Logic final

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Instructions. Answers must be written on the provided sheets in the space below the question and on the back of the page; scrap paper should not be handed in and will not be taken into any account. Theoretical justification must be provided for each answer in concise but complete form. Results should be properly simplified.

1 Prove soundness of resolution for first order logic.

SOUNDNESS RESOLUTION CALCULUS IS SOUND: $F \vdash \varphi \rightarrow F \models \varphi$

ASSUMPTION, FALSEHOOD, DOUBLE NEGATION, α AND β FORMULAS HAS THE SAME PROOF AS IN P.L.

WE FIRST HAVE TO SHOW THAT γ AND δ FORMULAS (UNIVERSAL AND EXISTENTIAL RESPECTIVELY) PRESERVE SATISFIABILITY OF CLOSED ~~TERMS~~ ~~FORMULAS~~ FORMULAS

- $M \models \gamma(x)$ SINCE t IS CLOSED, $\exists c \in M \mid \forall \overset{\text{VALUATION}}{v}: x \rightarrow M, v(t) = c$

} REMEMBER THAT $v \models \gamma(t) \leftrightarrow v[\frac{t}{x}] \models \gamma(x) \leftrightarrow v[\frac{c}{x}] \models \gamma(x)$

$M \models \gamma(x) \rightarrow M \models \forall x \gamma(x) \rightarrow \forall v: x \rightarrow M, v \models \gamma(x) \rightarrow \forall v: x \rightarrow M, \alpha \in M$

$v[\frac{\alpha}{x}] \models \gamma(x) \xrightarrow{\forall v} v[\frac{c}{x}] \models \gamma(x) \xrightarrow{\forall v} v \models \gamma(t) \rightarrow M \models \gamma(t)$ ✓

- $M \models \delta(x)$

$M \models \delta(x) \rightarrow M \models \exists x \delta(x) \rightarrow \exists v: x \rightarrow M, v \models \delta(x) \rightarrow \exists u: x \rightarrow M, u \models \delta(x) \rightarrow$

WHERE u IS AN x-VARIANT OF v. DEFINE M_p AS THE EXPANSION OF M BY

$p := u(x)$

$\rightarrow \exists u: x \rightarrow M_p, u \models \delta(x) \rightarrow \exists u: x \rightarrow M_p, u \models \delta(p) \rightarrow M \models \delta(p)$ ✓

NOW, $F \vdash \varphi \rightarrow$ ~~$F \cup \{\neg\varphi\}$~~ ~~Res~~ $\bot \in Res(F \cup \{\neg\varphi\}) \rightarrow \forall v \in M^*$,

$v \not\models Res(F \cup \{\neg\varphi\}) \rightarrow v \not\models F \cup \{\neg\varphi\} \rightarrow F \cup \{\neg\varphi\} \models \bot \rightarrow F \models \varphi$ ✓

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2

A binary relation R has the lifting property if whenever Ryx and Rzx, then Ryz as in the diagram below.

$$\begin{tikzcd} & z \arrow[d] \\ y \arrow[r] \arrow[ur, dashed] & x \end{tikzcd}$$

Prove, using resolution a Herbrand style, that if R is reflexive and has the lifting property, then it is symmetric:

$$\forall x.Rxx,\ \forall xyz(Ryx \wedge Rzx \to Ryz) \vdash \forall xy(Rxy \to Ryx).$$

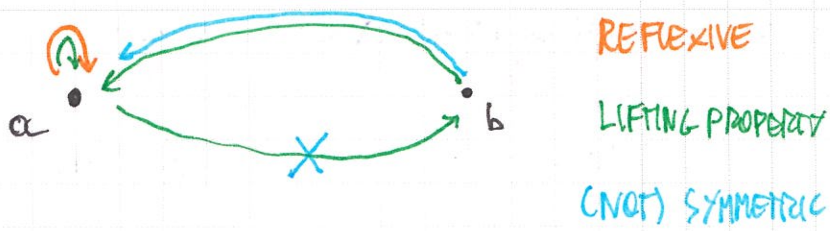
$F = \{\forall x Rxx, \forall xyz(Ryx \wedge Rzx \to Ryz), \neg \forall xy(Rxy \to Ryx)\}$

STEP	FORMULA	RULE
1	$\{\forall x Rxx\}$	ASSUMPTION
2	$\{\forall xyz(Ryx \wedge Rzx \to Ryz)\}$	ASSUMPTION
3	$\{\neg(\forall xy(Rxy \to Ryx))\}$	ASSUMPTION — not a good idea
4	$\{Raa\}$	1, γ-EXPANSION
5	$\{\forall yz(Rya \wedge Rza \to Ryz)\}$	2, γ-EXPANSION
6	$\{\forall z(Raa \wedge Rza \to Raz)\}$	5, γ-EXPANSION
7	$\{Raa \wedge Rba \to Rab\}$	6, γ-EXPANSION
8	$\{\neg(Raa \wedge Rba), Rab\}$	7, β-EXPANSION
9	$\{\neg Raa, \neg Rba, Rab\}$	8, β-EXPANSION
10	$\{\exists xy.\neg(Rxy \to Ryx)\}$	3, δ-RULES — not a δ-exp
11	$\{\exists y.\neg(Rby \to Ryb)\}$	10, δ-EXPANSION
12	$\{\neg(Rba \to Rab)\}$	11, δ-EXPANSION
13	$\{Rba\}$	12, α-EXPANSION
14	$\{\neg Rab\}$	12, α-EXPANSION

can not reuse a param in a δ-formula

15	$\{\neg Rba, Rab\}$	9,4 RESOLUTION
16	$\{Rab\}$	15,13 RESOLUTION
17	$\emptyset$	16,14 RESOLUTION

WE HAVE PROVED THAT $F \cup \{\neg\psi\}$ HAS A CLOSED EXPANSION. THEREFORE IT IS UNSATISFIABLE AND $F \vdash \varphi$ IS SATISFIABLE ✓



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3

Prove that the subset $R \subseteq \mathrm{Mat}_2(\mathbf{R})$ of matrices of type

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

is a commutative subring and that the subsets

$$\mathfrak{a}_1 = \{A \in R : b = a\}, \qquad \mathfrak{a}_2 = \{A \in R : b = -a\}$$

are ideals of R. Prove that every $A \in R$ admits a unique decomposition $A = x_1P_1 + x_2P_2$ with $x_1, x_2 \in \mathbf{R}$ and

$$P_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathfrak{a}_1, \qquad P_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \mathfrak{a}_2.$$

Deduce that there is a ring isomorphism $h : R \to \mathbf{R}^2$ with $h(A) = (x_1, x_2)$, where $\mathbf{R}^2$ has pointwise operations

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \qquad (x_1, x_2)\,(y_1, y_2) = (x_1y_1, x_2y_2)\,.$$

ⓐ

FIRST, WE PROVE THAT R IS A SUBRING OF $Mat_2(\mathbb{R})$ BY VERIFYING THE FOLLOWING PROPERTIES:

- NEUTRAL ELEMENT $0 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \in R$ $(a=0, b=0)$
- OPPOSITE $-A = \begin{vmatrix} -a & -b \\ -b & -a \end{vmatrix} \in R$ ✓
- NEUTRAL ELEMENT FOR PRODUCT $I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \in R$ $(a=1, b=0)$
- PRODUCT: GIVEN $A = \begin{vmatrix} a & b \\ b & a \end{vmatrix}$, $A' = \begin{vmatrix} a' & b' \\ b' & a' \end{vmatrix}$, $AA' \in R$, $A'A \overset{?}{\in} R$?

SINCE, IN GENERAL, COMMUTATIVITY FOR PRODUCT DOESN'T HOLD FOR RINGS, WE HAVE TO PROVE BOTH

$$A \cdot A' = \begin{vmatrix} a & b \\ b & a \end{vmatrix} \cdot \begin{vmatrix} a' & b' \\ b' & a' \end{vmatrix} = \begin{vmatrix} \overset{1}{aa' + bb'} & \overset{2}{ab' + ba'} \\ \overset{3}{ba' + ab'} & \overset{4}{bb' + aa'} \end{vmatrix} \quad 1=4 \text{ AND } 2=3$$

$\rightarrow A \cdot A' \in R$ ✓

$$A' \cdot A = \begin{vmatrix} a' & b' \\ b' & a' \end{vmatrix} \cdot \begin{vmatrix} a & b \\ b & a \end{vmatrix} = \begin{vmatrix} \overset{1}{a'a + b'b} & \overset{2}{a'b + b'a} \\ \overset{3}{b'a + a'b} & \overset{4}{b'b + a'a} \end{vmatrix} \quad 1=4 \text{ AND } 2=3 \rightarrow A'A \in R ✓$$

TO SEE IF IT IS COMMUTATIVE, SHOW THAT $\forall A, A' \in R$, $AA' = A'A$

$$\begin{cases} aa' + bb' = a'a + b'b \\ ab' + ba' = a'b + b'a \\ = \\ = \end{cases} \qquad \begin{cases} aa' - a'a = b'b - bb' \\ ab' - b'a = a'b - ba' \end{cases} \qquad \begin{cases} 0 = 0 \\ 0 = 0 \end{cases} ✓$$

✓

(b) $A \in a_1 \Rightarrow A = \begin{vmatrix} a & a \\ a & a \end{vmatrix} \in a_1$ IDEAL IF

- $0 \in a_1$ $\checkmark$ $(a=0)$
- $A \in a_1 \rightarrow -A \in a_1$ $\begin{vmatrix} -a & -a \\ -a & -a \end{vmatrix} \in a_1 \checkmark$
- $A, A' \in a_1 \rightarrow A + A' \in a_1$ $\begin{vmatrix} a & a \\ a & a \end{vmatrix} + \begin{vmatrix} a' & a' \\ a' & a' \end{vmatrix} = \begin{vmatrix} a+a' & a+a' \\ a+a' & a+a' \end{vmatrix} \in a_1 \checkmark$
- $A \in a_1, A' \in R \rightarrow AA', A'A \in a$ SINCE WE PROVED COMMUTATIVITY, WE JUST NEED TO PROVE ONE OF THE 2 PRODUCTS

$$A = \begin{vmatrix} a & a \\ a & a \end{vmatrix} \quad A' = \begin{vmatrix} a' & b' \\ b' & a' \end{vmatrix} \quad AA' = \begin{vmatrix} \overset{1}{aa'+ab'} & \overset{2}{ab'+aa'} \\ \overset{3}{aa'+ab'} & \overset{4}{ab'+aa'} \end{vmatrix} \in a_1 \checkmark$$

$1=2=3=4$ $\checkmark$

$A = \begin{vmatrix} a & -a \\ -a & a \end{vmatrix} \in a_2$. SIMILARLY:

- $0 \in a_2$ $\checkmark$ $(a=0)$
- $-A = \begin{vmatrix} -a & a \\ a & -a \end{vmatrix} \in a_2 \checkmark$
- $A + A' = \begin{vmatrix} a & -a \\ -a & a \end{vmatrix} + \begin{vmatrix} a' & -a' \\ -a' & a' \end{vmatrix} = \begin{vmatrix} a+a' & -a-a' \\ -a-a' & a+a' \end{vmatrix} \in a_2 \checkmark$
- $A \cdot A' = \begin{vmatrix} a & -a \\ -a & a \end{vmatrix} \cdot \begin{vmatrix} a' & b' \\ b' & a' \end{vmatrix} = \begin{vmatrix} \overset{1}{aa'-ab'} & \overset{2}{ab'-aa'} \\ \overset{3}{-aa'+ab'} & \overset{4}{-ab'+aa'} \end{vmatrix} \in a_2 \checkmark$

$1=4$, $2=3$ $\quad 1=-2$ $\checkmark$

(d) (c)

~~$h\left(\begin{vmatrix} a & b \\ b & a \end{vmatrix}\right) =$~~ $h(A) = h\left(x_1 \cdot \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + x_2 \cdot \frac{1}{2}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}\right) = (x_1, x_2)$

ISOMORPHIC IF IT IS BOTH:

- INJECTIVE $\rightarrow \forall (x_1, x_2) \in \mathbb{R}^2, |h^*(x_1, x_2)| \leq 1$

$$\left.\begin{aligned} h(x_1 \cdot P_1 + x_2 \cdot P_2) &= (x_1, x_2) \\ h(x_1' \cdot P_1 + x_2' \cdot P_2) &= (x_1, x_2) \end{aligned}\right\} \Leftrightarrow x_1 P_1 + x_2 P_2 = x_1' P_1 + x_2' P_2$$

$(x_1 - x_1')P_1 + (x_2 - x_2')P_2 = 0 \Leftrightarrow$

$\Leftrightarrow x_1 = x_1' \wedge x_2 = x_2' \Leftrightarrow (x_1, x_2) = (x_1', x_2') \checkmark \checkmark$

- SURJECTIVE $\forall (x_1, x_2) \in \mathbb{R}^2, |h^*(x_1, x_2)| \geq 1$ * $\checkmark$

* IT HOLDS BECAUSE, FOR EXAMPLE, SETTING $x_1 = K$, THERE IS AT LEAST AN ELEMENT IN R SUCH THAT $KP_1 + x_2 P_2 \in R$ $\forall x_2 \in \mathbb{R}$.