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PHYS414 Final Project

Table of Contents

Newton

Part A)

We have:

$$\circledast \ \frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \quad (1)$$

$$\circledast \ \frac{dP(r)}{dr} = -\frac{G\, m(r)\, \rho(r)}{r^2} \quad (2)$$

$$\circledast \ P = K\rho^{\gamma} = K\rho^{(1+\frac{1}{n})} \quad (3)$$

Substitute from (3) into (2):

$$\circledast \ \frac{d}{dr}\left[K\rho(r)^{(1+\frac{1}{n})}\right] = \frac{-G\, m(r)\, \rho(r)}{r^2}$$

$$\circledast \ K\left(1+\frac{1}{n}\right)\rho(r)^{\frac{1}{n}}\frac{d\rho(r)}{dr} = -\frac{G\, m(r)\, \rho(r)}{r^2}$$

$$\circledast \ -m(r) = \left[K\left(1+\frac{1}{n}\right)\rho(r)^{(\frac{1}{n}-1)}\frac{d\rho(r)}{dr}r^2\right]/G$$

$$\circledast \ -m(r) = r^2\frac{K(n+1)}{G}\frac{d}{dr}\rho(r)^{\frac{1}{n}} \quad ; \text{ use } (1)$$

$$\circledast \ -\frac{dm(r)}{dr} = -4\pi r^2 \rho(r) = \frac{d}{dr}\left[r^2\frac{K(n+1)}{G}\frac{d}{dr}\rho(r)^{\frac{1}{n}}\right]$$

$$\circledast \ \frac{K(n+1)}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}(\rho(r))^{\frac{1}{n}}\right] + \rho(r) = 0$$

Let $\alpha\xi := r$ such that $\frac{K(n+1)}{4\pi G}\rho_c^{(\frac{1}{n}-1)} = \alpha^2$, then:

$$\circledast \ \frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d}{d\xi}\left(\frac{\rho(\xi)}{\rho_c}\right)^{\frac{1}{n}}\right] + \frac{\rho(\xi)}{\rho_c} = 0$$

Let $\theta(\xi) = \left(\frac{\rho(\xi)}{\rho_c}\right)^{\frac{1}{n}}$ such that $\theta(0) = 1$ since $\rho(0) = \rho_c$

Therefore; $\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d}{d\xi}\theta(\xi)\right] + \theta(\xi)^n = 0$; Q.E.D

We have:

$$\circledast \; \frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2 \frac{d}{d\xi}\theta(\xi)\right] + \theta(\xi)^n = 0$$

Want to find an expression for values around zero:

$$\circledast \; \theta(\xi) = \theta(0) + \theta'(0)\xi + \frac{\theta''(0)}{2!}\xi^2 + \frac{\theta'''}{3!}\xi^3 + \dots$$

$$\circledast \; \frac{d}{d\xi}\theta(\xi) = \theta'(0) + \theta''(0)\xi + \frac{\theta'''(0)}{2!}\xi^2 + \dots$$

$$\circledast \; \frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d}{d\xi}\theta(\xi)\right] = \frac{2}{\xi}\theta'(0) + 3\theta''(0) + \frac{4}{2!}\theta'''(\xi)\xi + \dots \quad (1)$$

$$\circledast \; \theta(\xi)^n = \theta(0)^n + \binom{n}{1}\theta'(0)\theta(0)^{(n-1)}\xi + \binom{n}{1}\frac{\theta''(0)}{2!}\theta(0)^{(n-1)}\xi^2 + \dots (2)$$

We require that $(1) + (2) = 0$, coefficient of each term has to be 0

From these equations, we find that;

$$\circledast \; \theta(0) = 1 \;\&\; \theta'(0) = 0 \;\&\; \theta''(0) = \frac{-1}{3}$$
$$\&\; \theta'''(0) = 0 \;\&\; \theta^{(4)}(0) = \frac{n}{5}$$

Therefore, we have:

$$\circledast \; \theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n\xi^4}{120} + \dots \quad ; \; Q.E.D$$

For $n = 1$;

$$\circledast \; \theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} + \dots$$

We have:

$$\circledast \quad \frac{d}{dr} m(r) = 4\pi r^2 \rho(r)$$

Want to find the total mass of the star.

Use the following change of variables:

$$\circledast \quad r \to \alpha \xi \quad \& \quad \rho(r) \to \rho_c \, \theta(\xi)^n$$

We get:

$$\circledast \quad \frac{d}{d\xi} m(\xi) = 4\pi \xi^2 \rho_c \, \theta(\xi)^n \alpha^3$$

Total mass M is given by;

$$\circledast \quad M = \int_0^{\xi_f} 4\pi \rho_c \alpha^3 \xi^2 \theta(\xi)^n \, d\xi \text{ ; where } \alpha \xi_f = R$$

Using Lane-Emden equation to substitute for $\xi^2 \theta(\xi)^n$ gives:

$$\circledast \quad M = 4\pi \rho_c \alpha^3 \int_0^{\xi_f} -\frac{d}{d\xi}\left[\xi^2 \frac{d}{d\xi} \theta(\xi)\right] d\xi \text{ ; use FTC}$$

$$\circledast \quad M = 4\pi \rho_c \alpha^3 \xi_f^3 \left[\frac{-1}{\xi_f} \frac{d}{d\xi} \theta(\xi_f)\right] \text{; use } \alpha^3 \xi_f^3 = R^3$$

$$\circledast \quad M = 4\pi \rho_c R^3 \left[\frac{-1}{\xi_f} \frac{d}{d\xi} \theta(\xi_f)\right] \text{ ; Q.E.D}$$

We have;

$$\circledast \quad M = 4\pi \rho_c R^3 \left[\frac{-1}{\xi_f} \frac{d}{d\xi} \theta(\xi_f) \right] \qquad (1)$$

Want to express mass as a function of radius.

$$\circledast \quad \alpha^2 = \frac{K(n+1)}{4\pi G} \rho_c^{\left(\frac{1}{n}-1\right)} \quad ; \text{ substitute to } (1)$$

$$\circledast \quad M = 4\pi^{\left(\frac{1}{1-n}\right)} R^{\left(\frac{3-n}{1-n}\right)} \left[\frac{-d}{d\xi} \theta(\xi_f) \right] \left(\frac{K}{G}\right)^{\left(\frac{n}{n-1}\right)} (n+1)^{\left(\frac{n}{n-1}\right)} \xi_f^{\left(\frac{n+1}{n-1}\right)}$$

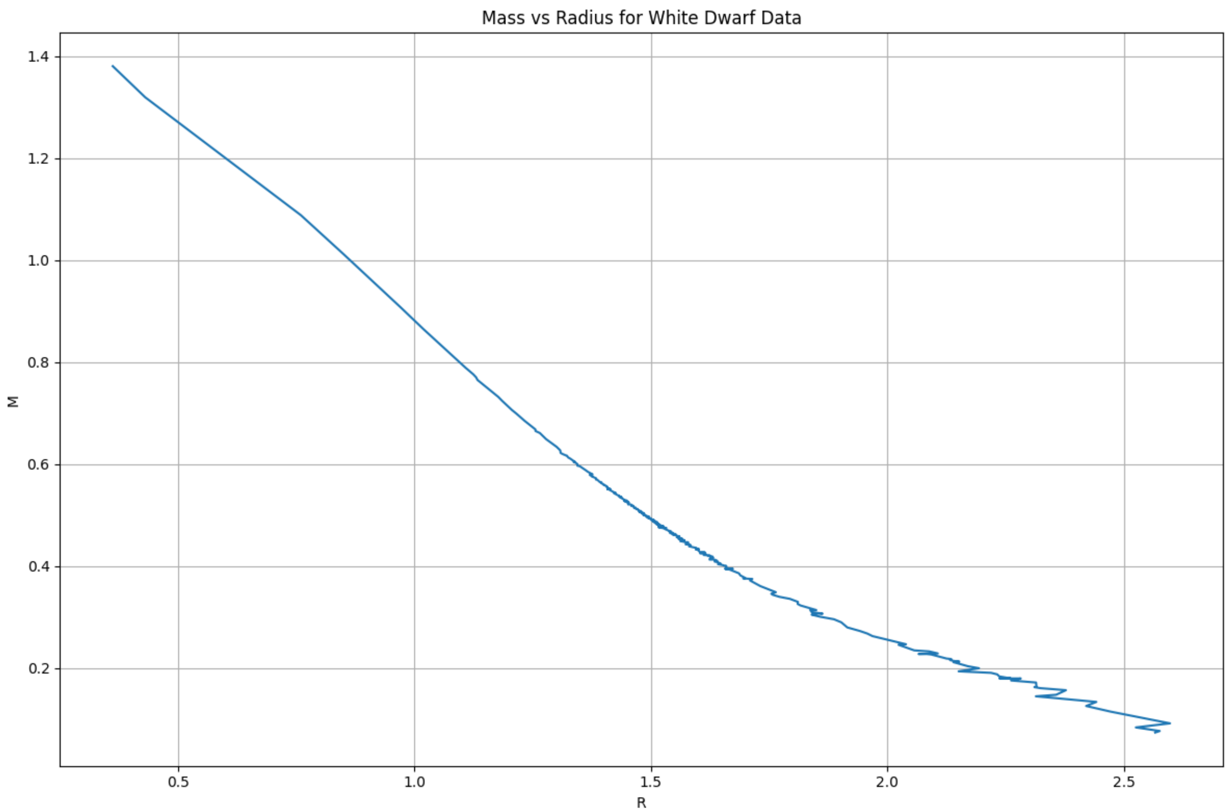
So, we can see that;

$$\circledast \quad M \propto R^{\left(\frac{3-n}{1-n}\right)} \quad ; \text{ Q.E.D}$$

with a proportionality constant of:

$$\circledast \quad 4\pi^{\left(\frac{1}{1-n}\right)} \left[\frac{-d}{d\xi} \theta(\xi_f) \right] \left(\frac{K}{G}\right)^{\left(\frac{n}{n-1}\right)} (n+1)^{\left(\frac{n}{n-1}\right)} \xi_f^{\left(\frac{n+1}{n-1}\right)}$$

Part B)



Plot 1 Mass vs Radius for White Dwarf Data

Part C)

We have;

$$P = C\left[x(2x^2-3)(x^2+1)^{\frac{1}{2}} + 3\sinh^{-1}(x)\right] \quad (1)$$

$$x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}} \quad (2)$$

$$P = K^* \rho^{1+\frac{1}{n^*}} \quad (3)$$

Considering $x \ll 1$;

$$(x^2+1)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots$$

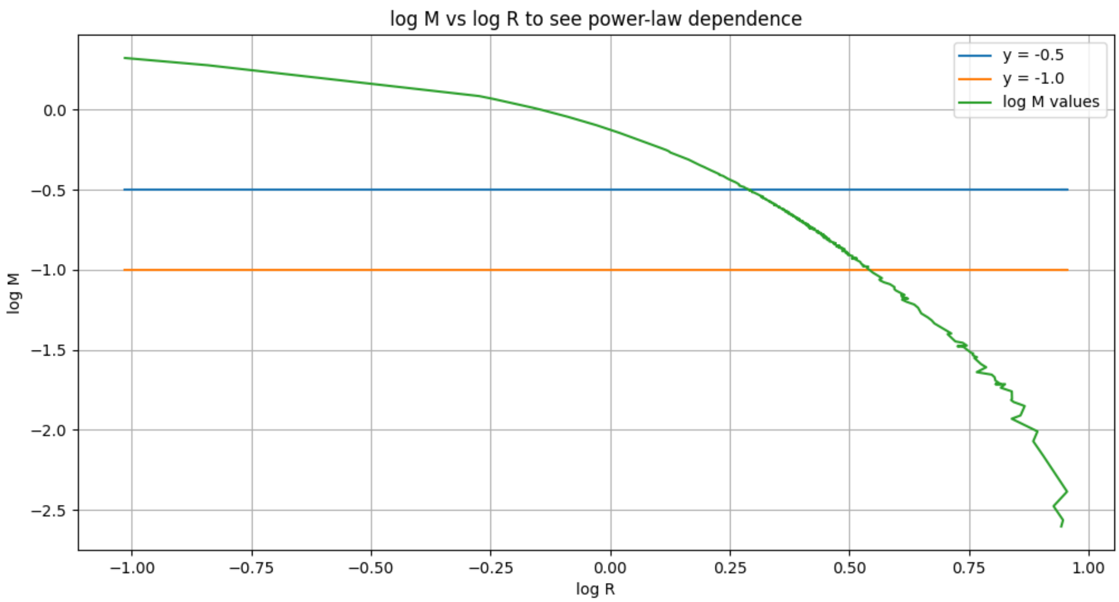
$$\sinh^{-1}(x) \simeq x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \dots$$

$$P \simeq C\left[\frac{11}{8}x^5 + \frac{1}{2}x^3 - 3x + 3\left(x - \frac{1}{6}x^3 + \frac{3}{40}x^5\right) + O(x^7)\right]$$

$$P \simeq C\left[\frac{8}{5}x^5 + O(x^7)\right]$$

$$P = \frac{8}{5}\frac{C}{D^{\left(\frac{5}{q}\right)}}\rho^{\left(\frac{5}{q}\right)}$$; fitting for (3) gives:

$$K^* = \frac{8C}{5D^{\left(\frac{5}{q}\right)}} \quad \& \quad n^* = \frac{q}{5-q}$$; Q.E.D



Plot 2 Demonstration of power-law dependence

From the above graph, the cutoff value is chosen to be -1.1 for the calculations regarding the rest of this section.

The calculated value of q from the fit is: 2.9576226797905165 and the closest integer value 3 is used for q in the remaining part of this section.

Since n = q / (5 - q) using q = 3 gives n_star as 3/2

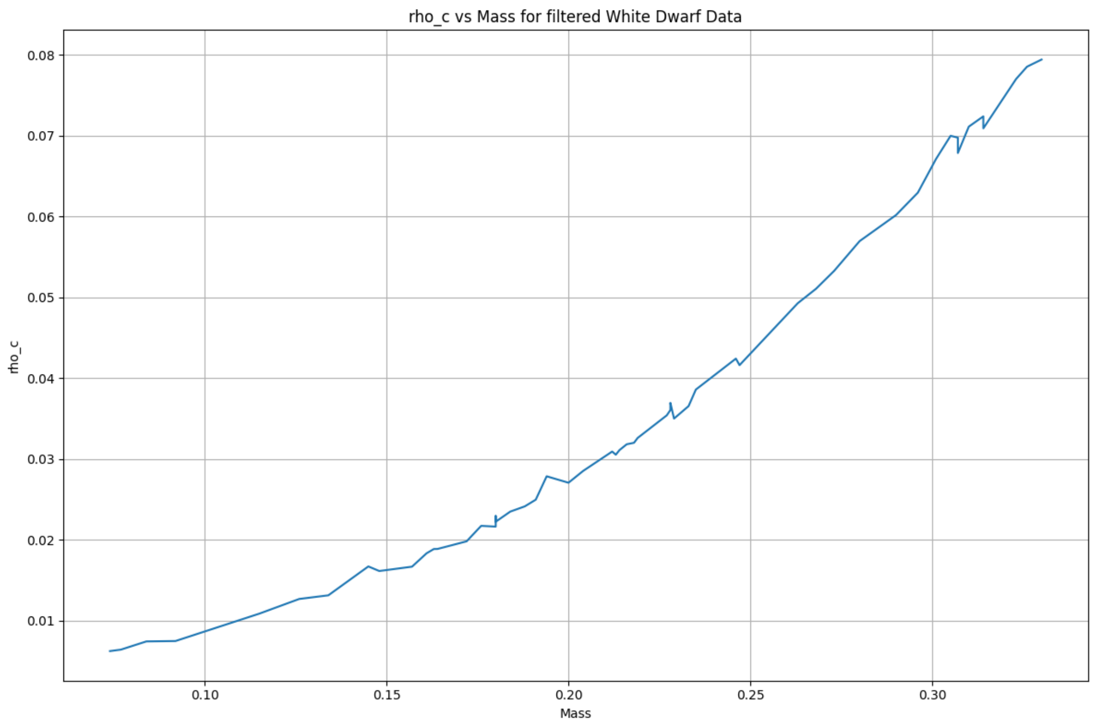
Then, the calculated value for K in the scaled units is: 0.3059168066552295

If we revert the value to SI units, we have K_SI = 3187788.5335691073

The analytical values for C, D, and K are calculated as:

- C: 6.002334828410536e+21
- D: 1947864981.4433918
- K: 3161128.260832043

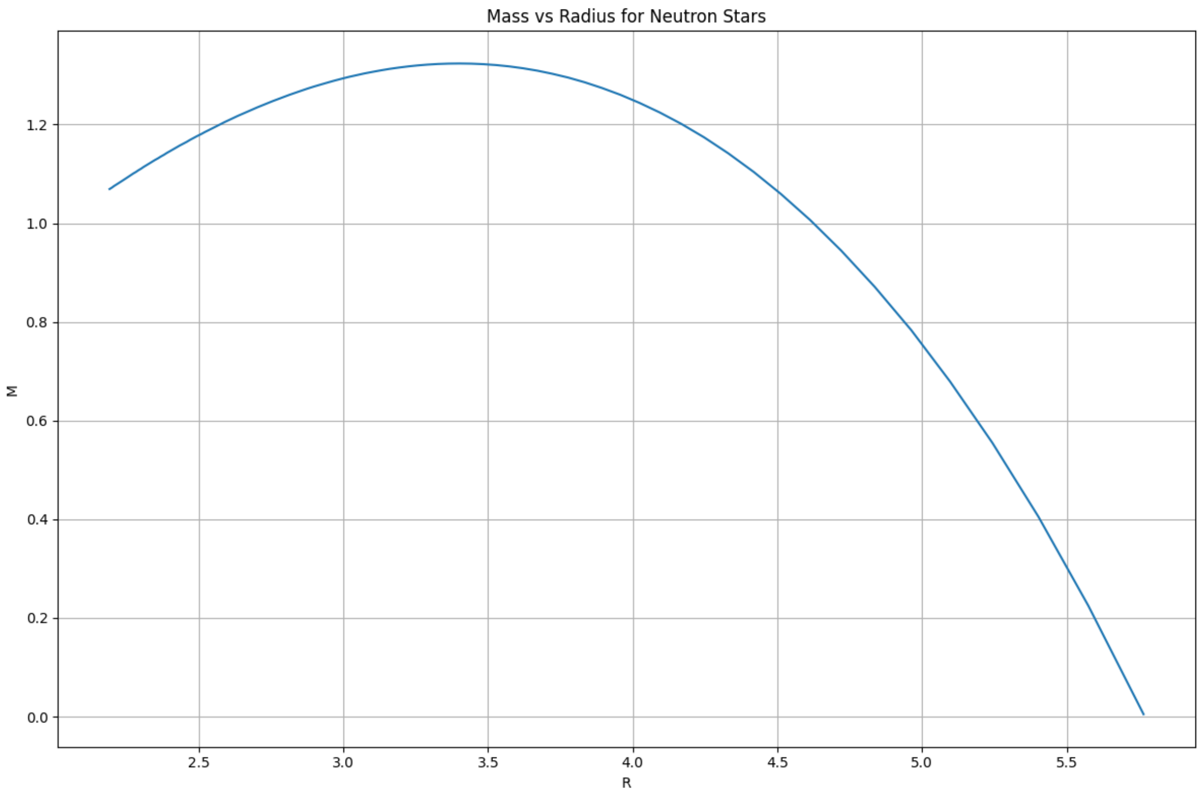
Therefore, the error for K is 0.84%



Plot 3 rho_c vs Mass for Filtered White Dwarf Data

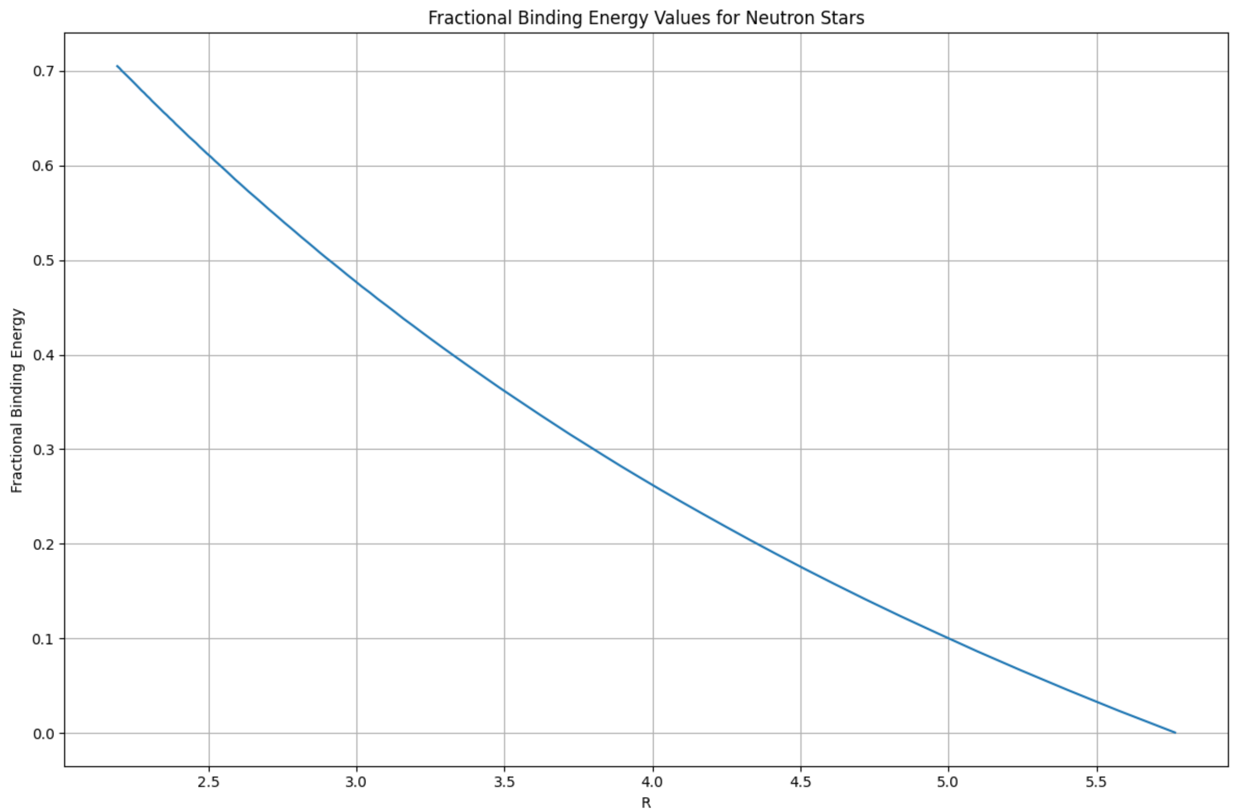
Einstein

Part A)



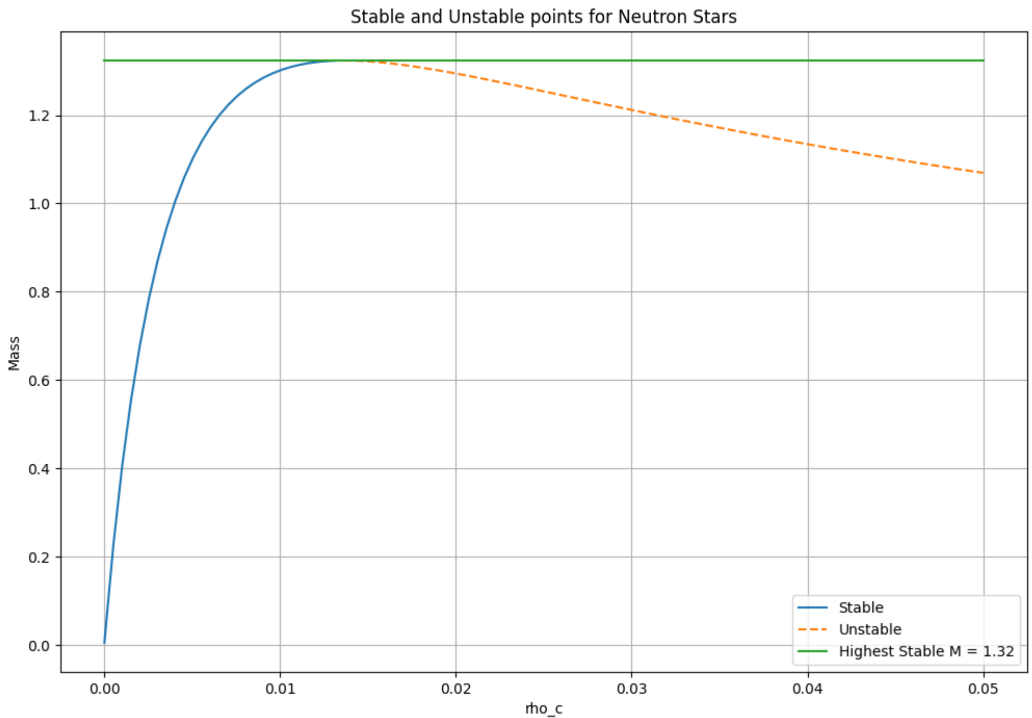
Plot 4 Mass vs Radius for Neutron Stars

Part B)



Plot 5 Fractional Binding Energy Values for Neutron Stars

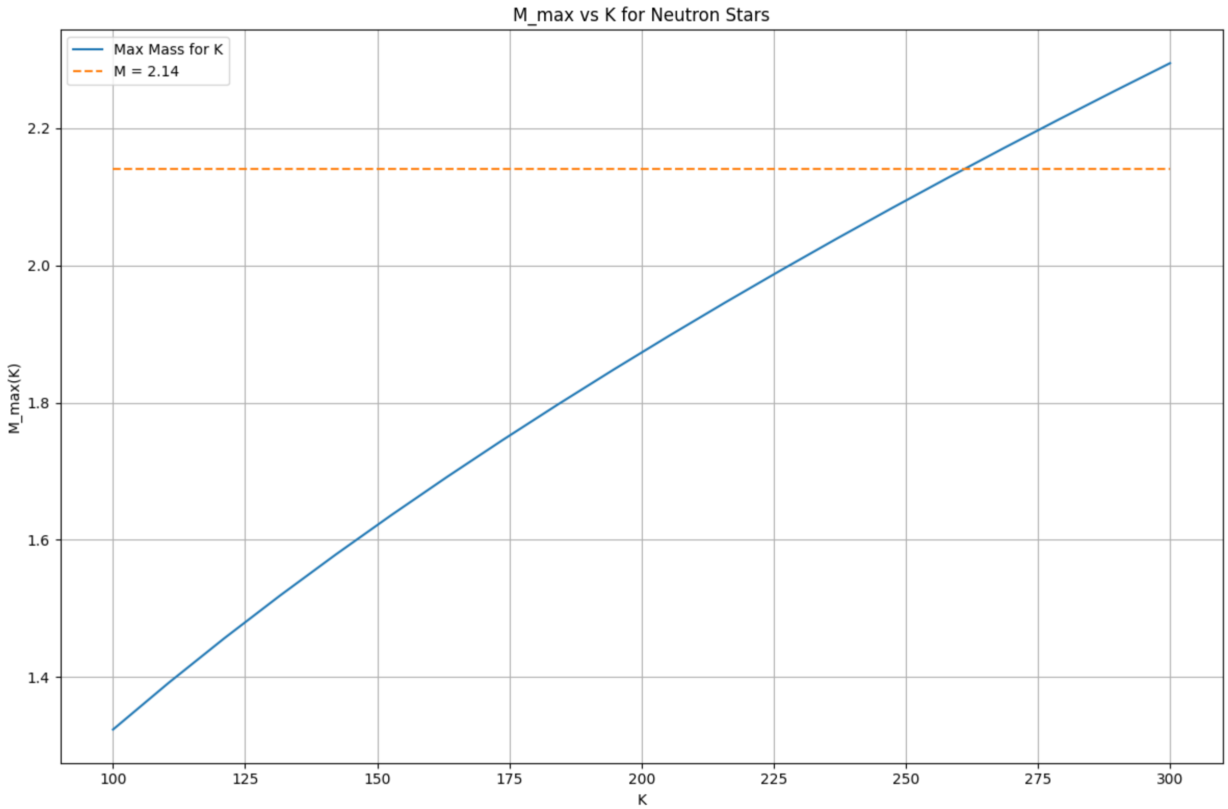
Part C)



Plot 6 Stable and Unstable points for Neutron Stars

The highest stable mass allowed by this Equation of State is 1.32 in scaled units.

Part D)



Plot 7 M_max vs K for Neutron Stars

Above is the graph for the highest stable mass for the Neutron Star corresponding to different K values.

Below are the values of the maximum masses for different K:

- Mass: 1.32344 for k=100.00
- Mass: 1.39154 for k=110.53
- Mass: 1.45647 for k=121.05
- Mass: 1.51863 for k=131.58
- Mass: 1.57834 for k=142.11
- Mass: 1.63589 for k=152.63
- Mass: 1.69148 for k=163.16
- Mass: 1.74530 for k=173.68
- Mass: 1.79752 for k=184.21
- Mass: 1.84826 for k=194.74
- Mass: 1.89765 for k=205.26
- Mass: 1.94579 for k=215.79
- Mass: 1.99277 for k=226.32
- Mass: 2.03867 for k=236.84
- Mass: 2.08355 for k=247.37
- Mass: 2.12750 for k=257.89
- Mass: 2.17055 for k=268.42 (first one above maximum)
- Mass: 2.21277 for k=278.95
- Mass: 2.25420 for k=289.47
- Mass: 2.29488 for k=300.00

Part E)

We have;

⊛ $\nu' = \dfrac{2M}{r(r-2M)}$, $(r > R)$

⊛ $\int_R^r \nu' \, dr' = \int_R^r \dfrac{2M}{r'(r'-2M)} dr'$

⊛ $\nu(r) - \nu(R) = \int_R^r \dfrac{2M}{(r')^2} \dfrac{1}{1 - \frac{2M}{r'}} dr'$; Use change of variables

⊛ $1 - \dfrac{2M}{r'} = u \Rightarrow \dfrac{2M}{(r')^2} dr' = du$

⊛ $\nu(r) - \nu(R) = \int_{1-\frac{2M}{R}}^{1-\frac{2M}{r}} \dfrac{1}{u} du$

⊛ $\nu(r) - \nu(R) = \ln\left(1 - \frac{2M}{r}\right) - \ln\left(1 - \frac{2M}{R}\right)$

⊛ $\nu(r) = \ln\left(1 - \frac{2M}{r}\right) - \ln\left(1 - \frac{2M}{R}\right) + \nu(R)$; $(r > R)$ Q.E.D