Interesting applications of the Taylor-Lagrange theorem in finding special limits.

October 29, 2023

1 Introduction

The main purpose of this article is to present a method with which we can find the following limit :

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right)$$

Consider f:[0,1]→ R *a function of class* C^{n+1}. *Let's begin with the limit of the following sequence :*

$$x_n = n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) (2)$$

Let $a_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$

We define :

$\Delta := (x_0, x_1, \ldots, x_n)$, *where* $x_k = \frac{k}{n}$, $(\forall)$ $k = \overline{0,n}$ *as a partition of* $[0,1]$ *with subintervals* $I_k = [x_{k-1}, x_k]$ *determined by the set of endpoints of the intervals* $0 = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = 1$.

$\xi := (\xi_1, \xi_2, ..., \xi_n)$ *where* $\xi_k = \frac{k}{n}$, $(\forall)$ $k = \overline{1,n}$, *the system of intermediate points associated with the partition* Δ.

Then

$$\lim_{n\to\infty} a_n = \sum_{k=1}^{n} f\left(\xi_k\right)\left(\Delta_k - \Delta_{k-1}\right) = \sigma(f,\Delta,\xi) = \int_0^1 f(x)\,dx$$

Now let's try to rewrite x_n :

$$x_n = n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx\right) = n\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f\left(\frac{k}{n}\right) - f(x)\,dx\right) \quad (3)$$

Let $x \in [\frac{k-1}{n}, \frac{k}{n})$ *be a fixed point.*

Applying the "Taylor – Lagrange Theorem" we obtain that there exist a point $c_x \in (x, \frac{k}{n})$ *such that* :

$$f(x) = f\left(\frac{k}{n}\right) + \frac{(x-\frac{k}{n})}{1!}f'\left(\frac{k}{n}\right) + \frac{(x-\frac{k}{n})^2}{2!}f''(c_x) \quad (4)$$

Replacing relation (2) *in* (3) *we obtain that*:

$$x_n = n\left(\sum_{k=1}^{n} - \int_{\frac{k-1}{n}}^{\frac{k}{n}} \frac{(x-\frac{k}{n})}{1!}f'\left(\frac{k}{n}\right) + \frac{(x-\frac{k}{n})^2}{2!}f''(c_x)dx\right) = n\left(\sum_{k=1}^{n}\frac{1}{2n^2}f'\left(\frac{k}{n}\right) - \frac{1}{6n^3}f''(c_x)\right) \quad (5)$$

The function f'' *is continuous. Applying Karl Weierstrass' Theorem, we find that* f'' *is bounded and reaches its boundaries. So there is an* $M > 0$ *such that* $|f(x)| \leq M\ (\forall)\ x \in [0,1]$.

$$0 \leq \left| n\sum_{k=1}^{n}\frac{1}{6n^3}f''(c_x)\right| \leq \left|\sum_{k=1}^{n}\frac{1}{6n^2}M\right| = \frac{M}{6n} \quad (6)$$

Passing to the limit in (5) *and applying the Sandwich Theorem we obtain that* :

$$\lim_{n\to\infty}\left| n\sum_{k=1}^{n}\frac{1}{6n^3}f''(c_x)\right| = 0$$

(7)

$Using$ (6) $we\ find\ that:$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} n\sum_{k=1}^{n} \frac{1}{2n^2} f'\left(\frac{k}{n}\right) = \frac{1}{2}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f'\left(\frac{k}{n}\right) = \frac{1}{2}\sigma(f,\Delta,\xi) = \frac{1}{2}\int_0^1 f'(x)dx = \frac{f(1)-f(0)}{2}$$

(8)

$Therefore\ the\ sequence\ x_n\ is\ convergent\ and\ we\ will\ denote\ its\ limit\ l_1.$

$An\ immediate\ application\ of\ the\ result\ obtained\ above\ is\ the\ following\ problem:$

Find:

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)$$

We note that:

$$x_n = n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n\int_0^1 f(x)\,dx$$

(9)

$$x_{n+1} = \sum_{k=1}^{n+1} f\left(\frac{k}{n}\right) - (n+1)\int_0^1 f(x)\,dx$$

(10)

$Subtracting$ (8) and (9) $we\ obtain:$

$$x_{n+1} - x_n = \sum_{k=1}^{n+1} f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx$$

(11)

Passing to the limit in (10) *we obtain that :*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right)-\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)=\int_0^1 f(x)\,dx$$

Now we want to find the main result of this article, more precisely, to find :

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right)-\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,dx\right) \tag{12}$$

For that we will consider the sequence y_n *:*

$$y_n=n\left(n\left(\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)-\int_0^1 f(x)\,dx\right)-\frac{f(1)-f(0)}{2}\right)=n\left(x_n-\frac{f(1)-f(0)}{2}\right)$$

Let $x\in[\frac{k-1}{n},\frac{k}{n})$ *be a fixed point.*

Applying the "Taylor – Lagrange Theorem" we obtain that there exist a point $d_x\in(x,\frac{k}{n})$ *such that :*

$$f(x)=f\left(\frac{k}{n}\right)+\frac{(x-\frac{k}{n})}{1!}f'\left(\frac{k}{n}\right)+\frac{(x-\frac{k}{n})^2}{2!}f''\left(\frac{k}{n}\right)+\frac{(x-\frac{k}{n})^3}{3!}f'''(d_x) \tag{14}$$

So

$$y_n=n\left(n\left(\sum_{k=1}^{n}-\int_{\frac{k-1}{n}}^{\frac{k}{n}}\frac{(x-\frac{k}{n})}{1!}f'\left(\frac{k}{n}\right)+\frac{(x-\frac{k}{n})^2}{2!}f''\left(\frac{k}{n}\right)+\frac{(x-\frac{k}{n})^3}{3!}f'''(d_x)\right)-\frac{f(1)-f(0)}{2}\right)$$

$$y_n = n\left(n\left(\sum_{k=1}^{n}\frac{1}{2n^2}f'\left(\frac{k}{n}\right)-\frac{1}{6n^3}f''\left(\frac{k}{n}\right)+\frac{1}{24n^4}f''(d_x)\right)-\frac{f(1)-f(0)}{2}\right)$$

The function f''' *is continuous. Applying Karl Weierstrass' Theorem, we find that* f'' *is bounded and reaches its boundaries. So there is an* $L>0$ *such that* $|f(x)|\le L$ $(\forall)$ $x\in[0,1]$.

$$0\le\left|n^2\sum_{k=1}^{n}\frac{1}{24n^4}f'''(d_x)\right|\le\left|\sum_{k=1}^{n}\frac{1}{24n^2}L\right|=\frac{L}{24n} \tag{17}$$

Passing to the limit in (16) *and applying the Sandwich Theorem we obtain that* :

$$\lim_{n\to\infty}\left|n\sum_{k=1}^{n}\frac{1}{6n^3}f'''(d_x)\right|=0$$

Also :

$$\lim_{n\to\infty}n\left(n\sum_{k=1}^{n}\frac{-1}{6n^3}f''\left(\frac{k}{n}\right)\right)=\frac{-1}{6}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f''\left(\frac{k}{n}\right)=\frac{-1}{6}\sigma(f'',\Delta,\xi)=\frac{-1}{6}\int_0^1 f''(x)dx=-\frac{f'(1)-f'(0)}{6} \tag{19}$$

Taking into account the relations (7) *and* (18) *we can write* :

$$\begin{aligned}\lim_{n\to\infty}y_n &= \frac{f'(0)-f'(1)}{6}+\lim_{n\to\infty}n\left(n\sum_{k=1}^{n}\frac{1}{2n^2}f'\left(\frac{k}{n}\right)-\frac{f(1)-f(0)}{2}\right)\\ &= \frac{f'(0)-f'(1)}{6}+\lim_{n\to\infty}\frac{n}{2}\left(\frac{1}{n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-(f(1)-f(0))\right)\\ &= \frac{f'(0)-f'(1)}{6}+\lim_{n\to\infty}\frac{n}{2}\left(\frac{1}{n}\sum_{k=1}^{n}f'\left(\frac{k}{n}\right)-\int_0^1 f'(x)dx\right)\\ &= \frac{f'(0)-f'(1)}{6}+\lim_{n\to\infty}\frac{n}{2}\left(\sum_{k=1}^{n}\int_{\frac{k-1}{n}}^{\frac{k}{n}} f'\left(\frac{k}{n}\right)-f'(x)\,dx\right)\end{aligned}$$

Applying again the "Taylor – Lagrange Theorem" we obtain that there exist a point $t_x\in(x,\frac{k}{n})$ *such that* :

$$f'(x)=f'\left(\frac{k}{n}\right)+\frac{(x-\frac{k}{n})}{1!}f''\left(\frac{k}{n}\right)+\frac{(x-\frac{k}{n})^2}{2!}f'''(t_x)$$

$Therefore,$

$$\lim_{n\to\infty} y_n = \frac{f'(0)-f'(1)}{6} + \lim_{n\to\infty} \frac{n}{2}\left(\sum_{k=1}^{n} - \int_{\frac{k-1}{n}}^{\frac{k}{n}} \frac{(x-\frac{k}{n})}{1!} f''\left(\frac{k}{n}\right) + \frac{(x-\frac{k}{n})^2}{2!} f'''(t_x)\right)$$

$$= \frac{f'(0)-f'(1)}{6} + \lim_{n\to\infty} \frac{n}{2}\left(\sum_{k=1}^{n} \frac{1}{2n^2} f''\left(\frac{k}{n}\right) + \frac{1}{6n^3} f'''(t_x)\right)$$

$But,$

$$0 \le \left| n \sum_{k=1}^{n} \frac{1}{6n^3} f'''(t_x) \right| \le \left| \sum_{k=1}^{n} \frac{1}{6n^2} L \right| = \frac{L}{6n}$$

$By\ Sandwich\ Theorem\ we\ obtain\ that:$

$$\lim_{n\to\infty} \left| n \sum_{k=1}^{n} \frac{1}{6n^3} f'''(t_x) \right| = 0$$

$So,$

$$\lim_{n\to\infty} y_n = \frac{f'(0)-f'(1)}{6} + \frac{1}{4} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f''\left(\frac{k}{n}\right) = \frac{f'(0)-f'(1)}{6} + \frac{1}{4}\sigma(f'', \Delta, \xi)$$

$$= \frac{f'(0)-f'(1)}{6} + \frac{1}{4}\int_0^1 f''(x)dx = \frac{f'(1)-f'(0)}{4} - \frac{f'(1)-f'(0)}{6} = \frac{f'(1)-f'(0)}{12}$$

$Now\ note\ that:$

$$y_n = n\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n^2 \int_0^1 f(x)\,dx - n\frac{f(1)-f(0)}{2} = n\sum_{k=1}^{n} f\left(\frac{k}{n}\right) - n^2 \int_0^1 f(x)\,dx - \frac{n}{2}\int_0^1 f'(x)\,dx$$

$$y_{n+1} = (n+1)\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - (n+1)^2 \int_0^1 f(x)\,dx - \frac{n+1}{2}\int_0^1 f'(x)\,dx$$

$So,$

$$y_{n+1} - y_n = n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - 2\int_0^1 f(x)\,dx\right) + \sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \frac{f(1)-f(0)}{2}$$

$$= n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) + \sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - n\int\limits_0^1 f(x)\,dx - \frac{f(1)-f(0)}{2}$$

$$= n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) + \sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - (n+1)\int\limits_0^1 f(x)\,dx - \frac{f(1)-f(0)}{2} + \int\limits_0^1 f(x)\,dx$$

$$= n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) + (n+1)\left(\frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \int\limits_0^1 f(x)\,dx\right) -$$

$$- \frac{f(1)-f(0)}{2} + \int\limits_0^1 f(x)\,dx$$

$$= n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) + x_{n+1} - \frac{f(1)-f(0)}{2} + \int\limits_0^1 f(x)\,dx$$

Finally,

$$y_{n+1} - y_n = n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) + x_{n+1} - \frac{f(1)-f(0)}{2} + \int\limits_0^1 f(x)\,dx$$

Passing to the limit we obtain :

$$0 = \lim_{n\to\infty} n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) + 0 + \int\limits_0^1 f(x)\,dx$$

So, we just proved that :

$$\lim_{n\to\infty} n\left(\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int\limits_0^1 f(x)\,dx\right) = -\int\limits_0^1 f(x)\,dx$$