New Geometries: systematic use of transformations

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Rather of continuing with the story's details at this point, the reader may wish to know what it means to claim that Lobachevsky and Bolyai "built a geometry." What does it mean to build a geometry? This is the 'Copernican' component of the finding; no one has before attempted to do so. Following the Euclidean paradigm, it would be reasonable to request a set of rules or axioms that are complete enough for a major theory to be inferred from them. Assume, like the pioneers did, that you simply refute the assumption. This means that you are adopting what Saccheri called the hypothesis of the acute angle. The simple negation of postulate 5, though, is: There exists a straight line, falling on two other straight lines, which makes the two interior angles on one side less than two right angles, such that the two straight lines, produced indefinitely, never meet.

This is so impenetrably ambiguous that it cannot be used to draw any meaningful conclusions. To have an usable geometry, one would need rules for when triangles are congruent, rules for angle addition, rules for figure areas, etc., at least in concepts known in the 1820s. In other words, one would need some kind of measurement, and constructing a new geometry required defining a new method of measurement. This was to be made clear by Riemann in the 1850s, but Lobachevsky and Bolyai did not continue in this manner. Their explanations were surprisingly similar, with each having its own merits; Bolyai's is likely the clearest, while Lobachevsky's is the most comprehensive. It is simplest to begin with Lobachevsky's explanation of the preceding imprecise remark. According to his analysis, things must go as follows:

All straight lines in a plane that extend from a given point may be classified as either cutting or non-cutting with respect to a given straight line in the same plane. The boundary lines of the first and second classes of lines shall be referred to as parallel to the supplied line.

From the point A, let the perpendicular AD fall onto the line BC, and then draw the perpendicular AE (Fig. 9)... In order to transition from the cutting lines, denoted by AF, to the non-cutting lines, denoted by AG, we must reach a line AH, parallel to DC, which serves as a boundary line. On one side of line AH, all lines AG are such that they do not intersect line DC, whereas on the other side, every straight line AF cuts line DC. The angle HAD between the parallel HA and the perpendicular AD is known as the parallel angle (angle of parallelism), and will be denoted as (p) for AD = p in the following.

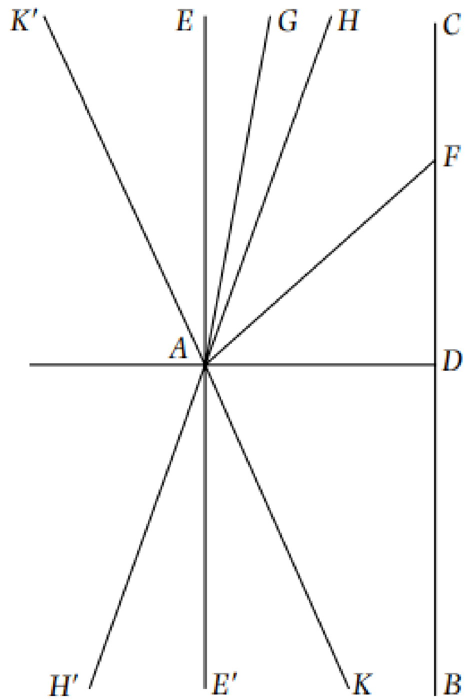

Figure 1: Lambert's quadrilateral

The geometry addressed — what kind of aworld do we inhabit? — was not taken into account. In this regard, the new geometries deviated from the mainstream alluded to before. In 1837, one of Lobachevsky's explanations of his method, the 'Géométrie Imaginaire,' was published in the prestigious Journal für die reine and angewandte Mathematik, but received no reaction. This was the year Chasles's 'History' was published, in which all the significant contemporary advancements were in projective geometry, which was then seen as an extension of Euclidean geometry (as we have shown). In his 1855 article titled "On the Hypotheses that Underlie Geometry," Gauss's pupil Bernhard Riemann notably broached the topic of the geometry of space as a potential source of doubt in a fresh and ultimately immensely significant way. This too had a delay; it was not published until 1866, and it is not an easy read while being a crucial text of modern mathematics. It is both a foundational work of contemporary physics, notably Einstein's General Theory of Relativity, and an introduction to the modern mathematical viewpoint, which separates the study of geometries from any preconceived notions of what the universe may be like.

Here, Riemann's goals need a deeper examination. According to his statements, they will decide the The basic facts from which we may learn about the geometry of space. Such verifiable facts Examples are the criteria for triangle congruence and the possibility of extending lines. endlessly; even the parallel axiom. Likewise, they may not exist, in which case one would need to substitute something else for them. While there is no doubt that geometry is the study of space, He wanted to evaluate, experimentally, what assumptions we bring to this subject and to what extent. We are able to defend our intuitions and utilize them to deduce what space must be. It had three dimensions; this is the meaning of the disapproving word 'triply' One of them featured criteria for measuring the lengths of curves contained inside it. Informed by Riemann conceived of "straight lines" in space as curves in relation to Gauss's work on surfaces. shortest length and provided guidelines, at least in principle, for finding such lines. He too Considered the issue of what geometry space would need to meet one reasonable requirement. The presumption that hard objects could be moved without altering form. (This is in order to talk in mechanical terms. From a geometrical perspective, standard norms for congruence The explanation is that what Riemann referred to as the 'curvature' of space must remain constant. from one location to another, and that this is met in three instances:

1. Euclidean geometry;
2. Geometry on a sphere, or something like it (which Riemann considered)—this is Saccheri's HOA;
3. Lobachevsky–Bolyai or 'hyperbolic' geometry. This was not considered by Riemann, but when his ideas came to be publicized, particularly by Helmholtz in the 1870s, it had become widely known and could be seen as another candidate.

Helmholtz wrote a number of articles setting out his view that alternative models for space should be considered (and tested). In particular, he wrote for the new English journals Nature and Mind; and an extract from one of his articles is included as Appendix C. By this time it had been established that hyperbolic geometry was free from contradiction (the model argument). However, this did not settle the question of whether it was worth considering, which hinged on whether space could conceivably have such a geometry. Lobachevsky had already considered the question of measurements to determine this, and Helmholtz clarified the point:

All practical systems of measuring the angles of large rectilinear triangles, and in particular all astronomical systems that make the parallax of immeasurably distant fixed stars equal to zero (in pseudospherical space, the parallax would be positive even for infinitely distant points), empirically confirm the axiom of parallels and demonstrate that the measure of curvature of our space is currently indistinguishable from zero. As Riemann noted, there is still the issue of whether the outcome would not be different if we were able to utilize alternative basis lines than the principal axis of the earth's orbit.

The 'parallax' of a star S is the angle $\alpha+\beta$ in the picture, which in Euclidean geometry equals γ (and is thus negligibly tiny when the star's distance is much more than p, the diameter of the Earth's orbit). In non-Euclidean geometry, the minimum value of $\alpha+\beta$is (almost) $\pi/2 + \pi(p)$. Interestingly, the fact that this is, practically speaking, zero was cited as an argument against Copernicanism; it was said that if the Earth moved, the parallax of the stars would be observable. By the 19th century, it was widely believed that the Earth moved, but the parallax was too tiny to measure.

References:

1. D. Jacquette, "The Soundness Paradox," in Logic Journal of the IGPL, vol. 11, no. 5, pp. 547-556, Sept. 2003, doi: 10.1093/jigpal/11.5.547.

2. L. John, J. Boardman and B. Sauser, "Technology and policy: Opposite ends of the paradox spectrum?," 2008 IEEE International Conference on System of Systems Engineering, 2008, pp. 1-6, doi: 10.1109/SYSOSE.2008.4724162.

3. D. Jacquette, "The Soundness Paradox," in Logic Journal of the IGPL, vol. 11, no. 5, pp. 547-556, Sept. 2003, doi: 10.1093/jigpal/11.5.547.