1 Overview of exercises (PART I)

1. limb-darkening scattering exercise we did during the course. — You can look into your notes from that, and I attach here also a sample program which you can use a base. After you have familiarised yourself with this, you can start to think bout how you would go about to extend this to a 3D setting (assuming isotropic scattering).

2. (As prep for Monte-Carlo school) here is a script computing a UV resonance P-Cygni line in spherically symmetric wind with v beta-law. At top of routine, a few exercises are given, where you can modify and play around with code. Monte-Carlo program which computes a UV resonance spectral line from a fast outflowing spherically symmetric stellar wind (if you were not cc'd on that email, let me know so that I can send you the files as well). At the top of that little script, there are a few suggestions for exercises (additions) you could do to that program, in order to learn a bit more about the general workings of Monte-Carlo radiative transfer in this context. — So that might be a good idea for you to do as well ! (And you can also ask the others in the group for some tips etc. then.)

3. Some background reading:
   - Attached mc manual by Puls.
   - Paper by Sundqvist+ 2010 (Appendix, I think).

2 Overview of exercises (PART II)

1. Calculate the probability distribution to sample from in the case of Eddington limb darkening for the initial distribution (see Section 8.3).
   - finished + Ok
2. Calculate analytical solution for simplified problem in the case that `mu = 1` (see Section 8.1).
   - finished + Ok + can be further studied
3. Perform convergence analysis (see Section 8.5).

3 Overview of exercises (PART III)

1. Revisit 3D limb darkening. ϕ should be sampled between 0 and 2π (see Section 7.5). (OK)
2. Revisit convergence analysis: adapt plot formatting and standard deviation is defined as square root of variance (see Section 8.5).
3. Test variance reduction technique (see Section 8.6).
4. Some general considerations about the definitiion of specific intensity (see Section **??**). (OK)
5. For the Monte Carlo approximation of the diffusion equation, why do wo have $N \sim \tau$ for low optical depth $\tau \ll 1$ (see Section 11).
6. Revisit the radial streaming approximation in `pcyg.f90` for lower optical depth (e.g. `xk0=0.5`). (see Section 8.1).
7. What happens when you add a line (e.g. $x = 0.5 = a$)? How would you do that? (see Section 10.1)
8. Towards a mathematical description of the problem.

4 Overview of exercises (PART IV)

1. Convergence analysis: also fit a line through the points. Formally, we write $V = CN^x$ and determine both C and X from experimental data. Correspondingly, $\log(V) = \log(C) + x\log(N)$. This is fitted using least-squares (see Section 8.5).

2. Variance reduction technique

   - averaging over different stochastic realizations?
   - take `xk0=0.5`
   - try to also discretize μ

3. Adding a second line: develop computer code in the radial streaming assumption (use analytic formulas) $\mu = 1$ (see Section 10).

   - a following improvement is the use of a grid instead of using the bisection method.

4. Limb darkening. Have a look at section 7.3.1.

5 Multiline transfer

- What happens when you add a line (e.g. $x = 0.5 = a$)? How would you do that? (see section 10)

6 Introductory exercises

6.1 Analytical exercises

From course material from (prof. Sundqvist - CMPAA course).

1. introduction
2. radiation quantities
   - exercise p.3:
     - on one hand, we know that $\Delta\epsilon \sim C/r^2$
     - on the other hand, from the definition we know that $\Delta\epsilon = I_\nu A_1 A_2/r^2 \Delta\nu\Delta t$
     - combining these equations shows that I_ν is independent from r
   - exercise p.4:
     -
   - exercise 1:
     - $F_x = \int_0^\pi \left[I_\nu(\theta)\sin^2(\theta) \int_0^{2\pi} \cos(\phi) \right] d\theta d\phi = 0$
     - the same reasoning for $F_y = 0$
   - exercise 2:
     - the equation follows from $d\mu = d\cos(\theta) = \sin(\theta)d\theta$
   - exercise 3:
     - isotropic radiation field (i.e. $I(\mu) = I$) then we have $F_\nu = 2\pi \int_{-1}^{1} I\mu d\mu = 2\pi I \left. \frac{x^2}{2} \right|_{-1}^{1} = 0$
   - exercise 4:
     - $F_\nu = 2\pi \int_{-1}^{1} I(\mu)\mu d\mu = 2\pi \int_{-1}^{0} I_\nu^- \mu d\mu + 2\pi \int_0^1 I_\nu^+ \mu d\mu = 2\pi I_\nu^+$
   - exercise p.7:
     - isotropic radiation field:
       * although the radiation pressure is a tensor, we will denote it as a scalar $P_\nu = \frac{4\pi I_\nu}{c}$
       * the radiation energy density $E_\nu = \frac{12\pi I_\nu}{c}$
       * thus $f_\nu = \frac{1}{3}$
     - very strongly peaked in radial direction (beam): $I_\nu = I_0\delta(\mu - \mu_0)$ with $\mu_0 = 1$
       * pressure tensor $P_{nu} = \frac{1}{c}\int I_0\delta(\mu - \mu_0) nn d\Omega$
       * energy density $E_\nu = \frac{1}{c}\int I_\nu d\Omega$
       * in this case $P_\nu = E_\nu$ thus $f_\nu = 1$
3. radiation transport vs. diffusion vs. equilibrium
   - exercise p. 12: 1D, Cartesian geometry, plane-parallel, frequency-independent and isotropic emission/extinction
     - radiation energy equation
       * The equation follows by integrating Equation (**??**)
       * By definition, $E = \frac{1}{c}\iint I_\nu d\nu d\Omega$
       * thus $\frac{dE}{dr} = \int (j - kI) d\nu d\Omega$ thus $\boxed{\frac{dE}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)}{c}}$

    - work out the integral taking into account frequency-independent and isotropic coefficients:
  - zeroth momentum equations
    - One must also take into account the specific form of the flux vector $F = \iint I_\nu n d\nu d\Omega = 2\pi \int_{-1}^{1} I_\nu(\mu)\mu d\mu$
    - thus $\frac{dF}{dr} = \frac{1}{c}\int (j - kI) n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j-kI)4\pi(\nu_1 - \nu_0)n}{c}}$
  - first moment equation
    - similar reasoning
    - $\frac{dP}{dr} = \int (j - kI) n.n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j-kI)4\pi(\nu_1 - \nu_0)n}{c}}$
- first exercise p. 15
  - $P = \frac{1}{c}\iint I_\nu \mu^2 d\Omega d\nu = \frac{2\pi}{c}\int_\nu \int_{-1}^{1} I_\nu \mu^2 d\mu d\nu = \frac{4\pi}{3c}\int B_\nu d\nu = \frac{aT^4}{3} = \frac{E}{3}$
- second exercise p.15
  - assuming the diffusion limit,
  - flux-weighted mean opacity $\kappa_F = \frac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}$
  - Rosseland mean opacity $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$.
    - in the diffusion limit, $F_\nu = -\frac{4\pi}{3}\frac{dB_\nu}{k_\nu dz}$ thus $\frac{dB_{nu}}{dT} =$
    -
- third exercise p.15

4. the equations of radiation-hydrodynamics
5. numerical techniques for the radiative diffusion approximation
6. applications and approximations for a dynamically important radiative force in supersonic flows
   - exercise p.27: $L_{SOB} = \Delta r = \frac{v_{th}}{dv/dr} = \frac{10[km/s]}{1000[km/s]/R_*} = 0.01R_*$
7. Appendix A: properties of equilibrium black-body radiation
   - exercise p. 29
     - this should be satisfied: $B_\nu d\nu = -B_\lambda d\lambda$ and also $\nu = \frac{c}{\lambda}$
     - this is equivalent to saying that $0 = \nu d\lambda + \lambda d\nu$ or $d\lambda = -\frac{\lambda}{\nu}d\nu$ thus $B_\lambda = \frac{\nu}{\lambda}B_\nu$
     - $B_\lambda(T) = \frac{\nu}{\lambda}\frac{2h\nu^3}{(\lambda\nu)^2}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2h\nu^2}{\lambda^3}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$
   - first exercise p.31
     - derive that $\lambda_{max}T = 2897.8[\mu m K]$
     - ...
   - second exercise p.31
     - this is about the spectra of (unknown) stars
   - first exercise p.32
     - see exercise 7
   - second exercise p.32

  - BB radiation: $I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kt}-1}$
  - the radiative flux for isotropic BB radiation is zero. See also exercise 3. This dus also holds for BB radiation.
- exercise p. 33
  - HR-diagram

8. Appendix B: Simple examples to the radiative transfer equation

- first exercise p. 34
  - start from radiative transport equation $\mu\frac{dI}{ds} = \alpha - \eta I$ in which $\eta = 0$ thus $\boxed{\mu\frac{dI}{ds} = \alpha}$
  - solving the ODE in the general case that $\alpha(s)$ is not constant:
    * integrate the equation $\mu I = \int_0^D \alpha ds$
    * ...
  - second exercise p. 34
    * case $\tau(D) >> 1$: then $I(D) \approx S$
    * case $\tau(D) << 1$: then $I(D) \approx I(0) + S(1-1) = I(0)$
  - first exercise p.35
    * is the plane-parallel approximation valid for the solar photosphere?
  - second exercise p.35
    * goal: find a solution to the equation $\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$ where $I(\tau, \mu)$
    * solution
- second exercise p.35

9. Appendix C: connecting random walk of photons with radiative diffusion model

- exercise p. 38. Computing the average photon mean-free path inside the Sun. $l = \frac{1}{\kappa\rho} = \frac{V_o}{\kappa M_o}[cm]$
- exercise p.39. Computing the random-walk time (diffusion time) for photons

6.2 Numerical exercises

6.2.1 Implicit 1D solver

Exercise from (20-11-2018).

Goal Implement implicit solver for time-dependent diffusion equation

$$\partial_t u = \partial_{xx} u \tag{1}$$

Solution The convergence behaviour of the method is shown in Figure 15.

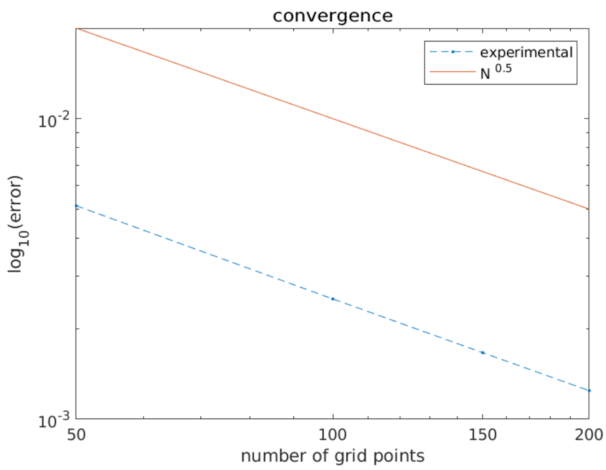


Figure 1: Convergence behaviour for 1D implicit solver (diffusion equation)

6.2.2 ADI 2D Solver

Goal Implement implicit solver for time-dependent diffusion equation

$$\partial_t u(t,x,y) = \partial_{xx} u(t,x,y) + \partial_{yy} u(t,x,y) \tag{2}$$

Solution There is still an error in the code.

6.2.3 Area of a circle

Goal Develop Monte Carlo code

Solution

6.3 Other Exercises

From course material from Ivan Milic.

6.3.1 Lecture 7

1. Derive expressions for the emergent radiation when properties are the following:
    - optically thin slab at all wavelengths
    - wavelength-independent incident radiation

    Solution: see slide 14?

2. Derive ralations between Einstein coefficients.

3. Calculate electron density in atmosphere from FALC model

7 Limb darkening

7.1 Formulation of the problem

- The radiative transfer equation **??** in this situtation becomes an integro-differential equation with $S(\tau) = \frac{1}{4\pi}\int I(\tau,\mu)d\Omega$

$$\begin{aligned}\mu\frac{dI(\tau,\mu)}{d\tau} &= -I(\tau,\mu) + S(\tau)\\ &= -I(\tau,\mu) + \frac{1}{4\pi}\int I(\tau,\mu)d\Omega\end{aligned} \tag{3}$$

- The difficulty resides in the (evaluation of) the source function. Monte Carlo simulation avoids explicit calculation source function: source function implicit in Monte Carlo simulation. There the physics are simulated IN BETWEEN TWO CONSECUTIVE SCATTERING EVENTS as follows

$$\frac{dI}{dz} = -\alpha I \tag{4}$$

thus $I = I_0 e^{-\delta\tau}$ and then τ is sampled according to $\tau = -\log(X_{\text{random}})$

7.2 Solving the (integro-differential) radiative transfer equation

Analytical Solution of Equation (3) Ik heb de mosterd gehaald op [**Dublin˙limb˙darkening**].

$$I(0,\mu) = \int_0^\infty S(\tau)exp\left(\frac{-\tau}{\mu}\right)d\left(\frac{\tau}{\mu}\right) \tag{5}$$

Numerical Solution of Equation (3) First rewrite the equation

$$\begin{aligned}\mu\frac{dI(\tau,\mu)}{d\tau} &= -I(\tau,\mu) + \frac{1}{4\pi}\int I(\tau,\mu)\sin(\theta)d\theta d\phi\\ &= -I(\tau,\mu) + \frac{1}{4\pi}\int I(\tau,\mu)d\mu d\phi\\ &= -I(\tau,\mu) + \frac{1}{2}\int I(\tau,\mu)d\mu\end{aligned} \tag{6}$$

Discretization scheme:

$$??? \tag{7}$$

7.3 Eddington-Barbier approximation

$$J(\tau) = 3H\left(\tau + \frac{2}{3}\right) \tag{8}$$

Together with the time-independent radiative transfer equation (**??**) in a gray (frequency-independent) planar medium gives

$$\mu\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - 3H\left(\frac{2}{3} + \tau\right) \tag{9}$$

The emergent intensity $I(0,\mu)$ is a solution of Equation (9). Its solution for $\tau = 0$ equals

$$I(\tau = 0,\mu) = I_1\left(\frac{2}{5} + \frac{3\mu}{5}\right) \tag{10}$$

with $a = \frac{\sigma}{2\pi}T_{eff}^4$ and $b = \frac{3\sigma}{4\pi}T_{eff}^4$

7.3.1 Validity of the Eddington-Barbier approximation

If we assume Equuation (8) then $I = I_1(a + b\mu)$ thus $J = \frac{1}{2}\int(\tau, \mu)d\mu = \frac{1}{2}\int_0^1(a + b\mu)d\mu$

dat ziet er hier niet goed uit

7.4 2D Case

We again have $\mu = \cos(\theta)$. The solution of the radiative transfer equation in plane-parallel syummetry with frequency-independent absorption and emission, is

$$I(\mu) = I_1(0.4 + 0.6\mu) \tag{11}$$

In the Monte Carlo code, the photons are sorted according to the direction that they leave the atmosphere.

Goal Calculates the angular dependence of photon's emitted from a plane-parallel, grey atmosphere of radial optical depth `taumax`. The value of `tau` determines the position of the photon

Variables and Algorithm

- `muarray` contains emergent photons
- `na` number of channels
- `dmu` = 1/`na` width of channels
- `nphot` number of photons
- `taumax` maximum optical depth

Algorithm 1 Limb darkening: compute quantitiy of photons

initialization
  radial optical depth τ
  direction μ
for all photons **do**
  $\boxed{\tau = \tau_{max}}$
  while `tau` ≥ 0 **do**
    compute scattering angle `mu`
    if tau $\geq$ taumax **then** $\boxed{mu = sqrt(x)}$ (initial distribution)
    else $\boxed{mu = 2 * x = 1}$ (isotropic scattering)
    tau_i = -log(x2)
    tau = tau - tau_i*mu
  end while
  now we know that the photon has left the photosphere
  compute the distribution of all angles `mu` at which the photon left the photosphere
end for
visualisation:

- plot photon numbers from $\mu d\mu$ against `mu`
- plot specific intensity from $d\mu$ against `mu` against

Figure 2 is according to what is expected $I = I_0(0.4 + 0.6\mu)$. The input parameters are as follows `Limb_Darkening(number_of_channels = 20, number_of_photons =` 10^5`,`
`maximum_optical_depth = 10)`.

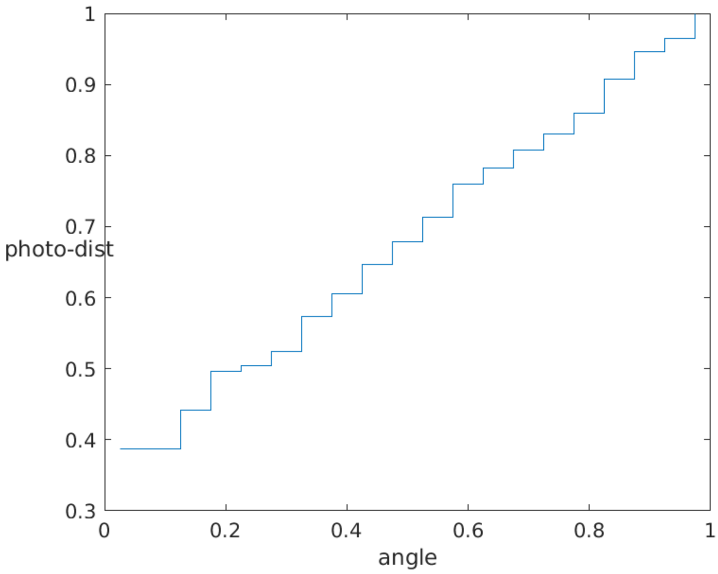


Figure 2: histogram for `mu`

7.5 3D Case

What changes is this:

- introduction of a new angle ϕ
- the optical depth is not updated with respect to ϕ

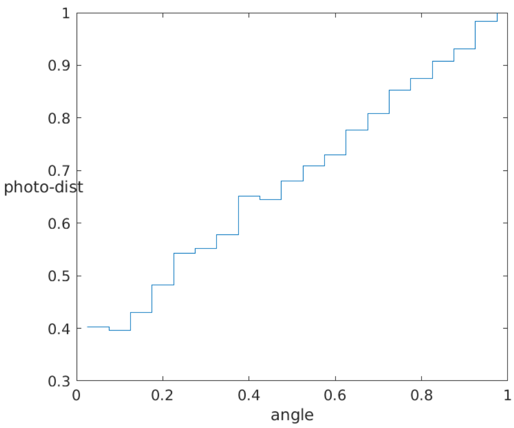


Figure 3: histogram for `mu`

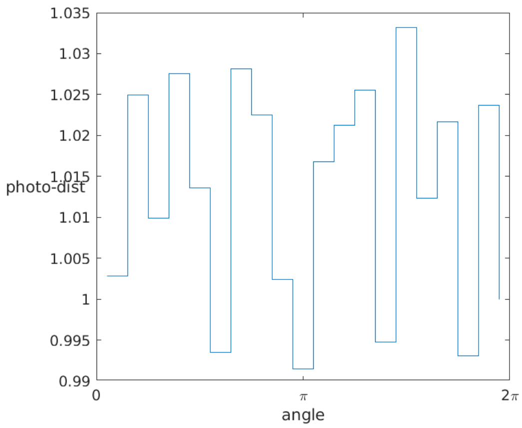


Figure 4: histogram for `phi`

Figure 3 and Figure 4 are the result of the function `Limb_Darkening_3D` with the following input parameters: `Limb_Darkening_3D(number_of_channels = 20, number_of_photons =` 10^5`, maximum_optical_depth = 10)`. The results according to what is expected, namely $I = I_0(0.4 + 0.6\mu)$ and ϕ follows a uniform distribution.

Extension: make version where the optical depth is updated with respect to ϕ

Via this link, you can go back to the exercises overview: Section 3.

8 Spectral line formation: pcyg.f90

This section is about the study of line formation in an expanding wind.

Background

Overview of variables

name	explanation
paramaters	
xk0	
alpha	velocity profile parameter
beta	velocity profile parameter
start frequency of the photon	
xstart	start frequency
vmin	
vmax	
angle of the photon	
xmuestart	start angle
xmuein	incident angle
xmueou	outward angle
pstart	impact parameter
xnew	new photon frequency
optical depth	
tau	optical depth
number of photons admin	
nphot	number of photons
nin	photons scattered back into core
nout	photons escaped
functions	
func	velocity profile distance from center of star r
xmueout	outwards (scattered) angle xk0 alpha r v sigma
nchan	amount of bins

$$\xrightarrow{\texttt{xstart}} \boxed{\texttt{pcyg.f90}} \xrightarrow{\texttt{xnew}}$$

The photons are sorted according to `xnew`. In general, the flux is dependent on μ and the frequency x.

Practical formula

- emission angle $\mu = \cos(\theta)$
- according p-ray $p = \sqrt{1-\mu^2} = \sin(\theta)$
- incident angle `xmuein` $= \sqrt{1 - \left(\frac{pstart}{r}\right)^2}$

Geometry & Symmetry assumptions

- spherical geometry

8.1 First adaptation: what if all photons are released radially from photosphere?

8.1.1 Release photons radially: numerical MC experiments

What would happen with line-profile, if you assumed all photons were released radially from photopshere?

- In other words `xmuestart` $= 1$.
- This is implemented under the test case `test_number=1`.
- Results in Figure 16 for opacity `xk0 = 100`.

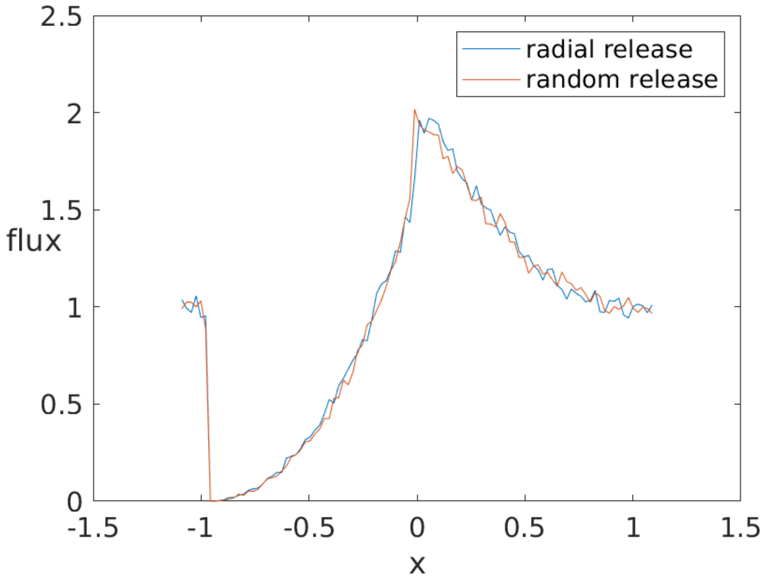


Figure 5: The number of photons equals 10^5, `xk0=100`

8.1.2 Derive analytic expression

See also slide 26/49 [Sundqvist course material].

- since `xmuein = 1` we have for the velocity profile

$$v = v_\infty (1 - b/r)^\beta \tag{12}$$

  A scaled version of Equation (12) yields

$$u = \frac{v(r)}{v_\infty} = \left(1 - \frac{r_\infty}{r}\right)^\beta \tag{13}$$

  with $u \in [0..1]$

- Doppler shift for the frequency of the photons: $x_{CMF} = x_{REF} - \mu u$.
- Condition for resonance from Sobolov approximation (to be studied later): $\boxed{x_{CMF} = 0}$ thus

$$x_{REF} = \mu u \tag{14}$$

  or thus $x_{REF} = \boxed{u_{\text{interaction}}}$ and than solve Equation 13 for $r_{\text{interaction}}$

- If $\mu = 1$ then

$$x = \left(1 - \frac{r_\infty}{r}\right)^\beta \tag{15}$$

$$x^{1/\beta} = 1 - \frac{r_\infty}{r}$$

$$r(1 - x^{1/\beta}) = r_\infty$$

$$\boxed{r(x) = \frac{r_\infty}{1 - x^{1/\beta}}} \tag{16}$$

attention, here was something wrong!

- From the location of interaction r, the incident angle can be calculated

$$\mathtt{xmuein} = \sqrt{1 - \left[\frac{\mathtt{pstart}}{r}\right]^2} = \sqrt{1 - \left[\frac{\sqrt{1 - \mathtt{xmuestart}^2}}{r}\right]^2} \tag{17}$$

Now also taking into account that `xmuestart = 1` then yields

$$\mathtt{xmuein\ =\ 1} \tag{18}$$

- The calculation of the optical depth goes as follows:

$$\tau = \frac{\mathtt{xk0}}{rv^{2-\alpha}(1 + \mathtt{xmuein}^2\sigma)} \tag{19}$$

Now also taking into account that `xmuestart = 1` gives

$$\tau = \frac{\mathtt{xk0}}{rv^2(1 + \sigma)} \tag{20}$$

where $\boxed{v(x) = \left(1 - \frac{b}{r}\right)^\beta}$ and $\frac{dv}{dr} = \frac{\beta b}{r^2}\left(1 - \frac{b}{r}\right)^{\beta - 1}$
and $\sigma(x) = \frac{dv}{dr}\frac{r}{v} - 1$ thus $\boxed{\sigma(x) = \frac{\beta b}{r}\left(1 - \frac{b}{r}\right)^{-1}}$

- Assuming that $\beta = 1$ then $\boxed{v(x) = 1 - \frac{b}{r}}$ and $\frac{dv}{dr} = \frac{\beta b}{r^2}$ and $\boxed{\sigma(x) = \frac{\beta b}{r}}$.
- Conclusion: $\tau(x)$ is only dependent on x and not on `xmuestart` or `xmuein`.
- `xmueou` follows the distribution as given by the function `xmueout`, namely

$$p(x) = \frac{1 - e^{-\tau}}{\tau} \tag{21}$$

with $\tau = \frac{\mathtt{tau0}}{1 + \mathtt{X}^2\sigma}$ where `X` is a random number, so actually this comes down to

$$\boxed{p(x) = \frac{1 - e^{-\frac{\tau_0}{1 + x^2\sigma(x)}}}{\frac{\tau_0}{1 + x^2\sigma(x)}}} \tag{22}$$

- Finally one can combine these results to get the distribution of the photons according to the frequency x via the relation

$$\texttt{xnew = xstart + v(xmueou-xmuein) = xstart + v(xmueou -1)} \tag{23}$$

In words, we initially have an isotropic distribution for `xstart`. The number of photons that are leaving the atmosphere at different frequencies is however not isotropic through complex interactions that are incorporated into $p(x)$. One must also take into account that not all of the photons that are released actually escape from the atmosphere and also that sometimes no resonance is possible, and then Equation (23) is not applicable.

TO DO: proceed from this to the analytical expression for the flux. Here I am stuck for the moment.

8.1.3 Experiments with other opacities

The results for `xk0=0.5` are shown in Figure 6.

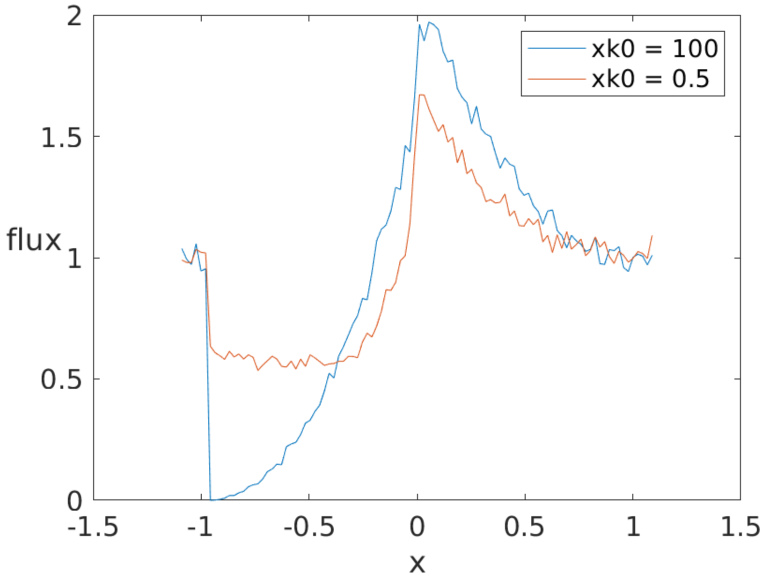


Figure 6: The number of photons equals 10^5, `xk0=0.5`

Via this link, you can go back to the exercises overview: Section 3.

8.2 Second adaptation: isotropic scattering

What would happen to line-profile, is you assumed scattering was isotropic (i.e., NOT following Sobolev-distrobution)

- in the implementation, `test_number = 2`
- the results are shown in Figure 7.

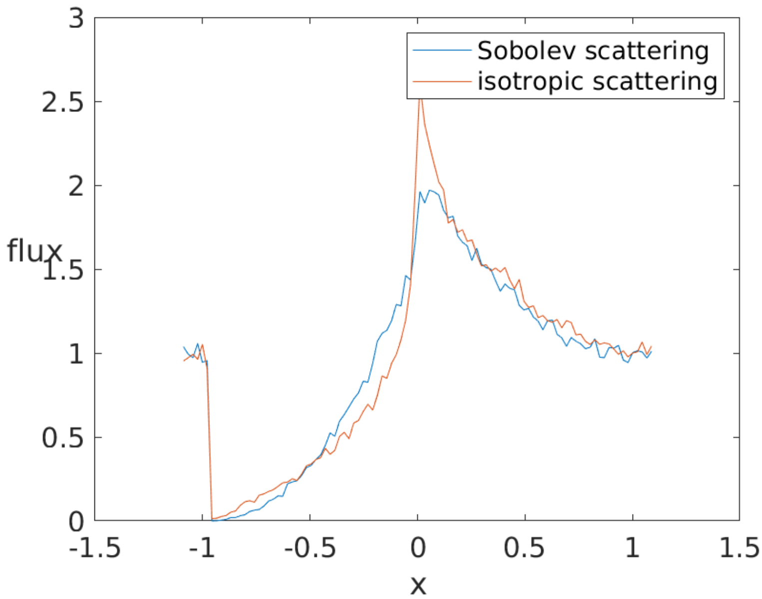


Figure 7: The number of photons equals 10^5

It is clear from Figure 7 that the peak around $x = 0$ is higher and sharper.

$\boxed{\text{Analyse this behaviour more closely}}$

8.3 Third adaptation: introduction of Eddington limb-darkening

Goal Put Eddington limb-darkening in. What happens?

8.3.1 Construction of probability distribution corresponding to Eddington limb darkening

For a general (introductory) discussion about Eddington limb darkening, please refer to Section **??**

1. Let us thus first review the emmission case where the flux in each direction is isotropic i.e. $I(\theta) = I$ (as experimented in paragraph 8.2)

   - the specific intensity is defined as $I_\nu(\mu) = \dfrac{dE_\nu}{\cos(\theta) dA dt d\nu d\Omega} = \dfrac{dE_\nu}{\mu dA dt d\nu d\Omega}$
   - the flux $F_\nu = \displaystyle\int_\Omega I_\nu \cos(\theta) d\Omega$ is in this case isotropic thus

   $$\xi = \int_0^\mu F_\nu d\mu = \int_0^\mu \int_\Omega I_\nu \cos(\theta) d\Omega d\mu = A \int_0^\mu \mu d\mu \tag{24}$$

   together with the condition that μ satisfies a probability distribution:

   $$1 = \int_{-1}^1 F_\nu d\mu = \int_{-1}^1 \int_\Omega I_\nu \cos(\theta) d\Omega d\mu = \frac{A}{2} \tag{25}$$

   thus $A = 2$. Photons need to be sampled according to $\mu d\mu$.

2. Now we look at a new case where the photons need to be emitted following a distrubution that corresponds to $I(\theta) = I(0)(0.4 + 0.6\cos(\theta))$.

   - in this case the flux $F_\nu = \displaystyle\int_\Omega I_\nu \cos(\theta) d\Omega$ is isotropic but also satisfies

   $$F_\nu = \int_\Omega I_\nu(0)[0.4 + 0.6\cos(\theta)] \cos(\theta) d\Omega \tag{26}$$

I am not sure about the correctness of the assumption of isotropy of the flux

   $$\xi = \int_0^\mu F_\nu d\mu = A \int_0^\mu (0.4 + 0.6\mu)\mu d\mu \tag{27}$$

   subject to the normalisation condition -very similar to Equation (25) - that

   $$1 = \int_0^1 F_\nu d\mu = \frac{2A}{5} \tag{28}$$

   thus $A = \dfrac{5}{2}$. Photons need to be sampled according to

   $$\frac{5}{2}(0.4 + 0.6\mu)\mu d\mu \tag{29}$$

   In the code `pcyg.f90` this corresponds to `test_number = 3` (not yet implemented).

   The results of an accept-reject method that samples the probability distribution in Equation (29).

Via this link, you can go back to the exercises overview: Section 2.

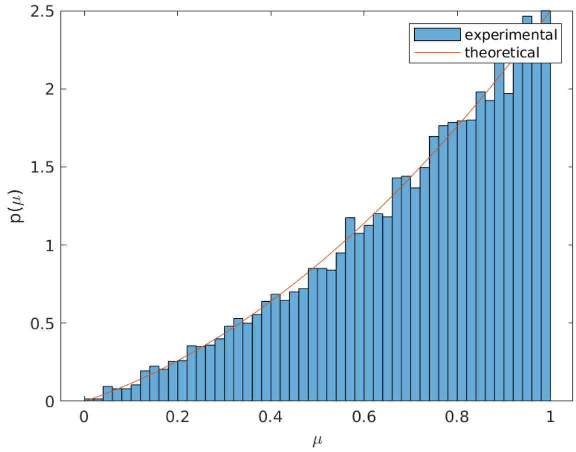


Figure 8: Accept-reject method for Eddington limb darkening

8.4 Fourth adaptaion: photospheric line-profile

Challening: Put photospheric line-profile (simple Gaussian) in. What happens? Test on `xk0=0` (opacity = 0) case.

- test case number 4
- This is still to be implemented.

8.5 Convergence analysis

Zero opacity The convergence of the Monte Carlo method is tested with the following input parameters

`kx0`	alpha	beta	`test_number`
0	0	1	0

for a varying amount of photons, as shown in Figure 9. We expect the method to have $\frac{1}{\sqrt{N}}$ convergence, where N is the number of photons. However, the methods strangely seems to have a faster convergence rate.

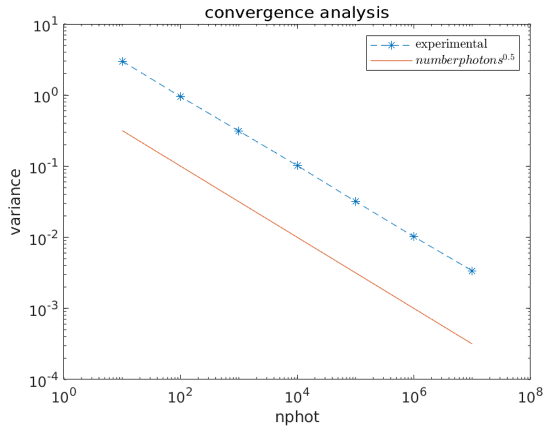


Figure 9: Original version of the code: convergence analysis (xk0=0)

Nonzero opacity The convergence test is set up as follows: different Monte Carlo simulations (with increasing number of photons) are compared to an *expensive* simulation with 10^7 photons. As can be seen in Figure 10, the spectrum profile behaves according to a $N^{0.5}$ law.

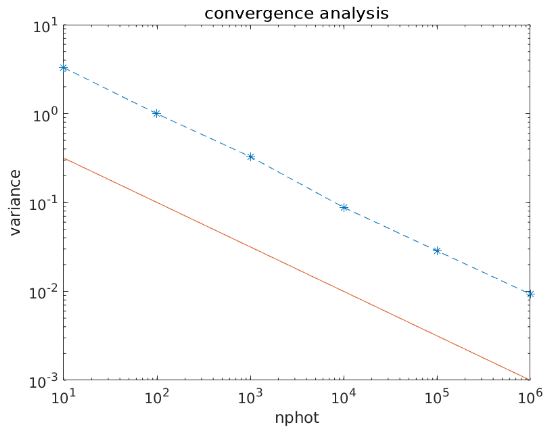


Figure 10: Original version of the code: convergence analysis (xk0=100)

Via this link, you can go back to the exercises overview: Section 2.

8.6 Variance reduction experiment

We will set up the test as follows

- run the code with `xk0=100` and number of photons $N = 10^7$
- run the code again for lower number of photons (e.g. $N = 10^3$), both with random sampling and pseudo-random sampling
- compute variance w.r.t. *expensive* simulation and compare
- `test_number = 5`

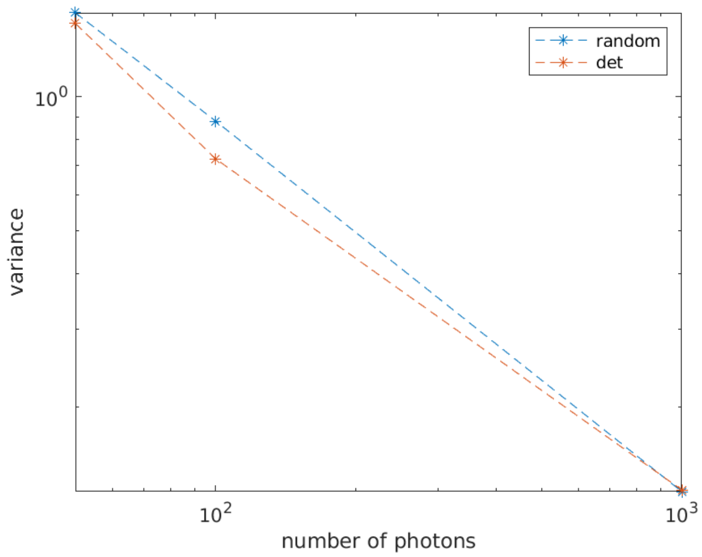


Figure 11: Original version of the code: convergence analysis (xk0=0)

`xk0=100` | Possible improvement: average over different stochastic realizations. |

Via this link, you can go back to the exercises overview: Section 3.

8.7 Mathematical description of the problem & Looking at literature

Have a look at [**NoebauerUlrichM˙2019MCRT**] (see Appendix).

9 Transferring the code to Matlab

9.1 Limit ariables

	`xmin`	`xmax`	`vmin`	`vmax`
Fortran	-1.1	1.1	0.01	0.98
Fortran (reverse order for scattering distribution)	-1.1	1.1	-0.98	-0.01
Matlab (with `resonance_x = 0`	-1	1	-0.8	0

9.2 Comparison

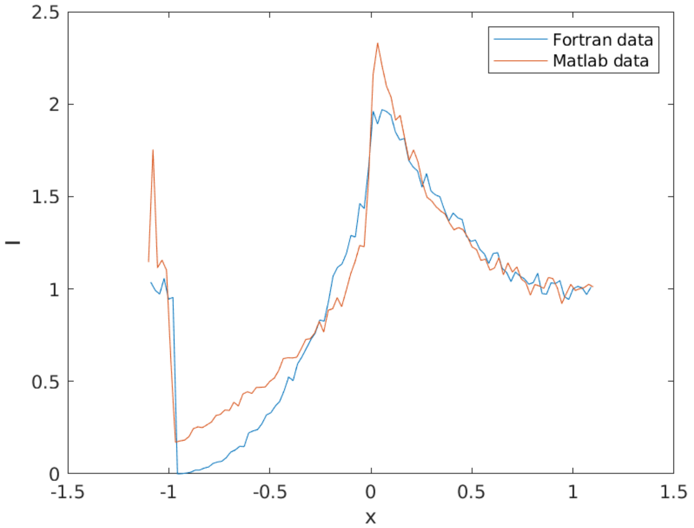


Figure 12: Original version of the code: convergence analysis (xk0=0)

10 Dual spectral line formation

10.1 Introduction of second line: theoretical

What happens when you add a line (e.g. $x = 0.5 = a$)? How would you do that?

10.1.1 Single line

Algorithm 2 `pcyg.f90: one resonance line`

for all photons **do**

1. Release photon with frequency x
2. Check if interaction is uberhaupt possible.
3. Solve for distance (radius r) of interaction using Sobolev approximation $x_{CMF} = x_{REL} - \mu v(r)$ with $\boxed{x_{CMF} = 0}$ and compute Sobolev optical depth
4. Check whether the photon is scattered:

   if $\tau_S > -log(\xi)$ **then**
   Interaction: the photon is scattered. Update the frequency
   else
   No interaction

4. update the frequency according to the scattering event

end for
collect photons and perform visualisation

10.1.2 Introduction of second line

Needs to be updated

10.1.3 Algorithm

1. release photon
2. $x_{CMF} = x_{REL} - muv$
3. scattering
4. interaction?
5. scattering
6. collect photons

Ranges and limits Assuming $\mu_{start} = 1$. Note that in the code $r_\infty = b$

10.2 Development of computer code

10.2.1 Implementation in Matlab: user's manual

Run the function `test_function(test_number)`.

`test_number`	parameter settings
0	original version
1	first adaptation: radial release
2	isotropic scattering – higher peak
3	Eddington limb darkening
4	photospheric line-profile
5	simple well
6	other resonance frequency (thus introducing shift)
7	formation of two lines, only radially streaming photons (thus also radial release
8	formation of two lines, with radial release
9	formation of two lines, full scattering possibilities

Via this link, you can go back to the exercises overview: Section 4.

10.2.2 Keeping track of the photon path

$$\begin{bmatrix} \texttt{xstart} \\ \texttt{xmuestart} \\ \texttt{r\_new} \\ \ldots \\ \texttt{xmueou} \end{bmatrix} \tag{30}$$

10.2.3 Dirty tricks

- when `xnew > xmax` then set `xnew` to `xmax`.

10.3 Random things

10.3.1 General things

$$\texttt{xnew = xstart + u(xmueou - xmuein)} \tag{31}$$

10.3.2 Specific things

Case when `xmuestart = 1`

- FIRST SCATTTERING: from Equation (31) we have that `xstart = u` and then

$$\begin{aligned} \texttt{xnew} &= \texttt{xstart + u(xmueou- 1)} \\ &= \texttt{xstart.xmueou} \end{aligned} \tag{32}$$

  Thus, since $\texttt{xmueou} \in [-1, 1]$ and $\texttt{xmin} \leq \texttt{xstart} \leq \texttt{xmax}$

$$\texttt{xmin} \leq \texttt{xnew} \leq \texttt{xmax} \tag{33}$$

  See also Section 10.4.5.

10.4 Experiments and results

10.4.1 About the scattering probability distribution

10.4.2 Visualisation of the photon path

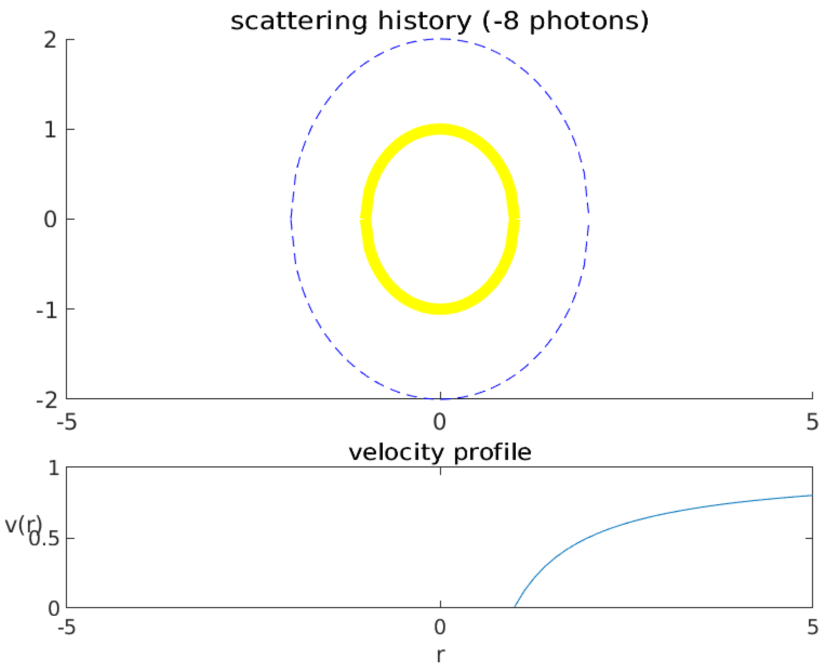


Figure 13: Convergence behaviour for 1D implicit solver (diffusion equation)

10.4.3 Is superposition valid?

From numerical experiments

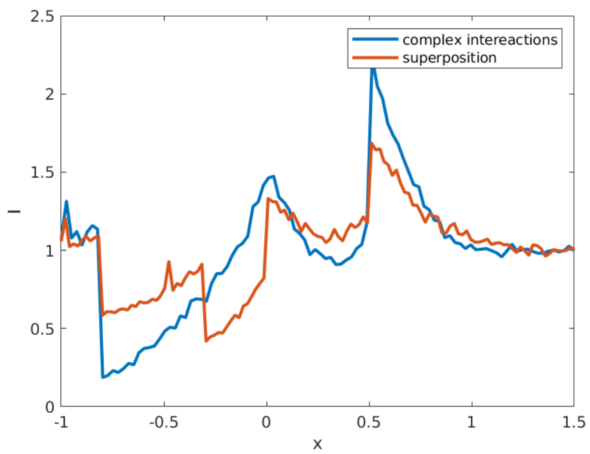


Figure 14: Convergence behaviour for 1D implicit solver (diffusion equation)

10.4.4 Sobolev escape angle from the resonance zone

10.4.5 Exerpiments with `xmuestart = 1`

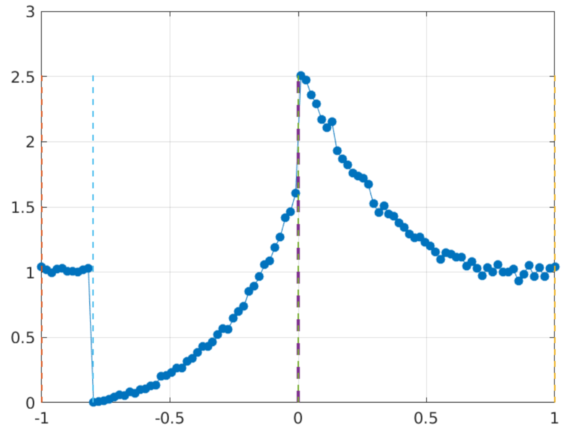
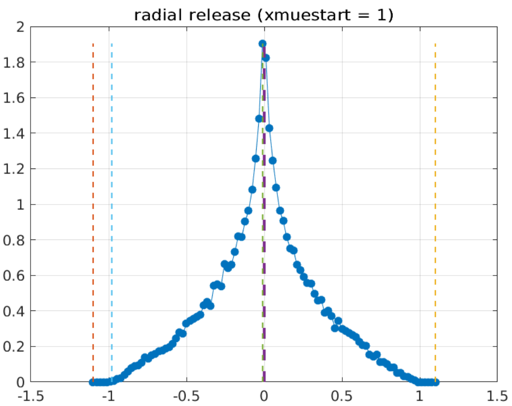
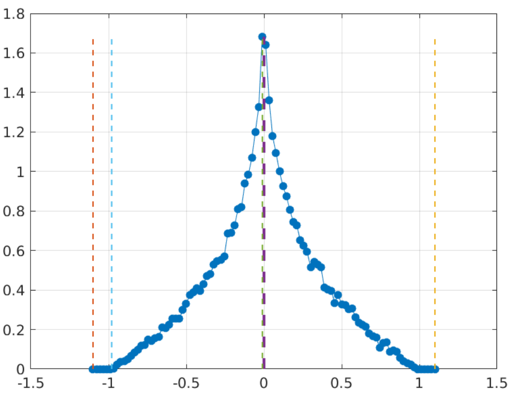


Figure 15: Convergence behaviour for 1D implicit solver (diffusion equation)

10.4.6 Scattering distribution



(a) Radial release



(b) Random release

Figure 16: Scattering distribution

11 Closer look at Monte Carlo simulations

11.1 Random walk (diffusion equation)

A more simple experiment that simulates the diffusion equation (1D random walk) is also set up. The results are shown in Figure 17. We observe that $N \sim \tau^2$, as can also be derived from theory.

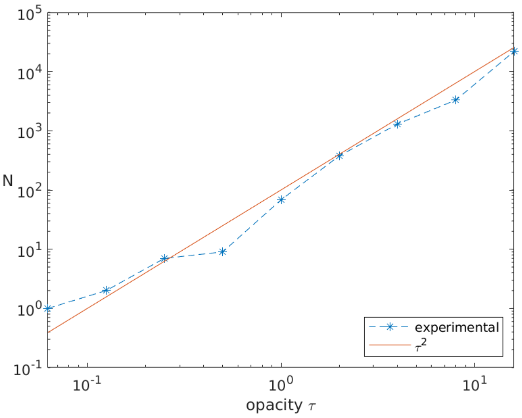


Figure 17: Number of interactions (scattering events) versus opacity, random walk

- When starting from an initial condition $x_0 = 0$ and

$$x_N = x_{N-1} \pm l \tag{34}$$

  we have for the variance that $\langle x_N \rangle^2 = Nl^2$

- If we require a photon to cover a distance R then $N = \frac{R^2}{l^2}$ and
  - the relation between mean-free path l and opacity α is $l = \frac{1}{\alpha}$
  - with $\tau = \int_0^R \alpha ds = \frac{R}{l}$

  then we have that $N = \tau^2$. This corresponds with the observations in Figure 17.

11.2 Iets

11.3 Limb darkening

We first look at results from the limb darkening program, as studied in Section 7. In Figure 18, the number of scattering events is plotted versus the opacity of the medium.

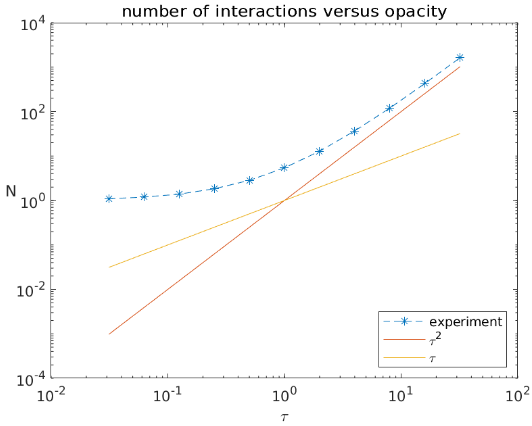


Figure 18: Number of interactions (scattering events) versus opacity, kimb darkening

- For high opacity $\tau \gg 1$ we observe that $N \sim \tau$.
- Bridging regime.
- For opacity $\tau \ll 1$ we observe that $N \sim 1$: namely the photons travels very far during the first emission event.

The splitting scheme from [**Dimarco2018**] can perfectly be applied to the used Monte Carlo code.

If you assume constant opacity then $\tau = \alpha z$