MATH REVIEW

① COMPUTE THE DERIVATIVE OF THE FOLLOWING FUNCTIONS W.R.T. X

$f(x)=3x^2+2x \quad f:\mathbb{R}\to\mathbb{R} \quad f'(x)=6x+2$

$f(x)=\ln(\cosh(x)) \quad f:\mathbb{R}\to\mathbb{R} \quad f'(x)=\frac{1}{\cosh(x)}\cdot \sinh(x)=\frac{\sinh(x)}{\cosh(x)}$

$f(x)=\sin(x_1^2)+3x_2^3+e^{x_3} \quad x=(x_1,x_2,x_3) \quad f:\mathbb{R}^3\to\mathbb{R}$

$f'(x)=\nabla f(x)=\left(\frac{\partial f}{\partial x_1};\frac{\partial f}{\partial x_2};\frac{\partial f}{\partial x_3}\right)^T=\left(2x\cos(x_1^2);\ 9x_2^2;\ e^{x_3}\right)^T$

$f(x):\mathbb{R}^n\to\mathbb{R} \quad f(x)=x^T\cdot A\cdot x;\ A\in\mathcal{M}_{n\times n}$

$$\begin{vmatrix} x_1 & \cdots & x_n\end{vmatrix}\cdot A\cdot\begin{vmatrix}x_1\\ \vdots\\ x_n\end{vmatrix}=\begin{vmatrix}Ax_1 & \cdots & Ax_n\end{vmatrix}\cdot\begin{vmatrix}x_1\\ \vdots\\ x_n\end{vmatrix}=A(x_1^2+\dots+x_n^2)=Ax^2 \quad f'(x)=\frac{\partial f}{\partial x}=2Ax$$

② SHOW THAT, FOR SMALL X, ~~log~~ $\ln(1+x)=x-\frac{x^2}{2}$

$\lim_{x\to 0}\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+o(x^4)$ BY TAYLOR EXPANSION

③ CONSIDER TWO DISCRETE RANDOM VARIABLES X AND Y. SHOW THAT $P(X=x, Y=y)\leq P(X=x)$

- $P(X=x,Y=y)=P(X=x)\cdot P(Y=y)$ WHEN X AND Y ARE INDEPENDENT. MORE GENERALLY, $P(X=x,Y=y)$ IS THE PROBABILITY THAT BOTH X AND Y OCCUR AT THE SAME TIME
- $P(X=x)=\sum_y P(X=x,\underline{Y=y})$ X OCCURS INDEPENDENTLY FROM Y ⇒ SUM ALL THE CASES W.R.T. Y

IT IS IMMEDIATE THAT $P(X=x,Y=y)\leq P(X=x)$, SINCE THE PROBABILITY THAT X AND Y OCCUR AT THE SAME TIME CAN BE AT MOST THE PROBABILITY THAT X OCCURS. IN CASE X AND Y ARE INDEPENDENT, FOR EXAMPLE, $P(X=x,Y=y)=\overbrace{P(X=x)}^{0\leq\ \leq 1}\cdot\overbrace{P(Y=y)}^{0\leq\ \leq 1}\leq P(X=x)$

④ THIS IS A GAUSSIAN DENSITY WITH MEAN μ AND VARIANCE σ^2: $f(x|\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}$

WHAT IS THE POINT x^* WITH THE HIGHEST PROBABILITY DENSITY? BACK UP YOUR CLAIM BY FINDING THE STATIONARY POINTS OF THE DENSITY FUNCTION $\quad f'(x)\geq 0$

$$f'(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}\cdot\left(-\frac{1}{2}\cdot\frac{2(x-\mu)}{\sigma^2}\right)=\cancel{\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\left(\frac{(x-\mu)^2}{2\sigma^2}\right)}}\cdot\left(-\frac{x-\mu}{\cancel{\sigma^2}}\right)\geq 0$$

$\Rightarrow -(x-\mu)\geq 0 \Rightarrow x^*\leq\mu$

f': + μ −

f: ↗ MAX ↘

⑤ LET $\vec{x}, \vec{y}$ AND $\vec{z}$ BE $n \times 1$ COLUMN VECTORS AND A BE AN $n \times n$ MATRIX. ASSUME THAT YOU KNOW THAT $\vec{x}^T A \vec{y} = 1$. FOR EACH OF THE FOLLOWING STATEMENTS, ANSWER TRUE, FALSE OR DIMENSIONALLY INCONSISTENT (IF ANY OF THE EQUATIONS IS UNDEFINED IN TERMS OF MATRIX ALGEBRA)

a) $\vec{y}^T A \vec{x} = 1$ FALSE ~~DIMENSIONALLY INCONSISTENT~~. WE DON'T HAVE ANY INFORMATION REGARDING THE SYMMETRY OF A AND OF THE BEHAVIOR OF X^T/Y^T AND Y/X W.R.T. A

IN GENERAL, $\vec{x}^T A \vec{y} \neq \vec{y}^T A \vec{x}$

b) $A \vec{x}^T \vec{y} = 1$ DIMENSIONALLY INCONSISTENT $\vec{x}^T \cdot \vec{y} \in \mathcal{M}_{1 \times 1} \Rightarrow A(\vec{x}^T \cdot \vec{y}) \in \mathcal{M}_{n \times n}$ $\Rightarrow$ IT CANNOT MATCH A SCALAR

$$\cancel{A \cdot \vec{x}^T = \begin{vmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix} = |A_{11}x_1 + \dots + A_{1n}x_n \quad \cdots \quad A_{n1}x_1 + \dots + A_{nn}x_n|}$$

$$\cancel{A \cdot \vec{x}^T \cdot \vec{y} = |A_{11}x_1 + \dots + A_{1n}x_n \quad \cdots \quad A_{n1}x_1 + \dots + A_{nn}x_n| \cdot |y_1 \cdots y_n|}$$ ~~CANNOT BE COMPUTED~~

c) $\vec{y}^T A^{-1} \vec{x} = 1$

- ~~$A^{-1} \cdot \vec{x}^T A \vec{x} = A^{-1}$~~

$A^{-1} \cdot \vec{x}^T A \vec{y} = \vec{x}^T A^{-1} A \vec{y} = \vec{x}^T I \vec{y} = \vec{x}^T \vec{y}$

- $\vec{x}^T A \vec{y} = 1$ TRUE

$\underbrace{\vec{y}^T A^{-1} \vec{x}^T A \vec{y} \vec{x}}_{=1?} = \vec{y}^T A^{-1} \vec{x}$

$\rightarrow \vec{y}^T A^{-1} \vec{x}^T A \vec{y} \vec{x} = \vec{y}^T \vec{x}^T A^{-1} A \vec{y} \vec{x} = \vec{y}^T \vec{x}^T I \vec{y} \vec{x}^T = 1$

d) $\vec{y}^T A^T \vec{x} = 1$ TRUE

$\vec{x}^T A \vec{y} = 1 \qquad (\vec{x}^T A \vec{y})^T = 1^T \Rightarrow \vec{y}^T A^T \vec{x}^{T^T} = 1 \Rightarrow \vec{y}^T A^T \vec{x} = 1$

e) $\vec{x}^T A (\vec{y} - \vec{z}) = 1 - \vec{x}^T A \vec{z}$ TRUE

$\vec{x}^T A (\vec{y} - \vec{z}) = \vec{x}^T A \vec{y} - \vec{x}^T A \vec{z} = 1 - \vec{x}^T A \vec{z}$

f) $\mathrm{Trace}(A \vec{y} \vec{x}^T) = 1$ TRUE

Trace $\rightarrow$ SUM OF THE ELEMENTS IN THE DIAGONAL $\mathrm{Trace}(A \vec{y} \vec{x}^T) = \mathrm{Trace}(\underbrace{\vec{x}^T A \vec{y}}_{1})$

⑥ LET P BE ...

(2; 6; 3) ...

PARAMETR...

$\vec{v}_1 = (2 \dots$

$r(s,t) = \dots$

$\begin{cases} x = 1 \dots \\ y = 2 \dots \\ z = 4 \dots \end{cases}$

NORMAL ...

$a(x - x \dots$

$\vec{v}_1 \times \vec{v}_2 = \dots$

$\vec{n} = \det \dots$

$\Rightarrow -17 \dots$

⑦ THE WEATHER ...

WEATHER IS ...

THE NOTATIO...

OF x_i", CO...

a) E[x ...

b) E[x ...

c) var ...

d) E[...

⑧ WHAT I ...

$\frac{x - \mu}{\sigma \sqrt{2}} = \dots$

$\int_0^{+\infty} e \dots$

$I^2 = \dots$

$= - \dots$

⑥ LET P BE A PLANE IN THREE DIMENSIONAL SPACE WHICH GOES THROUGH THE POINTS (1; 2; 4), (2; 6; 3) AND (5; 1; 0). WRITE DOWN ITS EQUATION IN PARAMETRIC FORM AND IN NORMAL FORM

PARAMETRIC FORM

$\vec{v}_1 = (2;6;3) - (1;2;4) = (1;4;-1)$ $\quad \vec{v}_2 = (5;1;0) - (1;2;4) = (4;-1;-4)$

$r(s,t) = (1;2;4) + s\vec{v}_1 + t\vec{v}_2 = (1+s+4t; 2+4s-t; 4-s-4t)$

$$\begin{cases} x = 1+s+4t \\ y = 2+4s-t \\ z = 4-s-4t \end{cases}$$

NORMAL FORM

$a(x-x_p) + b(y-y_p) + c(z-z_p) = 0 \quad P = (1;2;4) \quad (a;b;c) = \vec{n}$

$$\vec{n} = \underbrace{\det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & -1 \\ 4 & -1 & -4 \end{vmatrix}}_{\vec{v}_1 \times \vec{v}_2 =} = -17\hat{i} + 0 - 17\hat{k} \Rightarrow \vec{n} = (-17; 0; -17)$$

$\Rightarrow -17(x-1) - 17(z-4) = 0 \Rightarrow x + z - 5 = 0$

⑦ THE WEATHER IN LONDON HAS PROBABILITY P OF BEING RAINY ON ANY GIVEN DAY. ASSUME THE WEATHER IS INDEPENDENT ACROSS DAYS. LET $X_i = 1$ IF IT IS RAINY ON DAY i AND 0 OTHERWISE. USING THE NOTATION $E(X_i)$ TO MEAN THE "EXPECTED VALUE OF X_i" AND $Var(X_i)$ TO MEAN THE "VARIANCE OF X_i", COMPUTE THE VALUES (IN TERMS OF P) OF THE FOLLOWING QUANTITIES:

a) $E[X_i]$ $\quad E[X_i] = (PX_1 + (1-P)X_0)_i = P\cdot 1 + (1-P)\cdot 0 = P$

b) $E[X_i^2]$ $\quad E[X_i^2] = (PX_1^2 - (1-P)X_0^2)_i = P\cdot 1^2 + (1-P)\cdot 0^2 = P$

c) $var(X_i)$ $\quad var(X_i) = (E[X_i^2] - E[X]^2)_i = P - P^2 = P(1-P)$

d) $E[\sum_{i=1}^{n} P_i]$ $\quad E[\sum_{i=1}^{n} P_i] = (PX_1 + (1-P)X_0)_{i=1} + \dots + (PX_1 + (1-P)X_0)_{i=n} = nP$

⑧ WHAT IS $\int_{\mu}^{+\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$?

$\frac{x-\mu}{\sigma\sqrt{2}} = t \rightarrow dx = \sigma\sqrt{2}\,dt \quad x = \mu \rightarrow t = 0 \quad x = +\infty \rightarrow t = +\infty$

$$\int_0^{+\infty} e^{-t^2} \cdot \sigma\sqrt{2}\,dt = \sigma\sqrt{2} \underbrace{\int_0^{+\infty} e^{-t^2} dt}_{I} =$$

$$I^2 = \int_0^{+\infty} e^{-x^2} dx \int_0^{+\infty} e^{-y^2} dy = \int_0^{+\infty} e^{-(x^2+y^2)} dx\,dy = -\frac{1}{2}\int_0^{\pi} d\theta \int_0^{+\infty} -e^{-\rho^2} 2\rho\,d\rho$$

$$= -\frac{\pi}{4}\left[e^{-\rho^2}\right]_0^{+\infty} = -\frac{\pi}{4}(0-1) = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2} \quad = \frac{\sigma\sqrt{2\pi}}{2}$$