القيم الذاتية والمتجهات الذاتية

Eigenvalues and eigenvectors

تعريف: اذا كان A مصفوفه من النوع nxn فان المتجه غير الصفري $X \in R^n$ يسمى متجه ذاتي Eigen vector للمصفوفة A اذا كان

$$AX = \lambda X$$

حيث λ عدد قياسي ويسمى القيمة الذاتية للمصفوفة A

والمتجهX يسمى المتجه الذاتي للمصفوفة A المناظر للقيمه الذاتية λ

Definition 8.1. If A is an $n \times n$ matrix, then a nonzero vector $X \in \mathbb{R}^n$ is called an **eigenvector** of A, if AX is a scalar multiple of X; that is,

$$AX = \lambda X$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and X is said to be an eigenvector corresponding to λ.

Example 8.1. *For the matrix* $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, *the vector* $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *is an eigenvector of* A *and* $\lambda = 3$ *is an eigenvalue of* A, *since,*

$$AX = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda X.$$

EXAMPLE

Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, $\quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\quad$ and $\quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Verify that $\lambda_1 = 5$ is an eigenvalue of A corresponding to $\mathbf{x}_1$ and that $\lambda_2 = -1$ is an eigenvalue of A corresponding to $\mathbf{x}_2$.

SOLUTION To verify that $\lambda_1 = 5$ is an eigenvalue of A corresponding to $\mathbf{x}_1$, multiply the matrices A and $\mathbf{x}_1$, as follows.

$$A\mathbf{x}_1 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \mathbf{x}_1$$

Similarly, to verify that $\lambda_2 = -1$ is an eigenvalue of A corresponding to $\mathbf{x}_2$, multiply A and $\mathbf{x}_2$.

$$A\mathbf{x}_2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \lambda_2 \mathbf{x}_2.$$

From this example, you can see that it is easy to verify whether a scalar λ and an $n \times 1$ matrix $\mathbf{x}$ satisfy the equation $A\mathbf{x} = \lambda\mathbf{x}$. Notice also that if $\mathbf{x}$ is an eigenvector corresponding to λ, then so is any nonzero multiple of $\mathbf{x}$. For instance, the column matrices

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -7 \\ -7 \end{bmatrix}$$

are also eigenvectors of A corresponding to $\lambda_1 = 5$.

حساب القيم الذاتية و المتجهات الذاتية

Computing Eigenvalues and Eigenvectors

لايجاد طريقه عامه للحصول على القيم الذاتية و المتجهات الذاتية

لمصفوفه A من النوع $n \times n$ ، نكون المعادلة $AX = \lambda X$

و التي يمكن كتابتها في الصيغة

$$(\lambda IX - AX) = (\lambda I - A)X = 0,$$

حيث I مصفوفه الوحده من النوع $n \times n$

To obtain a general procedure for finding eigenvalues and eigenvectors of an $n \times n$ matrix A, note that the equation $AX = \lambda X$ can be rewritten as,

$$(\lambda IX - AX) = (\lambda I - A)X = 0,$$

where I is the $n \times n$ identity matrix.

لاي متجه غير صفري X ، من المعادله السابقه نحصل على

$$\det(\lambda I - A) = 0.$$

و تسمى المعادله المميزه للمصفوفه A

For a nonzero eigenvector $\breve{X}$, the above equation gives,

$$\det(\lambda I - A) = 0.$$

This is called the **characteristic equation** of A.

EXAMPLE

Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

SOLUTION The characteristic equation of A is

$$\begin{aligned} |\lambda I - A| &= \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \right| \\ &= \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} \\ &= \lambda^2 - 4\lambda + 3 - 8 \\ &= \lambda^2 - 4\lambda - 5 \\ &= (\lambda - 5)(\lambda + 1) = 0. \end{aligned}$$

This yields two eigenvalues, $\lambda_1 = 5$ and $\lambda_2 = -1$.

To find the corresponding eigenvectors, solve the homogeneous linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$. For $\lambda_1 = 5$, the coefficient matrix is

$$5I - A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 - 1 & -4 \\ -2 & 5 - 3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -2 & 2 \end{bmatrix},$$

which row reduces to

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

The solutions of the homogeneous system having this coefficient matrix are all of the form

$$\begin{bmatrix} t \\ t \end{bmatrix},$$

where t is a real number. So, the eigenvectors corresponding to the eigenvalue $\lambda_1 = 5$ are the nonzero scalar multiples of

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Similarly, for $\lambda_2 = -1$, the corresponding coefficient matrix is

$$-I - A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1-1 & -4 \\ -2 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -2 & -4 \end{bmatrix},$$

which row reduces to

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

where t is a real number. So, the eigenvectors corresponding to the eigenvalue $\lambda_2 = -1$ are the nonzero scalar multiples of

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

The solutions of the homogeneous system having this coefficient matrix are all of the form

$$\begin{bmatrix} 2t \\ -t \end{bmatrix},$$

Example: Find the Eigen values and Eigen vectors

$$\textbf{(i)}.\ A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}.$$

$$\textbf{(ii)}.\ B = \begin{bmatrix} 1 & 5 \\ 3 & -1 \end{bmatrix}.$$

$$\textbf{(iii)}.\ C = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solution: Part (i). Consider the characteristic equation, $\det(\lambda I - A) = 0$.

$$\Rightarrow \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 3 & 0 \\ -8 & \lambda + 1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 1) = 0 \Rightarrow\ \lambda = 3 \text{ or } \lambda = -1,$$

are the required eigenvalues of A.

For every λ we find its own vector(s).

الحاله الأولى عندما $\lambda = -1$..

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \quad -1I_2 - A = \begin{bmatrix} -4 & 0 \\ -8 & 0 \end{bmatrix}$$

$$x_1 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The eigenvectors corresponding to the eigenvalue1 are the **non-zero** solutions to the equation $x_1 = 0$. The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

عندما $\lambda = 3$.

$$3I_2 - A = \begin{bmatrix} 0 & 0 \\ -8 & 4 \end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 3 are the **non-zero** solutions to the equation $x_1 - \frac{1}{2}x_2 = 0$. The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

(ii). $B = \begin{bmatrix} 1 & 5 \\ 3 & -1 \end{bmatrix}.$

Consider the characteristic equation, $\det(\lambda I - B) = 0$.

$$\Rightarrow \det\left(\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}1 & 5\\ 3 & -1\end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix}\lambda - 1 & -5\\ -3 & \lambda + 1\end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) - 15 = 0 \Rightarrow \lambda^2 - 16 = 0 \Rightarrow \lambda = \pm 4,$$

$$\lambda = -4.$$

$$-4I_2 - A = \begin{bmatrix}-5 & -5\\ -3 & -3\end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}.$$

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$x_1 + x_2 = 0$$

$$x_2 = t ، x_1 = -x_2$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-t\\ t\end{bmatrix} = t\begin{bmatrix}-1\\ 1\end{bmatrix}.$$

$$\lambda = 4.$$

Here the matrix

$$4I_2 - A = \begin{bmatrix}3 & -5\\ -3 & 5\end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix}1 & -\frac{5}{3}\\ 0 & 0\end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 3 are the **non-zero** solutions to the equation $x_1 - \frac{5}{3}x_2 = 0$. The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{5}{3} \\ 1 \end{bmatrix}.$$

Exercise 8.3. *Find the eigenvalues and eigenvectors for all following matrices,*

(i). $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

(ii). $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

Solution: Part (i). Consider the characteristic equation, $\det(\lambda I - A) = 0$.

$$\Rightarrow \det\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}\right) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda - 3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda - 3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda - 15\lambda - 8 = 0 \Rightarrow (\lambda+1)(\lambda^2 - 7\lambda - 8) = 0 \Rightarrow (\lambda+1)(\lambda+1)(\lambda-8) = 0$$

$$\Rightarrow \lambda = -1(\text{twice}) \text{ or } \lambda = 8,$$

are the required eigenvalues of A.

For every λ we find its own vector(s).
Case 1: $\lambda = -1$.

Here the matrix

$$-1I_3 - A = \begin{bmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvectors corresponding to the eigenvalue -1 are the **non-zero** solutions to the equation $x_1 + \frac{1}{2}x_2 + x_3 = 0$.
The solutions to the equation are the vectors of the form

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t - s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 8 are the **non-zero** solutions to the equations

$$x_1 - x_3 = 0$$
$$x_2 - \frac{1}{2}x_3 = 0.$$

The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Case 2: $\lambda = 8$.
Here the matrix

$$8I_3 - A = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}$$

can be row reduced to the matrix:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvectors corresponding to the eigenvalue 8 are the **non-zero** solutions to the equations

$$x_1 - x_3 = 0$$

$$x_2 - \frac{1}{2}x_3 = 0.$$

The solutions to the equation are the vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$