

1 Glossary

- SED: spectral energy distribution
- (spectral) line-force: force on material in stellar atmosphere
- LASER: Light Amplification by Stimulated Emission of Radiation

2 General equations - first year overview

2.0.1 Hydrodynamics

Euler equations, together with closing relation (e.g. ideal gas law).

primitive variables			
mass density	velocity	gas energy density	gas pressure
ρ	v	e	p

2.0.2 Radiation

Radiative transfer equation: intensity along a ray while interacting with medium. Photons are massless.

$$\left[\frac{1}{c} \partial_t + \vec{n} \cdot \vec{\nabla} \right] I_\nu = \eta_\nu - \chi_\nu I_\nu \quad (1)$$

frequency	intensity	emissivity	total absorption
ν	I_ν	η_ν	χ_ν

These deliver two equations

- the radiative energy equation (diffusion flux \vec{F})

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot \vec{F} = \iint \dots d\nu d\Omega \quad (2)$$

- radiative momentum equation

$$\frac{d\vec{F}}{dt} = \iint \dots \vec{n} d\nu d\Omega \quad (3)$$

(after **integrating over all frequencies**). Depending on the geometry simplifications, one can e.g. integrate over all solid angles.

2.0.3 Radiation-Hydrodynamics

Combination delivers integral-diffusion equation

$$\begin{aligned} \frac{dI}{d\tau} &= S - I \\ &= \int I d\Omega - I \end{aligned} \quad (4)$$

2.0.4 Challenges

- combination with hydrodynamics
- current analysis: simplified geometries (symmetry). E.g. in 2D, an ADI method is used and now also a multigrid method.
- complex geometry difficult to show in ray-tracing scheme
- steady-state vs. time dependent
- focus on radiation equations

3 Very broad introduction & Summary

The material here originates from the master thesis of Nicolas Moens [MoensNicolas] and from the course notes *Introduction to numerical methods for radiation in astrophysics* from professor Sundqvist.

3.1 Definitions and equations

3.1.1 Radiation equations

Specific intensity $I(s, \lambda, x, y, t)$

- restrict ourselves to time-independent, one-dimensional (1D) case $I(s, \theta, \lambda)$ where s is the direction of the light ray

- it satisfies Radiation Transfer Equation (RTE) $\boxed{\frac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda}$

- with 'formal' solution $\boxed{I(\lambda, \tau_\lambda) = I_0(\lambda)e^{-\tau_\lambda} \int_0^{\tau_\lambda} S(t)e^{-t} dt}$

- no emissivity $S = 0$ then $I(\lambda)I_0(\lambda)e^{-\tau_\lambda}$
- no opacity then $I_0(\lambda) = \int_0^s \eta_\lambda(s) ds$
- constant source function $I(\lambda, \tau) = I_0(\lambda)e^{-\tau_\lambda} + S(1 - e^{-\tau_\lambda})$
- if $S = a + b\tau$ then $I(\lambda) = a + \frac{b}{k_\lambda}$ with k_λ the opacity. A jump in opacity leads to the jump in intensity of the opposite sign.

3.1.2 RHD equations

The full RHD equations consist of

- five partial differential equations
- one HD closure equation, e.g. (i) variable Eddington tensor method or (ii) flux limited diffusion

Heat flux The heat flow rate density $\vec{\phi}$ satisfies the Fourier law $\vec{\phi} = -k\nabla T$. More information can be found for instance on [WikiHeat].

Specific intensity and its angular moments

specific intensity	$\Delta\epsilon = \boxed{I_\nu} A_1 A_2 / r^2 \Delta\nu \Delta t$
energy density	$E = \frac{1}{c} \iint I_\nu d\nu d\Omega$
flux vector	$F = \iint I_\nu n d\nu d\Omega$
pressure tensor	$P = \iint I_\nu n n d\nu d\Omega$
mean intensity	$J_\nu = \frac{c}{4\pi} E_\nu$
Eddington flux	$H_\nu = \frac{1}{4\pi} F_\nu$
Eddington's K	$K_\nu = \frac{c}{4\pi} P_\nu$

Eddington factor In general, the Eddington factor is a tensor, for 1D systems it is reduced to a scalar.

$$f_\nu = \frac{K_\nu}{J_\nu} = \frac{P_\nu}{E_\nu} \quad (5)$$

- isotropic radiation field
- radiation field strongly peaked in radial (i.e. vertical in cartesian) direction

3.1.3 Radiation transport equations, diffusion, equilibrium

- black body radiation (Planck function $I_\nu = J_\nu = B_\nu$)
- in general, extinction (absorption, scattering) and emission

$$\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu \quad (6)$$

– Cartesian coordinates:

$$\boxed{\frac{\partial I_{n,\nu}}{\partial t} \frac{1}{c} + n \nabla I_{n,\nu} = j_\nu - k_{n,\nu} I_{n,\nu}} \quad (7)$$

– spherical coordinates

– 1D-problem with only variation along z-axis $\mu \frac{dI}{dz} = j - kI$

– spherical symmetry $\mu \frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} = j - kI$

– plane-parallel approximation

$$\boxed{\mu \frac{dI}{dr} = j - kI} \quad (8)$$

The angle μ is constant throughout the computational domain. Dividing by k_ν , this yields

$$\mu \frac{dI}{k_\nu dr} = \mu \frac{dI}{k_\nu dz} = S - I \quad (9)$$

- 0th moment equation: integrate Equation (7) over ν and Ω , i.e. $\int d\nu d\Omega$. Conservation of energy
- first multiply Equation (7) with $\frac{n}{c}$ and then do integration

3.1.4 Radiative Diffusion Approximation

The radiative diffusion approximation bridges two regimes: regimes with ...

- on one hand, large optical depth $\tau \gg 1$: diffusion equation: temperature structure in a static stellar atmosphere
- on the other hand, where radiative *transport* is important

The diffusive approximation is the following: replace $\boxed{I = B}$ or $I_\nu = B_\nu$.

$$I_\nu = B_\nu - \mu \frac{dB_\nu}{k_\nu dz} \quad (10)$$

This equation can be derived as a random walk of photons!

3.1.5 Applications and approximations for radiative forces

- definition of general radiative acceleration vector $g = \frac{1}{\rho c} \int \int n k_\nu I_\nu d\Omega d\nu$
 - continuum Thomson scattering
 - spectral line with extinction
 - * furthermore assume central continuum source
 - * then $g_{line} = \frac{F_\nu^0 k_L}{\rho c}$
- Sobolev approximation
- CAK theory

3.1.6 Optical depth (recap)

Optical depth: physical understanding Optical depth is the ratio of incident radiant power to transmitted radiant power ([WikiOpticalDepth]).

optical depth	optical depth along ray	line optical depth	Sobolev optical depth
$d\tau = k_\nu ds = \sigma_{nu} n ds = \kappa \rho ds$ $\tau_\nu = \int k_\nu ds = \int \sigma_\nu n ds$	$\tau_{\mu,\nu} = \int_z^{z_{max}} \frac{\alpha_{nu}(z')}{\mu} dz' = \frac{\tau_\nu(z)}{\mu}$	$\tau_\nu = \int k_L \phi_\nu dl = \int \kappa \rho ds$	

with

- σ cross-section
- n number density
- κ mass absorption density
- ρ mass density
- k_ν extinction coefficient

3.2 Overview of symmetry assumptions

plane-parallel	1D atmosphere bounded by horizontal surfaces	
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4 Introduction: course material (Sundqvist - CMPAA course)

4.1 EXERCISES: Introduction to numerical methods for radiation in astrophysics

1. introduction

2. radiation quantities

- exercise p.3:

- on one hand, we know that $\Delta\epsilon \sim C/r^2$
- on the other hand, from the definition we know that $\Delta\epsilon = I_\nu A_1 A_2 / r^2 \Delta\nu \Delta t$
- combining these equations shows that I_ν is independent from r

- exercise p.4:

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- exercise 1:

- $F_x = \int_0^\pi \left[I_\nu(\theta) \sin^2(\theta) \int_0^{2\pi} \cos(\phi) \right] d\theta d\phi = 0$
- the same reasoning for $F_y = 0$

- exercise 2:

- the equation follows from $d\mu = d\cos(\theta) = \sin(\theta)d\theta$

- exercise 3:

- isotropic radiation field (i.e. $I(\mu) = I$) then we have $F_\nu = 2\pi \int_{-1}^1 I \mu d\mu = 2\pi I \left. \frac{x^2}{2} \right|_{-1}^1 = 0$

- exercise 4:

- $F_\nu = 2\pi \int_{-1}^1 I(\mu) \mu d\mu = 2\pi \int_{-1}^0 I_\nu^- \mu d\mu + 2\pi \int_0^1 I_\nu^+ \mu d\mu = 2\pi I_\nu^+$

- exercise p.7:

- isotropic radiation field:

- * although the radiation pressure is a tensor, we will denote it as a scalar $P_\nu = \frac{4\pi I_\nu}{c}$

- * the radiation energy density $E_\nu = \frac{12\pi I_\nu}{c}$

- * thus $f_\nu = \frac{1}{3}$

- very strongly peaked in radial direction (beam): $I_\nu = I_0 \delta(\mu - \mu_0)$ with $\mu_0 = 1$

- * pressure tensor $P_{nu} = \frac{1}{c} \int I_0 \delta(\mu - \mu_0) n n d\Omega$

- * energy density $E_\nu = \frac{1}{c} \int I_\nu d\Omega$

- * in this case $P_\nu = E_\nu$ thus $f_\nu = 1$

3. radiation transport vs. diffusion vs. equilibrium

- exercise p. 12: 1D, Cartesian geometry, plane-parallel, frequency-independent and isotropic emission/extinction

- radiation energy equation

- * The equation follows by integrating Equation (8)

- * By definition, $E = \frac{1}{c} \iint I_\nu d\nu d\Omega$

- * thus $\frac{dE}{dr} = \int (j - kI) d\nu d\Omega$ thus $\boxed{\frac{dE}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)}{c}}$

- * work out the integral taking into account frequency-independent and isotropic coefficients:
 - zeroth momentum equations
 - * One must also take into account the specific form of the flux vector

$$F = \iint I_\nu n d\nu d\Omega = 2\pi \int_{-1}^1 I_\nu(\mu) \mu d\mu$$
 - * thus $\frac{dF}{dr} = \frac{1}{c} \int (j - kI) n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
 - first moment equation
 - * similar reasoning
 - * $\frac{dP}{dr} = \int (j - kI) n \cdot n d\nu d\Omega$ thus $\boxed{\frac{dP}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
 - first exercise p. 15
 - $P = \frac{1}{c} \iint I_\nu \mu^2 d\Omega d\nu = \frac{2\pi}{c} \int_{-1}^1 \int_{-1}^1 I_\nu \mu^2 d\mu d\nu = \frac{4\pi}{3c} \int B_\nu d\nu = \frac{aT^4}{3} = \frac{E}{3}$
 - second exercise p.15
 - assuming the diffusion limit,
 - flux-weighted mean opacity $\kappa_F = \frac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}$
 - Rosseland mean opacity $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$.
 - * in the diffusion limit, $F_\nu = -\frac{4\pi}{3} \frac{dB_\nu}{k_\nu dz}$ thus $\frac{dB_{nu}}{dT} =$
 - *
 - third exercise p.15
4. the equations of radiation-hydrodynamics
 5. numerical techniques for the radiative diffusion approximation
 6. applications and approximations for a dynamically important radiative force in supersonic flows
 - exercise p.27: $L_{SOB} = \Delta r = \frac{v_{th}}{dv/dr} = \frac{10[km/s]}{1000[km/s]/R_*} = 0.01 R_*$
 7. Appendix A: properties of equilibrium black-body radiation
 - exercise p. 29
 - this should be satisfied: $B_\nu d\nu = -B_\lambda d\lambda$ and also $\nu = \frac{c}{\lambda}$
 - this is equivalent to saying that $0 = \nu d\lambda + \lambda d\nu$ or $d\lambda = -\frac{\lambda}{\nu} d\nu$ thus $B_\lambda = \frac{\nu}{\lambda} B_\nu$
 - $B_\lambda(T) = \frac{\nu}{\lambda} \frac{2h\nu^3}{(\lambda\nu)^2} \frac{1}{e^{hc/\lambda kT} - 1} = \frac{2h\nu^2}{\lambda^3} \frac{1}{e^{hc/\lambda kT} - 1} = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$
 - first exercise p.31
 - derive that $\lambda_{max} T = 2897.8[\mu m K]$
 - ...
 - second exercise p.31
 - this is about the spectra of (unknown) stars
 - first exercise p.32
 - see exercise 7
 - second exercise p.32

- BB radiation: $I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$
- the radiative flux for isotropic BB radiation is zero. See also exercise 3. This also holds for BB radiation.
- exercise p. 33
 - HR-diagram

8. Appendix B: Simple examples to the radiative transfer equation

- first exercise p. 34
 - start from radiative transport equation $\mu \frac{dI}{ds} = \alpha - \eta I$ in which $\eta = 0$ thus $\boxed{\mu \frac{dI}{ds} = \alpha}$
 - solving the ODE in the general case that $\alpha(s)$ is not constant:
 - * integrate the equation $\mu I = \int_0^D \alpha ds$
 - * ...
 - second exercise p. 34
 - * case $\tau(D) \gg 1$: then $I(D) \approx S$
 - * case $\tau(D) \ll 1$: then $I(D) \approx I(0) + S(1 - 1) = I(0)$
 - first exercise p.35
 - * is the plane-parallel approximation valid for the solar photosphere?
 - second exercise p.35
 - * goal: find a solution to the equation $\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$ where $I(\tau, \mu)$
 - * solution
- second exercise p.35

9. Appendix C: connecting random walk of photons with radiative diffusion model

- exercise p. 38. Computing the average photon mean-free path inside the Sun.

$$l = \frac{1}{\kappa \rho} = \frac{V_o}{\kappa M_o} [cm]$$
- exercise p.39. Computing the random-walk time (diffusion time) for photons

4.2 Implicit 1D solver (20-11-2018)

See computer code

4.3 ADI 2D Solver

See computer code

4.4 Area of a circle

See computer code

4.5 Limb Darkening

See Section ??.

5 The mathematics of Radiative Transfer

The material in this section is based on the book [Busbridge].

5.1 Auxiliary mathematics

- $\cos(\Theta) = \cos(\theta) \cos(\theta') + \sin(\theta) \sin(\theta') \cos(\phi - \phi')$

- phase function
$$p(\mu, \phi, \mu', \phi', \tau) = \sum_{n=0}^N \omega_n P_n(\cos(\Theta))$$

- isotropic scattering $p(\tau) = \omega_0(\tau)$

- equation of transfer
$$\mu \frac{\partial I(\tau, \mu, \phi)}{\partial \tau} = I(\tau, \mu, \phi) - \mathcal{S}(\tau, \mu, \phi)$$

with $\mathcal{S}(\tau, \mu, \phi) = B_1(\tau) + \frac{1}{4\pi} \int_{-1}^1 d\mu' \int_0^{2\pi} I(\tau, \mu', \phi') p(\mu, \phi, \mu', \phi') d\phi'$

- axially symmetric with isotropic scattering

$$\mathcal{S}(\tau) = \frac{\omega_0(\tau)}{2} \int_{-1}^1 I(\tau, \mu') d\mu' = B_1(\tau) + \frac{\omega_0(\tau)}{2} \int_0^{\tau_1} \mathcal{S}(t) E_1(|t - \tau|) dt$$

- the Milne equation of the problem $(1 - \omega_0 \bar{\Lambda}) \{ \text{mahtcal} S(t) \} = B(\tau)$

- * solve for $\mathcal{S}(t)$

- * then find $I(\tau, \mu)$

5.2 The H-functions

- characteristic equation

6 Monte Carlo and Radiative Transfer (Puls)

6.1 basic definitions and facts

6.2 about random numbers

6.3 MC integration

6.4 MC simulation

Radiative transfer in stellar atmospheres

- GOAL: spatial radiation energy density $E(\tau)$ in an atmospheric layer
 - only photon-electron scattering
 - τ is the optical depth

- Milne's integral equation
$$E(\tau) = \frac{1}{2} \int_0^\infty E(t) E_1(|t - \tau|) dt$$

- analytical solution $\frac{E(\tau)}{E(0)} = \sqrt{3}(\tau + q(\tau))$
- MC simulation
 - * emission angle
 - * optical depth until next scattering event
 - * scattering angle

- HOW DOES THIS WORK?

Algorithm 1 Limb darkening: compute quantity of photons

create photons

probability distribution for emission angle $\mu = \cos(\theta)$: $p(\mu)d\mu = \mu d\mu$

optical depth until next scattering event: $p(\tau)d\tau \approx e^{-\tau} d\tau$

isotropic scattering angle at low energies: $p(\mu)d\mu \approx d\mu$

follow all photons until they leave the atmosphere or are scattered back into stellar interior

6.5 Exercise 1: RNG

6.6 Exercise 2: Planck-function

1. analytical method
2. MC method

6.7 limb darkening

See section ??.

7 Introduction to Monte Carlo Radiation Transfer (Wood+)

The material is taken from

- (Wood, Wittney, Bjorkman, Wolff - 2001)
- (Wood, Wittney, Bjorkman, Wolff - 2013)

7.1 Elementary principles

specific intensity	I_ν
radiant energy	dE_ν
surface area	dA
angle	θ
solid angle	$d\Omega$
frequency range	$d\nu$
time	dt
flux	F_ν
cross section	σ
scattering angle	χ $\mu = \cos(\chi)$
mean intensity	J
flux	H
radiation pressure	K

intensity	$I_\nu(l) = I_\nu(0)e^{n\sigma l}$
angular phase function of the scattering particle	$P(\cos(\chi))$

inverse method	$\xi = \int_0^{x_0} P(x)dx$ with $\xi \in \mathcal{U}(0, 1)$
rejection method	

7.2 Eddington factors

7.3 Example: plane parallel atmosphere

1. emission of photons: select two angles (3D space). In isotropic scattering

- θ met $\mu = \cos(\theta)$
 - $\mu = 2\xi - 1$ (isotropic scattering)
 - $\mu = \sqrt{\xi}$ (A slab is heated from below. Then $P(\mu) = \mu$)
- $\phi = 2\pi\xi$

2. propagation of photons

- sample optical depth from $\tau = -\log(\xi)$
- distance travelled $L = \frac{\tau z_{max}}{\tau_{max}}$

3. conclusion of emission and propagation

$$\begin{aligned}x &= x + L \sin(\theta) \cos(\phi) \\y &= y + L \sin(\theta) \sin(\phi) \\z &= z + L \cos(\theta)\end{aligned}\tag{11}$$

4. Binning: once the photon exists the slab. Produce histograms of the distribution function. Finally, we wish to compute the output flux or the intensity.

I have seen that a newer version of the paper is available, which was also used in these notes (which contains amongst other up-to-date references to code fragments).

A Plane Parallel, Isotropic Scattering Monte Carlo Code

7.4 Monte Carlo Radiative Transfer

From a macroscopic perspective, RT calculations rest on the transfer equation

- emissivity η (how much energy is added to radiation field due to emission)
- opacity χ (how much energy is removed due to absorption)
- the source function $S = \frac{\eta}{\chi}$
- optical depth τ captures the opaqueness of a medium

$$\left(\frac{1}{c} \frac{\partial}{\partial t} + \nabla \cdot \mathbf{n} \right) I = \eta - \chi I \quad (12)$$

$$d\epsilon = I d\nu dt d\Omega dA \cdot n \quad (13)$$

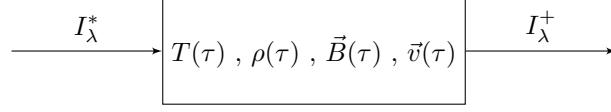
7.5 P Cygni profile for beta-velocity law and given opacity Monte Carlo simulation

7.5.1 Structure of the code

- module common
- module my_inter
- program pcyg
 - INPUT xk0, alpha, beta
 - OUTPUT
 - PROGRAM FLOW: loop over all photons
 - * get xstart and vstart
 - *
 - then do normalisation
- function func(r)
- function xmueout(xk0,alpha,r,v,sigma)
- function rtbis(func,x1,x2,xacc)

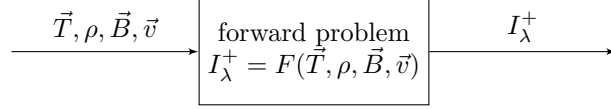
8 Challenges in Radiative Transfer (Ivan Milic)

8.1 Overview of the problem



Forward problem

The forward problem is schematically represented



In fact solve for intensity vector $\vec{I} = \begin{pmatrix} I \\ Q \\ \alpha \\ V \end{pmatrix}$ obeying the equation

$$\frac{d\vec{I}}{d\tau} = -X(\vec{T}, \rho, \vec{B}, \vec{v})\vec{I} - \vec{j}(\vec{T}, \rho, \vec{B}, \vec{v}) \quad (14)$$

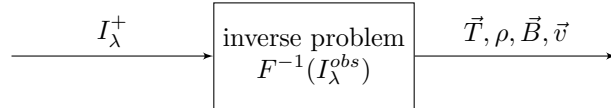
and the solution

$$I_{\lambda}^+ = I_0^+ e^{-\int} + \int \vec{j} e^{-\int} d\tau \quad (15)$$

Example Source function $S = a\tau + b$ then $\int_0^{\tau_{max}} (a\tau + b)e^{-\tau} d\tau = \dots$

Inverse problem

The inverse problem is schematically represented



Via least-squares approximation

$$\min_{\vec{T}, \rho, \vec{B}, \vec{v}} \sum \left(I_{\lambda}^{obs} - I_{\lambda}(\vec{T}, \rho, \vec{B}, \vec{v}) \right)^2 \quad (16)$$

8.2 Challenging domains of application

- Lyman alpha in Galaxy Halos
- Dusty torii (AGD)
- protoplanetary disks
- circumstellar disks
- atmospheres

9 Asymptotic Preserving Monte Carlo methods for transport equations in the diffusive limit (Dimarco+2018)

10 Splitting methods

From notes by professor Frank.

10.1 Exercises

10.1.1 Exercise 1