Box #______
Math 60 Section 1
Homework 12
31 May 2018

Collaborators:

Colley 7.3 #4 Verify Stoke's Theorem for S which is defined by $x^2 + y^2 + z^2 = 4$, $z \leq 0$, oriented by downward normal and

$$\mathbf{F} = (2y - z)\mathbf{i} + (x + y^2 - z)\mathbf{j} + (4y - 3x)\mathbf{k}.$$

$\blacksquare$

Colley 7.3 #6 Verify Gauss's Theorem for

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

$$D = \left\{(x, y, z) | 0 \leq z \leq 9 - x^2 - y^2\right\}.$$

$\blacksquare$

Colley 7.4 #6 Use Gauss's Theorem to derive the **heat equation**,

$$\sigma\rho\frac{\partial T}{\partial t} = k\nabla^2 T.$$

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Colley 7.4 #10 Consider the three-dimensional heat equation

$$\nabla^2 u = \frac{\partial u}{\partial t} \tag{1}$$

for functions $u(x, y, z, t)$. (Here $\nabla^2 u$ denotes the Laplacian $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.) In this exercise, show that any solution $T(x, y, z, t)$ to the heat equation is unique in the following sense: Let D be a bounded solid region in $\mathbb{R}^3$ and suppose that the functions $\alpha(x, y, z)$ and $\phi(x, y, z, t)$ are given. Then there exists a unique solution $T(x, y, z, t)$ to equation (1) that satisfies the conditions

$$T(x, y, z, 0) = \alpha(x, y, z), \quad \text{for } (x, y, z) \in D, \tag{2}$$

and

$$T(x, y, z, t) = \phi(x, y, z, t), \quad \text{for } (x, y, z) \in \partial D \text{ and } t \geq 0.$$

To establish uniqueness, let T_1 and T_2 be two solutions to equation (1) satisfying the conditions in (2) and set $w = T_1 - T_2$.

(a) Show that w must also satisfy equation (1), plus the conditions that

$$w(x, y, z, 0) = 0 \quad \text{for all } (x, y, z) \in D,$$

and

$$w(x, y, z, t) = 0 \quad \text{for all } (x, y, z) \in \partial D \text{ and } t \geq 0.$$

(b) For $t \geq 0$, define the "energy function"

$$E(t) = \frac{1}{2} \iiint_D [w(x, y, z, t)]^2 \, dV.$$

Use Green's first formula in Theorem 4.1 to show that $E'(t) \leq 0$ (i.e., that E does not increase with time).

(c) Show that $E(t) = 0$ for all $t \geq 0$. (Hint: Show that $E(0) = 0$ and use part (b).)

(d) Show that $w(x, y, z, t) = 0$ for all $t \geq 0$ and $(x, y, z) \in D$, and thereby conclude the uniqueness of solutions to equation (1) that satisfy the conditions in (2).

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Colley 7.4 #18 Suppose that $\mathbf{J} = \sigma \mathbf{E}$ (This is a version of Ohm's law that obtains in some electric conductors—here σ is a positive constant known as the **conductivity**) If $\rho = 0$, show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the so-called **telegrapher's equation**,

$$\nabla^2 \mathbf{F} = \mu_0 \sigma \frac{\partial \mathbf{F}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{F}}{\partial t^2}.$$

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True/False Questions

1. The function $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{X}(s,t) = (2s+3t+1, 4s-t, s+2t-7)$ parametrizes the plane $9x - y - 14z = 107$.

2. The function $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\mathbf{X}(s,t) = (s^2+3t-1, s^2+3, -2s^2+t)$ parametrizes the plane $x - 7y - 3z + 22 = 0$.

3. The function $\mathbf{X} : (-\infty, \infty) \times (-\pi/2, \pi/2) \to \mathbb{R}^3$ given by $\mathbf{X}(s,t) = (s^3 + 3\tan t - 1, s^3 + 3, -2s^3 + \tan t)$ parametrizes the plane $x - 7y - 3z + 22 = 0$.

4. The surface $\mathbf{X}(s,t) = (s^2 t, st^2, st)$ is smooth.

5. The area of the portion of the surface $z = xe^{xy}$ lying over the disk of radius 2 centered at the origin is given by

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{1 + e^{2xy}\left(x^4 + x^2y^2 + 2xy + 1\right)}\, dy\, dx.$$

6. If S is the unit sphere centered at the origin, then $\int\int_S x^3\, dS = 0$.

7. If S is the cube with the eight vertices $(\pm 1, \pm 1, \pm 1)$, then $\int\int_S \left(1 + x^3 y\right) dS = 0$.

8. If S denotes the rectangular box with faces given by the planes $x = \pm 1, \quad y = \pm 2, \quad z = \pm 3$, then $\int\int_S xys\, dS = 0$.

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True/False Questions (Continued)

9. If S denotes the sphere of radius a centered at the origin, then

$$\int\int_S (z^3 - z + 2)\, dS = \int\int_S (x - y^5 + 2)\, dS.$$

10. $\iint_S (-y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{S} = 0$, where S is the cylinder $x^2 + y^2 = 9$, $0 \leq z \leq 5$.

11. Let S denote the closed cylinder with lateral surface given by $y^2 + z^2 = 4$, front by $x = 7$, and back by $x = -1$, and oriented by outward normals. Then $\int_S x\mathbf{i} \cdot d\mathbf{S} = 24\pi$.

12. If S is the portion of the cylinder $x^2 + y^2 = 16$, $-2 \leq z \leq 7$, then $\int\int_S \nabla \times (()\, y\mathbf{i}) \cdot d\mathbf{S} = 0$.

13. $\iint_S \mathbf{F} \cdot d\mathbf{S} = 6\pi$, where S is the closed hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, together with the surface $x^2 + y^2 \leq 1$, $z = 0$ and $\mathbf{F} = yz\mathbf{i} - xz\mathbf{i} + 3\mathbf{k}$.

14. If S is the level set of a function $f(x, y, z)$ and $\nabla f \neq \mathbf{0}$, then the flux of ∇f across S is never zero.

15. A smooth surface has at most two orientations.

16. A smooth, connected surface is always orientable.

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True/False Questions (Continued)

17. If $\mathbf{F}$ is a vector field of class C^1 and S is the ellipsoid $x^2 + 4y^2 + 9z^2 = 36$, then $\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$.

18. $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ has the same value for all piecewise smooth, oriented surfaces S that have the same boundary curve C.

19. If $\mathbf{F}$ is a constant vector field, then $\oiint_S \mathbf{F} \cdot d\mathbf{S} = 0$, where S is any piecewise smooth, closed, orientable surface.

20. $\oiint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$, where S is any closed, orientable, smooth surface in $\mathbb{R}^3$ and $\mathbf{F}$ is of class C^1.

21. Suppose that $\mathbf{F}$ is a vector field of class C^1 whose domain contains the solid region D in $\mathbb{R}^3$ and is such that $\|\mathbf{F}(x, y, z)\| \leq 2$ at all points on the boundary surface S of D. Then $\iiint_D \nabla \cdot \mathbf{F} dV$ is twice the surface area of S.

22. If S is an orientable, piecewise smooth surface and $\mathbf{F}$ is a vector field of class C^1 that is everywhere tangent to the boundary of S, then $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0$.

23. If S is an orientable, piecewise smooth surface and $\mathbf{F}$ is a vector field of class C^1 that is everywhere perpendicular to the boundary of S, then $\iint_S \nabla \times \mathbf{F} \times d\mathbf{S} = 0$.

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True/False Questions (Continued)

24. If $\mathbf{F}$ is tangent to a closed surface S that bounds a solid region D in $\mathbb{R}^3$, then $\iiint_D \nabla \cdot \mathbf{F} dV = 0$.

25. Let S be a piecewise smooth, orientable surface and $\mathbf{F}$ a vector field of class C^1. Then the flux of $\mathbf{F}$ across S is equal to the circulation of $\mathbf{F}$ around the boundary of S.

26. Let D be a solid region in $\mathbb{R}^3$ and $\mathbf{F}$ a vector field of class C^1. Then the flux of $\mathbf{F}$ across the boundary of D is equal to the integral of the divergence of $\mathbf{F}$ over D.

27. Suppose that f and g are of class C^2 and D is a solid region in $\mathbb{R}^3$ with piecewise smooth boundary surface S that is oriented away from D. If g is harmonic, then $\iiint_D \nabla f \cdot \nabla g \, dV = \oiint_S f \nabla g \cdot d\mathbf{S}$.

28. Suppose that f and g are of class C^2 and D is a solid region in $\mathbb{R}^3$ with piecewise smooth boundary surface S that is oriented away from D. If f and g are harmonic, then $\oiint_S f \nabla g \cdot d\mathbf{S} = -\oiint_S g \nabla f \cdot d\mathbf{S}$.

29. If $\nabla^2 f$ is known, then f is uniquely determined up to a constant.

30. If S is a closed, orientable surface, then

$$\oiint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = 0.$$

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