1 Glossary

- SED: spectral energy distribution

- (spectral) line-force: force on material in stellar atmosphere

- LASER: Light Amplification by Stimulated Emission of Radiation

2 General equations - first year overview

2.0.1 Hydrodynamics

Euler equations, together with closing relation (e.g. ideal gas law).

primitive variables			
mass density	velocity	gas energy density	gas pressure
ρ	v	e	p

2.0.2 Radiation

Radiative transfer equation: intensity along a ray while interacting with medium. Photons are massless.

$$\left[\frac{1}{c}\partial_t + \vec{n}.\vec{\nabla}\right] I_\nu = \eta_\nu - \chi_\nu I_\nu \tag{1}$$

frequency	intensity	emissivity	total absorption
ν	I_ν	η_ν	χ_ν

These deliver two equations

- the radiative energy equation (diffusion flux $\vec{F}$

$$\frac{\partial E}{\partial t} + \vec{\nabla}.\vec{F} = \iint ...d\nu d\Omega \tag{2}$$

- radiative momentum equation

$$\frac{d\vec{F}}{\partial t} = \iint ...\vec{n}d\nu d\Omega \tag{3}$$

(after **integrating over all frequencies**). Depending on the geometry simplifcations, one can e.g. integrate over all solid angles.

2.0.3 Radiation-Hydrodynamics

Combination delivers integral-diffusion equation

$$\begin{aligned}\frac{dI}{d\tau} &= S - I \\ &= \int I d\Omega - I\end{aligned} \tag{4}$$

2.0.4 Challenges

- combination with hydrodynamics
- current analysis: simplified geometries (symmetry). E.g. in 2D, an ADI method is used and now also a multigrid method.
- complex geometry difficult to show in ray-tracing scheme
- steady-state vs. time dependent
- focus on radiation equations

3 Very broad introduction & Summary

The material here originates from the master thesis of Nicolas Moens [**MoensNicolas**] and from the course notes *Introduction to numerical methods for radiation in astrophsyics* from professor Sundqvist.

3.1 Definitions and equations

3.1.1 Radiation equations

Specific intensity $I(s, \lambda, x, y, t)$

- restrict oursevels to time-independent, one-dimensional (1D) case $I(s, \theta, \lambda)$ where s is the direction of the light ray

- it satisfies Radiation Transfer Equation (RTE) $\boxed{\dfrac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda}$

- with 'formal' solution $\boxed{I(\lambda, \tau_\lambda) = I_0(\lambda)e^{-\tau_\lambda} \displaystyle\int_0^{\tau_\lambda} S(t)e^{-t}dt}$

    - no emissivity $S = 0$ then $I(\lambda)I_0(\lambda)e^{-\tau_\lambda}$

    - no opacity then $I_0(\lambda) = \displaystyle\int_0^s \eta_\lambda(s)ds$

    - constant source function $I(\lambda, \tau) = I_0(\lambda)e^{-\tau_\lambda} + S(1 - e^{-\tau_\lambda})$

    - if $S = a + b\tau$ then $I(\lambda) = a + \dfrac{b}{k_\lambda}$ with k_λ the opacity. A jump in opacity leads to the jump in intensity of the opposite sign.

3.1.2 Radiation equations (bis)

Material from [**TheoryStellarAtmospheres2014**]

$$\frac{\delta I(q, t)}{\delta s} = \eta(q, t) - \chi(q, t)I(q, t) \tag{5}$$

In cartesian coordinates (with propagation vector $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}$):

$$\frac{1}{c}\frac{\partial I}{\partial t} + \sin(\theta)\cos(\phi)\frac{\partial I}{\partial x} + \sin(\theta)\sin(\phi)\frac{\partial I}{\partial y} + \cos(\theta)\frac{\partial I}{\partial z} = \eta - \chi I \tag{6}$$

- 1D planar atmosphere: $\dfrac{\partial I}{\partial x} = \dfrac{\partial I}{\partial y} = 0$:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial z} = \eta - \chi I \tag{7}$$

- diffusion limit

3.1.3 RHD equations

The full RHD equations consist of

- five partial differential equations

- one HD closure equation, e.g. (i) variable Eddington tensor method or (ii) flux limited diffusion

Heat flux The heat flow rate density $\vec{\phi}$ satisfies the Fourier law $\vec{\phi} = -k\nabla T$. More information can be found for instance on [**WikiHeat**].

Specific intensity and its angular moments

specific intensity	$\Delta\epsilon = \boxed{I_\nu} A_1 A_2 / r^2 \Delta\nu\Delta t$
energy density	$E = \dfrac{1}{c}\displaystyle\iint I_\nu d\nu d\Omega$
flux vector	$F = \displaystyle\iint I_\nu n\, d\nu d\Omega$
pressure tensor	$P = \displaystyle\iint I_\nu nn\, d\nu d\Omega$
mean intensity	$J_\nu = \dfrac{c}{4\pi} E_\nu$
Eddington flux	$H_\nu = \dfrac{1}{4\pi} F_\nu$
Eddington's K	$K_\nu = \dfrac{c}{4\pi} P_\nu$

Eddington factor In general, the Eddington factor is a tensor, for 1D systems it is reduced to a scalar.

$$f_\nu = \frac{K_\nu}{J_\nu} = \frac{P_\nu}{E_\nu} \tag{8}$$

- isotropic radiation field
- radiation field stronly peaked in radial (i.e. vertical in cartesian) direction

3.1.4   Radiation transport equations, diffusion, equilibrium

- black body radiation (Planck function $I_\nu = J_\nu = B_\nu$)
- in general, extinction(absorption,scattering) and emission

$$\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu \tag{9}$$

  - Cartesian coordinates:

$$\boxed{\frac{\partial I_{n,\nu}}{\partial t}\frac{1}{c} + n\nabla I_{n,\nu} = j_\nu - k_{n,\nu} I_{n,\nu}} \tag{10}$$

  - spherical coordinates
  - 1D-problem with only variation along z-axis $\mu\dfrac{dI}{dz} = j - kI$
  - spherical symmetry $\mu\dfrac{\partial I}{\partial r} + \dfrac{1-\mu^2}{r}\dfrac{\partial I}{\partial \mu} = j - kI$
  - plane-parallel approximation

$$\boxed{\mu\frac{dI}{dr} = j - kI} \tag{11}$$

  The angle μ is constant throughout the computational domain. Dividing by k_ν, this yields

$$\mu\frac{dI}{k_\nu dr} = \mu\frac{dI}{k_\nu dz} = S - I \tag{12}$$

- 0th moment equation: integrate Equation (10) over ν and Ω, i.e. $\displaystyle\int d\nu d\Omega$. Conservation of energy

- first multiply Equation (10) with $\dfrac{n}{c}$ and then do integration

3.1.5 Radiative Diffusion Approximation

The radiative diffusion approximation bridges two regimes: regimes with ...

- on one hand, large optical depth $\tau \gg 1$: diffusion equation: temperature structure in a static stellar atmosphere

- on the other hand, where radiative *transport* is important

The diffusive approximation is the following: replace $\boxed{I = B}$ or $I_\nu = B_\nu$.

$$I_\nu = B_\nu - \mu \frac{dB_\nu}{k_\nu dz} \tag{13}$$

This equation can be derived as a random walk of photons!

3.1.6 Applications and approximations for radiative forces

- definition of general radiative acceleration vector $g = \dfrac{1}{\rho c} \displaystyle\int \int n k_\nu I_\nu d\Omega d\nu$

  - continuum Thomson scattering
  - spectral line with extinction
    * furhtermore assume central continuum source
    * then $g_{line} = \dfrac{F_\nu^0 k_L}{\rho c}$

- Sobolev approximation

- CAK theory

3.1.7 Optical depth (recap)

Optical depth: physical understanding Optical depth is the ratio of incident radiant power to transmitted radiant power ([**WikiOpticalDepth**]).

optical depth	optical depth along ray	line optical depth	Sobolev optical depth
$d\tau = k_\nu ds = \sigma_{nu} n ds = \kappa\rho ds$ $\tau_\nu = \int k_\nu ds = \int \sigma_\nu n ds$	$\tau_{\mu,\nu} = \int_z^{z_{max}} \frac{\alpha_{nu}(z')}{\mu} dz' = \frac{\tau_\nu(z)}{\mu}$	$\tau_\nu = \int k_L \phi_\nu dl = \int \kappa\rho ds$	

with

- σ cross-section

- n number density

- κ mass absorption density

- ρ mass density

- k_ν extinction coefficient

3.2 Overview of symmetry assumptions

plane-parallel	1D atmosphere	
	bounded by horizontal surfaces	

3.3 Overview of units

opacity $\alpha = k_\nu$	$\left[\dfrac{m^2}{kg}\right]$
specific intensity I_ν	$\left[\dfrac{ergs}{cm^2.sr.Hz.s}\right] = \left[\dfrac{J}{cm^2.sr.Hz.s}\right]$
optical depth τ	$\boxed{\tau = 0}$ leave atmosphere

3.4 Things to know

- expanding flow: redshift (lower frequency)

- compressing flow: blueshift (higher frequency)

3.5 Definition of specific intensity

The definition of the specific intensity is

$$I_\nu = \frac{dE_\nu}{\cos(\theta)d\Omega dt d\nu} = \frac{dE_\nu}{\mu d\Omega dt d\nu} \tag{14}$$

On the other hand, for the total energy of a collection of N photons holds that

$$E_\nu = N E_{\nu,\text{photon}} \tag{15}$$

To the point From this we deduce that

$$I_\nu \mu = \frac{N(\mu)dE_{\nu,\text{photon}}}{d\Omega dt d\nu} \tag{16}$$

and thus

$$\boxed{I_{nu}\mu d\mu \sim N(\mu)d\mu} \tag{17}$$

Considering the solid angle In spherical geometry $d\Omega = \sin(\theta)d\theta d\phi = d\mu d\phi$.

4 Introduction: course material (Sundqvist - CMPAA course)

4.1 EXERCISES: Introduction to numerical methods for radiation in astrophysics

1. introduction

2. radiation quantities

   - exercise p.3:
     - on one hand, we know that $\Delta\epsilon \sim C/r^2$
     - on the other hand, from the definition we know that $\Delta\epsilon = I_\nu A_1 A_2 / r^2 \Delta\nu\Delta t$
     - combining these equations shows that I_ν is independent from r
   - exercise p.4:
     -
   - exercise 1:
     - $F_x = \int_0^\pi \left[I_\nu(\theta)\sin^2(\theta) \int_0^{2\pi} \cos(\phi) \right] d\theta d\phi = 0$
     - the same reasoning for $F_y = 0$
   - exercise 2:
     - the equation follows from $d\mu = d\cos(\theta) = \sin(\theta)d\theta$
   - exercise 3:
     - isotropic radiation field (i.e. $I(\mu) = I$) then we have $F_\nu = 2\pi \int_{-1}^1 I\mu d\mu = 2\pi I \left.\frac{x^2}{2}\right|_{-1}^1 = 0$
   - exercise 4:
     - $F_\nu = 2\pi \int_{-1}^1 I(\mu)\mu d\mu = 2\pi \int_{-1}^0 I_\nu^- \mu d\mu + 2\pi \int_0^1 I_\nu^+ \mu d\mu = 2\pi I_\nu^+$
   - exercise p.7:
     - isotropic radiation field:
       * although the radiation pressure is a tensor, we will denote it as a scalar $P_\nu = \dfrac{4\pi I_\nu}{c}$
       * the radiation energy density $E_\nu = \dfrac{12\pi I_\nu}{c}$
       * thus $f_\nu = \dfrac{1}{3}$
     - very strongly peaked in radial direction (beam): $I_\nu = I_0\delta(\mu - \mu_0)$ with $\mu_0 = 1$
       * pressure tensor $P_{nu} = \dfrac{1}{c}\int I_0\delta(\mu - \mu_0)nnd\Omega$
       * energy density $E_\nu = \dfrac{1}{c}\int I_\nu d\Omega$
       * in this case $P_\nu = E_\nu$ thus $f_\nu = 1$

3. radiation transport vs. diffusion vs. equilibrium

   - exercise p. 12: 1D, Cartesian geometry, plane-parallel, frequency-independent and isotropic emission/extinction
     - radiation energy equation
       * The equation follows by integrating Equation (11)
       * By definition, $E = \dfrac{1}{c}\iint I_\nu d\nu d\Omega$
       * thus $\dfrac{dE}{dr} = \int (j - kI)d\nu d\Omega$ thus $\boxed{\dfrac{dE}{dr} = \dfrac{(j - kI)4\pi(\nu_1 - \nu_0)}{c}}$

* work out the integral taking into account frequency-independent and isotropic coefficients:

– zeroth momentum equations

* One must also take into account the specific form of the flux vector

$$F = \iint I_\nu n d\nu d\Omega = 2\pi \int_{-1}^{1} I_\nu(\mu)\mu d\mu$$

* thus $\dfrac{dF}{dr} = \dfrac{1}{c}\int (j - kI)n d\nu d\Omega$ thus $\boxed{\dfrac{dF}{dr} = \dfrac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$

– first moment equation

* similar reasoning

* $\dfrac{dP}{dr} = \int (j - kI)n.n d\nu d\Omega$ thus $\boxed{\dfrac{dF}{dr} = \dfrac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$

• first exercise p. 15

– $P = \dfrac{1}{c}\iint I_\nu \mu^2 d\Omega d\nu = \dfrac{2\pi}{c}\int_\nu \int_{-1}^{1} I_\nu \mu^2 d\mu d\nu = \dfrac{4\pi}{3c}\int B_\nu d\nu = \dfrac{aT^4}{3} = \dfrac{E}{3}$

• second exercise p.15

– assuming the diffusion limit,

– flux-weighted mean opacity $\kappa_F = \dfrac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}$

– Rosseland mean opacity $\dfrac{1}{\kappa_R} = \dfrac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$.

* in the diffusion limit, $F_\nu = -\dfrac{4\pi}{3}\dfrac{dB_\nu}{k_\nu dz}$ thus $\dfrac{dB_{nu}}{dT} =$

*

• third exercise p.15

4. the equations of radiation-hydrodynamics

5. numerical techniques for the radiative diffusion approximation

6. applications and approximations for a dynamically important radiative force in supersonic flows

• exercise p.27: $L_{SOB} = \Delta r = \dfrac{v_{th}}{dv/dr} = \dfrac{10[km/s]}{1000[km/s]/R_*} = 0.01R_*$

7. Appendix A: properties of equilibrium black-body radiation

• exercise p. 29

– this should be satisfied: $B_\nu d\nu = -B_\lambda d\lambda$ and also $\nu = \dfrac{c}{\lambda}$

– this is equivalent to saying that $0 = \nu d\lambda + \lambda d\nu$ or $d\lambda = -\dfrac{\lambda}{\nu}d\nu$ thus $B_\lambda = \dfrac{\nu}{\lambda}B_\nu$

– $B_\lambda(T) = \dfrac{\nu}{\lambda}\dfrac{2h\nu^3}{(\lambda\nu)^2}\dfrac{1}{e^{hc/\lambda kT} - 1} = \dfrac{2h\nu^2}{\lambda^3}\dfrac{1}{e^{hc/\lambda kT} - 1} = \dfrac{2hc^2}{\lambda^5}\dfrac{1}{e^{hc/\lambda kT} - 1}$

• first exercise p.31

– derive that $\lambda_{max}T = 2897.8[\mu mK]$

– ...

• second exercise p.31

– this is about the spectra of (unknown) stars

• first exercise p.32

– see exercise 7

• second exercise p.32

- BB radiation: $I_\nu = \dfrac{2h\nu^3}{c^2} \dfrac{1}{e^{h\nu/kt} - 1}$
- the radiative flux for isotropic BB radiation is zero. See also exercise 3. This dus also holds for BB radiation.

- exercise p. 33

  - <u>HR-diagram</u>

8. Appendix B: Simple examples to the radiative transfer equation

- first exercise p. 34

  - start from radiative transport equation $\mu\dfrac{dI}{ds} = \alpha - \eta I$ in which $\eta = 0$ thus $\boxed{\mu\dfrac{dI}{ds} = \alpha}$
  - solving the ODE in the general case that $\alpha(s)$ is not constant:
    * integrate the equation $\mu I = \displaystyle\int_0^D \alpha\, ds$
    * ...
  - second exercise p. 34
    * case $\tau(D) >> 1$: then $I(D) \approx S$
    * case $\tau(D) << 1$: then $I(D) \approx I(0) + S(1-1) = I(0)$
  - first exercise p.35
    * is the plane-parallel approximation valid for the solar photosphere?
  - second exercise p.35
    * goal: find a solution to the equation $\mu\dfrac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$ where $I(\tau, \mu)$
    * solution

- second exercise p.35

9. Appendix C: connecting random walk of photons with radiative diffusion model

- exercise p. 38. Computing the average photon mean-free path inside the Sun.
  $l = \dfrac{1}{\kappa\rho} = \dfrac{V_o}{\kappa M_o}[cm]$
- exercise p.39. Computing the random-walk time (diffusion time) for photons

4.2   Implicit 1D solver (20-11-2018)

See computer code

4.3   ADI 2D Solver

See computer code

4.4   Area of a circle

See computer code

4.5   Limb Darkening

See Section **??**.

5 The mathematics of Radiative Transfer

The material in this section is based on the book [**Busbridge**].

5.1 Auxiliary mathematics

- $\cos(\Theta) = cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$

- phase function $\boxed{p(\mu,\phi,\mu',\phi',\tau) = \sum_{n=0}^{N} \omega_n P_n(\cos(\Theta))}$

  - isotropic scattering $p(\tau) = \omega_0(\tau)$

- equation of transfer $\boxed{\mu\dfrac{\partial I(\tau,\mu,\phi)}{\partial\tau} = I(\tau,\mu,\phi) - \mathcal{S}(\tau,\mu,\phi)}$

  with $\mathcal{S}(\tau,\mu,\phi) = B_1(\tau) + \dfrac{1}{4\pi}\displaystyle\int_{-1}^{1} d\mu' \int_{0}^{2\pi} I(\tau,\mu',\phi')p(\mu,\phi,\mu',\phi')d\phi'$

  - axially symmetric with isotropic scattering
    $$\mathcal{S}(\tau) = \frac{\omega_0(\tau)}{2}\int_{-1}^{1} I(\tau,\mu')d\mu' = B_1(\tau) + \frac{\omega_0(\tau)}{2}\int_{0}^{\tau_1} \mathcal{S}(t)E_1(|t-\tau|)dt$$
  - the Milne equation of the problem $(1 - \omega_0\bar{\Lambda})\{ mahtcalS(t)\} = B(\tau)$
    * solve for $\mathcal{S}(t)$
    * then find $I(\tau,\mu)$

5.2 The H-functions

- characteristic equation

6   Monte Carlo and Radiative Transfer (Puls)

6.1   basic definitions and facts

6.2   about random numbers

6.3   MC integration

6.4   MC simulation

Radiative transfer in stellar atmospheres

- GOAL: spatial radiation energy density $E(\tau)$ in an atmospheric layer
    - only photon-electron scattering
    - τ is the optical depth

- Milne's integral equation $\boxed{E(\tau) = \frac{1}{2} \int_0^\infty E(t) E_1(|t - \tau|) dt}$

    - analytical solution $\dfrac{E(\tau)}{E(0)} = \sqrt{3}(\tau + q(\tau))$
    - MC simulation
        * emission angle
        * optical depth until next scattering event
        * scattering angle

- HOW DOES THIS WORK?

Algorithm 1 Limb darkening: compute quantitiy of photons
---
create photons

probability distribution for emission angle $\mu = \cos(\theta)$: $\boxed{p(\mu)d\mu = \mu d\mu}$

optical depth until next scattering event: $\boxed{p(\tau)dt \approx e^{-\tau} d\tau}$

isotropic scattering angle at low energies: $\boxed{p(\mu)d\mu \approx d\mu}$

follow all photons until they leave the atmosphere or are scattered back into stellar interior

---

6.5   Exercise 1: RNG

6.6   Exercise 2: Planck-function

1. analytical method
2. MC method

6.7   limb darkening

See section .

7 Introduction to Monte Carlo Radiation Transfer (Wood+)

The material is taken from

- (Wood, Wittney, Bjorkman, Wolff - 2001)

- (Wood, Wittney, Bjorkman, Wolff - 2013)

7.1 Elementary principles

specific intensity	I_ν
radiant energy	dE_ν
surface area	dA
angle	θ
solid angle	$d\Omega$
frequency range	$d\nu$
time	dt
flux	F_ν
cross section	σ
scattering angle	χ
	$\mu = \cos(\chi)$
mean intensity	J
flux	H
radiation pressure	K

intensity	$I_\nu(l) = I_\nu(0)e^{n\sigma l}$
angular phase function of the scattering particle	$P(\cos(\chi))$

inverse method	$\xi = \int_0^{x_0} P(x)dx$ with $\xi \in \mathcal{U}(0,1)$
rejection method	

7.2 Eddington factors

7.3 Example: plane parallel atmosphere

1. emission of photons: select two angles (3D space). In isotropic scattering

   - θ met $\mu = \cos(\theta)$
     - $\mu = 2\xi - 1$ (isotropic scattering)
     - $\mu = \sqrt{\xi}$ (A slab is heated from below. Then $P(\mu) = \mu$)
   - $\phi = 2\pi\xi$

2. propagation of photons

   - sample optical depth from $\tau = -\log(\xi)$
   - distance travelled $L = \dfrac{\tau z_{max}}{\tau_{max}}$

3. conclusion of emission and propagation

$$\begin{aligned}
x &= x + L\sin(\theta)\cos(\phi) \\
y &= y + L\sin(\theta)\sin(\phi) \\
z &= z + L\cos(\theta)
\end{aligned} \tag{18}$$

4. Binning: once the photon exists the slab. Produce histograms of the distribution function. Finally, we wish to compute the output flux or the intensity.

I have seen that a newer version of the paper is available, which was also used in these notes (which contains amongst other up-to-date references to code fragments).

A Plane Parallel, Isotropic Scattering Monte Carlo Code

7.4   Monte Carlo Radiative Transfer

From a macroscopic perspective, RT calculations rest on the transfer equation

- emissivity η (how much energy is added to radiation field due to emission)

- opacity χ (how much energy is removed due to absorption)

- the source function $S = \dfrac{\eta}{\chi}$

- optical depth τ captures the opaqueness of a medium

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla.n\right) I = \eta - \chi I \tag{19}$$

$$d\epsilon = I d\nu dt d\Omega dA.n \tag{20}$$

7.5 P Cygni profile for beta-velocity law and given opacity Monte Carlo simulation

7.5.1 Structure of the code

- module common

- module my_inter

- program pcyg

    - INPUT xk0, alpha, beta
    - OUTPUT
    - PROGRAM FLOW: loop over all photons
        * get xstart and vstart
        *
    - then do normalisation

- function func(r)

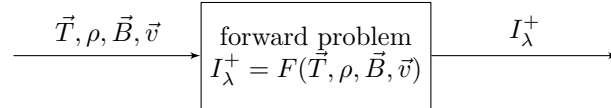- function xmueout(xk0,alpha,r,v,sigma)

- function rtbis(func,x1,x2,xacc)

8 Challenges in Radiative Transfer (Ivan Milic)

8.1 Overview of the problem

$$\xrightarrow{\quad I_\lambda^* \quad} \boxed{T(\tau) \,,\, \rho(\tau) \,,\, \vec{B}(\tau) \,,\, \vec{v}(\tau)} \xrightarrow{\quad I_\lambda^+ \quad}$$
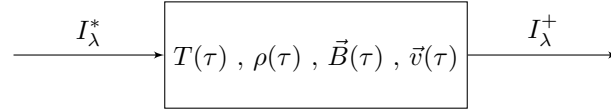
Forward problem

The forward problem is schematically represented

$$\xrightarrow{\quad \vec{T}, \rho, \vec{B}, \vec{v} \quad} \boxed{\begin{array}{c} \text{forward problem} \\ I_\lambda^+ = F(\vec{T}, \rho, \vec{B}, \vec{v}) \end{array}} \xrightarrow{\quad I_\lambda^+ \quad}$$

In fact solve for intensity vector $\vec{I} = \begin{pmatrix} I \\ Q \\ \alpha \\ V \end{pmatrix}$ obeying the equation

$$\frac{d\vec{I}}{d\tau} = -X(\vec{T}, \rho, \vec{B}, \vec{v})\vec{I} - \vec{j}(\vec{T}, \rho, \vec{B}, \vec{v}) \tag{21}$$
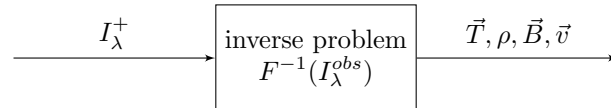
and the solution

$$I_\lambda^+ = I_0^+ e^{-\int} + \int \vec{j} e^{-\int} d\tau \tag{22}$$

> **Example** Source function $S = a\tau + b$ then $\displaystyle\int_0^{\tau_{max}} (a\tau + b)e^{-\tau}d\tau = ...$

Inverse problem

The inverse problem is schematically represented

$$\xrightarrow{\quad I_\lambda^+ \quad} \boxed{\begin{array}{c} \text{inverse problem} \\ F^{-1}(I_\lambda^{obs}) \end{array}} \xrightarrow{\quad \vec{T}, \rho, \vec{B}, \vec{v} \quad}$$

Via least-squares approximation

$$\min_{\vec{T}, \rho, \vec{B}, \vec{v}} \sum \left(I_\lambda^{obs} - I_\lambda(\vec{T}, \rho, \vec{B}, \vec{v}) \right)^2 \tag{23}$$

8.2 Challenging domains of application

- Lyman alpha in Galaxy Halos
- Dusty torii (AGD)
- protoplanetary disks
- circumstellar disks
- athmospheres

9 Asymptotic Preserving Monte Carlo methods for transport equations in the diffusive limit (Dimarco+2018)

10 Fluid and hybrid Fluid-Kinetic models (for neutral particles in plasma edge) (Horsten2019)

The material is mainly taken from [**HorstenNiels2019**].

- Kinetic Boltzmann equation: neutral velocity distribution $f_n(r, v)$

- If you taken into account (e.g. microscopic processes for atomic deuterium) then the kinetic Boltzmann equation becomes

$$v\nabla f_n(r, v) = S_r(r, v) + S_{cx}(r, v) - f_n(r, v)(R_{cx}(r, v) + R_i(r)) \tag{24}$$

- Numerical solution strategies

  - finite differences/volumes/elements :computationally infeasible
  - spectral methods (series expansion of $f_n(r, v)$: not suitable for modelling discontinuties
  - stochatic approach: the whole velocity distribution is discretized by finite set of particles

- from Equation (24), the fluid model and the hybrid model is derived.

  - Fluid model: 3 state equations (continuity - momentum - energy) with boundary conditions
    * pure-pressure equation: maximum error of 10 - 28 %
    * with parallel momentum source: error 10 %
    * with ion energy source: error 30 %
  - hybrid model based on micro-macro decomposition

11 plitting methods

From notes by professor Frank.

11.1 Exercises

11.1.1 Exercise 1

12 Overview of existing (Monte Carlo) radiative transfer codes

12.1 Synthesis codes

As is pointed out in [**PinteChristophe2015CRT**], there are basically two methods to solve the radiative transfer problem: ray-tracing and Monte Carlo methods.

- RADICAL [**RADICAL**] (Ray-tracing, 2D, multi-purpose)

- MULTI [**MULTI**] [**Carlsson1986**] (computer program for solving multi-level non-LTE radiative transfer problems in moving or static atmospheres, very old: Uppsala 1986 - 1995)

- SKIRT [**Camps2015**] (continuum (Monte Carlo) radiation transfer in dusty astrophysical systems, such as spiral galaxies and accretion disks, from Ugent)

- TORUS [**Harries2019**] (Monte Carlo radiation transfer and hydrodynamics code. Adopts 1D, 2D, 3D adaptive mesh refinement. Suitable for radiative equilibrium and creation of synthetic images and SED)

- RADMC-3D [**RADMC3D**] (Monte Carlo code that is especially applicable for dusty molecular clouds, protoplanetary disks, circumstellar envelopes, dusty tori around AGN and models of galaxies. Python interface with Fortran main code)

- TLUSTY and SYNSPEC [**Hubeny2017**], [**Hubeny2017a**], [**Hubeny2017b**].

12.2 Inversion codes

- VFISV

- ASP/HAO

- HeLIx+

- SNAPI (not publicly available, created by Ivan Milic)

- multiple codes available from Instituto de Astrofyiica de Canarias (IAC)

- STiC: the Stockholm inve rsion code

13   Integral equations

Based on the book [**Mmfp**].

1. integral equation from differential equation

2. types of integral equations

3. operator notation and existence of solutions

4. closed-form solutions

   - separable kernels
   - integral transform method (Fourier transform)
   - differentiation

5. Neumann series

6. Fredholm theory

7. Schmidt-Hilbert theory

Fredholm equation first kind

$$0 = f + \lambda \mathcal{K} y \tag{25}$$

Fredholm equation second kind

$$y = f + \lambda \mathcal{K} y \tag{26}$$

14 Master thesis discussions

limb darkening

- the integro-differential equation describing radiative transfer

$$
\begin{aligned}
\frac{dI}{d\tau} &= -I + S \\
&= -I + \frac{1}{4\pi} \int I d\Omega
\end{aligned}
\tag{27}
$$

- The difficulty resides in the source function

- Monte Carlo simulation avoids explicit source function: source function implicit in Monte Carlo simulation

- in the Monte Carlo program, the physics are simulated IN BETWEEN TWO CONSECUTIVE SCATTERING EVENTS as follows

$$
\frac{dI}{dz} = -\alpha I
\tag{28}
$$

thus $\dfrac{dI}{I} = -\alpha dz = -\delta\tau$ and $I = I_0 e^{-\delta\tau}$ and thus τ is sampled according to $\tau = -\log(X_{\text{random}})$