Applications of Second-Order Differnetial Equations

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Second-order linear differential equations have a variety of applications in science and engineering. In this section we explore two of them: the vibration of springs and electric circuits.

1 Vibrating Springs

We consider the motion of an object with mass m at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level sirface (as in Figure 2).
In Section 6.5 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the **spring constant**). if we ignore any external resistance forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have:

$$m\frac{d^2x}{dt^2} = -kx \text{ or } m\frac{d^2x}{dt^2} + kx = 0 \tag{1}$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$ where $\omega = \sqrt{k/m}$. Thus, the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

which can also be written as

$$x(t) = A \cos(\omega t + \delta)$$

where

$$\omega = \sqrt{\frac{k}{m}} \qquad \text{(frequency)}$$

$$A = \sqrt{c_1^2 + c_2^2} \qquad \text{(amplitude)}$$

$$\cos \delta = \frac{c_1}{A} \qquad \sin \delta = -\frac{c_2}{A} \ (\delta \text{ is the phase angle})$$

(See Excercise 17.) This type of motion is called **simple harmonic motion**

EXAMPLE 1 A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

SOLUTION From Hooke's Law, the force required to stretch the spring is

$$k(0.2) = 25.6$$

so $k = 25.6/0.2 = 128$. Using this value of the spring constant k, together with m = 2 in Equation 1, we have

$$2\frac{d^2x}{dt^2} + 128x = 0$$

As in the earlier general discussion, the solution of this equation is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t \tag{2}$$

We are given the initial condition that $x(0) = 0.2$. But, from equation 2, $x(0) = c_1$. Therefore, $c_1 = 0.2$. Differentiating Equation 2, we get

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$$

Since the initial velocity is give as $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

$$x(t) = \frac{1}{5}\cos 8t$$

2 Damped Vibration

We next consider the motion of a spring that is subject to a frictional force (in the case of horizontal spring of Figure 2) or a damping force (in the case where a vertical spring moves through a fluid as in Figure 3). An example is the dmaping force supplied by a shock absorber in a car or a bicycle.
We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite of the motion. (This has benn confirmed, at least approximately, by some physical experiments.) Thus

$$\text{damping force } = -c\frac{dx}{dt}$$

where c is a positive constant, called the **damping constant.** Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dtx^2} = \text{restoring force} + \text{damping force} = -kx - c\frac{dx}{dt}$$

or

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = c} \tag{3}$$

Equation 3 is a second-order linear differential equation and its auxiliary equation is $mr^2 + cr + k = 0$. The roots are

$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \; r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \tag{4}$$

We need to discuss three cases.

CASE I $c^2 - 4mk > 0$ **(overdamping)**

In this case r_1 and r_2 are distinct real roots and

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so the roots r_1 and r_2 given by Equations 4 must both be negative. This shows that $x \to 0$ as $t \to x$. Typical graphs of x as a function of t are shown in Figure 4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means there is a strong damping force (high-viscosity oil or grase) compared with a weak spring or small mass.

CASE II $c^2 - 4mk = 0$ **(critical damping)**
This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

and the solution is given by

$$x = (c_1 + c_2 t)e^{-(c/2m)t}$$

It is similar to Case I, and typical graphs resemble those in Figure 4 (see Exercise 12), but the damping is just sufficient to surpress vibrations. Any decrease in the viscosity of the fluid leads to the vibrations of the following case.

CASE III $c^2 - 4mk < 0$ **(underdamping)**

Here the roots are complex:

$$\begin{matrix} {}^{1} \\ r_2 \end{matrix} = -\frac{c}{2m} \pm \omega i$$