Spherical Geometry: investigating of the world of Mathematics

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The mathematics underlying non-Euclidean geometry may be conceptually and purely computationally challenging at times. This is the article's central issue. To function, Lobachevsky and Bolyai's theories need their formulas, which are not obvious. Following a typical practice, we will first investigate the 'geometry' defined on a sphere (think, as usual, of the Earth) by considering'straight lines' to be the lines with the smallest distance, that is, great circles. This is the same as Saccheri's HOA. By what may be termed It has always been disregarded as opposed to what is evident, as the HAA has not been.

It contains Euclid's proof along with a diagram illustrating how it fails to operate on a sphere. Gray asks in his consideration of the "conventional account" why spherical geometry, which was "well known during the whole time," was not considered a solution to the inquiry (whether the fifth postulate could be drawn from other self-evident facts). It is possible that basic Euclidean postulates, such as the existence of circles with arbitrary radii, were false in spherical geometry, or that other simple flaws, such as those shown by the failure of argument I.16, were to blame. In any event, the purpose of this section is not so much to analyze this 'what if?' topic (why did geometers not consider the sphere as a solution?) as it is to examine what was known about that geometry and how it inspired subsequent thought.

Already in Greek times, it was understood that a line of shortest distance on a sphere (let's call it S) was an arc of a 'great circle'—the intersection of S with a plane passing through its center. Due to their significance in astronomy, the Greeks, particularly Ptolemy, focused on understanding spherical triangles (triangles whose sides are the shortest lines on a sphere), and Islamic mathematicians who had (roughly) our trigonometric functions were able to derive the key formulae for'solving' them.

Ancient and contemporary geographers have relied on these formulas to navigate on a sphere. Albert Girard's discovery of the angle-sum of spherical triangles in the seventeenth century is of considerably less relevance to geographers than it is to mathematicians: that the angle-sum of spherical triangles is a constant.

1. the angle-sum $A + B + C$ is always greater than π (so much is obvious);
2. the 'excess' $A + B + C - \pi$ increases with area; in fact, it is precisely equal to

$$\frac{1}{R^2} \cdot area(ABC)$$

.

This is evident for the triangle whose angles are all, constituting an eighth of the sphere (why?). To show that the excess is merely a multiple of the area is a trickier argument, but it is acceptable if you are willing to devote some effort to considering how triangles might be joined.

Johann Heinrich Lambert, whose posthumous Theory of parallels was published in 1786, appears to have been the first to consider how Girard's formula may lead to a greater grasp of what it means to reject postulate 5. Gray (1979, chapter 5) identifies Lambert as a pivotal transitional character for these and other reasons. By reasoning with quadrilaterals once again, it was Lambert who arrived at a crucial result of the HAA.

Second, he saw that in consequence an HAA geometry must, like a spherical geometry, have an absolute measure of length. This comes from reasoning with a quadrilateral ADGB (Fig. 1) in which AB = AD and the three angles A,B,D are right angles, but the fourth (G) may not be:

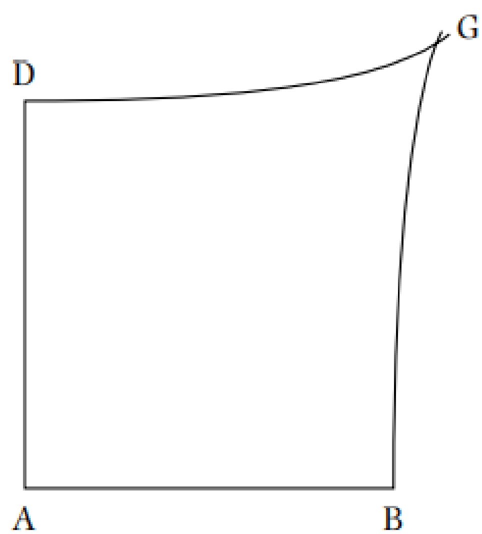

Figure 1: Lambert's quadrilateral

Last but not least, Lambert realized that the area formula he had discovered was the 'negative' of the area formula for spherical triangles; the defect $\pi - (A + B + C)$

replaced the excess as the area measurement. As Gray notes, he was almost there. In another sense, he was not there at all; he could see quite clearly what a non-Euclidean geometry must be like, but he never progressed beyond the unfortunate assertion that it may hold on an imaginary sphere, which no amount of current reinterpretation can make sense of. This may be the crucial moment at which Gray's (1979, p. 155) query is warranted: why did the evolution take so long, in this instance from the 1780s to the 1820s? Perhaps not too lengthy, and it should be recalled that the search of similarities was, as previously said, atypical. It was notably a problem for masochists, eccentrics, and the overly ambitious.

References:

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