Geometry and Space: Copernican revolution in astronomy

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Most people are unaware that a century and a half ago, a revolution in geometry occurred that was as scientifically profound as the Copernican revolution in astronomy and as philosophically significant as the Darwinian theory of evolution. Greenberg 1974. It is a widespread belief that geometry, with all of its truths, is universally applicable to all men, all times, and all peoples, and not just all historically factual ones, but all conceivable ones. Husserl (1989).

The goal of this article is to look at one of the classic "stories" in mathematics history: the origin of non-Euclidean geometry. Although it was a part of the modernization process,the story of the invention of non-Euclidean geometries by Lobachevsky and Bolyai in the 1820s has traditionally been told as one of a particular process of discovery; the problem of Euclid's parallel postulate, and the invention of non-Euclidean geometries by Lobachevsky and Bolyai in the 1820s. It has the characteristics of a good story: a connected thread, even a hero/heroes, searching for a solution, followed by an unexpected twist. Its defects, as historians are sometimes anxious to point out, are that history is more complex, and to construct such a story some important details must be left out or simplified. The 'classical' history (Bonola 1955) which is full, careful, and scholarly, is nearly 100 years old, and not surprisingly for some time there have appeared criticisms, attempts to tell the story differently, or to tell a different story altogether. The questions raised are typical ones in the history of scientific revolution, which were already discussed by Bachelard in the 1930s: when mathematicians discover a completely different way of doing mathematics (in this case, geometry), are they adding to the old mathematics, replacing it, or giving us a new perspective from which the old (Euclidean) is a special case of the new? To what extent is the previous pursuit of Euclidean geometry made invalid or irrelevant? And so on.

Before we begin, we must address several issues, one of which is geometry. Geometry has grown in popularity as a result of the'revolution' to which the quote refers.

At least in universities, it has gradually become a second-class subject. True, Pythagoras' theorem and the criteria for congruent triangles are still widely accepted, but their epistemological status is murky. So the reader might pause for a moment to consider two questions.

1. What is geometry about—what is its subject-matter?
2. How do we know that its results are true?

The answers to these will of course be influenced by your education as well as by personal opinion, but to have thought about them may help. In previous chapters, a too definite knowledge of modern mathematics was perhaps a handicap to evaluating the mathematics of the past. But it is an equal handicap to start with no view at all, and the questions above are meant to elicit one. To return to the story, the 'simplest' version, which still has wide currency and has the merit of simplicity, runs roughly as follows:

1. Dissatisfaction with Euclid's seemingly perfect system has been centred on the so-called 'parallel postulate' since the beginning of his geometry, and possibly even earlier. This states (in one version) that the linesl, l meet if the angles, in Fig. 1 add up to less than two right angles α and β. Another, perhaps more understandable version is 'Playfair's axiom': there is a unique straight line through A that is parallel to l (but does not meet it); and this line makes the angles add up to two right angles as stated. This was thought to be not intuitively obvious and should be demonstrated using the other axioms or from 'first principles.' AB, DE would not meet (they would be parallel). A quick way of 'seeing' this is as follows. If the angles on one side are two right angles, so are the angles on the other. If the lines meet on one side, then by symmetry they must meet on the other side too. But this implies that there is not a unique straight line joining two points (the two points where they meet), which is unreasonable.

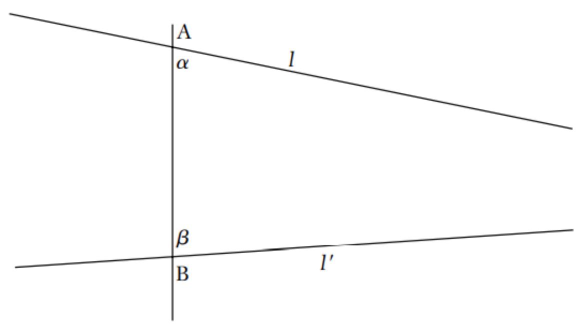

Figure 1: The figure for postulate 5.

2. There have been attempts to prove the postulate for roughly two millennia. Proclus (fifth century), Th abit ibn Qurra (ninth century), ibn al-Haytham (tenth century), Khayyam (eleventh century), Nas.ir al-D in al-T.us.i (thirteenth century), and a number of writers, some well-known, others obscure, made recorded efforts. It's worth noting that the "parallels problem" was never considered a critical question in mathematics. Obviously, it was more important to those who valued the Greek classics (such as the Arabs), but even so, it was frequently regarded as a blind alley pursued by eccentrics.

3. The last major serious ‘proof ’ within the context of classical geometry was due to an Italian priest, Gerolamo Saccheri, published in 1733. This refined a framework for the problem (division into three cases) which is perhaps originally due to al-T.us¯ . ¯i. We start by constructing a quadrilateral ABCD, (Fig. 2) with the angles at B and C both right angles, and the sides AB and CD equal. It is then easy to show that the angles at A and D are equal. If we have the parallels postulate, we can deduce that they are right angles (try to see how); but without it, we do not know this. Saccheri distinguished cases according to whether these two angles are right angles, acute, or obtuse; and describes these as the ‘hypothesis of the right (acute, obtuse) angle’—HRA, HAA, HOA for short

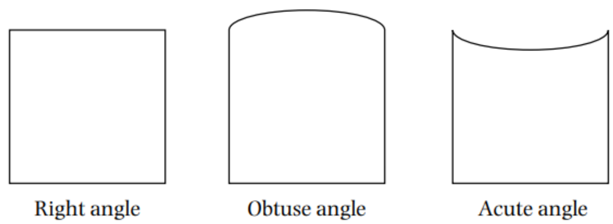

Figure 2: Saccheri’s three hypotheses.

HRA is equivalent to Euclid’s geometry with postulate 5 included; it is what we normally accept to be true. His plan was to create a contradiction by meticulously spelling out the consequences of the HOA and HAA, leaving the HRA as the only true geometry. The proof he thought he had was incorrect, but the concept of the three hypotheses proved very useful; and in developing his’proof,’ he deduced a great many consequences that must follow if we assume the HAA; this, as we will see, is the difficult case, which amounts to denying Postulate 5. Saccheri demonstrated that these are three mutually exclusive options: if the HAA is true for one quadrilateral, it is true for all. There are several other perspectives on this distinction. For example, the HOA has no parallels (we’ll look at how this can happen later), whereas the HAA has an infinite number of lines through a point P that do not meet a given line l. All triangles with the HOA (the HAA) have an angle-sum greater than (less than) two right angles.

4. After Saccheri, attempts at proof continued, but new elements involving explicit measurement (such as trigonometry) were gradually introduced—while an increasing tendency to doubt the possibility of effectively proving the postulate was observed. Gauss1 in particular became convinced (around 1800) that a consistent geometry in which the postulate was false could be constructed, but he kept his thoughts to himself.

5. In the 1820s, two separate scholars, N. I. Lobachevsky and Janos Bolyai, who had been attempting to verify the postulate, decided to develop a coherent geometry

based on the notion of sharp angles. Note, however, the closeness to Saccheri's program. In

Saccheri wanted to derive a contradiction, although Lobachevsky and Bolyai did not. In each instance, it was assumed that such a geometry was attainable. In the 1820s, both authors published their findings in obscure locations (Russia and Hungary) and both publications were mostly ignored. However, each of them demonstrated crucial and unexpected aspects of the alternative "non-Euclidean" geometry, so making it an intriguing subject of study in and of itself. This is the "Copernican revolution" mentioned in the introductory quotation.

6. Although Lobachevsky and Bolyai developed their non-Euclidean geometries, they were unable to demonstrate their consistency. Theoretically, it would still be able to demonstrate the inconsistency of non-Euclidean geometry and derive postulate 5 in spite of this. Riemann's 1854 seminal article established a larger range of geometries (more or fewer dimensions, various methods of measurement), which Helmholtz emphasized in the years that followed, and in particular, the notion of "non-Euclidean" was clarified. Evidence of consistency was achieved in stages through the later nineteenth century by Beltrami, Klein, and Poincaré among others, by the characteristically modern method of defining 'models' for the new geometry.

7. Of this again, more will follow later. As a result of these developments, it was realized that there was no unique geometry, paving the way for the modern understanding of 'geometry' as the study of an axiom-system that asserts certain properties of objects called (for instance) 'points,' 'lines,' etc., without regard to what these names may mean. The unique geometry of Euclid has been supplanted by other geometries that are equally legitimate as mathematical objects of study. Because they were the first to propose an alternative to Euclid, Lobachevsky and Bolyai are considered the revolution's founding fathers.

Note. The reader who has no idea of what non-Euclidean geometry is, let alone what a model of it might be, should consider the well-known picture 'Circle Limit III' by Moritz Escher (Fig. 3). In this, which is a picture of non-Euclidean geometry's version of a plane,

1. the curved lines are to be thought of as straight;

2. all the triangles (and all the fish) are to be considered as having the same size;

3. the bounding circle is 'at infinity', and lines which meet there are parallel.

I have provided the conventional plot framework, which is simple to critique. An up-to-date, serious history of mathematics, as this one purports to be, should be wary of a narrative that (a) assumes that a single project occupied mathematicians for over 2,000

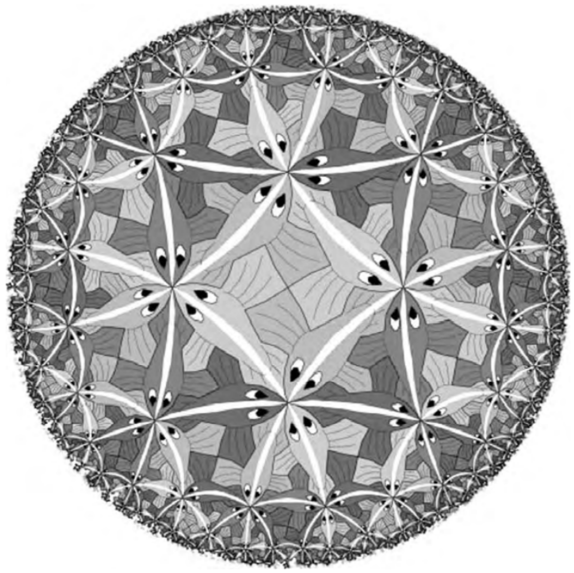

Figure 3: 'Circle Limit III' by Moritz Escher

years (from Euclid's time to the nineteenth century) and (b) identifies a single discovery at the end of this period as a founding event or revolution. The issue with the tale of non-Euclidean geometry is the issue with historical narratives in general. How much has a usually complex scenario been streamlined to make a coherent story? Has the definition of the phrases altered over time? What additional philosophical concerns or variations in the definition of the term geometry must be considered?

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