→ Second order differential equation :-

معادلة تفاضلية من الرتبة الثانية. $a y'' + b y' + c y = f(x)$

a, b, c constants

Solution $y_G = y_H + y_p$

$y_H \rightarrow$ 3 Cases when $a y'' + b y' + c y = \boxed{0}$ $\boxed{f(x) = 0}$

① $y_H = C_1 e^{m_1 x} + C_2 e^{m_2 x}$ if $m_1 \neq m_2$ & real

② $y_H = (C_1 + C_2 x) e^{m x}$ if $m_1 = m_2$ & real

③ $y_H = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$ if $m_{1,2}$ are complex $\alpha \pm \beta i$

$y_p \rightarrow$ الحل الخاص $f(x) \neq 0$

نفرض على حسب شكل $f(x)$

$f(x)$	العرض y_p
$f(x) =$ ثابت	$y_p = A$
$f(x) = 2x + 3$	$y_p = Ax + B$
$f(x) = x^2 + 2$	$y_p = Ax^2 + Bx + C$
$f(x) = \sin x$ or $\cos x$	$y_p = A \sin x + B \cos x$
$f(x) = e^x$	$y_p = A e^x$
$f(x) = x e^x$	$y_p = (Ax + B) e^x$
$f(x) = x \sin x$	$y_p = (A_1 x + B_1) \sin x + (A_2 x + B_2) \cos x$

نفرض اولاً y_p ونشتق مرتين ثم نعوض في المعادله الاساسيه لإيجاد الحل عن طريق مقارنة المعاملات

$\boxed{I}$

Q(1) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 24$ Find $\boxed{y_p}$

$f(x) = 24$

الحل الخاص $y_p = A$ ← ثابت

$y'_p = 0 \quad , y''_p = 0$

Sub into equation

$0 - 5(0) + 6A = 24$

$6A = 24 \implies A = \frac{6}{24} = \boxed{4}$

$y_p = 4$ #

Q(2) $\frac{d^2y}{dx^2} + 25y = 5x^2 + x$

y''

$f(x) = 5x^2 + x$

$y_p = Ax^2 + Bx + C$

$y'_p = 2Ax + B \quad , y''_p = 2A$

$2A + 25(Ax^2 + Bx + C) = 5x^2 + x$

$2A + 25Ax^2 + 25Bx + 25C = 5x^2 + x$ قارن المعاملات

$25A = 5 \rightarrow A = \frac{5}{25} = \frac{1}{5}$

$25B = 1 \rightarrow B = \frac{1}{25}$

$2A + 25C = 0$

$\frac{2}{5} + 25C = 0 \rightarrow C = -\frac{2}{125}$

$y_p = \frac{1}{5}x^2 + \frac{1}{25}x - \frac{2}{125}$ ✗

$\boxed{2}$

(3) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 4\sin x$

$f(x) = 4\sin x$

$y_p = A\sin x + B\cos x$

$y'_p = A\cos x - B\sin x$

$y''_p = -A\sin x - B\cos x$

$-A\sin x - B\cos x - 2A\cos x + 2B\sin x + A\sin x + B\cos x = 4\sin x$

$-\cancel{A} + 2B + \cancel{A} = 4 \qquad B = 4/2 = 2$

$-\cancel{B} - 2A + \cancel{B} = 0 \qquad A = 0$

$y_p = 2\cos x$ #

Solve the following equations:-

(1) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 2e^{-2x}$,

y_H : $y'' + 4y' + 5y = 0$

$m^2 + 4m + 5 = 0$

$m_{1,2} = -2 \pm i$

$\alpha = -2 \qquad B = 1$

$y_H = e^{-2x}[C_1\cos x + C_2\sin x]$

y_p : $f(x) = 2e^{-2x}$

$y_p = Ae^{-2x}$

$y'_p = -2Ae^{-2x}$

$y''_p = 4Ae^{-2x}$

$4A\cancel{e^{-2x}} - 8A\cancel{e^{-2x}} + 5A\cancel{e^{-2x}} = 2\cancel{e^{-2x}}$

$4A - 8A + 5A = 2$

$A = 2$ ⋇

$y_p = 2e^{-2x}$

$y_G = e^{-2x}[C_1\cos x + C_2\sin x] + 2e^{-2x}$

(2) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 54x + 18$

$y_G = y_H + y_P$

y_H : $y'' - 6y' + 9y = 0$

$m^2 - 6m + 9 = 0$

$m_{1,2} = 3$

$y_H = (C_1 + C_2 x) e^{3x}$

y_P : $f(x) = 54x + 18$

$y_P = Ax + B$

$y_P' = A$

$y_P'' = 0$

$0 - 6A + 9Ax + 9B = 54x + 18$

$9A = 54 \quad A = 6$

$-6A + 9B = 18$

$-6(6) + 9B = 18$

$9B = 54 \quad B = 6$

$y_P = 6x + 6$

$y_G = (C_1 + C_2 x) e^{3x} + 6x + 6$

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(3) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 100\sin 4x$

y_H : $y'' - 5y' + 6y = 0$

$m^2 - 5m + 6 = 0$

$m_1 = 3 \quad , m_2 = 2$

$y_H = C_1 e^{3x} + C_2 e^{2x}$

y_P : $f(x) = 100\sin 4x$

$y_P = A\sin 4x + B\cos 4x$

$y_P' = 4A\cos 4x - 4B\sin 4x$

$y_P'' = -16A\sin 4x - 16B\cos 4x$

$-16A\sin 4x - 16B\cos 4x - 20A\cos 4x + 20B\sin 4x$
$+ 6A\sin 4x + 6B\cos 4x = 100\sin 4x$

$\sin 4x[-16A + 20B + 6A] + \cos 4x[-16B - 20A + 6B] = 100\sin 4x$

$-16A + 20B + 6A = 100$

$-20A - 16B + 6B = 0$

$\left.\begin{array}{l} -10A + 20B = 100 \\ -20A - 10B = 0 \end{array}\right\}$

$-20A = 10B$

$A = -\frac{1}{2}B$

$5B + 20B = 100$

$25B = 100$

$B = 4$

$A = -2$

$y_P = -2\sin 4x + 4\cos 4x$

$y_G = C_1 e^{3x} + C_2 e^{2x} - 2\sin 4x + 4\cos 4x$ ⋕

$$\frac{d^2y}{dx^2} - 9y = \underbrace{\underbrace{e^{3x}}_{y_{p_1}} + \underbrace{\sin 3x}_{y_{p_2}}}_{f(x)}$$

$$y_G = \underline{\underline{y_H}} + y_p$$

y_H :
$$y'' - 9y = 0$$
$$m^2 - 9 = 0$$
$$m^2 = 9$$
$$m_{1,2} = \pm 3 \qquad y_H = c_1 e^{3x} + c_2 e^{-3x}$$

تكراره مع $f(x)$

y_p : $f(x) = \underline{\underline{e}}^{3x} + \sin \underline{\underline{3}} x$ نستخدم

e^{3x}

$$y_{p_1} = A x e^{3x} \quad \text{عدد مرات التكرار}$$
$$y'_{p_1} = 3Axe^{3x} + Ae^{3x}$$
$$y''_{p_1} = 9Axe^{3x} + 3Ae^{3x} + 3Ae^{3x}$$

$$\cancel{9Axe^{3x}} + 3Ae^{3x} + 3Ae^{3x} - \cancel{9Axe^{3x}} = e^{3x}$$
$$3A\cancel{e^{3x}} + 3A\cancel{e^{3x}} = \cancel{e^{3x}}$$
$$3A + 3A = \underline{\underline{1}}$$
$$6A = 1$$
$$\underline{\underline{A}} = \frac{1}{6}$$

$$y_{p_1} = \frac{xe^{3x}}{6}$$

$\sin 3x$

$$y_{p_2} = B\sin 3x + C\cos 3x$$
$$y'_{p_2} = 3B\cos 3x - 3C\sin 3x$$
$$y''_{p_2} = -9B\sin 3x - 9C\cos 3x$$

$$-9B\sin 3x - 9C\cos 3x - 9B\sin 3x - 9C\cos 3x = \boxed{1}\sin 3x$$

$$-9B - 9B = 1$$
$$-18B = 1 \qquad B = -\frac{1}{18}$$

$$-9C - 9C = \underline{\underline{0}}$$
$$-18C = 0 \qquad \underline{\underline{C = 0}}$$

$$y_{p_2} = -\frac{\sin(3x)}{18}$$

$$\therefore \; y_p = \frac{xe^{3x}}{6} - \frac{\sin(3x)}{18}$$

$$\therefore \; y_G = c_1 e^{3x} + c_2 e^{-3x} + \frac{xe^{3x}}{6} - \frac{\sin(3x)}{18} \quad \#$$

Solve the equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{-3t}$$

X_H :

$$m^2 + 4m + 3 = 0$$

$$m_1 = -1 \quad , \quad m_2 = -3$$

$$X_H = C_1 e^{-3t} + C_2 e^{-t}$$

X_p : $\quad f(t) = e^{-3t}$ لأنها مكررة

$$X_p = Ate^{-3t}$$

$$X_p' = -3Ate^{-3t} + Ae^{-3t}$$

$$X_p'' = 9Ate^{-3t} - 3Ae^{-3t} - 3Ae^{-3t}$$

$$9At\cancel{e^{-3t}} - 3A\cancel{e^{-3t}} - 3A\cancel{e^{-3t}} - 12At\cancel{e^{-3t}} + 4A\cancel{e^{-3t}} + 3At\cancel{e^{-3t}} = \cancel{e^{-3t}}$$

$$\cancel{9At} - 6A - \cancel{12At} + 4A + \cancel{3At} = 1$$

$$-2A = 1$$

$$A = \frac{1}{-2}$$

$$\therefore X_p = -\frac{te^{-3t}}{2}$$

$$\therefore X_G = C_1 e^{-3t} + C_2 e^{-t} - \frac{te^{-3t}}{2} \quad \#$$

Vairation of parameter

$$ay'' + by' + cy = f(x)$$

$$y_H = \boxed{C_1} y_1 + \boxed{C_2} y_2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$u_1 = \int \frac{f(x) \cdot w_1(x)}{a \cdot w(x)} \, dx$$

$$u_2 = \int \frac{f(x) \cdot w_2(x)}{a \cdot w(x)} \, dx$$

$$w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_2' y_1 - y_1' y_2$$

$$w_1(x) = \begin{vmatrix} 1 & y_2 \\ 0 & y_2' \end{vmatrix} = y_2'$$

$$w_2(x) = \begin{vmatrix} y_1 & 1 \\ y_1' & 0 \end{vmatrix} = -y_1'$$