sheet 1 solution (part 1): PDE

Give the order of each of the following PDEs:

a. $u_{xx} + u_{yy} = 0$

second order

b. $u_{xxx} + u_{xy} + a(x)u_y + log(u) = f(x, y)$

third order

c. $u_{xxx} + u_{xyyy} + a(x)u_{xxy} + u^2 = f(x, y)$

fourth order

d. $u_x + cu_y = d$

first order

State whether each of the following PDEs is linear or non-linear, homogeneous or non-homogeneous:

a. $u_{xx} + u_{yy} - 2u = x^2$

linear, non-homogeneous

b. $u_{xy} = u$

linear, homogeneous

c. $uu_x + xu_y = 0$

non-linear, homogeneous

d. $u_x^2 + log(u) = 2xy$

non-linear, non-homogeneous

e. $u_{xx} - 2u_{xy} + u_{yy} = cos(x)$

linear, non-homogeneous

f. $u_x(1 + u_y) = u_{xx}$

non-linear, homogeneous

g. $sin(u_x)u_x + u_y = e^x$

non-linear, non-homogeneous

h. $2u_{xx} - 4u_{xy} + 2u_{yy} + 3u = 0$

linear, homogeneous

i. $u_x + u_x u_y - u_{xy} = 0$

non-linear, homogeneous

Classify each of the following linear 2nd order PDEs as hyperbolic, parabolic or elliptic at every point (x,y) of the xy plane:

a. $xu_{xx} + u_{yy} = x^2$

$A = x \quad B = 0 \quad C = 1$
$\therefore B^2 - 4AC = -4x$

- x¿0: elliptic
- x=0: parabolic
- x¡0: hyberebolic

b. $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$

$A = x^2 \quad B = -2xy \quad C = y^2$
$\therefore B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0 \rightarrow$ **parabolic**

c. $e^xu_{xx} + e^yu_{yy} = u$

$A = e^x \quad B = 0 \quad C = e^y$
$\therefore B^2 - 4AC = 0 - 4e^{x+y} < 0 \rightarrow$ **elliptic**

Classify each of the following constant coefficient linear 2nd order PDEs as hyperbolic, parabolic or elliptic:

a. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

$A = 4 \quad B = 5 \quad C = 1$
$\therefore B^2 - 4AC = 25 - 16 > 0 \rightarrow$ **hyperbolic**

b. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$

$A = 1 \quad B = 1 \quad C = 1$
$\therefore B^2 - 4AC = 1 - 4 < 0 \rightarrow$ **elliptic**

c. $u_{xx} + u_{xy} + 7u_y = sin(x)$

$A = 1 \quad B = 1 \quad C = 0$
$\therefore B^2 - 4AC = 1 - 0 > 0 \rightarrow$ **hyperbolic**

show that $\mathbf{u(x,t) = cos(x - ct)}$is a solution of the PDE $\mathbf{u_t + cu_x = 0}$

$u_t = c\,sin(x - ct)$
$u_x = -sin(x - ct)$

$$\boxed{\therefore u_t + cu_x = c\,sin(x - ct) - c\,sin(x - ct) = 0}$$

Show that $\mathbf{u(x,y) = F(xy) + xG(\frac{x}{y})}$ is a general solution of the PDE $\mathbf{x^2 u_{xx} - y^2_{yy} = 0}$

let $\alpha = xy$ and $\beta = \frac{y}{x}$
$\alpha_x = y \quad \alpha_y = x \quad \beta_x = \frac{-y}{x^2} \quad \beta_y = \frac{1}{x}$
$\therefore u = F(\alpha) + xG(\beta)$
$\therefore u_x = y\dot{F} + G(\beta) - \dot{G}\frac{y}{x}$
$\therefore u_{xx} = y^2\ddot{F} + \frac{y^2}{x^3}\ddot{G}$
$\therefore u_y = x\dot{F} + \dot{G}$
$\therefore u_{yy} = x^2\dot{F} + \frac{1}{x}\ddot{G}$

$$\boxed{\therefore x^2 u_{xx} - y^2_{yy} = x^2y^2\ddot{F} + \frac{y^2}{x}\ddot{G} - x^2y^2\ddot{F} - \frac{y^2}{x}\ddot{G} = 0}$$