1 Glossary

- SED: spectral energy distribution
- (spectral) line-force: force on material in stellar atmosphere

2 General equations - preliminary overview

2.1 Hydrodynamics

Euler equations, together with closing relation (e.g. ideal gas law).

primitive variables			
mass density	velocity	gas energy density	gas pressure
ρ	v	e	p

2.2 Radiation

Radiative transfer equation: intensity along a ray while interagating with medium. Photons are massless.

$$\left[\frac{1}{c}\partial_t + \vec{n}.\vec{\nabla}\right] I_\nu = \eta_\nu - \chi_\nu I_\nu \tag{1}$$

frequency	intensity	emissivity	total absorbption
u	I_ν	η_ν	χ_ν

These deliver two equations

- the radiative energy equation (diffusion flux $\vec{F}$

$$\frac{\partial E}{\partial t} + \vec{\nabla}.\vec{F} = \iint ...d\nu d\Omega \tag{2}$$

- radiative momentum equation

$$\frac{d\vec{F}}{\partial t} = \iint ...\vec{n} d\nu d\Omega \tag{3}$$

(after **integrating over all frequencies**). Depending on the geometry simplifcations, one can e.g. integrate over all solid angles.

2.3 Radiation-Hydrodynamics

Combination delivers integral-diffusion equation

$$\begin{aligned}\frac{dI}{d\tau} &= S - I \\ &= \int I d\Omega - I\end{aligned} \tag{4}$$

2.4 Challenges

- combination with hydrodynamics
- current analysis: simplified geometries (symmetry). E.g. in 2D, an ADI method is used and now also a multigrid method.
- complex geometry difficult to show in ray-tracing scheme
- steady-state vs. time dependent
- focus on radiation equations

3 Very broad introduction: Radiation Hydrodynamics

The material here originates from the master thesis of Nicolas Moens and the course notes *Introduction to numerical methods for radiation in astrophsyics* from professor Sundqvist.

Heat flux diffusion equation $u_t = u_{xx}$. The flux

Specific intensity and its angular moments

specific intensity	$\Delta\epsilon = \boxed{I_\nu} A_1 A_2 / r^2 \Delta\nu\Delta t$
energy density	$E = \frac{1}{c}\iint I_\nu d\nu d\Omega$
flux vector	$F = \iint I_\nu n d\nu d\Omega$
pressure tensor	$P = \iint I_\nu n n d\nu d\Omega$
mean intensity	$J_\nu = \frac{c}{4\pi} E_\nu$
Eddington flux	$H_\nu = \frac{1}{4\pi} F_\nu$
Eddington's K	$K_\nu = \frac{c}{4\pi} P_\nu$

RHD equations The full RHD equations consist of

- five partial differential equations
- one HD closure equation, e.g. (i) variable Eddington tensor method or (ii) flux limited diffusion

Eddington factor In general, the Eddington factor is a tensor, for 1D systems it is reduced to a scalar.

$$f_\nu = \frac{K_\nu}{J_\nu} = \frac{P_\nu}{E_\nu} \tag{5}$$

- isotropic radiation field
- radiation field stronly peaked in radial (i.e. vertical in cartesian) direction

Radiation transport equations, diffusion, equilibrium

- black body radiation (Planck function $I_\nu = J_\nu = B_\nu$
- in general, extinction(absorption,scattering) and emission

$$\frac{dI_\nu}{ds} = j_\nu - k_\nu I_\nu \tag{6}$$

  - Cartesian coordinates:

$$\boxed{\frac{\partial I_{n,\nu}}{\partial t}\frac{1}{c} + n\nabla I_{n,\nu} = j_\nu - k_{n,\nu} I_{n,\nu}} \tag{7}$$

  - spherical coordinates
  - 1D-problem with only variation along z-axis $\mu\frac{dI}{dz} = j - kI$
  - spherical symmetry $\mu\frac{\partial I}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I}{\partial \mu} = j - kI$

- plane-parallel approximation

$$\boxed{\mu \frac{dI}{dr} = j - kI} \tag{8}$$

The angle μ is constant throughout the computational domain. Dividing by k_ν, this yields

$$\mu \frac{dI}{k_\nu dr} = \mu \frac{dI}{k_\nu dz} = S - I \tag{9}$$

- Oth moment equation: integrate Equation (3) over ν and Ω, i.e. $\int d\nu d\Omega$. Conservation of energy
- first multiply Equation (3) with $\frac{n}{c}$ and then do integration

Radiative Diffusion Approximation

1. Black-body radiation in perfect equilibrium
2. Radiative transfer equation in the *near-surface* limit.

The approximation is the following: replace $\boxed{I = B}$ or $I_\nu = B_\nu$, once but not twice.

$$I_\nu = B_\nu - \mu \frac{dB_\nu}{k_\nu dz} \tag{10}$$

Derive this equation as a random walk of photons!

3.1 Examples of radiation (diffusion equation)

1. Temperature structure in a static stellar atmosphere
2.

3.2 Applications and approximations for radiative forces

- definition of general radiative acceleration vector $g = \frac{1}{\rho c} \int \int n k_\nu I_\nu d\Omega d\nu$
  - continuum Thomson scattering
  - spectral line with extinction
    - furhtermore assume central continuum source
    - then $g_{line} = \frac{F_\nu^0 k_L}{\rho c}$
- Sobolev approximation
- CAK theory

3.3 Recap

optical depth	optical depth along ray
	$\tau_{\mu,\nu} = \int_z^{z_{max}} \frac{\alpha_{nu}(z')}{\mu} dz' = \frac{\tau_\nu(z)}{\mu}$

4 Introduction: course material from CMPAA (Sundqvist)

4.1 EXERCISES: Introduction to numerical methods for radiation in astrophysics

1. introduction

2. radiation quantities

   - exercise p.3:
     - on one hand, we know that $\Delta\epsilon \sim C/r^2$
     - on the other hand, from the definition we know that $\Delta\epsilon = I_\nu A_1 A_2 / r^2 \Delta\nu \Delta t$
     - combining these equations shows that I_ν is independent from r
   - exercise p.4:
     -
   - exercise 1:
     - $F_x = \int_0^\pi \left[I_\nu(\theta)\sin^2(\theta) \int_0^{2\pi} \cos(\phi) \right] d\theta d\phi = 0$
     - the same reasoning for $F_y = 0$
   - exercise 2:
     - the equation follows from $d\mu = d\cos(\theta) = \sin(\theta) d\theta$
   - exercise 3:
     - isotropic radiation field (i.e. $I(\mu) = I$) then we have $F_\nu = 2\pi \int_{-1}^{1} I\mu d\mu = 2\pi I \left. \frac{x^2}{2} \right|_{-1}^{1} = 0$
   - exercise 4:
     - $F_\nu = 2\pi \int_{-1}^{1} I(\mu)\mu d\mu = 2\pi \int_{-1}^{0} I_\nu^- \mu d\mu + 2\pi \int_0^1 I_\nu^+ \mu d\mu = 2\pi I_\nu^+$
   - exercise p.7:
     - isotropic radiation field:
       * although the radiation pressure is a tensor, we will denote it as a scalar $P_\nu = \frac{4\pi I_\nu}{c}$
       * the radiation energy density $E_\nu = \frac{12\pi I_\nu}{c}$
       * thus $f_\nu = \frac{1}{3}$
     - very strongly peaked in radial direction (beam): $I_\nu = I_0 \delta(\mu - \mu_0)$ with $\mu_0 = 1$
       * pressure tensor $P_{nu} = \frac{1}{c} \int I_0 \delta(\mu - \mu_0) nn d\Omega$
       * energy density $E_\nu = \frac{1}{c} \int I_\nu d\Omega$
       * in this case $P_\nu = E_\nu$ thus $f_\nu = 1$

3. radiation transport vs. diffusion vs. equilibrium

   - exercise p. 12: 1D, Cartesian geometry, plane-parallel, frequency-independent and isotropic emission/extinction
     - radiation energy equation
       * The equation follows by integrating Equation (4)
       * By definition, $E = \frac{1}{c} \iint I_\nu d\nu d\Omega$
       * thus $\frac{dE}{dr} = \int (j - kI) d\nu d\Omega$ thus $\boxed{\frac{dE}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)}{c}}$

    * work out the integral taking into account frequency-independent and isotropic coefficients:
  - zeroth momentum equations
    * One must also take into account the specific form of the flux vector $F = \iint I_\nu n d\nu d\Omega = 2\pi \int_{-1}^{1} I_\nu(\mu)\mu d\mu$
    * thus $\frac{dF}{dr} = \frac{1}{c}\int (j - kI) n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
  - first moment equation
    * similar reasoning
    * $\frac{dP}{dr} = \int (j - kI) n.n d\nu d\Omega$ thus $\boxed{\frac{dF}{dr} = \frac{(j - kI)4\pi(\nu_1 - \nu_0)n}{c}}$
- first exercise p. 15
  - $P = \frac{1}{c}\iint I_\nu \mu^2 d\Omega d\nu = \frac{2\pi}{c}\int_\nu \int_{-1}^{1} I_\nu \mu^2 d\mu d\nu = \frac{4\pi}{3c}\int B_\nu d\nu = \frac{aT^4}{3} = \frac{E}{3}$
- second exercise p.15
  - assuming the diffusion limit,
  - flux-weighted mean opacity $\kappa_F = \frac{\int F_\nu \kappa_\nu d\nu}{\int F_\nu d\nu}$
  - Rosseland mean opacity $\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}}{\int_0^\infty \frac{dB_\nu}{dT} d\nu}$.
    * in the diffusion limit, $F_\nu = -\frac{4\pi}{3}\frac{dB_\nu}{k_\nu dz}$ thus $\frac{dB_{nu}}{dT} =$
    *
- third exercise p.15

4. the equations of radiation-hydrodynamics

5. numerical techniques for the radiative diffusion approximation

6. applications and approximations for a dynamically important radiative force in supersonic flows

   - exercise p.27: $L_{SOB} = \Delta r = \frac{v_{th}}{dv/dr} = \frac{10[km/s]}{1000[km/s]/R_*} = 0.01 R_*$

7. Appendix A: properties of equilibrium black-body radiation

   - exercise p. 29
     - this should be satisfied: $B_\nu d\nu = -B_\lambda d\lambda$ and also $\nu = \frac{c}{\lambda}$
     - this is equivalent to saying that $0 = \nu d\lambda + \lambda d\nu$ or $d\lambda = -\frac{\lambda}{\nu} d\nu$ thus $B_\lambda = \frac{\nu}{\lambda} B_\nu$
     - $B_\lambda(T) = \frac{\nu}{\lambda}\frac{2h\nu^3}{(\lambda\nu)^2}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2h\nu^2}{\lambda^3}\frac{1}{e^{hc/\lambda kT} - 1} = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$
   - first exercise p.31
     - derive that $\lambda_{max} T = 2897.8[\mu m K]$
     - ...
   - second exercise p.31
     - this is about the spectra of (unknown) stars
   - first exercise p.32
     - see exercise 7
   - second exercise p.32

- BB radiation: $I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kt}-1}$
- the radiative flux for isotropic BB radiation is zero. See also exercise 3. This dus also holds for BB radiation.

- exercise p. 33
  - HR-diagram

8. Appendix B: Simple examples to the radiative transfer equation

- first exercise p. 34
  - start from radiative transport equation $\mu\frac{dI}{ds} = \alpha - \eta I$ in which $\eta = 0$ thus $\boxed{\mu\frac{dI}{ds} = \alpha}$
  - solving the ODE in the general case that $\alpha(s)$ is not constant:
    * integrate the equation $\mu I = \int_0^D \alpha ds$
    * ...
  - second exercise p. 34
    * case $\tau(D) >> 1$: then $I(D) \approx S$
    * case $\tau(D) << 1$: then $I(D) \approx I(0) + S(1-1) = I(0)$
  - first exercise p.35
    * is the plane-parallel approximation valid for the solar photosphere?
  - second exercise p.35
    * goal: find a solution to the equation $\mu\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$ where $I(\tau, \mu)$
    * solution
- second exercise p.35

9. Appendix C: connecting random walk of photons with radiative diffusion model

- exercise p. 38. Computing the average photon mean-free path inside the Sun. $l = \frac{1}{\kappa\rho} = \frac{V_o}{\kappa M_o}[cm]$
- exercise p.39. Computing the random-walk time (diffusion time) for photons

4.2 Implicit 1D solver (20-11-2018)

4.3 ADI 2D Solver

4.4 Area of a circle

4.5 Limb Darkening

See Section 9.1.

5 Computational Methods in Astrophysics: MC and RT (Puls)

5.1 basic definitions and facts

5.2 about random numbers

5.3 MC integration

5.4 MC simulation

Radiative transfer in stellar atmospheres

- GOAL: spatial radiation energy density $E(\tau)$ in an atmospheric layer
  - only photon-electron scattering
  - τ is the optical depth
- Milne's integral equation $\boxed{E(\tau) = \frac{1}{2}\int_0^\infty E(t)E_1(|t-\tau|)dt}$
  - analytical solution $\frac{E(\tau)}{E(0)} = \sqrt{3}(\tau + q(\tau))$
  - MC simulation
    * emission angle
    * optical depth until next scattering event
    * scattering angle
- HOW DOES THIS WORK?

Algorithm 1 Limb darkening: compute quantitiy of photons

create photons
probability distribution for emission angle $\mu = \cos(\theta)$: $\boxed{p(\mu)d\mu = \mu d\mu}$
optical depth until next scattering event: $\boxed{p(\tau)dt \approx e^{-\tau}d\tau}$
isotropic scattering angle at low energies: $\boxed{p(\mu)d\mu \approx d\mu}$
follow all photons until they leave the atmosphere or are scattered back into stellar interior

5.5 Exercise 1: RNG

5.6 Exercise 2: Planck-function

1. analytical method

2. MC method

5.7 limb darkening

See section 9.1.

6 Splitting methods

From notes by professor Frank.

6.1 Exercises

6.1.1 Exercise 1

7 Monte Carlo Radiation Transport

7.1 Limb Darkening

7.1.1 1D Code

We again have $\mu = \cos(\theta)$. The solution of the radiative transfer equation in plane-parallel syummetry with frequency-independent absorption and emission, is

$$I(\mu) = I_1(0.4 + 0.6\mu) \tag{11}$$

In the Monte Carlo code, the photons are sorted according to the direction that they leave the atmosphere.

Goal Calculates the angular dependence of photon's emitted from a plane-parallel, grey atmosphere of radial optical depth `taumax`. The value of `tau` determines the position of the photon

Variables and Algorithm

- `muarray` contains emergent photons
- `na` number of channels
- `dmu` = 1/`na` width of channels
- `nphot` number of photons
- `taumax` maximum optical depth

Algorithm 2 Limb darkening: compute quantitiy of photons

---

initialization
  radial optical depth τ
  direction μ
for all photons **do**
  $\boxed{\tau = \tau_{max}}$
  while `tau` ≥ 0 **do**
    compute scattering angle `mu`
    if tau $\geq$ taumax **then** $\boxed{mu = sqrt(x)}$ (initial distribution)
    else $\boxed{mu = 2 * x = 1}$ (isotropic scattering)
    tau_i = -log(x2)
    tau = tau - tau_i*mu
  end while
  now we know that the photon has left the photosphere
  compute the distribution of all angles `mu` at which the photon left the photosphere
end for
visualisation:

- plot photon numbers from $\mu d\mu$ against `mu`
- plot specific intensity from $d\mu$ against `mu` against

7.1.2 3D Code

What changes is this:

- introduction of a new angle ϕ
- the optical depth has to be updated according to ϕ also

7.2 Introduction to Monte Carlo Radiation Transfer

- (Wood, Wittney, Bjorkman, Wolff - 2001)
- (Wood, Wittney, Bjorkman, Wolff - 2013)

7.2.1 Elementary principles

specific intensity	I_ν
radiant energy	dE_ν
surface area	dA
angle	θ
solid angle	$d\Omega$
frequency range	$d\nu$
time	dt
flux	F_ν
cross section	σ
scattering angle	χ $\mu = \cos(\chi)$
mean intensity	J
flux	H
radiation pressure	K

intensity	$I_\nu(l) = I_\nu(0)e^{n\sigma l}$
angular phase function of the scattering particle	$P(\cos(\chi))$

inverse method	$\xi = \int_0^{x_0} P(x)dx$ with $\xi \in \mathcal{U}(0,1)$
rejection method	

7.2.2 Eddington factors

7.2.3 Example: plane parallel atmosphere

1. emission of photons: select two angles (3D space). In isotropic scattering
   - θ met $\mu = \cos(\theta)$
     - $\mu = 2\xi - 1$ (isotropic scattering)
     - $\mu = \sqrt{\xi}$ (A slab is heated from below. Then $P(\mu) = \mu$)
   - $\phi = 2\pi\xi$
2. propagation of photons
   - sample optical depth from $\tau = -\log(\xi)$
   - distance travelled $L = \frac{\tau z_{max}}{\tau_{max}}$
3. conclusion of emission and propagation

$$\begin{aligned} x &= x + L\sin(\theta)\cos(\phi) \\ y &= y + L\sin(\theta)\sin(\phi) \\ z &= z + L\cos(\theta) \end{aligned} \tag{12}$$

4. Binning: once the photon exists the slab. Produce histograms of the distribution function. Finally, we wish to compute the output flux or the intensity.

I have seen that a newer version of the paper is available, which was also used in these notes (which contains amongst other up-to-date references to code fragments).

A Plane Parallel, Isotropic Scattering Monte Carlo Code

7.3 Monte Carlo Radiative Transfer

From a macroscopic perspective, RT calculations rest on the transfer equation

- emissivity η (how much energy is added to radiation field due to emission)
- opacity χ (how much energy is removed due to absorption)
- the source function $S = \dfrac{\eta}{\chi}$
- optical depth τ captures the opaqueness of a medium

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \nabla . n\right) I = \eta - \chi I \tag{13}$$

$$d\epsilon = I d\nu dt d\Omega dA . n \tag{14}$$

7.4 P Cygni profile for beta-velocity law and given opacity Monte Carlo simulation

7.4.1 Structure of the code

- module common
- module my_inter
- program pcyg
  - INPUT xk0, alpha, beta
  - OUTPUT
  - PROGRAM FLOW: loop over all photons
    - get xstart and vstart
    -
  - then do normalisation
- function func(r)
- function xmueout(xk0,alpha,r,v,sigma)
- function rtbis(func,x1,x2,xacc)

8 The mathematics of Radiative Transfer

8.1 Auxiliary mathematics

- $\cos(\Theta) = cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$
- phase function $\boxed{p(\mu, \phi, \mu', \phi', \tau) = \sum_{n=0}^{N} \omega_n P_n(\cos(\Theta))}$
  - isotropic scattering $p(\tau) = \omega_0(\tau)$
- equation of transfer $\boxed{\mu \frac{\partial I(\tau, \mu, \phi)}{\partial \tau} = I(\tau, \mu, \phi) - \mathcal{S}(\tau, \mu, \phi)}$
  with $\mathcal{S}(\tau, \mu, \phi) = B_1(\tau) + \frac{1}{4\pi} \int_{-1}^{1} d\mu' \int_0^{2\pi} I(\tau, \mu', \phi') p(\mu, \phi, \mu', \phi') d\phi'$
  - axially symmetric with isotropic scattering
    $\mathcal{S}(\tau) = \frac{\omega_0(\tau)}{2} \int_{-1}^{1} I(\tau, \mu') d\mu' = B_1(\tau) + \frac{\omega_0(\tau)}{2} \int_0^{\tau_1} \mathcal{S}(t) E_1(|t - \tau|) dt$
  - the Milne equation of the problem $(1 - \omega_0 \bar{\Lambda})\{$
    $mahtcal S(t)\} = B(\tau)$
    * solve for $\mathcal{S}(t)$
    * then find $I(\tau, \mu)$

8.2 The H-functions

- characteristic equation

9 Overview of symmetry assumptions

plane-parallel	1D atmosphere bounded by horizontal surfaces	

10 Challenges in Radiative Transfer (Ivan Milic)

10.1 Overview of the problem

$$\xrightarrow{I^*_\lambda} \boxed{T(\tau)\ ,\ \rho(\tau)\ ,\ \vec{B}(\tau)\ ,\ \vec{v}(\tau)} \xrightarrow{I^+_\lambda}$$

Forward problem

The forward problem is schematically represented

$$\xrightarrow{\vec{T}, \rho, \vec{B}, \vec{v}} \boxed{\begin{array}{c}\text{forward problem}\\ I^+_\lambda = F(\vec{T}, \rho, \vec{B}, \vec{v})\end{array}} \xrightarrow{I^+_\lambda}$$

In fact solve for intensity vector $\vec{I} = \begin{pmatrix} I \\ Q \\ \alpha \\ V \end{pmatrix}$ obeying the equation

$$\frac{d\vec{I}}{d\tau} = -X(\vec{T}, \rho, \vec{B}, \vec{v})\vec{I} - \vec{j}(\vec{T}, \rho, \vec{B}, \vec{v}) \tag{15}$$

and the solution

$$I^+_\lambda = I^+_0 e^{-\int} + \int \vec{j} e^{-\int} d\tau \tag{16}$$

Example Source function $S = a\tau + b$ then $\int_0^{\tau_{max}} (a\tau + b)e^{-\tau} d\tau = ...$

Inverse problem

The inverse problem is schematically represented

$$\xrightarrow{I^+_\lambda} \boxed{\begin{array}{c}\text{inverse problem}\\ F^{-1}(I^{obs}_\lambda)\end{array}} \xrightarrow{\vec{T}, \rho, \vec{B}, \vec{v}}$$

Via least-squares approximation

$$\min_{\vec{T}, \rho, \vec{B}, \vec{v}} \sum \left(I^{obs}_\lambda - I_\lambda(\vec{T}, \rho, \vec{B}, \vec{v}) \right)^2 \tag{17}$$

10.2 Challenging domains of application

- Lyman alpha in Galaxy Halos
- Dusty torii (AGD)
- protoplanetary disks
- circumstellar disks
- athmospheres