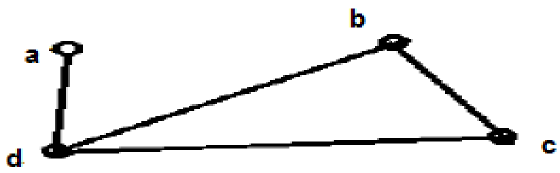
Graphs and Matrices

<u>Adjacency Matrix</u>:

Let G be a graph with n vertices. Labeled $v_1, v_2, \ldots, v_n$. For each $i \text{ and } j$ with $1 \leq i, j \leq n$, defined $a_{i,j} = \begin{cases} 1, & if\ v_i v_j \in E(G), \\ 0, & if\ v_i v_j \notin E(G). \end{cases}$

The adjacency matrix of G is the n× n matrix $A = [a_{ij}]$, where $(i,j)\ entry\ is\ a_{ij}$ and is denoted by $A(G)$.

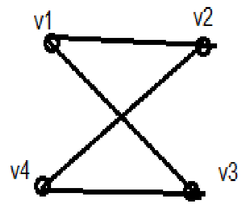
<u>Example:</u> Find the adjacency matrix of the given graph G.



$$A(G)= \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example: Draw the graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



If G is a simple graph , then the adjacency matrix has the following properties:

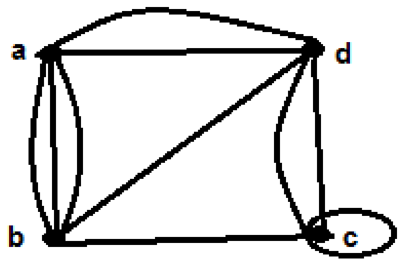
1. The diagonal entries are all 0.
2. The adjacency matrix is symmetric.
3. $\deg(v_i)$ is the sum of the entries in the row i(or column j).

Example: Use an adjacency matrix to represent the pseudo graph.

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Solution:

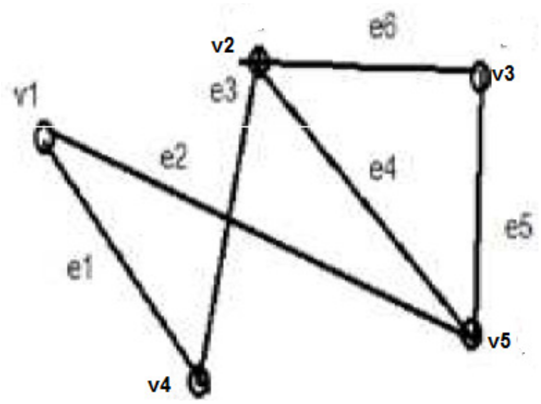
$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$



<u>Incidence matrix:</u> Let $G = (V, E)$ be a graph . Suppose that $v_1, v_2, \dots, v_n$ are the vertices of G and $e_1, e_2, \dots, e_m$ are the edges of G. Then the incident matrix is the $n \times m$ matrix

$$M = [m_{ij}], \qquad where\ \ m_{ij} = \begin{cases} 1, & if\ v_i\ is\ incident\ with\ e_j \\ 0, & otherwise. \end{cases}$$

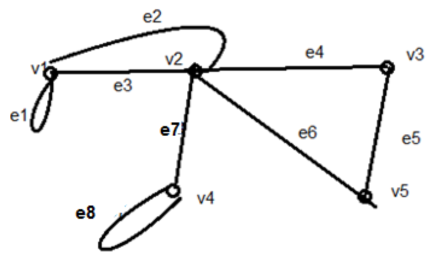
Example: Represent the following graph with an incident matrix.



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

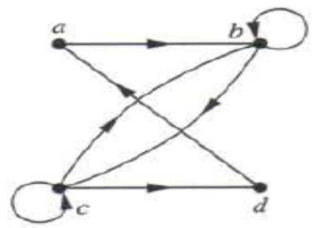
Example: Draw the pseudo graph with the following incident matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$



<u>Adjacency Matrix of a directed graph</u>:

Example: Find the adjacency matrix of the given multi graph



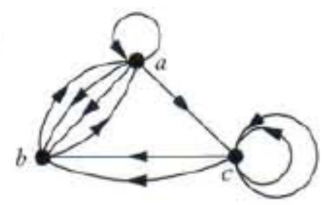
Solution:

$$\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Example: Draw the graph represented by the following matrix.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

Solution:



Example:Draw the graph with the given matrix

$$\begin{bmatrix} 1\,1\,1\,0 \\ 0\,0\,1\,0 \\ 1\,0\,1\,0 \\ 1\,1\,1\,0 \end{bmatrix}$$

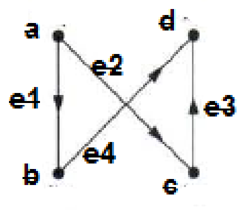
Incidence Matrix for direct graph:

Let $G = (V, E)$ be a directed graph with no loops . Suppose that $v_1, v_2, \dots, v_n$ are the vertices of G and $e_1, e_2, \dots, e_m$ are the edges of G. Then the incident matrix is the $n \times m$ matrix $\mathrm{B} = \left[\mathrm{b_{ij}}\right]$, where

,

$$\boldsymbol{b_{ij}} = \begin{cases} \mathbf{1}, \; if\ the\ edge\ e_j\ is\ \ directed\ away\ from\ the\ vertex\ v_i \\ \mathbf{-1}, \qquad if the\ edge\ e_j\ \ is\ \ directed\ \ towords\ the\ vertex\ v_i \\ \mathbf{0}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad otherwise. \end{cases}$$

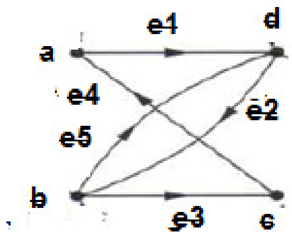
Example :Find the incidence matrix of the given graph



Solution:

$$\boldsymbol{B} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} e1 & e2 & e3 & e4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Example: Find the incidence matrix of the given multi graph



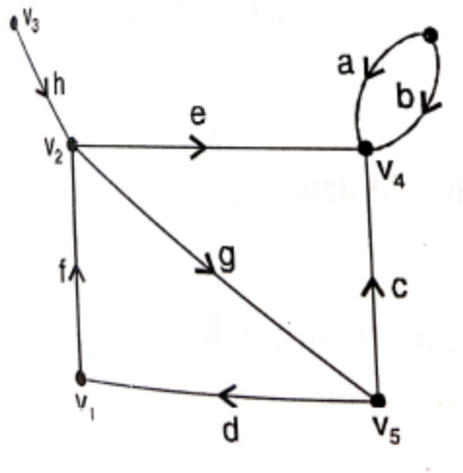
Solution:

$$
\begin{array}{c c}
 & \begin{array}{ccccc} e1 & e2 & e3 & e4 & e5 \end{array} \\
B = \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 & -1 \end{bmatrix}
\end{array}
$$

EXAMPLE: Draw the graph with the matrix

$$
\begin{array}{c c}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

Solution:

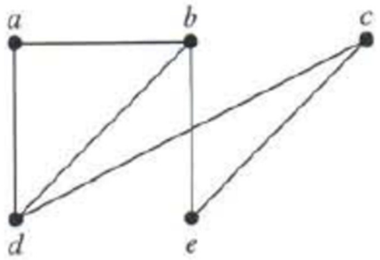


2.12 Exercise

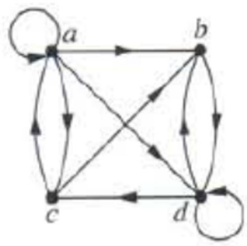
In Exercises 1–4 use an adjacency list to represent the given graph.

1.

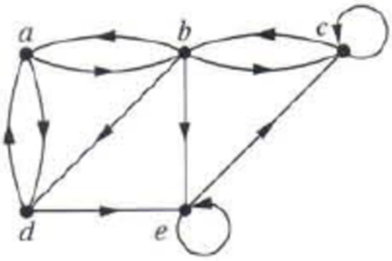
2.



3.



4.



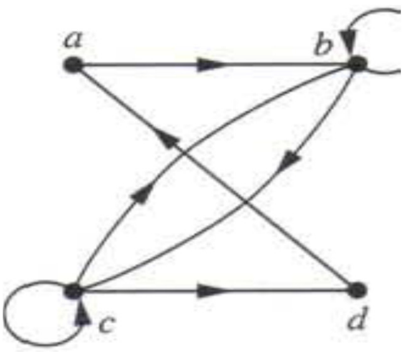
Represent each of these graphs with an adjacency matrix.

a) K_4 b) $K_{1,4}$ c) $K_{2,3}$
d) C_4 e) W_4 f) Q_3

Draw a graph with the given adjacency matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In Exercises 19–21 find the adjacency matrix of the given directed multigraph.



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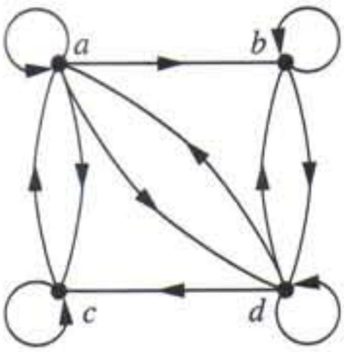

In Exercises 22–24 draw the graph represented by the given adjacency matrix.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$