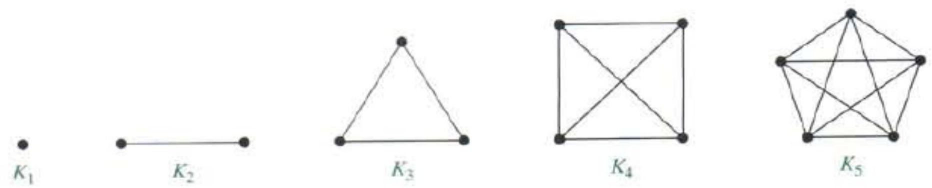
Some special graphs

<u>Complete graphs</u>: The complete graph of order n, denoted by K_n is the simple graph that contains exactly one edge between each pair of distinct vertices.



So, in K_n, every vertex has degree $n-1$.

Theorem : The number of edges of K_n is $\frac{n(n-1)}{2}$.

Proof: Since the number of vertices of K_n is n and for all $v \in V(K_n)$,

$\deg(v) = n-1$, then by the first theorem of graph theory, we have

$$n(n-1) = 2q$$

Implies that $q = \frac{n(n-1)}{2}$.

<u>Cycles:</u> The cycle $C_{n,}$ $n \geq 3$ consists of n vertices $v_1, v_2, \dots, v_n$ and edges $v_1v_{2,}v_2v_3, v_3v_4, \dots, v_{n-1}v_n, v_nv_1$. The number of edges of the cycle $C_{n,}$ $n \geq 3$, is n edges.

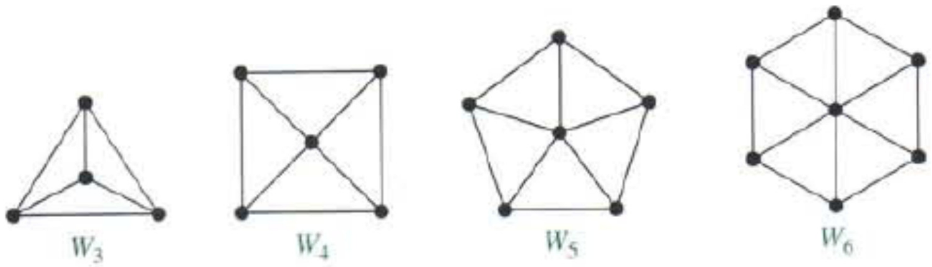
Theorem : $|\boldsymbol{E}(\boldsymbol{C_n})| = \boldsymbol{n}$.

Proof: The number of vertices of C_n is $\boldsymbol{n}$ and each vertex is of degree 2. Then by the first theorem of graph theory, theorem follows.

<u>**Odd and Even cycles**</u>: A cycle $C_{n,}$ $n \geq 3$ is called an odd cycle if its size is odd and is called even if its size is even.

<u>**Example:**</u> **$\boldsymbol{C_3}$ is an odd cycle while $\boldsymbol{C_4}$ is an even cycle.**

<u>**Wheels**</u>: We obtain the wheel W_n, when we add an additional vertex to the cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges.



Theorem : $|\boldsymbol{E}(\boldsymbol{W_n})| = \boldsymbol{2n}$

Proof: The number of vertices of W_n is $\boldsymbol{n+1}$ and there are n each vertex is of degree 3 and one vertex is of degree n. Then by the first theorem of graph theory, $3n + 1.n = 2q \Rightarrow q = 2n$.

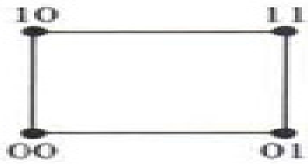
<u>***n − Cube graphs***</u>:

The n- dimensional hyper cubes or n- cubes, denoted by Q_n, is the graph has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that are represent differ in exactly one bit position.

For $n = 1$, then Q_1 has 2^1 vertices, namely 0 and 1

Q1

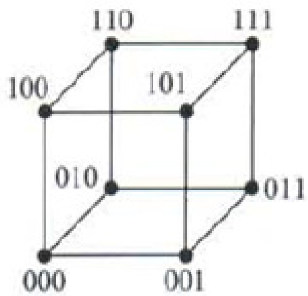
For For $n = 2$, then Q_2 has 2^2 vertices, namely 00 , $01,11,01$.



Q 2

For For $n = 3$, then Q_3 has 2^3 vertices, namely 000 , $001,011,101,111,110,$

$100,010.$



Q 3

Theorem : $|\boldsymbol{E(Q_n)}| = n2^{n-1}$.

Proof: The number of vertices of Q_n is 2^n and each vertex is of degree n. Then by the first theorem of graph theory, we have

$$n2^n = 2q \Rightarrow q = n2^{n-1}.$$

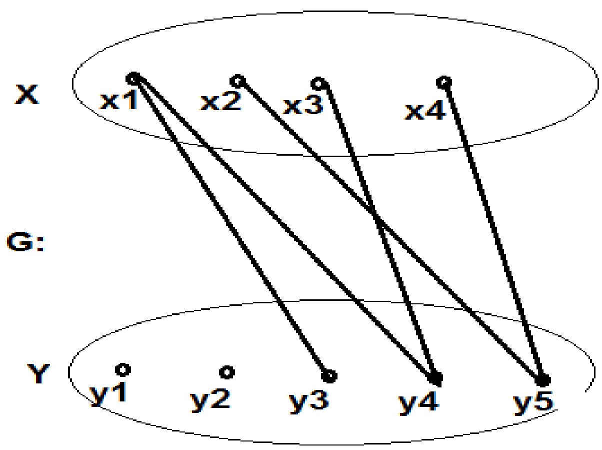
Bipartite Graphs and Complete Bipartite Graphs

Bipartite graph: A simple graph $G$is bipartite if its vertex set V can be portioned into two disjoint subsets X and Y such that every edge joins a vertex of X and a vertex of$Y$.

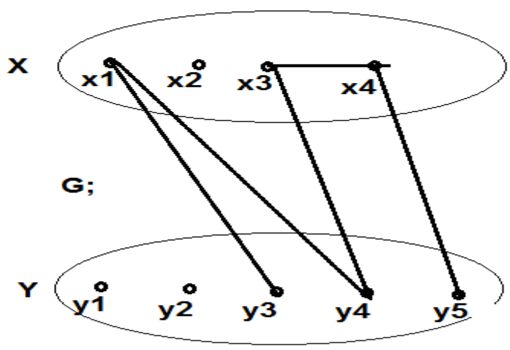
Example: Let G be a graph with vertex set

$V(G) = \{x_1, y_1, x_2, y_2, x_3, y_3, x_{4,,} y_4, y_5\}$.

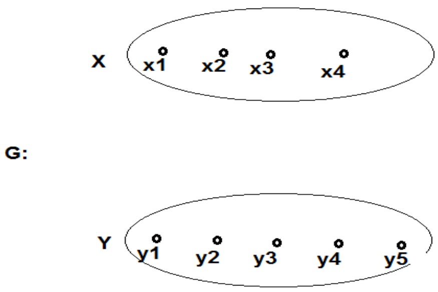
We partition the vertex set of Ginto two subsets, say $X = \{x_1, x_2, x_{3,} x_4\}$ and $V_2 = \{y_1, y_2, y_3, y_4, y_5\}$.



So, G is a bipartite graph.

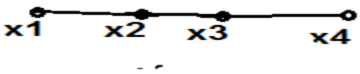


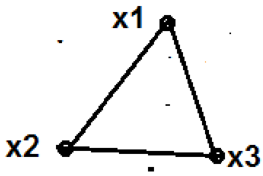
G is not a bipartite graph

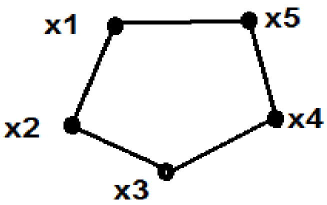


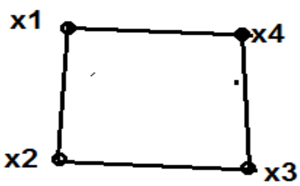
G is a bipartite graph

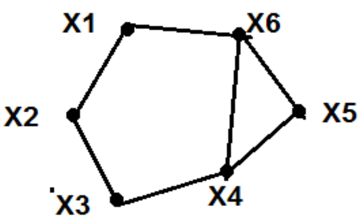
Example: Determine whether the following graphs are bipartite, if so redraw them with their partitions.

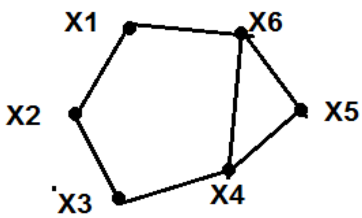


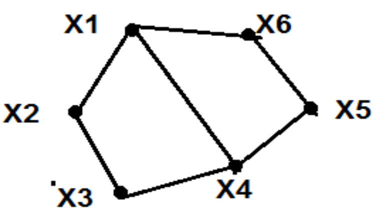




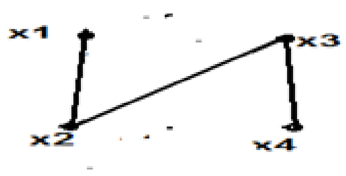






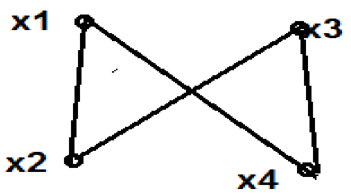


Solution: (a) is bipartite graph

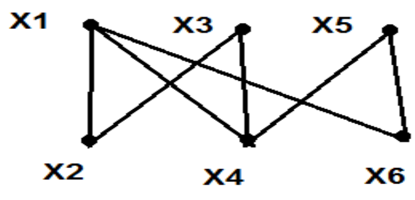


(b) is not bipartite.

(c) is bipartite



(a) is not bipartite.
(b) is not bipartite.
(c) is bipartite.

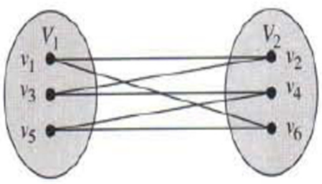


Theorem: *Let G be a nontrivial graph. If G is bipartite ,then G contains no odd cyles.*

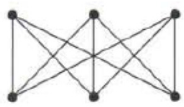
Theorem:*Let G be a nontrivial graph. If G contains no odd cycles, then G is bipartite .*

<u>Complete bipartite graph:</u> The complete bipartite graph $K_{m,n}$ is the graph that has its vertex set portioned into, two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex in the second subset.

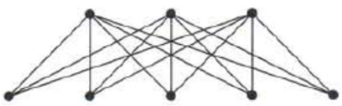
Examples:



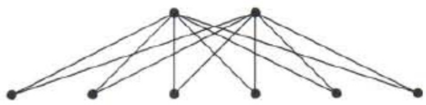
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

Theorem : $\left|E(K_{m,n})\right| = mn$.

Proof: The number of vertices of $K_{m,n}$ is $m + n$ and there are m vertices each of degree n and there are n vertices each of degree m Then by the first theorem of graph theory, we have

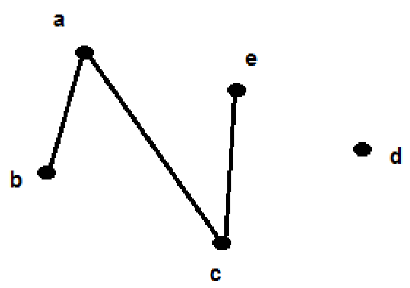
$mn + nm = 2q \Rightarrow q = mn$.

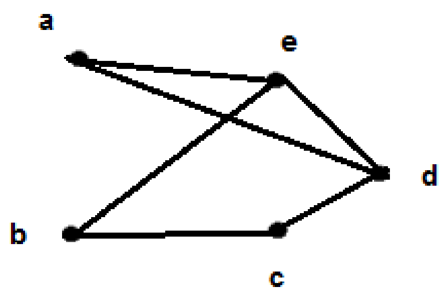
Regular Graphs: A graph G is regular if all its vertices have the same degrees.

Examples of regular graphs: K_n **and** C_n **are regular graphs.**

The Complement of a graph: The complementary graph $\bar{G}$ of a graph G has the same vertices as G and two vertices are adjacent in $\bar{G}$ if and only they are not adjacent in G.

Example: Find the complement of the graph

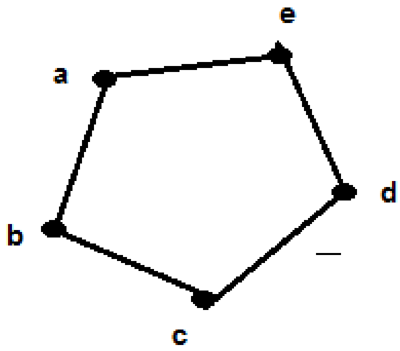




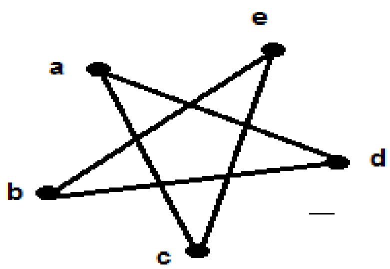
Solution:

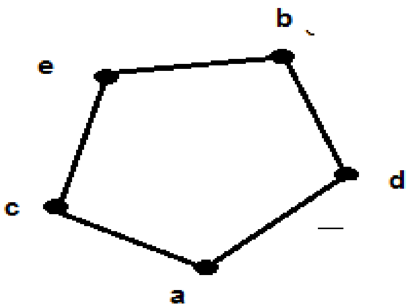
Example:

Find the complement of the graph $C_{5.}$



Solution:





So, $C_{5.}$ is self –complementary graph.

Example: Find $\overline{K_n}$.

Solution: The graph $\overline{K_n}$ has n vertices and no edges and it is called the trivial graph of order n

Theorem : Let $|V(G)| = n$.Then $|E(G)| + \left|E\overline{(G)}\right| = |E(K_n)| = \frac{n(n-1)}{2}$

The graph $\overline{K_n}$ has n vertices and no edges and it is called the trivial graph of order n.

Example: Let G be a graph of order 10 and size 30 . Find the size of $\bar{G}$.

Solution: Since $|E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2}$ Then

$$30 + |E(\overline{G})| = \frac{10(10-1)}{2}$$

Or

$$|E(\overline{G})| = 45 - 30$$

OR

$$|E(\overline{G})| = 15$$

Example: Let G be a graph with 30 edges and $\bar{G}$ with 15 edges. Find the number of vertices of G.

Solution: Since $|E(G)| + |E(\overline{G})| = \frac{n(n-1)}{2}$ Then

$$30 + 15 = \frac{n(n-1)}{2}$$

$$\Rightarrow 45 = \frac{n(n-1)}{2}$$

$$\Rightarrow n^2 - n - 90 = 0$$

$$\Rightarrow n = 10.$$

Theorem: Let G be a graph of order n and $v \in V(G)$ such that $\deg(v)_G\,(v) = k$. Then

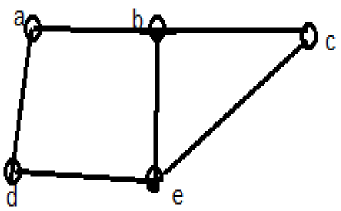
$$deg_{\bar{G}}(v) = n - k - 1.$$

Example : Let G be a graph with vertex degree 4.3.3.2.2. Find the degrees of $\bar{G}$.

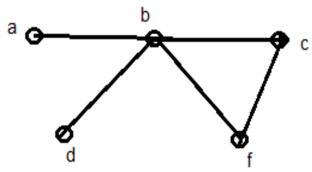
The Union of two Simple graphs: The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

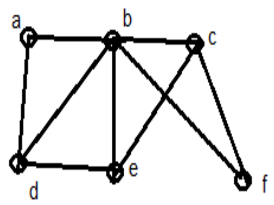
Examples:

G1 G1



G 2:



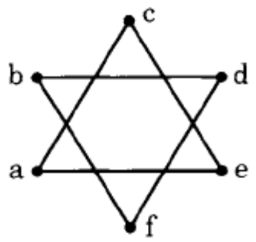


$G_1 \cup G_2$.

2.7 sub graphs

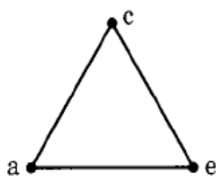
Sub graphs: A sub graph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where W is a sub set of V and F is a sub set of E.

Examples:

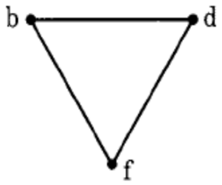


Graph G.

G_1 and G_2 are sub graphs of the graph G.



Graph G_1



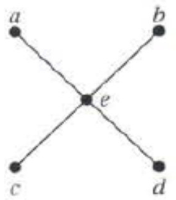
Graph G_2

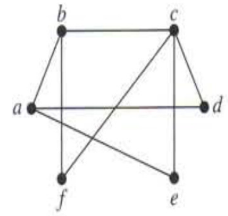
The simplest type of sub graphs of a graph is that obtained by deletion a vertex or an edge. If $v \in V(G)$ and $|V(G)| \geq 2$, then $G\backslash v$ denotes the sub graph with vertex set $V\backslash\{v\}$ and whose edge set are all those of G not incident at v.

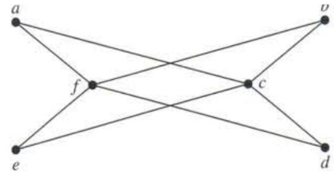
Similarly if $e \in E(G)$, then $G\backslash e$ is the sub graph having vertex set $V(G)$ and e$dge\ set\ E(G)\backslash\{e\}$.

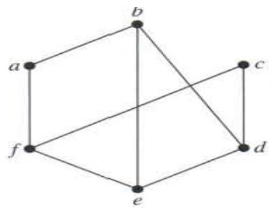
Exercises

1. Draw the following graphs: $K_7, C_7, K_{1,8}, K_{4,4}, W_7, Q_4.$
2. Determine which of the following graphs is bipartite





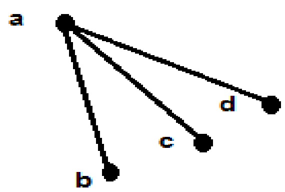




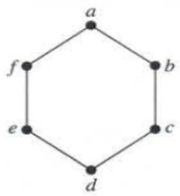
3. How many vertices and how many edges do these graphs have?

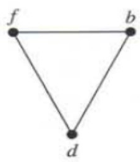
$$K_n, C_n, W_n, K_{m,n}, Q_n.$$

4. How many sub graphs with at least one vertex does K_2 have?
5. Draw all sub graphs of this graph



6. Find the union of the following two graphs





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