Box #_____
Math 65 Section 1
Homework 7
25 May 2018

1. Suppose A is a matrix with eigenvalues $\lambda_1 = -1, \lambda_2 = 2$ and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use the given eigendata to provide a qualitative sketch of the phase portrait for the associated linear system $\mathbf{x}' = A\,\mathbf{x}$.

∎

2. Let $M = \begin{bmatrix} 0 & q \\ -q & 0 \end{bmatrix}$. Use the power series definition to show

$$e^{Mt} = \begin{bmatrix} \cos qt & \sin qt \\ -\sin qt & \cos qt \end{bmatrix}.$$

■

3. In this example we show that the property $e^{a+b} = e^a e^b$ does not extend to matrices. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Compute e^{At}, e^{Bt}, and $e^{(A+B)t}$.

(b) Show $e^{At}e^{Bt} \neq e^{(A+B)t}$.

(c) Show $e^{At}e^{Bt} \neq e^{Bt}e^{At}$.

If $AB = BA$ then $e^{At+Bt} = e^{At}e^{Bt} = e^{Bt}e^{At}$. You can freely use this fact when needed in this course, including in the next few exercises where it will come in handy!

■

4. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Compute e^{At} by first expressing A in the form $A = I + M$ for some matrix M. Be sure to justify your steps.

(b) Solve the initial-value problem $\mathbf{x}' = A\,\mathbf{x}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

■

5. Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}$.

(a) Show that A is diagonalizable and determine a matrix P such that $P^{-1}AP = D$.

(b) Use your result from part (a) to compute the matrix exponential e^{At}.

Simplify your answer by using our hyperbolic trig friends $\cosh x = \frac{e^x+e^{-x}}{2}$ and $\sinh x = \frac{e^x-e^{-x}}{2}$ (it may help to remember these by recalling $\sinh 0 = 0$ and $\cosh 0 = 1$, like their cousins).

■

6. Compute the matrix exponential for the system

$$\mathbf{x}' = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mathbf{x}, \tag{1}$$

where $a, b \in \mathbb{R}$ with $b \neq 0$. Hint: $A = aI + N$ for some matrix N. What is N^2?
This is an example where the matrix is not diagonalizable (a is an eigenvalue of algebraic multiplicity 2 but geometric multiplicity 1), so the ansatz method from yesterday's homework does not produce a full basis for the solution space (but now with e^{At} we're all good!).

■

7. Compute the matrix exponential for the system

$$\mathbf{x}' = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix} \mathbf{x}, \tag{2}$$

where $\alpha \in \mathbb{R}$ is a parameter.

■

8. Consider the initial-value problem for (2) with initial state $\mathbf{x}(0) = \mathbf{x}_0$.

(a) We know the solution of the IVP has the form $\mathbf{x}(t) = e^{At}\,\mathbf{x}_0$. Show that this can be expressed in the form $\mathbf{x}(t) = e^{\alpha t}R(t)\mathbf{x}_0$ where $R(t)$ is a (clockwise) rotation matrix. This shows directly how the matrix exponential flows the initial state by a combination of rotation (by $R(t)$) and dilation (expansion or contraction controlled by the $e^{\alpha t}$ scaling factor).

(b) Provide a qualitative sketch for the phase portrait in each case $\alpha < 0$, $\alpha > 0$, or $\alpha = 0$.

∎