Abstraction and Bourbaki: origin of Mathematical abstraction

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The movement for a more abstract perspective of mathematics, which has been both praised and criticized as a peculiarity of the twentieth century, has its origins in the founding endeavor. The schools of axiom-builders frequently asserted that it did not matter what their axioms referred to: Hilbert was quoted as saying that one should be able to replace the words 'points, lines, planes' in the axioms of geometry with 'tables, chairs, beer-mugs'; and Russell described mathematics as the science 'in which we do not know what we are talking about, nor whether what we are saying is true' However, if we consider the abstract viewpoint to be one in which one systematically seeks to lose sight of any actual real-world objects to which the discourse refers and instead focuses on the relations and structures which connect those objects, then the zenith of abstraction occurred in the 1940s and 1950s, and the driving forces behind a program to make all mathematics more abstract were a strange revolutionary group of young French university professors. Typically, the term derives from a youthful prank perpetrated by a mathematician whose name was taken from a Greek commander serving under Napoleon III.

The Bourbakists (Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonné, André Weil, and a later'second generation' after the war), a group of males from usually French elite schools, did not intend to attack the foundations of mathematics. Their objective was more obvious and understandable: they believed they had received dreadful and antiquated instruction. As an alternative, they had learnt of new concepts from Germany, notably those of Hilbert and Emmy Noether as contained in van der Waerden's ground-breaking new algebra textbook; and they planned to construct a set of textbooks that would constitute a whole new course of mathematics. Did they intend for their novels to be required reading?

Possibly. Even though practically all of the group's founders are deceased, the group's folklore is vast and increasing. There are interviews and more or less gossipy histories (e.g. Mashaal, 2002) that show images of young men gathering and presumably fighting over their goal in the French countryside. The contrast between the icy impersonal words of the Éléments de mathématiques and the allegedly chaotic and filthy environment, part party, part student gathering, in which their substance was hammered out could not be greater.

This is an excerpt from the introduction of Bourbaki's Algebra to demonstrate with How well he articulated his objectives and strategy. Before analysis could begin, actual numbers had to be gathered. be defined; before real numbers, topological elements; before that, the theory of sets. The The student had a lengthy journey before arriving at the least upper limit theorem, which we now possess. seen generating similar complications in the past. Pierre Cartier, a Bourbakist of the "second generation," supplies A fair analysis of the project's merits and weaknesses:

Bourbaki had a clear objective: to provide the basis for mathematics. All mathematics had to conform to Hilbert's design; van der Waerden's work on algebra had to be repeated for the rest. mathematical. What should be included was rather obvious. The first six novels of Bourbaki constitute the fundamental The historical context of a contemporary graduate student.

Many individuals believed that it should be taught exactly as it was written in the book. books. You might consider of the early volumes of Bourbaki as a mathematical encyclopedia, since they include all the required information. information. This is an accurate depiction. Considered as a textbook, it is a catastrophe. (Senechal 1998) This is little dishonest; it is uncommon to find an encyclopedia with this feature. A comprehensive collection of exercises. Nonetheless, there were few who employed the full Éléments as their primary musical source. textbook. What was considerably more significant was that its presence had a dramatic effect on the approach in which many mathematicians thought about their topic, and some of them Utilized their reasoning to compose more comprehensible textbooks, using the concepts of structures. The mapping took precedence over the item, etc., and became the focal point. The author The author of a textbook may, and often does, pick and choose from accessible content without overt plagiarism. In any event, Bourbaki's work was difficult to plagiarize, but a watered-down version was easier. France grew immensely powerful both inside and outside its borders.

The Bourbakists were unconcerned with axiom systems as a way of preserving mathematics. from contradiction.5 They did see axiom systems as the foundational instrument for defining'structures', which are free from contradiction. were to be the focal point of the presentation of the Éléments. As has been emphasized Among a multitude of other concepts, the concept of structure was never defined; Nonetheless, various distinct structures (group, ring, topological space, uniform space, etc.) pervaded the system. The book and its significance were crucial to its unique way of thinking. It nearly became a reflex in France, if One had fallen under the enchantment of the Elements, speaking not of defining a group but of "giving" (provide) a collection having a group structure.

Let us look at an example. To define the sine and cosine functions in chapter VIII (general topology), Section 2 ('measure of angles'). [6] Bourbaki used the group isomorphism from the (multiplicative) unit circle U in the complex numbers $\mathbf{C} = \mathrm{R}^2$ to the quotient group $\mathbf{R}/\mathbf{Z}$. He then 'endowed' the set A of "half-line angles' $\left(\widehat{\Delta_1, \Delta_2}\right)$ (see Fig. 1) with a group structure, and showed it was isomorphic to $\mathbf{U}$. Finally: 1. If θ is an angle in A, you define $\cos\theta$ to be the real part of the complex number in $\mathbf{U}$ corresponding to θ : 2. If x is a real number, you define $\cos x$ using some homomorphism from R to A. At this point, Bourbaki points out the need to decide the least positive value of $x \in \mathbf{R}$ which corresponds to $1 \in$ U, and discusses the merits of 360,400 , and a number called 2π ('we will prove the existence of such a number later'). [This account is of course simplified; in Bourbaki it takes four pages, with everything proved.] As a reward, you finally get one of the author's rare picturesthe graphs of the trigonometric functions (Fig. 2).

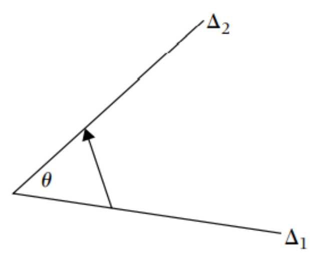

Figure 1: The 'half-line angle' $\left(\widehat{\Delta_1 \cdot \Delta_2}\right)$ is the angle of rotation from the first line to the second.

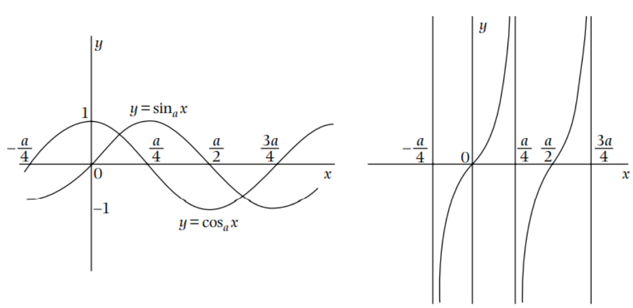

Figure 2: The graphs of $\sin x, \cos x$, and $\tan x$ from Bourbaki's Topologie Générule VIII

This hardly qualified as a philosophy in the sense that the great systems of Russell, Brouwer, and Hilbert did, but it was certainly a practical ideology, and defined an orthodoxy about what one liked or disliked in mathematics. Much has been written, particularly by opponents, about the hegemony of Bourbakist ideas in France from 1945 on, and their insistence that their way of teaching was the only right one. It never extended even over all of France, but where it was entrenched, it could be fairly intolerant. If installed as head of an unreformed mathematics department, a Bourbakist was capable of purging the library of most of its books and writing out orders for replacements—Bourbaki, naturally, but also the seminars of Henri Cartan, the works of van der Waerden, Eilenberg, MacLane, Steenrod, and subscriptions to the Annals of Mathematics, and the publications of the American Mathematical Society. More disastrous was the brief incursion of Bourbakism into the French high-school curriculum in the 1960s (paralleled by what, in the United States was called 'new math'); the idea of replacing times tables by theorems about Z caused confusion among teachers and students and was eventually withdrawn.

We have expressed doubts about the responsibility of mathematicians for 'modernism'. In the more minor case of the philosophical movement called structuralism, the case is clearer. In a text for the 'Que-sais-je' series explaining structuralism to the general public (1970), Jean Piaget cited Bourbaki as his first example before proceeding to the social sciences, adding that 'the structural models of Lévi-Strauss, the acknowledged master of present-day social and cultural anthropology, are a direct adaptation of general algebra'. The influence appears directly in an anecdote of André Weil

References:

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