Connectivity

Paths: A path is a sequence of edges that begins at a vertex of a graph and travels from vertex to a vertex along edges of the graph.

Definition:

Let n be a nonnegative integer and G an undirected graph. A path of length n from a vertex u to a vertex v in G is a sequence of n edges $e_1, e_2, \dots, e_n$ of G such that $e_1 = x_0x_1, e_{2=}x_1x_2, \dots, e_n = x_{n-1}x_n$, where $u = x_0$ and $v = x_n$.

When a graph is simple, we denote this path by its vertex sequence $x_0, x_1, \dots, x_n$.

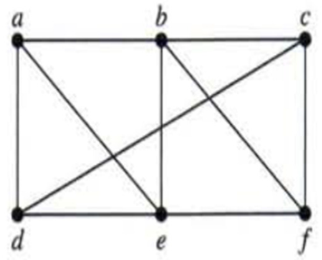
The path is circuit if it begins and ends at the same vertex, that is , if $u = v$ and has length greater than zero.

A path or circuit is simple if it does not contain the same edge more than once.

Example1:

In the simple graph shown in the figure , a, d, c, f, e is a simple path of length 4, because ad, dc, cf, and fe are all edges. However , d, e, c, a is not a path , because ec is not an edge.

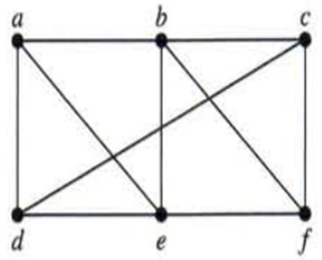
Note that b, c, f, e, b is a circuit of length 4.



<u>Example</u>: (a) Find a simple path of length 5 and a simple path of length 6 from a to f.

(b)Find a longest path from$a$ to f.

(c)Find a circuit of length 7 (d) Find a cycle of length 6.



<u>Solution: (a)</u> ab, bc, cd, de, ef,is a path of length 5 and ab, bc, cd, de, eb,cd, is a path of length6.

(b)one of the longest paths is the path $ad, de, ef, fb, be, ea, ab, bc, cf$ of length 9.

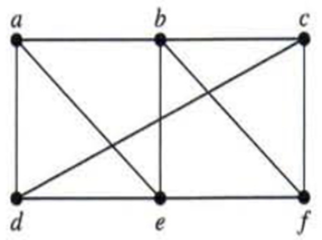
(c) $ad, dc, cb, bf, fe, eb, ba$ is a circuit of length 7.

(d) ad, dc, cf, fb, be, ea is a cycle of length 6.

<u>Definition</u>: A vertex u in a graph G is said to be connected to a vertex v if there is a path in G from u to v.

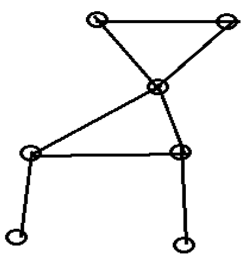
<u>Connected graph</u>: A graph is connected if every two of its vertices are connected. A graph is connected if and only if it has exactly one component.

Example: Show that the given graph is connected



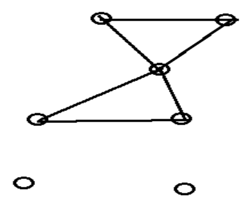
Proof: Since between any two vertices in the given graph , there is a path . Then the graph is connected.

Example:



Definition: A graph that is not connected is called disconnected.

Example:

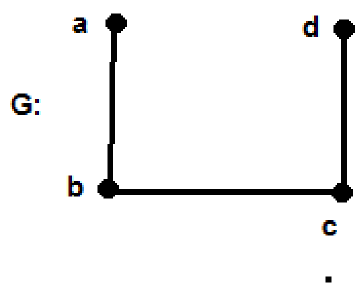


Disconnected graph

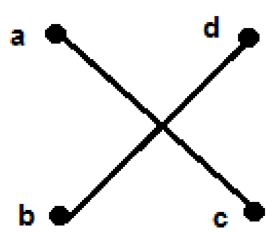
The statement If a graph is connected ,then its complement need not connected.

Also, If If a graph is connected ,then its complement need not disconnected.

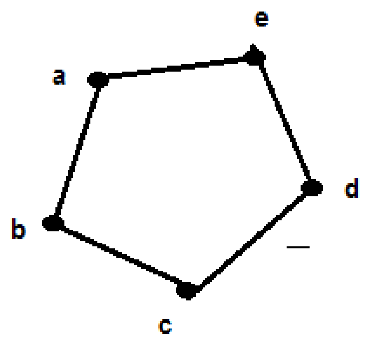
Examples:



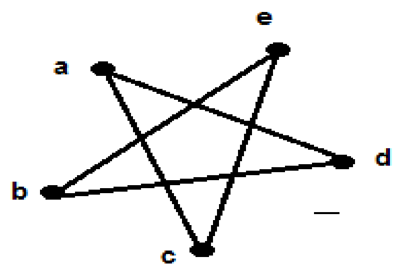
G is a connected graph. But its complement $\bar{G}$ is disconnected graph.



The graph C_5 is connected.



and its complement $\acute{C}_5$ is connected



$\boldsymbol{Theorem}$: $if\ \ G\ is\ disconnected\ graph, then\ \overline{G}\ is\ connected.$

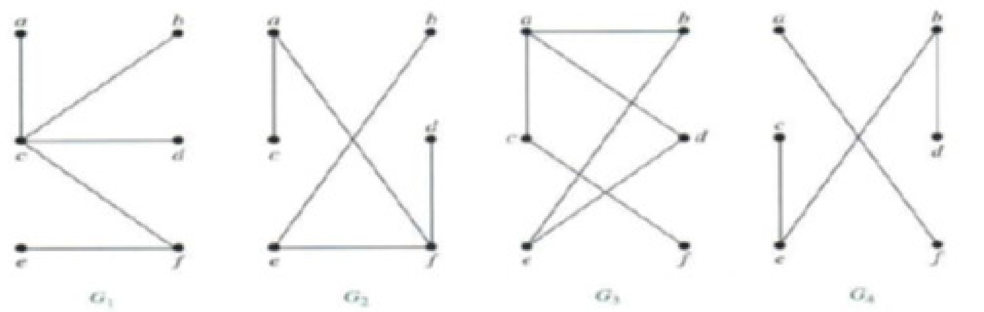
TREES

Definition:

A tree is a conneted undirected graph with no cycles.

Example:

Which of the following graphs are trees?



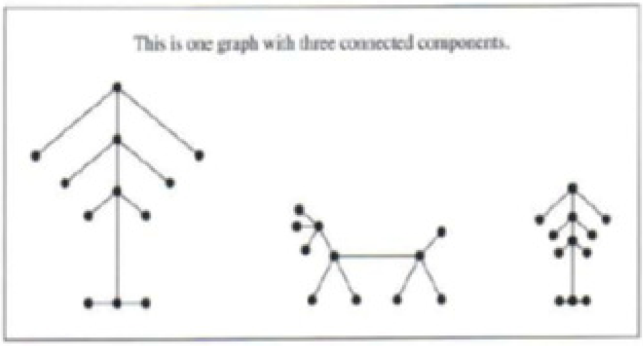
Examples of Trees and Graphs That Are Not Trees.

Solution: G_1 and G_2 are trees, because both are connected graphs with no

Cycles. G_3 is not a tree because e e, b, a, e is a cycle. And G_4 is not a tree because it is not connected.

Theorem:

A graph is a tree if and only if there is a unique path between any two of its vertices.



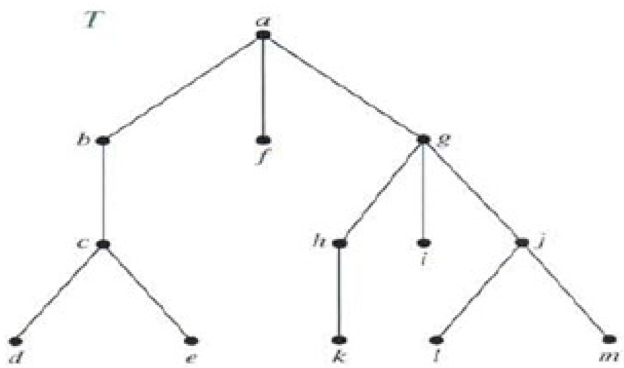
Example of a Forest.

In many applications of a tree , a particular vertex of a tree is designated as the **root.** Once we specify a root, we can assign a direction to each edge as follows. Because there is a unique path from the root to each vertex of the graph (by Theorem 1),we direct each edge away from the root. Thus , a tree togather with its root produces a directed graph a **rooted tree**.

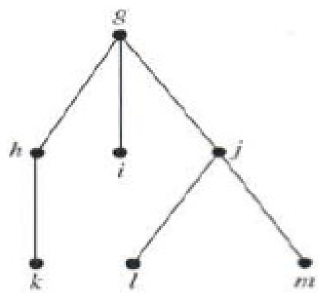
Definition:

A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

The direction of the edge in a root tree can be omitted because the choice of the root determines the direction of the edges



A Rooted Tree T.



Subtree Rooted at g.

3.2 Terminology for trees:

The terminology for trees has botanical and genealogical origin. Suppose that T is a root tree . If v is a vertex in Tother than the root, the $\boldsymbol{parent}$ of v is the unique vertex u such that there is a directed edge from u to v. When u is the parent of v is called a $\boldsymbol{child}$of u.

Vertices with the same parent are called $\boldsymbol{siblings}$**.** The $\boldsymbol{ancestors}$ of a vertex

Other than the root are the vertices in the path from the root to this vertex .

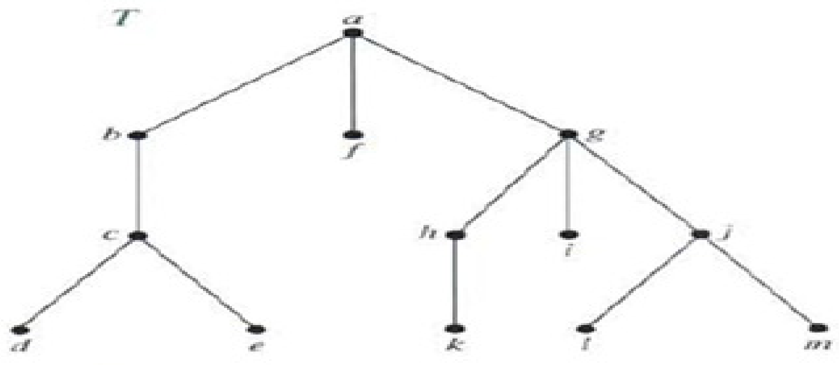
Excluding the vertex itself and including the root(that is ,its parent, its parent's parent and so on , until the root is reached).

The $\boldsymbol{descendants}$ of a vertex$v$ are those vertices that have v as an ancestor. A vertex of a tree is called a $\boldsymbol{leaf}$ if it has no children. Vertices that have children are $\boldsymbol{internal\ vertices}$**.** The root is an internal vertex unless it is the only vertex in the graph, in which case it is a leaf.

If a is a vertex in a tree , the $\boldsymbol{subtree}$ with a as its root is the sub graph of the tree consisting of a and its descendants and all edges are incident to these descendants.

Example:

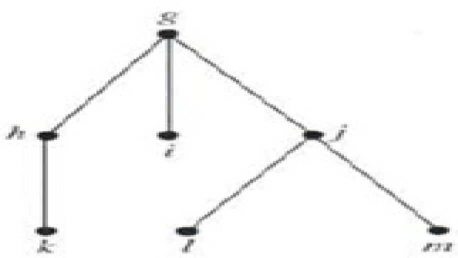
In the given rooted tree T with root a, find the parent of c, the children of g, the siblings of h, all ancestors of e, all descendants of b, all internal vertices, and all leafs. What is the sub tree rooted at g?



A Rooted Tree T.

Solution: The parent of c is b. The children of g are h, i, and j. The siblings of h are i and j. The ancestors of e are c, b and a. The descendants of b are c, d and e. The internals vertices are a, b, c, h and j. The leafs are d, e, f, i, k, l and m.

The sub tree at g is shown in the figure



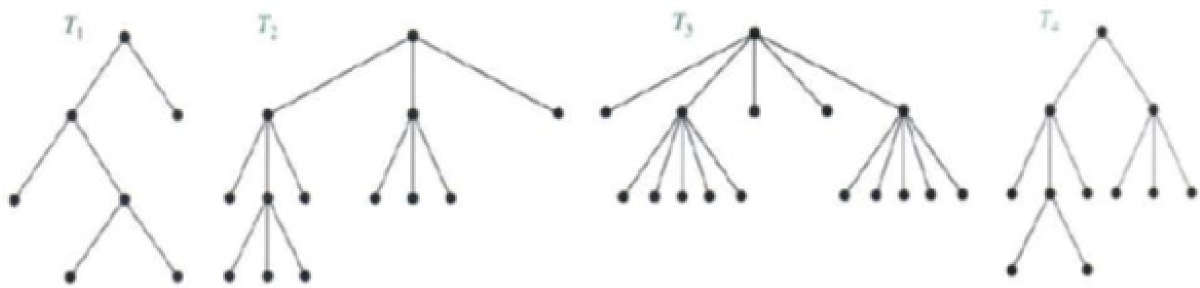
Subtree Rooted at g.

Definition:

A rooted tree is called an $\boldsymbol{m-ary\ tree}$ if every internal vertex has no more than m children. The tree is called a$\boldsymbol{full\ m-ary\ tree}$ if every internal vertex has exactly m children. An $m-ary$ tree with $m=2$ is called a $\boldsymbol{binary\ tree}$.

Example:

Are the given trees “full m-ary tree for some positive integer m?



Four Rooted Trees.

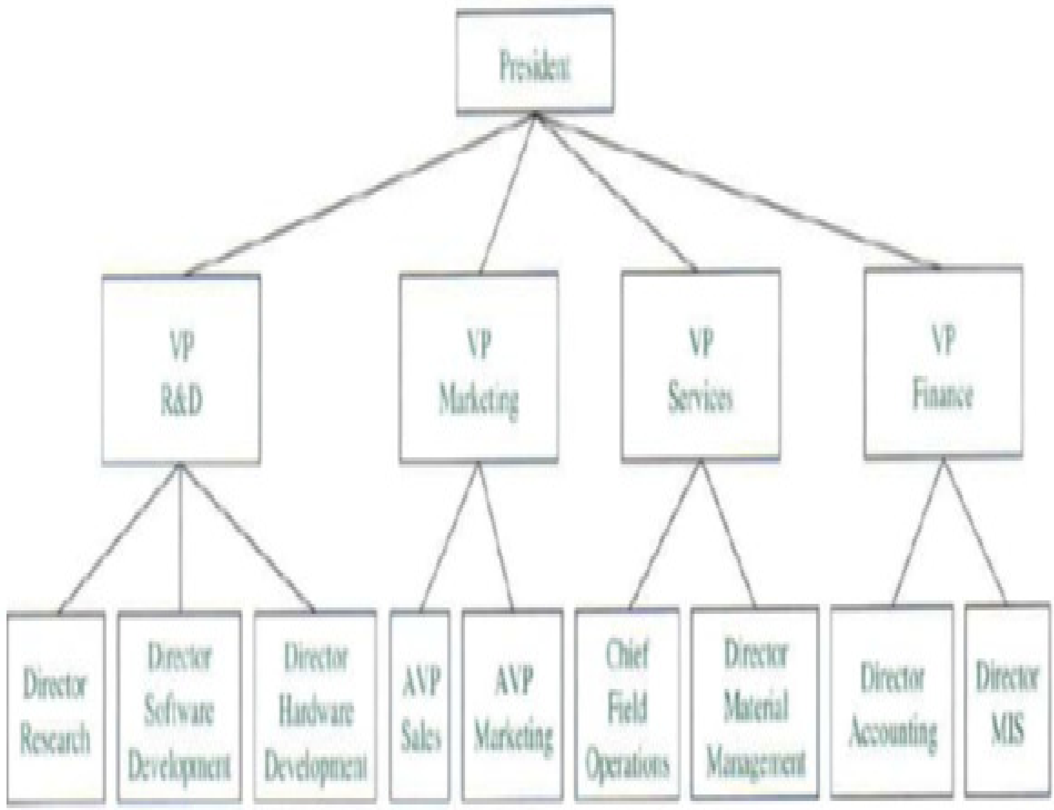
Solution: T_1 is a full binary tree , because each of its internal vertices has two children. T_2 is a full 3-ary tree because each internal vertex has three children. In T_3 each internal vertex has five children, so T_3 is a full 5-ary tree. T_4 is not a full m-ary tree for any m because its internal vertices have two children and other have three children.

3.3 Trees as Models

Trees are used as model in such diverse areas as computer science, chemistry, geology , botany and psychology.

Representing Orginatinos

The stucture of a large organization can be model using a roored tree. Each vertex in this tree represents a postion in the organization. An edge from vertex to another indicates that the person represented by initial vertex is the (direct) boss of the person represented by the terminal vertex.



An Organizational Tree for a Computer Company.

Theorm :

A tree with n vertices has $n-1$ edges.

3.4.Exercises:

1. Which of these graphs are trees?

a)


b)

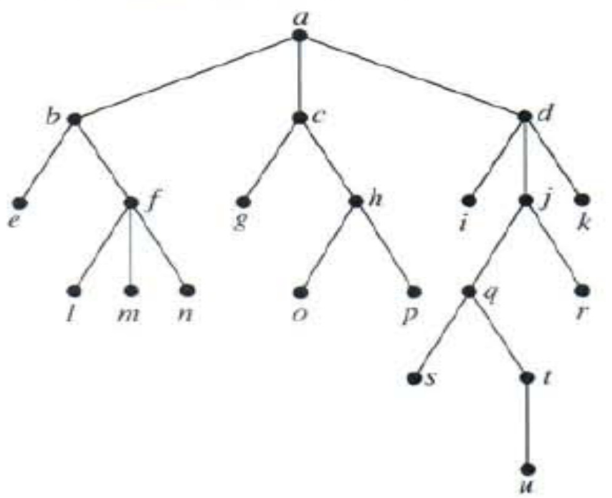

c)


d)


e)


f)


3. Answer these questions about the rooted tree illustrated.



a) Which vertex is the root?
b) Which vertices are internal?
c) Which vertices are leaves?
d) Which vertices are children of j?
e) Which vertex is the parent of h?
f) Which vertices are siblings of o?
g) Which vertices are ancestors of m?
h) Which vertices are descendants of b?

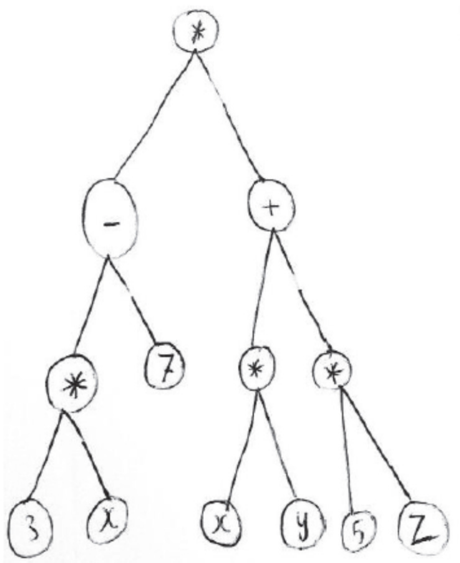
Binary trees and binary operations

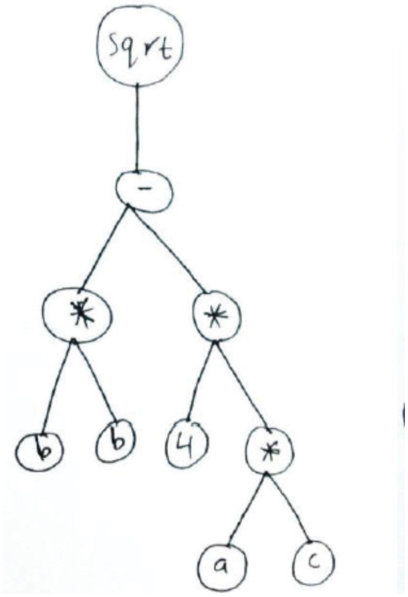
Example: Draw the generated binary tree representations of

a) $(3x-7)(xy+5z)$ b) $\sqrt{b^2-4ac}$ c) $(x+y)/(2x-y)$

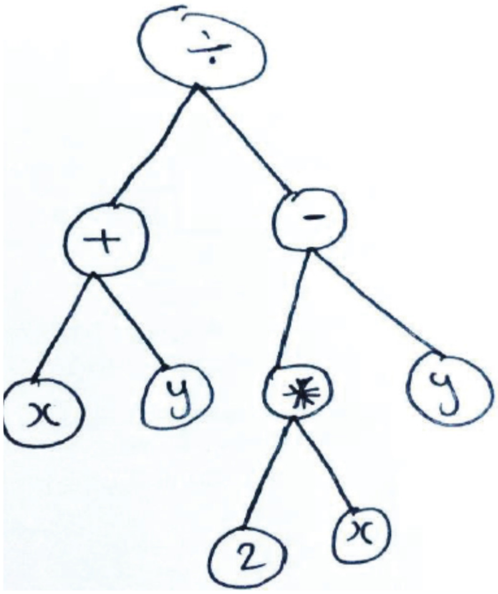
Solution:

a) $(3x-7)(xy+5z)$ b) $\sqrt{b^2-4ac}$





c) $(x+y)/(2x-y)$

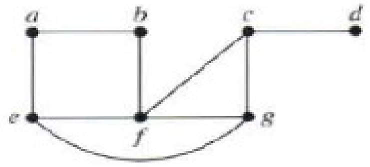


3.5 Spanning Trees

Definition:

Let G be a simple graph. A spanning tree of G is a sub graph of G that is a tree containing every vertex of G.

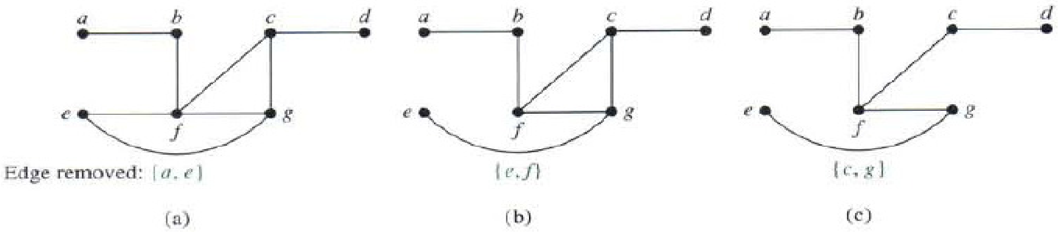
Example: Find a spanning tree of the simple graph G showen in the figure.

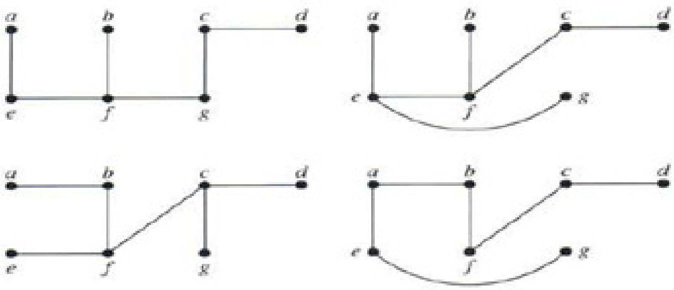


Simple Graph G.

Solution: The graph G is conntected, but it is not a tree because it contains a cycle, Remove the edge ae. This eliminates one cycle, and the

Resulting subgraph is still connected and still contains every vertex of G. Next remove the edge ef to eliminate a second cycle. Finally remove the edge cg to produce a simkple graph with no cycles. This sub graph is a spanning tree, because it is a tree contains every vertex of G.

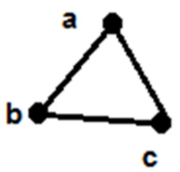




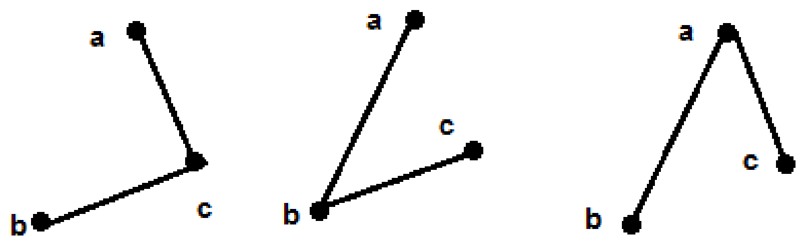
Spanning Trees of G.

The number of Spanning trees:

Example: Find all the spanning trees of the graph K_3



Solution:



Example: Find all the spanning trees of $K_{4.}$

Solution:$K_{4.}$Containing 16 spanning trees.

<u>Cayley Tree Formula</u>:

Theorem: The number of spanning trees of K_n is n^{n-2}.

Cofactors of an $n \times n$ matrix:

Let $A = \begin{bmatrix} a_{11} a_{12} \dots\dots\dots. a_{1n} \\ a_{21}\, a_{22} \dots\dots\dots\dots a_{2n} \\ . \\ . \\ . a_{n1} a_{n2} \dots\dots\dots\dots\dots a_{nn} \end{bmatrix}$

By a cofactor of an $n \times n$ matrix $A = \lfloor a_{ij} \rfloor$, we mean $(-1)^{i+j}\, det(A_{ij})$, where $det(A_{ij})$ indicates the determinant of the $(n-1) \times (n-1)$ submatrix A_{ij} of A obtaing by deleting row i and column $j$of A.

Example: $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 2 & 3 & -2 \\ 4 & 2 & 1 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$,

The (1,1) cofactor of A is $(-1)^{1+1}\det\begin{bmatrix} 2 & 3 & -2 \\ 2 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$

$$= 1.\left(2\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + (-2)\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right)$$

$$= 1.(\mathbf{2}(\mathbf{3}-\mathbf{1}) - \mathbf{3}(\mathbf{6}-\mathbf{0}) + (-\mathbf{2})(2-0)) = -18.$$

The (2,1) cofactor of A is $(-1)^{2+1}\det\begin{bmatrix} 2 & -1 & 4 \\ 4 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

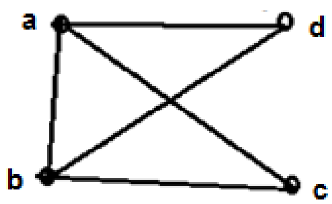
$$= -\left(2\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} + 4\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\right)$$

$$= -(\mathbf{2}(\mathbf{3}-\mathbf{1}) - (-\mathbf{1})(\mathbf{12}-\mathbf{1}) + \mathbf{4}(4-1)) = -27.$$

Kirchhoff Theorem(Matrix Tree Theorem)

Theorem: Let M be the matrix obtained from the adjacency matrix of a simple connected graph G by changing all 1' to -1' and each diagonal 0 to the degree of the corresponding vertex. Then the number of spanning trees of G is the number of any cofactor of M.

Example: Find the number of spanning tress of the given graph



Solution: The adjacency matrix is

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

The (1,1) cofactor of M is $\det\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

$$= 3\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}$$

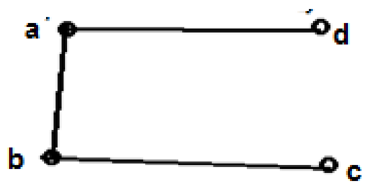
$$= 3(4-0) + (-2-0) - 1(0+2) = 12 - 4 = 8.$$

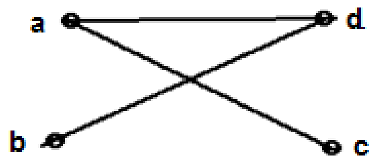
The (2,1) cofactor of $M = (-1)^{2+1} \det\begin{bmatrix} -1 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

$$= (-1)\left[-1\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}\right]$$

$$= (-1)[(-1)(4-0) + (-2-0) - 1(0+2)]$$

$$= (-1)[-4-2-2] = 8$$


a
d
b
c


a
d
b
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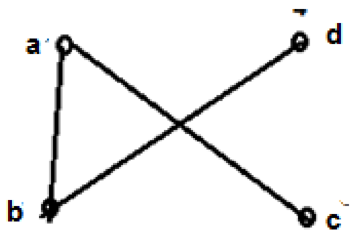
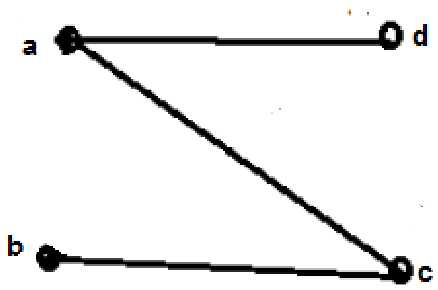

a
d
b
c


a
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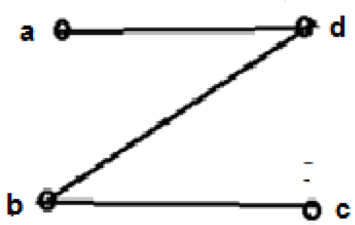
a
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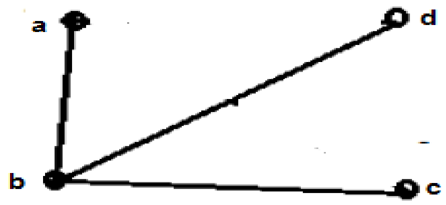
a
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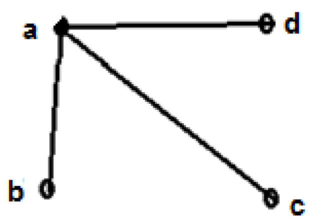
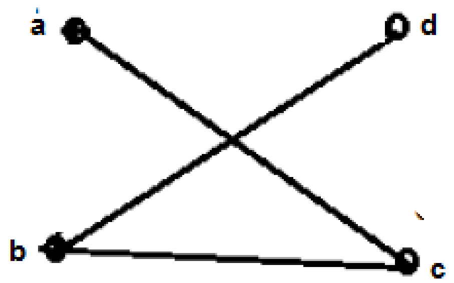
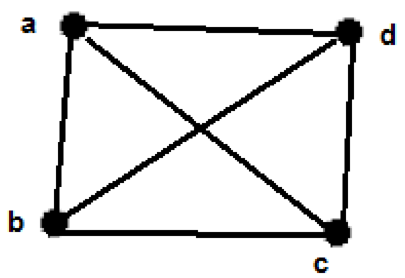

a
d
b
c



Example: use Krichooff theorem to find number the spanning trees of K_4.



SOLUTION:

The adjacency matrix of $\boldsymbol{K_4}$.

$$A(K_4) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

The (1,1) cofactor of M is $\det\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$

$$= 3\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 3 \\ -1 & -1 \end{vmatrix}$$

$$= 3(9-1) + (-3+1) - (1+3) = 24 - 2 - 4 = 16.$$