1 General equations - first year overview

This is made in May 2019.

1.0.1 Hydrodynamics

Euler equations, together with closing relation (e.g. ideal gas law).

primitive variables			
mass density	velocity	gas energy density	gas pressure
ρ	v	e	p

1.0.2 Radiation

Radiative transfer equation: intensity along a ray while interacting with medium. Photons are massless.

$$\left[\frac{1}{c}\partial_t + \vec{n}.\vec{\nabla}\right] I_\nu = \eta_\nu - \chi_\nu I_\nu \tag{1}$$

frequency	intensity	emissivity	total absorption
ν	I_ν	η_ν	χ_ν

These deliver two equations

- the radiative energy equation (diffusion flux $\vec{F}$

$$\frac{\partial E}{\partial t} + \vec{\nabla}.\vec{F} = \iint ...d\nu d\Omega \tag{2}$$

- radiative momentum equation

$$\frac{d\vec{F}}{\partial t} = \iint ...\vec{n} d\nu d\Omega \tag{3}$$

(after **integrating over all frequencies**). Depending on the geometry simplifcations, one can e.g. integrate over all solid angles.

1.0.3 Radiation-Hydrodynamics

Combination delivers integral-diffusion equation

$$\begin{aligned} \frac{dI}{d\tau} &= S - I \\ &= \int I d\Omega - I \end{aligned} \tag{4}$$

1.0.4 Challenges

- combination with hydrodynamics
- current analysis: simplified geometries (symmetry). E.g. in 2D, an ADI method is used and now also a multigrid method.
- complex geometry difficult to show in ray-tracing scheme
- steady-state vs. time dependent
- focus on radiation equations

2 Very broad introduction & Summary

The material here originates from the master thesis of Nicolas Moens [**MoensNicolas**] and from the course notes *Introduction to numerical methods for radiation in astrophsyics* from professor Sundqvist.

2.1 Definition of specific intensity

The definition of the specific intensity is

$$I_\nu = \frac{dE_\nu}{\cos(\theta) d\Omega dt d\nu} = \frac{dE_\nu}{\mu d\Omega dt d\nu} \tag{5}$$

On the other hand, for the total energy of a collection of N photons holds that

$$E_\nu = N E_{\nu,\text{photon}} \tag{6}$$

To the point From this we deduce that

$$I_\nu \mu = \frac{N(\mu) dE_{\nu,\text{photon}}}{d\Omega dt d\nu} \tag{7}$$

and thus

$$\boxed{I_{nu} \mu d\mu \sim N(\mu) d\mu} \tag{8}$$

Considering the solid angle In spherical geometry $d\Omega = \sin(\theta) d\theta d\phi = d\mu d\phi$.

2.2 Radiation equations

Material from [**TheoryStellarAtmospheres2014**]

Specific intensity $I(s, \lambda, x, y, t)$

$$\frac{\delta I(q,t)}{\delta s} = \eta(q,t) - \chi(q,t) I(q,t) \tag{9}$$

In cartesian coordinates (with propagation vector $\vec{n} = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}$):

$$\frac{1}{c}\frac{\partial I}{\partial t} + \sin(\theta)\cos(\phi)\frac{\partial I}{\partial x} + \sin(\theta)\sin(\phi)\frac{\partial I}{\partial y} + \cos(\theta)\frac{\partial I}{\partial z} = \eta - \chi I \tag{10}$$

- 1D planar atmosphere: $\frac{\partial I}{\partial x} = \frac{\partial I}{\partial y} = 0$:

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial z} = \eta - \chi I \tag{11}$$

- diffusion limit
- Definition of J in Equation (3.15)

Plane parallel geometry

- restrict oursevels to time-independent, one-dimensional (1D) case $I(s, \theta, \lambda)$ where s is the direction of the light ray
- it satisfies Radiation Transfer Equation (RTE) $\boxed{\frac{dI_\lambda}{d\tau_\lambda} = S_\lambda - I_\lambda}$

- with 'formal' solution $\boxed{I(\lambda,\tau_\lambda) = I_0(\lambda)e^{-\tau_\lambda}\int_0^{\tau_\lambda} S(t)e^{-t}dt}$
  - no emissivity $S = 0$ then $I(\lambda)I_0(\lambda)e^{-\tau_\lambda}$
  - no opacity then $I_0(\lambda) = \int_0^s \eta_\lambda(s)ds$
  - constant source function $I(\lambda,\tau) = I_0(\lambda)e^{-\tau_\lambda} + S(1-e^{-\tau_\lambda})$
  - if $S = a + b\tau$ then $I(\lambda) = a + \frac{b}{k_\lambda}$ with k_λ the opacity. A jump in opacity leads to the jump in intensity of the opposite sign.

Specific intensity and its angular moments

specific intensity	$\Delta\epsilon = \boxed{I_\nu} A_1A_2/r^2\Delta\nu\Delta t$
energy density	$E = \frac{1}{c}\iint I_\nu d\nu d\Omega$
flux vector	$F = \iint I_\nu n d\nu d\Omega$
pressure tensor	$P = \iint I_\nu nn d\nu d\Omega$
mean intensity	$J_\nu = \frac{c}{4\pi}E_\nu$
Eddington flux	$H_\nu = \frac{1}{4\pi}F_\nu$
Eddington's K	$K_\nu = \frac{c}{4\pi}P_\nu$

Eddington factor In general, the Eddington factor is a tensor, for 1D systems it is reduced to a scalar.

$$f_\nu = \frac{K_\nu}{J_\nu} = \frac{P_\nu}{E_\nu} \tag{12}$$

- isotropic radiation field
- radiation field stronly peaked in radial (i.e. vertical in cartesian) direction

2.3 Radiative Diffusion Approximation

The radiative diffusion approximation bridges two regimes: regimes with ...

- on one hand, large optical depth $\tau \gg 1$: diffusion equation: temperature structure in a static stellar atmosphere
- on the other hand, where radiative *transport* is important

The diffusive approximation is the following: replace $\boxed{I = B}$ or $I_\nu = B_\nu$.

$$I_\nu = B_\nu - \mu\frac{dB_\nu}{k_\nu dz} \tag{13}$$

This equation can be derived as a random walk of photons!

2.4 Applications and approximations for radiative forces

- definition of general radiative acceleration vector $g_{\mathrm{rad}} = \frac{1}{\rho c}\int\int nk_\nu I_\nu d\Omega d\nu$

2.5 RHD equations

The full RHD equations consist of

- five partial differential equations
- one HD closure equation, e.g. (i) variable Eddington tensor method or (ii) flux limited diffusion

Heat flux The heat flow rate density $\vec{\phi}$ satisfies the Fourier law $\vec{\phi} = -k\nabla T$. More information can be found for instance on [**WikiHeat**].

2.6 Overview of symmetry assumptions

plane-parallel	1D atmosphere bounded by horizontal surfaces	

2.7 Overview of units

opacity $\alpha = k_\nu$	$\left[\frac{m^2}{kg}\right]$
specific intensity I_ν	$\left[\frac{ergs}{cm^2.sr.Hz.s}\right] = \left[\frac{J}{cm^2.sr.Hz.s}\right]$
optical depth τ	$\boxed{\tau = 0}$ leave atmosphere

2.7.1 Things to know

- expanding flow: redshift (lower frequency)
- compressing flow: blueshift (higher frequency)

3 The mathematics of Radiative Transfer

The material in this section is based on the book [**Busbridge**].

3.1 Auxiliary mathematics

- $\cos(\Theta) = cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$
- phase function $\boxed{p(\mu, \phi, \mu', \phi', \tau) = \sum_{n=0}^{N} \omega_n P_n(\cos(\Theta))}$
  - isotropic scattering $p(\tau) = \omega_0(\tau)$
- equation of transfer $\boxed{\mu \frac{\partial I(\tau, \mu, \phi)}{\partial \tau} = I(\tau, \mu, \phi) - \mathcal{S}(\tau, \mu, \phi)}$

  with $\mathcal{S}(\tau, \mu, \phi) = B_1(\tau) + \frac{1}{4\pi} \int_{-1}^{1} d\mu' \int_0^{2\pi} I(\tau, \mu', \phi') p(\mu, \phi, \mu', \phi') d\phi'$
  - axially symmetric with isotropic scattering
    $\mathcal{S}(\tau) = \frac{\omega_0(\tau)}{2} \int_{-1}^{1} I(\tau, \mu') d\mu' = B_1(\tau) + \frac{\omega_0(\tau)}{2} \int_0^{\tau_1} \mathcal{S}(t) E_1(|t - \tau|) dt$
  - the Milne equation of the problem $(1 - \omega_0 \bar{\Lambda})\{$
    $mahtcalS(t)\} = B(\tau)$
    * solve for $\mathcal{S}(t)$
    * then find $I(\tau, \mu)$

3.2 The H-functions

- characteristic equation

3.3 Integral equations

Based on the book [**Mmfp**].

1. integral equation from differential equation
2. types of integral equations
3. operator notation and existence of solutions
4. closed-form solutions
   - separable kernels
   - integral transform method (Fourier transform)
   - differentiation
5. Neumann series
6. Fredholm theory
7. Schmidt-Hilbert theory

Fredholm equation first kind

$$0 = f + \lambda \mathcal{K} y \tag{14}$$

Fredholm equation second kind

$$y = f + \lambda \mathcal{K} y \tag{15}$$

4 Challenges in Radiative Transfer

The material here originates from an oral discussion with Ivan Milic.

4.1 Overview of the problem

$$\xrightarrow{\quad I_\lambda^* \quad} \boxed{T(\tau)\ ,\ \rho(\tau)\ ,\ \vec{B}(\tau)\ ,\ \vec{v}(\tau)} \xrightarrow{\quad I_\lambda^+ \quad}$$

Forward problem

The forward problem is schematically represented

$$\xrightarrow{\quad \vec{T}, \rho, \vec{B}, \vec{v} \quad} \boxed{\begin{array}{c}\text{forward problem}\\ I_\lambda^+ = F(\vec{T}, \rho, \vec{B}, \vec{v})\end{array}} \xrightarrow{\quad I_\lambda^+ \quad}$$

In fact solve for intensity vector $\vec{I} = \begin{pmatrix} I \\ Q \\ \alpha \\ V \end{pmatrix}$ obeying the equation

$$\frac{d\vec{I}}{d\tau} = -X(\vec{T}, \rho, \vec{B}, \vec{v})\vec{I} - \vec{j}(\vec{T}, \rho, \vec{B}, \vec{v}) \tag{16}$$

and the solution

$$I_\lambda^+ = I_0^+ e^{-\int} + \int \vec{j} e^{-\int} d\tau \tag{17}$$

Example Source function $S = a\tau + b$ then $\int_0^{\tau_{max}} (a\tau + b)e^{-\tau} d\tau = ...$

Inverse problem

The inverse problem is schematically represented

$$\xrightarrow{\quad I_\lambda^+ \quad} \boxed{\begin{array}{c}\text{inverse problem}\\ F^{-1}(I_\lambda^{obs})\end{array}} \xrightarrow{\quad \vec{T}, \rho, \vec{B}, \vec{v} \quad}$$

Via least-squares approximation

$$\min_{\vec{T}, \rho, \vec{B}, \vec{v}} \sum \left(I_\lambda^{obs} - I_\lambda(\vec{T}, \rho, \vec{B}, \vec{v}) \right)^2 \tag{18}$$

4.2 Challenging domains of application

- Lyman alpha in Galaxy Halos
- Dusty torii (AGD)
- protoplanetary disks
- circumstellar disks
- athmospheres

5 Glossary

- SED: spectral energy distribution
- (spectral) line-force: force on material in stellar atmosphere
- LASER: Light Amplification by Stimulated Emission of Radiation