Methods of Proving Theorems

Definition:

-1 An integer n is called even if $n = 2m, m \in z$

-2 An integer n is called odd if $n = 2m + 1, m \in z$.

Direct Proofs

A Direct proof is used to show $p \to q$ $true$ whenever p is true. Thus in a direct proof we assume that p is true and use axioms, definitions to show q must be true.

Example1:

Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Proof: By definition of an odd integer, it follows that $n = 2k + 1$,Where k is an integer. We want to show that n^2 is odd. We can square both sides of the equation $n = 2k + 1$ $we\ get$

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Therefore n^2 is odd.

Example 2: If a and b are both odd integers , then $a + b$ is an even integer.

Proof: Since a and d are odd integers, then $a = 2m + 1$, $b = 2n + 1$ so, $a + b = (2m + 1) + (2n + 1) = 2(m + n + 1)$ and

$a + b = 2k, where\ k = m + n + 1.$

Therefore $a + b$ is an even integer.

<u>Proof by contraposition (indirect proof)</u>

This method is very useful and a powerful at all levels of mathematics and computer science. Contrapositive method follows

from the logical equivalence $p \to q \equiv \neg p \to \neg q$.This means that the conditional statement $p \to q$ can be proved by showing that its contrapositive $\neg q \to \neg$p is true .

Example 1:

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution :$p \to q$ means that if $3n + 2$ is odd , then n is odd.

Its contrapositive $\neg q \to \neg$p that is, n is even implies $3n + 2$ is even .

Assume that n is even

$$n = 2k, \quad k \in Z$$

$$\Rightarrow 3n = 3(2k)$$

$$\Rightarrow 3n + 2 = 6k + 2$$

$$= 2(3k + 1)$$

$\Rightarrow 3n + 2$ is even.

Therefore if $3n + 2$ is odd, then n is odd.

Example 2: For any integer $n > 2$, prove that n is prime implies n is odd.

Proof: (by contrapositive)

Let p: n is prime and $q{:}\,n$ is odd. Then $\neg q{:}\,n$ is even and $\neg p{:}\,n$ is not prime.

We know that $p \to q \equiv \neg q \to \neg p$.

we prove that $\neg q \to \neg p$. So, if n is an integer number greater than 2. Then

$n = 2m, m \in Z$. Thus n is divisible by 2 and $n \neq 2$.

Therefore n can not be a prime. Thus $\neg q \to \neg p$.

Hence n is prime $\Rightarrow n$ is odd.

Proof by contradiction:

Example1: Prove that $\sqrt{2}$ is irrational number using proof by contradiction.

Proof: Suppose that $\sqrt{2}$ is rational number.

$\Rightarrow \sqrt{2} = \frac{a}{b}$ Where a and b have no common factor.

$$\Rightarrow 2 = \frac{a^2}{b^2}$$

$$\Rightarrow 2b^2 = a^2$$

$\Rightarrow a^2$ is an even number

$\Rightarrow a$ is an even number

$$\Rightarrow a = 2c \ \ , c \in Z$$

$$\Rightarrow 2b^2 = 4c^2$$

$$\Rightarrow b^2 = 2c^2$$

$\Rightarrow b^2$ is an even integer

$\Rightarrow b$ is an even integer.

This contradicts a and b have no common factor.

Example 2: Give a proof by contradiction of the statement. *If* $3n + 2$ *is odd, then* n *is odd.*

Proof: Assume that n is even. Then $n = 2k, k \in Z$

$$\Rightarrow \ 3n = 6k$$

$$\Rightarrow \ 3n + 2 = 6k + 2$$

$$\Rightarrow \ 3n + 2 = 2(3k + 1)$$

$\Rightarrow$ $3n + 2$ is even, a contradiction that $3n + 2$ *is odd*.

Therefore *If* $3n + 2$ *is odd, then* n *is odd*.

Proof by Counter example

Example: **Prove or disprove If x *and* y *are real numbers , then***

$$(\boldsymbol{x^2 = y^2}) \Leftrightarrow (\boldsymbol{x = y}).$$

Counter example : **-3 and 3 are real numbers and $(-3)^2 = 3^2$**

But -3$\neq$ 3.

Example 2: **Prove or disprove** Let a and b be real numbers .

If $a > b, then\ a^2 > b^2$

Solution: The statement is false. Since -2> -3 , but $(-2)^2 < (-3)^2$.

Mathematical Induction:

In general mathematical induction can be used to prove statements that assert that $p(n)$ is true for all positive integers n, where $p(n)$ is a proposition function, A proof by mathematical induction has two parts ,**a basis step** , where we show that $p(1)$ is true and **inductive step** , where we show that for all positive integers k, if $p(k)$ is true , then $p(k+1)$

is true.

Example 1: Show that if n is positive integer , then

$$1 + 2 + 3 + \cdots .. + n = \frac{n(n+1)}{2} \qquad ....(1)$$

Proof: By induction. Let $p(n)$: $1 + 2 + 3 + \cdots .. + n = \frac{n(n+1)}{2}$

Basis step: $p(1)$, *that is* $n = 1$, we get $1 = \frac{1(1+1)}{2}$ =1 which is true.

Inductive step: We assume that $p(k)$ is **true.**

Assume that $1+2+3+\cdots..+k=\frac{k(k+1)}{2}$ is true ...... (2)

We must show that

$1+2+3+\cdots..+k+(k+1)=\frac{(k+1)(k+1+1)}{2}=\frac{(k+1)(k+2)}{2}$ is true ....(3)

Add $(k+1)$ to both sides of equation (2)

$1+2+3+\cdots..+k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{k(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$

Which is equation (3).

Hence, the proposition is true for all values of n.

Example 2: Use mathematical induction to show that

$1+2+2^2+3^2+\cdots.+2^n=2^{n+1}-1$. For all n$\in N$.

Proof: let $p(n)$ be the proposition that $1+2+2^2+3^2+\cdots.+2^n=2^{n+1}-1$.

<u>**Basis step**</u>: $p(0)$ is true because $2^0=2^1-1=1$.

<u>Inductive **step**</u>**:** We assume that $p(k)$ **is true . That is**

$1+2+2^2+3^2+\cdots.+2^k=2^{k+1}-1$ **is true. ....(1)**

We show that the proposition is true for $p(k+1)$. That is we must show that

$1+2+2^2+3^2+\cdots.+2^k+2^{k+1}=2^{(k+1)+1}-1=\mathbf{2^{k+1}-1}$ **is true.**

Add 2^{k+1} to both sides of equation (1).

$(1+2+2^2+3^2+\cdots.+2^k)+2^{k+1}=2^{k+1}-1+2^{k+1}$

$$=(1+2+2^2+3^2+\cdots.+2^k)+2^{k+1}$$

$$=2^{k+1}-1+2^{k+1}$$

$$=\mathbf{2}2^{k+1}-1$$

$$=2^{k+2}-1.$$

Therefore $p(k+1)$ is true and hence $p(n)$ is true for all $n\in N$.