Introduction to Floquet Theory in QM
-Tutorial-

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1 Linear System of ODEs

1.1 Formal Solution

We investigate the problem of a general linear system of ordinary differential equations

$$\frac{d}{dt}\mathbf{X}(t) = \hat{\mathbf{M}}(t)\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{X}_0, \tag{1}$$

where $\mathbf{X}(t)$ is a vector of time dependent variables $x_i(t)$, $\hat{\mathbf{M}}(t)$ is a time-dependent linear operator, and $\mathbf{X}_0$ is a given initial condition.

Because this is a linear system, we know that the time evolution is a linear transformation from an initial state to a final one, we refer to this matrix as the *fundamental operator* $\hat{\mathbf{\Phi}}(t)$ that takes the state at $t = 0$ and transform it to a final state at a given instant t

$$\mathbf{X}(t) = \hat{\mathbf{\Phi}}(t)\mathbf{X}(0). \tag{2}$$

where clearly $\hat{\mathbf{\Phi}}(0) = \hat{\mathbf{I}}$. For arbitrary initial state, we simply take the arbitrary initial state $\mathbf{X}(t_0)$ and transform it to $\mathbf{X}(0)$ with $\mathbf{\Phi}^{-1}(t_0)$, then evolve it to $\mathbf{X}(t)$ with $\mathbf{\Phi}(t)$. We can express this compactly as $\mathbf{X}(t) = \hat{\mathbf{\Phi}}(t, t_0)\mathbf{X}(t_0)$ where $\hat{\mathbf{\Phi}}(t, t_0) \equiv \hat{\mathbf{\Phi}}(t)\hat{\mathbf{\Phi}}^{-1}(t_0)$, such that $\hat{\mathbf{\Phi}}(t_0, t_0) = \mathbf{I}$. By substituting [2] into [1], we get

$$\frac{d}{dt}\hat{\mathbf{\Phi}}(t) = \hat{\mathbf{M}}(t)\hat{\mathbf{\Phi}}(t), \quad \hat{\mathbf{\Phi}}(0) = \mathbf{I}. \tag{3}$$

By brute force integration we find

$$\begin{aligned}\hat{\mathbf{\Phi}}(t) &= \hat{\mathbf{I}} + \int_0^t dt' \hat{\mathbf{M}}(t')\hat{\mathbf{\Phi}}(t') \\ &= \hat{\mathbf{I}} + \int_0^t dt' \hat{\mathbf{M}}(t') + \int_0^t dt' \hat{\mathbf{M}}(t') \int_0^{t'} dt'' \hat{\mathbf{M}}(t'') + \ldots\end{aligned} \tag{4}$$

not that if we exponentiate the solution as we normally do with functions we get

$$\begin{aligned}\hat{\mathbf{\Phi}}(t) &= \exp\{\int_0^t dt' \hat{\mathbf{M}}(t)\} \\ &= \hat{\mathbf{I}} + \int_0^t dt' \hat{\mathbf{M}}(t') + \frac{1}{2}\int_0^t dt' \hat{\mathbf{M}}(t') \int_0^t dt'' \hat{\mathbf{M}}(t'') + \ldots\end{aligned} \tag{5}$$

unless $[\hat{\mathbf{M}}(t'), \hat{\mathbf{M}}(t'')] = 0$ for all times, orders higher than one in the exponentiation does not match the natural ordering of the evolution imposed by the differential equation, therefore we write the solution formally as

$$\hat{\mathbf{\Phi}}(t) = \mathcal{T}\exp\left[\int_0^t \hat{\mathbf{M}}(t')dt'\right], \tag{6}$$

where $\mathcal{T}$ is the time order operator.

1.2 Floquet Theory

If $\hat{\mathbf{M}}(t)$ is periodic $\hat{\mathbf{M}}(t + T) = \hat{\mathbf{M}}(t)$ with period T, the solution of the fundamental matrix $\hat{\mathbf{\Phi}}(t)$ is of the *Floquet normal form*

$$\hat{\mathbf{\Phi}}(t) = \hat{\mathbf{P}}(t)e^{\hat{\mathbf{R}}t}, \quad \hat{\mathbf{\Phi}}(0) = \hat{\mathbf{I}}, \tag{7}$$

where $\hat{\mathbf{P}}(t)$ is periodic matrix $\hat{\mathbf{P}}(t + T) = \hat{\mathbf{P}}(t)$ with the same periodicity T, and $\hat{\mathbf{R}}$ is a constant matrix.

For a periodic $\hat{\mathbf{M}}(t)$ with period T, the differential equation after one period reads

$$\frac{d}{dt}\hat{\mathbf{\Phi}}(t + T) = \hat{\mathbf{M}}(t)\hat{\mathbf{\Phi}}(t + T), \quad \hat{\mathbf{\Phi}}(0) = \mathbf{I}, \tag{8}$$

implying that $\mathbf{\Phi}(t + T) = \mathbf{\Phi}(t)\hat{\mathbf{C}}$ up to a constant invertible matrix which satisfies $\mathbf{\Phi}(T) = \mathbf{\Phi}(0)\hat{\mathbf{C}} \to \mathbf{\Phi}(T) = \hat{\mathbf{C}}$, so the solution of the fundamental matrix must be of the form

$$\mathbf{\Phi}(t + T) = \mathbf{\Phi}(t)\mathbf{\Phi}(T) \tag{9}$$

where $\mathbf{\Phi}(T)$ is referred to as the *monodromy matrix.* Note that for n periods we find

$$\mathbf{\Phi}(t + nT) = \mathbf{\Phi}(t)\mathbf{\Phi}(nT) = \mathbf{\Phi}(t)[\mathbf{\Phi}(T)]^n, \tag{10}$$

where the monodromy matrix satisfies the relation $\mathbf{\Phi}(nT) = [\mathbf{\Phi}(T)]^n$, this suggets the general exponential form $\mathbf{\Phi}(T) = e^{\hat{\mathbf{R}}T}$ where $\hat{\mathbf{R}}$ is a constant matrix. Any time translation t can be decomposed into an integer multiple of the period plus a remainder $t \to (t \bmod T) + nT, n \in \mathbb{Z}$, such that we can write the evolution operator as

$$\mathbf{\Phi}(t) = \mathbf{\Phi}(t \bmod T)e^{\hat{\mathbf{R}}T} \equiv \hat{\mathbf{P}}'(t)e^{\hat{\mathbf{R}}T}, \tag{11}$$

the evolution between the periods $\hat{\mathbf{P}}'(t+T) = \hat{\mathbf{P}}'(t)$ is clearly periodic, however, it is conventional to write the exponential part with a continuous time parameter $T = t - (t \bmod T)$

$$\mathbf{\Phi}(t) = \hat{\mathbf{P}}'(t)e^{-\hat{\mathbf{R}}(t \bmod T)}e^{\hat{\mathbf{R}}t} \equiv \hat{\mathbf{P}}(t)e^{\hat{\mathbf{R}}t}, \tag{12}$$

and absorb the extra part in the definition of the evolution between periods since it does not affect its periodicity. For a general initial the solution read

$$\mathbf{\Phi}(t; t_0) = \hat{\mathbf{P}}(t)e^{\hat{\mathbf{R}}(t-t_0)}\hat{\mathbf{P}}^{-1}(t_0), \tag{13}$$

we can go further and insert an identity $\hat{\mathbf{I}} = \hat{\mathbf{P}}^{-1}(t_0)\hat{\mathbf{P}}(t_0)$ such that we are able to re-write this more compactly as

$$\mathbf{\Phi}(t; t_0) = \hat{\mathbf{P}}(t)[t_0]e^{\hat{\mathbf{R}}[t_0](t-t_0)}, \tag{14}$$

where $\hat{\mathbf{P}}(t)[t_0] \equiv \hat{\mathbf{P}}(t)\hat{\mathbf{P}}^{-1}(t_0)$ and $\hat{\mathbf{R}}[t_0] = \hat{\mathbf{P}}(t_0)\hat{\mathbf{R}}\hat{\mathbf{P}}^{-1}(t_0)$, this introduces the *initial time gauge freedom* in the solution. *Floquet theory is the statement that for a periodic drive, the evolution operator is decomposition of a time-independent generator evolution followed by a periodic (time-dependent) evolution.* This is a powerful statement and we can already see how it invokes the time-independent frame in quantum mechanics.

2 Time Dependent Schrodinger Equation

2.1 General Solution

The Schrodinger equation written in terms of the fundamental operator, or the *evolution operator* $\hat{\mathcal{U}}(t)$ which is unitary in quantum mechanics $\hat{\mathcal{U}}^{-1}(t) = \hat{\mathcal{U}}^{\dagger}(t)$ (because it is the exponential of a Hermitian operator, i.e. the Hamiltonian $\hat{\mathbf{H}} = \hat{\mathbf{H}}^{\dagger}$) such that for any instantaneous state $|\psi(t)\rangle = \hat{\mathcal{U}}(t)\hat{\mathcal{U}}^{\dagger}(t_0)|\psi(t_0)\rangle$ is given by

$$\frac{d}{dt}\mathcal{U}(t) = -\frac{i}{\hbar}\hat{\mathbf{H}}(t)\mathcal{U}(t), \quad \mathcal{U}(0) = \hat{\mathbf{I}}. \tag{15}$$

The formal solution as disccused in the previous section is $\hat{\mathcal{U}}(t) = \mathcal{T}\exp\{\int_0^t dt'\hat{\mathbf{H}}(t')\}$.

2.2 Periodic Hamiltonian

Periodic Hamiltonian $\hat{\mathbf{H}}(t+nT) = \hat{\mathbf{H}}(t), n \in \mathbb{Z}$, where $T = \frac{2\pi}{\omega}$ is the time period, can always be expanded it as

$$\hat{\mathbf{H}}(t) = \hat{\mathbf{H}}_0 + \sum_{\substack{n=-\infty \\ \neq 0}}^{\infty} \hat{\mathbf{V}}^{(n)} e^{in\omega t}. \tag{16}$$

The solution for the periodic Hamiltonian must be of the normal Floquet form

$$\hat{\mathcal{U}}(t) = \hat{\mathbf{U}}(t)e^{-\frac{i}{\hbar}\hat{\mathbf{H}}_F t}, \tag{17}$$

where $\hat{\mathbf{H}}_F$ is time independent, and $\hat{\mathbf{U}}(t) = \hat{\mathbf{U}}(t+T)$ is periodic with the same periodicity as the Hamiltonian.

If the initial state is prepared in the bases that diagonlaizes the effective Hamiltonian $\hat{\mathbf{H}}_F = \sum_\alpha \varepsilon_\alpha \left|u_\alpha^F\right\rangle\left\langle u_\alpha^F\right|$ as

$$|\Psi(0)\rangle = \sum_\alpha c_\alpha \left|u_\alpha^F(0)\right\rangle. \tag{18}$$

Now we evolve this state using the evolution operator

$$\begin{aligned} |\Psi(t)\rangle &= \sum_\alpha c_\alpha e^{-\frac{i}{\hbar}\varepsilon_\alpha t}\hat{\mathbf{U}}(t)\left|u_\alpha^F\right\rangle \\ &\equiv \sum_\alpha c_\alpha e^{-\frac{i}{\hbar}\varepsilon_\alpha t}\left|u_\alpha^F(t)\right\rangle \\ &\equiv \sum_\alpha c_\alpha \left|\psi_\alpha^F(t)\right\rangle, \end{aligned} \tag{19}$$

we can nicely say that any state can be written as a time-independent superposition of time-dependent *Floquet states* $\left|\psi_\beta^F(t)\right\rangle$, with *quasi-energies* ε_β. Now we note that $\hat{\mathbf{U}}(t)$ is exactly the frame transformation that take us to a time-indpendent evolution with $\hat{\mathbf{H}}_F$. If we insert the Floquet form into the Schrodinger equation, we get

$$\hat{\mathbf{H}}_F = \hat{\mathbf{U}}^{\dagger}(t)\Big(\hat{\mathbf{H}}(t) - i\hbar\frac{d}{dt}\Big)\hat{\mathbf{U}}(t) \quad \hat{\mathbf{U}}(t+T) = \hat{\mathbf{U}}(t) \tag{20}$$

where the operator $\left(\hat{\mathbf{H}}(t) - i\hbar\frac{d}{dt}\right)$ is referred to as the *quasi energy* operator, which is diagonalized by the unitary frame transformation $\hat{\mathbf{U}}(t)$. This is our main equation to solve.

Recall that if a system is initially in an eigenstate of a time-independent Hamiltonian, the observables become time-independent at all times

$$|\psi(t)\rangle = e^{-iE_\alpha t}|\psi(0)\rangle \quad \Rightarrow \quad \langle\mathcal{O}\rangle \neq f(t), \tag{21}$$

and therefore we refer to these states as stationary ones. Similarly, if a system is initially in an eigenstate (Floquet states) of a time-dependent *periodic* Hamiltonian, the observables become time-independent at *stroboscopic times*

$$\left|\psi_\alpha^F(nT)\right\rangle = e^{-i\varepsilon_\alpha nT}\left|\psi_\alpha^F(0)\right\rangle \quad \Rightarrow \quad \langle\mathcal{O}\rangle \neq f(nT). \tag{22}$$

This is a powerful statement, because if the periods are small compared to the problem's time scale, then any observable you read is indeed agnostic to the in-between periods dynamics, i.e. Floquet states are stroboscopically stationary states.

2.2.1 Extended Fourier Space (Sambe's Space)

The physical Hilbert space $\mathcal{H}_{phys}$ of dimension d spanned by a finite time-dependent physical Floquet modes $\sum_\alpha \left|u_\alpha^F(t)\right\rangle\left\langle u_\alpha^F(t)\right| = \hat{\mathbf{I}}$ can be equivalently described in the extended Hilbert space, which is decomposed of the physical space and the Fourier harmonics $\mathcal{H}_{phys}\otimes\mathcal{H}_{Fourier}$ of dimension $d\otimes\infty$ spanned by an infinite time-independent states $\sum_{\alpha n}|u_\alpha, n\rangle\langle u_\alpha, n| = \hat{\mathbf{I}}$. This is referred to as *Sambe's space* in the literature.

As mentioned earlier our main goal is to find $\hat{\mathbf{U}}(t)$ that diagonalizes the quasi energy operator $\left(\hat{\mathbf{H}}(t) - i\hbar\frac{d}{dt}\right)$, we approach this by first projecting onto the Floquet Hamiltonian's eigenbases

$$\left\langle u_\alpha^F\right|\left(\hat{\mathbf{H}}(t) - i\hbar\frac{d}{dt}\right)\hat{\mathbf{U}}(t)\left|u_\beta^F\right\rangle = \varepsilon_\beta\left\langle u_\alpha^F\right|\hat{\mathbf{U}}(t)\left|u_\beta^F\right\rangle, \tag{23}$$

and then expand $\hat{\mathbf{U}}(t) = \sum_m \hat{\mathbf{U}}^m e^{im\omega t}$ and $\hat{\mathbf{H}}(t) = \sum_m \hat{\mathbf{H}}^m e^{im\omega t}$, and then solve for these Fourier coefficients, we arrive at

$$\sum_{k=-\infty}^{\infty}\sum_{\gamma}^{d}\Big[H_{\alpha\gamma}^{(n-k)} + k\omega\delta_{nk}\delta_{\alpha\gamma}\Big]U_{\gamma\beta}^{*(k)} = \varepsilon_\beta U_{\alpha\beta}^{*(n)}. \tag{24}$$

We see that if we add $l\omega$ to the quasi energy where $l \in \mathbb{Z}$, the equation remains invariant under the shift $U_{\gamma\beta}^{k+l}$ and renaming $k - l \to k$, as we can absorb it into the infinite sum. We can further introduce the space of both Fourier and physical such that

$$\begin{aligned} &\hat{\mathbf{U}}(t)\left|u_\alpha^F\right\rangle = \sum_{\beta,n} U_{\alpha\beta}^{(n)} e^{in\omega t}\left|u_\beta^F\right\rangle \Leftrightarrow |\mathcal{F}_\alpha\rangle = \sum_{\beta,n} U_{\alpha\beta}^{(n)}\left|u_\beta^F, n\right\rangle, \\ &\left\langle u_\alpha^F, n\middle|u_\beta^F, m\right\rangle \equiv \delta_{\alpha\beta}\delta_{mn}, \\ &\sum_\alpha^d\sum_{n=-\infty}^{\infty}\left|u_\alpha^F, n\right\rangle\left\langle u_\alpha^F, n\right| = \mathbf{I}. \quad \text{(Sambe's Space)} \end{aligned} \tag{25}$$

where now $U^{(n)}_{\alpha\beta} = \left\langle u^F_\beta \middle| \hat{\mathbf{U}}^n \left| u^F_\alpha \right\rangle = \left\langle u^F_\beta, n \middle| \mathcal{F}_\alpha \right\rangle$. In that extended space the eigenvalue problem reads

$$\mathcal{H}_F \left|\mathcal{F}_{\alpha l}\right\rangle = \varepsilon_{\alpha l} \left|\mathcal{F}_{\alpha l}\right\rangle, \tag{26}$$

where $\varepsilon_{\alpha l} = \varepsilon_\alpha + l\omega$. The extended Floquet Hamiltonian now reads $\left\langle u^F_\alpha, n\right| \hat{\mathcal{H}}_F \left|u^F_\gamma, k\right\rangle = \hat{H}^{n-k}_{\alpha\gamma} + k\omega\delta_{nk}\delta_{\alpha\gamma}$. The evolution operator expanded onto the Floquet bases reads

$$\begin{aligned}
\mathcal{U}_{\alpha\beta}(t;t_0) &= \left\langle u^F_\alpha\right| \hat{\mathbf{U}}(t) e^{-i\hat{\mathbf{H}}_F\tau} \hat{\mathbf{U}}^\dagger(t_0) \left|u^F_\beta\right\rangle \\
&= \sum_\sigma \left\langle u^F_\alpha\right| \hat{\mathbf{U}}(t) \left|u^F_\sigma\right\rangle e^{-i\varepsilon_\sigma\tau} \left\langle u^F_\sigma\right| \hat{\mathbf{U}}^\dagger(t_0) \left|u^F_\beta\right\rangle \\
&= \sum_{nm\sigma} U^n_{\alpha\sigma} e^{in\omega t} e^{-i\varepsilon_\sigma\tau} (U^m_{\beta\sigma})^* e^{-im\omega t_0} \\
&= \sum_{nm\sigma} \left\langle u^F_\alpha, n \middle| \mathcal{F}_\sigma\right\rangle e^{in\omega t} e^{-i\varepsilon_\sigma\tau} \left\langle \mathcal{F}_\sigma \middle| u^F_\beta, m\right\rangle e^{-im\omega t_0} \\
&= \sum_{nm\sigma} \left\langle u^F_\alpha, n \middle| \mathcal{F}_\sigma\right\rangle e^{in\omega t} e^{-i(\varepsilon_\sigma + m\omega)\tau} \\
&\qquad\qquad \left\langle \mathcal{F}_{\sigma(-m)} \middle| u^F_\beta, 0\right\rangle e^{im\omega t} \\
&= \sum_{nm\sigma} \left\langle u^F_\alpha, n-m \middle| \mathcal{F}_{\sigma(-m)}\right\rangle e^{i(n-m)\omega t} \\
&\qquad\qquad e^{-i(\varepsilon_\sigma - m\omega)\tau} \left\langle \mathcal{F}_{\sigma(-m)} \middle| u^F_\beta, 0\right\rangle \\
&= \sum_{lk\sigma} \left\langle u^F_\alpha, k \middle| \mathcal{F}_{\sigma l}\right\rangle e^{ik\omega t} e^{-i\varepsilon_{\sigma l}\tau} \left\langle \mathcal{F}_{\sigma l} \middle| u^F_\beta, 0\right\rangle \\
&= \sum_k \left\langle u^F_\alpha, k\right| e^{-i\mathcal{H}_F\tau} \left|u^F_\beta, 0\right\rangle e^{ik\omega t}.
\end{aligned} \tag{27}$$

where $\tau \equiv t - t_0$. Now the time-dependent transitions of the bare states is transformed to a time-independent transitions between Floquet states. Using this evolution operator we can calculate any observable

$$\langle\mathcal{O}\rangle(t) = \langle\Psi(t_0)|\,\hat{\mathcal{U}}(t;t_0)\hat{\mathcal{O}}(t_0)\hat{\mathcal{U}}^\dagger(t;t_0)\,|\Psi(t_0)\rangle \tag{28}$$

and for the probability of transition $P_{\alpha\beta}(t;t_0) = |\hat{\mathcal{U}}_{\alpha\beta}(t;t_0)|^2$ between Floquet states read

$$\begin{aligned}
P_{\alpha\beta}(t;t_0) &= \sum_k \left\langle u^F_\alpha, k\right| e^{-i\mathcal{H}_F\tau} \left|u^F_\beta, 0\right\rangle e^{ik\omega t} \\
&\quad \sum_{k'} \left\langle u^F_\beta, 0\right| e^{i\mathcal{H}_F\tau} \left|u^F_\alpha, k'\right\rangle e^{-ik'\omega t} \\
&= \sum_{k\sigma l} \left\langle u^F_\alpha, k \middle| \mathcal{F}_{\sigma l}\right\rangle \left\langle \mathcal{F}_{\sigma l} \middle| u^F_\beta, 0\right\rangle e^{i(\varepsilon_{\sigma l} - \varepsilon_{\sigma' l'})\tau} \\
&\quad \sum_{\sigma' l' k'} \left\langle u^F_\beta, 0 \middle| \mathcal{F}_{\sigma' l'}\right\rangle \left\langle \mathcal{F}_{\sigma' l'} \middle| u^F_\alpha, k'\right\rangle e^{i(k-k')\omega t}
\end{aligned} \tag{29}$$

and for the average transition probability $\bar{P}_{\alpha\beta} = \frac{1}{T}\int_{t_0}^{t_0+T} P_{\alpha\beta}(t;t_0)$, we get

$$\bar{P}_{\alpha\beta} = \sum_{k\sigma l} \left| \left\langle u^F_\alpha, k \middle| \mathcal{F}_{\sigma l}\right\rangle \left\langle \mathcal{F}_{\sigma l} \middle| u^F_\beta, 0\right\rangle \right|^2 \tag{30}$$

note that the validity of the average here assumes that $\varepsilon_{\sigma l} - \varepsilon_{\sigma' l'} \gg t - t_0$ such that the interferences dynamics are negligible within that interval.

2.2.2 Brillouin-Wigner Approach

The Brillouin-Wigner theorem makes use of the time-independent problem of the extended space, and solves for the Floquet Hamiltonian using the Feshbach projection formalism which gives us a perturbative expansion in terms of m-photon processes, we start by the time-independent problem

$$\hat{\mathcal{H}}_F \left|\mathcal{F}_{\alpha n}\right\rangle = \varepsilon_{\alpha n} \left|\mathcal{F}_{\alpha n}\right\rangle, \tag{31}$$

and now if we define a projector $\hat{P}$ onto the $n = 0$ subspace, and the rest space of the space is $\hat{Q} = \hat{\mathbf{I}} - \hat{P}$ such that

$$\hat{P} = \sum_\alpha \left|u_\alpha, 0\right\rangle \left\langle u_\alpha, 0\right|, \quad \left(\hat{P} + \hat{Q}\right) \left|\mathcal{F}_{\alpha n}\right\rangle = \left|\right\rangle \tag{32}$$

Now you can eliminate the m-photon couplings as an effective corrections onto that subspace using Feshbach formalism

$$\mathcal{H}_F \left|\mathcal{F}_{\alpha n}\right\rangle \tag{33}$$

2.2.3 Magnus Expansion

Here, we introduce the explicit form of the unitary transformation $\hat{\mathbf{U}}(t) = e^{-i\hat{\mathbf{K}}(t)}$ with the *kick operator*, which clearly satisfies $\hat{\mathbf{K}}(nT) = 0$. So now the unitary evolution operator reads

$$\hat{\mathcal{U}}(t_0 + T; t_0) = e^{-i\hat{\mathbf{K}}(t)} e^{-i\hat{\mathbf{H}}_F(t-t_0)} = e^{\Omega(t)}, \tag{34}$$

now at stroboscopic times we get the time-independent evolution

$$\hat{\mathcal{U}}(T) = e^{-i\hat{\mathbf{H}}_F T} \tag{35}$$

3 Examples

3.1 Qubit

Here we investigate a two-level system that lives in a $d = 2$ dimensional Hilbert space spanned by the bases $\{|g\rangle, |e\rangle\}$ as

$$|\Psi(t)\rangle = c_g(t)\,|g\rangle + c_e(t)\,|e\rangle, \tag{36}$$

the state's time evolution is generated by the Hamiltonian

$$\hat{\mathbf{H}}(t) = \frac{\omega_0}{2}\sigma_z + \frac{\Omega_0}{2}[e^{i\omega t} + e^{-i\omega t}]\sigma_x, \tag{37}$$

where ω_0 is the transition energy, Ω_0 is the coupling strength, and ω is the field's frequency. This is the so-called semi-classical Rabi model.

3.1.1 Dyson Series

We start with a perturbative approach for the small parameter $\frac{\Omega_0}{\omega_0} \ll 1$. We first go to the interaction picture with $\hat{\mathbf{U}}(t) = e^{\frac{i}{\hbar}\hat{\mathbf{H}}_0 t} = e^{i\frac{\omega_0}{2}t\sigma_z}$, where the dynamics in that frame read

$$\begin{aligned}\hat{\mathbf{H}}_I(t) =& \frac{\Omega_0}{2}\Big(\Big(e^{-i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}\Big)\hat{\sigma} \\ &+ \Big(e^{i(\omega_0-\omega)t} + e^{i(\omega_0+\omega)t}\Big)\hat{\sigma}^\dagger\Big).\end{aligned} \tag{38}$$

The evolution operator reads

$$\hat{\mathcal{U}}(t;t_0) = e^{-i\hat{\mathbf{H}}_0 t}\mathcal{T}\exp[\int_0^t \hat{\mathbf{H}}_I(t')dt']e^{i\hat{\mathbf{H}}_0 t_0}, \tag{39}$$

and if the system initially at $t_0 = 0$ in the ground state then the evolved state up to 1st order in Ω_0 reads

$$\begin{aligned}|\Psi(t)\rangle =& e^{-i\omega_0 t}|g\rangle \\ &+ \frac{\Omega_0}{2}\left(\frac{e^{-i(\omega_0-\omega)t}-1}{(\omega-\omega_0)} - \frac{e^{i(\omega_0+\omega)t}-1}{(\omega+\omega_0)}\right)e^{i\omega_0 t}|e\rangle \\ &+ \mathcal{O}[(\frac{\Omega_0}{2})^2].\end{aligned} \tag{40}$$

We see that the rotating and counter rotating terms perturb the bare states with the same order of the coupling Ω_0. However, the driving frequency ω clearly determines the dominant term. Also, the truncation here doesn't guarantee unitary evolution, so besides that $\frac{\Omega_0}{\omega_0} \ll 1$, both ω and Ω_0 should lie within the regime that makes $|c_e(t)|^2 \leq 1$ at all times.

3.1.2 Rotating Wave Approximation

We saw earlier that if $|\omega_0 - \omega| \ll |\omega_0 + \omega|$ then the counter rotating terms become negligible compared to the rotating ones. We first go to the rotating frame $\hat{\mathbf{U}}(t) = e^{i\frac{\omega}{2}t\hat{\sigma}_z}$

$$\begin{aligned}\hat{\mathbf{H}}_{\mathrm{RF}}(t) &= \frac{\Delta}{2}\hat{\sigma}_z + \frac{\Omega_0}{2}\Big(\hat{\sigma} + \hat{\sigma}^\dagger + \hat{\sigma}e^{i2\omega t} + \hat{\sigma}^\dagger e^{-i2\omega t}\Big) \\ &= \hat{\mathbf{H}}_{\mathrm{RF}}^{\mathrm{RWA}} + \hat{\mathbf{H}}_{2\omega}(t)\end{aligned} \tag{41}$$

Let's define that the full cycle of transition probability by T_{sys} defined by the time-independent part and average the Hamiltonian

$$\begin{aligned}&\frac{1}{T_{sys}}\int_0^{T_{sys}} \hat{\mathbf{H}}_{\mathrm{RF}}(t) \\ &= \hat{\mathbf{H}}_{\mathrm{RF}}^{\mathrm{RWA}} + \frac{\Omega_0}{2i\pi}\frac{T_\omega}{T_{sys}}\Big((e^{i\pi\frac{T_{sys}}{T_\omega}}-1)\hat{\sigma} - (e^{-i\pi\frac{T_{sys}}{T_\omega}}-1)\hat{\sigma}^\dagger\Big) \\ &= \hat{\mathbf{H}}_{\mathrm{RF}}^{\mathrm{RWA}} + \langle\hat{\mathbf{H}}_{2\omega}(t)\rangle\end{aligned} \tag{42}$$

if $T_{sys} \gg T_\omega$, we safely get the time-independent part, where $\Delta \equiv \omega_0 - \omega$ is the detunning. If the drive's period $T_w = \frac{2\pi}{\omega}$ is small compared to the time-scale of the system's evolution $T_{sys} = \frac{2\pi}{\Omega}$, i.e. $\frac{\Omega}{\omega} \ll 1$, we can eliminate the fast rotating terms 2ω as their effects average out

$$\hat{\mathbf{H}}_{\mathrm{RF}}^{\mathrm{RWA}} = \frac{\Delta}{2}\hat{\sigma}_z + \frac{\Omega_0}{2}\Big(\hat{\sigma} + \hat{\sigma}^\dagger\Big), \tag{43}$$

now this evolution $e^{i\hat{\mathbf{H}}_{\mathrm{RF}}^{\mathrm{RWA}}t}$ defines our T_{sys}, that we used earlier, this is the so-called rotating wave approximation, which allow us to obtain a time-independent Hamiltonian, and if the system initially in the ground state, the evolution in the lab frame reads

$$\hat{\mathcal{U}}(t,t_0) = e^{i\frac{\omega}{2}\sigma_z t}e^{-i\hat{\mathbf{H}}_{\mathrm{RF}}^{\mathrm{RWA}}t}e^{-i\frac{\omega}{2}\sigma_z t_0}, \tag{44}$$

and if the system initially at $t_0 = 0$ in the ground state then the evolved state reads

$$\begin{aligned}|\Psi(t)\rangle = e^{i\frac{\omega}{2}\sigma_x t}\Big[&\Big(\cos\Big(\frac{\Omega_R t}{2}\Big) + i\frac{\Delta}{\Omega_R}\sin\Big(\frac{\Omega_R t}{2}\Big)\Big)|g\rangle \\ &+ \Big(-i\frac{\Omega}{\Omega_R}\sin\Big(\frac{\Omega_R t}{2}\Big)\Big)|e\rangle\Big]\end{aligned} \tag{45}$$

where $\Omega_R = \sqrt{\Delta^2 + \Omega^2}$. However, this approach fails if $\frac{\Omega_0}{\omega} \simeq 1$ and also if $\frac{\Omega_0}{\omega_0} \simeq 1$ because although the fast rotations will still average out, but their amplitude will alter the evolutions significantly.

3.1.3 Magnus Expansion

3.1.4 Floquet theory

Here we make use of the periodicity of the Hamiltonian

$$\hat{\mathbf{H}}(t) = \hat{\mathbf{H}}^{(0)} + \hat{\mathbf{H}}^{(-1)} e^{-i\omega t} + \hat{\mathbf{H}}^{(+1)} e^{i\omega t} \tag{46}$$

where $\hat{\mathbf{H}}^{(0)} = \frac{\omega_0}{2}\hat{\sigma}_z$, $\hat{\mathbf{H}}^{(\pm)} = \frac{\Omega_0}{2}\hat{\sigma}_x$. For illustration, we will show how the structure is built for truncation $M = 1$, the eigenvalue problem in the extended space reads

$$\begin{bmatrix} \hat{\mathbf{H}}^{(0)} - \omega & \hat{\mathbf{H}}^{(+1)} & 0 \\ \hat{\mathbf{H}}^{(-1)} & \hat{\mathbf{H}}^{(0)} & \hat{\mathbf{H}}^{(+1)} \\ 0 & \hat{\mathbf{H}}^{(-1)} & \hat{\mathbf{H}}^{(0)} + \omega \end{bmatrix} \begin{bmatrix} \begin{bmatrix} U_{\alpha 1}^{(-1)} \\ U_{\alpha 2}^{(-1)} \end{bmatrix} \\ \begin{bmatrix} U_{\alpha 1}^{(0)} \\ U_{\alpha 2}^{(0)} \end{bmatrix} \\ \begin{bmatrix} U_{\alpha 1}^{(+1)} \\ U_{\alpha 2}^{(+1)} \end{bmatrix} \end{bmatrix} = \varepsilon_\alpha \begin{bmatrix} \begin{bmatrix} U_{\alpha 1}^{(-1)} \\ U_{\alpha 2}^{(-1)} \end{bmatrix} \\ \begin{bmatrix} U_{\alpha 1}^{(0)} \\ U_{\alpha 2}^{(0)} \end{bmatrix} \\ \begin{bmatrix} U_{\alpha 1}^{(+1)} \\ U_{\alpha 2}^{(+1)} \end{bmatrix} \end{bmatrix}. \tag{47}$$

Now we can construct the evolution operator as

$$\hat{\mathcal{U}}(t) = e^{-i\varepsilon_\alpha t} \sum_{\beta m} e^{im\omega t} U_{\alpha\beta}^{m} \left| u_\alpha^F \right\rangle \tag{48}$$

After we solve for these eigenvalues (quasi energies) and the eigenvectors (Floquet modes), we construct the Floquet states as

$$\left| \psi_\alpha(t) \right\rangle = e^{-i\varepsilon_\alpha t} \left| u_\alpha^F(t) \right\rangle = e^{-i\varepsilon_\alpha t} \sum_{m}^{M} e^{im\omega t} \left| u_{\alpha m} \right\rangle. \tag{49}$$

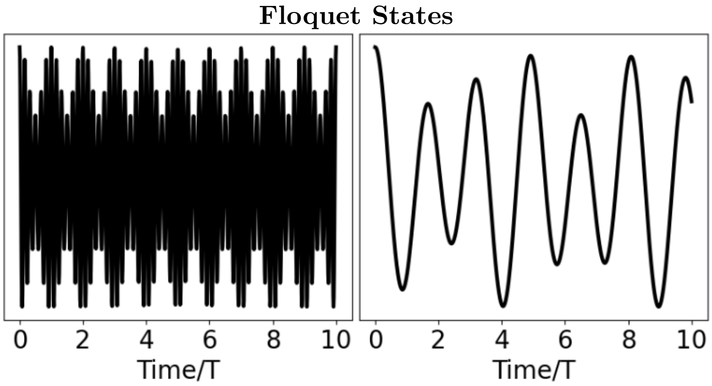

Figure 1: Floquet mode (left) $\left| u_\alpha^F(t+nT) \right\rangle = \left| u_\alpha^F(t) \right\rangle$ and Floquet states (right) $\left| \psi_\alpha^F(t+nT) \right\rangle = e^{-i\varepsilon_\alpha nT} \left| \psi_\alpha^F(t) \right\rangle$.

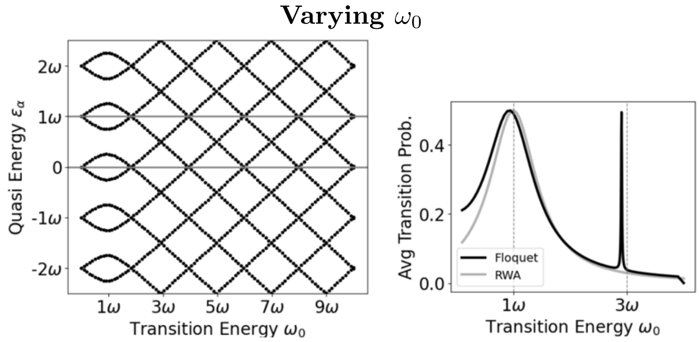

Figure 2: $\Omega_0 = 0.5\omega_0$.

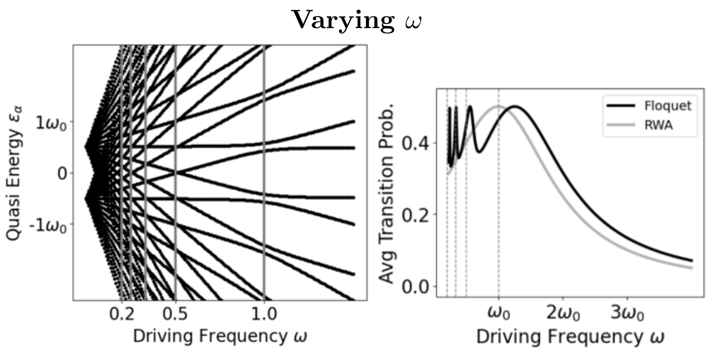

Figure 3: $\Omega_0 = 0.15\omega_0$ (left) and $\Omega_0 = \omega_0$ (right).

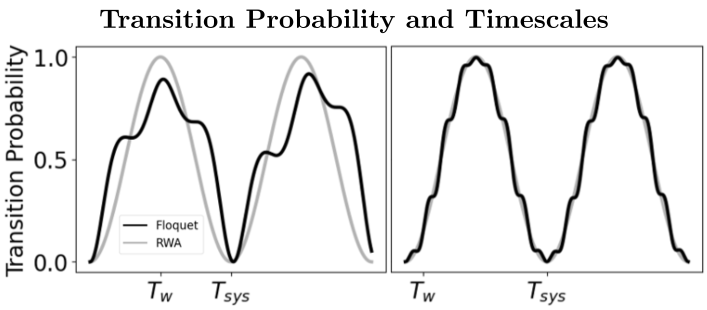

Figure 4: $\Delta = 0$, $\Omega_0 = 1$ (left) $\Omega_0 = 0.1$ (right), and $T_{sys} = \frac{2\pi}{\Omega_R}$ where $\Omega_R = \sqrt{\Omega_0^2 + \Delta^2}$.

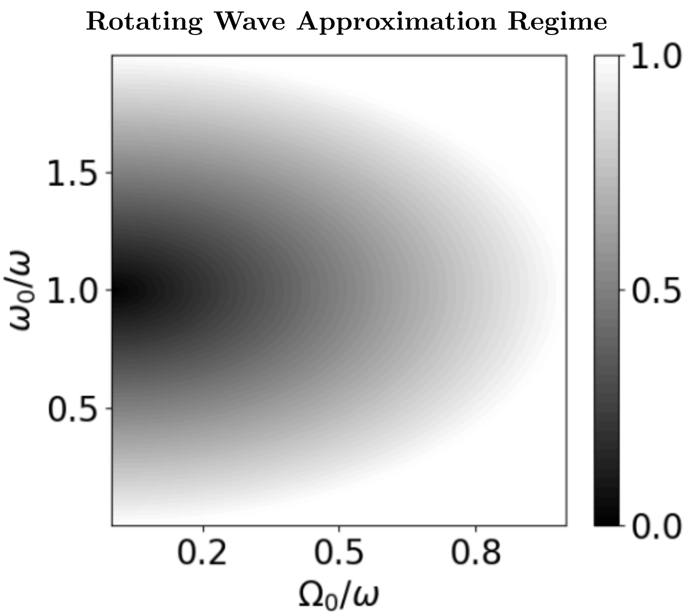

Figure 5: RWA validity $\frac{T_w}{T_{sys}} = \sqrt{(\frac{\Omega_0}{\omega})^2 + (1 - \frac{\omega_0}{\omega})^2} \ll 1$.