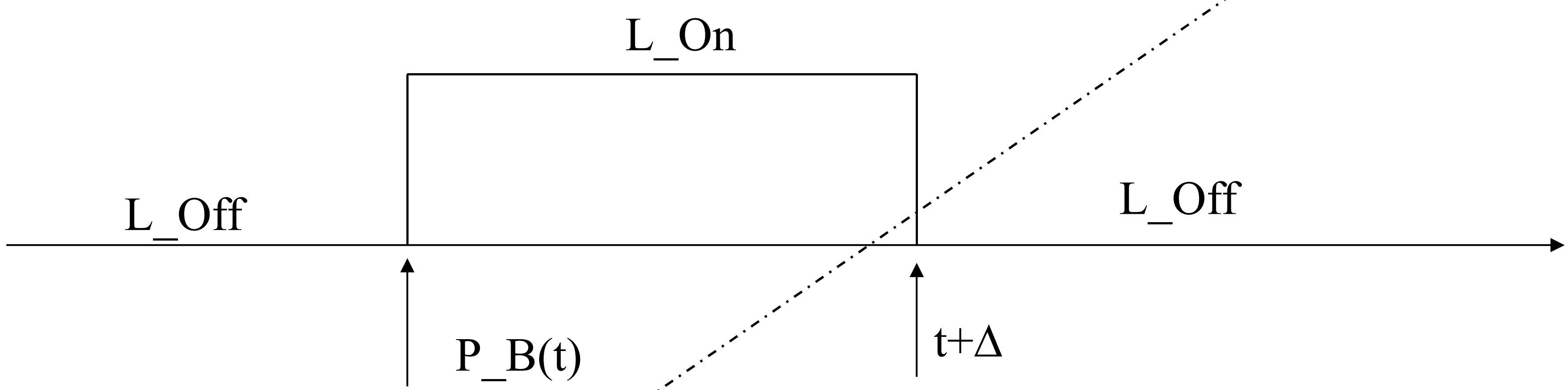
$$\forall t(P\_B(t) \rightarrow \exists t_1((t \leq t_1 \leq t + \Delta) \wedge L\_On(t_1)))$$

This specification is correct but the system requirement is not realistic.

-nondeterminism

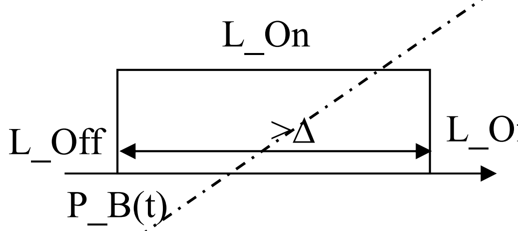
-timed lamps usually stay on for a given time interval

- If the button is pushed the light is on *for* Δ t.u. (similar requirements for an alarm or an electronic safe): here is a typical ***intended interpretation*** (assuming ∀t(L_On(t)↔¬L_Off(t)))

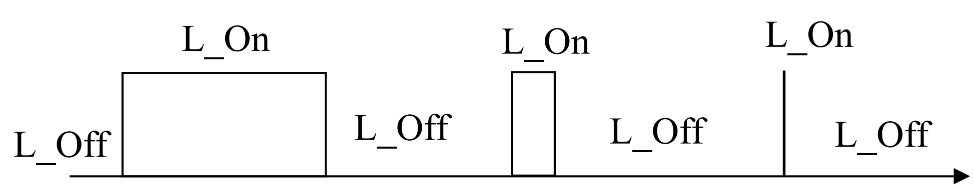


$$\forall t\big(P\_B(t) \rightarrow \forall t_1\big((t \le t_1 \le t+\Delta) \rightarrow L\_On(t_1)\big)\big)$$

- But here are some ***unintended*** interpretations



lamp is on for longer time

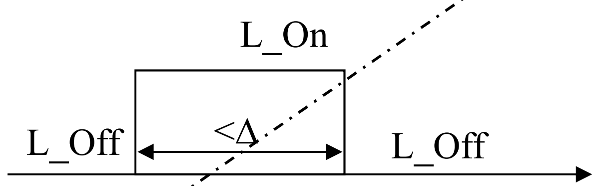


spurious lamp activation (no button press)

- Previous formula provides a ***sufficient*** condition of the lamp to be on. But using only sufficient conditions we get spurious L_On values
- We need a necessary and sufficient condition: try this

$$\forall t\big(P\_B(t) \leftrightarrow \forall t_1\big((t \le t_1 \le t+\Delta) \to L\_On(t_1)\big)\big)$$

- But this still admits spurious intepretations, because an interval with L_On implies P_B only if it lasts at least Δ t.u.



- A correct necessary condition for the light to be on, to be added to (conjoined with) the sufficient condition in the previous slide):

$$\forall t\big(L\_On(t) \to \exists d\big((0 < d < \Delta) \wedge P\_B(t-d)\big)\big)$$

Variations on the theme:

- Button for turning off combined with timeout
- Light maintained on by button pressure
- Opening closing of tents/windows/gates, ...
  - keep button pressure or not
  - interrupt movement or not
  - …..

- Logic approach for specification is extremely flexible but needs some method, to be effectively mastered

- Towards specification *languages* and *methods*

Theory of computation

- Which problems can we solve
  - With some (any given) type of machine
  - In the broadest possible meaning
- At first sight the question seems too general:
  - What do we mean by "problem"?
    A mathematical computation; taking a decision in a meeting of people; cash withdrawal from an ATM …?
  - Which abstract machines should we consider?
  - What does it mean to be able to solve a problem:
    If I am not able to solve the problem by some means I might be able to solve it by some other one.

In fact we can formalize the problem in its broader generality

- The notion of language allows us to formalize any "computer science problem" :
  $x \in L$? (Language recognition problem)
  $y = \tau(x)$? (Function computation problem)
  The two above formulations can be reduced to each other:
  - If I can find a machine to solve the problem of computing any function $y = \tau(x)$ and I wish to use this to solve the problem $x \in$ L, it suffices to define the *predicate* function $\tau(x) = 1$ if $x \in L$, $\tau(x)=0$ if $x \notin L$.
  - Viceversa, if I wish to compute function $y = \tau(x)$ I could define the language

$$L_\tau = \{x\$y \mid y = \tau(x)\}$$

  - Assuming that I can recognize L_τ using some machine, then, for a fixed x, I could enumerate all possible strings y over the output alphabet and for each of them ask the machine if $x\$y \in L_\tau$: soon or late, ***if $\tau(x)$ is defined***, I will find the string for which the machine answers positively: this is a way to compute $y=\tau(x)$.
    The procedure is "a bit long" but at the moment we are not concerned about the length of computations

- Concerning the machine … in fact there exist many, besides the ones we know; and many more can be invented, so we might be able to get results of the kind

  $\{a^n b^n | n > 0\}$ is accepted by a PDA and a TM but not by a FA.

- However we noticed that it is not so easy to overcome the computing power of the TM: adding tapes, heads, nondeterminism, … does not increase the power, (i.e., the class of accepted languages);
  It is not so difficult to have the TM do what a normal computer does: It suffices to simulate the memory of one of the two by the other one

$\longrightarrow$ ***HENCE***

there is a ultimate generalization:

Church Thesis (back to 1930's!)

- There does not exist a computational device more powerful than the TM or any other formalism that is equivalento to it
  It is not a theorem (in principle, it should be checked every time anyone comes up with a new computational model)
- No *algorithm*, independently from the tool used to implement it, can solve problems that cannot be solved by a TM: the TM is the most powerful computer that we have and will ever have!
- Then the question "Which are the problems that can be solved algorithmically (or, with an equivalent term, "automatically")?" can be answered:
  These are the problems that can be solved by the (relatively simple) TM

Then let us focus on the TM

- Are there problems that cannot be solved by a TM?
- How can we find them?

The answers that we find hold also for C programs, Java programs, supercomputers ….

First, relevant fact:

- We can *algorithmically enumerate* TM's
- Enumeration of a set S:
- $\mathcal{E}$: S $\longleftrightarrow$ $\mathcal{N}$
- Algorithmic Enumeration: $\mathcal{E}$ can be computed by an algorithm, hence (church Thesis…) by a TM
- Algorithmic Enumeration of $\{a, b\}^*$:
- {ε, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, …}
  $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$
- {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ….}
  - Usually called the "lexicographical ordering"

- Now we define an algorithmic enumeration of TM's
- For simplicity, without loss of generality:
- Let us fix a unique alphabet A
(in the examples |A| = 2, A = {0, 1})
- Single tape TM
- Let us ignore one state TM's … and consider those with two states:

	0	1
q_0	$\perp$	$\perp$
q_1	$\perp$	$\perp$

TM_0

	0	1
q_0	$\perp$	$\perp$
q_1	$\perp$	$\langle q_0, 0, S\rangle$

TM_1

……

- How many two-states TM's there exist ?
  $\delta: Q \times A \to Q \times A \times \{R,L,S\} \cup \{\perp\}$
- In general: how many functions of type f: $D \to R$ are there?
- $|R|^{|D|}$ (for each $x \in D$ we have $|R|$ choices}
- With $|Q| = 2$, $|A| = 2$, $(2 \cdot 2 \cdot 3+1)^{(2 \cdot 2)} = 13^4$ two-state TM's
- Let us sort these TM's: $\{M_0, M_1, \ldots M_{13^4-1}\}$
- Then let us sort similarly the $(3 \cdot 2 \cdot 3+1)^{(3 \cdot 2)}$ 3-state TM's and so on.
- We obtain an enumeration $\mathcal{E}$: $\{TM\} \longleftrightarrow \mathcal{N}$
- $\mathcal{E}$ is algorithmic (or *effective*): we can write a C program (i.e., a TM…) which, given *n*, produces the *n*-th TM (for instance by providing a table defining δ) and vice versa, given a (table describing a) TM M, tells the position $\mathcal{E}$(M) of M in the enumeration.
- $\mathcal{E}$(M) is called the ***Goedel number*** of M, $\mathcal{E}$ a goedelization

- A further convention: since we are speaking of numbers from now on we identify (for what concerns computability)
- Solving a problem = computing a function $f: \mathcal{N} \rightarrow \mathcal{N}$
- f_y = function computed by the y-th TM
- NB: $f_y(x) = \perp$ by definition ***if*** M_y does not stop when it takes x as input
- Conversely we stipulate (with no loss of generality) the reverse
  - if $f_y(x) = \perp$ then it means that the y-th TM does not stop on x
  - it suffices to stipulate that any TM M_y that, on input x, stops in a **non final** state (that does not define any significant output value $f_y(x)$), is by convention equivalent to a TM that enters a new state and continues forever, for instance by moving indefinitely the head right
- Therefore, in conclusion, $f_y(x) = \perp$ ***if and only if*** M_y does not stop when it takes x as input

A brief digression on terminology (to avoid misunderstandings)

- for function f, being total/partial or computable/uncomputable are distinct (orthogonal) matters
- total (resp., partial) = defined for every (resp., undefined for some) value of its domain
- computable = there is a TM that computes it
- All combinations are possible
  - $\forall x \in \mathbb{N}$, f_x is computable by definition

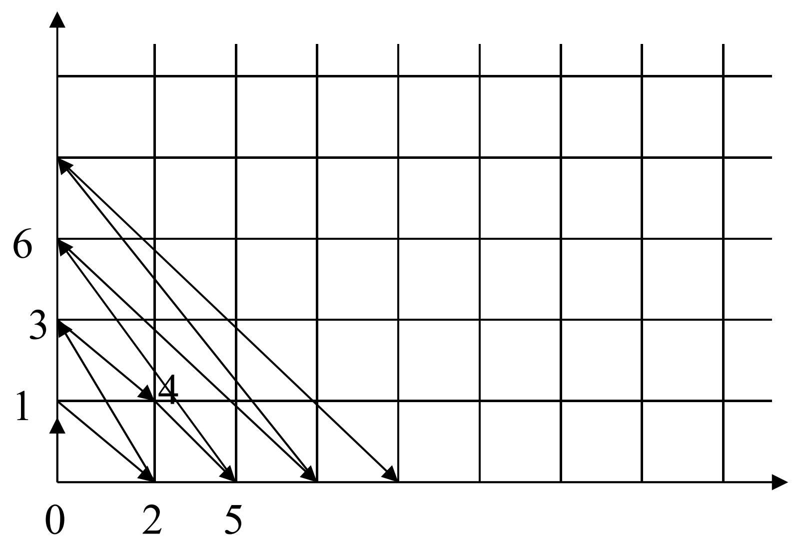
	total	**partial**
computable	1	2
not computable	3	4

1) function f(x)=x+1 is total and computable
2) the everywhere undefined function $f(x)=\perp$ for all x is partial and computable
   function $f(x)$ = if (x is even) then x else $\perp$ is partial and computable
3) the (predicate) function f(x)=if $f_x(y)=2\cdot y$ then 1 else 0 is total, not computable
4) $f(x)$= if (x is even and $f_{x/2}$ is total) then 1 elsif (x is even and $f_{x/2}$ is partial) then 0 else $\perp$
   is partial and not computable (this will be an easy exercise …)

- 3) and 4) to be explained later… as well as many other nontrivial, interesting cases

Second relevant fact:

- There exists a ***Universal Turing Machine*** (UTM): the TM that computes the function $g(y, x) = f_y(x)$
- The UTM seems not to belong to the family $\{M_y\}$ because f_y is a function of one variable, while g is a function of two variables
- But we know that integer pairs can be effectively enumerated, i.e., $\mathcal{N} \times \mathcal{N} \leftrightarrow \mathcal{N}$: an example enumeration:



$$d(x, y) = \frac{(x+y)(x+y+1)}{2} + x$$

- So we can devise a suitable $g\^(n)$ such that $g\^(n) = g(d^{-1}(n)) = g(y, x)$; $g(y,x)$ is encoded as $g\^(n)$, with $n=d(y,x)$, i.e., $<y,x>=d^{-1}(n)$
  Notice that d and d^{-1} are both computable
- Sketch of the operation of the UTM that computes $g\^$ (NB from now on for simplicity we will simply write g instead of $g\^$):
  - Given n, the UTM computes $d^{-1}(n) = <y,x>$
  - Then it builds the transition function of M_y (by computing $\mathcal{E}^{-1}(y)$) and stores it on some portion of a tape:

	\$	q	a	q’	a’	S	\$	..	..							

- In another tape portion is stores an encoding of the configuration of M_y

	#	0	1		0			..	1	q	1		1		0	#

NB: The special symbols #, \$ and other ones are coded as binary strings

At the end the UTM leaves on the tape only $f_y(x)$ if and only if M_y terminates its computation on x

- The TM is a very abstract and simple model of a computer
- Let us pursue further the analogy:
- TM: computer with a single, built-in program
  An “ordinary” TM always executes the same algorithm, i.e., it always computes the same function
- UTM: computer with memory-stored program:
  y = program
  x = input to the program

Back to the question "which problems can be solved algorithmically?

Read very carefully textbook p.4-5!!

- How many and which are the computable functions f_y: $\mathcal{N} \to \mathcal{N}$?
- First, "how many" functions (not necessarily computable) are there?
- $\{f: \mathcal{N} \to \mathcal{N}\} \supseteq \{f: \mathcal{N} \to \{0,1\}\} \Rightarrow$
  $|\{f: \mathcal{N} \to \mathcal{N}\}| \geq |\{f: \mathcal{N} \to \{0,1\}\}| = |\wp(\mathcal{N})| = 2^{\aleph_0}$
- On the other hand, the set $\{f_y: \mathcal{N} \to \mathcal{N}\}$ of ***computable*** functions is by definition denumerable:
  NB: $\mathcal{E}$: $\{M_y\} \leftrightarrow \mathcal{N}$ induces $\mathcal{E}^\wedge$: $\mathcal{N} \to \{f_y\}$ not one-to-one (in many cases $f_y = f_z$, with $z \neq y$) but (*a fortiori*) it allows us to state that
- $|\{f_y: \mathcal{N} \to \mathcal{N}\}| = \aleph_0 < 2^{\aleph_0} \Rightarrow$
- "most" of the functions (problems) cannot be solved algorithmically!
- There are (very many) more problems than programs!

Is this such a bad shame?

- In fact, how many problems can be ***defined***?
- To define a problem we typically use a phrase (a string) of some language:
  - $f(x) = x^2$

    $$f(x) = \int_a^x g(z)dz$$

  - "the number that multiplied by itself is equal to y"
  - …
- But any language (over some alphabet A) is a subset of A^*, which also is a denumerable set $\Rightarrow$
- Hence the set of the problems that can be *defined* is also denumerable, just like the set of the problems that can be (algorithmically) solved
- Therefore we can still hope that they are the same
  Certainly {Solvable Problems} $\subseteq$ {Definable Problems}

(BTW, a TM *defines* a function, besides computing it)

Next we turn again to the question: "Which problems can be solved?"

- **<u>The problem of termination</u>**
  (it has quite "practical" implications):
  - One can build a program
  - One can provide input data for it
  - One knows that in general the program might not terminate its execution (in jargon: "run into a loop")
  - Can one determine if this will occur?
    - NB: determine **«in advance»**, not by running the program on the input and ...
- Stated in –completely equivalent– terms of TM:
  - Given (predicate) function $\boldsymbol{g(y, x) = 1}$ **if** $\boldsymbol{f_y(x) \neq \perp}$, $\boldsymbol{g(y,x) = 0}$ **if** $\boldsymbol{f_y(x) = \perp}$
  - Does there exist a TM that computes g? (i.e., is g computable?)

Answer: **NO**

- That's why a computer (which is a program) cannot warn us that the program we just wrote will run into an endless execution ***on a given input datum*** (while it easily signals a missing "}"):
- Determining if an arithmetic expression is well parenthesized is a solvable (decidable) problem;
- Determining if a given program will run into an endless execution on a given input is an algorithmically unsolvable (undecidable) problem [we will see many more ones: there are many things that a computer cannot do]

Proof

- It employs a typical *diagonal technique* (adopted also in the Cantor theorem to show that $\aleph_0 < 2^{\aleph_0}$)
- Let us ***assume*** (by contradiction) that the total function :

$$g(y,x) = 1 \text{ if } f_y(x) \neq \perp, \qquad g(y,x) = 0 \text{ if } f_y(x) = \perp$$

  is computable
- Then also the partial function

$$h(x) = \begin{cases} 1 \text{ if } g(x,x) = 0 \text{ (i.e., if } f_x(x) = \perp), \\ \perp \text{ if } g(x,x) = 1 \text{ (i.e., if } f_x(x) \neq \perp) \end{cases}$$

  is computable

  NB: we went on the *diagonal* $y=x$, we changed the no answer ($g(x,x) = 0$) into a yes answer ($h(x)=1$), and we turned the yes ($g(x,x) = 1$) into a nontermination ($h(x)=\perp$), which can always be easily done by modifying the TM that (supposedly) computes g
- If h is computable then $h = f_{xh}$ for some xh.
- Question: $h(xh) = 1$ or $h(xh) = \perp$?

RECALL $h(x) = 1$ if $g(x,x) = 0$ (i.e., if $f_x(x) = \perp$), $h(x) = \perp$ if $g(x,x) = 1$ (i.e., if $f_x(x) \neq \perp$)

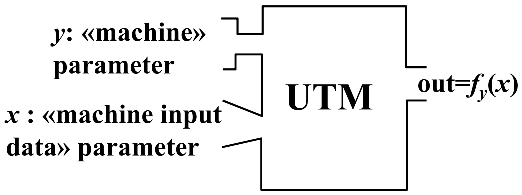
- Let us assume that $h(xh) = f_{xh}(xh) = 1$
- Then $g(xh,xh) = 0$, that is, $f_{xh}(xh) = \perp$:
- A contradiction

- Then let us assume the opposite: $h(xh) = f_{xh}(xh) = \perp$
- Then $g(xh,xh) = 1$, that is, $f_{xh}(xh) \neq \perp$:
- Another contradiction

then the assumption was wrong

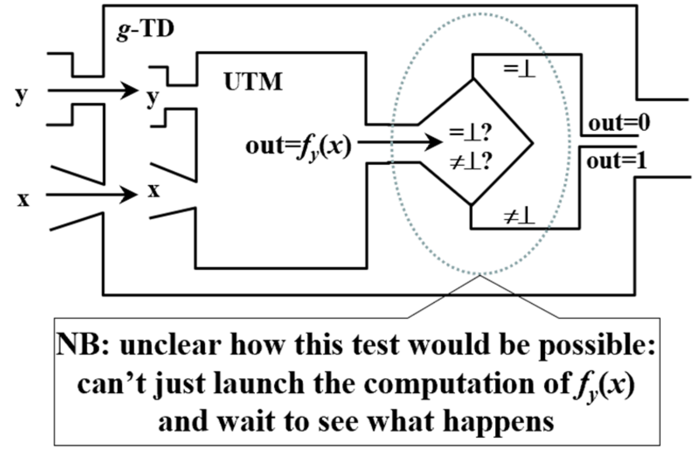
QED

PICTORIAL REPRESENTATION OF THE UNSOLVABILITY OF HALTING



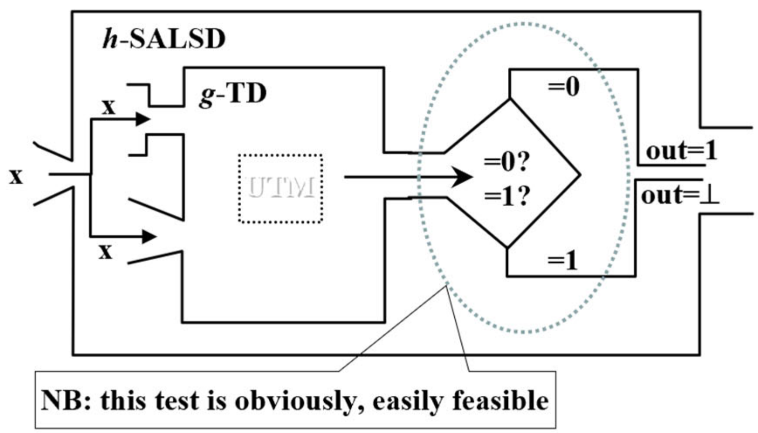
g(y,x)= 1 if $f_y(x) \neq \perp$, g(y,x)=0 if $f_y(x) = \perp$

***g* is a "Termination Detecting" function (*g*-TD)**



We introduce function $h(x) = 1$ if $g(x, x)=0$, $h(x) = \perp$ if $g(x, x)=1$

***h* detects if a function, applied to itself, loops forever (the opposite of terminating), but if the function terminates *h* is undefined (the TM that computes *h* goes into a loop); we say that it "semidetects" loops: it answers only in case a loop occurs; therefore we call *h* "Self Application Loop Semi Detector" function (*h*-SALSD)**



out=1 means: answer "YES", i.e., *x* applied to itself does not terminate

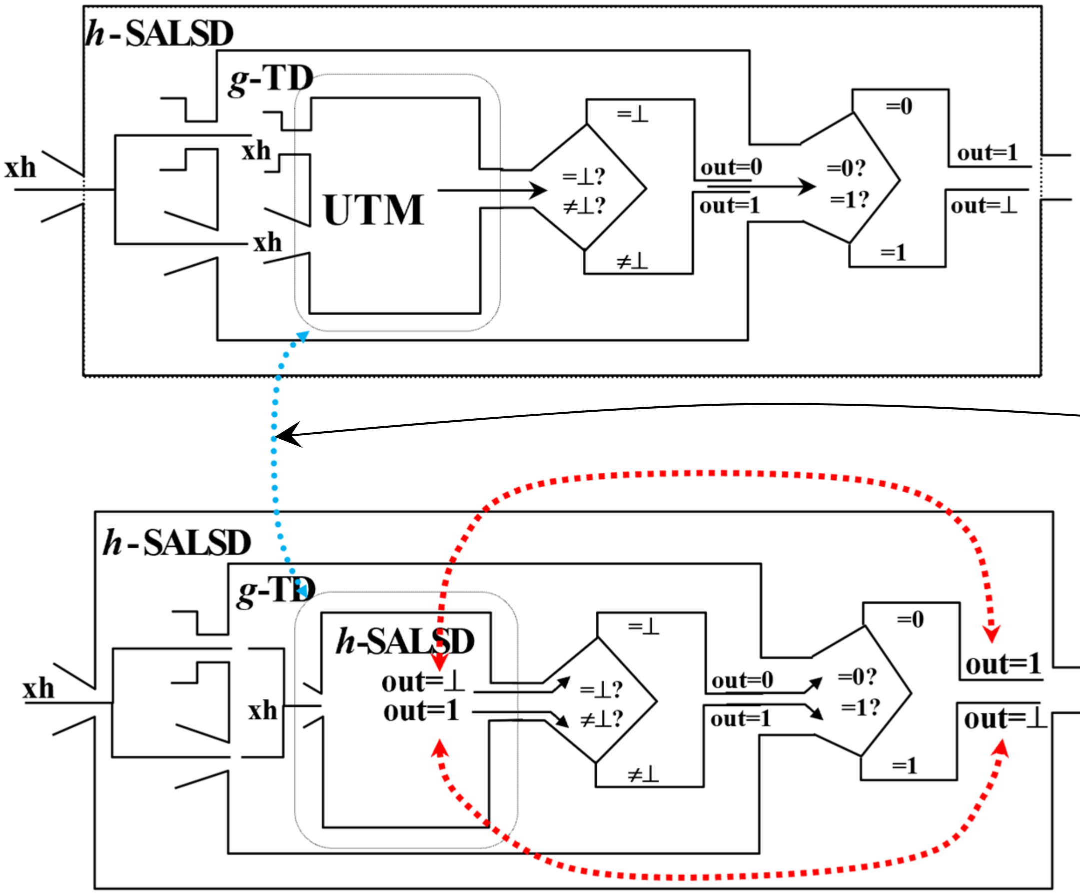
out=⊥ means: *h*-SALSD TM loops forever (provides no answer)

h*-SALSD is a variant of *g*-TD that is very easy to compute *if TD is computable

assume *h*-SALSD = f_{xh} , i.e., *xh* is the Goedel # of *h*-SALSD

PICTORIAL REPRESENTATION OF THE UNSOLVABILITY OF HALTING

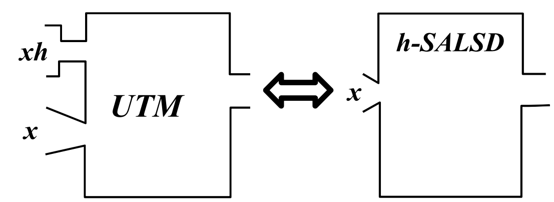
now let us apply *h-SALSD* to itself



applied to itself, *h*-SALSD:

- returns 1 iff TD says that SALSD applied to itself does not terminate
- returns ⊥ (i.e., its TM loops forever) iff TD says that SALSD applied to itself terminates

but the UTM applied to *xh* computes function *h*: $f_{xh}(xh)=h(xh)$, i.e., function *h*-SALSD



therefore a contradiction follows : **if $h(xh)=1$ then $h(xh)=\perp$**

if $h(xh)=\perp$ then $h(xh)=1$

- A first corollary of the unsolvability of the halting problem for the TM
  - The predicate (hence total function) characterizing the set of computable functions that are defined when applied to themselves : $\boldsymbol{h(x)=1}$ **if** $\boldsymbol{f_x(x) \neq \perp}$**;** $\boldsymbol{h(x)=0}$ **if** $\boldsymbol{f_x(x) = \perp}$ is not computable (proof on textbook, Corollary 2.7 p.168)
  - By itself the assertion is not very meaningful (but it will be so later, see slide 193)
  - Notice that $h(x)$ is a special case of function $g(y, x)$ (the function of slide 172)

    $$g(y,x) = 1 \text{ if } f_y(x) \neq \perp,\ g(y,x) = 0 \text{ if } f_y(x) = \perp$$

    [because $h(x)=g(x,x)$], and g was just proved to be uncomputable
  - Notice that the uncomputability of $h(x)$ is ***not*** a necessary consequence of the uncomputability of $g(y,x)$:
  - *NB: in general, if a problem is unsolvable, then a special case of it might be solvable (e.g., some properties that cannot be decided for any language can be decided for regular languages); instead a more general case of an unsolvable problem is necessarily unsolvable.*
    On the contrary if a problem is solvable, a generalization of it ***might*** *be unsolvable, while any specialization of it is certainly solvable.*

Another important unsolvable problem

- The (predicate, hence total) function
  $k(y) = 1$ if f_y is total, i.e., $f_y(x) \neq \perp \ \forall\ x \in \mathcal{N}$; $\quad k(y) = 0$ otherwise
  is not computable (NB: this result is a trivial consequence of the coming Rice Theorem, because function k is a predicate characterizing total computable functions)
- NB: it is a problem *similar to but different* from the previous one. Here we have a quantification w.r.t. all possible input data.
  For this problem, **testing is useless**: In some cases one could be able to establish, for a large set of values of variabile x, that $f_y(x) \neq \perp$, without however being able to answer the question "is f_y a total function?" (Obviously, if one finds an x such that $f_y(x) = \perp$, one can conclude that f_y is not total, but what if one does not find it?).
  Vice versa, one could be able to conclude that f_y is not total and however be unable to decide whether $f_y(x) \neq \perp$ for a given single x. (However, if one was able to conclude that f_y is total there would be no doubt on whether $f_y(x) \neq \perp$ for a given x)
- From a practical viewpoint, this problem is perhaps even more relevant than the halting problem: given a program, one wants to know if it will terminate the execution *for every input* datum or if it may, *for some* datum, run into an endless execution. In the *problem of termination*, instead, one was interested to know if *a given program with some given input datum* would terminate.

Proof

- Standard technique: diagonal + contradiction, with some more technical detail.
- Hypothesis: $k(y) = 1$ if f_y *is total*, i.e., if $f_y(x) \neq \perp \ \forall\ x \in \mathcal{N}$; otherwise $k(y) = 0$ is computable and obviously, by definition, total
- Then define $g(x) = w =$ index (Goedel number) of the x-th TM (in $\mathcal{E}$) that computes a total function.
- If k is computable and total, then so is g:
  - compute $k(0), k(1), \ldots,$ let w_0 the first value such that $k(w_0) = 1$, then let $g(0) = w_0$;
  - then let $g(1) = w_1$, w_1 being the second value such that $k(w_1) = 1$; …
  - the procedure is algorithmic; furthermore, being total functions infinite in number, $g(x)$ is certainly defined for each x, hence it is total.
- g is also strictly monotonic: $w_{x+1} > w_x$;
- hence g^{-1} is also a function, strictly monotonic too, though not total: $g^{-1}(w)$ is defined only if w is the Goedel number of a total function.
- next define
  - (α) $h(x) = f_{g(x)}(x) + 1 = f_w(x) + 1$: f_w is computable and total hence so is $h \Rightarrow$
  - (β) $h = f_{w0}$ for some w_0; since h is total, $g^{-1}(w_0) \neq \perp$, let $g^{-1}(w_0) = x_0$ (hence w_0=g(x_0))

RECALL

- (α) $h(x) = f_{g(x)}(x) + 1 = f_w(x) + 1$: f_w is computable and total hence so is $h \Rightarrow$
- (β) $h = f_{w0}$ for some w_0; since h is total, $g^{-1}(w_0) \neq \perp$, let $g^{-1}(w_0) = x_0$ (hence $w_0 = g(x_0)$)

w_0 is the Goedel number of h

- What is the value of $h(x_0)$?
  - $h(x_0) = f_{g(x0)}(x_0) + 1 = f_{w0}(x_0) + 1$ (from (α))
  - $h = f_{w0}$ hence $h(x_0) = f_{w0}(x_0)$ (from (β))
- *Contradiction!*

A crucial remark: knowing that a problem is solvable does not mean being able to solve it!

- In mathematics we often have non-constructive proofs: one shows that a mathematical object exists without providing a way to actually find (and exhibit) it
- In our case:
  - a problem is solvable if *there exists* a TM that solves it
  - for some problems we can reach the conclusion that *there exists* a TM that solves them, but, despite our knowledge of this, we are unable to find (build) it or we do not know which one it is in a set of TM that certainly includes the "right one"
- Let us start with a trivial case
  - The "problem" consists of answering a question with a yes/no answer (a so-called ***closed question***, whose answer does not depend on an input value/parameter):
    - Is it true that the number of atoms in the universe is $10^{10^{10^{10^{10}}}}$?
    - Is it true that the "perfect chess game" will end in parity?
    - (30 years ago ...) Is it true that $\neg\exists x, y, z, w \in \mathcal{N} (x^y + z^y = w^y \wedge y > 2)$? (proved in 1994)
    - ….

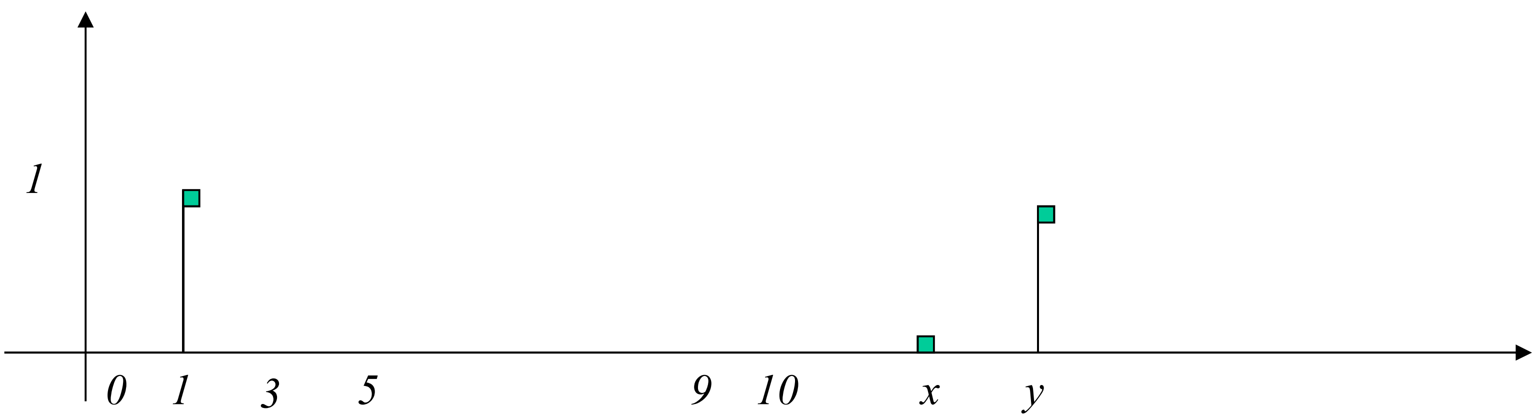
- In such cases one knows *a priori* that the answer is either ***Yes*** or ***No***, though one does not know (or did not until some point in time) which one it is
- This fact is less surprising if we consider that
  Problem = function; solve a problem = compute a function
  What function can one associate to the above problems?
  If one encodes TRUE=1; FALSE=0, *all* the above problems are expressed by one of the following two functions: $f1(x) = 1 \;\; \forall x$, or $f0(x) = 0 \;\; \forall x$
  Both functions are trivially computable (all constant functions are…), hence
  Whatever the answer is, it is ***computable***, though ***not necessarily known***.
- More abstractly, considering for instance function $g(y, x)$ of the halting problem:
  $g(10,20) = 1$ if $f_{10}(20) \neq \perp$, $g(10,20) = 0$ if $f_{10}(20) = \perp$
  $g(100,200) = 1$ if $f_{100}(200) \neq \perp$, $g(100,200) = 0$ if $f_{100}(200) = \perp$
  $g(7,28) = 1$ if $f_{7}(28) \neq \perp$, $g(7,28) = 0$ if $f_{7}(28) = \perp$
  ….
  Hence $g(10,20)$ (i.e.: does TM M_{10} stop on input 20?), $g(100,200)$, $g(7,28)$ *etc.* are all solvable problems, though we do not necessarily know the solution (i.e., the value of g for those arguments).

- Let us now consider less trivial and more instructive cases:
  - *f*(*x*) = *x*-th digit of the decimal expansion of π.
    f is certainly computable (we know algorithms (TM's) to compute it)
  - Instead, *g(x) = 1 if somewhere in π there exist* ***exactly x*** *consecutive digits* 5,
    0 otherwise

    might be computable or not (NB: it is total). How can we try to compute ***g***?
  - Using our ability to compute *f* (we *currently* have no other knowledge)
  - By computing the sequence (notice that π=3.14159…)

    $\{f(0) = 3, f(1) = 1, f(2) = 4, f(3) = 1, f(4) = 5, f(5) = 9, \ldots\}$

We get *g*(1) = 1 (i.e., there is a sequence of exactly 1 consecutive digit 5)
In general the plot of function g will be something like:



- For some value of x we might find that $g(x) = 1$;
- better, if $g(x)=1$, soon or late we will find it, if we are patient
- but what if $g(x)=0$?
  - Computing $f(y)$ for all $y \in [1..1000]$ or for all $y \in [1..10^{1000}]$ or … is useless
    - A side remark (forward pointing): the set $\{ x \mid g(x) = 1 \}$ is ***semidecidable***
- If the following conjecture
  "For every x, by producing a sufficiently long sequence of π, soon or late we will find exactly x consecutive 5's"

  was true, then g would be the constant function $g(x) = 1 \; \forall \, x$

  hence g would be computable
- Otherwise g could be some possibly very irregular function, maybe computable maybe not …
- In conclusion, *at the current state of the art*, we cannot conclude that ***g*** is computable, nor that it is not

- Now consider the following "slight" modification of g:
  h(x) = 1 *if in* π $\exists$ ***at least*** *x consecutive* 5*'s,* 0 *otherwise*

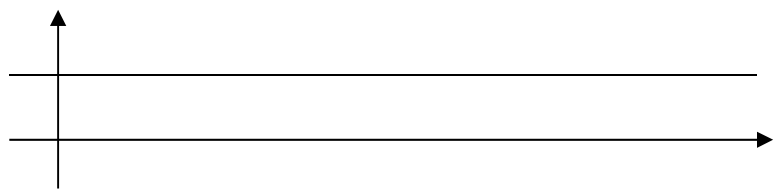
  (obviously, if $g(x) = 1$ then also $h(x) = 1$)

  Notice that, for any x,

if $h(x) = 1$, then $h(y)=1$ $\forall y \leq x$ (*h* is "downward closed for value 1"), and

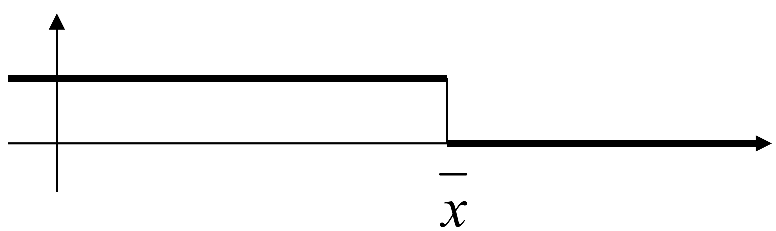
if $h(x)=0$ then $h(y)=0$ $\forall y > x$ (*h* is "upward closed for value 0")

  - Hence the plot of *h* is like one the following two:
  - 1)



($h(x) = 1 \; \forall \; x$)

  - 2)



$$h(x) = 1 \; \forall \; x \leq \bar{x}$$

$$h(x) = 0 \; \forall \; x > \bar{x}$$

- Then *h* certainly is in the following set of functions

$$\{h_{\bar{x}} \mid h_{\bar{x}}(x) = 1\ \forall\ x \le \bar{x} \wedge h_{\bar{x}}(x) = 0\ \forall\ x > \bar{x}\} \cup \{\bar{h} \mid \bar{h}(x) = 1\ \forall\ x\}$$

  Notice that each one of the functions in the above set is trivially computable (for any fixed $\bar{x}$ it is immediate to construct a TM that computes $h_{\bar{x}}$; similarly for $\bar{h}$)
- Hence *h* is certainly computable: there *exists* a TM that computes it
- Do we know which of the functions in the above set is *h*?

  No (at the moment): among the infinite TM's that compute the functions of the above set we do not know which is the right one

Decidability and semidecidability

Or: 1/2 + 1/2 = 1

- Let us focus on the problems stated in such a way that the answer is binary: Problem = " given set $\boldsymbol{S} \subseteq \mathcal{N}$ and $x \in \mathcal{N}$, $x \in S$? "

  (NB: *all* problems can be (re)phrased in such a way,

  because they can all be viewed as a language - see slide 158)
- **characteristic function** or **characteristic *predicate*** of a set ***S***: it is the predicate characterizing the set

$$\boldsymbol{c_S(x) = 1 \textit{ if } x \in S,\ c_S(x) = 0 \textit{ if } x \notin S}$$

(NB: $\boldsymbol{c_S}$ is total by definition)

- A set ***S*** is ***recursive*** (R) *or **decidable*** if and only if its characteristic function is computable
  - Note on terminology: It is also customary to say that solvable problems are ***decidable***

- ***S*** is ***recursively enumerable*** (RE) (or ***semidecidable***) if and only if:
  - S is the empty set, *or*
  - S is the ***image*** of a ***total, computable*** function $\boldsymbol{g_S}$, the ***generating function*** of ***S***:

    $$S = I_{g_S} = \{x \mid x = g_S(y), y \in N\}$$

    notice that this implies

    $$S = \{g_S(0), g_S(1), g_S(2), g_S(3), \ldots\}$$

    the term "recursively (i.e., algorithmically) enumerable" comes from this "enumeration"
  - The term "semidecidable" can also be explained intuitively: given the question "$x \in S$?", if $x \in S$ then*, by enumerating the elements of* S, **soon or late** one finds x and is able to get a correct (yes) answer to the question;

    but what if $\boldsymbol{x \notin S}$? In this case the above procedure does not work: using function $\boldsymbol{g_S}$ one continues indefinitely to generate elements of S without coming to a conclusion.
  - A formal characterization of this matter comes from the following ...

Theorem

- A) If S is recursive, it is also RE
  (i.e., decidable is more than –not less than- semidecidable)
- B) S is recursive if and only if both S itself and its complement $S^\wedge = \mathcal{N} - S$ are RE
  (two "semidecidabilities" make a "decidability"; or, when answering NO is equivalent to (i.e., it is equally difficult as) answering Yes
- (Corollary: the class of decidable sets (languages, problems, …) is closed under complement)
- Proof:

A): *S* recursive implies *S* RE

- If S is empty it is RE by definition
- Let us then assume $S \neq \emptyset$ and call c_s its characteristic predicate: note that, since $S \neq \emptyset$,

  $\exists\, k \in S$, that is $\exists\, k\; c_s(k) = 1$
- Let us define the generating function g_s as follows: $g_s(x) = x$ *if* $c_s(x) = 1$, otherwise $g_s(x) = k$ (the k above)
- g_s is total, computable (because so is c_s), and $I_{gs} = S$
- $\rightarrow S$ is RE
- NB: it is a *non-constructive* proof:
  do we know if $S \neq \emptyset$? not necessarily…
  We only know that if $S \neq \emptyset$ *there exists* g_s : this is enough for us!

B) *S* is recursive if and only if both *S* and $S^\wedge = \mathcal{N}$-*S* are RE

B) equivalent to: (B.1 S recursive $\rightarrow$ both S and $S^\wedge$ RE) **and**
(B.2 both S and $S^\wedge$ RE $\rightarrow$ S recursive)

- B.1.1) S recursive $\rightarrow$ S RE (already proved in part A)
- B.1.2) S recursive $\rightarrow c_S(x)$ (= 1 *if* $x \in S$; = 0 *if* $x \notin S$) computable
  $\rightarrow c_{S^\wedge}(x)$ (= 0 *if* $x \in S$; = 1 *if* $x \notin S$) computable
  $\rightarrow S^\wedge$ recursive $\rightarrow S^\wedge$ RE
- B.2) S RE $\rightarrow$ construct the enumeration $S = \{g_S(0), g_S(1), g_S(2), g_S(3), \ldots\}$
  $S^\wedge$ RE $\rightarrow$ construct $S^\wedge = \{g_{S^\wedge}(0), g_{S^\wedge}(1), g_{S^\wedge}(2), g_{S^\wedge}(3), \ldots\}$
  But $S \cup S^\wedge = \mathcal{N}$, $S \cap S^\wedge = \emptyset$

  hence $\forall x \in \mathcal{N}$, x belongs to exactly one of the two enumerations $\rightarrow$
  The following enumeration

  $$\{g_S(0), g_{S^\wedge}(0), g_S(1), g_{S^\wedge}(1), g_S(2), g_{S^\wedge}(2), g_S(3), g_{S^\wedge}(3), \ldots\}$$

  certainly includes any x in exactly one position: if x is at an odd position, then $x \in S$, if it is at an even position then $x \in S^\wedge$. Hence c_S can be computed.

Other very important results

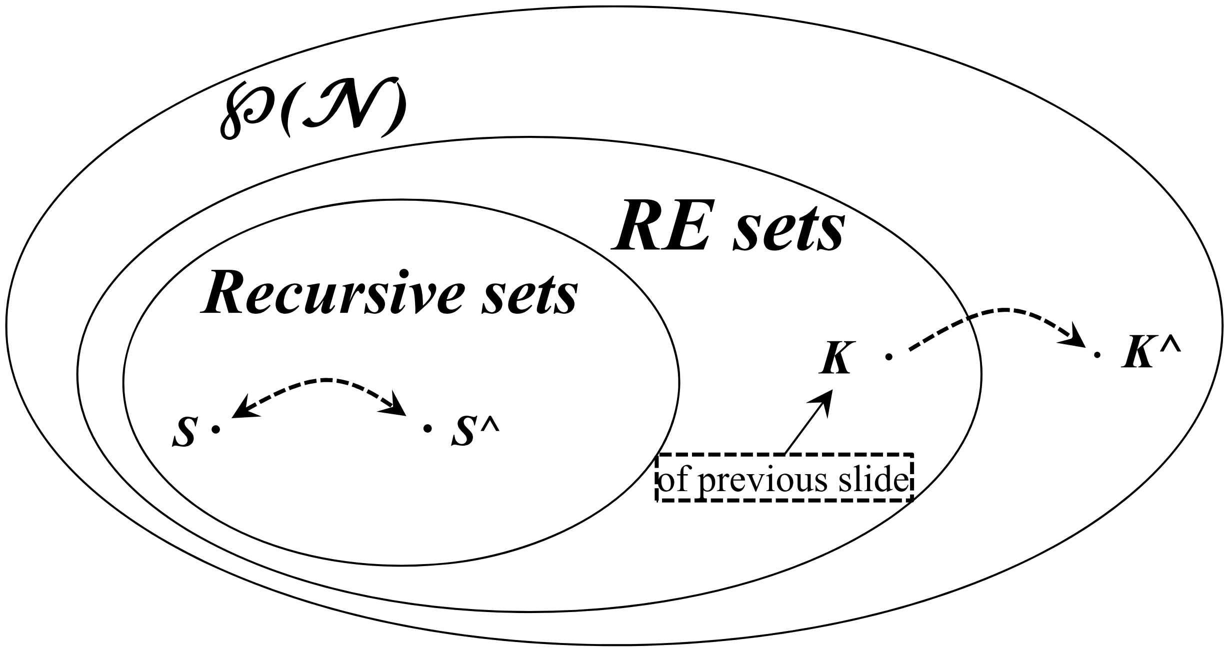
- $\boldsymbol{S}$ is RE $\leftrightarrow \boldsymbol{S} = \boldsymbol{D_h}$, with $\boldsymbol{h}$ computable and partial: $\boldsymbol{S} = \boldsymbol{D_h} = \{x \mid h(x) \neq \perp\}$
  and

  $\boldsymbol{S}$ is RE $\leftrightarrow \boldsymbol{S} = \boldsymbol{I_g}$, with $\boldsymbol{g}$ computable and partial: $\boldsymbol{S} = \boldsymbol{I_g} = \{x \mid \exists y \in \mathcal{N}: \ x = g(y)\}$
- Proof is omitted here: it uses a quite useful and significant technique
- The above theorem allows us to view RE sets as characterizing precisely the languages ***recognized/accepted*** by the Turing Machines

  (NB not ***decided*** **: decide** and **recognize/accept** differ slightly)
- It can also serve as a Lemma to prove that:
- There exist semidecidable sets that are not decidable:

  $K = \{x \mid f_x(x) \neq \perp\}$ is semidecidable because $K = D_h$ with $h(x) = f_x(x)$.
  We know however that the characteristic function of K,
  ($c_K(x) = 1$ *if* $f_x(x) \neq \perp$, 0 otherwise) is not computable (see function $h(x)$, slide 178)

  $\Rightarrow K$ is not decidable
- Conclusion:



BTW: notice that the relation «is the complement of» among sets is symmetric

Inclusions are all strict

Corollary: the class of RE sets (i.e., languages recognized by TM's) is *not* closed under complement

Why? Because if RE sets were closed under complement then all RE sets would also be recursive (also their complement would be RE…), but we know (from the previous slide) that this is not the case

[a brief digression]

Undecidability of the Halting problem as a key to **R ≠ RE**

- Undecidability of halting is a fundamental fact (not just a bizarre anomaly)
  – It is a key to the distinction between **R** and **RE**
- Theorem: if Halting was decidable, then **R = RE**
  – Suppose S is semidecidable: ⇒ (slide 189) ∃ a TM M_S that ***semidecides*** or ***recognizes*** it (slide 193), that is, for any x, M_S executed with input x
    - terminates and accepts if $x \in S$
    - does not terminate if $x \notin S$
  – if Halting decidable ⇒ possible to tell whether M_S would halt on x or not ***in advance*** (without actually executing M_S)
    - if (we know that) M_S halts on x ⇒ accept x (determine that $x \in S$)
    - if (we know that) M_S does not halt on x ⇒ reject x (determine that $x \notin S$)
  – therefore (under the assumption that halting is decidable) S would be not only RE but also R

The mighty Rice theorem

- Let $\boldsymbol{F}$ be a set of ***computable*** functions
  The set $\boldsymbol{S}$ of (the indices of) TM's that compute the functions of $\boldsymbol{F}$
  $$\boldsymbol{S} = \{ \boldsymbol{x} \mid \boldsymbol{f_x} \in \boldsymbol{F} \}$$
  is decidable if and only if $\boldsymbol{F} = \emptyset$ or $\boldsymbol{F}$ is the set of all computable functions
- $\Rightarrow$ (alas!) in *all non trivial cases* ($\exists f$ s.t. $f \in F$ and $\exists f$ s.t. $f \notin F$) S is not decidable!
- $\Rightarrow$ e.g., the following very interesting problems are unsolvable
  - Program correctness: does P solve a given problem (identified by a computable function f)? i.e., $F=\{f\}$, $P \approx M_p$ does $\boldsymbol{p}$ belong to the set $\boldsymbol{S} = \{ \boldsymbol{x} \mid \boldsymbol{f_x} \in \{\boldsymbol{f}\} \}$?
  - Program equivalence (given $\boldsymbol{M_y}$, same problem as above, with the set $\boldsymbol{F=\{f_y\}}$)
  - Does a program have any specified property concerning the function it computes (function with only even values in the image, function with a limited image, …)?
  - …
- There is an endless list of interesting problems whose unsolvability follows trivially from the Rice theorem

How can we, in practice, determine that a problem/set is (semi)decidable or not?

- If we find an algorithm that always terminates → decidable
- If we find an algorithm that may not terminate, but it always terminates when the answer is positive → semidecidable
- If we think that the problem/set is not (semi)decidable, how can we prove it?
- Do we have to build a new diagonal proof every time? … no way!
- There are easier means:
- A first, very powerful tool is the Rice theorem
  - Applies to many «problems» concerning computation, e.g., software features
- Though implicitly, we have already used another very natural and general technique:

Problem reduction

- If one has an algorithm to solve problem P one can use it to solve problem P':
  - TRIVIAL EXAMPLE: One can compute the *product* of two numbers *a* and *b* if one can compute the operations: *sum*, *difference*, *division by 2*, *square*. It suffices to use the formula $a \times b = ((a+b)^2 - a^2 - b^2)/2$. Hence *multiplication* is reduced to {*sum*, *difference*, *division by 2*, *square*}
  - In general if there is an algorithm that, given a instance of a problem P' builds its solution by producing (algorithmically) an instance of another problem P that is solvable, and such that from the solution of P one can obtain algorithmically that of P', then **P' has been *reduced to* P**.
  - Using the ***set inclusion formulation*** of a problem:
    - I want to solve $x \in S'$
    - I can solve $y \in S$ for all possible y
    - If I have a computable, total function t such that $x \in S' \leftrightarrow t(x) \in S$ then I can answer algorithmically the question $x \in S'$
    - i.e., I have reduced the problem $x \in S'$ to the problem $y \in S$

- The method can work also in the opposite way:
  - I want to know ***if*** I can solve $x \in S$
  - I know I cannot solve $y \in S'$ (S' is *not* decidable)
  - If I reduce S' to S, that is,
    I find a computable total function t such that $y \in S' \leftrightarrow t(y) \in S$
    then I can conclude that $x \in S$ is not solvable (otherwise I could show that $x \in S'$ is solvable by reducing it to $x \in S$)
- In fact, we already used implicitly this way of reasoning several times:
  - From the undecidability of the halting problem for the TM we derived in general the undecidability of the problem of termination of any computation on any computer; for instance, concerning the termination of C programs:
    - Consider a TM M_y and an integer x
    - I can build a C program, P, that simulates M_y and I can store x in an input file f
    - program P terminates its computation on file f if and only if $f_y(x) \neq \perp$
    - If I could decide if P terminates its computation on f then I could solve also the halting problem for the TM.

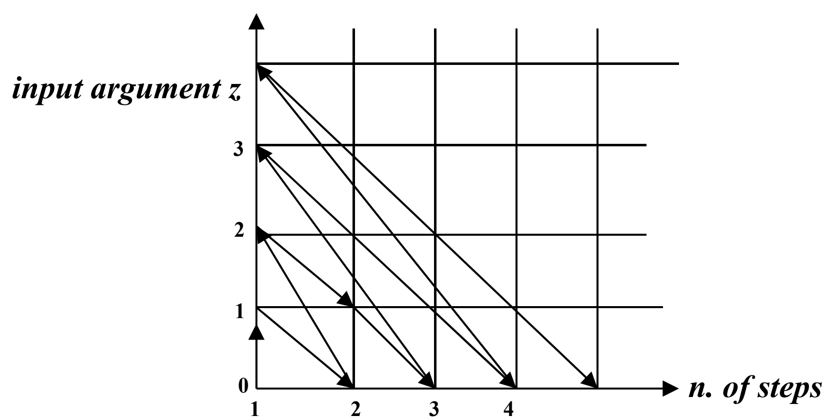
Reduction is a general, powerful technique

- Is the problem "does a generic program P access an uninitialized variable?" decidable?
  - Let us assume (by contradiction) that it is decidable and let us show by reduction that in that case the halting problem would be decidable
  - We can consider and instance P of the **halting** problem and **reduce** it **to** the following "**uninitialized variable access**" problem P^:
    - P^:
      ```
      begin var   x, y: …
                  P;
                  y := x
      end
      ```
      making sure that identifiers *x* and *y* are "fresh variables", not used in P
    - It is clear that the assignment statement y := x results in accessing an uninitialized variable, because *x* and *y* do not occur in P
    - Hence the uninitialized variable *x* is accessed in P^ if and only if P terminates.
  - Then if I could solve the problem of "diagnosis of uninitialized variable access" then I could solve also the termination problem, which cannot be.

- The same technique can be applied to prove the undecidability of many other typical properties of program execution:
  - Array indices out of bounds
  - Division by 0
  - Dynamic type compatibility
  - …
  - Typical ***run time errors***: concerning this issue ...

- Let us consider again the previous examples
  – halt of the TM
  – Division by 0 and other run-time errors, …
- The related sets are undecidable, but they are semidecidable:
  a. if the TM stops, soon or late I find it;
  b. if there exists any datum x, input of a program P, such that P, executed on input x, eventually executes a division by 0, then soon or late I can find it …
- Let us make a digression on the latter problem:
  – How can I ensure the above result (b.):
    - if I start to execute program P on x and P does not stop on x, how can I find that P, executed on $y \neq x$, will execute a division by 0?

- In general: Theorem (abstract formulation of the various concrete cases above):
  - The set of values x for which $\exists\, z$ such that $f_x(z) \neq \perp$ is semidecidable
    (NB: here the "input parameter" of the problem is the TM index x)
  - Sketch of the proof
    - If I compute $f_x(0)$ and find it $\neq \perp$ then I can answer;
    - But what if the computation of $f_x(0)$ does not terminate and $f_x(1) \neq \perp$: how can I find it?
    - Then I use the following trick, known as ***dovetailing***:
      - I simulate 1 execution step of $f_x(0)$: if it stops, then I have answered positively the question;
      - Otherwise I can simulate one computation step of $f_x(1)$;
      - Again if it does not stop I can simulate 2 steps of $f_x(0)$; next 1 step of $f_x(2)$; 2 steps of $f_x(1)$; 3 of $f_x(0)$; and so on, according to the scheme in the figure:



This way, if $\exists\, z$ s.t. $f_x(z) \neq \perp$, eventually I find it because eventually I will simulate enough computation steps of $f_x(z)$ to terminate

- Concluding the digression:
- we have therefore a significant number of problems/sets (typically, related to run time errors in programs) that are not decidable, but are semidecidable.
- We must however pay attention to which is precisely the ***semi***decidable problem:
  - detecting the ***presence*** of the error (i.e., if there is one I can find it)
    - NB: the semidecidable set is the set of erroneous programs
  - Not its absence! The set in question is that of error-free programs
- Notice however that, since the complement $\neg$**s** of a set **s** $\in$ (RE – R) is not even RE (otherwise they would both be decidable),

  The absence of errors (i.e., the ***correctness*** of a program with respect to an error) is not only not decidable, but it is not even semidecidable!
- Important implications on verification by ***testing***
  - Famous statement by Dijkstra: testing can prove the ***presence*** of errors, ***not*** their ***absence***

- Hence, as an additional result, we obtain a systematic technique to prove that a (unsolvable) problem is not RE: by proving that its complement is RE.

The complexity of computing

- We do not analyze individual algorithms (refer to courses on data structures and algorithms)
- We do not study advanced algorithms (refer to successive courses)
- Rather:
  - A critical examination of the problem and an approach to its solution
  - We strive for general principles
  - To be able to set individual problems in the correct framework

Complexity as a refinement of the notion of computability

- We are not satisfied to know that we can (algorithmically) solve a problem: we want to know how much it costs to solve it
- A critical analysis of the notion of costs (and benefits):
  - Cost of execution (necessary physical resources), in turn consisting of:
    - Time
      - compilation
      - execution
    - Space
  - Development cost
  - …
  - Objective and subjective evaluations, trade-off among contrasting goals …
  - … towards problems and approaches typical of Software Engineering
- We limit ourselves to notions of cost that are objective and can be formalized in a quantitative fashion: typical resources are ***memory*** and ***execution time***

We would like a hypothesis similar to the one on computability:

- The questions we ask and the answers we find do ***not*** depend on how we state the problem nor on the model we use to analyze it (Church Thesis).
- But:
  - Computing the sum in unary notation is quite different than in base k
  - Reducing the problem of computing a function $\tau(x)$ to that of deciding the language problem $x \in L_\tau = \{x\$y \mid y=\tau(x)\}$ could change completely the complexity: I might have to decide the language inclusion problem an unbound number of times to solve the original function-computing problem
  - Is it likely that, when the computer (or TM) changes, the execution times does not change? Obviously not, but...

- We will be able, by paying sufficient care, to obtain results that have a very broad validity ….
- … a sort of "Church Thesis for complexity"
- But for the moment …

… let us start with a complexity analysis for the TM

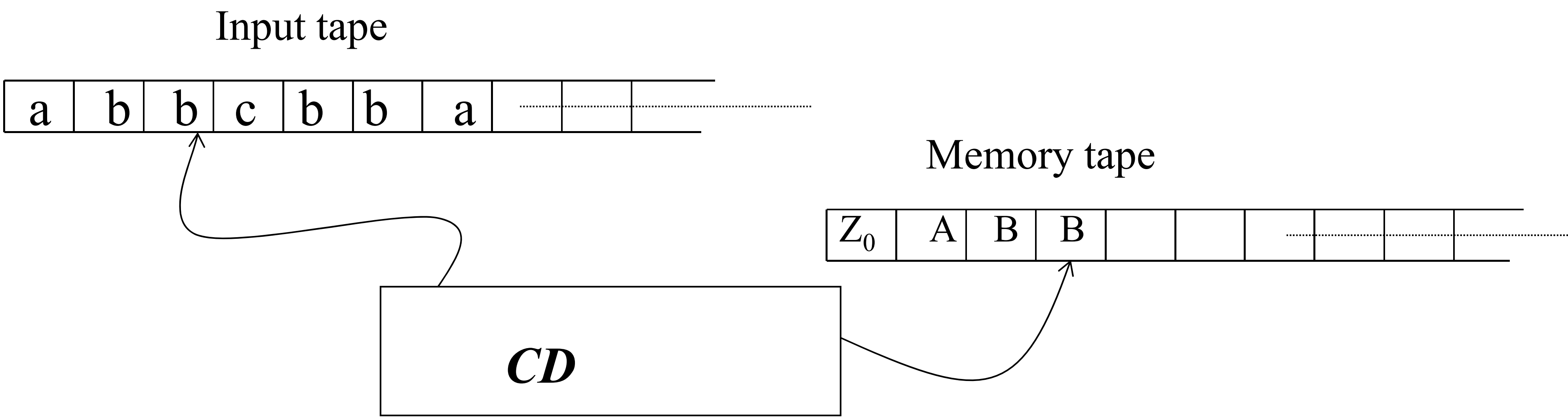
- Time complexity: let a computation be represented by a sequence of configurations (relation $\vdash$)
  $c = c_0 \vdash c_1 \vdash c_2 \vdash c_3 \ldots \vdash c_r$
  $T_M(x) = r$ if the computation stops at c_r, ∞ otherwise
- Space complexity:
  $c = c_0 \vdash c_1 \vdash c_2 \vdash c_3 \ldots \vdash c_r$

  $$S_M(x) = \sum_{j=1}^{k} \max\{|\alpha_{ij}|, i = 1,\ldots,r\} \quad \alpha_{ij} = \text{content of tape j at i - th step}$$

NB: *k* is the number of tapes; sum of maximal occupation of each tape, possibly at different times

- NB, very relevant fact: $\forall\, x \;\; S_M(x) \leq k \cdot T_M(x)$
  because $T_M(x)$ is also the distance the head can reach

A first example: accepting $\{wcw^R\}$, $w \in \{a,b\}^*$



$T_M(x) = |x| + 1$ if $x \in L$

$T_M(x) = |w|+1$ if $x = sa\overbrace{ucu^R}b\ p$ (=> the TM rejects x upon reading the b after u^R)

and $w = saucu^R$, with $u,s,p \in \{a,b\}^*$

$T_M(x) = |x| + 1$ if $x \in \{a,b\}^*$ i.e., x does not contain any c

…

$S_M(x) = |x|$ if $x \in \{a,b\}^*$, $\lfloor |x|/2 \rfloor$ if $x \in L$, ...

- Too many details …
- useful/necessary?

- Let us try to simplify and focus on the essentials:

- From complexity as ***f*(*x*)** (x input string) to complexity as ***f*(*n*)** (n “size” of the input datum x):
  ***n* = |*x*|** (string length), or rows/columns of a matrix, or number of records in a file, …
  But in general $|x_1| = |x_2|$ does not imply $T_M(x_1) = T_M(x_2)$ (idem for S_M), therefore ...
- if $|x_1| = |x_2|$ but $T_M(x_1) \neq T_M(x_2)$ which one do we choose?

- Choice of the **worst case**:
  $T_M(n) = \max\{T_M(x), |x| = n\}$ (idem for $S_M(n)$)
- Choice of the **average case**:

$$T_M(n) = \frac{\sum_{|x|=n} T_M(x)}{k^n}, \quad k = \text{cardinality of the alphabet}$$

- We will mostly adopt the worst case:
  - Most relevant from an engineering viewpoint (for certain applications)
  - Mathematically, the simplest (the average case should consider probabilistic hypotheses on data distribution: e.g., names on a phonebook are not equally likely)

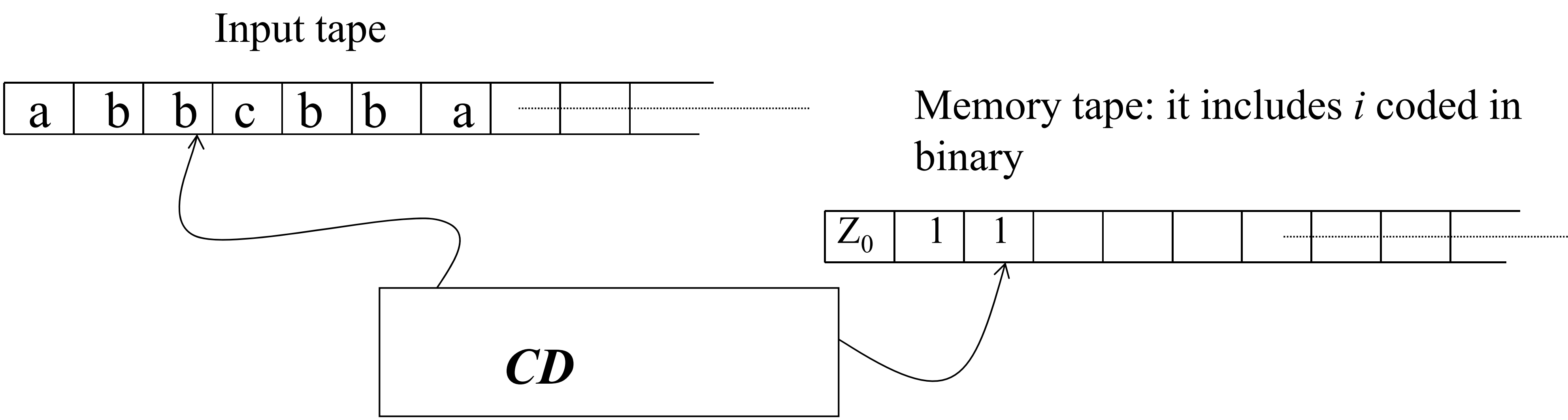
- We want a simple, concise, precise and practical way to indicate how a complexity function $f(n)$ grows with its argument n
- Use of the Θ notation to
  - estimate the ***dominant factor*** in the growth of a (complexity) function and its asymptotic behavior
  - Say when two different functions can be considered equivalent

$$f \,\Theta\, g \leftrightarrow \exists c \; such\; that \; \lim_{n\to\infty}\frac{f(n)}{g(n)} = c, c \neq 0, c \neq \infty$$

- Θ is an ***equivalence relation*** therefore…
- For any function $f(n)$ we may say that T_M is $\Theta(f)$, or $T_M \in \Theta(f)$
  ($\Theta(f)$) is viewed as an equivalence class) …
- For instance, taking $T_M(n)=f(n)=5n^3+3n^2$, we may say T_M is $\Theta(n^3)$, or $T_M \in \Theta(n^3)$
- (Saying that T_M is $\Theta(n)$ is like saying that $T_M(n)$ is linear?)
- Successively we will see that the Θ notation not only shows the dominant growth factor, but it also, in a way, describes the part that is “independent from the power of the computing device”

Let's return to the example $\{wcw^R\}$

- $T_M(n)$ is $\Theta(n)$, $S_M(n)$ is also $\Theta(n)$
- Can one do anything better?
- Concerning $T_M(n)$ it is difficult (in general one has to at least read all the input string)
- Concerning $S_M(n)$:



Store in the memory tape only the position i of the symbol to examine; then move the scanning head in position i and n-i+1 to compare the two read symbols ===>

- One gets: (let $n=|w|$)
- $S_M(n)$: $\Theta(\log(n))$ (input is ***not*** copied to the memory tape)
  but
- $T_M(n)$: $\Theta(n^2 \cdot \log(n))$: use 2 tapes T_1 (main counter) and T_2 (auxiliary counter),
  - $\forall i$, starting from 0
    - increment i (coded in binary on T_1) ($\Theta(\log(i))$ to implement $i := i+1$);
    - read next input symbol and store in the state
    - copy i on an auxiliary T_2 ($j := i$) ($\Theta(\log(i))$); then go to opposite end of input ($\Theta(i)$)
    - decrement (i times) j by 1 and move by 1 position the scanning head (towards the center) ($\Theta(i \cdot \log(i))$);
    - check for equality current input and symbol stored in the state
  - dominating factor is $\sum_{i=1}^{n} i \cdot \log(i) = n^2 \cdot \log(n)$
- A typical ***time-space trade-off***

- BTW: The example shows us why in the k-tape TM the scanning head can move in the two directions: otherwise one would lose important cases of sub-linear **spatial** complexity

Concerning k-tape TM ...

- Let us consider a change in the computing model:
  - For FA always $S_A(n)$ is $\Theta(k)$ and $T_A(n)$ is $\Theta(n)$, or even more precisely $T_A(n) = n$ (FA are ***real-time*** machines…);
  - For PDA always $S_A(n) \leq \Theta(n)$ and $T_A(n) \in \Theta(n)$ (number of ε-moves bounded *a priori*);
  - For ***single tape*** TM?
    - Accepting $\{wcw^R\}$ requires, at first sight, $\Theta(n^2)$ (head goes up and back n times)
    - Space complexity will never be $< \Theta(n)$
      (which provides an additional explanation of choosing k-tape TM as the principal model)
    - Can one do better than $\Theta(n^2)$? NO: the proof is technically complex as it is often the case with non-trivial complexity lower bounds.
    - (NB: a PDA accepts $\{wcw^R\}$ in $\Theta(n)$)

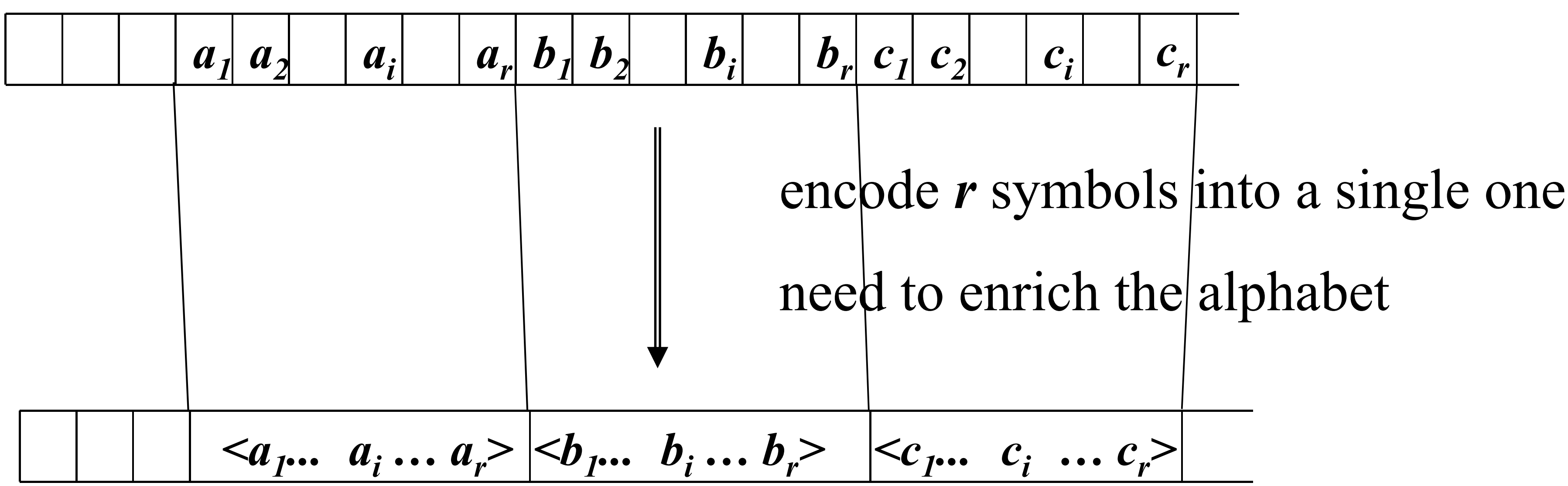
$\Rightarrow$

  - Hence, single tape **TM more powerful than PDA but sometimes less efficient**
  - What about von Neumann computers?
    We'll see in a while ...

Linear "speed-up" theorems

- If L is accepted by a k-tape TM M with complexity $S_M(n)$, then for each c > 0 one can build a k-tape TM M' with complexity $\mathbf{S_{M'}(n) < c \cdot S_M(n)}$

  (=> one can "reduce space")



$r \cdot c \geq 2$ **e.g. to reach half the complexity (c=1/2) one must take $r \geq 4$**

- If L is accepted by a **k-tape** TM M with space complexity $S_M(n)$, one can build a **1-tape** (NB: not a single tape) TM M’ with complexity $S_{M'}(n) = S_M(n)$.

  (=> one can “reduce #tapes”)

- If L is accepted by a k-tape TM M with space complexity $S_M(n)$, then for each c > 0 one can build a 1-tape TM M’ with complexity $S_{M'}(n) < c \cdot S_M(n)$.

  (=> one can “reduce space + #tapes”)

- If L is accepted by a k-tape TM M with ***time*** complexity $T_M(n)$, then for every $c > 0$ one can devise a (k+1-tape) TM M' with complexity
  $T_{M'}(n) = \max\{n+1, c \cdot T_M(n)\}$
  (=> one can "reduce time")
- Proof schema is the same as the one for spatial complexity (collapse *r* adjacent cells of M into one cell of M'), with some additional technical detail:
  - First, one must read and translate all the input (which requires *n* moves)
  - (This will create some problems within the class $\Theta(n)$)
  - Then M' can use a fixed number of moves (i.e., 6) to simulate groups of *r* moves of M
  - then by suitably choosing *r* one can reduce the complexity by an arbitrary (linear) factor

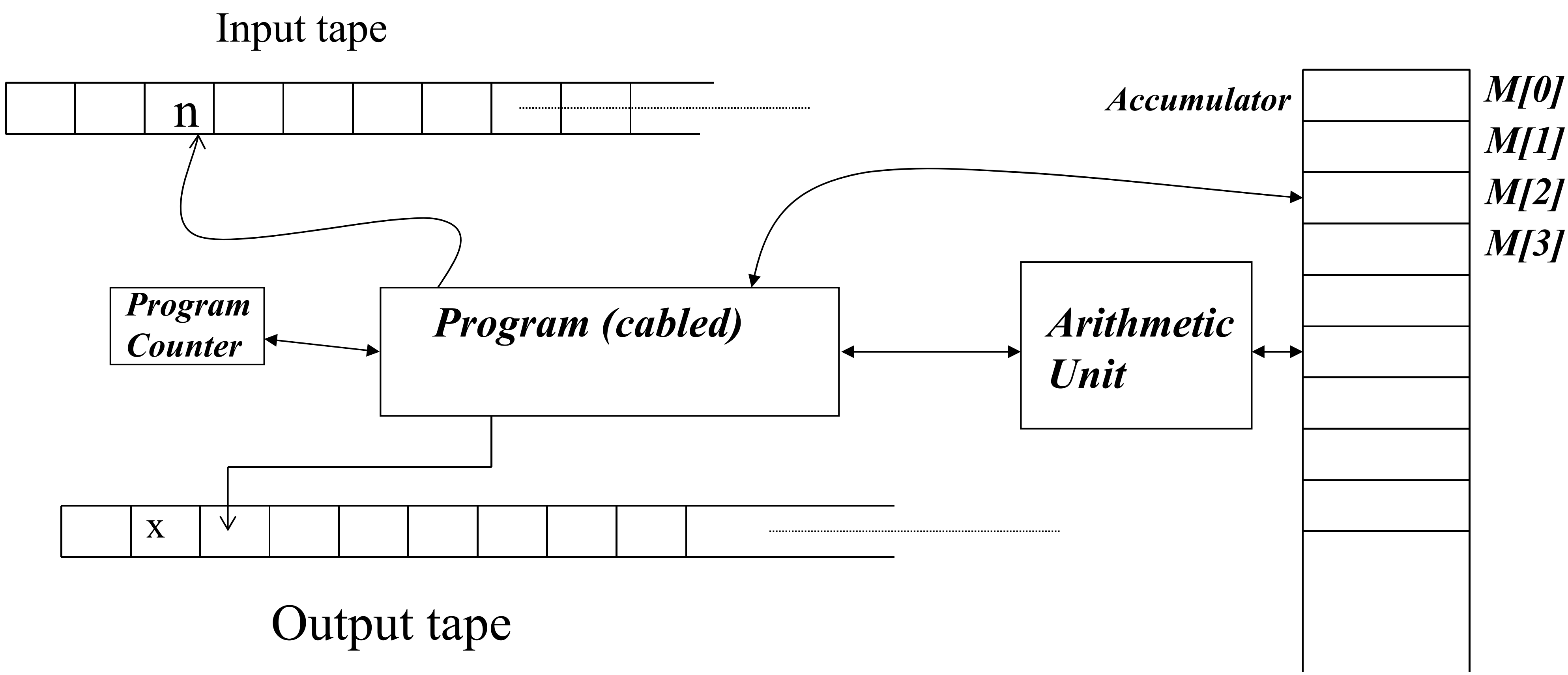
Practical implications of the linear speed-up theorems

- The proof scheme is valid for every computation model: also for real computers:
- This amounts to increasing the **physical parallelism** (from 16 bit to 32, to 64…)
- So, as long as one can increase the computing power in terms of available resources, one can increase *ad libitum* the execution speed
- This performance increase is however limited to improvements that are at most ***linear***: one cannot change the **order of magnitude** (i.e., the Θ class)
- Improvements in the order of magnitude can be obtained only by changing the algorithm and not in a mechanical/automatic way:
- E.g., for sufficiently large values of n sorting a sequence of n elements with merge sort will always be more efficient than sorting it through insertion sort or bubble sort, even if the more efficient algorithm is executed on a computer of modest power and the less efficient one on a supercomputer:
- Moral of the story: Brains can still overcome brute force!

Let us go back to comparing TM and real computers

- At first sight the comparison is unfair …:
  - To compute the sum of two numbers a TM needs $\Theta(n)$ (n is the length of –the string that encodes– the two numbers) while a computer provides this operation as an elementary operation (executed in a single machine step)
  - A computer can directly access any memory cell, while the TM has only a sequential access:
    - For instance, if we try to implement binary search through a TM we even get a complexity worse than linear, that is $\Theta(n.\log(n)) > \Theta(\mathrm{n})$
- We therefore cannot get along with complexity estimates that are bound only to the TM

A very abstract computer model: the RAM machine



Every cell contains an integer, not a symbol (NB!)

- The RAM instruction repertoir:

– LOAD [=, *] X	M[0] := M[X], X, M[M[X]]
– i.e., LOAD X	M[0] := M[X]
LOAD =X	M[0] := X
LOAD * X	M[0] := M[M[X]]

– STORE [*] X	M[X] := M[0], M[M[X]] := M[0]
– ADD [=, *] X	M[0] := M[0]+M[X], …
– SUB, MULT, DIV	…
– READ [*] X	
– WRITE [=, *] X	
– JUMP lab	PC := b(lab)
– JZ, JGZ, ... lab	
– HALT	

A RAM program computing the function
is_prime(n) = ***if*** *n is prime* ***then*** *1* ***else*** *0*

Consider in sequence all values from 1 to *n* and check whether they are divisors of *n*

	READ	1	The input value *n* is stored in cell M[1]
	LOAD=	1	If *n* = 1, it is trivially prime ...
	SUB	1	
	JZ	YES	
	LOAD=	2	the counter in M[2] is initialized to 2
	STORE	2	
LOOP:	LOAD	1	If M [1] = M[2] (i.e., counter = *n*) then *n* is prime
	SUB	2	
	JZ	YES	
	LOAD	1	If M [1] = (M[1] DIV M[2]) * M[2] then
	DIV	2	M[2] is a divisor of M[1];
	MULT	2	hence M[1] is not prime
	SUB	1	
	JZ	NO	
	LOAD	2	the counter in M[2] is incremented by 1 and the loop is repeated
	ADD=	1	
	STORE	2	
	JUMP	LOOP	
YES	WRITE=		1
	HALT		
NO	WRITE=		0
	HALT		

How much does it cost to execute the above RAM program?

- Obviously:
  - $S_R(n)$ is $\Theta(2)$
  - $T_R(n)$ is $\Theta(n)$
  - (Attention, however: what is n?? It is ***not*** the length of the input string! BTW, the input size is constant, independent of the value of n …
    Pay attention to the "data size" parameter!)
- Also obviously:
  - Accepting wcw^R:
    - $S_R(n)$ is $\Theta(n)$
    - $T_R(n)$ is $\Theta(n)$
  - Binary search: $T_R(n)$ is $\Theta(\log(n))$ (…assuming the set to be searched is in memory)
  - Sorting: …
  - … these complexity figures seem reasonable …
- However ...

Let us compute $2^{(2^n)}$ using a RAM (or any similar) machine

- read n;
- x := 2;
- for i :=1 to n do x := x*x;
- write x

- One gets $((2)^2)^2)$ … n times, i.e., 2^{2^n}

- What is the time complexity?
- $\Theta(n)$
- Are we sure?!
- How can it be? As a matter of fact just to ***write*** the result in binary one needs at least 2^n bits (hence time 2^n) !

- The above analysis is definitely not realistic!

What's the problem? The RAM (and also the von Neumann machine) is a bit too... **abstract**

- Can a memory cell (=memory unit) store an arbitrary number?
- Can an arithmetic operation (on ***any*** value) be a unit-cost elementary operation?
- This is correct only as long as the abstract machine is an adequate model of the actual machine (manipulated values are small compared to the numbers that can be encoded into 16, 32, 64, … bits)
- Otherwise … double precision etc. ---> the operations are not elementary any more and must be ***programmed***
- ---->
- Should we re-do all the algorithms and complexity analysis based on the adopted level of precision (i.e., number of bits)?
- Conceptually, yes, but in practice, and more easily:

- Logarithmic cost criterion: based on a "microscopic" analysis (see the "microcode") of the HW operations:

- What is the cost of copying the number i from a store cell to another one?
  As many elementary micro-operations as the number of bits necessary to encode i :
  $$\log(i)$$
- What is the cost of accessing the store cell in the i-th position?:
  the opening of $\log(i)$ access "gates" to the same number of memory blocks
- What is the cost of executing the operation
  LOAD i ?
- …
- By means of a simple, systematic analysis we get the following ...
  for brevity $l(x)$ stands for $log(x)$ (but $l(x)$ may as well be read as "***length***" of $\boldsymbol{x}$)

Table of logarithmic RAM costs

LOAD=	x	l(x)
LOAD	x	l(x) + l(M[x])
LOAD*	x	l(x) + l(M[x]) + l(M[M[x]])
STORE	x	l(x) + l(M[0])
STORE *	x	l(x) + l(M[x]) + l(M[0])
ADD=	x	l(M[0]) + l(x)
ADD	x	l(M[0]) + l(x) + l(M[x])
ADD *	x	l(M[0]) + l(x) + l(M[x]) + l(M[M[x]])
…		
READ	x	l(value of current input) + l(x)
READ*	x	l(value of current input) + l(x) + l(M[x])
WRITE=	x	l(x)
WRITE	x	l(x) + l(M[x])
WRITE *	x	l(x) + l(M[x]) + l(M[M[x]])
JUMP	lab	1
JGZ	lab	l(M[0])
JZ	lab	l(M[0])
HALT		1

Let us apply the new cost criterion

- To the computation of ***is-prime*(*n*)** (only main points)

M[1] stores *n*
M[2] stores the counter

LOOP:	LOAD	1	$1+l(n)$	
	SUB	2	$l(n) +2 + l(M[2])$	
	JZ	YES	$l(M[0])$	
	LOAD	1	$1+l(n)$	
	DIV	2	$l(n) +2 + l(M[2])$	
	MULT	2	$l(n/M[2]) +2 + l(M[2])$	$(< 2 \cdot l(n))$
	SUB	1	$l(M[0]) +1 + l(n)$	$(< 2 \cdot l(n) +1)$
	JZ	NO	$\ldots \leq l(n)$	
	LOAD	2	$\ldots \leq l(n) + k$	
	ADD=	1	...	
	STORE	2		
	JUMP	LOOP		

- In conclusion, one can easily put an upper bound on the cost of the individual loop iteration as $\Theta(\log(n))$
- Hence the overall time complexity is $\Theta(n \cdot \log(n))$

- Similarly we obtain:

- $\Theta(n.\log(n))$ for accepting wcw^R (NB: greater than the TM! Is it possible to do better?)
- $\Theta(\log^2(n))$ for binary search
- …
- ? overall (logarithmic criterion) cost = overall (constant criterion) cost * log(n)?
- ? overall (logarithmic criterion) cost = overall (constant criterion) cost * log(overall constant criterion cost)?
- Often, but not always:
- For the case of computing 2^{2^n} overall logarithmic cost $\geq 2^n$ (time complexity $\geq$ space complexity)
- Is there a criterion to choose which criterion to adopt?
  - Common sense (!):
  - If the computation does not change the order of magnitude of the input data, the initially (statically?) allocated memory may not change at run time ---> it does not depend on data ---> an individual cell can be considered elementary and with it all the relative operations ---> constant cost criterion is OK
  - Otherwise (computing a factorial, 2^{2^n}, "heavy" recursion, …) one needs the logarithmic criterion, the only one that is certainly correct.

Relations among the complexity measures obtained w.r.t. different computation models

- The same problem, solved with different kinds of machines can have different complexity
- It may happen that for P1 model M1 is better than model M2 but for P2 the opposite holds (binary search, direct access, sequential access and storing, accepting wcw^R)
- There is no model that is superior in all cases
- There is no result similar to the Church Thesis for complexity …
- However:
- It is possible to establish ***a priori*** at least a relation among the complexity of various computational models - an upperbound on the ratio of figures
- Polynomial Correlation Theorem (thesis) (analogy with the Church Thesis):
  - Under "reasonable" cost criterion hypotheses (the constant criterion for the RAM is not "reasonable" in all cases!) if a problem can be solved by a computation model M_1 with (space/time) complexity $\mathbf{C_1(\mathit{n})}$ then there exists a a suitable ***polynomial*** $\mathbf{P_2}$ such that the same problem can be solved by ***any*** other computation model M_2 with complexity $\mathbf{C_2(\mathit{n}) \leq P_2(C_1(\mathit{n}))}$

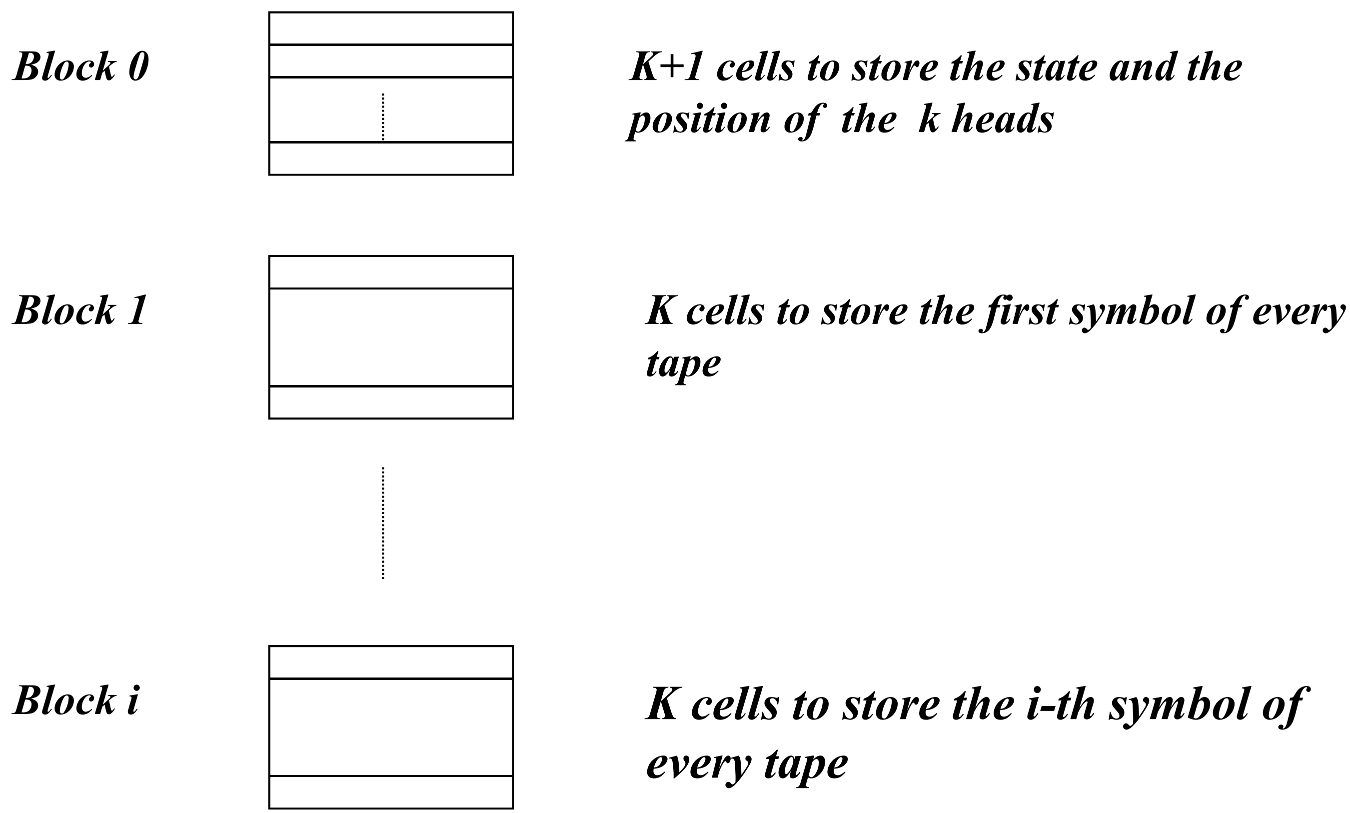
Before proving the theorem (not a thesis any more!) for the TM-RAM case, let us evaluate its impact:

- Though a polynomial can be as large as n^{1000}, it is always better than the exponential "abyss" ($\boldsymbol{n^k}$ against $\mathbf{2}^{\boldsymbol{n}}$)
- Thanks to the polynomial correlation theorem we can speak of the class of problems that can be solved in polynomial time/space (not of the quadratic ones!): this class is not affected by the adopted model (it is ***invariant*** w.r.t. the model)
- Thanks to this result –and other important theoretical facts– the following analogy has been adopted:
  - Class $\mathcal{P}$ of practically "tractable" problems = class of problems solvable in polynomial time
  - $\mathcal{P}$ includes also the problems with complexity n^{1000} (better, at any rate, than the exponential ones), but practical experience shows that the problems with any practical applications (searches, paths, optimizations, …) that are in $\mathcal{P}$ also admit solutions with an acceptable degree (exponent) in the polynomial
  - (similarly we will see shortly that the complexity relation between TM and RAM is "close")

(Time) correlation between TM and RAM:

1: How a RAM can simulate a (k-tape) TM

- The RAM memory simulates the TM memory:
  1 RAM cell for each cell of the TM tape
  However, instead of using blocks of adjacent RAM memory cells to simulate each tape, we associate a -k cells- block to each k-tuple of cells taken for each position of the tape, + a “base” block:



- A move of the TM is simulated by the RAM:

Block 0

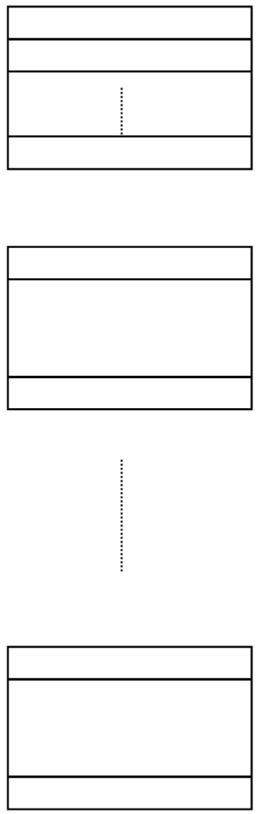
state and heads' positions

Block 1

k cells for first symbol on each tape

Block i

k cells for i-th symbol on each tape



- ***Reading:***
- The content of the 0 block is checked
  (a packet of k+1 accesses, c*(k+1) moves)
- k indirect accesses to k blocks to examine the content of the cells corresponding to the heads
- ***Writing:***
- The state is changed (in block 0)
- The content of the cells corresponding to the head positions are updated through indirect STORE operations
- The values of the k heads positions are updated (in block 0)

A move of the TM requires h*k RAM moves:

With a constant cost criterion: $\boldsymbol{T_R \in \Theta(T_M)}$

With a logarithmic cost criterion [the "serious" one]: $\boldsymbol{T_R \in \Theta(T_M.log(T_M))}$ (an indirect access to position *i* costs log(i), and the value of the position is bound by the time complexity, i.e., $\boldsymbol{S_M} \le \boldsymbol{T_M}$)

Time correlation between TM and RAM: (Theorem 3.17 on the textbook)
2: How a TM can simulate a RAM
(in a simple but central case: accepting languages (hence no output tape) without using MULT nor DIV: the generalization is trivial)

- The TM has 3 tapes: the first tape encodes the content of the RAM memory:

.....	**$**	i_j	#	$M[i_j]$	**$**	.....	**$**	i_k	#	$M[i_k]$	**$**	.....

- NB:
  - The various RAM cells are kept ordered (i.e., $i_j < i_{j+1}$)
  - Initially the tape is empty ---> at a generic time it includes only the cells that have been assigned a value (through a STORE)
  - i_j and $M[i_j]$ are represented in binary encoding
- 2 further tapes:
  - A tape includes the "accumulator register" M[0] (in binary)
  - A "service" tape

- How a RAM step is simulated by the TM:

…	$\$$	i_j	#	$M[i_j]$	$\$$	…	$\$$	i_k	#	$M[i_k]$	$\$$	…

- Let us consider a few representative examples:
- LOAD h:
  - Look for the value of h on the main tape (if it is not found: error)
  - The part next to h, M[h], it is copied into (the tape that stores) M[0]
- STORE h:
  - Look for h. If it is not found, “make a hole” using the service tape
  - Store h and copy M[0] in the part next to it (M[h]); copy the successive part from the service tape
  - If h already exists copy M[0] in the part next to it (M[h]); this can require the use of the service tape if the number of cells already occupied is not equal to that of M[0].
- ADD* h:
  - Look for h; look for M[h]; …
- With an easy generalization:
- ***Simulating a RAM move can take the TM a number of moves with an upper bound c*(length of the tape that stores the content of the RAM memory)***.

- Now a Lemma (Lemma 3.18 on the textbook):
  the length of the main tape has an upper bound equal to a function not greater than $\Theta(T_R)$

....	$\$$	i_j	#	$M[i_j]$	$\$$	....	$\$$	i_k	#	$M[i_k]$	$\$$	....

Each "i_j-th cell" of the RAM requires in the TM tape $l(i_j) + l(M[i_j])$ (+2) tape cells.

Each "i_j-th cell" exists in the tape if and only if the RAM has executed at least a STORE on it.

The STORE costs for the RAM is $l(i_j) + l(M[i_j])$ ---->

To fill r cells, of total length

$$\sum_{j=1,r} l(i_j) + l(M[i_J])$$

the <u>RAM</u> needs a time (w.r.t.* logarithmic *cost criterion) that is <u>at least</u> proportional to the same value.

Hence to simulate a RAM move, the TM needs a time ***at most*** $\Theta(T_R)$;

a RAM move costs *at least* 1;

if the RAM has complexity T_R, the TM executes *at most* T_R moves ---> the complete simulation of the RAM by the TM costs *at most* $\Theta(T^2_R)$.

Some remarks and concluding warnings

- Pay attention to the "data dimension" parameter:
  - Input string length (absolute value)
  - Value of the datum (n)
  - Number of elements of a table, of nodes in a graph, of rows in a matrix, …
  - …
  - There are relations among these parameters, but they are not always linear (the number n requires an input string of length $\log(n)$).
- Does binary search implemented on a TM (whose complexity is $\Theta(n.\log(n))$) violate the polynomial correlation theorem?
- Pay attention to the hypothesis: accepting a language ---> data not already in memory ---> complexity at least linear.
- *in practice* the average case is often used (Quicksort, compilation, …) but not for critical, real-time applications

A brief digression on some advanced –but extremely relevant– aspects of computational complexity

- Some important questions:
  - Are there any lower bounds on complexity?
  - By increasing the complexity class does the class of solvable problems (always) increase?
    (i.e., do I get more by spending more?)
  - Does there exist a sort of "universal complexity class" (all solvable problems belong to that class)?
  - Does it make any sense to consider the complexity of nondeterministic computing models? If yes, how can this complexity be defined?
  - Does the introduction of nondeterminism change the complexity of solving problems?
  - ...

Let us focus on the complexity of nondeterministic computations

- *First of all*: how can it be defined?
  - w.r.t. the fastest computation?
  - w.r.t. the slowest?
  - What if some computations terminate and others do not?
  - w.r.t. only the accepting computations?
  - The fastest computation among the accepting ones, … if there is any !
- What is the practical meaning of nondeterministic computations as long as real computations are deterministic?
- To answer, let us consider in general nondeterministic computations as a model for parallelism: "blind" search among various ways, …
- The great practical impact of this theme derives from the very fact that ….

- … that many problems of great practical impact have a simple, natural and “efficient” solution in nondeterministic terms:
  - Hamiltonian path on a graph (a path that touches all nodes exactly once)
  - Satisfiability of logical propositional formulas (→requirements verification of finite state systems)
  - ….
- What is common to the solution of all such problems is the ***difficulty of finding*** the solution*,* as opposed to the ***ease of checking*** that a possible (candidate) solution is in fact correct:
  - If a “little devil” would tell me “try this: see if this solution is correct”, then checking if the advice is right or not is not difficult ---->
  - Typical problems of this kind are solved exhaustively (i.e., by “trying all possibilities”) in a nondeterministic fashion: choose (ND) a possible solution (in polynomial, often linear complexity), then check if it is so (also polynomial complexity).
  - Obviously when one goes to the deterministic version, trying all ways is very onerous (combinatorial effects ⇒ exponential complexity or worse)
- Based on this approach one can derive a substantial class of problems (including the above examples and thousands of other ones):

- $\mathcal{NP}$: the class of problems that can be solved ***non***deterministically in polynomial time
- $\mathcal{P}$: the class of problems that can be solved deterministically in polynomial time (the ***tractable*** problems)
- A momentous question : $\mathcal{P} = \mathcal{NP}$?
- ***Most likely*** not. However, surprisingly … this crucial question has not been answered!
- If $\mathcal{P} = \mathcal{NP}$ we might solve efficiently an enormous quantity of problems that now are intractable, and must be addressed by means of heuristics, by special cases, etc.
- The notion of ($\mathcal{NP}$) ***completeness***: a “representative” of a class incorporates the essence of all problems of the class: if we find the solution for it we have it for all!
  - $\mathcal{NP}$ completeness is based on the existence of ***polynomial-time reductions***: a problem is reduced to another one by means of a polynomial-time transformation (total computable function)
- It is interesting to notice that in the enormous set of $\mathcal{NP}$ problems, a great number is also $\mathcal{NP}$-complete: it would suffice to solve one of then in (deterministic) polynomial time and we would have $\mathcal{P} = \mathcal{NP}$; it would suffice to prove that one of them is intractable and all other ones would also be so!

- Finally: nondeterministic is not a synonym for random, but … random computations are very effective and promising.