Infinities: Mathematicians Measure Infinities, and Find They're Equal

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August 6, 2022

Around 1600, more or less independently of the work in algebra, we see the first systematic use of 'the infinite' in European mathematics; by the middle of the century, it had become common, and Pascal, a mathematical mystic, used it in a number of metaphorical statements (such as his famous 'wager') and an early version of the calculus. Physical applications and the recognition that infinities in some sense underlie Archimedes' work appear to be the primary impetus, although it may be necessary to be less cautious than he was in what one permits.

And indeed, In the early stages of what will eventually become calculus, there is a general sense of exploration, of trying out statements to hear how they sound, and (in contrast to algebra) of a loosening rather than a tightening of definition. While there are traces in Stevin's work, the most interesting introduction is found in Kepler's Astronomia Nova, which presents itself (only semi-realistically) as an account of the various false trails he followed before discovering his famous planetary laws. (This account relies heavily on Bruce Stephenson's detailed analysis) (1987).

Any analysis of a text as complex as the Astronomia Nova is debatable, but the broad outlines appear convincing.) Almost immediately, Kepler was confronted with a new obstacle. Both Ptolemy and Copernicus explained the motion of planets, Mars in particular, as consisting of uniform motions in a circle around a point that was not the actual centre (the sun for Copernicus and the earth for Ptolemy), but rather a point in empty space, the 'equant'. 15 Using uniform motion in a circle makes it relatively simple to calculate using such a scheme. A contemporary physicist would note that this is "unphysical" because one is postulating a force between the planet and the equant, where there is no matter. Kepler's version of this, which stemmed from his own ideas on planets, was that the souls animating them could perceive the sun (for example) and its distance and adjust their movement accordingly; however, it was implausible to assume that they could perceive the equant. Therefore, he had to assume that motion was dependent on distance from the sun and slowed as the planet moved away from the sun.

Currently, there are two issues with velocity. The first is the age-old question of whether Kepler or his contemporaries could define or even conceive of "velocity in an instant," in the same way that we recognise in an instant that a car is travelling at 45 miles per hour. It has been suggested that Th abit ibn Qurra and al-B ir un I employed this concept (see, for instance, Hartner and Schramm 1963), but it does not appear to have been widely used by Islamic astronomers. Galileo took the first steps toward understanding what this could mean; see below. The second issue was that even velocity over a time interval was the ratio of different types of quantities — distance and time — and was therefore incompatible with a Greek framework. Kepler avoided these issues by employing the 'delay' - the time required for a planet to travel along a small segment of its orbit - when the segments were approximately equal. He had to formulate a hypothesis regarding how the delay depended on the distance, and he tried several; however, in each case, he was faced with the challenge of adding a large number of very small delays to arrive at the observable time-intervals. Here, he introduces the difficulty, assuming the planet orbits the sun in a circular eccentric orbit:

Since, then, the delays of the planet in equal parts of the eccentric are in the inverse proportion to the distances of those parts [from the eccentre], but the individual points change their distance from the eccentre all around the semicircle, I believed it would be difficult to determine how to calculate the sum of the individual distances.

For without the sum of all, which would be infinite, we could not determine the delay of each individual. We should therefore not know the equation. As the total sum of the distances corresponds to the total period, so does each portion of the total sum correspond to its own time.

Kepler initially divided the circle into 360 degrees but found the calculation tedious; he then had an idea.] For I recalled that Archimedes, in his search for the ratio of the circumference to the diameter, once divided the circle into an infinite number of triangles, but his proof by contradiction obscured this strategy. Whereas previously I had divided the circumference into 360 parts, I now divide the eccentric's plane into the same number of lines from the point from which the eccentricity was calculated. [Fig. 1].

This is a relatively early stage in the research; Kepler is still following a 'wrong' theory and discovers a result that does not match his data; and even before this, he must modify his method because his ideas on summing triangles are ineffective. However, he has made a very audacious claim about Archimedes, namely that his proofs by contradiction conceal infinite methods. This is not what the ancient Greek texts say (as far as we are aware), nor was it the orthodox view during Kepler's time period. (At the time, it was unknown that Archimedes' Method utilised an infinite process (see Chapter 3)). The notion that a circle could be viewed as a polygon with an infinite number of sides was utilised by

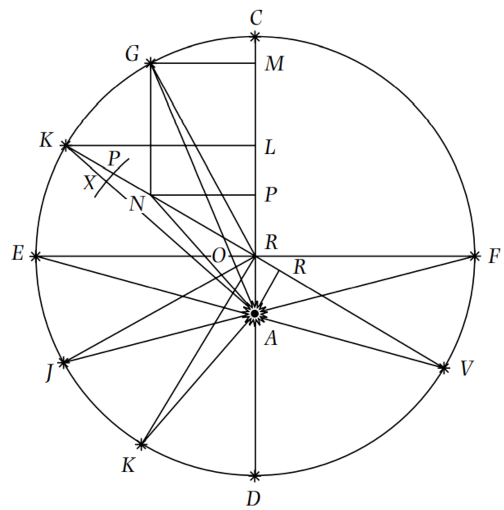

Figure 1: Kepler's diagram from 'Astronomia Nova'.

the mystic Nicholas of Cusa, whose mathematical/theological thinking greatly influenced Kepler. It is repeated in his work on the measurement of wine barrels, which I reproduce in Appendix C; there, the recourse to the infinite is justified because it is simply faster.

References:

1. A. G. Aghdam, "A Method to Obtain the Infinity-Norm of Systems using the Routh Table," 2006 World Automation Congress, 2006, pp. 1-6, doi: 10.1109/WAC.2006.375964.

2. A. Wendt et al., "Work in progress — engineering case studies inspired by society's Grand Challenges for Engineering: An integrated approach to middle school stem instruction," 2012 Frontiers in Education Conference Proceedings, 2012, pp. 1-2, doi: 10.1109/FIE.2012.6462261.

3. T. Simley et al., "Assessing the Efficacy of Integrating Computer Science, Math, and Science in a Middle School Sphero Robotics Summer Program," 2020 Research on Equity and Sustained Participation in Engineering, Computing, and Technology (RESPECT), 2020, pp. 1-8, doi: 10.1109/RESPECT49803.2020.9272479.

4. D. N. Abara and A. Lanzon, "H-infinity-PID controller for an open-loop unstable system," 2017 IEEE 3rd International Conference on Electro-Technology for National Development (NIGERCON), 2017, pp. 134-138, doi: 10.1109/NIGERCON.2017.8281886.