Lecture Notes1: PDE

Contents

1 Introduction

1.1 Mathematical model

The process of describing a real world problem in mathematical terms

1.2 differential equations

An equation that relates one or more unknown functions and their derivatives.

- **ODE:** An equation that contains one or several derivatives of unknown function and single variable.
- **PDE:** An equation that contains partial derivatives of unknown function of tow or more variables

1.3 Common examples of PDEs

- $u_t + cu_x = 0$
- $u_{xx} + u_{yy} = f(x, y)$
- $a(x, y)u_{xx} + 2uxy + 3x^2u_{yy} = 4e^x$
- $u_xu_{xx} + (u_y)^2 = 0$
- $(u_{xx})^2 + u_{yy} + a(x, y)u_x + b(x, y)u = 0$

2 Basic concepts and definitions

2.1 Definition 1

The order of a PDE is the order of the highest order derivative in the equation.

Examples

- **first order**
  1. $u_t + cu_x = 0$
- **second order**
  1. $u_{xx} + u_{yy} = f(x, y)$
  2. $u_xu_{xx} + (uy)^2 = 0$

2.2 Definition 2

A PDE is linear if it's linear in the unknown function and all its derivatives with coefficients depending only on the independent variables.

Examples

- **linear**

  1. $u_t + cu_x = 0$
  2. $u_{xx} + u_{yy} = f(x, y)$
  3. $a(x, y)u_{xx} + 2uxy + 3x^2 u_{yy} = 4e^x$

- **Non-linear**

  1. $u_x u_{xx} + (uy)^2 = 0$
  2. $(u_{xx})^2 + u_{yy} + a(x, y)u_x + b(x, y)u = 0$

2.3 Definition 3

A PDE is homogeneous if the equation does not contain a term independent of the unknown function and its derivatives.

Examples

- **linear**

  1. $u_t + cu_x = 0$
  2. $u_x u_{xx} + (uy)^2 = 0$
  3. $(u_{xx})^2 + u_{yy} + a(x, y)u_x + b(x, y)u = 0$

- **Non-linear**

  1. $u_{xx} + u_{yy} = f(x, y)$
  2. $a(x, y)u_{xx} + 2uxy + 3x^2 u_{yy} = 4e^x$

3 Basic exercises

Example 1

Show that $\mathbf{u(x,y) = F(xy) + xG(\frac{x}{y})}$ is a general solution of the PDE $\mathbf{x^2 u_{xx} - y^2_{yy} = 0}$

let $\alpha = xy$ and $\beta = \frac{y}{x}$
$\alpha_x = y \quad \alpha_y = x \quad \beta_x = \frac{-y}{x^2} \quad \beta_y = \frac{1}{x}$
$\therefore u = F(\alpha) + xG(\beta)$
$\therefore u_x = y\dot{F} + G(\beta) - \dot{G}\frac{y}{x}$
$\therefore u_{xx} = y^2\ddot{F} + \frac{y^2}{x^3}\ddot{G}$
$\therefore u_y = x\dot{F} + \dot{G}$
$\therefore u_{yy} = x^2\ddot{F} + \frac{1}{x}\ddot{G}$

$$\boxed{\therefore x^2 u_{xx} - y^2_{yy} = x^2y^2\ddot{F} + \frac{y^2}{x}\ddot{G} - x^2y^2\ddot{F} - \frac{y^2}{x}\ddot{G} = 0}$$

Example 2: PDE can have many solutions

Find three possible solutions to any 2D Laplace Equation under specific initial conditions and boundary conditions $\mathbf{u_{xx} + u_{yy} = 0}$

first sol: $u(x,y) = x^2 - y^2$
second sol: $u(x,y) = ln(x^2 + y^2)$
second sol: $e^x cos(y)$

4 Fundamental Theorem (Superposition)

Let D be linear differential operators (in the variables $x_1, x_2, ..., x_n$), let f1 and f2 be functions (in the same variables), and let c1 and c2 be constants.

- If u_1 solves the linear PDE $Du = f_1$ and u_2 solves $Du = f_2$, then $u = c_1u_1 + c_2u_2$ solves $Du = c_1f_1 + c_2f_2$. In particular, if u_1 and u_2 both solve the same homogeneous linear PDE, so does $u = c_1u_1 + c_2u_2$.

- If u_1 satisfies the linear boundary condition $Du|_A = f_1|_A$ and u_2 satisfies $Du|_A = f_2|_A$, then $u = c_1u_1 + c_2u_2$ satisfies $Du|_A = c_1f_1 + c_2f_2|_A$. In particular, if u_1 and u_2 both satisfy the same homogeneous linear boundary conditions, so does $u = c_1u_1 + c_2u_2$.

5 Classification of linear 2nd order PDEs

- Linear Second order PDEs are used to model many systems in many different fields
- The general form of a linear 2nd order PDE in two variables x and y is given by: $\mathbf{Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G}$, Where A, B, C,D, E, F, and G depend only on x and y.
- A discriminant at some point is defined as: $\mathbf{\Delta = B^2 - 4AC}$

The PDE is classified as follows:

- $\mathbf{\Delta > 0}$: Hyperbolic
- $\mathbf{\Delta = 0}$: Parabolic
- $\mathbf{\Delta < 0}$: Elliptic