# *Proposed problem*

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*Let* $x_1 > 0$, and let

$$x_{n+1} = (\sqrt{x_n} + \frac{1}{\sqrt{x_n}})^2$$

when $n \geq 1$.For $n \geq 1$, let

$$y_n = x_n - 2n - \frac{log(n)}{2}$$

*As in Problem* 12210 (*by P.Bracken A.M.M* 9/2020), *but with* $x_1$ *arbitrary and positive, the sequence* $(y_n)_{n\geq 1}$ *converges and let*

$$y = \lim_{n\to\infty} y_n.$$

*Prove that* :

$$\lim_{n\to\infty} \frac{n}{log(n)} \cdot (y_n - y) = \frac{1}{8}$$

## Solution.

**(Step 1)**

First, we *will prove that*

$$\lim_{n\to\infty} \frac{n^2}{log(n)} \cdot (y_{n+1} - y_n) = -\frac{1}{8}$$

The following results will be useful in our proof:

$i)$Appling Cesaro-Stolz we find that:

$$\lim_{n\to\infty}\frac{x_n}{2n}=\lim_{n\to\infty}\frac{x_{n+1}-x_n}{2(n+1)-2n}=\lim_{n\to\infty}\frac{2+\frac{1}{x_n}}{2}=1$$

ii)

$$\lim_{n\to\infty}\frac{x_n-2n}{log(\sqrt{n})}=2\cdot\lim_{n\to\infty}\frac{x_n-2n}{log(n)}=2\cdot\lim_{n\to\infty}\frac{x_{n+1}-2(n+1)-x_n+2n}{log(n+1)-log(n)}=$$

$$=2\cdot\lim_{n\to\infty}\frac{\frac{1}{x_n}}{log(\frac{n+1}{n})}=2\cdot\lim_{n\to\infty}\frac{\frac{n}{x_n}}{log(\frac{n+1}{n})^n}=2\cdot\frac{\frac{1}{2}}{loge}=1$$

iii)

$$\lim_{n\to\infty}(x_n-2n)=\lim_{n\to\infty}\frac{(x_n-2n)}{log(\sqrt{n})}\cdot log(\sqrt{n})=\infty$$

iv)

Factoring 2n in ii)

$$\lim_{n\to\infty}\frac{n}{log(\sqrt{n})}\cdot(\frac{x_n}{2n}-1)=\frac{1}{2}$$

v) Using iv) and i) we obtain:

$$\lim_{n\to\infty}\frac{\sqrt{n}}{log(\sqrt{n})}\cdot(\sqrt{x_n}-\sqrt{2n})=\lim_{n\to\infty}\frac{n}{log(\sqrt{n})}\cdot\frac{(\sqrt{x_n}-\sqrt{2n})}{\sqrt{n}}=$$

$$=\sqrt{2}\cdot\lim_{n\to\infty}\frac{n}{log(\sqrt{n})}\cdot(\sqrt{\frac{x_n}{2n}}-1)=\sqrt{2}\cdot\lim_{n\to\infty}\frac{n}{log(\sqrt{n})}\cdot(\frac{x_n}{2n}-1)\cdot\frac{1}{\sqrt{\frac{x_n}{2n}}+1}=\sqrt{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2\sqrt{2}}$$

vi) Using ii) and i) we obtain:

$$\lim_{n\to\infty}\frac{n^2}{log(n)}\cdot(\frac{1}{x_n}-\frac{1}{2n})=\lim_{n\to\infty}\frac{n^2}{log(n)}\cdot(\frac{2n-x_n}{2n\cdot x_n})=-\frac{1}{4}\cdot\lim_{n\to\infty}\frac{n}{log(\sqrt{n})}\cdot\frac{x_n-2n}{x_n}$$

$$=-\frac{1}{4}\cdot\lim_{n\to\infty}\frac{n}{x_n}\cdot\frac{(x_n-2n)}{log(\sqrt{n})}=-\frac{1}{8}$$

$$y_{n+1}-y_n=x_{n+1}-x_n-2-log(\sqrt{\frac{n+1}{n}})=\frac{1}{x_n}-log(\sqrt{\frac{n+1}{n}})=\frac{1}{x_n}-\frac{1}{2n}+\frac{1}{2n}-log(\sqrt{\frac{n+1}{n}})$$

Then,

$$\lim_{n\to\infty} \frac{n^2}{log(n)} \cdot (y_{n+1} - y_n) = \lim_{n\to\infty} \frac{n^2}{log(n)} \cdot (\frac{1}{x_n} - \frac{1}{2n}) + \frac{n^2}{log(n)} \cdot (\frac{1}{2n} - \frac{1}{2} \cdot log(\frac{n+1}{n}) =$$

$$= \lim_{n\to\infty} \frac{n^2}{log(n)} \cdot (\frac{1}{x_n} - \frac{1}{2n}) + \frac{1}{2} \cdot \lim_{n\to\infty} [\frac{1}{log(n)} \cdot \frac{\frac{1}{n} - log(1+\frac{1}{n})}{\frac{1}{n^2}}] = -\frac{1}{8} + 0 = -\frac{1}{8}$$

**(Step 2)**

*The sequence* $(y_n)_{n\geq 1}$ *converges. This is a consequence of the last limit of step* 1.

From the result we just proved above and taking into consideration that $\sum_{n\geq 1} \frac{log(n)}{n^2}$ *converges* (It can be proved by using *Cauchy condensation test*), *we find out that the series* $\sum_{n\geq 1} (y_n - y_{n+1})$ *converges.*

So, the sequence $y_n$ *converges.*

**(Step 3)**

*We now prove that,*

$$\lim_{n\to\infty} \frac{n}{log(n)} \cdot (y_n - y) = \frac{1}{8}$$

Let,

$$L := \lim_{n\to\infty} \frac{y_n - y}{\frac{log(n)}{n}}$$

*Appling Stolz – Cesaro(case* $\left[\frac{0}{0}\right]$) *we obtain:*

$$L = \lim_{n\to\infty} \frac{y_{n+1} - y_n}{\frac{log(n+1)}{n+1} - \frac{log(n)}{n}} = \lim_{n\to\infty} \frac{n^2}{log(n)} \cdot (y_{n+1} - y_n) \cdot \frac{log(n)}{n^2} \cdot \frac{1}{\frac{log(n+1)}{n+1} - \frac{log(n)}{n}} =$$

$$= -\frac{1}{8} \cdot \lim_{n\to\infty} \cdot \frac{log(n)}{n^2} \cdot \frac{c_n^2}{1 - log(c_n)}$$

$Consider\ \ f:[n,n+1]\rightarrow R,\ \ f(x):=\frac{log(x)}{x},\ \ (\forall)\ n\in N^{*}.\ Appling\ \ Mean\ \ Value\ \ Theorem\ \ on\ \ the\ \ interval\ \ [n,n+1]\ \ we\ \ find\ \ that\ \ there\ \ is\ \ a\ \ point\ \ c_n\ \in\ (n,n+1)\ \ such\ \ that:$

$$f'(c_n)=\frac{log(n+1)}{n+1}-\frac{log(n)}{n} <=> \frac{1-log(c_n)}{c_n^2} == \frac{log(n+1)}{n+1}-\frac{log(n)}{n}$$

$Also,\ \ we\ \ notice\ \ that:$

$$n<c_n<n+1\ \ and\ 1-log(n+1)<1-log(c_n)<1-log(n)$$

$By\ \ the\ \ Squeeze\ \ theorem\ \ we\ \ find\ \ that$:

$$1)\ \lim_{n\rightarrow\infty}\frac{c_n}{n}=1 \tag{23}$$

$$2)\ \lim_{n\rightarrow\infty}\frac{1-log(c_n)}{log(n)}=-1$$

$So,$

$$L=-\frac{1}{8}\cdot\lim_{n\rightarrow\infty}\cdot\frac{log(n)}{1-log(c_n)}\cdot(\frac{c_n}{n})^2=-\frac{1}{8}\cdot(-1)\cdot 1=\frac{1}{8}$$

$Which\ \ leads\ \ us\ \ to:$

$$\lim_{n\rightarrow\infty}\frac{n}{log(n)}\cdot(y_n-y)=\frac{1}{8}$$