Master thesis

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Contents

Part I
Questions

1 Questions for professor Sundqvist

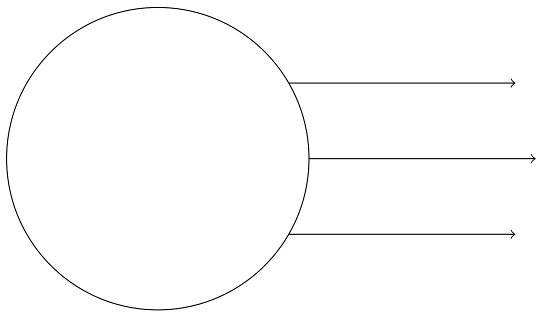
- pcyg.f90: for `test_number` = 2, why do we call it isotropic since isotropy of `mu` does not imply isotropy of `theta`?

2 Questions for professor Samaey

- what is the difference between Monte Carlo and equation-free computing?

3 Solved questions

- Sundqvist+ 2009: what is thermal velocity (see Wikipedia)
- Sundqvist+ 2009: what is line force (see explanation Dylan)
- unclassified: what is a flux limiter? (see course notes)
- unclassified: what is cross section of scattering (see Wikipedia)
- Puls manual: p.26: how does the Milne equation appear? (see library book)
- pcyg.f90: what are p-rays? (see anwser professor Sundqvist)
  - parallel rays leaving the atmosphere (of, e.g. a star)



- pcyg.f90: what is meant by Eddington limb-darkening? (see answer professor Sundqvist)
  - standard limb darkening
- Sundqvist+ 2009: what is the geometry of a *slice*?
- CMFAA course notes p.13 (the example) what is understood by plane-parallel geometry and is it 1D or 2D? (see answer professor Sundqvist)
  -
- CMFAA course notes p.15: why is this called diffusion $F = T^3 \frac{dT}{dx}$ (flux proportional to local gradient in temperature)?
- unclassified: what is the terminal velocity v_∞?
- unclassified: what is Sobo-distribution?

4 Interesting problems

- inverse radiative transfer problem

Part II
Monte Carlo Radiative Transfer

5 Glossary

- SED: spectral energy distribution
- (spectral) line-force: force on material in stellar atmosphere
- LASER: Light Amplification by Stimulated Emission of Radiation

6 Very broad introduction: Radiation Hydrodynamics

The material here originates from the master thesis of Nicolas Moens [Moe18] and from the course notes *Introduction to numerical methods for radiation in astrophsyics* from professor Sundqvist.

6.1 Definitions and equations

6.1.1 RHD equations

The full RHD equations consist of

- five partial differential equations
- one HD closure equation, e.g. (i) variable Eddington tensor method or (ii) flux limited diffusion

Heat flux The heat flow rate density $\vec{\phi}$ satisfies the Fourier law $\vec{\phi} = -k\nabla T$. More information can be found for instance on [Wik18].

Specific intensity and its angular moments

specific intensity	$\Delta\epsilon = \boxed{I_\nu} A_1 A_2 / r^2 \Delta\nu\Delta t$
energy density	$E = \frac{1}{c} \iint I_\nu d\nu d\Omega$
flux vector	$F = \iint I_\nu n d\nu d\Omega$
pressure tensor	$P = \iint I_\nu n n d\nu d\Omega$
mean intensity	$J_\nu = \frac{c}{4\pi} E_\nu$
Eddington flux	$H_\nu = \frac{1}{4\pi} F_\nu$
Eddington's K	$K_\nu = \frac{c}{4\pi} P_\nu$

Eddington factor In general, the Eddington factor is a tensor, for 1D systems it is reduced to a scalar.

$$f_\nu = \frac{K_\nu}{J_\nu} = \frac{P_\nu}{E_\nu} \tag{1}$$

- isotropic radiation field
- radiation field stronly peaked in radial (i.e. vertical in cartesian) direction

6.1.2 Radiation transport equations, diffusion, equilibrium

- black body radiation (Planck function $I_\nu = J_\nu = B_\nu$)
- in general, extinction(absorption,scattering) and emission

$$\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu \tag{2}$$

  - Cartesian coordinates:

$$\boxed{\frac{\partial I_{n,\nu}}{\partial t}\frac{1}{c} + n\nabla I_{n,\nu} = j_\nu - k_{n,\nu} I_{n,\nu}} \tag{3}$$

  - spherical coordinates
  - 1D-problem with only variation along z-axis $\mu\frac{dI}{dz} = j - kI$
  - spherical symmetry $\mu\frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = j - kI$
  - plane-parallel approximation

$$\boxed{\mu\frac{dI}{dr} = j - kI} \tag{4}$$

The angle μ is constant throughout the computational domain. Dividing by k_ν, this yields

$$\mu\frac{dI}{k_\nu dr} = \mu\frac{dI}{k_\nu dz} = S - I \tag{5}$$

- Oth moment equation: integrate Equation (3) over ν and Ω, i.e. $\int d\nu d\Omega$. Conservation of energy
- first multiply Equation (3) with $\frac{n}{c}$ and then do integration

6.1.3 Radiative Diffusion Approximation

The radiative diffusion approximation bridges two regimes: regimes with ...

- on one hand, large optical depth $\tau \gg 1$: diffusion equation: temperature structure in a static stellar atmosphere
- on the other hand, where radiative *transport* is important

The diffusive approximation is the following: replace $\boxed{I = B}$ or $I_\nu = B_\nu$.

$$I_\nu = B_\nu - \mu\frac{dB_\nu}{k_\nu dz} \tag{6}$$

This equation can be derived as a random walk of photons!

6.1.4 Applications and approximations for radiative forces

- definition of general radiative acceleration vector $g = \frac{1}{\rho c}\int\int n k_\nu I_\nu d\Omega d\nu$
  - continuum Thomson scattering
  - spectral line with extinction
    * furhtermore assume central continuum source
    * then $g_{line} = \frac{F_\nu^0 k_L}{\rho c}$
- Sobolev approximation
- CAK theory

6.1.5 Optical depth (recap)

Optical depth: physical understanding Optical depth is the ratio of incident radiant power to transmitted radiant power ([**WikiOpticalDepth**]).

optical depth	optical depth along ray	line optical depth	Sobolev optical depth
$d\tau = k_\nu ds = \sigma_{nu} nds = \kappa\rho ds$ $\tau_\nu = \int k_\nu ds = \int \sigma_\nu nds$	$\tau_{\mu,\nu} = \int_z^{z_{max}} \frac{\alpha_{nu}(z')}{\mu} dz' = \frac{\tau_\nu(z)}{\mu}$	$\tau_\nu = \int k_L \phi_\nu dl = \int \kappa\rho ds$	

with

- σ cross-section
- n number density
- κ mass absorption density
- ρ mass density
- k_ν extinction coefficient

6.2 Overview of symmetry assumptions

plane-parallel	1D atmosphere bounded by horizontal surfaces	

7 General equations - first year overview

7.0.1 Hydrodynamics

Euler equations, together with closing relation (e.g. ideal gas law).

primitive variables			
mass density	velocity	gas energy density	gas pressure
ρ	v	e	p

7.0.2 Radiation

Radiative transfer equation: intensity along a ray while interacting with medium. Photons are massless.

$$\left[\frac{1}{c}\partial_t + \vec{n}.\vec{\nabla}\right] I_\nu = \eta_\nu - \chi_\nu I_\nu \tag{7}$$

frequency	intensity	emissivity	total absorption
ν	I_ν	η_ν	χ_ν

These deliver two equations

- the radiative energy equation (diffusion flux $\vec{F}$

$$\frac{\partial E}{\partial t} + \vec{\nabla}.\vec{F} = \iint ...d\nu d\Omega \tag{8}$$

- radiative momentum equation

$$\frac{d\vec{F}}{\partial t} = \iint ...\vec{n}d\nu d\Omega \tag{9}$$

(after **integrating over all frequencies**). Depending on the geometry simplifcations, one can e.g. integrate over all solid angles.

7.0.3 Radiation-Hydrodynamics

Combination delivers integral-diffusion equation

$$\begin{aligned}\frac{dI}{d\tau} &= S - I \\ &= \int I d\Omega - I\end{aligned} \tag{10}$$

7.0.4 Challenges

- combination with hydrodynamics
- current analysis: simplified geometries (symmetry). E.g. in 2D, an ADI method is used and now also a multigrid method.
- complex geometry difficult to show in ray-tracing scheme
- steady-state vs. time dependent
- focus on radiation equations

8 Introduction: course material (Sundqvist - CMPAA course)

8.1 EXERCISES: Introduction to numerical methods for radiation in astrophysics

1. introduction

2. radiation quantities

   - exercise p.3:
     - on one hand, we know that $\Delta\epsilon \sim C/r^2$
     - on the other hand, from the definition we know that $\Delta\epsilon = I_\nu A_1 A_2/r^2 \Delta\nu\Delta t$
     - combining these equations shows that I_ν is independent from r
   - exercise p.4:
     -
   - exercise 1:
     - $F_x = \int_0^\pi \left[I_\nu(\theta)\sin^2(\theta) \int_0^{2\pi} \cos(\phi) \right] d\theta d\phi = 0$
     - the same reasoning for $F_y = 0$
   - exercise 2:
     - the equation follows from $d\mu = d\cos(\theta) = \sin(\theta)d\theta$
   - exercise 3:
     - isotropic radiation field (i.e. $I(\mu) = I$) then we have $F_\nu = 2\pi \int_{-1}^{1} I\mu d\mu = 2\pi I \left.\frac{x^2}{2}\right|_{-1}^{1} = 0$
   - exercise 4:
     - $F_\nu = 2\pi \int_{-1}^{1} I(\mu)\mu d\mu = 2\pi \int_{-1}^{0} I_\nu^- \mu d\mu + 2\pi \int_0^1 I_\nu^+ \mu d\mu = 2\pi I_\nu^+$
   - exercise p.7:
     - isotropic radiation field:
       * although the radiation pressure is a tensor, we will denote it as a scalar $P_\nu = \frac{4\pi I_\nu}{c}$
       * the radiation energy density $E_\nu = \frac{12\pi I_\nu}{c}$
       * thus $f_\nu = \frac{1}{3}$
     - very strongly peaked in radial direction (beam): $I_\nu = I_0\delta(\mu - \mu_0)$ with $\mu_0 = 1$
       * pressure tensor $P_{nu} = \frac{1}{c}\int I_0\delta(\mu - \mu_0) nn d\Omega$
       * energy density $E_\nu = \frac{1}{c}\int I_\nu d\Omega$
       * in this case $P_\nu = E_\nu$ thus $f_\nu = 1$

3. radiation transport vs. diffusion vs. equilibrium

   - exercise p. 12: 1D, Cartesian geometry, plane-parallel, frequency-independent and isotropic emission/extinction
     - radiation energy equation
       * The equation follows by integrating Equation (4)
       * By definition, $E = \frac{1}{c}\iint I_\nu d\nu d\Omega$
       * thus $\frac{dE}{dr} = \int (j - kI) d\nu d\Omega$ thus $\boxed{\frac{dE}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)}{c}}$

    * work out the integral taking into account frequency-independent and isotropic coefficients:
  - zeroth momentum equations
    * One must also take into account the specific form of the flux vector $F = \iint I_\nu n d\nu d\Omega = 2\pi \int_{-1}^{1} I_\nu(\mu)\mu d\mu$
    * thus $\frac{dF}{dr} = \frac{1}{c}\int (j - kI) n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
  - first moment equation
    * similar reasoning
    * $\frac{dP}{dr} = \int (j - kI) n.n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
- first exercise p. 15
  - $P = \frac{1}{c}\iint I_\nu \mu^2 d\Omega d\nu = \frac{2\pi}{c}\int_\nu \int_{-1}^{1} I_\nu \mu^2 d\mu d\nu = \frac{4\pi}{3c}\int B_\nu d\nu = \frac{aT^4}{3} = \frac{E}{3}$
- second exercise p.15
  - assuming the diffusion limit,
  - flux-weighted mean opacity $\kappa_F = \frac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}$
  - Rosseland mean opacity $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$.
    * in the diffusion limit, $F_\nu = -\frac{4\pi}{3}\frac{dB_\nu}{k_\nu dz}$ thus $\frac{dB_{nu}}{dT} =$
    *
- third exercise p.15

4. the equations of radiation-hydrodynamics
5. numerical techniques for the radiative diffusion approximation
6. applications and approximations for a dynamically important radiative force in supersonic flows
   - exercise p.27: $L_{SOB} = \Delta r = \frac{v_{th}}{dv/dr} = \frac{10[km/s]}{1000[km/s]/R_*} = 0.01R_*$
7. Appendix A: properties of equilibrium black-body radiation
   - exercise p. 29
     - this should be satisfied: $B_\nu d\nu = -B_\lambda d\lambda$ and also $\nu = \frac{c}{\lambda}$
     - this is equivalent to saying that $0 = \nu d\lambda + \lambda d\nu$ or $d\lambda = -\frac{\lambda}{\nu}d\nu$ thus $B_\lambda = \frac{\nu}{\lambda}B_\nu$
     - $B_\lambda(T) = \frac{\nu}{\lambda}\frac{2h\nu^3}{(\lambda\nu)^2}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2h\nu^2}{\lambda^3}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$
   - first exercise p.31
     - derive that $\lambda_{max}T = 2897.8[\mu mK]$
     - ...
   - second exercise p.31
     - this is about the spectra of (unknown) stars
   - first exercise p.32
     - see exercise 7
   - second exercise p.32

  - BB radiation: $I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kt}-1}$
  - the radiative flux for isotropic BB radiation is zero. See also exercise 3. This dus also holds for BB radiation.
- exercise p. 33
  - HR-diagram

8. Appendix B: Simple examples to the radiative transfer equation

- first exercise p. 34
  - start from radiative transport equation $\mu\frac{dI}{ds} = \alpha - \eta I$ in which $\eta = 0$ thus $\boxed{\mu\frac{dI}{ds} = \alpha}$
  - solving the ODE in the general case that $\alpha(s)$ is not constant:
    * integrate the equation $\mu I = \int_0^D \alpha ds$
    * ...
  - second exercise p. 34
    * case $\tau(D) >> 1$: then $I(D) \approx S$
    * case $\tau(D) << 1$: then $I(D) \approx I(0) + S(1-1) = I(0)$
  - first exercise p.35
    * is the plane-parallel approximation valid for the solar photosphere?
  - second exercise p.35
    * goal: find a solution to the equation $\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$ where $I(\tau,\mu)$
    * solution
- second exercise p.35

9. Appendix C: connecting random walk of photons with radiative diffusion model

- exercise p. 38. Computing the average photon mean-free path inside the Sun. $l = \frac{1}{\kappa\rho} = \frac{V_o}{\kappa M_o}[cm]$
- exercise p.39. Computing the random-walk time (diffusion time) for photons

8.2 Implicit 1D solver (20-11-2018)

See computer code

8.3 ADI 2D Solver

See computer code

8.4 Area of a circle

See computer code

8.5 Limb Darkening

See Section 17.

9 The mathematics of Radiative Transfer

The material in this section is based on the book [Bus60].

9.1 Auxiliary mathematics

- $\cos(\Theta) = cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$
- phase function $\boxed{p(\mu, \phi, \mu', \phi', \tau) = \sum_{n=0}^{N} \omega_n P_n(\cos(\Theta))}$
  - isotropic scattering $p(\tau) = \omega_0(\tau)$
- equation of transfer $\boxed{\mu \frac{\partial I(\tau, \mu, \phi)}{\partial \tau} = I(\tau, \mu, \phi) - \mathcal{S}(\tau, \mu, \phi)}$
  with $\mathcal{S}(\tau, \mu, \phi) = B_1(\tau) + \frac{1}{4\pi} \int_{-1}^{1} d\mu' \int_0^{2\pi} I(\tau, \mu', \phi') p(\mu, \phi, \mu', \phi') d\phi'$
  - axially symmetric with isotropic scattering
    $\mathcal{S}(\tau) = \frac{\omega_0(\tau)}{2} \int_{-1}^{1} I(\tau, \mu') d\mu' = B_1(\tau) + \frac{\omega_0(\tau)}{2} \int_0^{\tau_1} \mathcal{S}(t) E_1(|t - \tau|) dt$
  - the Milne equation of the problem $(1 - \omega_0 \bar{\Lambda})\{$
    $mahtcal S(t)\} = B(\tau)$
    * solve for $\mathcal{S}(t)$
    * then find $I(\tau, \mu)$

9.2 The H-functions

- characteristic equation

10 Monte Carlo and Radiative Transfer (Puls)

10.1 basic definitions and facts

10.2 about random numbers

10.3 MC integration

10.4 MC simulation

Radiative transfer in stellar atmospheres

- GOAL: spatial radiation energy density $E(\tau)$ in an atmospheric layer
  - only photon-electron scattering
  - τ is the optical depth
- Milne's integral equation $\boxed{E(\tau) = \frac{1}{2}\int_0^\infty E(t)E_1(|t-\tau|)dt}$
  - analytical solution $\frac{E(\tau)}{E(0)} = \sqrt{3}(\tau + q(\tau))$
  - MC simulation
    - * emission angle
    - * optical depth until next scattering event
    - * scattering angle
- HOW DOES THIS WORK?

Algorithm 1 Limb darkening: compute quantitiy of photons

create photons
probability distribution for emission angle $\mu = \cos(\theta)$: $\boxed{p(\mu)d\mu = \mu d\mu}$
optical depth until next scattering event: $\boxed{p(\tau)dt \approx e^{-\tau}d\tau}$
isotropic scattering angle at low energies: $\boxed{p(\mu)d\mu \approx d\mu}$
follow all photons until they leave the atmosphere or are scattered back into stellar interior

10.5 Exercise 1: RNG

10.6 Exercise 2: Planck-function

1. analytical method
2. MC method

10.7 limb darkening

See section 17.

11 Introduction to Monte Carlo Radiation Transfer (Wood+)

The material is taken from

- (Wood, Wittney, Bjorkman, Wolff - 2001)
- (Wood, Wittney, Bjorkman, Wolff - 2013)

11.1 Elementary principles

specific intensity	I_ν
radiant energy	dE_ν
surface area	dA
angle	θ
solid angle	$d\Omega$
frequency range	$d\nu$
time	dt
flux	F_ν
cross section	σ
scattering angle	χ $\mu = \cos(\chi)$
mean intensity	J
flux	H
radiation pressure	K

intensity	$I_\nu(l) = I_\nu(0)e^{n\sigma l}$
angular phase function of the scattering particle	$P(\cos(\chi))$

inverse method	$\xi = \int_0^{x_0} P(x)dx$ with $\xi \in \mathcal{U}(0,1)$
rejection method	

11.2 Eddington factors

11.3 Example: plane parallel atmosphere

1. emission of photons: select two angles (3D space). In isotropic scattering
   - θ met $\mu = \cos(\theta)$
     - $\mu = 2\xi - 1$ (isotropic scattering)
     - $\mu = \sqrt{\xi}$ (A slab is heated from below. Then $P(\mu) = \mu$)
   - $\phi = 2\pi\xi$
2. propagation of photons
   - sample optical depth from $\tau = -\log(\xi)$
   - distance travelled $L = \frac{\tau z_{max}}{\tau_{max}}$

3. conclusion of emission and propagation

$$\begin{aligned} x &= x + L\sin(\theta)\cos(\phi) \\ y &= y + L\sin(\theta)\sin(\phi) \\ z &= z + L\cos(\theta) \end{aligned} \tag{11}$$

4. Binning: once the photon exists the slab. Produce histograms of the distribution function. Finally, we wish to compute the output flux or the intensity.

I have seen that a newer version of the paper is available, which was also used in these notes (which contains amongst other up-to-date references to code fragments).

A Plane Parallel, Isotropic Scattering Monte Carlo Code

11.4 Monte Carlo Radiative Transfer

From a macroscopic perspective, RT calculations rest on the transfer equation

- emissivity η (how much energy is added to radiation field due to emission)
- opacity χ (how much energy is removed due to absorption)
- the source function $S = \frac{\eta}{\chi}$
- optical depth τ captures the opaqueness of a medium

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla.n\right) I = \eta - \chi I \tag{12}$$

$$d\epsilon = I d\nu dt d\Omega dA.n \tag{13}$$

11.5 P Cygni profile for beta-velocity law and given opacity Monte Carlo simulation

11.5.1 Structure of the code

- module common
- module my_inter
- program pcyg
  - INPUT xk0, alpha, beta
  - OUTPUT
  - PROGRAM FLOW: loop over all photons
    - get xstart and vstart
    -
  - then do normalisation
- function func(r)
- function xmueout(xk0,alpha,r,v,sigma)
- function rtbis(func,x1,x2,xacc)

12 Challenges in Radiative Transfer (Ivan Milic)

12.1 Overview of the problem

$$\xrightarrow{I^*_\lambda} \boxed{T(\tau)\ ,\ \rho(\tau)\ ,\ \vec{B}(\tau)\ ,\ \vec{v}(\tau)} \xrightarrow{I^+_\lambda}$$

Forward problem

The forward problem is schematically represented

$$\xrightarrow{\vec{T},\rho,\vec{B},\vec{v}} \boxed{\begin{array}{c}\text{forward problem}\\ I^+_\lambda = F(\vec{T},\rho,\vec{B},\vec{v})\end{array}} \xrightarrow{I^+_\lambda}$$

In fact solve for intensity vector $\vec{I} = \begin{pmatrix} I \\ Q \\ \alpha \\ V \end{pmatrix}$ obeying the equation

$$\frac{d\vec{I}}{d\tau} = -X(\vec{T},\rho,\vec{B},\vec{v})\vec{I} - \vec{j}(\vec{T},\rho,\vec{B},\vec{v}) \tag{14}$$

and the solution

$$I^+_\lambda = I^+_0 e^{-\int} + \int \vec{j} e^{-\int} d\tau \tag{15}$$

Example Source function $S = a\tau + b$ then $\int_0^{\tau_{max}} (a\tau + b)e^{-\tau} d\tau = ...$

Inverse problem

The inverse problem is schematically represented

$$\xrightarrow{I^+_\lambda} \boxed{\begin{array}{c}\text{inverse problem}\\ F^{-1}(I^{obs}_\lambda)\end{array}} \xrightarrow{\vec{T},\rho,\vec{B},\vec{v}}$$

Via least-squares approximation

$$\min_{\vec{T},\rho,\vec{B},\vec{v}} \sum \left(I^{obs}_\lambda - I_\lambda(\vec{T},\rho,\vec{B},\vec{v}) \right)^2 \tag{16}$$

12.2 Challenging domains of application

- Lyman alpha in Galaxy Halos
- Dusty torii (AGD)
- protoplanetary disks
- circumstellar disks
- athmospheres

13 Asymptotic Preserving Monte Carlo methods for transport equations in the diffusive limit (Dimarco+2018)

14 Splitting methods

From notes by professor Frank.

14.1 Exercises

14.1.1 Exercise 1

Part III

Practical work and Exercises

15 Overview of exercises (PART I)

1. limb-darkening scattering exercise we did during the course. — You can look into your notes from that, and I attach here also a sample program which you can use a base. After you have familiarised yourself with this, you can start to think bout how you would go about to extend this to a 3D setting (assuming isotropic scattering).

2. (As prep for Monte-Carlo school) here is a script computing a UV resonance P-Cygni line in spherically symmetric wind with v beta-law. At top of routine, a few exercises are given, where you can modify and play around with code. Monte-Carlo program which computes a UV resonance spectral line from a fast outflowing spherically symmetric stellar wind (if you were not cc'd on that email, let me know so that I can send you the files as well). At the top of that little script, there are a few suggestions for exercises (additions) you could do to that program, in order to learn a bit more about the general workings of Monte-Carlo radiative transfer in this context. — So that might be a good idea for you to do as well ! (And you can also ask the others in the group for some tips etc. then.)

3. Some background reading:
   - Attached mc manual by Puls.
   - Paper by Sundqvist+ 2010 (Appendix, I think).

16 Overview of exercises (PART II)

1. calculate the probability distribution to sample from in the case of Eddington limb darkening for the initial distribution (see 18.3.4).

2. calculate analytical solution for simplified problem in 18.3.2 in the case that `mu = 1`.

3. perform convergence analysis

17 Limb darkening

17.0.1 2D Case

We again have $\mu = \cos(\theta)$. The solution of the radiative transfer equation in plane-parallel syummetry with frequency-independent absorption and emission, is

$$I(\mu) = I_1(0.4 + 0.6\mu) \tag{17}$$

In the Monte Carlo code, the photons are sorted according to the direction that they leave the atmosphere.

Goal Calculates the angular dependence of photon's emitted from a plane-parallel, grey atmosphere of radial optical depth `taumax`. The value of `tau` determines the position of the photon

Variables and Algorithm

- `muarray` contains emergent photons
- `na` number of channels
- `dmu` = 1/`na` width of channels
- `nphot` number of photons
- `taumax` maximum optical depth

Algorithm 2 Limb darkening: compute quantitiy of photons

---

initialization
  radial optical depth τ
  direction μ
for all photons **do**
  $\boxed{\tau = \tau_{max}}$
  while `tau` ≥ 0 **do**
    compute scattering angle `mu`
    if tau $\geq$ taumax **then** $\boxed{mu = sqrt(x)}$ (initial distribution)
    else $\boxed{mu = 2 * x = 1}$ (isotropic scattering)
    tau_i = -log(x2)
    tau = tau - tau_i*mu
  end while
  now we know that the photon has left the photosphere
  compute the distribution of all angles `mu` at which the photon left the photosphere
end for
visualisation:

- plot photon numbers from $\mu d\mu$ against `mu`
- plot specific intensity from $d\mu$ against `mu` against

Figure 1 is according to what is expected $I = I_0(0.4 + 0.6\mu)$

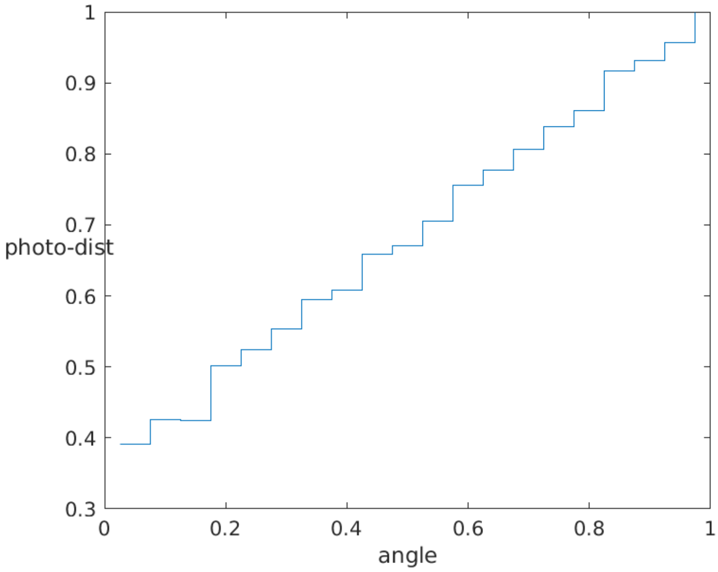


Figure 1: histogram for `mu`

17.0.2 3D Code

What changes is this:

- introduction of a new angle ϕ
- the optical depth has to be updated according to ϕ also

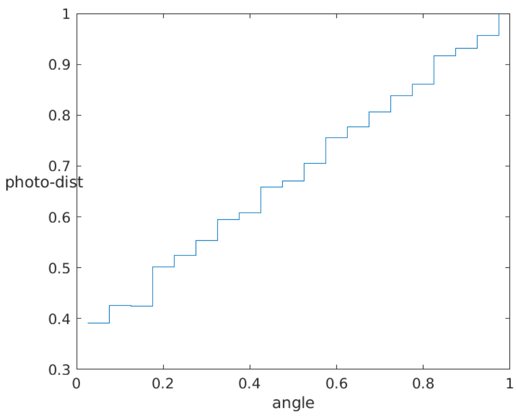


Figure 2: histogram for `mu`

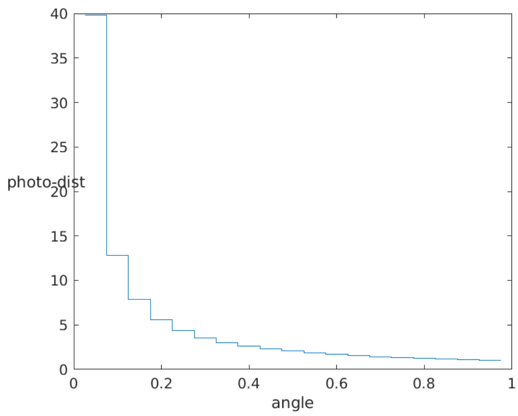


Figure 3: histogram for `phi`

Figure 2 and Figure 3 are according to what is expected, namely $I = I_0(0.4 + 0.6\mu)$ and a uniform distribution for *phi*, which corresponds to a $I \sim \dfrac{1}{\phi}$

18 Investigation of program: pcyg.f90

18.1 Overview of variables

name	explanation	scope
paramaters		
xk0		
alpha		
beta		
start frequency of the photon		
xstart	start frequency	
vmin		
vmax		
angle of the photon		
xmuestart	start angle	
xmuein	incident angle	
xmueou	outward angle	
pstart	impact parameter	
xnew	new photon frequency	
optical depth		
tau	optical depth	
number of photons admin		
nphot	number of photons	
nin	photons scattered back into core	
nout	photons escaped	
functions		
func	velocity profile	
	r	distance from center of star
xmueout	sign of outwards angle xk0 alpha r v sigma	

18.2 Mathematical things that are noteworthy

General working

$$\xrightarrow{\texttt{xstart}} \boxed{\texttt{pcyg.f90}} \xrightarrow{\texttt{xnew}}$$

The photons are sorted according to `xnew`.

Practical formula

- emission angle $\mu = \cos(\theta)$
- according p-ray $p = \sqrt{1 - \mu^2} = \sin(\theta)$
- incident angle $xmuin = \sqrt{1 - \left(\frac{pstart}{r}\right)^2}$

Geometry & Symmetry assumptions

- spherical geometry

18.3 Exercises

18.3.1 Investigation of original code

In original version of the code, all photons are released isotropially from the photosphere.

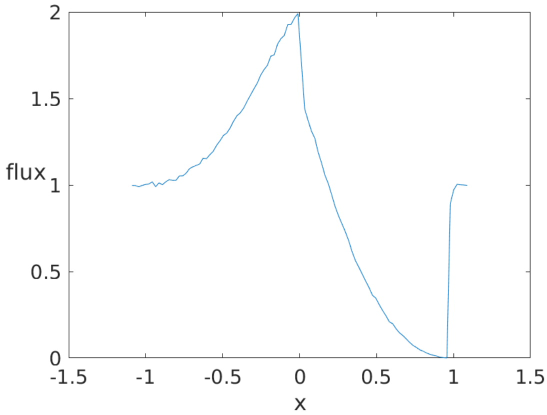


Figure 4: Original version of the code

18.3.2 First adaptation: what if all photons are released radially from photosphere?

Release photons radially: experiments What would happen with line-profile, if you assumed all photons were released radially from photopshere?

- In other words `xmuestart` $= 1$. Results in Figure 5.
- This is implemented under the test case `test_number=1`.

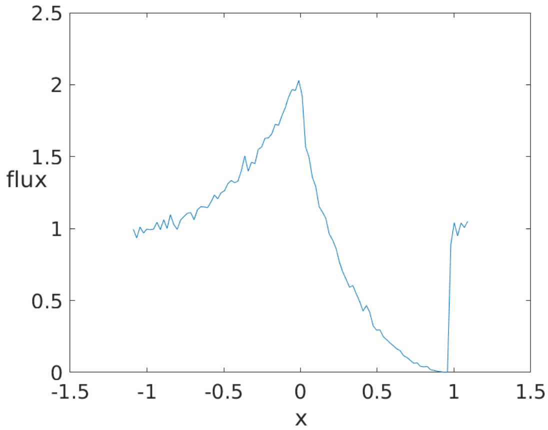


Figure 5: First adaptation

Derive analytic expression See also slide Sundqvist 26/49.

- since `mu = 1` we have for the velocity profile that $v = v_\infty(1 - b/r)^\beta$
- a scaled version of the above yields

$$u = \frac{v(r)}{v_\infty} = \left(1 - \frac{r_\infty}{r}\right)^\beta \in [0..1] \tag{18}$$

- Doppler shift: $x_{CMF} = x_{REF} - \mu u$
- condition for resonance from Sobolov approximation (to be studied later): $x_{CMF} = 0$ thus

$$x_{RF} = \mu u \tag{19}$$

  or thus $u_{int} = x_{REF}$ and than solve Equation 18 for r_{int}

- Equation 19 can be solved for the frequency, namely

$$x = \left(1 - \frac{r_\infty}{r}\right)^\beta \tag{20}$$

- thus

$$x^{-\beta} = 1 - \frac{r_\infty}{r}$$

$$r(1 - x^{-\beta}) = r_\infty$$

  thus

$$r = \frac{r_\infty}{1 - x^{-\beta}} \tag{21}$$

- then the calculation of the optical depth proceeds as follows:

$$\tau = \frac{\texttt{xk0}}{rv^{2-\alpha}(1 + \texttt{muein}^2\sigma)} \tag{22}$$

  where

  - $v = \left(1 - \frac{b}{r}\right)^{\beta}$
  - $\frac{dv}{dr} = \beta \frac{b}{r^2} \left(1 - \frac{b}{r}\right)^{\beta - 1}$
  - $\sigma = \frac{dv}{dr}\frac{r}{v} - 1$

18.3.3 Second adaptation: isotropic scattering

What would happen to line-profile, is you assumed scattering !was isotropic (i.e., NOT following Sobolev-distrobution)

- test case number 2

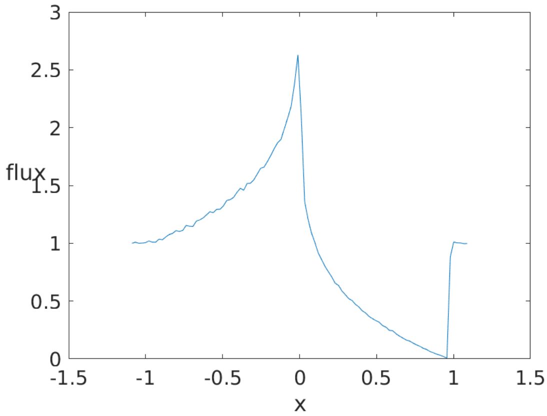


Figure 6: Second adaptation

The *grosso modo* form has not changed, although the scaling has changed.

18.3.4 Third adaptation: introduction of Eddington limb-darkening

Put Eddington limb-darkening in. What happens?

General discussion: Eddington limb darkening The data are taken from Christensen, 2015.

- the source function $S =< I >= a + b\tau_\nu$ with $a = \frac{\sigma}{2\pi}T_{eff}^4$ and $b = \frac{3\sigma}{4\pi}T_{eff}^4$
- solve the equation
- this yields $\frac{I(\theta)}{I(0)} = \frac{a + b\cos(\theta)}{a + b} = \frac{2}{5} + \frac{3}{5}\cos(\theta)$

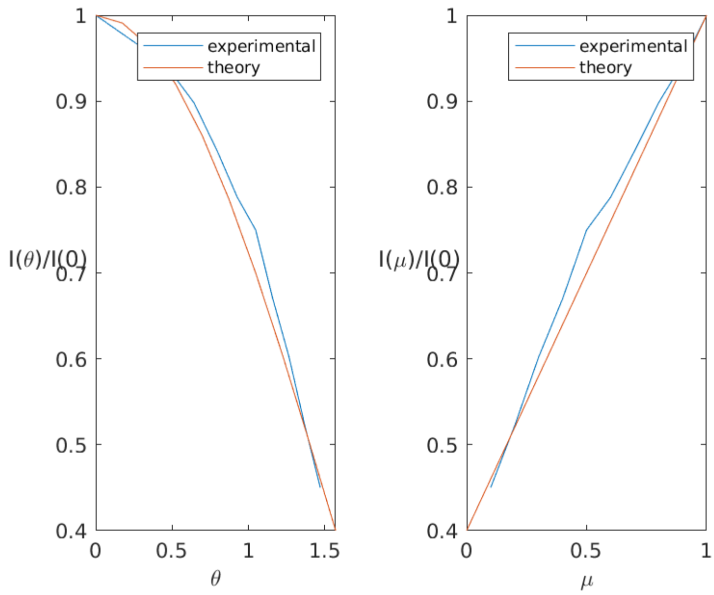


Figure 7: Eddington limb darkening (two times the same plot with $\mu = \cos(\theta)$

for which star are the exerpimental data and what assumptions are used in the theory?

Let us thus first review the emmission case where the flux in each directino is isotropic (paragraph 18.3.3) $I(\theta) = I$

- the specific intensity $I_\nu(\mu) = \frac{dE_\nu}{\cos(\theta)dAdtd\nu d\Omega} = \frac{dE_\nu}{\mu dAdtd\nu d\Omega}$
- the flux $F_\nu = \int_\Omega I_\nu \cos(\theta)d\Omega$ is in this case isotropic thus

$$\xi = \int_0^\mu F_\nu d\mu = \int_0^\mu \int_\Omega I_\nu \cos(\theta)d\Omega d\mu = A\int_0^\mu \mu d\mu \tag{23}$$

with the condition that μ satisfies a probability distribution:

$$1 = \int_{-1}^1 F_\nu d\mu = \int_{-1}^1 \int_\Omega I_\nu \cos(\theta)d\Omega d\mu = \frac{A}{2} \tag{24}$$

thus $A = 2$.

Application to exercise Now we look at a new case where the photons need to be emitted following a distrubution that corresponds to $I(\theta) = I(0)(0.4 + 0.6\cos(\theta))$. In the code this corresponds to `test_number = 3`.

- in this case the flux $F_\nu = \int_\Omega I_\nu \cos(\theta) d\Omega$ is also isotropic but satisfies

$$F_\nu = \int_\Omega I_\nu(0)[0.4 + 0.6\cos(\theta)]\cos(\theta)d\Omega \tag{25}$$

$$\xi = \int_0^\mu F_\nu d\mu = A\int_0^\mu (0.4 + 0.6\mu)\mu d\mu \tag{26}$$

subject to the condition -very similar to Equation 24 - that

$$1 = \int_{-1}^1 F_\nu d\mu = \frac{2A}{5} + \frac{A}{3} = \frac{11A}{15} \tag{27}$$

thus $A = \dfrac{15}{11}$

18.3.5 Fourth adaptaion: photospheric line-profile

Challening: Put photospheric line-profile (simple Gaussian) in !What happens? Test on xk0=0 (opacity =0) case.

- test case number 4 (not yet implemented)

18.3.6 Convergence analysis

19 Mass loss from inhomogeneous hot star winds (Sundqvist)

- GOAL: synthesis of UV resonance lines from inhomogeneous 2D winds
  - clumped in density
  - clumped in velocity
  - effects of non-void inter-clump medium
- WIND MODELS
  - symmetry assumptions
    * 1D: spherical symmetry
    * 2D: symmetry in Φ
  - models
    1. time-dependent radiation-hydrodynamic from Puls and Owocki (POF)
       * 1D
       * isothermal flow
       * perturbations triggered by photospheric sound waves
    2. time-dependent radiation-hydrodynamic from Feldmeier (FPP)
       * 1D
       * treatment of energy equation
       * perturbations triggered by photospeheric sound waves or Langevin perturbagions (photospheric turbulence)
    3. stochastic model, clumped in density
       * smooth winds with $v_\beta = (1 - b/r)^\beta$ with $\beta = 1$
       * clumping factor f_{cl}
    4. stochastic model, clumped in density and in velocity (non-monotonic velocity field)
       * smooth winds with $v_\beta = (1 - b/r)^\beta$ with $\beta = 1$
       * clumping factor f_{cl}
- RADIATIVE TRANSFER (MC-2D)

20 Asymptotic preserving Monte Carlo methods for radiative transfer equation in diffusion limit (Dimarco+ 2018)

20.1 Goldstein-Taylor

20.2 Radiative transfer

Part IV
Equation meetings

21 Meeting of 10 April 2019

22 Meeting of 17 April 2019

23 Meeting of 14 August 2019

Part V
Thesis meetings

24 Meeting on 6 September 2019

- overview of Petnica summer institute on Astrophysics
- question: manual by Puls: why is isotropic distribution sampled from $\mu\mu$?
- `pcyg.f90` program
- practical arangements
- SKIRT code
- discussion of paper (Dimarco+2018)

Literature Study

- General guidelines for good practices in scientific computing are found in [Wil+14].
- I went to the 2019 Petnica Summer school in Petnica, Serbia. This was a good general introduction and overview to astrophysics.
- A very interesting article about Monte Carlo methods for radiative transfer problems, from a mathematical point of view, is [DPS18]. I am currently trying to reproduce the numerical experiments that are reported in the article.

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[DPS18] G. Dimarco, L. Pareschi, and G. Samaey. "Asymptotic-Preserving Monte Carlo Methods for Transport Equations in the Diffusive Limit". eng. In: *SIAM Journal on Scientific Computing* 40.1 (2018), pp. 504–528. ISSN: 1064-8275.

[Moe18] Nicolas Moens. *Radiation-Hydrodynamics with MPI-AMRVAC: Massive-Star Atmospheres and Winds*. Leuven, 2018.

[Wik18] Wikipedia contributors. *Heat flux — Wikipedia, The Free Encyclopedia*. [Online; accessed 14-September-2019]. 2018. URL: `https://en.wikipedia.org/w/index.php?title=Heat_flux&oldid=863233807`.

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