# Master thesis

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## Contents

**Exercises (part III)** Via this link, you can go back to the exercises overview: see section 13.

**Goal of the thesis** Compute multiline transfer. With Monte Carlo techniques.

# Part I
# Radiative Transfer: theory

## 1 Very broad introduction & Summary

The material here originates from the master thesis of Nicolas Moens [Moe18] and from the course notes *Introduction to numerical methods for radiation in astrophsyics* from professor Sundqvist.

### 1.1 Definition of specific intensity

The definition of the specific intensity is

$$I_\nu = \frac{dE_\nu}{\cos(\theta) d\Omega dt d\nu} = \frac{dE_\nu}{\mu d\Omega dt d\nu} \tag{1}$$

On the other hand, for the total energy of a collection of $N$ photons holds that

$$E_\nu = N E_{\nu,\text{photon}} \tag{2}$$

**To the point** From this we deduce that

$$I_\nu \mu = \frac{N(\mu) dE_{\nu,\text{photon}}}{d\Omega dt d\nu} \tag{3}$$

and thus

$$\boxed{I_{nu} \mu d\mu \sim N(\mu) d\mu} \tag{4}$$

**Considering the solid angle** In spherical geometry $d\Omega = \sin(\theta) d\theta d\phi = d\mu d\phi$.

### 1.2 Radiation equations

Material from [Iva14] Specific intensity $I(s, \lambda, x, y, t)$ satisfies the Radiative Transfer Equation:

$$\boxed{\frac{\delta I(q,t)}{\delta s} = \eta(q,t) - \chi(q,t) I(q,t)} \tag{5}$$

In cartesian coordinates (with propagation vector $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}$):

$$\frac{1}{c}\frac{\partial I}{\partial t} + \sin(\theta)\cos(\phi)\frac{\partial I}{\partial x} + \sin(\theta)\sin(\phi)\frac{\partial I}{\partial y} + \cos(\theta)\frac{\partial I}{\partial z} = \eta - \chi I \tag{6}$$

- 1D planar atmosphere: $\frac{\partial I}{\partial x} = \frac{\partial I}{\partial y} = 0$:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial z} = \eta - \chi I \tag{7}$$

- diffusion limit
- Definition of $J$ in Equation (3.15)

**Plane parallel geometry**

- restrict oursevels to time-independent, one-dimensional (1D) case $I(s, \theta, \lambda)$ where $s$ is the direction of the light ray
- it satisfies Radiation Transfer Equation (RTE) $\boxed{\dfrac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda}$
- with 'formal' solution $\boxed{I(\lambda, \tau_\lambda) = I_0(\lambda)e^{-\tau_\lambda} \int_0^{\tau_\lambda} S(t)e^{-t}dt}$
  - no emissivity $S = 0$ then $I(\lambda)I_0(\lambda)e^{-\tau_\lambda}$
  - no opacity then $I_0(\lambda) = \int_0^s \eta_\lambda(s)ds$
  - constant source function $I(\lambda, \tau) = I_0(\lambda)e^{-\tau_\lambda} + S(1 - e^{-\tau_\lambda})$
  - if $S = a + b\tau$ then $I(\lambda) = a + \dfrac{b}{k_\lambda}$ with $k_\lambda$ the opacity. A jump in opacity leads to the jump in intensity of the opposite sign.

**Specific intensity and its angular moments**

| | |
|---|---|
| specific intensity | $\Delta\epsilon = \boxed{I_\nu} A_1 A_2/r^2 \Delta\nu\Delta t$ |
| energy density | $E = \dfrac{1}{c} \iint I_\nu d\nu d\Omega$ |
| flux vector | $F = \iint I_\nu n d\nu d\Omega$ |
| pressure tensor | $P = \iint I_\nu n n d\nu d\Omega$ |
| mean intensity | $J_\nu = \dfrac{c}{4\pi} E_\nu$ |
| Eddington flux | $H_\nu = \dfrac{1}{4\pi} F_\nu$ |
| Eddington's K | $K_\nu = \dfrac{c}{4\pi} P_\nu$ |

**Eddington factor** In general, the Eddington factor is a tensor, for 1D systems it is reduced to a scalar.

$$f_\nu = \frac{K_\nu}{J_\nu} = \frac{P_\nu}{E_\nu} \tag{8}$$

- isotropic radiation field
- radiation field stronly peaked in radial (i.e. vertical in cartesian) direction

## 1.3 Radiative Diffusion Approximation

The radiative diffusion approximation bridges two regimes: regimes with ...

- on one hand, large optical depth $\tau \gg 1$: diffusion equation: temperature structure in a static stellar atmosphere
- on the other hand, where radiative *transport* is important

The diffusive approximation is the following: replace $\boxed{I = B}$ or $I_\nu = B_\nu$.

$$I_\nu = B_\nu - \mu \frac{dB_\nu}{k_\nu dz} \tag{9}$$

This equation can be derived as a random walk of photons!

## 1.4 Radiation Hydrodynamical equations

The full RHD equations consist of

- five partial differential equations
- one HD closure equation, e.g. (i) variable Eddington tensor method or (ii) flux limited diffusion

**Heat flux** The heat flow rate density $\vec{\phi}$ satisfies the Fourier law $\vec{\phi} = -k\nabla T$. More information can be found for instance on [Wik18].

## 1.5 Overview of symmetry assumptions

| | |
|---|---|
| plane-parallel | 1D atmosphere<br>bounded by horizontal surfaces |
| spherical symmetry | |

## 1.6 Overview of units and key variables

| | |
|---|---|
| specific intensity $I_\nu$ | $\left[\frac{ergs}{cm^2.sr.Hz.s}\right] = \left[\frac{J}{cm^2.sr.Hz.s}\right]$ |
| opacity $\alpha = k_\nu$ | $\left[\frac{m^2}{kg}\right]$ |
| optical depth $\tau = \int_0^l \alpha(s)ds$ | [/]<br>$\boxed{\tau = 0}$ leave atmosphere |

- expanding flow: photons are redshifted (lower frequency - higher wavelength)
- compressing flow: photons are blueshifted (higher frequency - lower wavelength)

# 2 The mathematics of Radiative Transfer

The material in this section is based on the book [Bus60].

## 2.1 Auxiliary mathematics

- $\cos(\Theta) = cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$
- phase function $\boxed{p(\mu, \phi, \mu', \phi', \tau) = \sum_{n=0}^{N} \omega_n P_n(\cos(\Theta))}$
  - isotropic scattering $p(\tau) = \omega_0(\tau)$
- equation of transfer $\boxed{\mu \frac{\partial I(\tau, \mu, \phi)}{\partial \tau} = I(\tau, \mu, \phi) - \mathcal{S}(\tau, \mu, \phi)}$

  with $\mathcal{S}(\tau, \mu, \phi) = B_1(\tau) + \frac{1}{4\pi}\int_{-1}^{1} d\mu' \int_0^{2\pi} I(\tau, \mu', \phi')p(\mu, \phi, \mu', \phi')d\phi'$
  - axially symmetric with isotropic scattering
    $\mathcal{S}(\tau) = \frac{\omega_0(\tau)}{2}\int_{-1}^{1} I(\tau, \mu')d\mu' = B_1(\tau) + \frac{\omega_0(\tau)}{2}\int_0^{\tau_1} \mathcal{S}(t)E_1(|t - \tau|)dt$
  - the Milne equation of the problem $(1 - \omega_0\bar{\Lambda})\{$
    $mahtcalS(t)\} = B(\tau)$
    - solve for $\mathcal{S}(t)$
    - then find $I(\tau, \mu)$

## 2.2 Background: integral equations

Based on the book [BHR02].

1. integral equation from differential equation
2. types of integral equations
3. operator notation and existence of solutions
4. closed-form solutions
   - separable kernels
   - integral transform method (Fourier transform)
   - differentiation
5. Neumann series
6. Fredholm theory
7. Schmidt-Hilbert theory

   Fredholm equation first kind
   $$0 = f + \lambda\mathcal{K}y \tag{10}$$
   Fredholm equation second kind
   $$y = f + \lambda\mathcal{K}y \tag{11}$$

# 3 Challenges in Radiative Transfer

The material here originates from an oral discussion with Ivan Milic.

## 3.1 Overview of the problem

$$\xrightarrow{I_\lambda^*} \boxed{T(\tau)\ ,\ \rho(\tau)\ ,\ \vec{B}(\tau)\ ,\ \vec{v}(\tau)} \xrightarrow{I_\lambda^+}$$

**Forward problem**

The forward problem is schematically represented

$$\xrightarrow{\vec{T},\rho,\vec{B},\vec{v}} \boxed{\begin{matrix}\text{forward problem}\\ I_\lambda^+ = F(\vec{T},\rho,\vec{B},\vec{v})\end{matrix}} \xrightarrow{I_\lambda^+}$$

In fact solve for intensity vector $\vec{I} = \begin{pmatrix} I \\ Q \\ \alpha \\ V \end{pmatrix}$ obeying the equation

$$\frac{d\vec{I}}{d\tau} = -X(\vec{T},\rho,\vec{B},\vec{v})\vec{I} - \vec{j}(\vec{T},\rho,\vec{B},\vec{v}) \tag{12}$$

and the solution

$$I_\lambda^+ = I_0^+ e^{-\int} + \int \vec{j} e^{-\int} d\tau \tag{13}$$

| |
|---|
| **Example** Source function $S = a\tau + b$ then $\int_0^{\tau_{max}} (a\tau + b)e^{-\tau} d\tau = ...$ |

**Inverse problem**

The inverse problem is schematically represented

$$\xrightarrow{I_\lambda^+} \boxed{\begin{matrix}\text{inverse problem}\\ F^{-1}(I_\lambda^{obs})\end{matrix}} \xrightarrow{\vec{T},\rho,\vec{B},\vec{v}}$$

Via least-squares approximation

$$\min_{\vec{T},\rho,\vec{B},\vec{v}} \sum \left( I_\lambda^{obs} - I_\lambda(\vec{T},\rho,\vec{B},\vec{v}) \right)^2 \tag{14}$$

## 3.2 Challenging domains of application

- Lyman alpha in Galaxy Halos
- Dusty torii (AGD)
- protoplanetary disks
- circumstellar disks
- athmospheres

# 4 Stellar Winds

From [**introStellarWindsLamersCassinelli1999**]

## 4.1 Chronology of stellar wind studies

1. early history: similarties between spectra of nova and luminous stars
2. diagnostics of structure of oouter atmospheres of the sun and stars
3. the development of the solar wind theory, further evidence for outflows
4. rocket and early satellite observations of stellar winds
5. instabilities and non-speherical effects in winds

There are still many things of stellar winds that are uncertain.

## 4.2 Observations & Formation of spectral lines in stellar winds

- line scattering
  - resonance scattering: from ground state of atom

### 4.2.1 Pcygni profiles

# 5 Glossary

- LASER: Light Amplification by Stimulated Emission of Radiation
- (spectral) Line-force: force on material in stellar atmosphere
- MASER: Microwave Amplification by Stimulated Emission of Radiation
- SED: spectral energy distribution

# Part II

# Radiative Transfer: numerical techniques

**forewarned is forearmed** General guidelines for good practices in scientific computing are found in [Wil+14].

## 6 Introduction to Monte Carlo Radiation Transfer

The material is taken from [**WWBW2001**] and from [**WWBW2013**].

### 6.1 Elementary principles

| | |
|---|---|
| specific intensity | $I_\nu$ |
| radiant energy | $dE_\nu$ |
| surface area | $dA$ |
| angle | $\theta$ |
| solid angle | $d\Omega$ |
| frequency range | $d\nu$ |
| time | $dt$ |
| flux | $F_\nu$ |
| cross section | $\sigma$ |
| scattering angle | $\chi$<br>$\mu = \cos(\chi)$ |
| mean intensity | $J$ |
| flux | $H$ |
| radiation pressure | $K$ |

### 6.2 Example: plane parallel atmosphere

1. emission of photons: select two angles (3D space). In isotropic scattering
   - $\theta$ met $\mu = \cos(\theta)$
     - $\mu = 2\xi - 1$ (isotropic scattering)
     - $\mu = \sqrt{\xi}$ (A slab is heated from below. Then $P(\mu) = \mu$)
   - $\phi = 2\pi\xi$
2. propagation of photons
   - sample optical depth from $\tau = -\log(\xi)$
   - distance travelled $L = \dfrac{\tau z_{max}}{\tau_{max}}$
3. conclusion of emission and propagation
$$\begin{aligned} x &= x + L\sin(\theta)\cos(\phi) \\ y &= y + L\sin(\theta)\sin(\phi) \\ z &= z + L\cos(\theta) \end{aligned} \tag{15}$$
4. Binning: once the photon exists the slab. Produce histograms of the distribution function. Finally, we wish to compute the output flux or the intensity.

# 7 Asymptotic Preserving Monte Carlo methods for transport equations in the diffusive limit

A very interesting article about Monte Carlo methods for radiative transfer problems, from a mathematical point of view, is [DPS18]. I am currently trying to reproduce the numerical experiments that are reported in the article.

# 8 Fluid and hybrid Fluid-Kinetic models (for neutral particles in plasma edge) (Horsten2019)

The material is mainly taken from [Hor19].

- Kinetic Boltzmann equation: neutral velocity distribution $f_n(r, v)$
- If you taken into account (e.g. microscopic processes for atomic deuterium) then the kinetic Boltzmann equation becomes

$$v\nabla f_n(r, v) = S_r(r, v) + S_{cx}(r, v) - f_n(r, v)(R_{cx}(r, v) + R_i(r)) \tag{16}$$

- Numerical solution strategies
  - finite differences/volumes/elements :computationally infeasible
  - spectral methods (series expansion of $f_n(r, v)$: not suitable for modelling discontinuties
  - stochatic approach: the whole velocity distribution is discretized by finite set of particles
- from Equation (16), the fluid model and the hybrid model is derived.
  - Fluid model: 3 state equations (continuity - momentum - energy) with boundary conditions
    * pure-pressure equation: maximum error of 10 - 28 %
    * with parallel momentum source: error 10 %
    * with ion energy source: error 30 %
  - hybrid model based on micro-macro decomposition

# 9 Overview of existing (Monte Carlo) radiative transfer codes

## 9.1 Synthesis codes

As is pointed out in [Chr15], there are basically two methods to solve the radiative transfer problem: ray-tracing and Monte Carlo methods.

- RADICAL [**RADICAL**] (Ray-tracing, 2D, multi-purpose)
- MULTI [Car] [Car86] (computer program for solving multi-level non-LTE radiative transfer problems in moving or static atmospheres, very old: Uppsala 1986 - 1995)
- SKIRT [CB2] (continuum (Monte Carlo) radiation transfer in dusty astrophysical systems, such as spiral galaxies and accretion disks, from Ugent)
- TORUS [Har+19] (Monte Carlo radiation transfer and hydrodynamics code. Adopts 1D, 2D, 3D adaptive mesh refinement. Suitable for radiative equilibrium and creation of synthetic images and SED)
- RADMC-3D [Dul17] (Monte Carlo code that is especially applicable for dusty molecular clouds, protoplanetary disks, circumstellar envelopes, dusty tori around AGN and models of galaxies. Python interface with Fortran main code)
- TLUSTY and SYNSPEC [HL17a], [HL17b], [HL17c].

## 9.2 Inversion codes

- VFISV
- ASP/HAO
- HeLIx+
- SNAPI (not publicly available, created by Ivan Milic)
- multiple codes available from Instituto de Astrofyiica de Canarias (IAC)
- STiC: the Stockholm inve rsion code

# Part III

# Practical work and Exercises

## 10 Overview of exercises (PART I)

1. limb-darkening scattering exercise we did during the course. — You can look into your notes from that, and I attach here also a sample program which you can use a base. After you have familiarised yourself with this, you can start to think bout how you would go about to extend this to a 3D setting (assuming isotropic scattering).
2. (As prep for Monte-Carlo school) here is a script computing a UV resonance P-Cygni line in spherically symmetric wind with v beta-law. At top of routine, a few exercises are given, where you can modify and play around with code. Monte-Carlo program which computes a UV resonance spectral line from a fast outflowing spherically symmetric stellar wind (if you were not cc'd on that email, let me know so that I can send you the files as well). At the top of that little script, there are a few suggestions for exercises (additions) you could do to that program, in order to learn a bit more about the general workings of Monte-Carlo radiative transfer in this context. — So that might be a good idea for you to do as well ! (And you can also ask the others in the group for some tips etc. then.)
3. Some background reading:
   - Attached mc manual by Puls.
   - Paper by Sundqvist+ 2010 (Appendix, I think).

## 11 Overview of exercises (PART II)

1. Calculate the probability distribution to sample from in the case of Eddington limb darkening for the initial distribution (see Section 20.3).
   - finished + Ok
2. Calculate analytical solution for simplified problem in the case that `mu = 1` (see Section 20.1).
   - finished + Ok + can be further studied
3. Perform convergence analysis (see Section 20.5).

## 12 Overview of exercises (PART III)

1. Revisit 3D limb darkening. $\phi$ should be sampled between 0 and $2\pi$ (see Section 19.5). (OK)
2. Revisit convergence analysis: adapt plot formatting and standard deviation is defined as square root of variance (see Section 20.5).
3. Test variance reduction technique (see Section 20.6).
4. Some general considerations about the definitiion of specific intensity (see Section 1.1). (OK)
5. For the Monte Carlo approximation of the diffusion equation, why do wo have $N \sim \tau$ for low optical depth $\tau \ll 1$ (see Section 27).
6. Revisit the radial streaming approximation in `pcyg.f90` for lower optical depth (e.g. `xk0=0.5`). (see Section 20.1).
7. What happens when you add a line (e.g. $x = 0.5 = a$)? How would you do that? (see Section **??**)
8. Towards a mathematical description of the problem.

# 13 Overview of exercises (PART IV)

1. Convergence analysis: also fit a line through the points. Formally, we write $V = CN^x$ and determine both $C$ and $X$ from experimental data. Correspondingly, $\log(V) = \log(C) + x\log(N)$. This is fitted using least-squares (see Section 20.5).
2. Variance reduction technique
   - averaging over different stochastic realizations?
   - take `xk0=0.5`
   - try to also discretize $\mu$
3. Adding a second line: develop computer code in the radial streaming assumption (use analytic formulas) $\mu = 1$ (see Section ??).
   - a following improvement is the use of a grid instead of using the bisection method.
4. Limb darkening. Have a look at section 19.3.1.

# 14 Multiline transfer (PART I)

1. What happens when you add a line (e.g. $x = 0.5 = a$)? How would you do that? (see section ??)

# 15 Multiline transfer (PART II)

1. calculate force from one line (see section 24).
   - discretize in shells
   - assume $\epsilon_i = cte$
2. compare to analytic expressions

# 16 Preparation for Equation meeting on 15 October

See Section 23.3.

# 17 Preparation for meeting on 21 October

I have been working on Section 22 and on Section 23.3.

**questions**

- also take into account the photons where no scattering takes place?
- language of master thesis?
- analytical expression for $L(r)$?

# 18 Introductory exercises

## 18.1 Analytical exercises

From course material from (prof. Sundqvist - CMPAA course).

1. introduction
2. radiation quantities
   - exercise p.3:
     - on one hand, we know that $\Delta\epsilon \sim C/r^2$
     - on the other hand, from the definition we know that $\Delta\epsilon = I_\nu A_1 A_2/r^2 \Delta\nu\Delta t$
     - combining these equations shows that $I_\nu$ is independent from $r$
   - exercise p.4:
     -
   - exercise 1:
     - $F_x = \int_0^\pi \left[ I_\nu(\theta)\sin^2(\theta) \int_0^{2\pi} \cos(\phi) \right] d\theta d\phi = 0$
     - the same reasoning for $F_y = 0$
   - exercise 2:
     - the equation follows from $d\mu = d\cos(\theta) = \sin(\theta)d\theta$
   - exercise 3:
     - isotropic radiation field (i.e. $I(\mu) = I$) then we have $F_\nu = 2\pi \int_{-1}^{1} I\mu d\mu = 2\pi I \left.\frac{x^2}{2}\right|_{-1}^{1} = 0$
   - exercise 4:
     - $F_\nu = 2\pi \int_{-1}^{1} I(\mu)\mu d\mu = 2\pi \int_{-1}^{0} I_\nu^- \mu d\mu + 2\pi \int_0^1 I_\nu^+ \mu d\mu = 2\pi I_\nu^+$
   - exercise p.7:
     - isotropic radiation field:
       * although the radiation pressure is a tensor, we will denote it as a scalar $P_\nu = \frac{4\pi I_\nu}{c}$
       * the radiation energy density $E_\nu = \frac{12\pi I_\nu}{c}$
       * thus $f_\nu = \frac{1}{3}$
     - very strongly peaked in radial direction (beam): $I_\nu = I_0\delta(\mu - \mu_0)$ with $\mu_0 = 1$
       * pressure tensor $P_{nu} = \frac{1}{c}\int I_0\delta(\mu - \mu_0) nn d\Omega$
       * energy density $E_\nu = \frac{1}{c}\int I_\nu d\Omega$
       * in this case $P_\nu = E_\nu$ thus $f_\nu = 1$
3. radiation transport vs. diffusion vs. equilibrium
   - exercise p. 12: 1D, Cartesian geometry, plane-parallel, frequency-independent and isotropic emission/extinction
     - radiation energy equation
       * The equation follows by integrating Equation (**??**)
       * By definition, $E = \frac{1}{c}\iint I_\nu d\nu d\Omega$
       * thus $\frac{dE}{dr} = \int (j - kI) d\nu d\Omega$ thus $\boxed{\frac{dE}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)}{c}}$

    - work out the integral taking into account frequency-independent and isotropic coefficients:
  - zeroth momentum equations
    - One must also take into account the specific form of the flux vector $F = \iint I_\nu n d\nu d\Omega = 2\pi \int_{-1}^{1} I_\nu(\mu)\mu d\mu$
    - thus $\frac{dF}{dr} = \frac{1}{c}\int (j - kI) n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
  - first moment equation
    - similar reasoning
    - $\frac{dP}{dr} = \int (j - kI) n.n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
- first exercise p. 15
  - $P = \frac{1}{c}\iint I_\nu \mu^2 d\Omega d\nu = \frac{2\pi}{c}\int_\nu \int_{-1}^{1} I_\nu \mu^2 d\mu d\nu = \frac{4\pi}{3c}\int B_\nu d\nu = \frac{aT^4}{3} = \frac{E}{3}$
- second exercise p.15
  - assuming the diffusion limit,
  - flux-weighted mean opacity $\kappa_F = \frac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}$
  - Rosseland mean opacity $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$.
    - in the diffusion limit, $F_\nu = -\frac{4\pi}{3}\frac{dB_\nu}{k_\nu dz}$ thus $\frac{dB_{nu}}{dT} =$
    - 
- third exercise p.15

4. the equations of radiation-hydrodynamics
5. numerical techniques for the radiative diffusion approximation
6. applications and approximations for a dynamically important radiative force in supersonic flows
   - exercise p.27: $L_{SOB} = \Delta r = \frac{v_{th}}{dv/dr} = \frac{10[km/s]}{1000[km/s]/R_*} = 0.01 R_*$
7. Appendix A: properties of equilibrium black-body radiation
   - exercise p. 29
     - this should be satisfied: $B_\nu d\nu = -B_\lambda d\lambda$ and also $\nu = \frac{c}{\lambda}$
     - this is equivalent to saying that $0 = \nu d\lambda + \lambda d\nu$ or $d\lambda = -\frac{\lambda}{\nu} d\nu$ thus $B_\lambda = \frac{\nu}{\lambda} B_\nu$
     - $B_\lambda(T) = \frac{\nu}{\lambda}\frac{2h\nu^3}{(\lambda\nu)^2}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2h\nu^2}{\lambda^3}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$
   - first exercise p.31
     - derive that $\lambda_{max} T = 2897.8[\mu m K]$
     - ...
   - second exercise p.31
     - this is about the spectra of (unknown) stars
   - first exercise p.32
     - see exercise 7
   - second exercise p.32

  - BB radiation: $I_\nu = \dfrac{2h\nu^3}{c^2}\dfrac{1}{e^{h\nu/kt}-1}$
  - the radiative flux for isotropic BB radiation is zero. See also exercise 3. This dus also holds for BB radiation.
- exercise p. 33
  - <u>HR-diagram</u>

8. Appendix B: Simple examples to the radiative transfer equation

- first exercise p. 34
  - start from radiative transport equation $\mu\dfrac{dI}{ds} = \alpha - \eta I$ in which $\eta = 0$ thus $\boxed{\mu\dfrac{dI}{ds} = \alpha}$
  - solving the ODE in the general case that $\alpha(s)$ is not constant:
    * integrate the equation $\mu I = \displaystyle\int_0^D \alpha ds$
    * ...
  - second exercise p. 34
    * case $\tau(D) >> 1$: then $I(D) \approx S$
    * case $\tau(D) << 1$: then $I(D) \approx I(0) + S(1-1) = I(0)$
  - first exercise p.35
    * is the plane-parallel approximation valid for the solar photosphere?
  - second exercise p.35
    * goal: find a solution to the equation $\mu\dfrac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$ where $I(\tau,\mu)$
    * solution
- second exercise p.35

9. Appendix C: connecting random walk of photons with radiative diffusion model

- exercise p. 38. Computing the average photon mean-free path inside the Sun.
  $l = \dfrac{1}{\kappa\rho} = \dfrac{V_o}{\kappa M_o}[cm]$
- exercise p.39. Computing the random-walk time (diffusion time) for photons

## 18.2 Numerical exercises

### 18.2.1 Implicit 1D solver

Exercise from (20-11-2018).

**Goal** Implement implicit solver for time-dependent diffusion equation

$$\partial_t u = \partial_{xx} u \tag{17}$$

**Solution** The convergence behaviour of the method is shown in Figure 1.

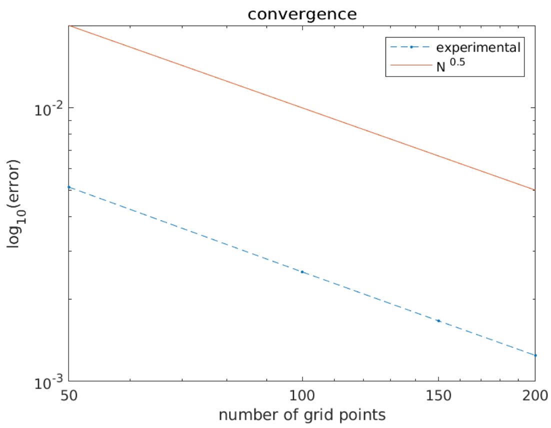



Figure 1: Convergence behaviour for 1D implicit solver (diffusion equation)

### 18.2.2 ADI 2D Solver

**Goal** Implement implicit solver for time-dependent diffusion equation

$$\partial_t u(t,x,y) = \partial_{xx} u(t,x,y) + \partial_{yy} u(t,x,y) \tag{18}$$

**Solution** There is still an error in the code.

### 18.2.3 Area of a circle

**Goal** Develop Monte Carlo code

**Solution**

## 18.3 Other Exercises

From course material from Ivan Milic.

### 18.3.1 Lecture 7

1. Derive expressions for the emergent radiation when properties are the following:
   - optically thin slab at all wavelengths
   - wavelength-independent incident radiation

   Solution: see slide 14?
2. Derive ralations between Einstein coefficients.
3. Calculate electron density in atmosphere from FALC model

# 19 Limb darkening

## 19.1 Formulation of the problem

- The radiative transfer equation 5 in this situtation becomes an integro-differential equation with $S(\tau) = \frac{1}{4\pi}\int I(\tau,\mu)d\Omega$

$$\begin{aligned} \mu\frac{dI(\tau,\mu)}{d\tau} &= -I(\tau,\mu) + S(\tau) \\ &= -I(\tau,\mu) + \frac{1}{4\pi}\int I(\tau,\mu)d\Omega \end{aligned} \tag{19}$$

- The difficulty resides in the (evaluation of) the source function. Monte Carlo simulation avoids explicit calculation source function: source function implicit in Monte Carlo simulation. There the physics are simulated IN BETWEEN TWO CONSECUTIVE SCATTERING EVENTS as follows

$$\frac{dI}{dz} = -\alpha I \tag{20}$$

thus $I = I_0 e^{-\delta\tau}$ and then $\tau$ is sampled according to $\tau = -\log(X_{\text{random}})$

## 19.2 Solving the (integro-differential) radiative transfer equation

**Analytical Solution of Equation (19)** Ik heb de mosterd gehaald op [Esp].

$$I(0,\mu) = \int_0^\infty S(\tau)exp\left(\frac{-\tau}{\mu}\right)d\left(\frac{\tau}{\mu}\right) \tag{21}$$

**Numerical Solution of Equation (19)** First rewrite the equation

$$\begin{aligned} \mu\frac{dI(\tau,\mu)}{d\tau} &= -I(\tau,\mu) + \frac{1}{4\pi}\int I(\tau,\mu)\sin(\theta)d\theta d\phi \\ &= -I(\tau,\mu) + \frac{1}{4\pi}\int I(\tau,\mu)d\mu d\phi \\ &= -I(\tau,\mu) + \frac{1}{2}\int I(\tau,\mu)d\mu \end{aligned} \tag{22}$$

Discretization scheme:

$$??? \tag{23}$$

## 19.3 Eddington-Barbier approximation

$$J(\tau) = 3H\left(\tau + \frac{2}{3}\right) \tag{24}$$

Together with the time-independent radiative transfer equation (5) in a gray (frequency-independent) planar medium gives

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - 3H\left(\frac{2}{3} + \tau\right) \tag{25}$$

The emergent intensity $I(0,\mu)$ is a solution of Equation (25). Its solution for $\tau = 0$ equals

$$I(\tau = 0,\mu) = I_1\left(\frac{2}{5} + \frac{3\mu}{5}\right) \tag{26}$$

with $a = \frac{\sigma}{2\pi}T_{eff}^4$ and $b = \frac{3\sigma}{4\pi}T_{eff}^4$

#### 19.3.1 Validity of the Eddington-Barbier approximation

If we assume Equuation (24) then $I = I_1(a + b\mu)$ thus $J = \frac{1}{2}\int(\tau,\mu)d\mu = \frac{1}{2}\int_0^1(a + b\mu)d\mu$

| dat ziet er hier niet goed uit |
|---|

### 19.4 2D Case

We again have $\mu = \cos(\theta)$. The solution of the radiative transfer equation in plane-parallel syummetry with frequency-independent absorption and emission, is

$$I(\mu) = I_1(0.4 + 0.6\mu) \tag{27}$$

In the Monte Carlo code, the photons are sorted according to the direction that they leave the atmosphere.

**Goal** Calculates the angular dependence of photon's emitted from a plane-parallel, grey atmosphere of radial optical depth `taumax`. The value of `tau` determines the position of the photon

**Variables and Algorithm**

- `muarray` contains emergent photons
- `na` number of channels
- `dmu` = 1/`na` width of channels
- `nphot` number of photons
- `taumax` maximum optical depth

**Algorithm 1** Limb darkening: compute quantitiy of photons

initialization
  radial optical depth $\tau$
  direction $\mu$
**for** all photons **do**
  $\boxed{\tau = \tau_{max}}$
  **while** `tau` $\geq 0$ **do**
    compute scattering angle `mu`
    **if** tau $\geq$ taumax **then** $\boxed{mu = sqrt(x)}$ (initial distribution)
    **else** $\boxed{mu = 2 * x = 1}$ (isotropic scattering)
    tau_i = -log(x2)
    tau = tau - tau_i*mu
  **end while**
  now we know that the photon has left the photosphere
  compute the distribution of all angles `mu` at which the photon left the photosphere
**end for**
visualisation:

- plot photon numbers from $\mu d\mu$ against `mu`
- plot specific intensity from $d\mu$ against `mu` against

Figure 2 is according to what is expected $I = I_0(0.4 + 0.6\mu)$. The input parameters are as follows `Limb_Darkening(number_of_channels = 20, number_of_photons = 10^5, maximum_optical_depth = 10)`.

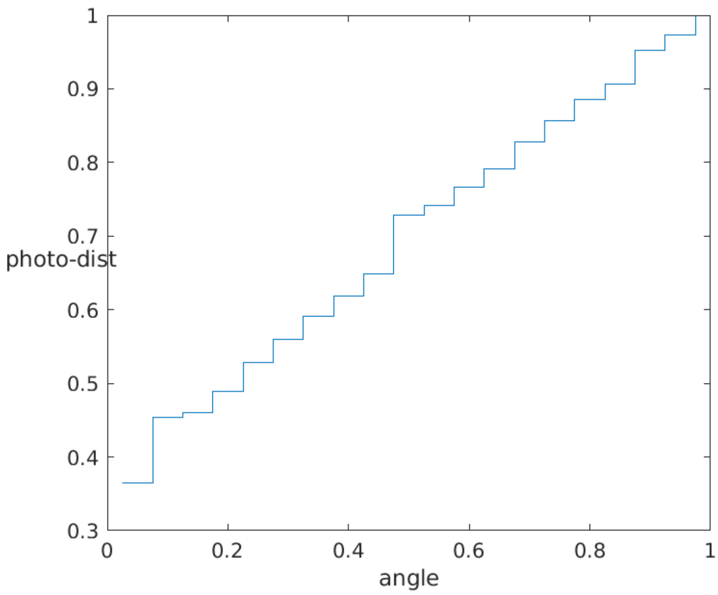



Figure 2: histogram for `mu`

## 19.5 3D Case

What changes is this:

- introduction of a new angle $\phi$
- the optical depth is not updated with respect to $\phi$

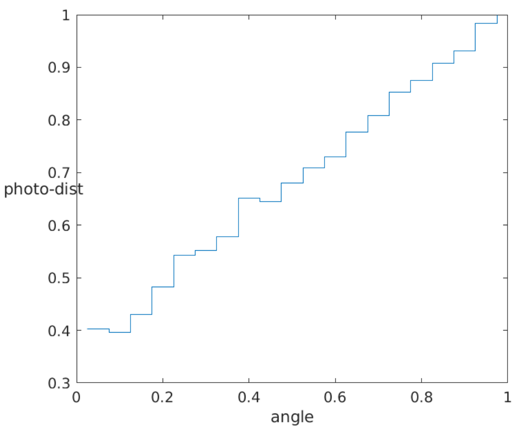



Figure 3: histogram for `mu`

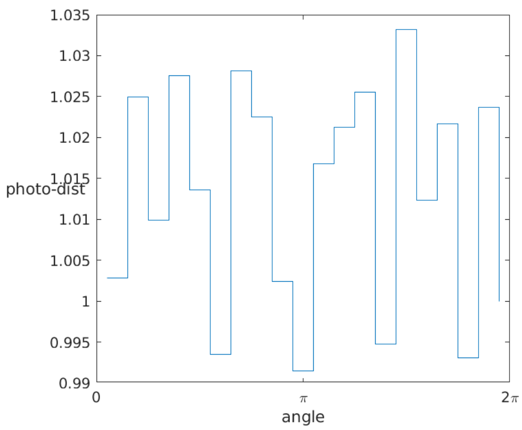



Figure 4: histogram for `phi`

Figure 3 and Figure 4 are the result of the function `Limb_Darkening_3D` with the following input parameters: `Limb_Darkening_3D(number_of_channels = 20, number_of_photons =` $10^5$`, maximum_optical_depth = 10)`. The results according to what is expected, namely $I = I_0(0.4 + 0.6\mu)$ and $\phi$ follows a uniform distribution.

**Extension**: make version where the optical depth is updated with respect to $\phi$

Via this link, you can go back to the exercises overview: Section 12.

# 20 Spectral line formation: pcyg.f90

This section is about the study of line formation in an expanding wind.

## Background

### Overview of variables

| name | explanation |
|---|---|
| paramaters | |
| xk0 | |
| alpha | velocity profile parameter |
| beta | velocity profile parameter |
| start frequency of the photon | |
| xstart | start frequency |
| vmin | |
| vmax | |
| angle of the photon | |
| xmuestart | start angle |
| xmuein | incident angle |
| xmueou | outward angle |
| pstart | impact parameter |
| xnew | new photon frequency |
| optical depth | |
| tau | optical depth |
| number of photons admin | |
| nphot | number of photons |
| nin | photons scattered back into core |
| nout | photons escaped |
| functions | |
| func | velocity profile<br>distance from center of star $r$ |
| xmueout | outwards (scattered) angle<br>xk0<br>alpha<br>r<br>v<br>sigma |
| nchan | amount of bins |

$$\xrightarrow{\texttt{xstart}} \boxed{\texttt{pcyg.f90}} \xrightarrow{\texttt{xnew}}$$

The photons are sorted according to `xnew`. In general, the flux is dependent on $\mu$ and the frequency $x$.

**Practical formula**

- emission angle $\mu = \cos(\theta)$
- according p-ray $p = \sqrt{1-\mu^2} = \sin(\theta)$
- incident angle `xmuein` $= \sqrt{1 - \left(\frac{pstart}{r}\right)^2}$

**Geometry & Symmetry assumptions**

- spherical geometry

## 20.1 First adaptation: what if all photons are released radially from photosphere?

### 20.1.1 Release photons radially: numerical MC experiments

What would happen with line-profile, if you assumed all photons were released radially from photopshere?

- In other words `xmuestart` $= 1$.
- This is implemented under the test case `test_number=1`.
- Results in Figure 18 for opacity `xk0 = 100`.

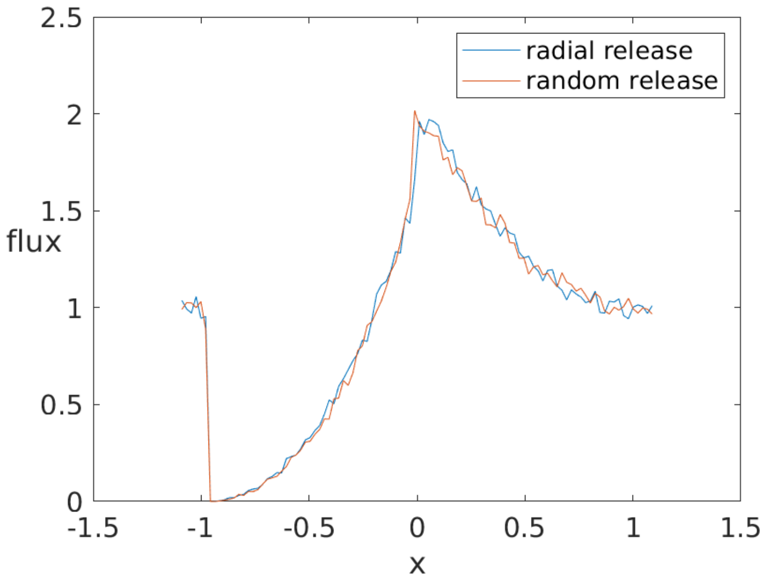



Figure 5: The number of photons equals $10^5$, `xk0=100`

### 20.1.2 Derive analytic expression

See also slide 26/49 [Sundqvist course material].

- since `xmuein = 1` we have for the velocity profile

$$v = v_\infty (1 - b/r)^\beta \tag{28}$$

  A scaled version of Equation (28) yields

$$u = \frac{v(r)}{v_\infty} = \left(1 - \frac{b}{r}\right)^\beta \tag{29}$$

  with $u \in [0..1]$

- Doppler shift for the frequency of the photons: $x_{CMF} = x_{REF} - \mu u$.
- Condition for resonance from Sobolov approximation (to be studied later): $\boxed{x_{CMF} = 0}$ thus

$$x_{REF} = \mu u \tag{30}$$

  or thus $x_{REF} = \boxed{u_{\text{interaction}}}$ and than solve Equation 29 for $r_{\text{interaction}}$

- If $\mu = 1$ then

$$x = \left(1 - \frac{b}{r}\right)^{\beta} \tag{31}$$

$$x^{1/\beta} = 1 - \frac{b}{r}$$

$$r(1 - x^{1/\beta}) = b$$

$$\boxed{r(x) = \frac{b}{1 - x^{1/\beta}}} \tag{32}$$

- From the location of interaction $r$, the incident angle can be calculated

$$\texttt{xmuein} = \sqrt{1 - \left[\frac{\texttt{pstart}}{r}\right]^2} = \sqrt{1 - \left[\frac{\sqrt{1 - \texttt{xmuestart}^2}}{r}\right]^2} \tag{33}$$

Now also taking into account that `xmuestart = 1`

$$\texttt{xmuein = 1} \tag{34}$$

- The calculation of the optical depth goes as follows:

$$\tau = \frac{\texttt{xk0}}{rv^{2-\alpha}(1 + \texttt{xmuein}^2\sigma)} \tag{35}$$

Now also taking into account that `xmuestart = 1` gives

$$\tau = \frac{\texttt{xk0}}{rv^2(1 + \sigma)} \tag{36}$$

where $\boxed{v(x) = \left(1 - \frac{b}{r}\right)^{\beta}}$ and $\frac{dv}{dr} = \frac{\beta b}{r^2}\left(1 - \frac{b}{r}\right)^{\beta - 1}$

and $\sigma(x) = \frac{dv}{dr}\frac{r}{v} - 1$ thus $\boxed{\sigma(x) = \frac{\beta b}{r}\left(1 - \frac{b}{r}\right)^{-1}}$

- Assuming that $\beta = 1$ then $\boxed{v(x) = 1 - \frac{b}{r}}$ and $\frac{dv}{dr} = \frac{\beta b}{r^2}$ and $\boxed{\sigma(x) = \frac{\beta b}{r}}$.

- `xmueou` follows the distribution as given by the function `xmueout`, namely

$$p(x) = \frac{1 - e^{-\tau}}{\tau} \tag{37}$$

with $\tau = \frac{\texttt{tau0}}{1 + \texttt{X}^2\sigma}$ where `X` is a random number, so actually this comes down to

$$\boxed{p(x) = \frac{1 - e^{-\frac{\tau_0}{1 + x^2\sigma(x)}}}{\frac{\tau_0}{1 + x^2\sigma(x)}}} \tag{38}$$

- Finally one can combine these results to get the distribution of the photons according to the frequency $x$ via the relation

$$\texttt{xnew = xstart + v(xmueou-xmuein) = xstart + v(xmueou -1)} \tag{39}$$

In words, we initially have an isotropic distribution for `xstart`. The number of photons that are leaving the atmosphere at different frequencies is however not isotropic through complex interactions that are incorporated into $p(x)$. One must also take into account that not all of the photons that are released actually escape from the atmosphere and also that sometimes no resonance is possible, and then Equation (39) is not applicable.

| |
|---|
| TO DO: proceed from this to the analytical expression for the flux. Here I am stuck for the moment. |

### 20.1.3 Experiments with other opacities

The results for `xk0=0.5` are shown in Figure 6.

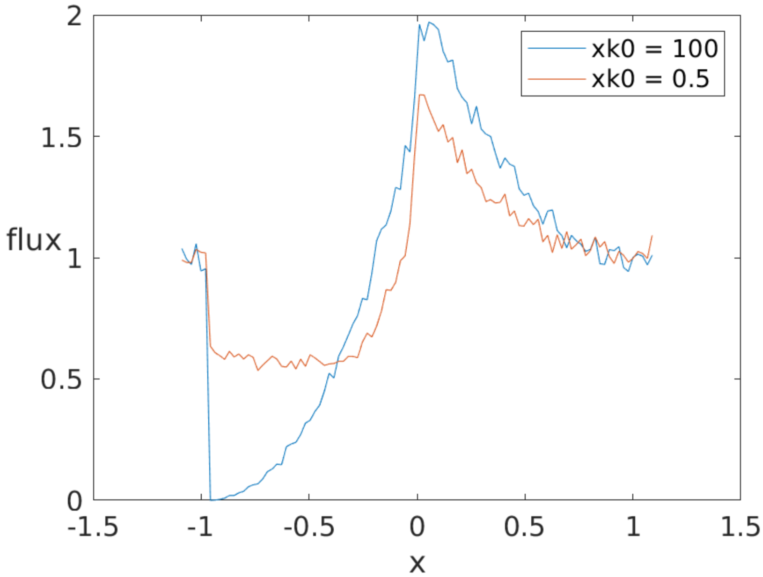



Figure 6: The number of photons equals $10^5$, `xk0=0.5`

Via this link, you can go back to the exercises overview: Section 12.

## 20.2 Second adaptation: isotropic scattering

What would happen to line-profile, is you assumed scattering was isotropic (i.e., NOT following Sobolev-distrobution)

- in the implementation, `test_number = 2`
- the results are shown in Figure 7.

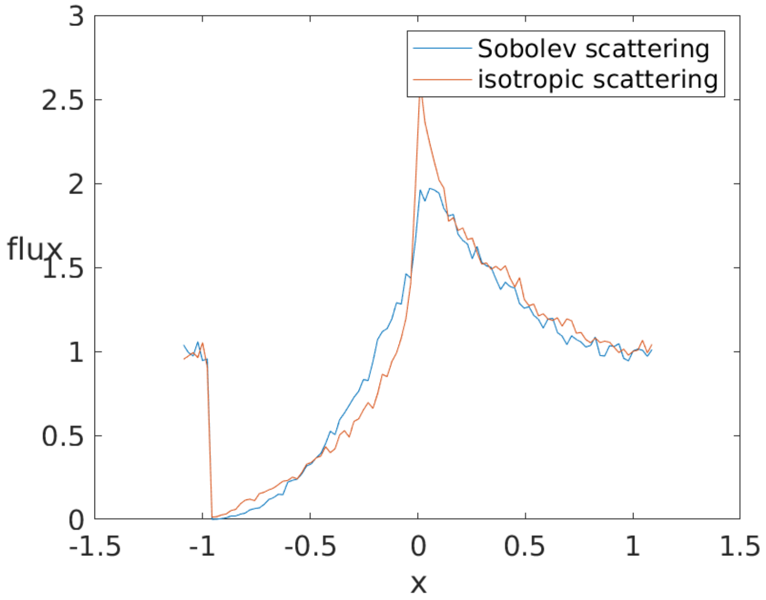



Figure 7: The number of photons equals $10^5$

It is clear from Figure 7 that the peak around $x = 0$ is higher and sharper.

Analyse this behaviour more closely

## 20.3 Third adaptation: introduction of Eddington limb-darkening

**Goal** Put Eddington limb-darkening in. What happens?

### 20.3.1 Construction of probability distribution corresponding to Eddington limb darkening

For a general (introductory) discussion about Eddington limb darkening, please refer to Section **??**

1. Let us thus first review the emmission case where the flux in each direction is isotropic i.e. $I(\theta) = I$ (as experimented in paragraph 20.2)

   - the specific intensity is defined as $I_\nu(\mu) = \dfrac{dE_\nu}{\cos(\theta) dA dt d\nu d\Omega} = \dfrac{dE_\nu}{\mu dA dt d\nu d\Omega}$
   - the flux $F_\nu = \int_\Omega I_\nu \cos(\theta) d\Omega$ is in this case isotropic thus

   $$\xi = \int_0^\mu F_\nu d\mu = \int_0^\mu \int_\Omega I_\nu \cos(\theta) d\Omega d\mu = A \int_0^\mu \mu d\mu \tag{40}$$

   together with the condition that $\mu$ satisfies a probability distribution:

   $$1 = \int_{-1}^1 F_\nu d\mu = \int_{-1}^1 \int_\Omega I_\nu \cos(\theta) d\Omega d\mu = \frac{A}{2} \tag{41}$$

   thus $A = 2$. Photons need to be sampled according to $\mu d\mu$.

2. Now we look at a new case where the photons need to be emitted following a distrubution that corresponds to $I(\theta) = I(0)(0.4 + 0.6\cos(\theta))$.

   - in this case the flux $F_\nu = \int_\Omega I_\nu \cos(\theta) d\Omega$ is isotropic but also satisfies

   $$F_\nu = \int_\Omega I_\nu(0)[0.4 + 0.6\cos(\theta)] \cos(\theta) d\Omega \tag{42}$$

   | I am not sure about the correctness of the assumption of isotropy of the flux |
   |---|

   $$\xi = \int_0^\mu F_\nu d\mu = A \int_0^\mu (0.4 + 0.6\mu) \mu d\mu \tag{43}$$

   subject to the normalisation condition -very similar to Equation (41) - that

   $$1 = \int_0^1 F_\nu d\mu = \frac{2A}{5} \tag{44}$$

   thus $A = \frac{5}{2}$. Photons need to be sampled according to

   $$\frac{5}{2}(0.4 + 0.6\mu)\mu d\mu \tag{45}$$

   In the code `pcyg.f90` this corresponds to `test_number = 3` (not yet implemented).

   The results of an accept-reject method that samples the probability distribution in Equation (45).

Via this link, you can go back to the exercises overview: Section 11.

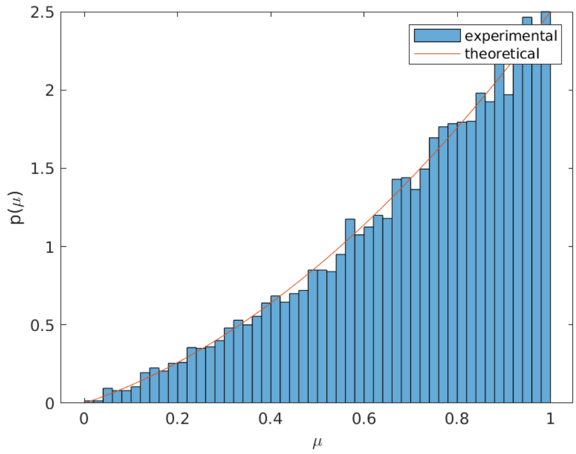



Figure 8: Accept-reject method for Eddington limb darkening

## 20.4 Fourth adaptaion: photospheric line-profile

Challening: Put photospheric line-profile (simple Gaussian) in. What happens? Test on `xk0=0` (opacity = 0) case.

- test case number 4
- $\boxed{\text{This is still to be implemented.}}$

## 20.5 Convergence analysis

**Zero opacity** The convergence of the Monte Carlo method is tested with the following input parameters

| kx0 | alpha | beta | test_number |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

for a varying amount of photons, as shown in Figure 9. We expect the method to have $\frac{1}{\sqrt{N}}$ convergence, where $N$ is the number of photons. However, the methods strangely seems to have a faster convergence rate.

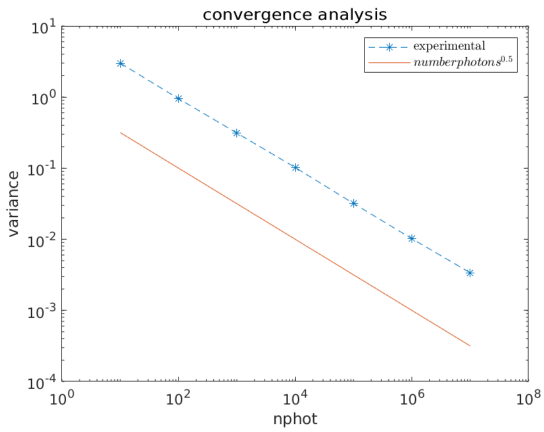



Figure 9: Original version of the code: convergence analysis (xk0=0)

**Nonzero opacity** The convergence test is set up as follows: different Monte Carlo simulations (with increasing number of photons) are compared to an *expensive* simulation with $10^7$ photons. As can be seen in Figure 10, the spectrum profile behaves according to a $N^{0.5}$ law.

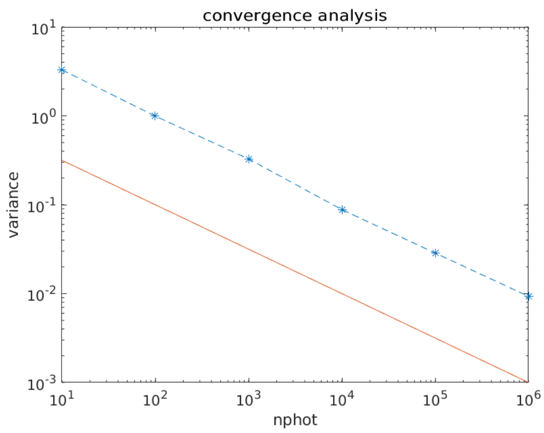



Figure 10: Original version of the code: convergence analysis (xk0=100)

Via this link, you can go back to the exercises overview: Section 11.

## 20.6 Variance reduction experiment

We will set up the test as follows

- run the code with `xk0=100` and number of photons $N = 10^7$
- run the code again for lower number of photons (e.g. $N = 10^3$), both with random sampling and pseudo-random sampling
- compute variance w.r.t. *expensive* simulation and compare
- `test_number = 5`

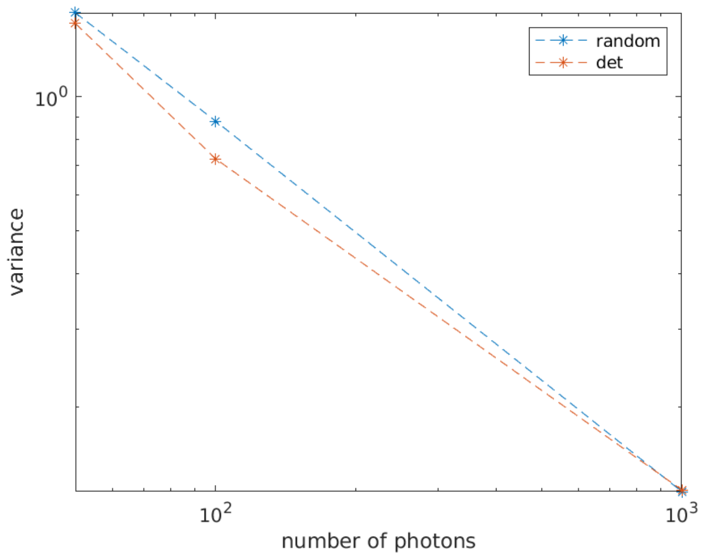



Figure 11: Original version of the code: convergence analysis (xk0=0)

`xk0=100` Possible improvement: average over different stochastic realizations.

Via this link, you can go back to the exercises overview: Section 12.

## 20.7 Mathematical description of the problem & Looking at literature

Have a look at [NS19] (see Appendix).

# 21 Transferring the code to Matlab

## 21.1 Limit variables

| | `xmin` | `xmax` | `vmin` | `vmax` |
|---|---|---|---|---|
| Fortran | -1.1 | 1.1 | 0.01 | 0.98 |
| Fortran (reverse order for scattering distribution) | -1.1 | 1.1 | -0.98 | -0.01 |
| Matlab (with `resonance_x = 0` | -1 | 1 | -0.8 | 0 |

## 21.2 Comparison

### 21.2.1 Literal Matlab version

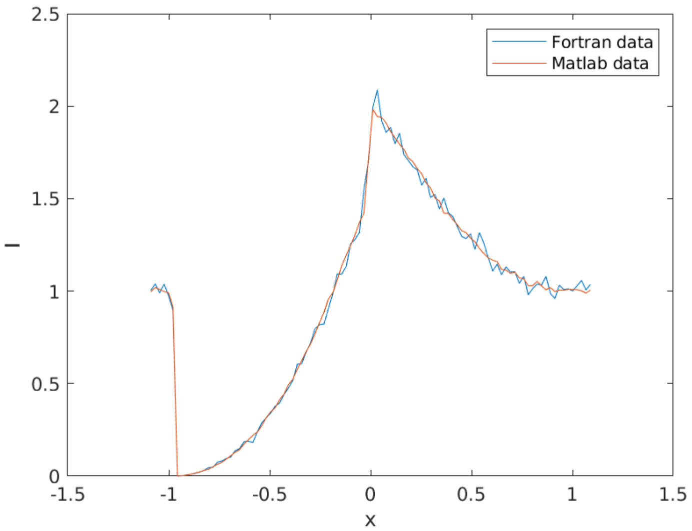



Figure 12: Comparison of Fortran code and Matlab code (unchanged version - `xk0 = 100`)

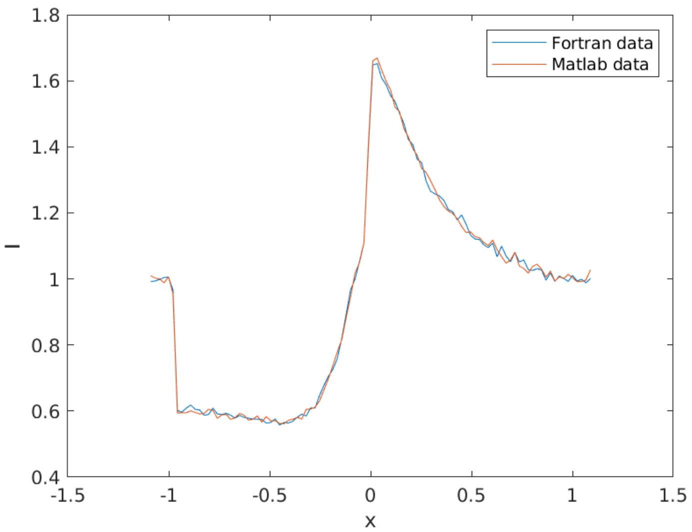



Figure 13: Comparison of Fortran code and Matlab code (unchanged version - `xk0 = 0.5`)

#### 21.2.2 More freely translated version

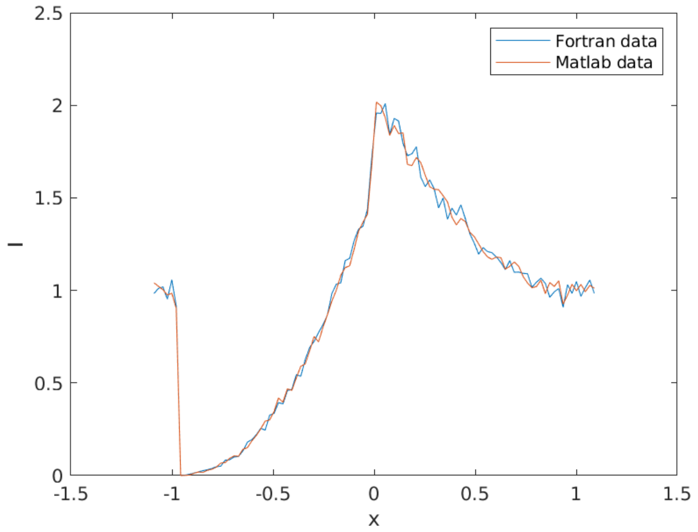



Figure 14: Comparison of Fortran code and Matlab code (freely adapted version)

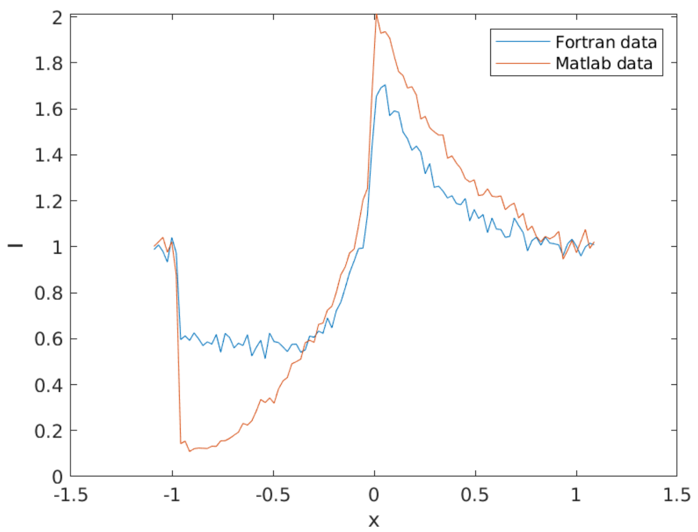



Figure 15: Comparison of Fortran code and Matlab code (unchanged version - `xk0 = 0.5`)

## 21.3 Optimizations

### 21.3.1 Already implemented

### 21.3.2 To be implemented

- generation of random numbers that undergo no scattering: replace with continuum profile

# 22 Theoretical background

**Algorithm 2** `pcyg.f90: one resonance line`

**for** all photons **do**

1. Release photon with frequency $x$
2. Check if interaction is uberhaupt possible.
3. Solve for distance (radius $r$) of interaction using Sobolev approximation $x_{CMF} = x_{REL} - \mu v(r)$ with $\boxed{x_{CMF} = 0}$ and compute Sobolev optical depth
4. Check whether the photon is scattered:

   **if** $\tau_S > -log(\xi)$ **then**
   
   Interaction: the photon is scattered. Update the frequency
   
   **else**
   
   No interaction

4. update the frequency according to the scattering event

**end for**

collect photons and perform visualisation

## 22.1 General things

- $\lambda\nu = c$. Mostly stellar spectra are recorded for increasing $\lambda$, thus decreasing $\nu$
- the $\beta$-velocity law

$$v = \left(1 - \frac{b}{r}\right)^{\beta} \tag{46}$$

where we want $v = \in [v_{\min}, v_{\max}]$. Thus on one hand $b = 1 - x_{v_{\min}}^{1/\beta} > 0$ and then we can compute the radius where $v = v_{\max}$, namely $r_{\max} = b/(1 - v_{\max}^{1/\beta})$

  – Minimal velocity. Without angle correction (see Section 22.2), the minimal velocity is given by $(1-b)^{\beta}$. With angle correction,

$$v = \cos(\alpha)\left(1 - \frac{b}{r}\right)^{\beta} \tag{47}$$

then the minimal velocity is slightly higher.

  – Maximal velocity equals $v_{max}$

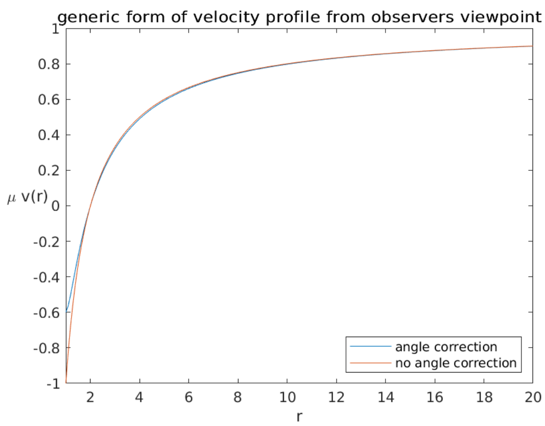



Figure 16: Example of velocity profile (give paramters)

## 22.2 Geometry

Spherical symmetry. Scattering at point $P$

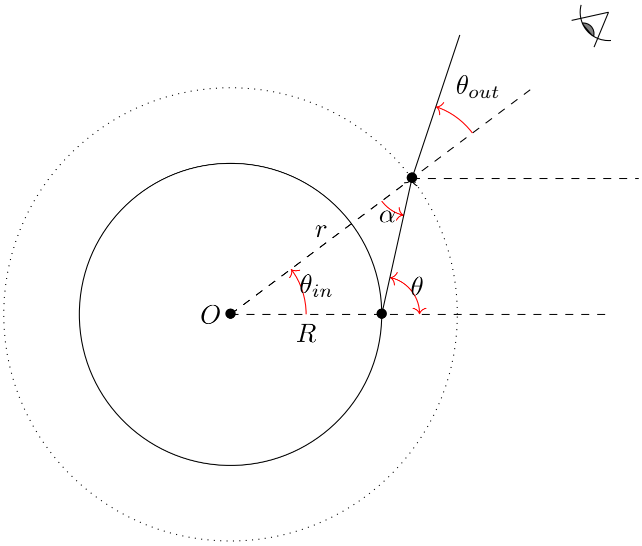



Figure 17: 2D Geometry for scattering

We can derive an expression, but first the sum of the angles in a triangle equals $\pi$ which leads to $\alpha = \theta_{in} - \theta$. We then have the sine rule, namely that, with

$$\frac{\sin(\alpha)}{R} = \frac{\sin(\pi - \theta)}{r} = \frac{\sin(\theta)}{r} \tag{48}$$

and thus, with $R = R_* = 1$,

$$\cos(\alpha) = \sqrt{1 - \left(\frac{\sin(\theta)}{r}\right)^2} \tag{49}$$

Per definition

- `xmuestart` $= \cos(\theta)$
- `pstart` $= R_* \sin(\theta)$
- `xmuein` $= \cos(\alpha)$
- `xmueou` $= \cos(\theta_{\text{out}})$

### 22.2.1 Condition for hitting the core

Check this only for inwards streaming photons!

$$\begin{aligned} \texttt{pcheck} &= \sqrt{r^2 \left(1 - \cos(\theta_{\text{out}})^2\right)} \\ &= r \sin(\theta_{\text{out}}) \end{aligned} \tag{50}$$

### 22.2.2 Slightly more general version

When $R_* \neq 1$, Equation (49) is replaced by

$$\cos(\alpha) = \sqrt{1 - \left(\frac{R_* \sin(\theta)}{r}\right)^2} \tag{51}$$

### 22.3 Sobolev approximation

- In fact, the absolute frequency of the photons does not change. However, in an observer's frame, we observe that the photon frequency changes at a scattering event. After a scattering event, the oberserver's frame frequency is updated as follows:

$$\texttt{xnew = xstart + u(xmueou - xmuein)} \tag{52}$$

- Sobolev condition for resonance:

$$x_{REF} - \mu u = 0 \tag{53}$$

with $u = \dfrac{v}{v_\infty} \in [0, 1]$ and $\mu = \cos(\theta) \in [-1, 1]$ then

$$x_{REF} \in [0, 1] \tag{54}$$

- The radius of interaction, we solve for $v_{photon} = x > 0$, for a specific $\nu_0$

$$\sqrt{1 - \left(\frac{p}{r}\right)^2} \left(1 - \frac{b}{r}\right)^\beta = \frac{\nu - \nu_0}{\nu_0} \frac{c}{v_\infty} \tag{55}$$

Now we are going to invest the effect of

  – increasing $\nu_0 \uparrow$. What then happens is that the RHS decreases, thus $r \downarrow$.
  – Inversely, $\nu_0 \downarrow$, then $r \uparrow$

### 22.4 Can resonance in the same resonance line happen twice?

After a first scattering event, the frequency is updated according to Equation (52). In my opinion, it is possible to have multiple scatterings.

### 22.5 Meaning of the parameters

- `xk0` is a characteristic scale of opacity $\chi = \dfrac{\texttt{xk0}}{rv^{2+\alpha}}$
- opacity $\chi = \dfrac{\texttt{xk0}}{rv^{2+\alpha}} \propto \kappa\rho \propto \dfrac{\kappa}{rv^2}$
- from this we deduce that $\kappa \propto \dfrac{1}{v^\alpha}$

### 22.6 Special case: `xmuestart = 1`

- FIRST SCATTTERING: from Equation (52) we have that `xstart = u` and then

$$\begin{aligned} \texttt{xnew} &= \texttt{xstart + u(xmueou- 1)} \\ &= \texttt{xstart.xmueou} \end{aligned} \tag{56}$$

Thus, since $\texttt{xmueou} \in [-1, 1]$ and $\texttt{xmin} \leq \texttt{xstart} \leq \texttt{xmax}$

$$\texttt{xmin} \leq \texttt{xnew} \leq \texttt{xmax} \tag{57}$$

# 23 Development of computer code (in `Matlab`)

## 23.1 Implementation in Matlab: user's manual

Run the function `test_function(test_number)`.

| `test_number` | parameter settings |
|---|---|
| 0 | original version |
| 1 | first adaptation: radial release |
| 2 | isotropic scattering – higher peak |
| 3 | Eddington limb darkening |
| 4 | photospheric line-profile |
| 5 | simple well |
| 6 | other resonance frequency (thus introducing shift) |
| 7 | formation of two lines, only radially streaming photons (thus also radial release |
| 8 | formation of two lines, with radial release |
| 9 | formation of two lines, full scattering possibilities |

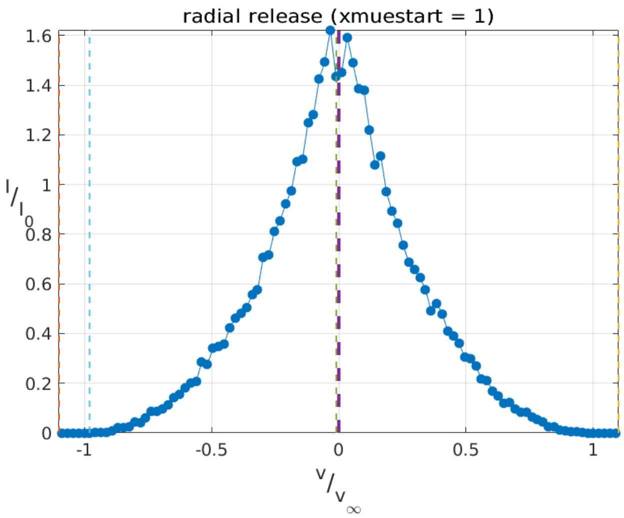
Via this link, you can go back to the exercises overview: Section 13.

## 23.2 Keeping track of the photon path

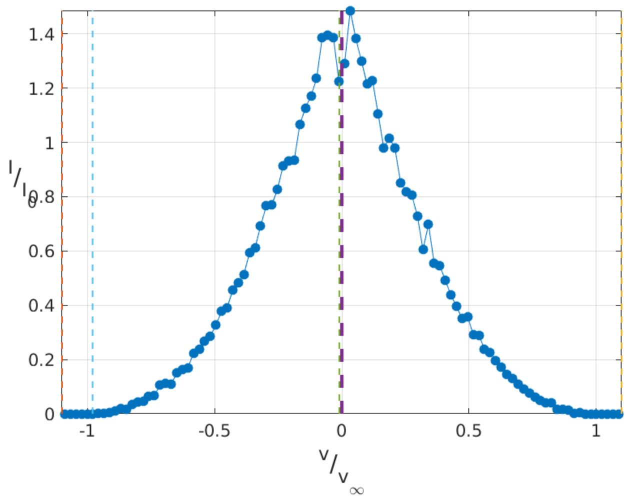
$$\left[\begin{array}{c} \texttt{xstart} \\ \texttt{xmuestart} \\ \texttt{goto\_end\_of\_loop} \\ \hline \texttt{r\_new} \\ \texttt{xmueou} \\ \texttt{xnew} \\ \texttt{x\_selected} \\ \hline \vdots \end{array}\right] \tag{58}$$

# 24 Experiments and results

## 24.1 About the scattering probability distribution



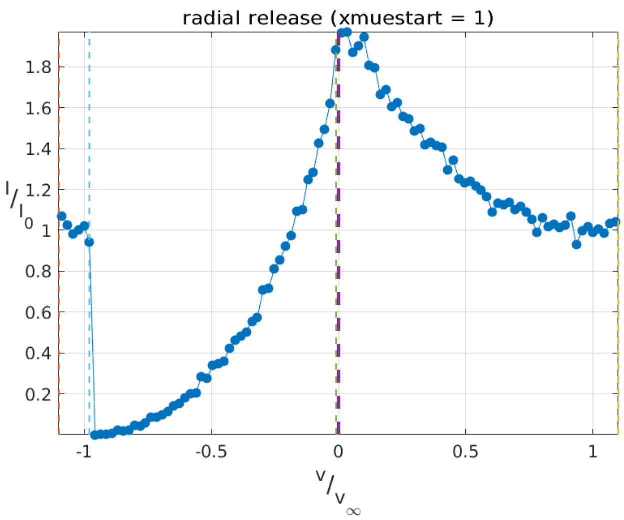

(a) Radial release



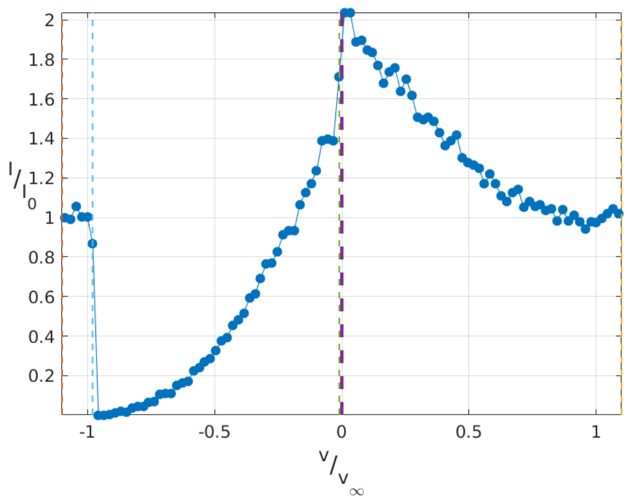

(b) Random release

Figure 18: Scattering distribution

## 24.2 Single resonance line




(a) Radial release




(b) Random release

Figure 19: Single line formation with Sobolev approximation

## 24.3 Multiple line formation

- NON-INTERACTING LINES

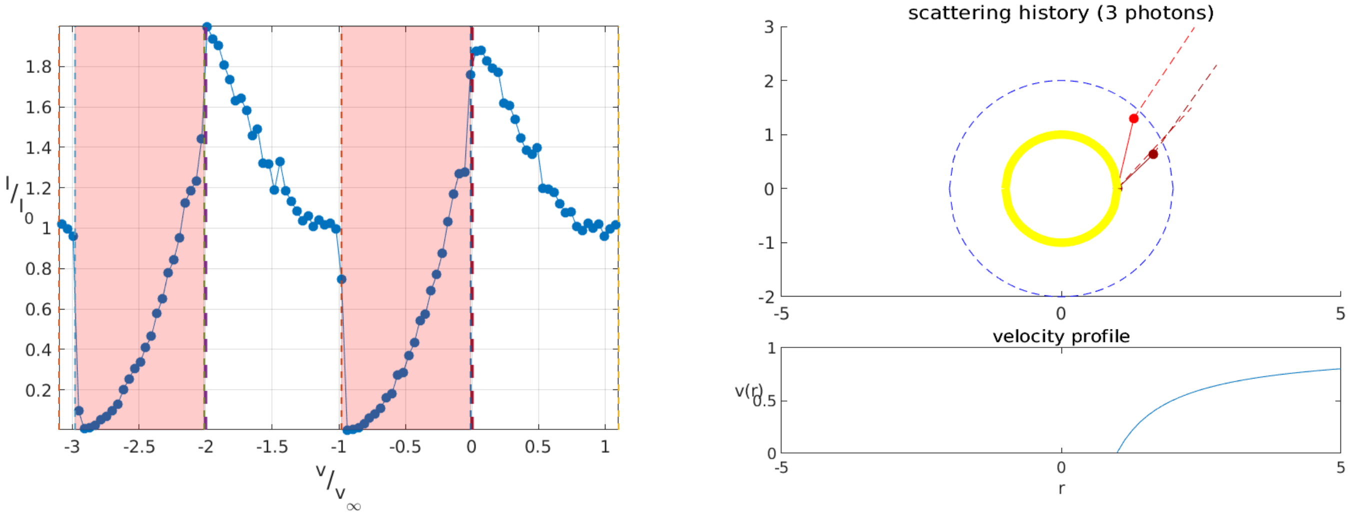



Figure 20: Multiple lines (distant lines, non-interacting)

- OVERLAPPING LINES

Figure 21: Multiple lines (distant lines, non-interacting)

## 24.4 Effect of the opacity

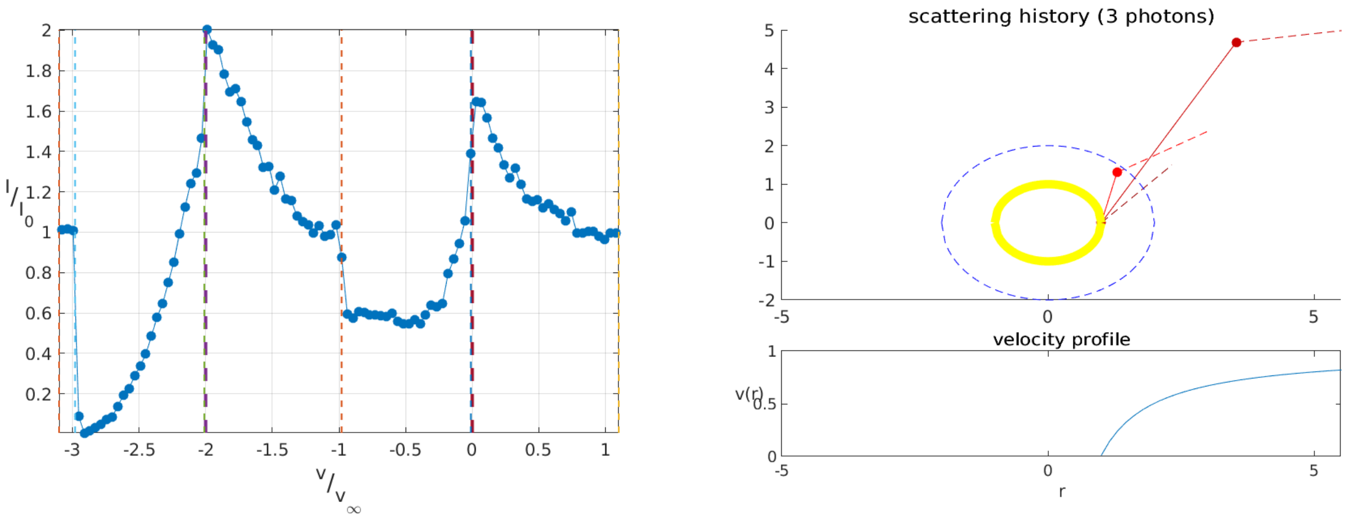



Figure 22: Multiple lines (left `xk0 = 100` and right `xk0 = 0.5`)

## 24.5 Observations

Figure **??** shows the profile when the scattering angle is allowed to lie in the interval $\mu_{\text{out}} \in [-1, 1]$. Figure **??** shows the same computations, but with $\mu_{\text{out}} \in [0, 1]$

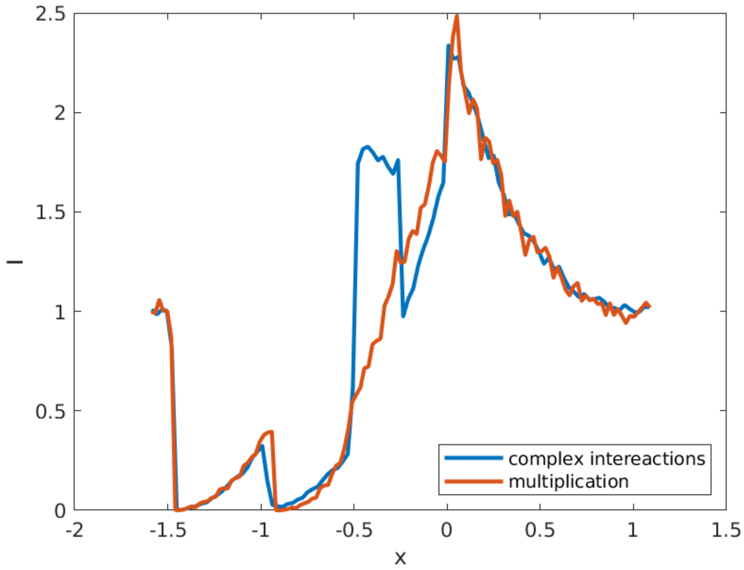



Figure 23: Comparison with multiplied profile ($\mu_{\text{out}} \in [-1, 1]$)

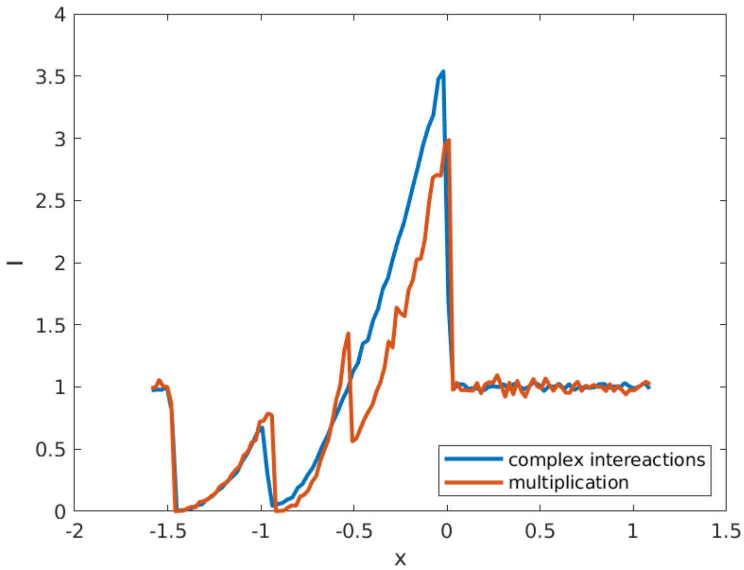



Figure 24: Comparison with multiplied profile ($\mu_{\text{out}} \in [0, 1]$)

Loosely speaking, one can understand this result because in Figure **??**, photons are not allowed to be backscattered to the first resonance line.

## 24.6 Convergence behaviour

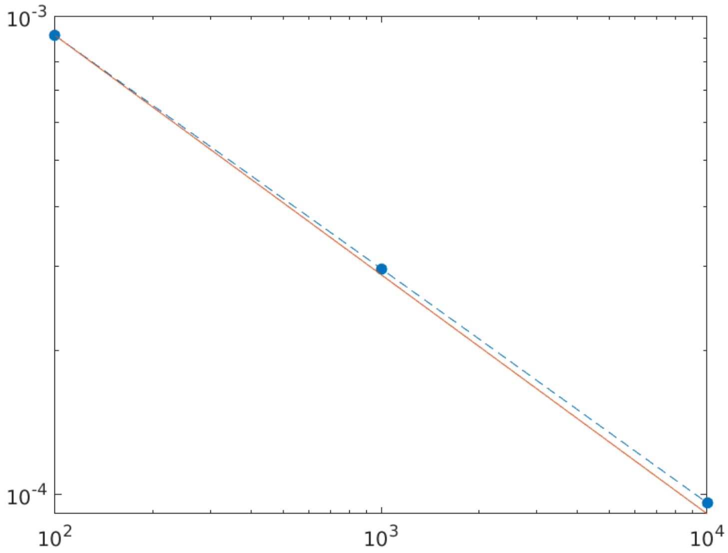



Figure 25: Convergence behaviour

## 24.7 Some theoretical observations

- The resonance frequencies need to lie close enough to each other, otherwise no resonance is possible.
  - if the lines are close enough to each other, resonance ends always at the rightmost frequency because you can indeed have a pumping up, but the leftmost peak is then in the absorption zone of the rightmost, thus it is scattered again! You can indeed see that the joint zone is depleted - and actaully makes the rightmost line stronger.
- Calculation of scattering probability for overlapping regions. Define ...
  - the 'one-time scattering probability' $p_1$.
  - probabiltiy of the overlapping region $p_2 = \frac{\text{length of overlapping region}}{\text{union of the length of the 'one-scattering' regions}}$.
  - HOWEVER take also into account that not all scattered photons are scattered.

  Then we have

$$p_{\text{total}} = p_1(1 + 2p_2(1 + 2p_2(1 + ...))) \tag{59}$$

# 25 Computing the radiation force & luminosity $L(r)$

This is based upon material from the text provided by professor Sundqvist, and the Phd thesis from Uwe Springmann [**UweSpringmannPHD**] where sperically symmetric Wolf-Rayet stars are discussed.

## 25.1 Theoretical formulas

1. Formal definition
$$g_{\text{radiation}} = \frac{\Delta p}{\Delta t \Delta m} = \frac{\Delta E}{v} \frac{1}{\Delta t} \frac{v}{M \Delta r} = -\frac{1}{\dot{M}} \frac{dL}{dr} \tag{60}$$

2. Numerical approximation, we count photons
$$L(r) = \frac{1}{\Delta t} \sum_i \epsilon_i \text{sign} \mu_i(r) \tag{61}$$

   where $L(r_*)$ is given and the photon energies $\epsilon_i$ are given by a black-body distribution.

## 25.2 Solution strategy

Loop over all photons. In that loop, loop over all $r$ values and assign the corresponding angle. Thus we need two additional arrays: `r_values` and `angles`. Update the array at each scattering event.

- create a grid for $r$
- for each $r$ on the grid, look at the
- calculate $L(r)$
- then calculate $\frac{dL}{dr}$ numerically on that grid

## 25.3 Checking the correctness of the program

Enforce algorithmically that all photons stream in one direction (no scattering with `xmueout < 0`). Comment out the line `xmueou = -xmueou` in the function `scatter.m`.

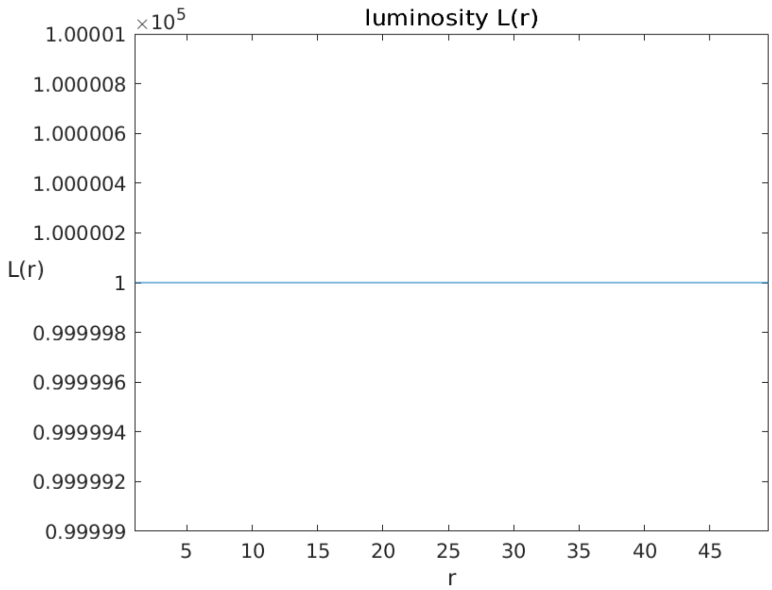



Figure 26: Luminosity $L(r)$ (for one resonance line) - test situation

## 25.4 Computing the radiation force for one line

Basic, well-known test situation.

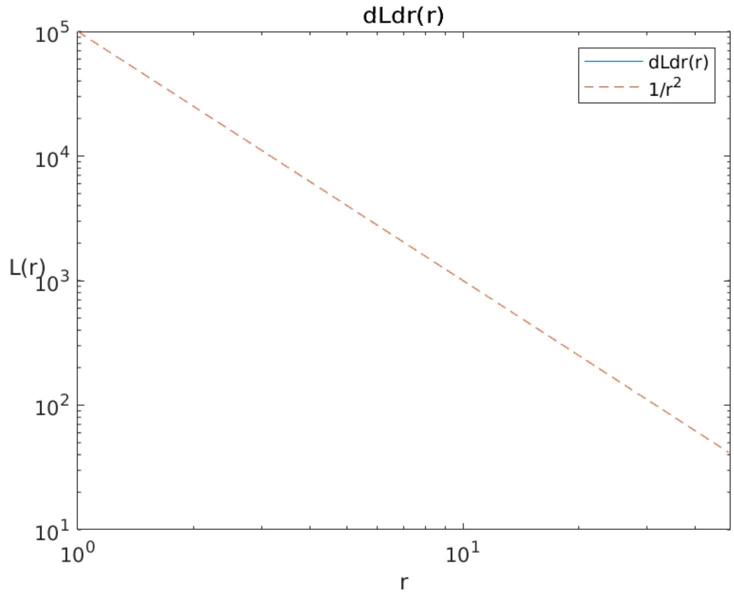



Figure 27: Luminosity $L(r)$ (for one resonance line)

## 25.5 Multiple lines

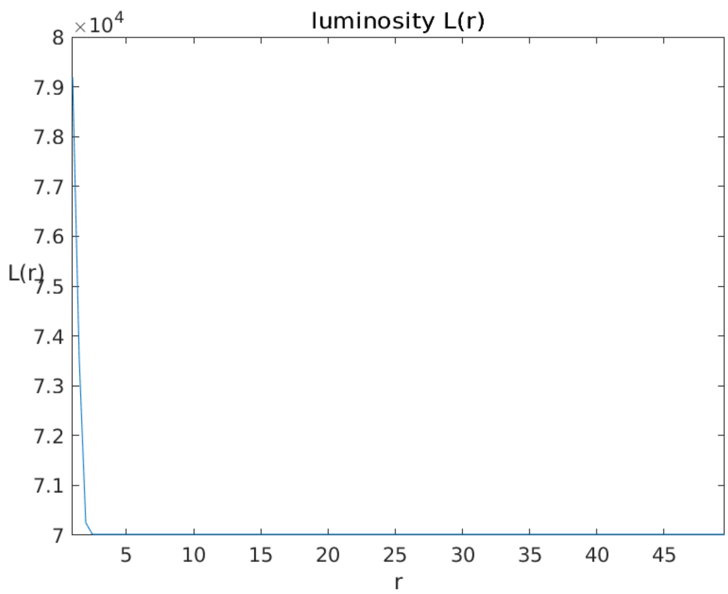



Figure 28: Luminosity $L(r)$ (for multiple resonance line)

## 25.6 Theoretical considerations

### 25.6.1 Analytical quantities

From the course notes on radiative processes

$$g_{\text{radiation}} = g_{\text{line}} = g_{\text{line}}^{\text{thin}} \frac{1 - e^{-\tau_S}}{\tau_S} \tag{62}$$

where

$$g_{\text{thin}} = g_e q = \frac{F_\nu^0 k_L}{\rho c} \tag{63}$$

with the Sobolev optical depth

$$\tau_S = \frac{\kappa_0}{r^2 v \left| \frac{dv}{dr} \right|} = \frac{\kappa_0}{r v^2 (\mu^2 \sigma + 1)} \tag{64}$$

and the density - mass loss relation

$$\dot{M} = \rho 4 \pi r^2 v \tag{65}$$

Combining these quantities gives

$$g_{\text{radiation}} = \frac{F_{\nu_0}}{c} \frac{k_L}{\rho} \frac{1 - e^{-\tau_S}}{\tau_S} \tag{66}$$

$$g_{\text{radiation}} = \frac{F_{\nu_0}}{c} \frac{\kappa_0 v_\infty}{\lambda_0 R_* r^2 v} \frac{4\pi r^2 v}{\dot{M}} \frac{r^2 v \left| \frac{dv}{dr} \right|}{\kappa_0} \left(1 - e^{-\tau_S}\right) \tag{67}$$

$$\boxed{g_{\text{radiation}} = \frac{F_{\nu_0} \nu_0}{\dot{M} c^2} \frac{dv_l}{dl} \left(1 - e^{-\tau_s}\right)} \tag{68}$$

### 25.6.2 From numerical computations

On the other hand we have

$$\boxed{g_{\text{radiation}} = -\frac{1}{\dot{M}} \frac{dL}{dr}} \tag{69}$$

### 25.6.3 These quantities should equal

For optically thick lines ($\tau_S \gg 1$)

$$g_{\text{radiation}}^{\text{numerical}} = -\frac{1}{\dot{M}} \frac{dL}{dr} = g_{\text{radiation}}^{\text{analytical}} = \frac{F_{\nu_0} \nu_0}{\dot{M} c^2} \frac{dv_l}{dl} \tag{70}$$

$$\frac{dL}{dr} = \frac{1}{4\pi r^2 c^2} \frac{F_{\nu_0}}{F} \frac{dv}{dr} \tag{71}$$

Combining that with a $\beta$-velocity profile

$$v = \left(1 - \frac{b}{r}\right)^\beta \tag{72}$$

Then we have

$$\boxed{\frac{dL}{dr} = \frac{F_{\nu_0}}{4\pi c^2} \frac{b}{r^3} \left(1 - \frac{b}{r}\right)^{\beta - 1}} \tag{73}$$

and for $\beta = 1$

$$\frac{dL}{dr} = \frac{F_{\nu_0}}{4\pi c^2} \frac{b}{r^4} \tag{74}$$

$$\left| \frac{dL}{dr} \right| = C^{te} \frac{1}{r^4} \tag{75}$$

### 25.7 With high `xk0`

TO BE UPDATED

# 26 Backup from theory

Transport step and collision step. There is no absorption.

# 27 Extension to higher dimensions

TO BE DONE

# 28 Closer look at Monte Carlo simulations

## 28.1 Random walk (diffusion equation)

A more simple experiment that simulates the diffusion equation (1D random walk) is also set up. The results are shown in Figure 27. We observe that $N \sim \tau^2$, as can also be derived from theory.

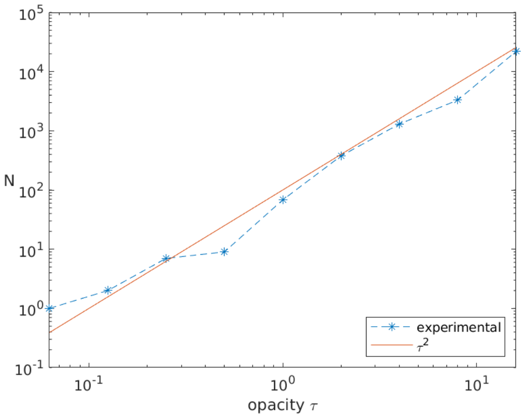



Figure 29: Number of interactions (scattering events) versus opacity, random walk

- When starting from an initial condition $x_0 = 0$ and

$$x_N = x_{N-1} \pm l \tag{76}$$

  we have for the variance that $\langle x_N \rangle^2 = Nl^2$

- If we require a photon to cover a distance $R$ then $N = \dfrac{R^2}{l^2}$ and

  - the relation between mean-free path $l$ and opacity $\alpha$ is $l = \dfrac{1}{\alpha}$
  - with $\tau = \displaystyle\int_0^R \alpha ds = \dfrac{R}{l}$

  then we have that $N = \tau^2$. This corresponds with the observations in Figure 27.

## 28.2 Limb darkening

We first look at results from the limb darkening program, as studied in Section 19. In Figure 28, the number of scattering events is plotted versus the opacity of the medium.

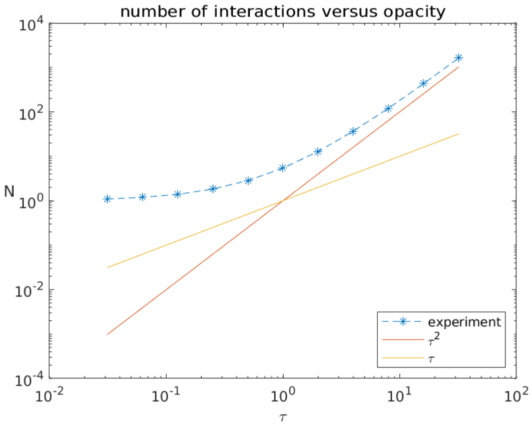



Figure 30: Number of interactions (scattering events) versus opacity, kimb darkening

- For high opacity $\tau \gg 1$ we observe that $N \sim \tau$.
- Bridging regime.
- For opacity $\tau \ll 1$ we observe that $N \sim 1$: namely the photons travels very far during the first emission event.

If you assume constant opacity then $\tau = \alpha z$

# Part IV

# Questions

## 29 Questions for professor Sundqvist

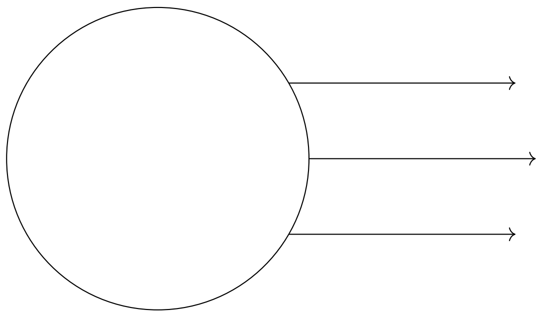
- What are the equations governing the processes in `pcyg.f90`
- What does this mean? `xnew=xstart+(v-sign(0.06,xmueou))*xmueou-v*xmuein`
- Pcygni profiles: Why don't we just take the absorption and the emission and add them together.
- why in `rtbis` we have that $rmin = 1$
- changed `xmuestart` from $-1$ to 1.

## 30 Questions for professor Samaey

- In [DPS18], Equation (31) why does it correspond to diffusion (more specifically the second term on the right hand side).
- what is the difference between Monte Carlo and equation-free computing?

# 31 Solved questions

- Sundqvist+ 2009: what is thermal velocity (see Wikipedia)
- Sundqvist+ 2009: what is line force (see explanation Dylan)
- unclassified: what is a flux limiter? (see course notes)
- unclassified: what is cross section of scattering (see Wikipedia)
- Puls manual: p.26: how does the Milne equation appear? (see library book)
- pcyg.f90: what are p-rays? (see anwser professor Sundqvist)
  - parallel rays leaving the atmosphere (of, e.g. a star)



- pcyg.f90: what is meant by Eddington limb-darkening? (see answer professor Sundqvist)
  - standard limb darkening
- Sundqvist+ 2009: what is the geometry of a *slice*?
- CMFAA course notes p.13 (the example) what is understood by plane-parallel geometry and is it 1D or 2D? (see answer professor Sundqvist)
  -
- CMFAA course notes p.15: why is this called diffusion $F = T^3 \frac{dT}{dx}$ (flux proportional to local gradient in temperature)?
- unclassified: what is the terminal velocity $v_\infty$?
- unclassified: what is Sobo-distribution? (Sobolev distribution)
- pcyg.f90: for `test_number` = 2, why do we call it isotropic since isotropy of `mu` does not imply isotropy of `theta`? (myself, see definition of intensity)
- (for which star are the exerpimental data and what assumptions are used in the theory?) (see ... and derive some formulas)
- book *Stellar Atmospheres* [Mihalas] (bought)
- ordening of array `freq` (adapted the code, experimented with it)
  - why `freq(1) = xmax-5*deltax`?
  - frequency binning: how are you sure that no lower/higher frequencies can occur?
    $\boxed{\text{derive this analytically}}$
- `Pcyg.f90` does it take into account that photons are scattered away from the observer? (via assumption of radial symmetry)

# 32 Interesting problems

- inverse radiative transfer problem

**might be interesting for looking at**

- splitting methods
- Eddington factors

# 33 Do not forget

- convergence plots

# Part V
# Thesis meetings

## 34 Meeting on 6 September 2019

- overview of Petnica summer institute on Astrophysics
- question: manual by Puls: why is isotropic distribution sampled from $\mu\mu$?
- `pcyg.f90` program
- practical arangements
- SKIRT code
- discussion of paper (Dimarco+2018)

## 35 Meeting on 23 September 2019

- convergence plots
- relation $N \sim \tau + \tau^2/2$
- limb darkening

## 36 Meeting on 30 September 2019

- discussion about $x_{\min}$ and $x_{\max}$
- discussion about introduction of second line.
  - take into account the Doppler shift because you have two different frames
  - make them radially streaming (release)
  - do step for step:
    - begin with creating a correct well
- normalized frequency: $x = \dfrac{\nu - \nu_0}{\nu_0} \dfrac{c}{v_{infty}}$
- adding other resonance frequencies, involves enlarging the frequency frame.
- about the goal of the master thesis. Making a master thesis is not doing a course where all is well-defined and *a priori* known.

# Part VI
# Equation meetings

- Meeting of 10 April 2019
- Meeting of 17 April 2019
- Meeting of 14 August 2019
- Meeting of 18 September 2019
- Meeting of 25 September 2019

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