2.2 Propositional Equivalence ($\equiv$)

Tautology, Contradiction and Contingency

Definition: A compound proposition that is always true no matter what the truth values of the propositions that occur in it is called a **tautology**. A compound proposition that is always false is called a **contradiction**. A compound proposition that is neither a tautology nor a contradiction is called a **contingency**.

Example: Show that $p \vee \neg q$ is a tautology and $p \wedge \neg q$ is a contradiction.

Solution:

$\mathbf{p}$	$\neg \mathbf{p}$	$\mathbf{p} \vee \neg \mathbf{p}$	$\mathbf{p} \wedge \neg \mathbf{p}$
T	F	T	F
T	F	T	F
F	T	T	F
F	T	T	F

Example: Show that $\mathbf{p} \rightarrow (\mathbf{p} \vee \mathbf{q})$ is a tautology.

Solution:

$\mathbf{p}$	$\mathbf{q}$	$\mathbf{p} \vee \mathbf{q}$	$\mathbf{p} \rightarrow (\mathbf{p} \vee \mathbf{q})$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Example: Show that $(\mathbf{p} \wedge \mathbf{q}) \rightarrow (\mathbf{p} \vee \mathbf{q})$ is a tautology.

Solution:

$\mathbf{p}$	$\mathbf{q}$	$\mathbf{p} \wedge \mathbf{q}$	$\mathbf{p} \vee \mathbf{q}$	$(\mathbf{p} \wedge \mathbf{q}) \rightarrow (\mathbf{p} \vee \mathbf{q})$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Example: Show that 1) $(\mathbf{p} \vee \mathbf{q}) \vee (\neg \mathbf{p})$ is a tautology.

2) $(\mathbf{p} \wedge \mathbf{q}) \wedge (\neg \mathbf{p})$ is a contradiction

Solution:

$\mathbf{p}$	$\mathbf{q}$	$\neg\mathbf{p}$	$\mathbf{p\vee q}$	$\mathbf{p\wedge q}$	$\mathbf{(p\vee q)\vee(\neg p)}$	$\mathbf{(p\wedge q)\wedge(\neg p)}$
T	T	F	T	T	T	F
T	F	F	T	F	T	F
F	T	T	T	F	T	F
F	F	T	F	F	T	F

Example: Show that $\mathbf{(p\wedge q)\rightarrow(p\rightarrow q)}$ is a tautology

Solution:

$\mathbf{p}$	$\mathbf{q}$	$\mathbf{p\wedge q}$	$\mathbf{p\rightarrow q}$	$\mathbf{(p\wedge q)\rightarrow(p\rightarrow q)}$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Definition: When two compound statements have the same truth values in all possible cases are called **logically equivalent.**

The notation $p\equiv q$ denotes that p and q are logically equivalent.

De Morgan's laws:

1) Show that $\neg\,\mathbf{(p\vee q)}$ and $\mathbf{\neg p\wedge\neg q}$ are logically equivalent**.**

(or) Show that $\neg\,\mathbf{(p\vee q)}\equiv\mathbf{\neg p\wedge\neg q}$

2) Show that $\neg\,\mathbf{(p\wedge q)}$ and $\mathbf{\neg p\vee\neg q}$ are logically equivalent**.**

(or) Show that $\neg\,\mathbf{(p\wedge q)}\equiv\mathbf{\neg p\vee\neg q}$

Solution:

1)

$\mathbf{p}$	$\mathbf{q}$	$\neg\mathbf{p}$	$\neg\mathbf{q}$	$\mathbf{p\vee q}$	$\neg\mathbf{(p\vee q)}$	$\mathbf{\neg p\wedge\neg q}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

2)

$\mathbf{p}$	$\mathbf{q}$	$\neg\mathbf{p}$	$\neg\mathbf{q}$	$\mathbf{p \wedge q}$	$\neg(\mathbf{p \wedge q})$	$\neg\mathbf{p} \vee \neg\mathbf{q}$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Example: Show that $\neg \mathbf{p} \vee \mathbf{q}$ and $\mathbf{p} \rightarrow \mathbf{q}$ are logically equivalent.

(or) Show that $\neg \mathbf{p} \vee \mathbf{q} \equiv \mathbf{p} \rightarrow \mathbf{q}$

Solution:

$\mathbf{p}$	$\mathbf{q}$	$\neg\mathbf{p}$	$\neg \mathbf{p} \vee \mathbf{q}$	$\mathbf{p} \rightarrow \mathbf{q}$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example: Show that $\mathbf{p} \rightarrow \mathbf{q}$ and $\neg\mathbf{q} \rightarrow \neg\mathbf{p}$ are logically equivalent.

(or) Show that $\mathbf{p} \rightarrow \mathbf{q} \equiv \neg\mathbf{q} \rightarrow \neg\mathbf{p}$

Solution:

$\mathbf{p}$	$\mathbf{q}$	$\neg\mathbf{p}$	$\neg\mathbf{q}$	$\mathbf{p} \rightarrow \mathbf{q}$	$\neg \mathbf{q} \rightarrow \neg\mathbf{p}$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Example: Show that $\neg\mathbf{p} \leftrightarrow \mathbf{q}$ and $\mathbf{p} \leftrightarrow \neg\mathbf{q}$ are logically equivalent.

(or) Show that $\neg\mathbf{p} \leftrightarrow \mathbf{q} \equiv \mathbf{p} \leftrightarrow \neg\mathbf{q}$

Solution:

$\mathbf{p}$	$\mathbf{q}$	$\neg\mathbf{p}$	$\neg\mathbf{q}$	$\neg\mathbf{p} \leftrightarrow \mathbf{q}$	$\mathbf{p} \leftrightarrow \neg\mathbf{q}$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

Logical Equivalances	
Equivalance	**Name**
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv (p \vee q) \vee r$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Exercises

1. Show that $\neg(\neg p)$ and p are logically equivalent.
2. Use a truth table to verify the first De Morgan law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

3. Show that $p \rightarrow (p \vee q)$conditional statements is a tautology by using truth table.
4. Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.
5. Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent.