1 Overview of exercises (PART I)

1. limb-darkening scattering exercise we did during the course. — You can look into your notes from that, and I attach here also a sample program which you can use a base. After you have familiarised yourself with this, you can start to think about how you would go about to extend this to a 3D setting (assuming isotropic scattering).

2. (As prep for Monte-Carlo school) here is a script computing a UV resonance P-Cygni line in spherically symmetric wind with v beta-law. At top of routine, a few exercises are given, where you can modify and play around with code. Monte-Carlo program which computes a UV resonance spectral line from a fast outflowing spherically symmetric stellar wind (if you were not cc'd on that email, let me know so that I can send you the files as well). At the top of that little script, there are a few suggestions for exercises (additions) you could do to that program, in order to learn a bit more about the general workings of Monte-Carlo radiative transfer in this context. — So that might be a good idea for you to do as well ! (And you can also ask the others in the group for some tips etc. then.)

3. Some background reading:
   - Attached mc manual by Puls.
   - Paper by Sundqvist+ 2010 (Appendix, I think).

2 Overview of exercises (PART II)

1. Calculate the probability distribution to sample from in the case of Eddington limb darkening for the initial distribution (see Section 11.3).
   - finished + Ok
2. Calculate analytical solution for simplified problem in the case that `mu = 1` (see Section 11.1).
   - finished + Ok + can be further studied
3. Perform convergence analysis (see Section 11.5).

3 Overview of exercises (PART III)

1. Revisit 3D limb darkening. ϕ should be sampled between 0 and 2π (see Section 10.6). (OK)
2. Revisit convergence analysis: adapt plot formatting and standard deviation is defined as square root of variance (see Section 11.5).
3. Test variance reduction technique (see Section 11.6).
4. Some general considerations about the definitiion of specific intensity (see Section **??**). (OK)
5. For the Monte Carlo approximation of the diffusion equation, why do wo have $N \sim \tau$ for low optical depth $\tau \ll 1$ (see Section 19).
6. Revisit the radial streaming approximation in `pcyg.f90` for lower optical depth (e.g. `xk0=0.5`). (see Section 11.1).
7. What happens when you add a line (e.g. $x = 0.5 = a$)? How would you do that? (see Section **??**)
8. Towards a mathematical description of the problem.

4 Overview of exercises (PART IV)

1. Convergence analysis: also fit a line through the points. Formally, we write $V = CN^x$ and determine both C and X from experimental data. Correspondingly, $\log(V) = \log(C) + x\log(N)$. This is fitted using least-squares (see Section 11.5).

2. Variance reduction technique

   - averaging over different stochastic realizations?
   - take `xk0=0.5`
   - try to also discretize μ

3. Adding a second line: develop computer code in the radial streaming assumption (use analytic formulas) $\mu = 1$ (see Section **??**).

   - a following improvement is the use of a grid instead of using the bisection method.

4. Limb darkening. Have a look at section 10.3.1.

5 Multiline transfer (PART I)

1. What happens when you add a line (e.g. $x = 0.5 = a$)? How would you do that? (see section **??**)

6 Multiline transfer (PART II)

1. calculate force from one line (see section 16).

   - discretize in shells
   - assume $\epsilon_i = cte$

2. compare to analytic expressions

7 Preparation for Equation meeting on 15 October

See Section 15.

8 Preparation for meeting on 21 October

I have been working on Section 13 and on Section 15.

questions

- also take into account the photons where no scattering takes place?
- language of master thesis?
- analytical expression for $L(r)$?

9 Introductory exercises

9.1 Analytical exercises

From course material from (prof. Sundqvist - CMPAA course).

1. introduction
2. radiation quantities
   - exercise p.3:
     - on one hand, we know that $\Delta\epsilon \sim C/r^2$
     - on the other hand, from the definition we know that $\Delta\epsilon = I_\nu A_1 A_2/r^2 \Delta\nu\Delta t$
     - combining these equations shows that I_ν is independent from r
   - exercise p.4:
     -
   - exercise 1:
     - $F_x = \int_0^\pi \left[I_\nu(\theta)\sin^2(\theta) \int_0^{2\pi} \cos(\phi) \right] d\theta d\phi = 0$
     - the same reasoning for $F_y = 0$
   - exercise 2:
     - the equation follows from $d\mu = d\cos(\theta) = \sin(\theta)d\theta$
   - exercise 3:
     - isotropic radiation field (i.e. $I(\mu) = I$) then we have $F_\nu = 2\pi \int_{-1}^{1} I\mu d\mu = 2\pi I \left. \frac{x^2}{2} \right|_{-1}^{1} = 0$
   - exercise 4:
     - $F_\nu = 2\pi \int_{-1}^{1} I(\mu)\mu d\mu = 2\pi \int_{-1}^{0} I_\nu^- \mu d\mu + 2\pi \int_0^1 I_\nu^+ \mu d\mu = 2\pi I_\nu^+$
   - exercise p.7:
     - isotropic radiation field:
       * although the radiation pressure is a tensor, we will denote it as a scalar $P_\nu = \frac{4\pi I_\nu}{c}$
       * the radiation energy density $E_\nu = \frac{12\pi I_\nu}{c}$
       * thus $f_\nu = \frac{1}{3}$
     - very strongly peaked in radial direction (beam): $I_\nu = I_0\delta(\mu - \mu_0)$ with $\mu_0 = 1$
       * pressure tensor $P_{nu} = \frac{1}{c}\int I_0\delta(\mu - \mu_0) nn d\Omega$
       * energy density $E_\nu = \frac{1}{c}\int I_\nu d\Omega$
       * in this case $P_\nu = E_\nu$ thus $f_\nu = 1$
3. radiation transport vs. diffusion vs. equilibrium
   - exercise p. 12: 1D, Cartesian geometry, plane-parallel, frequency-independent and isotropic emission/extinction
     - radiation energy equation
       * The equation follows by integrating Equation (**??**)
       * By definition, $E = \frac{1}{c}\iint I_\nu d\nu d\Omega$
       * thus $\frac{dE}{dr} = \int (j - kI) d\nu d\Omega$ thus $\boxed{\frac{dE}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)}{c}}$

    - work out the integral taking into account frequency-independent and isotropic coefficients:
  - zeroth momentum equations
    - One must also take into account the specific form of the flux vector $F = \iint I_\nu n d\nu d\Omega = 2\pi \int_{-1}^{1} I_\nu(\mu)\mu d\mu$
    - thus $\frac{dF}{dr} = \frac{1}{c}\int (j - kI)nd\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j-kI)4\pi(\nu_1-\nu_0)n}{c}}$
  - first moment equation
    - similar reasoning
    - $\frac{dP}{dr} = \int (j - kI)n.nd\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j-kI)4\pi(\nu_1-\nu_0)n}{c}}$
- first exercise p. 15
  - $P = \frac{1}{c}\iint I_\nu \mu^2 d\Omega d\nu = \frac{2\pi}{c}\int_\nu \int_{-1}^{1} I_\nu \mu^2 d\mu d\nu = \frac{4\pi}{3c}\int B_\nu d\nu = \frac{aT^4}{3} = \frac{E}{3}$
- second exercise p.15
  - assuming the diffusion limit,
  - flux-weighted mean opacity $\kappa_F = \frac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}$
  - Rosseland mean opacity $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$.
    - in the diffusion limit, $F_\nu = -\frac{4\pi}{3}\frac{dB_\nu}{k_\nu dz}$ thus $\frac{dB_{nu}}{dT} =$
    -
- third exercise p.15

4. the equations of radiation-hydrodynamics
5. numerical techniques for the radiative diffusion approximation
6. applications and approximations for a dynamically important radiative force in supersonic flows
   - exercise p.27: $L_{SOB} = \Delta r = \frac{v_{th}}{dv/dr} = \frac{10[km/s]}{1000[km/s]/R_*} = 0.01R_*$
7. Appendix A: properties of equilibrium black-body radiation
   - exercise p. 29
     - this should be satisfied: $B_\nu d\nu = -B_\lambda d\lambda$ and also $\nu = \frac{c}{\lambda}$
     - this is equivalent to saying that $0 = \nu d\lambda + \lambda d\nu$ or $d\lambda = -\frac{\lambda}{\nu}d\nu$ thus $B_\lambda = \frac{\nu}{\lambda}B_\nu$
     - $B_\lambda(T) = \frac{\nu}{\lambda}\frac{2h\nu^3}{(\lambda\nu)^2}\frac{1}{e^{hc/\lambda kT}-1} = \frac{2h\nu^2}{\lambda^3}\frac{1}{e^{hc/\lambda kT}-1} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$
   - first exercise p.31
     - derive that $\lambda_{max}T = 2897.8[\mu mK]$
     - ...
   - second exercise p.31
     - this is about the spectra of (unknown) stars
   - first exercise p.32
     - see exercise 7
   - second exercise p.32

- BB radiation: $I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kt}-1}$
- the radiative flux for isotropic BB radiation is zero. See also exercise 3. This dus also holds for BB radiation.

- exercise p. 33
  - HR-diagram

8. Appendix B: Simple examples to the radiative transfer equation

- first exercise p. 34
  - start from radiative transport equation $\mu\frac{dI}{ds} = \alpha - \eta I$ in which $\eta = 0$ thus $\boxed{\mu\frac{dI}{ds} = \alpha}$
  - solving the ODE in the general case that $\alpha(s)$ is not constant:
    * integrate the equation $\mu I = \int_0^D \alpha ds$
    * ...
  - second exercise p. 34
    * case $\tau(D) >> 1$: then $I(D) \approx S$
    * case $\tau(D) << 1$: then $I(D) \approx I(0) + S(1-1) = I(0)$
  - first exercise p.35
    * is the plane-parallel approximation valid for the solar photosphere?
  - second exercise p.35
    * goal: find a solution to the equation $\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$ where $I(\tau,\mu)$
    * solution
- second exercise p.35

9. Appendix C: connecting random walk of photons with radiative diffusion model

- exercise p. 38. Computing the average photon mean-free path inside the Sun. $l = \frac{1}{\kappa\rho} = \frac{V_o}{\kappa M_o}[cm]$
- exercise p.39. Computing the random-walk time (diffusion time) for photons

9.2 Numerical exercises

9.2.1 Implicit 1D solver

Exercise from (20-11-2018).

Goal Implement implicit solver for time-dependent diffusion equation

$$\partial_t u = \partial_{xx} u \tag{1}$$

Solution The convergence behaviour of the method is shown in Figure 1.

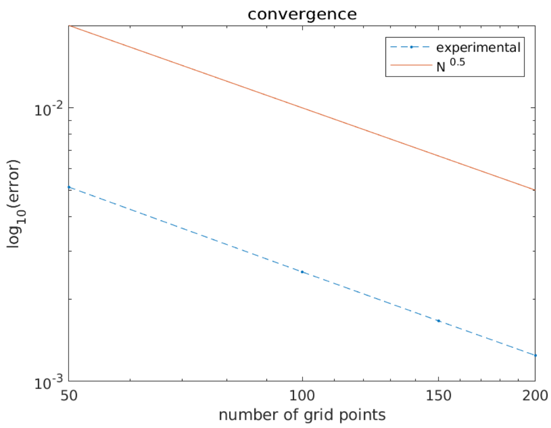


Figure 1: Convergence behaviour for 1D implicit solver (diffusion equation)

9.2.2 ADI 2D Solver

Goal Implement implicit solver for time-dependent diffusion equation

$$\partial_t u(t, x, y) = \partial_{xx} u(t, x, y) + \partial_{yy} u(t, x, y) \tag{2}$$

Solution There is still an error in the code.

9.2.3 Area of a circle

Goal Develop Monte Carlo code

Solution

9.3 Other Exercises

From course material from Ivan Milic.

9.3.1 Lecture 7

1. Derive expressions for the emergent radiation when properties are the following:
   - optically thin slab at all wavelengths
   - wavelength-independent incident radiation

   Solution: see slide 14?

2. Derive ralations between Einstein coefficients.

3. Calculate electron density in atmosphere from FALC model

10 Limb darkening

10.1 Formulation of the problem

- The radiative transfer equation **??** in this situtation becomes an integro-differential equation with $S(\tau) = \frac{1}{4\pi} \int I(\tau, \mu) d\Omega$

$$\begin{aligned} \mu \frac{dI(\tau, \mu)}{d\tau} &= -I(\tau, \mu) + S(\tau) \\ &= -I(\tau, \mu) + \frac{1}{4\pi} \int I(\tau, \mu) d\Omega \end{aligned} \tag{3}$$

- The difficulty resides in the (evaluation of) the source function. Monte Carlo simulation avoids explicit calculation source function: source function implicit in Monte Carlo simulation. There the physics are simulated IN BETWEEN TWO CONSECUTIVE SCATTERING EVENTS as follows

$$\frac{dI}{dz} = -\alpha I \tag{4}$$

thus $I = I_0 e^{-\delta\tau}$ and then τ is sampled according to $\tau = -\log(X_{\text{random}})$

10.2 Solving the (integro-differential) radiative transfer equation

Analytical Solution of Equation (3) Ik heb de mosterd gehaald op [**Dublin˙limb˙darkening**].

$$I(0, \mu) = \int_0^{\infty} S(\tau) exp\left(\frac{-\tau}{\mu}\right) d\left(\frac{\tau}{\mu}\right) \tag{5}$$

Numerical Solution of Equation (3) First rewrite the equation

$$\begin{aligned} \mu \frac{dI(\tau, \mu)}{d\tau} &= -I(\tau, \mu) + \frac{1}{4\pi} \int I(\tau, \mu) \sin(\theta) d\theta d\phi \\ &= -I(\tau, \mu) + \frac{1}{4\pi} \int I(\tau, \mu) d\mu d\phi \\ &= -I(\tau, \mu) + \frac{1}{2} \int I(\tau, \mu) d\mu \end{aligned} \tag{6}$$

Discretization scheme:

$$??? \tag{7}$$

10.3 Eddington-Barbier approximation

$$J(\tau) = S(\tau) = \left(\tau + \frac{2}{3}\right) 3H \tag{8}$$

Together with the time-independent radiative transfer equation (**??**) in a gray (frequency-independent) planar medium gives

$$\mu \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - \left(\frac{2}{3} + \tau\right) 3H \tag{9}$$

The emergent intensity $I(0, \mu)$ is a solution of Equation (9). Its solution for $\tau = 0$ equals

$$\boxed{I(\tau = 0, \mu) = I_1 \left(\frac{2}{5} + \frac{3\mu}{5}\right) = a + b\mu} \tag{10}$$

with $a = \frac{\sigma}{2\pi} T_{eff}^4$ and $b = \frac{3\sigma}{4\pi} T_{eff}^4$

10.3.1 Validity of the Eddington-Barbier approximation

If we assume Equuation (8) then $I = I_1(a + b\mu)$ thus

$$J = \frac{1}{2}\int I(\tau,\mu)d\mu = \frac{1}{2}\int_0^1 (a+b\mu)d\mu \tag{11}$$

But we can not really compute this since we only know $I(0,\mu)$.

The thing is that Eddington specifies $S(\tau)$ but not $S(\mu)$ directly. If that expression is used into the expression of a semi-infinite atmosphere (namely that the specific intensity is the Laplace transform of the source function), then we get the desired result AT THE OUTER EDGE OF THE ATMOSPHERE. See also [Mihalas] p. 571.

dat ziet er hier niet goed uit

10.4 2D Case

We again have $\mu = \cos(\theta)$. The solution of the radiative transfer equation in plane-parallel syummetry with frequency-independent absorption and emission, is

$$I(\mu) = I_1(0.4 + 0.6\mu) \tag{12}$$

In the Monte Carlo code, the photons are sorted according to the direction that they leave the atmosphere.

Goal Calculates the angular dependence of photon's emitted from a plane-parallel, grey atmosphere of radial optical depth `taumax`. The value of `tau` determines the position of the photon

Variables and Algorithm

- `muarray` contains emergent photons
- `na` number of channels
- `dmu` = 1/`na` width of channels
- `nphot` number of photons
- `taumax` maximum optical depth

Algorithm 1 Limb darkening: compute quantitiy of photons

---

initialization
  radial optical depth τ
  direction μ
for all photons **do**
  $\boxed{\tau = \tau_{max}}$
  while `tau` ≥ 0 **do**
    compute scattering angle `mu`
    if tau $\geq$ taumax **then** $\boxed{mu = sqrt(x)}$ (initial distribution)
    else $\boxed{mu = 2 * x = 1}$ (isotropic scattering)
    tau_i = -log(x2)
    tau = tau - tau_i*mu
  end while
  now we know that the photon has left the photosphere
  compute the distribution of all angles `mu` at which the photon left the photosphere
end for
visualisation:

- plot photon numbers from $\mu d\mu$ against `mu`
- plot specific intensity from $d\mu$ against `mu` against

Figure 2 is according to what is expected $I = I_0(0.4 + 0.6\mu)$. The input parameters are as follows `Limb_Darkening(number_of_channels = 20, number_of_photons = `10^5`, maximum_optical_depth = 10)`.

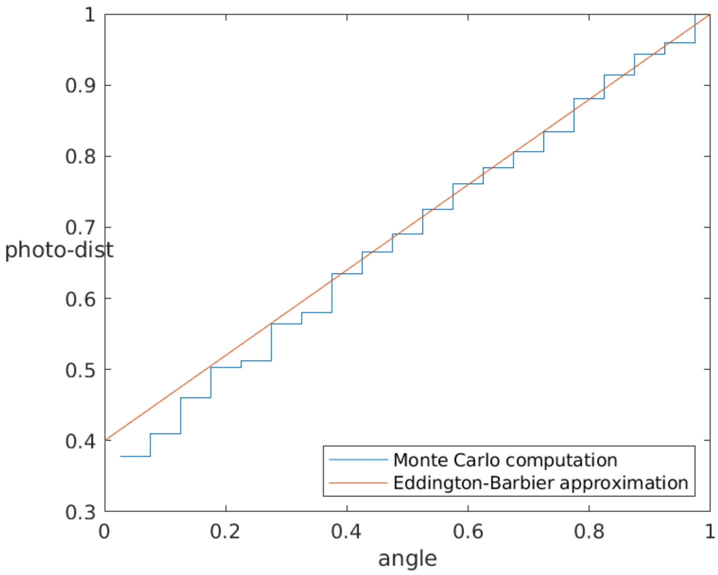


Figure 2: histogram for `mu`

10.5 Calculating the moments of intensity: study

Limb darkening. Expected results in Equation (12). The first angular moment $J(r)$ of the intensity should approximately equal Equation (8), if the Eddington-Barbier approximation is valid. I am now also interested in computing $J(r)$, numerically. Photons are counted using formula in Table 1.

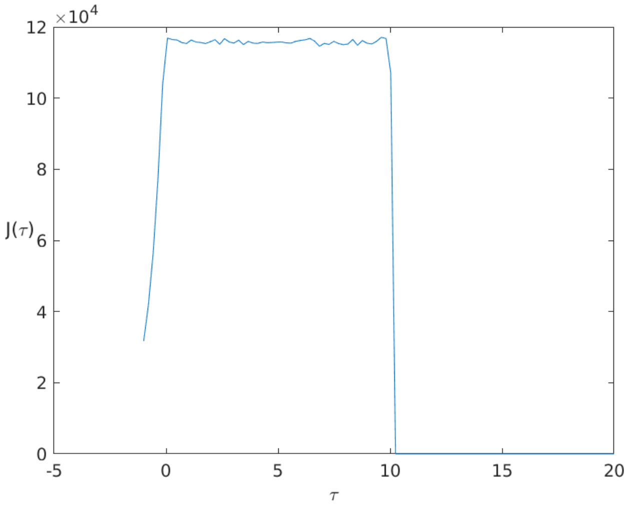


Figure 3: $J(\tau)$ as numerically computed with MC limb darkening algorithm

Problem From Figure 4 (output of Monte Carlo simulation), $J(\tau) \equiv J$ seems to be constant. This is not as predicted in Equation (8).

Explanation One can understand this result as follows: photons are counted according to the sign of their angle. But of course all photons eventually reach the zone where $\tau = 0$ and zones where a photon *goes for a little walk* does not contribute to the moment J.

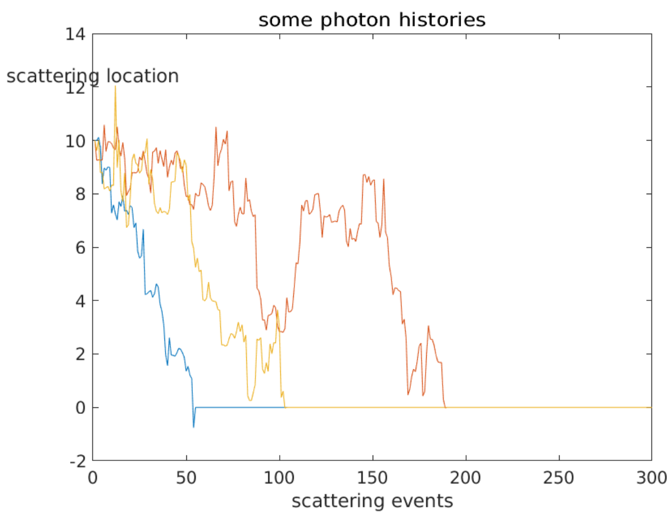


Figure 4: $J(\tau)$

Conclusion ... where do I make an error in my reasoning or coding?

Thank you for looking at this :) Ignace.

10.6 3D Case

What changes is this:

- introduction of a new angle ϕ
- the optical depth is not updated with respect to ϕ

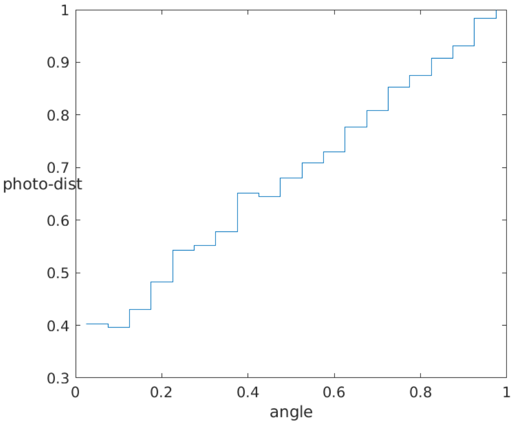


Figure 5: histogram for `mu`

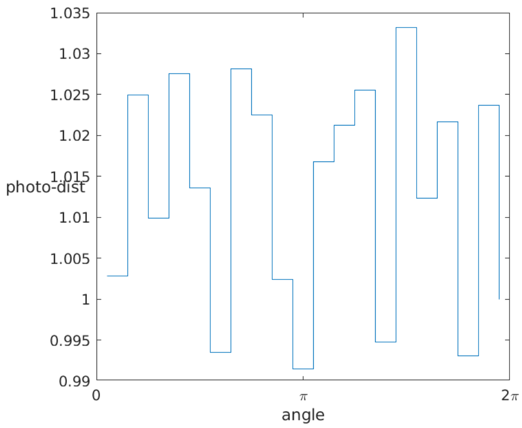


Figure 6: histogram for `phi`

Figure 5 and Figure 6 are the result of the function `Limb_Darkening_3D` with the following input parameters: `Limb_Darkening_3D(number_of_channels = 20, number_of_photons =` 10^5`, maximum_optical_depth = 10)`. The results according to what is expected, namely $I = I_0(0.4 + 0.6\mu)$ and ϕ follows a uniform distribution.

Extension: make version where the optical depth is updated with respect to ϕ

Via this link, you can go back to the exercises overview: Section 3.

11 Spectral line formation: pcyg.f90

This section is about the study of line formation in an expanding wind.

Background

name	explanation
paramaters	
xk0	
alpha	velocity profile parameter
beta	velocity profile parameter
start frequency of the photon	
xstart	start frequency
vmin	
vmax	
angle of the photon	
xmuestart	start angle
xmuein	incident angle
xmueou	outward angle
pstart	impact parameter
xnew	new photon frequency
optical depth	
tau	optical depth
number of photons admin	
nphot	number of photons
nin	photons scattered back into core
nout	photons escaped
functions	
func	velocity profile distance from center of star r
xmueout	outwards (scattered) angle xk0 alpha r v sigma
photon tallying	
nchan	amount of bins

$$\xrightarrow{\texttt{xstart}} \boxed{\texttt{pcyg.f90}} \xrightarrow{\texttt{xnew}}$$

The photons are sorted according to `xnew`. In general, the flux is dependent on μ and the frequency x.

Practical formula

- emission angle $\mu = \cos(\theta)$
- according p-ray $p = \sqrt{1-\mu^2} = \sin(\theta)$
- incident angle `xmuein` $= \sqrt{1 - \left(\frac{pstart}{r}\right)^2}$

Geometry & Symmetry assumptions

- spherical geometry

11.1 First adaptation: what if all photons are released radially from photosphere?

11.1.1 Release photons radially: numerical MC experiments

What would happen with line-profile, if you assumed all photons were released radially from photopshere?

- In other words `xmuestart` $= 1$.
- This is implemented under the test case `test_number=1`.
- Results in Figure 30 for opacity `xk0 = 100`.

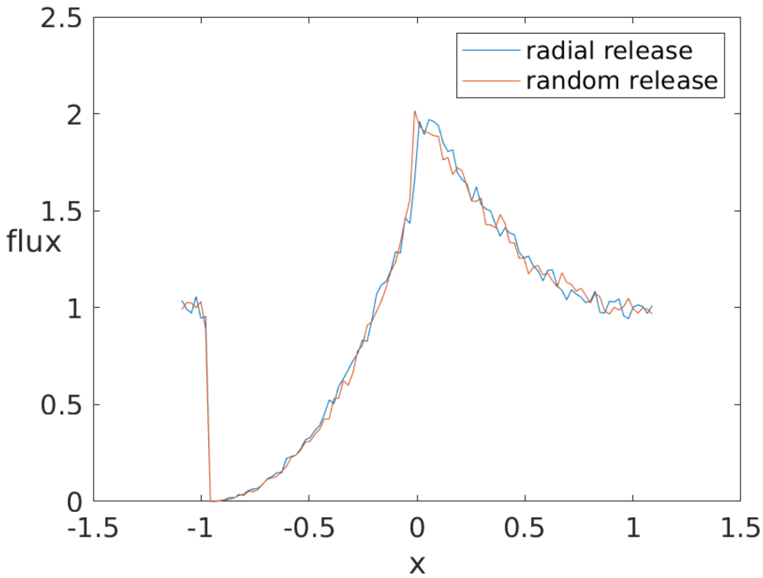


Figure 7: The number of photons equals 10^5, `xk0=100`

11.1.2 Derive analytic expression

See also slide 26/49 [Sundqvist course material].

- since `xmuein = 1` we have for the velocity profile

$$v = v_\infty(1 - b/r)^\beta \tag{13}$$

  A scaled version of Equation (13) yields

$$u = \frac{v(r)}{v_\infty} = \left(1 - \frac{b}{r}\right)^\beta \tag{14}$$

  with $u \in [0..1]$

- Doppler shift for the frequency of the photons: $x_{CMF} = x_{REF} - \mu u$.
- Condition for resonance from Sobolov approximation (to be studied later): $\boxed{x_{CMF} = 0}$ thus

$$x_{REF} = \mu u \tag{15}$$

  or thus $x_{REF} = \boxed{u_{\text{interaction}}}$ and than solve Equation 14 for $r_{\text{interaction}}$

- If $\mu = 1$ then

$$x = \left(1 - \frac{b}{r}\right)^{\beta} \tag{16}$$

$$x^{1/\beta} = 1 - \frac{b}{r}$$

$$r(1 - x^{1/\beta}) = b$$

$$\boxed{r(x) = \frac{b}{1 - x^{1/\beta}}} \tag{17}$$

- From the location of interaction r, the incident angle can be calculated

$$\texttt{xmuein} = \sqrt{1 - \left[\frac{\texttt{pstart}}{r}\right]^2} = \sqrt{1 - \left[\frac{\sqrt{1 - \texttt{xmuestart}^2}}{r}\right]^2} \tag{18}$$

Now also taking into account that `xmuestart = 1`

$$\texttt{xmuein = 1} \tag{19}$$

- The calculation of the optical depth goes as follows:

$$\tau = \frac{\texttt{xk0}}{rv^{2-\alpha}(1 + \texttt{xmuein}^2\sigma)} \tag{20}$$

Now also taking into account that `xmuestart = 1` gives

$$\tau = \frac{\texttt{xk0}}{rv^2(1 + \sigma)} \tag{21}$$

where $\boxed{v(x) = \left(1 - \frac{b}{r}\right)^{\beta}}$ and $\frac{dv}{dr} = \frac{\beta b}{r^2}\left(1 - \frac{b}{r}\right)^{\beta - 1}$

and $\sigma(x) = \frac{dv}{dr}\frac{r}{v} - 1$ thus $\boxed{\sigma(x) = \frac{\beta b}{r}\left(1 - \frac{b}{r}\right)^{-1}}$

- Assuming that $\beta = 1$ then $\boxed{v(x) = 1 - \frac{b}{r}}$ and $\frac{dv}{dr} = \frac{\beta b}{r^2}$ and $\boxed{\sigma(x) = \frac{\beta b}{r}}$.

- `xmueou` follows the distribution as given by the function `xmueout`, namely

$$p(x) = \frac{1 - e^{-\tau}}{\tau} \tag{22}$$

with $\tau = \frac{\texttt{tau0}}{1 + \texttt{X}^2\sigma}$ where `X` is a random number, so actually this comes down to

$$\boxed{p(x) = \frac{1 - e^{-\frac{\tau_0}{1+x^2\sigma(x)}}}{\frac{\tau_0}{1+x^2\sigma(x)}}} \tag{23}$$

- Finally one can combine these results to get the distribution of the photons according to the frequency x via the relation

$$\texttt{xnew = xstart + v(xmueou-xmuein) = xstart + v(xmueou -1)} \tag{24}$$

In words, we initially have an isotropic distribution for `xstart`. The number of photons that are leaving the atmosphere at different frequencies is however not isotropic through complex interactions that are incorporated into $p(x)$. One must also take into account that not all of the photons that are released actually escape from the atmosphere and also that sometimes no resonance is possible, and then Equation (24) is not applicable.

TO DO: proceed from this to the analytical expression for the flux. Here I am stuck for the moment.

11.1.3 Experiments with other opacities

The results for `xk0=0.5` are shown in Figure 8.

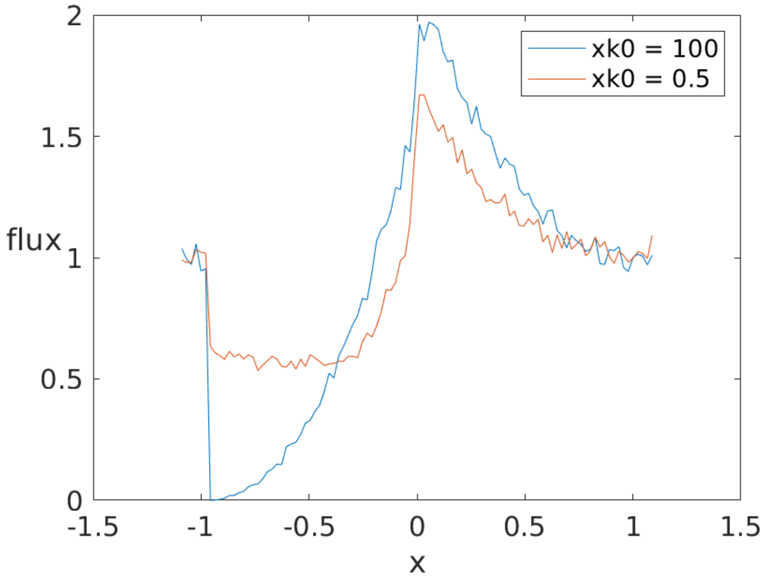


Figure 8: The number of photons equals 10^5, `xk0=0.5`

Via this link, you can go back to the exercises overview: Section 3.

11.2 Second adaptation: isotropic scattering

What would happen to line-profile, is you assumed scattering was isotropic (i.e., NOT following Sobolev-distrobution)

- in the implementation, `test_number = 2`
- the results are shown in Figure 9.

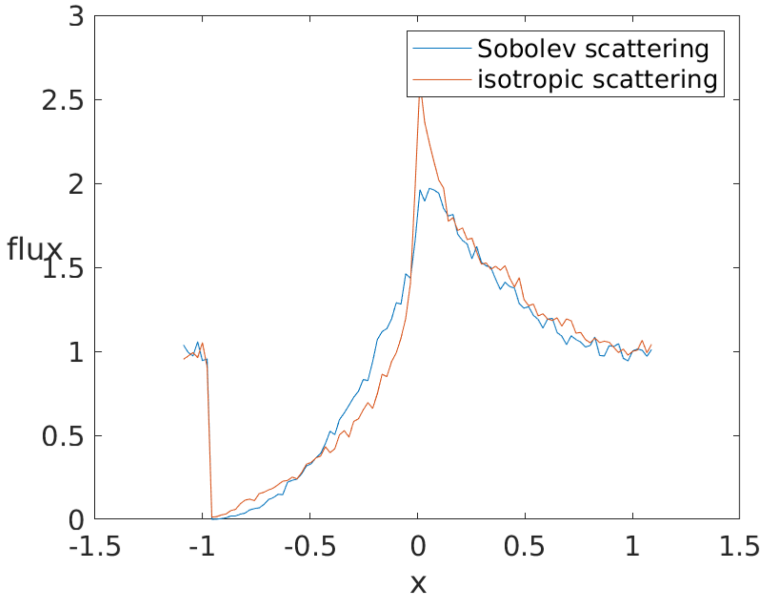


Figure 9: The number of photons equals 10^5

It is clear from Figure 9 that the peak around $x = 0$ is higher and sharper.

$\boxed{\text{Analyse this behaviour more closely}}$

11.3 Third adaptation: introduction of Eddington limb-darkening

Goal Put Eddington limb-darkening in. What happens?

11.3.1 Construction of probability distribution corresponding to Eddington limb darkening

For a general (introductory) discussion about Eddington limb darkening, please refer to Section **??**

1. Let us thus first review the emmission case where the flux in each direction is isotropic i.e. $I(\theta) = I$ (as experimented in paragraph 11.2)

   - the specific intensity is defined as $I_\nu(\mu) = \dfrac{dE_\nu}{\cos(\theta) dA dt d\nu d\Omega} = \dfrac{dE_\nu}{\mu dA dt d\nu d\Omega}$
   - the flux $F_\nu = \displaystyle\int_\Omega I_\nu \cos(\theta) d\Omega$ is in this case isotropic thus

   $$\xi = \int_0^\mu F_\nu d\mu = \int_0^\mu \int_\Omega I_\nu \cos(\theta) d\Omega d\mu = A \int_0^\mu \mu d\mu \tag{25}$$

   together with the condition that μ satisfies a probability distribution:

   $$1 = \int_{-1}^1 F_\nu d\mu = \int_{-1}^1 \int_\Omega I_\nu \cos(\theta) d\Omega d\mu = \frac{A}{2} \tag{26}$$

   thus $A = 2$. Photons need to be sampled according to $\mu d\mu$.

2. Now we look at a new case where the photons need to be emitted following a distrubution that corresponds to $I(\theta) = I(0)(0.4 + 0.6\cos(\theta))$.

   - in this case the flux $F_\nu = \displaystyle\int_\Omega I_\nu \cos(\theta) d\Omega$ is isotropic but also satisfies

   $$F_\nu = \int_\Omega I_\nu(0)[0.4 + 0.6\cos(\theta)] \cos(\theta) d\Omega \tag{27}$$

   I am not sure about the correctness of the assumption of isotropy of the flux

   $$\xi = \int_0^\mu F_\nu d\mu = A \int_0^\mu (0.4 + 0.6\mu)\mu d\mu \tag{28}$$

   subject to the normalisation condition -very similar to Equation (26) - that

   $$1 = \int_0^1 F_\nu d\mu = \frac{2A}{5} \tag{29}$$

   thus $A = \dfrac{5}{2}$. Photons need to be sampled according to

   $$\frac{5}{2}(0.4 + 0.6\mu)\mu d\mu \tag{30}$$

   In the code `pcyg.f90` this corresponds to `test_number = 3` (not yet implemented).

   The results of an accept-reject method that samples the probability distribution in Equation (30).

Via this link, you can go back to the exercises overview: Section 2.

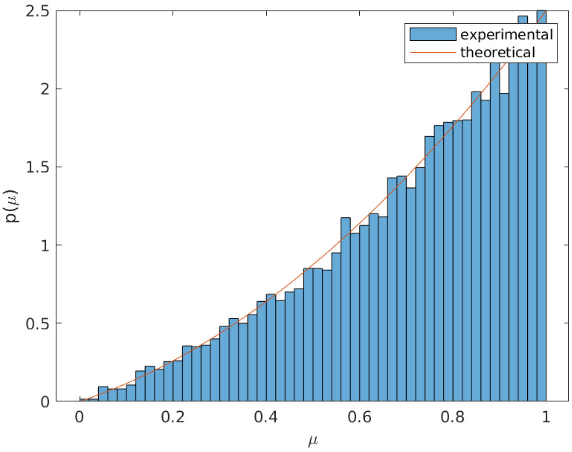


Figure 10: Accept-reject method for Eddington limb darkening

11.4 Fourth adaptaion: photospheric line-profile

Challening: Put photospheric line-profile (simple Gaussian) in. What happens? Test on `xk0=0` (opacity = 0) case.

- test case number 4
- $\boxed{\text{This is still to be implemented.}}$

11.5 Convergence analysis

Zero opacity The convergence of the Monte Carlo method is tested with the following input parameters

kx0	alpha	beta	test_number
0	0	1	0

for a varying amount of photons, as shown in Figure 11. We expect the method to have $\frac{1}{\sqrt{N}}$ convergence, where N is the number of photons. However, the methods strangely seems to have a faster convergence rate.

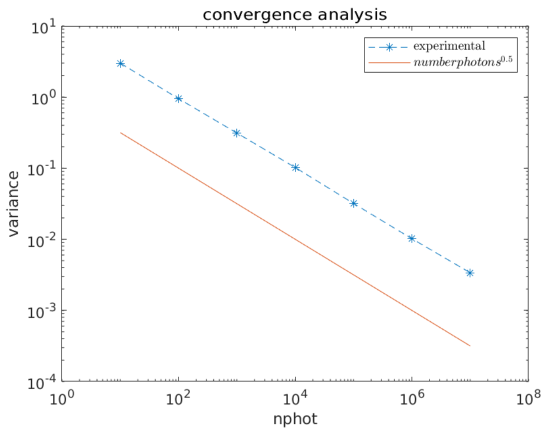


Figure 11: Original version of the code: convergence analysis (xk0=0)

Nonzero opacity The convergence test is set up as follows: different Monte Carlo simulations (with increasing number of photons) are compared to an *expensive* simulation with 10^7 photons. As can be seen in Figure 12, the spectrum profile behaves according to a $N^{0.5}$ law.

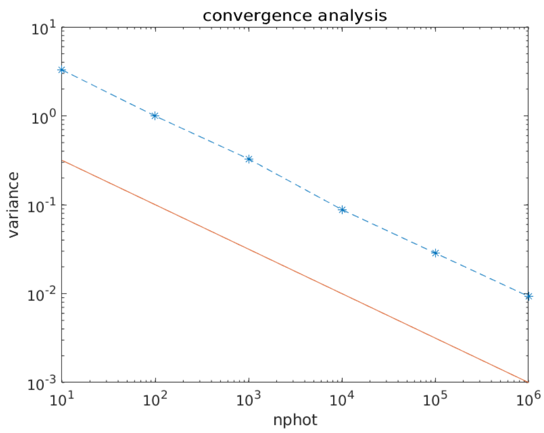


Figure 12: Original version of the code: convergence analysis (xk0=100)

Via this link, you can go back to the exercises overview: Section 2.

11.6 Variance reduction experiment

We will set up the test as follows

- run the code with `xk0=100` and number of photons $N = 10^7$
- run the code again for lower number of photons (e.g. $N = 10^3$), both with random sampling and pseudo-random sampling
- compute variance w.r.t. *expensive* simulation and compare
- `test_number = 5`

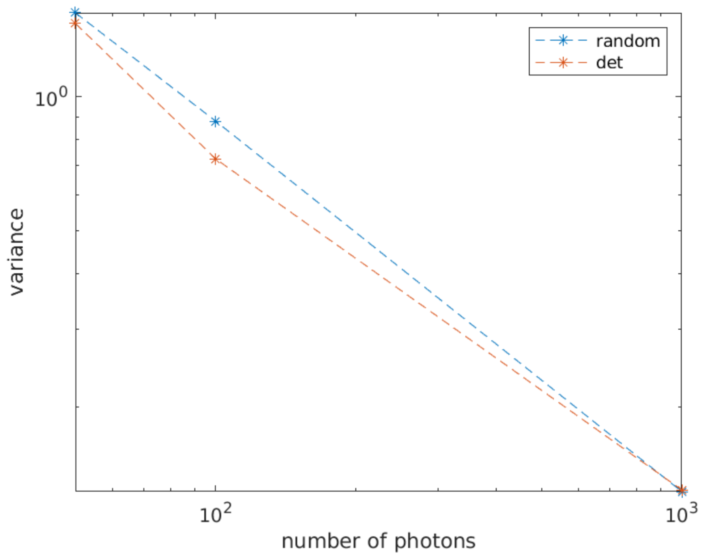


Figure 13: Original version of the code: convergence analysis (xk0=0)

`xk0=100` Possible improvement: average over different stochastic realizations.

Via this link, you can go back to the exercises overview: Section 3.

11.7 Mathematical description of the problem & Looking at literature

Have a look at [**NoebauerUlrichM˙2019MCRT**] (see Appendix).

12 Transferring the code to Matlab

12.1 Limit variables

	xmin	xmax	vmin	vmax
Fortran	-1.1	1.1	0.01	0.98
Fortran (reverse order for scattering distribution)	-1.1	1.1	-0.98	-0.01
Matlab (with `resonance_x = 0`	-1	1	-0.8	0

12.2 Comparison

12.2.1 Literal Matlab version

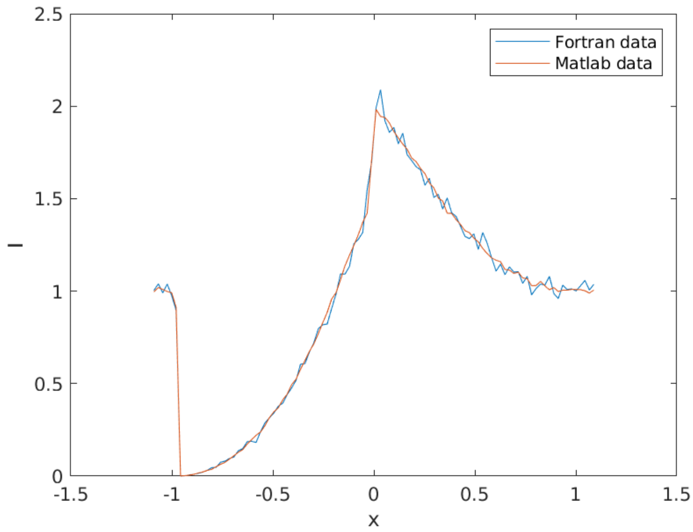


Figure 14: Comparison of Fortran code and Matlab code (unchanged version - `xk0 = 100`)

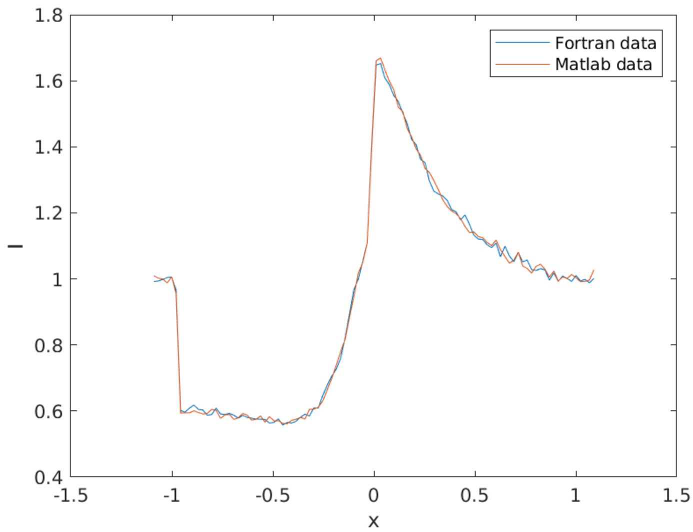


Figure 15: Comparison of Fortran code and Matlab code (unchanged version - `xk0 = 0.5`)

12.2.2 More freely translated version

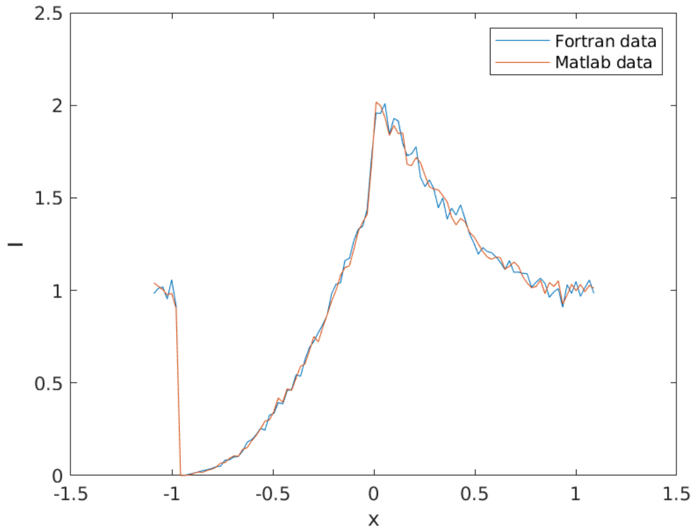


Figure 16: Comparison of Fortran code and Matlab code (freely adapted version)

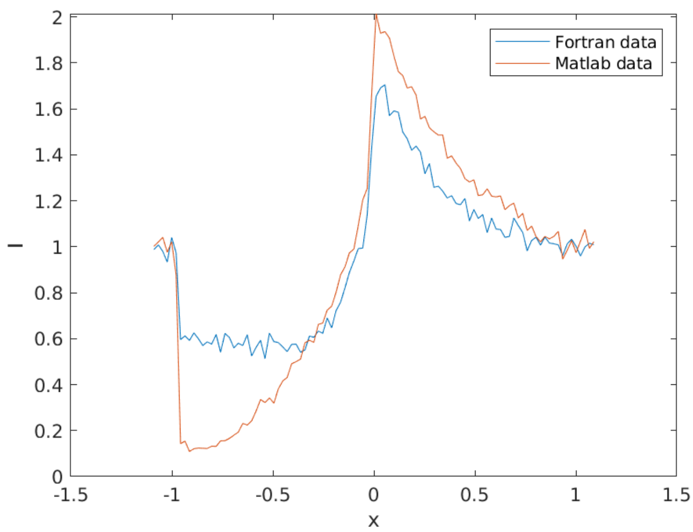


Figure 17: Comparison of Fortran code and Matlab code (unchanged version - `xk0 = 0.5`)

12.3 Optimizations

12.3.1 Already implemented

12.3.2 To be implemented

- generation of random numbers that undergo no scattering: replace with continuum profile

13 Theoretical background

Algorithm 2 `pcyg.f90:  one resonance line`

for all photons **do**

1. Release photon with frequency x
2. Check if interaction is uberhaupt possible.
3. Solve for distance (radius r) of interaction using Sobolev approximation $x_{CMF} = x_{REL} - \mu v(r)$ with $\boxed{x_{CMF} = 0}$ and compute Sobolev optical depth
4. Check whether the photon is scattered:

   if $\tau_S > -log(\xi)$ **then**

   Interaction: the photon is scattered. Update the frequency

   else

   No interaction
4. update the frequency according to the scattering event

end for

collect photons and perform visualisation

13.1 General things

- $\lambda\nu = c$. Mostly stellar spectra are recorded for increasing λ, thus decreasing ν
- the β-velocity law

$$v = \left(1 - \frac{b}{r}\right)^{\beta} \tag{31}$$

where we want $v = \in [v_{\min}, v_{\max}]$. Thus on one hand $b = 1 - x_{v_{\min}}^{1/\beta} > 0$ and then we can compute the radius where $v = v_{\max}$, namely $r_{\max} = b/(1 - v_{\max}^{1/\beta})$

  - Minimal velocity. Without angle correction (see Section 13.2), the minimal velocity is given by $(1-b)^{\beta}$. With angle correction,

$$v = \cos(\alpha)\left(1 - \frac{b}{r}\right)^{\beta} \tag{32}$$

then the minimal velocity is slightly higher.

  - Maximal velocity equals v_{max}

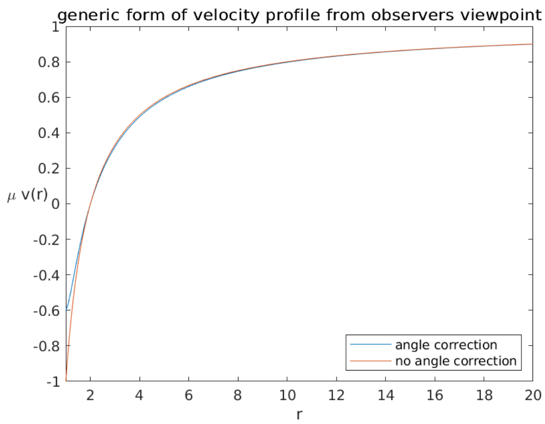


Figure 18: Example of velocity profile (give paramters)

13.2 Geometry

Spherical symmetry. Scattering at point P

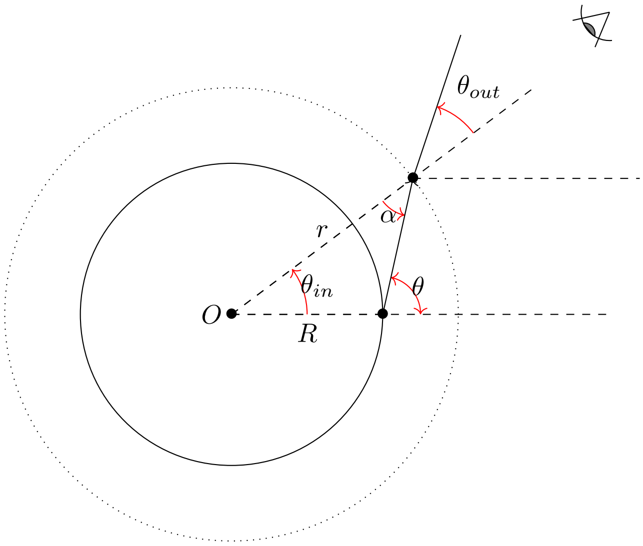


Figure 19: 2D Geometry for scattering

We can derive an expression, but first the sum of the angles in a triangle equals π which leads to $\alpha = \theta_{in} - \theta$. We then have the sine rule, namely that, with

$$\frac{\sin(\alpha)}{R} = \frac{\sin(\pi - \theta)}{r} = \frac{\sin(\theta)}{r} \tag{33}$$

and thus, with $R = R_* = 1$,

$$\cos(\alpha) = \sqrt{1 - \left(\frac{\sin(\theta)}{r}\right)^2} \tag{34}$$

Per definition

- `xmuestart` $= \cos(\theta)$
- `pstart` $= R_* \sin(\theta)$
- `xmuein` $= \cos(\alpha)$
- `xmueou` $= \cos(\theta_{\mathrm{out}})$

13.2.1 Condition for hitting the core

Check this only for inwards streaming photons!

$$\begin{aligned} \texttt{pcheck} &= \sqrt{r^2 \left(1 - \cos(\theta_{\mathrm{out}})^2\right)} \\ &= r \sin(\theta_{\mathrm{out}}) \end{aligned} \tag{35}$$

13.2.2 Slightly more general version

When $R_* \neq 1$, Equation (34) is replaced by

$$\cos(\alpha) = \sqrt{1 - \left(\frac{R_* \sin(\theta)}{r}\right)^2} \tag{36}$$

13.3 Sobolev approximation

- In fact, the absolute frequency of the photons does not change. However, in an observer's frame, we observe that the photon frequency changes at a scattering event. After a scattering event, the oberserver's frame frequency is updated as follows:

$$\texttt{xnew = xstart + u(xmueou - xmuein)} \tag{37}$$

- Sobolev condition for resonance:

$$x_{REF} - \mu u = 0 \tag{38}$$

with $u = \dfrac{v}{v_\infty} \in [0,1]$ and $\mu = \cos(\theta) \in [-1,1]$ then

$$x_{REF} \in [0,1] \tag{39}$$

- The radius of interaction, we solve for $v_{photon} = x > 0$, for a specific ν_0

$$\sqrt{1 - \left(\frac{p}{r}\right)^2}\left(1 - \frac{b}{r}\right)^\beta = \frac{\nu - \nu_0}{\nu_0}\frac{c}{v_\infty} \tag{40}$$

Now we are going to invest the effect of

  - increasing $\nu_0 \uparrow$. What then happens is that the RHS decreases, thus $r \downarrow$.
  - Inversely, $\nu_0 \downarrow$, then $r \uparrow$

13.4 Can resonance in the same resonance line happen twice?

After a first scattering event, the frequency is updated according to Equation (37). In my opinion, it is possible to have multiple scatterings.

13.5 Meaning of the parameters

- `xk0` is a characteristic scale of opacity $\chi = \dfrac{\texttt{xk0}}{rv^{2+\alpha}}$
- opacity $\chi = \dfrac{\texttt{xk0}}{rv^{2+\alpha}} \propto \kappa\rho \propto \dfrac{\kappa}{rv^2}$
- from this we deduce that $\kappa \propto \dfrac{1}{v^\alpha}$

13.6 Special case: `xmuestart = 1`

- FIRST SCATTTERING: from Equation (37) we have that `xstart = u` and then

$$\begin{aligned}\texttt{xnew} &= \texttt{xstart + u(xmueou- 1)} \\ &= \texttt{xstart.xmueou}\end{aligned} \tag{41}$$

Thus, since $\texttt{xmueou} \in [-1,1]$ and $\texttt{xmin} \leq \texttt{xstart} \leq \texttt{xmax}$

$$\texttt{xmin} \leq \texttt{xnew} \leq \texttt{xmax} \tag{42}$$

14 Development of computer code (in `Matlab`)

14.1 Implementation in Matlab: user's manual

Run the function `test_function(test_number)`.

`test_number`	parameter settings
0	original version
1	first adaptation: radial release
2	isotropic scattering – higher peak
3	Eddington limb darkening
4	photospheric line-profile
5	simple well
6	other resonance frequency (thus introducing shift)
7	formation of two lines, only radially streaming photons (thus also radial release
8	formation of two lines, with radial release
9	formation of two lines, full scattering possibilities

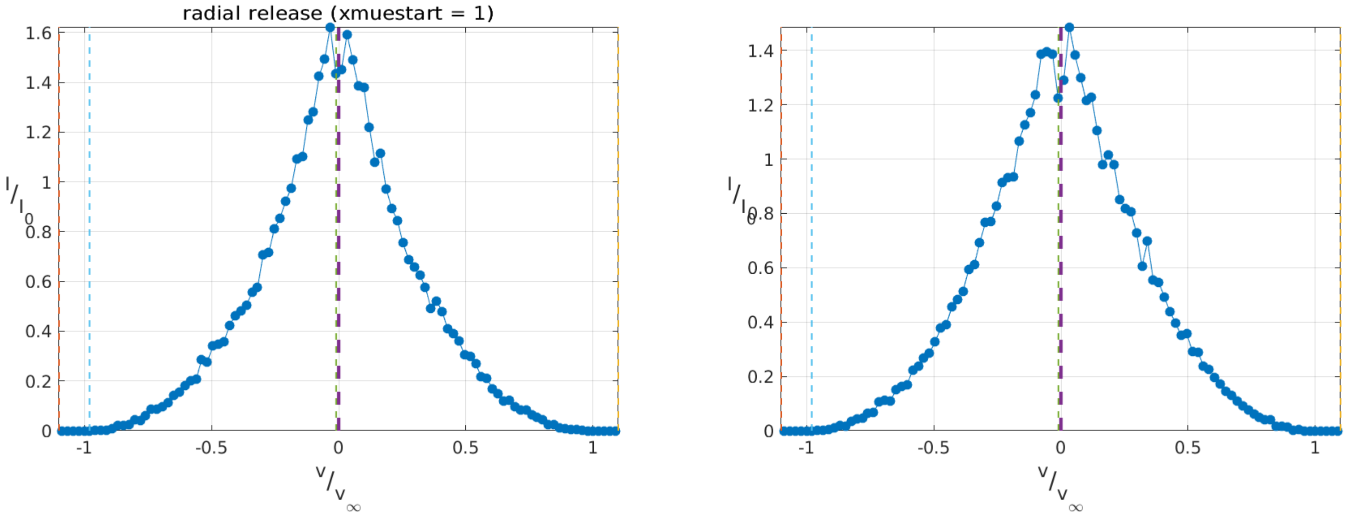
Via this link, you can go back to the exercises overview: Section 4.

14.2 Keeping track of the photon path

$$\left[\begin{array}{c} \texttt{xstart} \\ \texttt{xmuestart} \\ \texttt{goto\_end\_of\_loop} \\ \hline \texttt{r\_new} \\ \texttt{xmueou} \\ \texttt{xnew} \\ \texttt{x\_selected} \\ \hline \vdots \end{array}\right] \tag{43}$$

15 Experiments and results

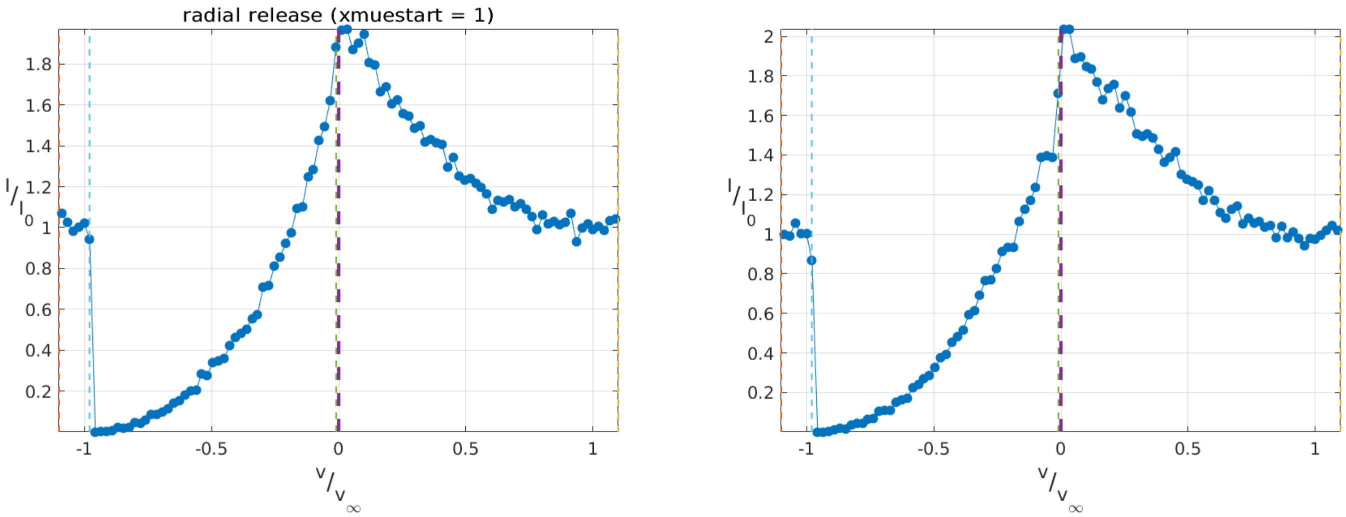
15.1 About the scattering probability distribution



(a) Radial release (b) Random release

Figure 20: Scattering distribution

15.2 Single resonance line



(a) Radial release (b) Random release

Figure 21: Single line formation with Sobolev approximation

15.3 Multiple line formation

- NON-INTERACTING LINES

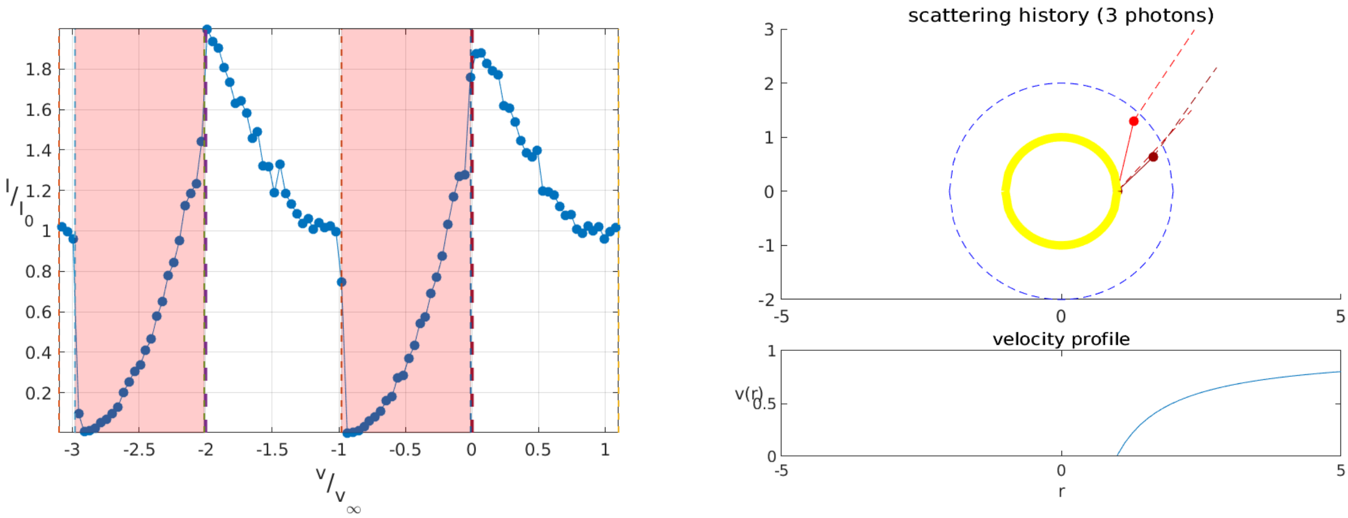


Figure 22: Multiple lines (distant lines, non-interacting)

- OVERLAPPING LINES

Figure 23: Multiple lines (distant lines, non-interacting)

15.4 Effect of the opacity

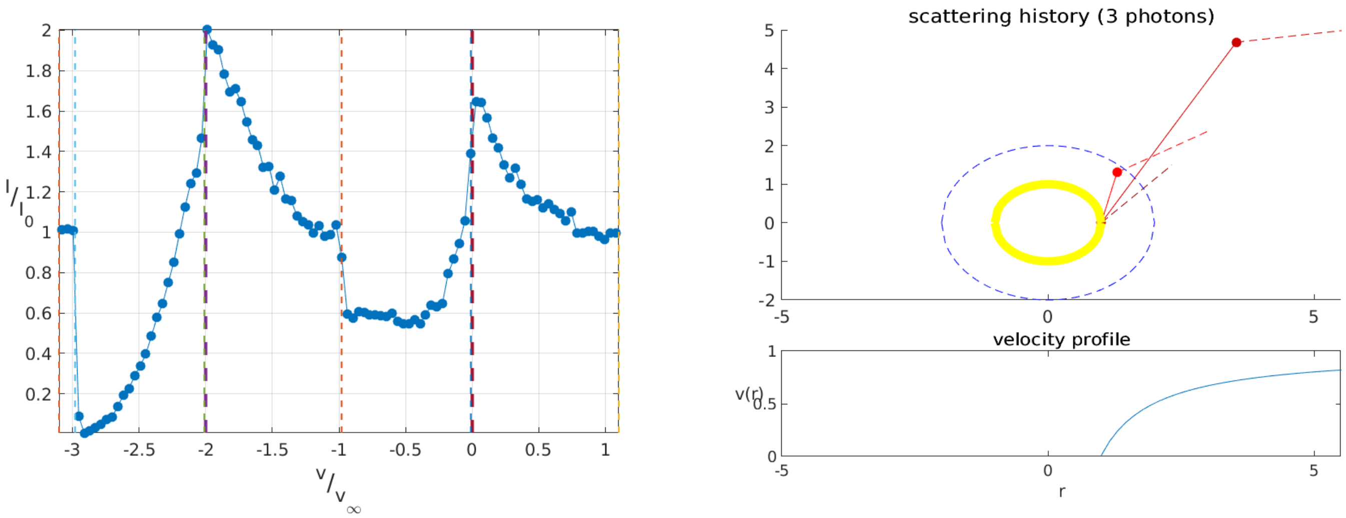


Figure 24: Multiple lines (left `xk0 = 100` and right `xk0 = 0.5`)

15.5 Observations

Figure 25 shows the profile when the scattering angle is allowed to lie in the interval $\mu_{\text{out}} \in [-1, 1]$. Figure 26 shows the same computations, but with $\mu_{\text{out}} \in [0, 1]$

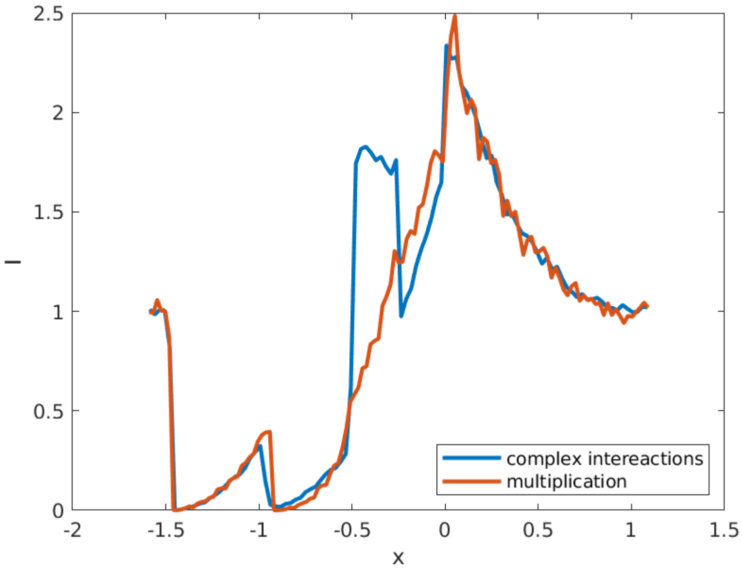


Figure 25: Comparison with multiplied profile ($\mu_{\text{out}} \in [-1, 1]$)

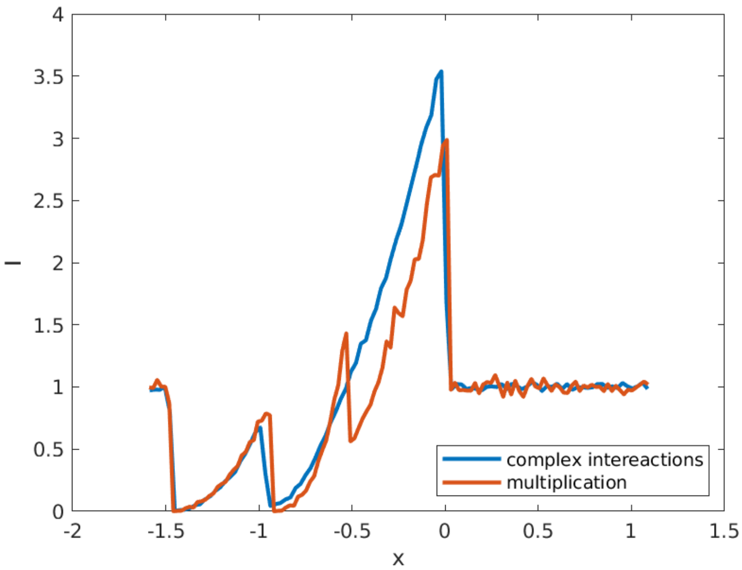


Figure 26: Comparison with multiplied profile ($\mu_{\text{out}} \in [0, 1]$)

Loosely speaking, one can understand this result because in Figure 26, photons are not allowed to be backscattered to the first resonance line.

15.6 Convergence behaviour

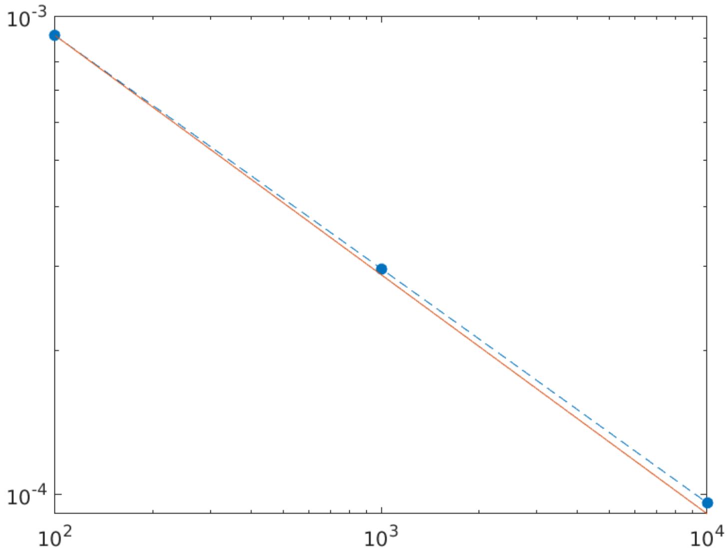


Figure 27: Convergence behaviour

15.7 Some theoretical observations

- The resonance frequencies need to lie close enough to each other, otherwise no resonance is possible.
  - if the lines are close enough to each other, resonance ends always at the rightmost frequency because you can indeed have a pumping up, but the leftmost peak is then in the absorption zone of the rightmost, thus it is scattered again! You can indeed see that the joint zone is depleted - and actaully makes the rightmost line stronger.
- Calculation of scattering probability for overlapping regions. Define ...
  - the 'one-time scattering probability' p_1.
  - probabiltiy of the overlapping region $p_2 = \frac{\text{length of overlapping region}}{\text{union of the length of the 'one-scattering' regions}}$.
  - HOWEVER take also into account that not all scattered photons are scattered.

  Then we have

$$p_{\text{total}} = p_1(1 + 2p_2(1 + 2p_2(1 + ...))) \tag{44}$$

16 Computing the radiation force & luminosity $L(r)$

This is based upon material from the text provided by professor Sundqvist, and the Phd thesis from Uwe Springmann [**UweSpringmannPHD**] where sperically symmetric Wolf-Rayet stars are discussed.

16.1 Theoretical formulas

1. Formal definition
$$g_{\text{radiation}} = \frac{\Delta p}{\Delta t \Delta m} = \frac{\Delta E}{v}\frac{1}{\Delta t}\frac{v}{M\Delta r} = -\frac{1}{\dot{M}}\frac{dL}{dr} \tag{45}$$

2. Numerical approximation, we count photons
$$L(r) = \frac{1}{\Delta t}\sum_i \epsilon_i \text{sign}(\mu_i(r)) \tag{46}$$

   where $L(r_*)$ is given and the photon energies ϵ_i are given by a black-body distribution.

16.2 Solution strategy

Loop over all photons. In that loop, loop over all r values and assign the corresponding angle. Update the array at each scattering event.

- create a grid for r
- for each r on the grid, look at the
- calculate $L(r)$
- then calculate $\frac{dL}{dr}$ numerically on that grid

In the case $\mu_{\text{out}} < 0$ then

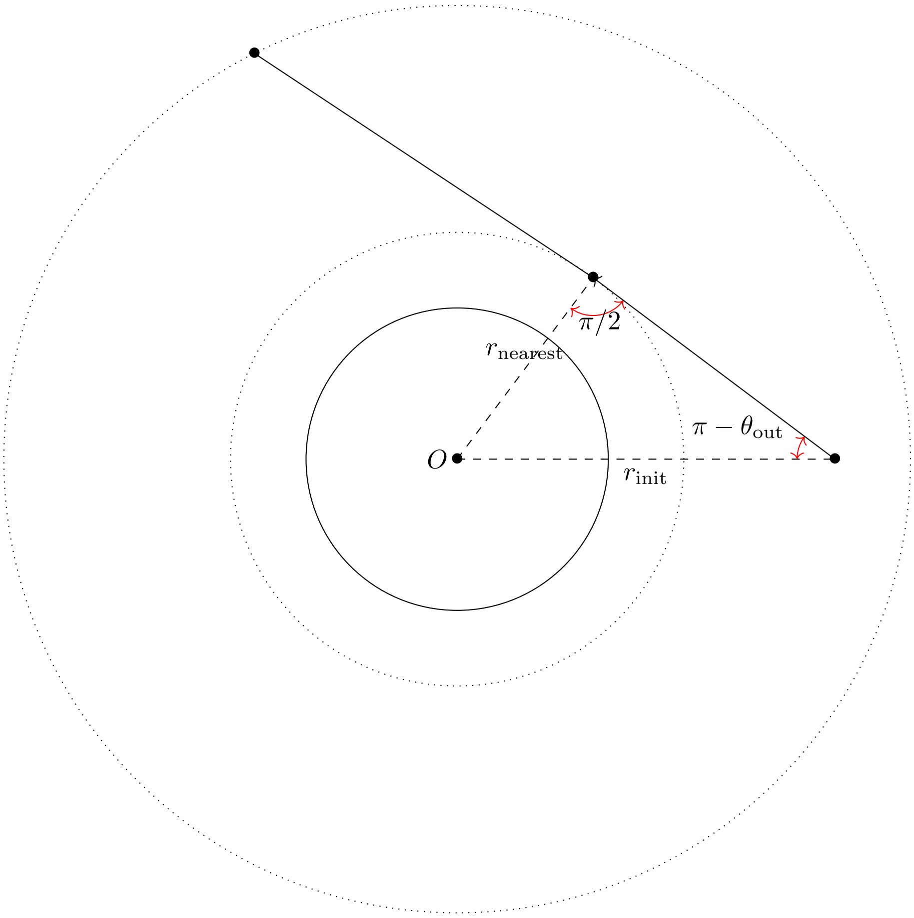


Figure 28: Trick when computin $L(r)$

16.3 Checking the correctness of the program

Enforce algorithmically that all photons stream in one direction (no scattering with `xmueout < 0`). Comment out the line `xmueou = -xmueou` in the function `scatter.m`.

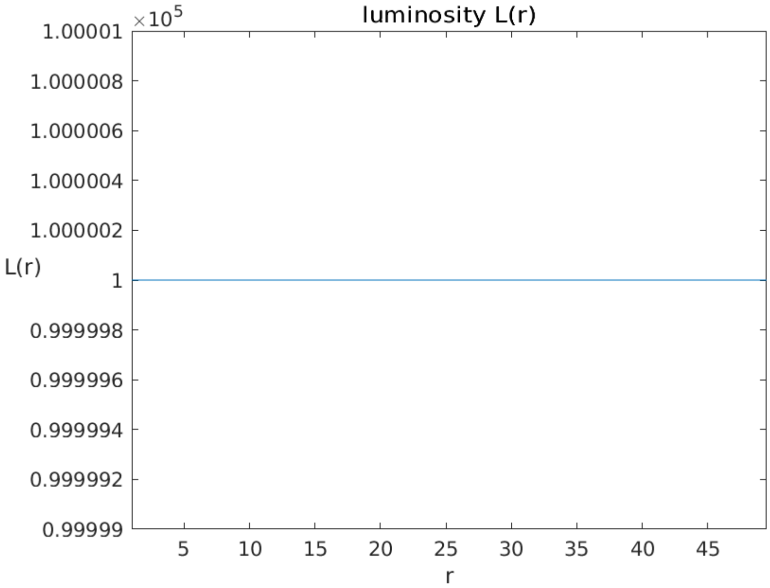
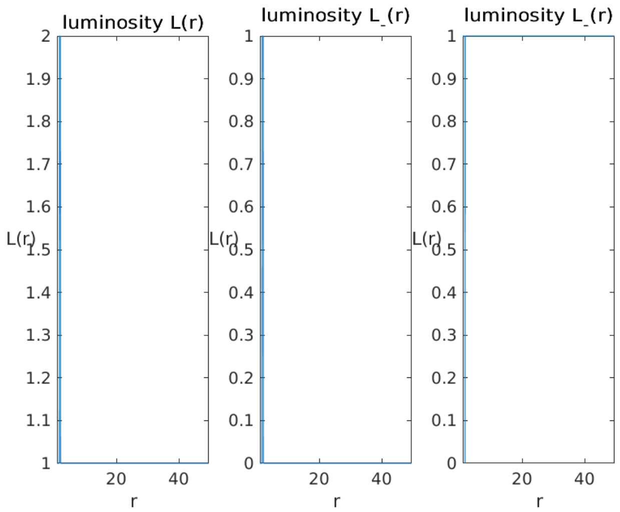


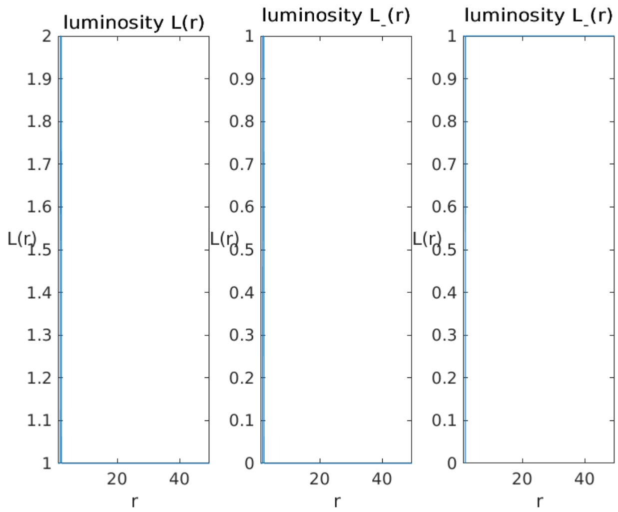
Figure 29: Luminosity $L(r)$ (for one resonance line) - test situation

16.4 Computing the radiation force for one line

Basic, well-known test situation.



(a) Luminosity $L(r)$ (for one resonance line)



(b) dL/dr

Figure 30: Scattering distribution

16.5 Multiple lines

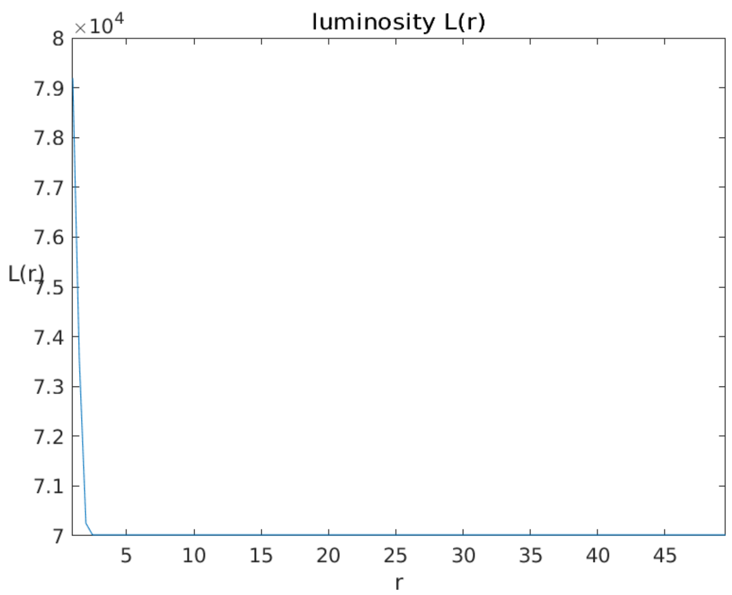


Figure 31: Luminosity $L(r)$ (for multiple resonance line)

16.6 Theoretical considerations

16.6.1 Analytical quantities

Theoretical From the course notes on radiative processes

$$\boxed{g_{\text{radiation}} = g_{\text{line}} = g_{\text{line}}^{\text{thin}} \frac{1 - e^{-\tau_S}}{\tau_S}} \tag{47}$$

- expression for g_{thin}
$$g_{\text{thin}} = g_e q = \frac{F_\nu^0 k_L}{\rho c} \tag{48}$$
- the Sobolev optical depth
$$\tau_S = \frac{\kappa_0}{r^2 v \left|\frac{dv_l}{dl}\right|} = \frac{\kappa_0}{r v^2(\mu^2\sigma + 1)} \tag{49}$$
$$\frac{dv_l}{dl} = \mu^2 \frac{dv_r}{dr} + (1 - \mu^2)\frac{v_r}{r} \tag{50}$$
- the density - mass loss relation
$$\dot{M} = \rho 4\pi r^2 v \tag{51}$$
$$\rho = \frac{\dot{M}}{4\pi r^2 v} \tag{52}$$
- expression for the extinction coefficient k_L

  @ Luka, do you know where this expression comes from?
$$k_L = \frac{\kappa_0 \rho 4\pi R_* v_\infty^2}{\dot{M}\lambda_0} = \frac{\kappa_0 R_* v_\infty^2}{r^2 v \lambda_0} \tag{53}$$

Conclusion Combining these quantities gives

$$g_{\text{radiation}} = \frac{F_{\nu_0}}{c} k_L \frac{1}{\rho} \frac{1 - e^{-\tau_S}}{\tau_S} \tag{54}$$

$$g_{\text{radiation}} = \frac{F_{\nu_0}}{c} \frac{\kappa_0 R_* v_\infty^2}{\lambda_0 r^2 v} \frac{4\pi r^2 v}{\dot{M}} \frac{r^2 v \left|\frac{dv_l}{dl}\right|}{\kappa_0} \left(1 - e^{-\tau_S}\right) \tag{55}$$

$$\boxed{g_{\text{radiation}} = \frac{F_{\nu_0}}{c\lambda_0} \frac{4\pi R_* v_\infty^2}{\dot{M}} r^2 v \left|\frac{dv_l}{dl}\right| \left(1 - e^{-\tau_S}\right)} \tag{56}$$

Now since we are assuming spherical symmetry

$$\boxed{g_{\text{radiation}} = \frac{F_{\nu_0}}{c\lambda_0} \frac{4\pi R_* v_\infty^2}{\dot{M}} v \left|\frac{dv_l}{dl}\right| \left(1 - e^{-\tau_S}\right)} \tag{57}$$

$$\boxed{g_{\text{radiation}} = \frac{F_{\nu_0} \nu_0}{c^2} \frac{4\pi R_* v_\infty^2}{\dot{M}} v \left|\frac{dv_l}{dl}\right| \left(1 - e^{-\tau_S}\right)} \tag{58}$$

We have that, since for $\beta = 1$ we have $\frac{dv_l}{dl} = \mathcal{O}(r^{-2})$ and $v = \mathcal{O}(r^{-1})$

$$\frac{g_{\text{rad}}}{\dot{M}} = \mathcal{O}(r^{-3}) \tag{59}$$

From numerical computations On the other hand we have

$$\boxed{g_{\text{radiation}} = -\frac{1}{\dot{M}} \frac{dL}{dr}} \tag{60}$$

These quantities should equal and also accounting for spherical symmetry. Also taking the limit $\tau_S \gg 1$ and assuming the β-velocity law, wet get

$$\frac{dL}{dr} = -\frac{C^{te}}{4\pi r^2} r^2 v \left| \frac{dv_l}{dl} \right| \tag{61}$$

$$\frac{dL}{dr} = \mathcal{O}(r^{-2}) \tag{62}$$

16.6.2 The same quantities in dimensionless units

Theoretical The dimensionless units (those used in the computer code) for the wind velocity and for the radius are $v' = \dfrac{v}{v_\infty}$ and $r' = \dfrac{r}{R_*}$. Thus $v = v'v_\infty$ and $r = r'R_*$.

- for Sobolev optical depth

$$\tau'_S = \frac{\kappa_0}{r'^2 v' \left|\frac{dv'_l}{dl'}\right|} = R_*^2 v_\infty \tau_S \tag{63}$$

$$\frac{dv'_l}{dl'} = \frac{v_\infty}{R_*}\frac{dv_l}{dl} \tag{64}$$

- extinction coefficient

$$k'_L = \frac{\kappa_0 v_\infty}{\lambda_0 R_* r'^2 v'} = R_*^2 v_\infty k_L \tag{65}$$

- density versus mass-loss rate

$$\rho' = \frac{\dot{M}}{R_*^2 v_\infty 4\pi r'^2 v'} \tag{66}$$

Putting these together

$$g_{\text{radiation}} = \frac{F_{\nu_0}}{c} k'_L \frac{1}{\rho'} \frac{1 - e^{-\tau'_S}}{\tau'_S} \tag{67}$$

$$g'_{\text{radiation}} = \frac{F_{\nu_0}}{c} \frac{\kappa_0 v_\infty}{\lambda_0 R_* r'^2 v'} \frac{R_*^2 v_\infty 4\pi r'^2 v'}{\dot{M}} \frac{r'^2 v' \left|\frac{dv'_l}{dl'}\right|}{R_*^2 v_\infty \kappa_0} \left(1 - e^{-\tau_S}\right) \tag{68}$$

$$\boxed{g'_{\text{radiation}} = \frac{1}{R_*}\frac{F_{\nu_0}}{c} \frac{\kappa_0}{\lambda_0 r'^2 v'} \frac{4\pi r'^2 v'}{\dot{M}} \frac{r'^2 v' \left|\frac{dv'_l}{dl'}\right|}{\kappa_0} \left(1 - e^{-\tau_S}\right)} \tag{69}$$

From numerical computations

$$g'_{\text{radiation}} = -\frac{R_*}{\dot{M}}\frac{dL}{dr'} \tag{70}$$

These quantities should equal

16.6.3 These quantities should equal (dimensionless units?)

For optically thick lines ($\tau_S \gg 1$)

$$g_{\text{radiation}}^{\text{numerical}} = -\frac{1}{\dot{M}}\frac{dL}{dr} = g_{\text{radiation}}^{\text{analytical}} = \frac{F_{\nu_0}\nu_0}{\dot{M}c^2}\frac{dv_l}{dl} \tag{71}$$

$$\frac{dL}{dr} = \frac{1}{4\pi r^2 c^2}\frac{F_{\nu_0}}{F}\frac{dv}{dr} \tag{72}$$

Combining that with a β-velocity profile

$$v = \left(1 - \frac{b}{r}\right)^{\beta} \tag{73}$$

Then we have

$$\boxed{\frac{dL}{dr} = \frac{F_{\nu_0}}{4\pi c^2}\frac{b}{r^3}\left(1 - \frac{b}{r}\right)^{\beta-1}} \tag{74}$$

and for $\beta = 1$

$$\frac{dL}{dr} = \frac{F_{\nu_0}}{4\pi c^2}\frac{b}{r^4} \tag{75}$$

$$\left|\frac{dL}{dr}\right| = C^{te}\frac{1}{r^4} \tag{76}$$

16.6.4 From random notes, global mass-loss rates

- definition of luminosity

$$g_{\rm rad} = -\frac{1}{\dot{M}}\frac{dL}{dr} \tag{77}$$

- steady-state equation of motion

$$v\frac{dv}{dr} = -\frac{GM}{r^2} + g_{\rm rad}(r) \tag{78}$$

  using Equation (77)

  – assuming that $\dot{M}(r) \equiv \dot{M}$

$$\int_{R_*}^{\infty} v\frac{dv}{dr}dr = -\int_{R_*}^{\infty}\frac{GM}{r^2}dr - \frac{1}{\dot{M}}\int_{R_*}^{\infty}\frac{dL}{dr}dr \tag{79}$$

  we then get

$$\frac{v^2}{2}\bigg|_{R_*}^{\infty} = -\frac{GM}{r}\bigg|_{R_*}^{\infty} - \frac{1}{\dot{M}}L\bigg|_{R_*}^{\infty} \tag{80}$$

  with $v(R_*) = 0$

$$\frac{v(\infty)^2}{2} - \frac{GM}{R_*} = \frac{L(R_*) - L(\infty)}{\dot{M}} \tag{81}$$

  – in general

$$\int_{R_*}^{\infty} v\frac{dv}{dr}dr = -\int_{R_*}^{\infty}\frac{GM}{r^2}dr - \frac{1}{4\pi\rho}\int_{R_*}^{\infty}\frac{1}{r^2 v}\frac{dL}{dr}dr \tag{82}$$

16.7 With high `xk0`

TO BE UPDATED

17 Backup from theory

Transport step and collision step. There is no absorption.

17.1 Moments of the intensity

definition	numerical estimation
I_ν	tally the number of photons (spherical symmetry)
$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega$	$J(r) = \frac{1}{16\pi^2 r^2}\frac{\epsilon_0}{\Delta t}\sum_i \frac{\epsilon_i}{\epsilon_0}\frac{1}{\lvert\mu_i(r)\rvert}$
$H_\nu = \frac{1}{4\pi}\int \mu I_\nu d\Omega$	$H(r) = \frac{1}{16\pi^2 r^2}\frac{\epsilon_0}{\Delta t}\sum_i \frac{\epsilon_i}{\epsilon_0}\mathrm{sign}(\mu_i(r))$
$K_\nu = \frac{1}{4\pi}\int \mu^2 I_\nu d\Omega$	$K(r) = \frac{1}{16\pi^2 r^2}\frac{\epsilon_0}{\Delta t}\sum_i \frac{\epsilon_i}{\epsilon_0}\lvert\mu_i(r)\rvert$

Table 1: Moments of intensity

We are interested

- in $F \propto H$ which is an average of I over $d\Omega$ (but only at 'shell of release')
- in L, which is then both an average of I over $d\Omega$ and over $d\nu$

17.2 Computing total photon distance and time that it has travelled

See figure 19.

18 Extension to higher dimensions

add angle ϕ

19 Closer look at Monte Carlo simulations

19.1 Random walk (diffusion equation)

A more simple experiment that simulates the diffusion equation (1D random walk) is also set up. The results are shown in Figure 32. We observe that $N \sim \tau^2$, as can also be derived from theory.

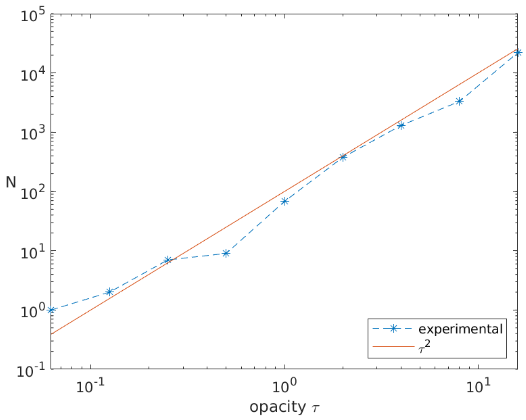


Figure 32: Number of interactions (scattering events) versus opacity, random walk

- When starting from an initial condition $x_0 = 0$ and

$$x_N = x_{N-1} \pm l \tag{83}$$

  we have for the variance that $\langle x_N \rangle^2 = Nl^2$

- If we require a photon to cover a distance R then $N = \dfrac{R^2}{l^2}$ and
  - the relation between mean-free path l and opacity α is $l = \dfrac{1}{\alpha}$
  - with $\tau = \displaystyle\int_0^R \alpha ds = \frac{R}{l}$

  then we have that $N = \tau^2$. This corresponds with the observations in Figure 32.

19.2 Limb darkening

We first look at results from the limb darkening program, as studied in Section 10. In Figure 33, the number of scattering events is plotted versus the opacity of the medium.

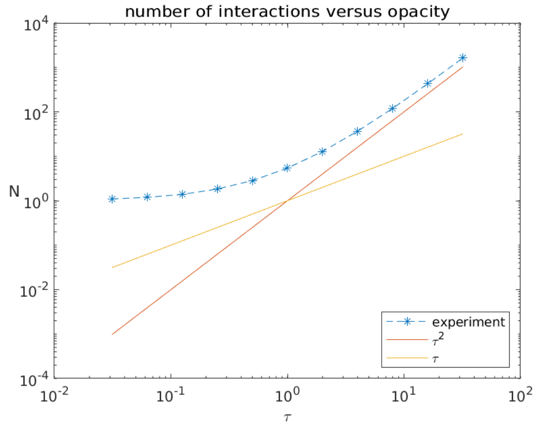


Figure 33: Number of interactions (scattering events) versus opacity, kimb darkening

- For high opacity $\tau \gg 1$ we observe that $N \sim \tau$.
- Bridging regime.
- For opacity $\tau \ll 1$ we observe that $N \sim 1$: namely the photons travels very far during the first emission event.

If you assume constant opacity then $\tau = \alpha z$