Logic midterm 2023-11-03

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Instructions. Answers must be written on the provided sheets in the space below the question and on the back of the page; scrap paper should not be handed in and will not be taken into any account. Theoretical justification must be provided for each answer in concise but complete form. Results should be properly simplified.

1 State and prove soundness of resolution

STATEMENT: INFERENCE RESOLUTION IS SOUND $F \vdash \varphi \Rightarrow F \vDash \varphi$

PROOF

WE FIRST PROVE THAT EVERY AXIOM OF THE INFERENCE RESOLUTION IS SOUND. ? WE WILL LOOK THAT THERE EXIST VALUATIONS α THAT, IF IT/THEY SATISFY F, α SATISFY φ

- FALSEHOOD SUPPOSE $\bot \in \varphi_i$. WE CAN DEFINE $\varphi_i \equiv \bot \vee \tilde{x}_i$, WHERE $\tilde{x}_i$ IS A GENERALIZED CLAUSE THAT DO NOT CONTAIN $\bot$. WE DEFINE THE SUBSTITUTION $\sigma = [\emptyset/\bot]$ AND, APPLYING SUBSTITUTION LEMMA, $\varphi\sigma = (\bot \vee \tilde{x}_i)[\emptyset/\bot] = \emptyset \vee \tilde{x}_i = \tilde{x}_i \equiv \varphi_i$
  IN CONCLUSION, $\alpha \vDash \varphi_i \Rightarrow \alpha \vDash \varphi_i[\emptyset/\bot]$
- DOUBLE NEGATION SUPPOSE $\neg\neg\psi \in \varphi_i$. WE CAN DEFINE (then) $\varphi_i \equiv \neg\neg\psi \vee \tilde{x}_i$, WHERE $\tilde{x}_i$ IS A GENERALIZED CLAUSE THAT DO NOT CONTAIN $\neg\neg\psi$. WE DEFINE THE SUBSTITUTION $\sigma = [\psi/\neg\neg\psi]$ AND, APPLYING SUBSTITUTION LEMMA, $\varphi\sigma = (\neg\neg\psi \vee \tilde{x}_i)[\psi/\neg\neg\psi] = \psi \vee \tilde{x}_i \equiv \varphi_i$ IN CONCLUSION, $\alpha \vDash \varphi_i \Rightarrow \alpha \vDash \varphi_i[\psi/\neg\neg\psi]$
- α- FORMULA SUPPOSE $\alpha \in \varphi_i$. $\varphi_i = \alpha \vee \tilde{x}_i$, WHERE $\tilde{x}_i$ IS A GENERALIZED CLAUSE THAT DO NOT CONTAIN α. WE DEFINE $\sigma_1 = [\alpha_1/\alpha]$ AND $\sigma_2 = [\alpha_2/\alpha]$ AND, APPLYING SUBSTITUTION LEMMA, $\varphi\sigma = \varphi_i\sigma_1 \wedge \varphi_i\sigma_2 = (\alpha_1 \vee \tilde{x}_i)[\alpha_1/\alpha] \wedge (\alpha_2 \vee \tilde{x}_i)[\alpha_2/\alpha] = (\alpha_1 \vee \tilde{x}_i) \wedge (\alpha_2 \vee \tilde{x}_i) = (\alpha_1 \wedge \alpha_2)\tilde{x}_i \equiv \varphi_i$?
  IN CONCLUSION, $\alpha \vDash \varphi_i \Rightarrow \alpha \vDash \varphi_i[\alpha_1/\alpha] \wedge \varphi_i[\alpha_2/\alpha]$
  $\Rightarrow \alpha \vDash \varphi_i[\alpha_1/\alpha]$

- **β-FORMULA** SUPPOSE $\beta \in \varphi_i$*. $\varphi_i = \beta \vee \kappa_i$, WHERE κ_i IS A GENERALIZED CLAUSE THAT DOESN'T CONTAIN β. WE DEFINE THE SUBSTITUTION $\sigma = [\beta_1, \beta_2 / \beta]$ AND, APPLYING SUBSTITUTION LEMMA,

$$\varphi\sigma = (\overset{(\beta_1,\beta_2)}{\beta_1 \vee \beta_2} \vee \kappa)[\beta_1, \beta_2 / \kappa] = \beta \vee \kappa = \varphi_i$$

(* β IS A β-FORMULA, DISJUNCTION BETWEEN β_1, β_2)

IN CONCLUSION, $\alpha \models \varphi_i \Rightarrow \alpha \models \varphi_i[\beta_1, \beta_2 / \kappa]$

- **RESOLUTION** SUPPOSE $\psi \in \varphi_i$ AND $\neg\psi \in \varphi_j$. $\varphi = \varphi_i \vee \varphi_j = (\psi \vee \kappa_i) \vee (\neg\psi \vee \kappa_j)$, WHERE κ_i AND κ_j ARE GENERALIZED CLAUSES THAT DO NOT CONTAIN, RESPECTIVELY, ψ AND $\neg\psi$. WE DEFINE THE SUBSTITUTIONS $\sigma = [\emptyset / \psi]$ AND $\tau = [\emptyset / \neg\psi]$ AND, APPLYING SUBSTITUTION LEMMA, $\varphi = \varphi_i\sigma \vee \varphi_j\tau = (\psi \vee \kappa_i)[\emptyset/\psi] \vee (\neg\psi \vee \kappa_j)[\emptyset/\neg\psi] = \kappa_i \vee \kappa_j$

SINCE $\varphi = \kappa_i \vee \kappa_j$

  - $\alpha \models \varphi_i \Rightarrow \alpha \not\models \varphi_j$. $\alpha \models \kappa_i \Rightarrow \alpha \models \varphi_i[\emptyset/\psi] \vee \varphi_j[\emptyset/\neg\psi]$
  - $\alpha \models \varphi_j \Rightarrow \alpha \not\models \varphi_i$ $\alpha \models \kappa_j \Rightarrow \alpha \models \varphi_i[\emptyset/\psi] \vee \varphi_j[\emptyset/\neg\psi]$

NOW:

- EVERY AXIOM OF INFERENCE RESOLUTION IS SOUND
- WE CAN DEFINE AT LEAST A VALUATION $\alpha \models F$ THAT SATISFIES THE SET OF PREMISES
- F HAS A CLOSED EXPANSION $\Rightarrow$ IT IS UNSATISFIABLE
- $F \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION $\Rightarrow$ IT IS UNSATISFIABLE $\Rightarrow F \vdash \varphi$
- $F \vdash \varphi \Rightarrow F \models \varphi$

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Consider the claim

$$x \wedge y \to z,\ (x \to z) \leftrightarrow (y \to z)\ \vDash x \vee y \to z.$$

1. Prove the claim semantically (either with a truth table or a semantic tree).
2. Prove the claim using resolution (with a Herbrand table)

① TRUTH TABLE

X	Y	Z	X∧Y	X∧Y→Z	X→Z	Y→Z	(X→Z)↔(Y→Z)	X∨Y	X∨Y→Z
0	0	0	0	1	1	1	1	0	1
0	0	1	0	1	1	1	1	0	1
0	1	0	0	1	1	0	0	1	0
0	1	1	0	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1	1
1	1	0	1	0	1	0	0	1	0
1	1	1	1	1	1	1	1	1	1

VALUATIONS v_1, v_2, v_4, v_8 SATISFY BOTH THE TWO PREMISES AND THE CONCLUSION $\Rightarrow$ $F \vDash \varphi$ ✓

not quite correct: since $\forall v,\ v \vDash F \Rightarrow v \vDash \varphi$, it follows $F \vDash \varphi$

② RESOLUTION

WE PROVE THAT $F \cup \{\neg\varphi\} = \{x \wedge y \to z, (x \to z) \leftrightarrow (y \to z), \neg(x \vee y \to z)\}$ HAS A CLOSED EXPANSION

α-FORMULAS			β-FORMULAS		
$\varphi \wedge \psi$	φ	ψ	$\neg(\varphi \wedge \psi)$	$\neg\varphi$	$\neg\psi$
$\neg(\varphi \vee \psi)$	$\neg\varphi$	$\neg\psi$	$\varphi \vee \psi$	φ	ψ
$\neg(\varphi \to \psi)$	φ	$\neg\psi$	$\varphi \to \psi$	$\neg\varphi$	ψ

STEP	FORMULA	RULE	STEP	FORMULA	RULE
1	$\{x \wedge y \to z\}$	ASSUMPTION	12	$\{\neg(y \to z), x \to z\}$	7, β-EXPANSION
2	$\{(x \to z) \leftrightarrow (y \to z)\}$	//	13	$\{\neg(y \to z), \neg x, z\}$	12, β-EXPANSION
3	$\{\neg(x \vee y) \to z\}$	//	14	$\{\neg x, y, z\}$	13, α-EXPANSION
4	$\{\neg(x \wedge y), z\}$	1, β-EXPANSION	15	$\{\neg x, z\}$	13, α-EXPANSION
5	$\{\neg x, \neg y, z\}$	4, β-EXPANSION	16	$\{x \vee y\}$	3, α-EXPANSION
6	$\{(x \to z) \to (y \to z)\}$	2, ~~BIIMPLICATION~~ α-exp.	17	$\{\neg z\}$	3, α-EXPANSION
7	$\{(y \to z) \to (x \to z)\}$	2, ~~BIIMPLICATION~~ α-exp	18	$\{x, y\}$	16, β-EXPANSION
8	$\{\neg(x \to z), y \to z\}$	6, β-EXPANSION	19	$\{\neg y\}$	11, 17 RESOLUTION
9	$\{\neg(x \to z), \neg y, z\}$	8, β-EXPANSION	20	$\{\neg x\}$	15, 17 RESOLUTION
10	$\{x, \neg y, z\}$	9, α-EXPANSION	21	$\{y\}$	18, 20 RESOLUTION
11	$\{\neg y, z\}$	9, α-EXPANSION	22	$\emptyset$	19, 21 RESOLUTION

$F \cup \{\neg\varphi\}$ HAS A CLOSED EXPANSION $\Longrightarrow F \vdash \varphi$ ✓

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Let $A = \{1,2,3,4,5\} = B$ and let $R : B \rightharpoonup B$ and $f : A \to B$ be defined by the matrices below.

$$M(R) = \begin{pmatrix} 1&1&1&1&1\\ 0&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix} \qquad M(f) = \begin{pmatrix} 0&0&0&0&1\\ 1&0&0&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 1&0&0&0&0 \end{pmatrix}$$

1. Compute the inverse image $S = f^*R : A \rightharpoonup A$ of R along f and the order U induced by S.
2. Compute the epi-mono factorization $f = ep$ of f and the inverse image $V = m^*R$ of R along m.
3. Show that $U = V$ and draw the Hasse diagram of this relation

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "S"'] & B \arrow[d, "R"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$S = f^*(R) = fRf^{op}$

(1) $f^{op} = \begin{vmatrix} 0&1&0&0&1\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 1&0&0&0&0 \end{vmatrix}$

$$S = fRf^{op} = \begin{vmatrix} 0&0&0&0&1\\ 1&0&0&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 1&0&0&0&0 \end{vmatrix} \bullet \begin{vmatrix} 1&1&1&1&1\\ 0&1&0&0&1\\ 0&0&1&0&1\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{vmatrix} \bullet \begin{vmatrix} 0&1&0&0&1\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 1&0&0&0&0 \end{vmatrix} =$$

$$= \begin{vmatrix} 0&0&0&0&1\\ 1&1&1&1&1\\ 0&0&0&1&1\\ 0&1&0&0&1\\ 1&1&1&1&1 \end{vmatrix} \cdot \begin{vmatrix} 0&1&0&0&1\\ 0&0&0&1&0\\ 0&0&0&0&0\\ 0&0&1&0&0\\ 1&0&0&0&0 \end{vmatrix} = \begin{vmatrix} 1&0&0&0&0\\ 1&1&1&1&1\\ 1&0&0&0&0\\ 1&0&0&1&0\\ 1&1&1&1&1 \end{vmatrix}$$

✓

$$U = p^{op}Sp = \begin{vmatrix} 1&0&0&0&0\\ 1&1&1&1&1\\ 1&0&1&0&0\\ 1&0&0&1&0\\ 1&1&1&1&1 \end{vmatrix} \cap \begin{vmatrix} 1&1&1&1&1\\ 0&1&0&0&1\\ 0&1&1&0&1\\ 0&1&0&1&1\\ 0&1&0&0&1 \end{vmatrix} = \begin{array}{c} [1]\ [2]\ [3]\ [4]\ [5] \\ \begin{vmatrix} 1&0&0&0&0\\ 0&1&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&1 \end{vmatrix} \end{array}$$

$p : A \to A/_E$

$[2] = [5]$

What is E?

$$
\begin{array}{c}
 \begin{array}{cccc} [1] & [2] & [3] & [4] \end{array} \\
P = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix}
\end{array}
\qquad
P^{op} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix}
$$

$$
V = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix}
$$

$$
= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}
$$

(2) $\cancel{K = f f^{op} =}$

$$
K = f f^{op} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix}
\qquad
e = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix}
$$

$$
m = e^{op} f = \begin{vmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix}
$$

$$
V = m^{*}R = m R m^{op} = \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}
$$

$$
= \begin{vmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}
$$

✓

(3)

$$
U = V = \begin{array}{c}
 \begin{array}{cccc} [1] & [2] & [3] & [4] \end{array} \\
\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}
\end{array}
$$