Machine Learning

RL in Finite Domains

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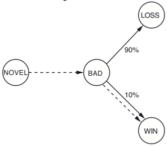
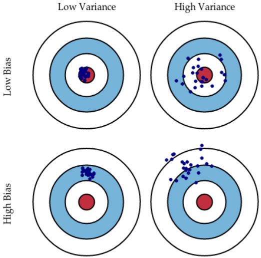




RL techniques

- Model–free vs Model–based
- On–policy vs Off–policy
- Online vs Offline
- Tabular vs Function Approximation
- Value–based vs Policy–based vs Actor–Critic

RL problems

- **Model–free Prediction**: Estimate the value function of an **unknown** MRP (MDP + policy)
- **Model-free Control**: Optimize the value function of an **unknown** MDP

Monte–Carlo Reinforcement Learning

- MC methods learn **directly** from episodes of **experience**
- MC is **model–free**: no knowledge of MDP transitions/rewards
- MC learns from **complete** episodes: no bootstrapping
- MC uses the simplest possible idea: **value = mean return**
- Caveat: can only apply MC to **episodic** MDPs
    - All episodes **must terminate**

Monte Carlo for Prediction and Control

- MC can be used for **prediction**:
  - **Input**: Episodes of experience $\{s_1, a_1, r_2, \ldots, s_T\}$ generated by following policy π in given MDP
  - or: Episodes of experience $\{s_1, a_1, r_2, \ldots, s_T\}$ generated by MRP
  - **Output**: Value function V^π
- Or for **control**:
  - **Input**: Episodes of experience $\{s_1, a_1, r_2, \ldots, s_T\}$ in given MDP
  - **Output**: Optimal value function V^*
  - **Output**: Optimal policy π^*

Estimation of Mean: Monte Carlo

- Let X be a random variable with mean $\mu = \mathbb{E}[x]$ and variance $\sigma^2 = Var[X]$. Let $x_i \sim X$, $i = 1, \dots, n$ be n i.i.d. realizations of X.
- **Empirical mean** of X:

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$$

- We have $\mathbb{E}[\hat{\mu}_n] = \mu$, $\text{Var}[\hat{\mu}_n] = \frac{\text{Var}[X]}{n}$
  - **Weak law of large numbers**: $\hat{\mu}_n \xrightarrow{P} \mu$ $(\lim_{n\to\infty} \mathbb{P}(|\hat{\mu}_n - \mu| > \epsilon) = 0)$
  - **Strong law of large numbers**: $\hat{\mu}_n \xrightarrow{a.s.} \mu$ $(\mathbb{P}(\lim_{n\to\infty} \hat{\mu}_n = \mu) = 1)$
  - **Central limit theorem**: $\sqrt{n}(\hat{\mu}_n - \mu) \xrightarrow{D} \mathcal{N}(0, \text{Var}[x])$

Monte–Carlo Policy Evaluation

- **Goal**: learn V^π from experience under policy π

$$s_1, a_1, r_2, \ldots, s_T \sim \pi$$

- Recall that the **return** is the total discounted reward:

$$v_t = r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^{T-1} r_{t+T}$$

- Recall that the **value function** is the expected return:

$$V^\pi(s) = \mathbb{E}[v_t | s_t = s]$$

- Monte Carlo policy evaluation uses **empirical mean** return instead of expected return
  - **first visit**: average returns only for the first time s is visited (**unbiased** estimator)
  - **every visit**: average returns for every time s is visited (**biased** but **consistent** estimator)

First–Visit Monte–Carlo Policy Evaluation

Initialize:
$\pi \leftarrow$ policy to be evaluated
$V \leftarrow$ an arbitrary state–value function
$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

loop
  Generate an episode using π
  for each state s in the episode **do**
    $R \leftarrow$ return following the first occurrence of s
    Append R to $Returns(s)$
    $V(s) \leftarrow$ average($Returns(s)$)
  end for
end loop

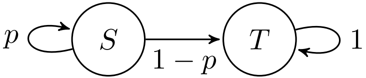
Every–Visit Monte–Carlo Policy Evaluation

Initialize:
$\pi \leftarrow$ policy to be evaluated
$V \leftarrow$ an arbitrary state–value function
$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

loop
  Generate an episode using π
  for each state s in the episode **do**
    for each occurrence of state s in the episode **do**
      $R \leftarrow$ return following this occurrence of s
      Append R to $Returns(s)$
      $V(s) \leftarrow$ average($Returns(s)$)
    end for
  end for
end loop

First–Visit vs Every–Visit

Model:

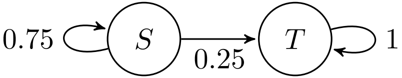


Reward is $+1$ on every step

Sample Path: $S \to S \to S \to S \to T$

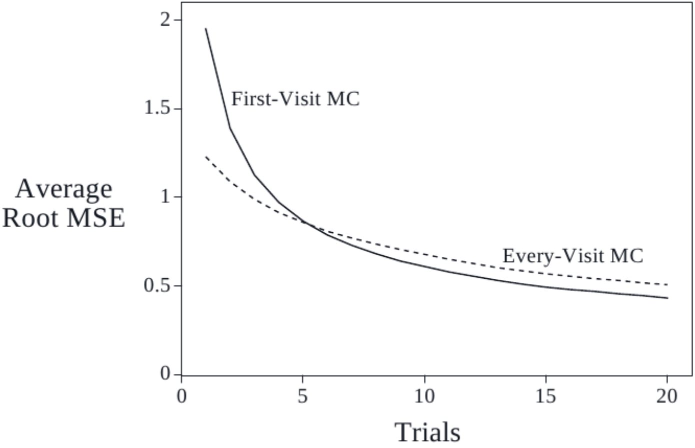
$$V^{FV}(S) = 4 \quad V^{EV}(S) = 2.5$$

Max Likelihood Model:



First–Visit vs Every–Visit

Crossover

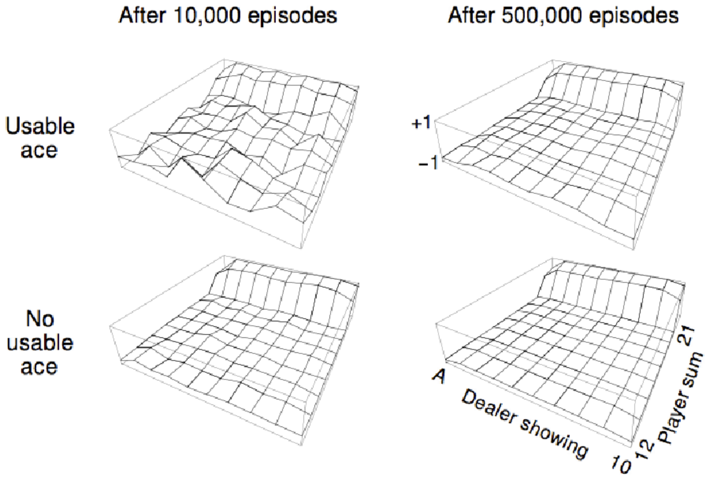


Blackjack Example

- **Goal**: Have your card sum be greater than the dealers without exceeding 21
- **States** (200 of them):
  - current sum (12–21)
  - dealer's showing card (ace–10)
  - do I have a usable ace?
- **Reward**: +1 for winning, 0 for a draw, -1 for losing
- **Actions**: stand (stop receiving cards), hit (receive another card)
- **Policy**: Stand if my sum is 20 or 21, else hit

Blackjack Example

After Monte–Carlo Learning



Incremental Mean

The mean $\hat{\mu}_1, \hat{\mu}_2, \dots$ of a sequence $x_1, x_2, \dots$ can be computed incrementally

$$\begin{aligned}
\hat{\mu}_k &= \frac{1}{k}\sum_{j=1}^{k} x_j \\
&= \frac{1}{k}\left(x_k + \sum_{j=1}^{k-1} x_j\right) \\
&= \frac{1}{k}\left(x_k + (k-1)\hat{\mu}_{k-1}\right) \\
&= \hat{\mu}_{k-1} + \frac{1}{k}\left(x_k - \hat{\mu}_{k-1}\right)
\end{aligned}$$

Incremental Monte–Carlo Updates

- Update $V(s)$ **incrementally** after episode $s_1, a_1, r_2, \dots, s_T$
- For each state s_t with return v_t

$$N(s_t) \leftarrow N(s_t) + 1$$

$$V(s_t) \leftarrow V(s_t) + \frac{1}{N(s_t)}(v_t - V(s_t))$$

- In **non–stationary** problems, it is useful to track a running mean, i.e., **forget** old episodes

$$V(s_t) \leftarrow V(s_t) + \alpha(v_t - V(s_t))$$

Stochastic Approximation

Estimation of Mean

Let X be a random variable in $[0, 1]$ with mean $\mu = \mathbb{E}[X]$. Let $x_i \sim X, i = 1, \ldots, n$ be n i.i.d. realizations of X.
Consider the estimator (**exponential average**)

$$\mu_i = (1 - \alpha_i)\mu_{i-1} + \alpha_i x_i,$$

with $\mu_1 = x_1$ and α_i's are **step–size parameters** or **learning rates**

Proposition

If $\sum_{i \geq 0} \alpha_i = \infty$ and $\sum_{i \geq 0} \alpha_i^2 < \infty$, then $\hat{\mu}_n \xrightarrow{a.s.} \mu$, i.e., the estimator $\hat{\mu}_n$ is **consistent**

Note: The step sizes $\alpha_i = \frac{1}{i}$ satisfy the above conditions. In this case, the exponential average gives us the empirical mean $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} x_i$, which is **consistent** according to the **strong law of large numbers**

Monte–Carlo Backups

- **Entire episode** included
- Only **one choice** at each state (unlike DP)
- MC **does not bootstrap**
- Time required to estimate one state **does not depend** on the total number of states

Temporal Difference Learning

- TD methods **learn directly** from episodes of experience
- TD is **model–free**: no knowledge of MDP transitions/rewards
- TD learns from **incomplete** episodes: **bootstrapping**
- TD updates a **guess** towards a **guess**

TD Prediction

- **Goal**: learn V^π online from experience under policy π
- **Recall**: incremental every–visit Monte Carlo

$$V(s_t) \leftarrow V(s_t) + \alpha(v_t - V(s_t))$$

- **Simplest** temporal–difference learning algorithm: TD(0)
  - Update value $V(s_t)$ towards **estimated return** $r_{t+1} + \gamma V(s_{t+1})$

$$V(s_t) \leftarrow V(s_t) + \alpha(r_{t+1} + \gamma V(s_{t+1}) - V(s_t))$$

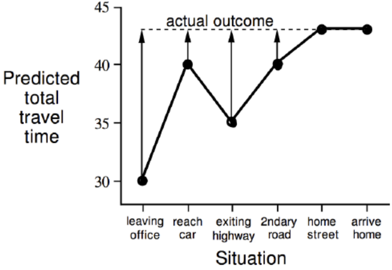
  - $r_{t+1} + \gamma V(s_{t+1})$ is called the **TD target**
  - $\delta_t = r_{t+1} + \gamma V(s_{t+1}) - V(s_t)$ is called the **TD error**

Driving Home Example

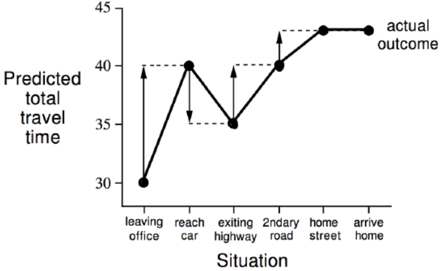
State	Elapsed Time (minutes)	Predicted Time to Go	Predicted Total Time
leaving office	0	30	30
reach car, raining	5	35	40
exit highway	20	15	35
behind truck	30	10	40
home street	40	3	43
arrive home	43	0	43

Driving Home Example: MC vd TD

Changes recommended by Monte Carlo methods ($\alpha = 1$)



Changes recommended by TD methods ($\alpha = 1$)



Comparison between MC and TD

- TD can learn **before** knowing the final outcome
  - TD can **learn online** after every step
  - MC must **wait until end of episode** before return is known
- TD can learn **without** the final outcome
  - TD can learn from **incomplete sequences**
  - MC can only learn form **complete sequences**
  - TD works in **continuing** (non–terminating) environments
  - MC only works for **episodic** (terminating) environments

Bias–Variance Trade–Off

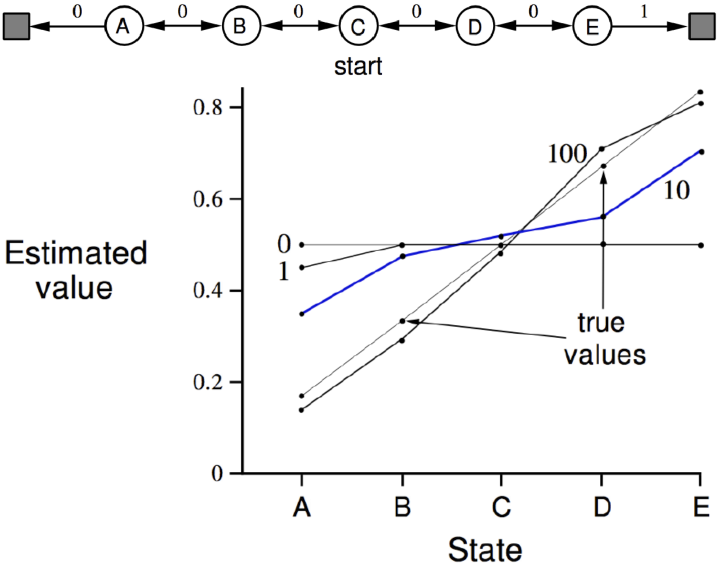
MC vs TD

- Return $v_t = r_{t+1} + \gamma r_{t+2} + \cdots + \gamma^{T-1} r_{t+T}$ is an **unbiased** esitmate of $V^\pi(s_t)$
- TD target $r_{t+1} + \gamma V(s_{t+1})$ is a **biased** estimate of $V^\pi(s_t)$
  - Unless $V(s_{t+1}) = V^\pi(s_{t+1})$
- But the TD target is much **lower variance**:
  - Return depends on **many** random actions, transitions, rewards
  - TD target depends on **one** random action, transition, reward

Bias–Variance comparison between MC and TD

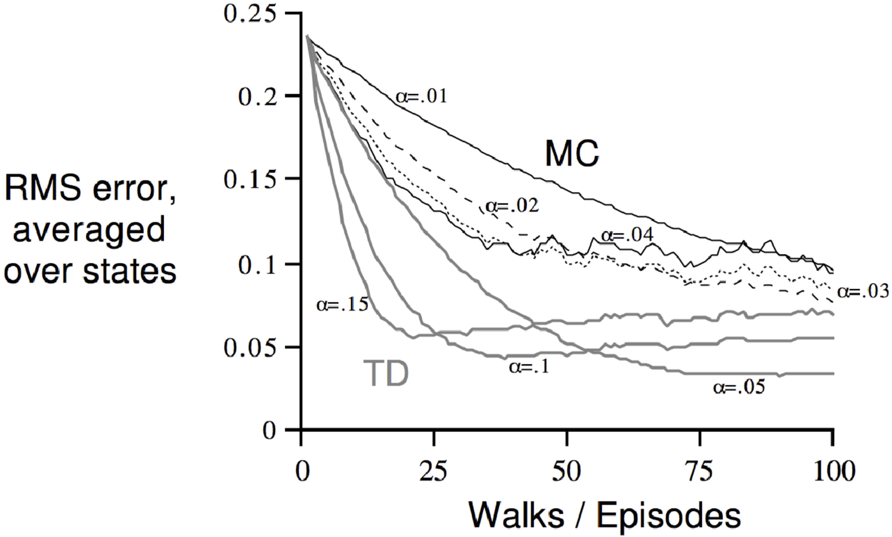
- MC has high variance, zero bias
  - **Good convergence** properties
  - Works well with **function approximation**
  - **Not** very sensitive to **initial value**
  - Very **simple** to understand and use
- TD has low variance, some bias
  - Usually **more efficient** than MC
  - TD(0) converges to $V^{\pi}(s)$
  - **Problem** with function approximation
  - More **sensitive** to initial values

Random Walk Example



Random Walk

MC vs TD



Comparison between MC and TD

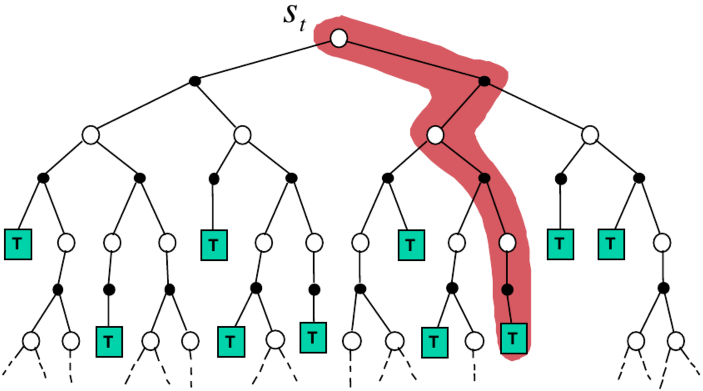
Markov Property

- TD **exploits** Markov property
    - Usually more efficient in **Markov environments**
- MC **does not exploit** Markov property
    - Usually more efficient in **non–Markov environments**

MC vs TD vs DP

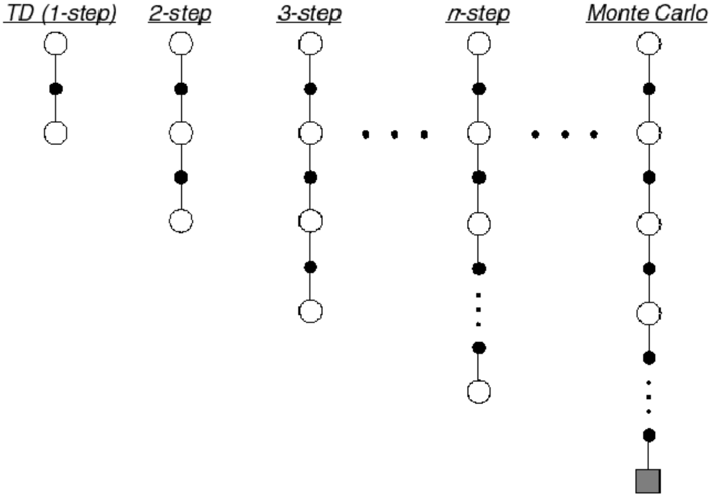
$$V(s_t) \leftarrow V(s_t) + \alpha[v_t - V(s_t)]$$

where R_t is the actual return following state s_t



n–Step Prediction

Let TD target look n steps into the future



n–Step Return

- Consider the following n–step returns for $n = 1, 2, \dots, \infty$:

$$\begin{array}{lll} n = 1 & (TD) & v_t^{(1)} = r_{t+1} + \gamma V(s_{t+1}) \\ n = 2 & & v_t^{(2)} = r_{t+1} + \gamma r_{t+2} + \gamma^2 V(s_{t+2}) \\ \vdots & & \vdots \\ n = \infty & (MC) & v_t^{(\infty)} = r_{t+1} + \gamma r_{t+2} + \dots + \gamma^{T-1} r_T \end{array}$$

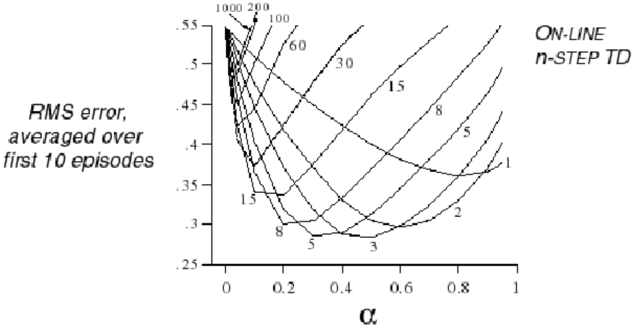
- Define the n–step return

$$v_t^{(n)} = r_{t+1} + \gamma r_{t+2} + \dots + \gamma^{n-1} r_{t+n} + \gamma^n V(s_{t+n})$$

- n–step temporal–difference learning

$$V(s_t) \leftarrow V(s_t) + \alpha(v_t^{(n)} - V(s_t))$$

Large Random Walk Example



Averaging n–step Returns

- We can **average** n–step returns over **different** n
- e.g., average the 2–step and 4–step returns

$$\frac{1}{2}v^{(2)} + \frac{1}{2}v^{(4)}$$

- **Combines information** from two different time–steps
- Can we **efficiently** combine information from all time–steps?

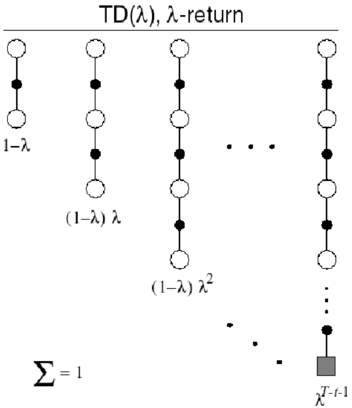
λ–return

- The λ–return v_t^λ combines **all** n–step returns $v_t^{(n)}$
- Using **weight** $(1-\lambda)\lambda^{n-1}$

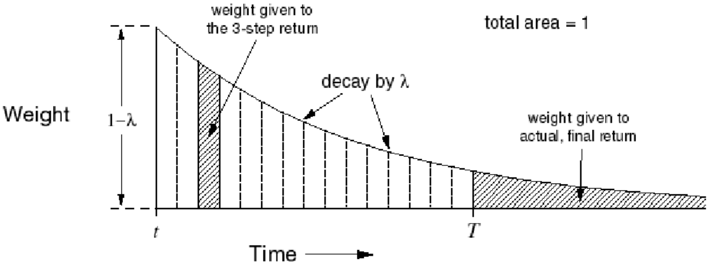
$$v_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty}\lambda^{n-1}v_t^{(n)}$$

- **Forward–view** TD(λ)

$$V(s_t) \leftarrow V(s_t) + \alpha\left(v_t^\lambda - V(s_t)\right)$$

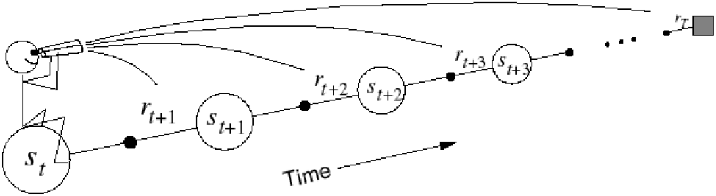


TD(λ) Weighting Function



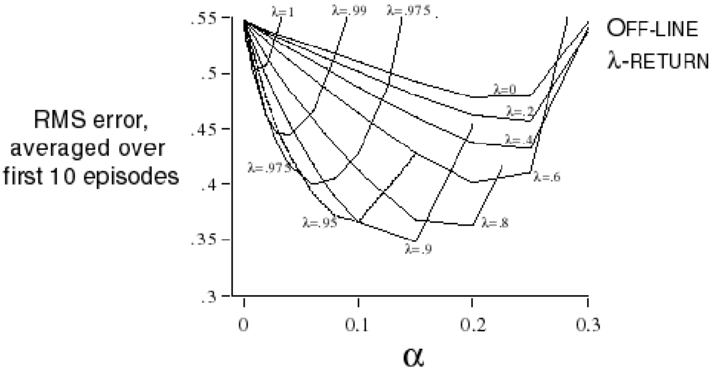
$$v_t^{\lambda} = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} v_t^{(n)}$$

Forward–view TD(λ)



- Update value function towards the λ**–return**
- Forward-view looks into the **future** to compute v_t^λ
- Like MC, can only be computed from **complete episodes**

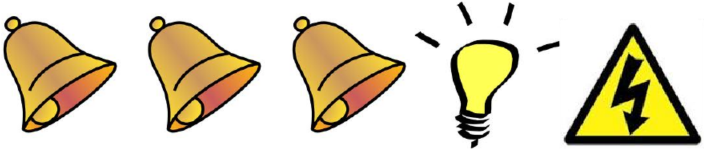
Forward–view TD(λ) on Large Random Walk



Backward–view TD(λ)

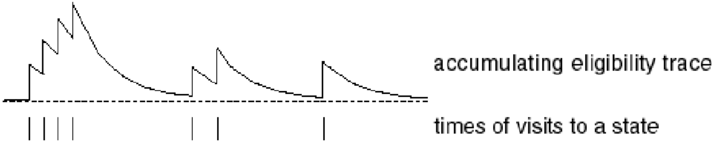
- Forward view provides **theory**
- Backward view provides **mechanism**
- Update **online**, every step, from **incomplete sequences**

Eligibility Traces



- **Credit assignment problem**: did bell or light cause shock?
- **Frequency heuristic**: assign credit to the most frequent states
- **Recency heuristics**: assign credit to the most recent states
- **Eligibility traces** combine both heuristics

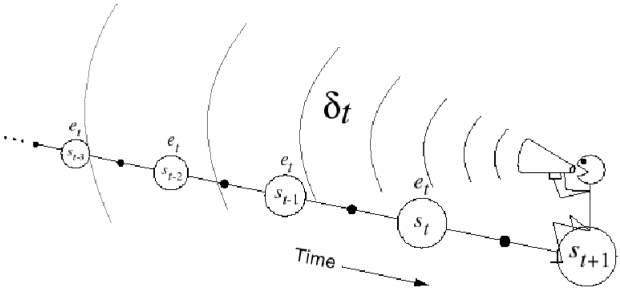
$$e_{t+1}(s) = \gamma\lambda e_t(s) + \mathbf{1}(s = s_t)$$



Backward–view TD(λ)

- **Update value** $V(s)$ for every state s
- In proportion to TD–error δ_t and **eligibility trace** $e_t(s)$

$$\begin{aligned} e_0(s) &= 0 \\ e_t(s) &= \gamma\lambda e_{t-1}(s) + \mathbf{1}(s = s_t) \\ V(s) &\leftarrow V(s) + \alpha\delta_t e_t(s) \end{aligned}$$



Backward–view TD(λ) Algorithm

Initialize $V(s)$ arbitrarily
for all episodes **do**
  $e(s) = 0, \quad \forall s \in \mathcal{S}$
  Initialize s
  repeat
    $a \leftarrow$ action given by π for s
    Take action a, observe reward r, and next state s'
    $\delta \leftarrow r + \gamma V(s') - V(s)$
    $e(s) \leftarrow e(s) + 1$
    for all $s \in \mathcal{S}$ **do**
      $V(s) \leftarrow V(s) + \alpha \delta e(s)$
      $e(s) \leftarrow \gamma \lambda e(s)$
    end for
    $s \leftarrow s'$
  until s is terminal
end for

TD(λ) and TD(0)

- When $\lambda = 0$, only current state is updated

$$\begin{aligned} e_t(s) &= \mathbf{1}(s = s_t) \\ V(s) &\leftarrow V(s) + \alpha \delta_t e_t(s) \end{aligned}$$

- This is exactly equivalent to TD(0) update

$$V(s_t) \leftarrow V(s_t) + \alpha \delta_t$$

Telescoping in TD(1)

When $\lambda = 1$, sum of TD errors telescopes into MC error

$$
\begin{aligned}
& \delta_t + \gamma\delta_{t+1} + \gamma^2\delta_{t+2} + \cdots + \gamma^{T-t}\delta_{T-1} \\
= \;& r_{t+1} + \gamma V(s_{t+1}) - V(s_t) \\
+ \;& \gamma r_{t+2} + \gamma^2 V(s_{t+2}) - \gamma V(s_{t+1}) \\
+ \;& \gamma^2 r_{t+3} + \gamma^3 V(s_{t+3}) - \gamma^2 V(s_{t+2}) \\
& \vdots \\
+ \;& \gamma^{T-1} r_{t+T} + \gamma^T V(s_{t+T}) - \gamma^{T-1} V(s_{t+T-1}) \\
= \;& r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots + \gamma^{T-1} r_{t+T} - V(s_t) \\
= \;& v_t - V(s_t)
\end{aligned}
$$

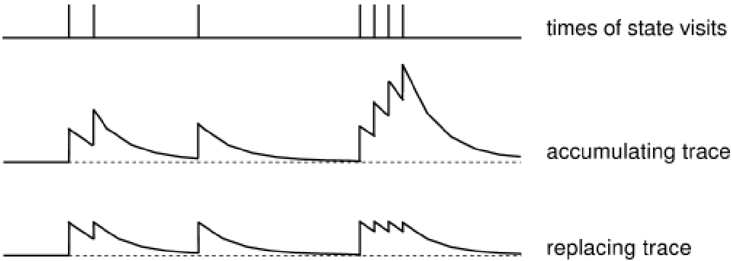
TD(λ) and TD(1)

- TD(1) is **roughly equivalent** to every–visit Monte–Carlo
- Error is accumulated online, **step–by–step**
- If value function is only **updated offline** at end of episode, then the total update is **exactly** the same as MC
- If value function is **updated online** after every step, then TD(1) may have **different** total update to MC

Replacing Traces

- Using **accumulating** traces, **frequently** visited states can have eligibilities greater than 1
  - This can be a **problem for convergence**
- **Replacing traces**: Instead of adding 1 when you visit a state, set that trace to 1

$$e_t(s) = \begin{cases} \gamma\lambda e_{t-1}(s) & \text{if } s \neq s_t \\ 1 & \text{if } s = s_t \end{cases}$$



Use of Model–Free Control

Some example problems that can be modeled as MDPs:

- Elevator
- Parallel Parking
- Ship Steering
- Bioreactor
- Helicopter
- Airplane Logistics
- Robocup Soccer
- Quake
- Portfolio management
- Protein folding
- Robot walking
- Game of Go

For most of these problems, either:

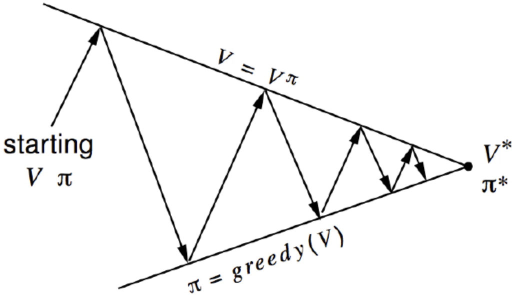
- MDP model is **unknown**, but experience can be **sampled**
- MDP model is **known**, but is **too big** to use, except by samples

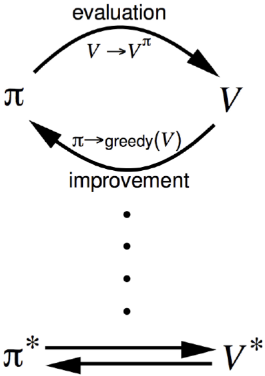
Model–free control can solve these problems

On and Off–Policy Learning

- **On–policy** learning
  - "Learn on the job"
  - Learn about policy π from experience sampled from π
- **Off–policy** learning
  - "Learn over someone's shoulder"
  - Learn about policy π from experience sampled from $\overline{\pi}$

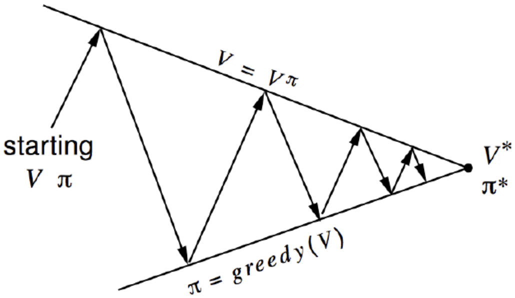
Generalized Policy Iteration (Refresher)





- **Policy evaluation**: Estimate V^π
  - e.g., Iterative policy evaluation
- **Policy improvement**: Generate $\pi' \geq \pi$
  - e.g., Greedy policy improvement

Generalized Policy Iteration with Monte–Carlo Evaluation



- **Policy Evaluation**: Monte–Carlo policy evaluation, $V = V^\pi$?
- **Policy Improvement**: Greedy policy improvement?

Model–Free Policy Iteration Using Action–Value Function

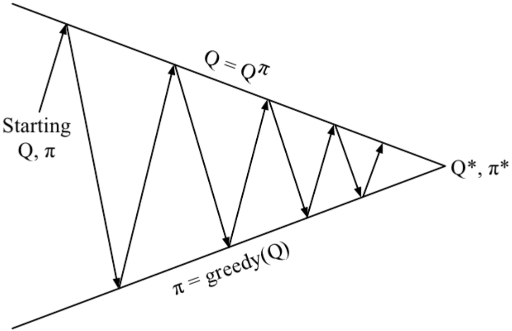
- Greedy policy improvement over $V(s)$ **requires model** of MDP

$$\pi'(s) = arg\max_{a\in\mathcal{A}} \left\{ R(s,a) + \gamma \sum_{s'\in\mathcal{S}} P(s'|s,a)V(s') \right\}$$

- Greedy policy improvement over $Q(s,a)$ is **model–free**

$$\pi'(s) = arg\max_{a\in\mathcal{A}} Q(s,a)$$

Generalized Policy Iteration with Monte–Carlo Evaluation



- **Policy Evaluation**: Monte–Carlo policy evaluation, $Q = Q^{\pi}$
- **Policy Improvement**: Greedy policy improvement?

On–Policy Exploration

- There are two doors in front of you
- You open the left door and get reward 0, $V(left) = 0$
- You open the right door and get reward $+1$, $V(right) = +1$
- You open the right door and get reward $+3$, $V(right) = +2$
- You open the right door and get reward $+2$, $V(right) = +2$

$\vdots$

-
- Are you sure you've chosen the **best** door?

ϵ–Greedy Exploration

- **Simplest** idea for ensuring **continual exploration**
- **All** m actions are tried with **non–zero** probability
- With probability $1-\epsilon$ choose the **greedy action**
- With probability ϵ choose an action **at random**

$$\pi(s,a) = \begin{cases} \frac{\epsilon}{m} + 1 - \epsilon & \text{if } a^* = arg\max_{a \in \mathcal{A}} Q(s,a) \\ \frac{\epsilon}{m} & \text{otherwise} \end{cases}$$

ϵ–Greedy Policy Improvement

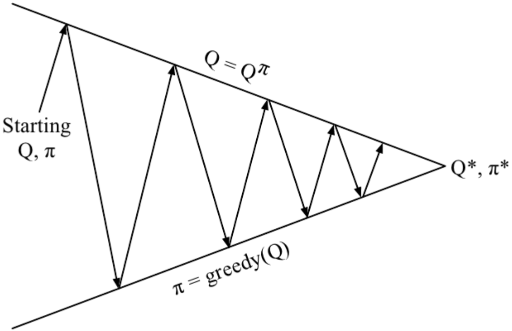
Theorem

For any ϵ–greedy policy π, the ϵ–greedy policy π' with respect to Q^π is an improvement

$$
\begin{aligned}
Q^\pi(s, \pi'(s)) &= \sum_{a \in \mathcal{A}} \pi'(a|s) Q^\pi(s,a) \\
&= \frac{\epsilon}{m} \sum_{a \in \mathcal{A}} Q^\pi(s,a) + (1-\epsilon) \max_{a \in \mathcal{A}} Q^\pi(s,a) \\
&\geq \frac{\epsilon}{m} \sum_{a \in \mathcal{A}} Q^\pi(s,a) + (1-\epsilon) \sum_{a \in \mathcal{A}} \frac{\pi(a|s) - \frac{\epsilon}{m}}{1-\epsilon} Q^\pi(s,a) \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) Q^\pi(s,a) = V^\pi(s)
\end{aligned}
$$

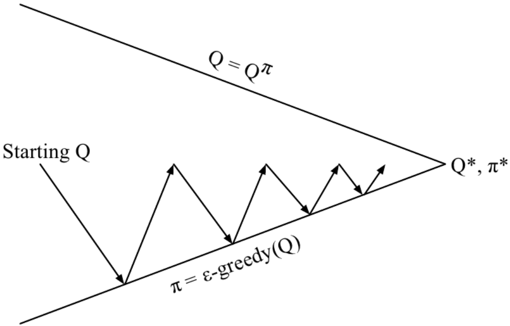
Therefore from policy improvement theorem, $V^{\pi'(s)} \geq V^\pi(s)$

Monte–Carlo Policy Iteration



- **Policy Evaluation**: Monte–Carlo policy evaluation, $Q = Q^\pi$
- **Policy Improvement**: ϵ–greedy policy improvement

Monte–Carlo Control



Every episode:

- **Policy Evaluation**: Monte–Carlo policy evaluation, $Q \approx Q^\pi$
- **Policy Improvement**: ϵ–greedy policy improvement

GLIE

Definition

Greedy in the Limit of Infinite Exploration (GLIE)

- **All** state–action pairs are explored **infinitely** many times

$$\lim_{k \to \infty} N_k(s, a) = \infty$$

- The policy **converges** on a **greedy** policy

$$\lim_{k \to \infty} \pi_k(a|s) = \mathbf{1}(a = arg \max_{a' \in \mathcal{A}} Q_k(s', a'))$$

GLIE Monte–Carlo Control

- Sample k–th episode using π: $\{s_1, a_1, r_2, \ldots, s_T\} \sim \pi$
- For each state s_t and action a_t in the episode,

$$N(s_t, a_t) \leftarrow N(s_t, a_t) + 1$$

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \frac{1}{N(s_t, a_t)}(v_t - Q(s_t, a_t))$$

- **Improve policy** based on new action–value function

$$\epsilon \leftarrow \frac{1}{k}$$

$$\pi \leftarrow \epsilon\text{–greedy}(Q)$$

Theorem

GLIE Monte–Carlo control ***converges*** *to the* ***optimal*** *action–value function,* $Q(s, a) \rightarrow Q^*(s, a)$

Relevant Time Scales

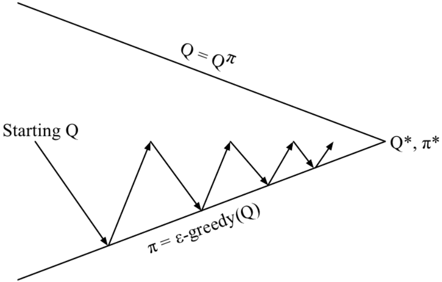
- There are three main time scales
  1. **Behavioral** time scale $\frac{1}{1-\gamma}$ (**discount factor**)
  2. **Sampling** in the estimation of the Q–function α (**learning rate**)
  3. **Exploration** ϵ (e.g., for ϵ–greedy strategy)
- $1-\gamma \gg \alpha \gg \epsilon$
- **Initially** $1-\gamma \approx \alpha \approx \epsilon$ is possible
- Then decrease ϵ **faster** than α
- **Practically**, you can choose number of trials $M < \infty$ and set $\alpha \sim 1 - \frac{m}{M}$ and $\epsilon \sim \left(1 - \frac{m}{M}\right)^2, \quad m = 1, \dots, M$
- In some cases, γ should be **initialized to low values** and then gradually moved towards its correct value

MC vs TD Control

- Temporal–Difference (TD) learning has several **advantages** over Monte–Carlo (MC)
  - Lower Variance
  - Online
  - Incomplete sequences
- Natural idea: use **TD** instead of MC in our **control loop**
  - Apply TD to $Q(s, a)$
  - Use ϵ–greedy policy improvement
  - Update every time–step

On–Policy Control with SARSA

$$Q(s,a) \leftarrow Q(s,a) + \alpha(r + \gamma Q(s',a') - Q(s,a))$$



Every time–step:

- **Policy Evaluation**: SARSA, $Q \approx Q^\pi$
- **Policy Improvement**: ϵ–greedy policy improvement

SARSA Algorithm for On–Policy Control

Initialize $Q(s,a)$ arbitrarily
loop
  Initialize s
  Choose a from s using policy derived from Q (e.g., ϵ–greedy)
  repeat
    Take action a, observe r, s'
    Choose a' from s' using policy derived from Q (e.g., ϵ–greedy)
    $Q(s,a) \leftarrow Q(s,a) + \alpha[r + \gamma Q(s',a') - Q(s,a)]$
    $s \leftarrow s'; a \leftarrow a';$
  until s is terminal
end loop

Convergence of SARSA

Theorem

SARSA converges to the optimal action–value function, $Q(s,a) \to Q^*(s,a)$, under the following conditions:

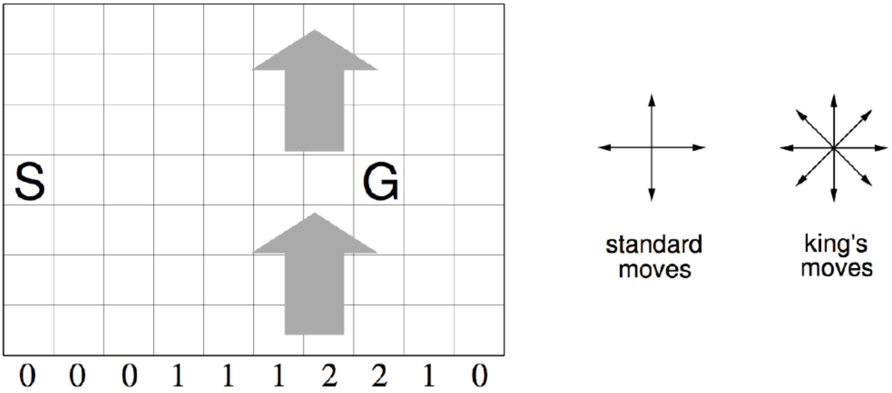
- *GLIE sequence of policies $\pi_t(s,a)$*
- *Robbins–Monro sequence of step–sizes α_t*

$$\sum_{t=1}^{\infty} \alpha_t = \infty$$

$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

SARSA Example

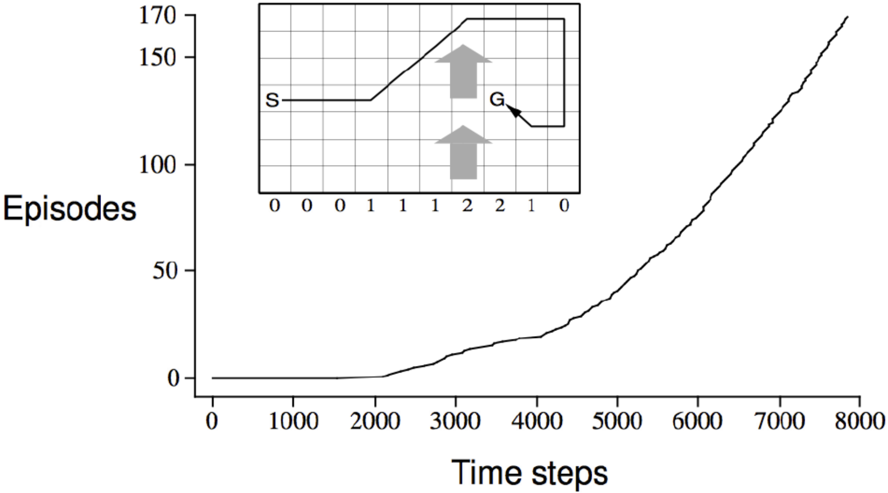
Windy Gridworld



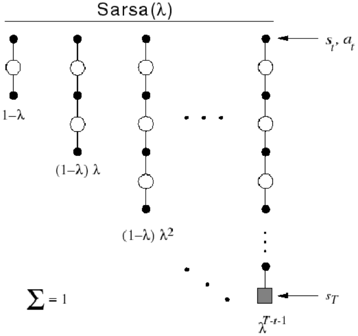
undiscounted, episodic, reward = −1 until goal

SARSA Example

Results in the Windy Gridworld



SARSA with Eligibility Traces



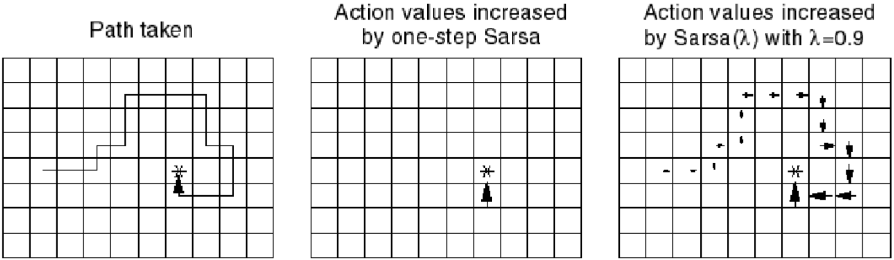
- **Forward view**: update action–value $Q(s, a)$ to λ–return v_t^{λ}
- **Backward view**: use eligibility traces for state–action pairs

$$e_t(s, a) = \gamma\lambda e_{t-1}(s, a) + \mathbf{1}(s_t, a_t = s, a)$$

SARSA(λ) Algorithm

Initialize $Q(s, a)$ arbitrarily
loop
  $e(s, a) = 0$, for all s, a
  Initialize s, a
  repeat
    Take action a, observe r, s'
    Choose a' from s' using policy derived from Q (e.g., ϵ–greedy)
    $\delta \leftarrow r + \gamma Q(s', a') - Q(s, a)$
    $e(s, a) \leftarrow e(s, a) + 1$
    for all s, a **do**
      $Q(s, a) \leftarrow Q(s, a) + \alpha\delta e(s, a)$
      $e(s, a) \leftarrow \gamma\lambda e(s, a)$
    end for
    $s \leftarrow s'; a \leftarrow a';$
  until s is terminal
end loop

SARSA(λ) Gridworld Example



Off–Policy Learning

- Learn about **target policy** $\pi(a|s)$
- While following **behavior policy** $\overline{\pi}(a|s)$
- Why is this important?
  - Learn from **observing** humans or other agents
  - **Re–use** experience generated from old policies $\pi_1, \pi_2, \ldots, \pi_{t-1}$
  - Learn about **optimal** policy while following **exploratory** policy
  - Learn about **multiple** policies while following **one** policy

Importance Sampling

Estimate the expectation of a **different** distribution w.r.t. the distribution used to **draw samples**

$$\begin{aligned} \mathbb{E}_{x\sim P}[f(x)] &= \sum P(x)f(x) \\ &= \sum Q(x)\frac{P(x)}{Q(x)}f(x) \\ &= \mathbb{E}_{x\sim Q}\left[\frac{P(x)}{Q(x)}f(x)\right] \end{aligned}$$

Importance Sampling for Off–Policy Monte–Carlo

- Use returns **generated** from $\overline{\pi}$ to **evaluate** π
- Weight return v_t according to **similarity** between policies
- Multiply **importance sampling corrections** along whole episode

$$v_t^{\mu} = \frac{\pi(a_t|s_t)}{\overline{\pi}(a_t|s_t)} \frac{\pi(a_{t+1}|s_{t+1})}{\overline{\pi}(a_{t+1}|s_{t+1})} \dots \frac{\pi(a_T|s_T)}{\overline{\pi}(a_T|s_T)} v_t$$

- Update value towards **corrected** return

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha({\color{red}v_t} - Q(s_t, a_t))$$

- **Cannot use** if $\overline{\pi}$ is zero where π is non–zero
- Importance sampling can dramatically **increase variance**

Importance Sampling for Off–Policy Monte–Carlo

Derivation

Off–policy MC is derived from **expected return**:

$$\begin{aligned}
Q^\pi(s,a) &= \mathbb{E}_\pi[v_t|s_t = s, a_t = a] \\
&= \sum \mathbb{P}[s_1, a_1, r_2, \dots, s_T] v_t \\
&= \sum \mathbb{P}[s_1] \left(\Pi_{t=1}^{T} \overline{\pi}(s_t, a_t) P(s_t|s_{t-1}, a_{t-1}) \frac{\pi(s_t|a_t)}{\overline{\pi}(s_t, a_t)} \right) v_t \\
&= \mathbb{E}_{\overline{\pi}} \left[\Pi_{t=1}^{T} \frac{\pi(s_t, a_t)}{\overline{\pi}(s_t, a_t)} v_t | s_t = s, a_t = a \right]
\end{aligned}$$

Off–Policy MC Control

Initialize, for all $s \in \mathcal{S}, a \in \mathcal{A}$:
$Q(s,a) \leftarrow$ arbitrary
$N(s,a) \leftarrow 0$
$D(s,a) \leftarrow 0$
$\pi \leftarrow$ an arbitrary deterministic policy
loop
  Using a policy $\overline{\pi}$, generate an episode $s_0, a_0, r_1, s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T, s_T$
  $\tau \leftarrow$ latest time at which $a_\tau \neq \pi(s_\tau)$
  for all pair s, a appearing in the episode after τ **do**
    $t \leftarrow$ the time of first occurence (after τ) of s, a
    $w \leftarrow \Pi_{k=t+1}^{T-1} \dfrac{1}{\overline{\pi}(s_k, a_k)}$
    $N(s,a) \leftarrow N(s,a) + wR_t$
    $D(s,a) \leftarrow D(s,a) + w$
    $Q(s,a) \leftarrow \frac{N(s,a)}{D(s,a)}$
  end for
  for all $s \in \mathcal{S}$ **do**
    $\pi(s) \leftarrow arg\max_{a \in \mathcal{A}} Q(s,a)$
  end for
end loop

Importance Sampling for Off–Policy SARSA

- Use **TD targets** generated from π to evaluate $\overline{\pi}$
- **Weight** TD target $r + \gamma Q(s', a')$ according to **similarity** between policies
- Only need a **single** importance sampling correction

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha \left(r_{t+1} + \gamma \frac{\pi(a_{t+1}|s_{t+1})}{\overline{\pi}(a_{t+1}|s_{t+1})} Q(s_{t+1}, a_{t+1}) - Q(s_t, a_t) \right)$$

- Much **lower variance** than Monte–Carlo importance sampling
- Policies only need to be similar over a **single step**

Importance Sampling for Off–Policy SARSA

Bellman expectation equation

Off–Policy SARSA comes from Bellman expectation equation for $Q^\pi(s,a)$

$$
\begin{aligned}
Q^\pi(s,a) &= \mathbb{E}_\pi\left[r_{t+1} + \gamma Q^\pi(s_{t+1}, a_{t+1}) | s_t = s, a_t = a\right] \\
&= R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a) \sum_{a' \in \mathcal{A}} \pi(a'|s') Q^\pi(s', a') \\
&= R(s,a) + \gamma \sum_{s' \in \mathcal{S}} P(s'|s,a) \sum_{a' \in \mathcal{A}} \overline{\pi}(a'|s') \frac{\pi(a'|s')}{\overline{\pi}(a'|s')} Q^\pi(s', a') \\
&= \mathbb{E}_\mu\left[r_{t+1} + \gamma \frac{\pi(a_{t+1}|s_{t+1})}{\overline{\pi}(a_{t+1}|s_{t+1})} Q^\pi(s_{t+1}, a_{t+1}) | s_t = s, a_t = a\right]
\end{aligned}
$$

Off–Policy Control with Q–learning

- Learn about **optimal policy** $\pi = \pi^*$
- From experience sampled from **behavior policy** $\overline{\pi}$
- Estimate $Q(s, a) \approx Q^*(s, a)$
- Behavior policy **can depend on** $Q(s, a)$
  - e.g., $\overline{\pi}$ could be ϵ–greedy with respect to $Q(s, a)$
  - As $Q(s, a) \to Q^*(s, a)$, behavior policy $\overline{\pi}$ **improves**

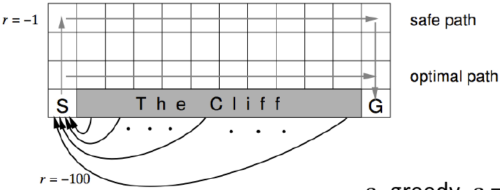
Q–learning Algorithm

$$Q(s,a) \leftarrow Q(s,a) + \alpha(r + \gamma \max_{a' \in \mathcal{A}} Q(s',a') - Q(s,a))$$

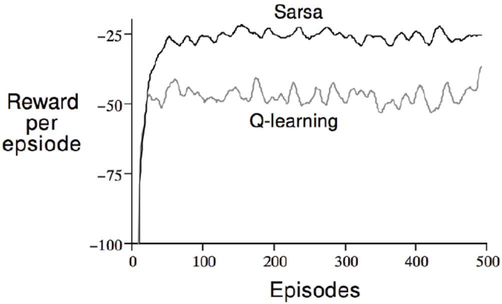
Initialize $Q(s,a)$ arbitrarily
loop
    Initialize s
    repeat
        Choose a from s using policy derived from Q (e.g., ϵ–greedy)
        Take action a, observe r, s'
        $Q(s,a) \leftarrow Q(s,a) + \alpha[r + \gamma \max_{a'} Q(s',a') - Q(s,a)]$
        $s \leftarrow s'$;
    until s is terminal
end loop

SARSA vs Q–learning

Cliffwalking



ε–greedy, ε = 0.1



Q–learning vs SARSA

- SARSA: $Q(s,a) \leftarrow Q(s,a) + \alpha[r + \gamma Q(s',a') - Q(s,a)]$ on–policy
- Q–learning: $Q(s,a) \leftarrow Q(s,a) + \alpha[r + \gamma \max_{a'} Q(s',a') - Q(s,a)]$ off–policy
- In the cliff–walking task:
  - Q–learning: learns **optimal policy** along edge
  - SARSA: learns a **safe non–optimal policy** away from edge
- ϵ–greedy algorithm
  - For $\epsilon \neq 0$ SARSA performs **better online**
  - For $\epsilon \to 0$ gradually, **both converge to optimal**

Relationship Between DP and TD

Full Backup (DP)	**Sample backup (TD)**
Iterative Policy Evaluation $V(s) \leftarrow \mathbb{E}_\pi[r + \gamma V(s')\vert s]$	TD Learning $V(s) \overset{\alpha}{\leftarrow} r + \gamma V(s')$
Q–Policy Iteration $Q(s,a) \leftarrow \mathbb{E}_\pi[r + \gamma Q(s',a')\vert s,a]$	SARSA $Q(s,a) \overset{\alpha}{\leftarrow} r + \gamma Q(s',a')$
Q–Value Iteration $Q(s,a) \leftarrow \mathbb{E}_\pi[r + \gamma \max_{a' \in \mathcal{A}} Q(s',a')\vert s,a]$	Q–learning $Q(s,a) \overset{\alpha}{\leftarrow} r + \gamma \max_{a' \in \mathcal{A}} Q(s',a')$

MDP Extensions

MDP: $\langle \mathcal{S}, \mathcal{A}, P, R, \gamma, \mu \rangle$

- States and actions can be **continous**: Function approximation in RL
- State can be **partially observable**: Partially Observable MDPs (POMDPs)
- Actions can be **temporally extended**: Semi MDPs (SMDPs) and Hierarchical Learning
- Rewards can be **multiple**: Multi–Objective RL
- Rewards can be **unknown**: Inverse RL
- Rewards can be **intrinsic**: Intrinsically Motivated RL
- Information can be **reused**: Transfer learning
- There can be **many learning agents**: Stochastic/Markov Games