GRAPH THEORY

Lecture one

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RAPHS

Basic Definitions

Definition: A graph is an order triple $G = (V(G), E(G), I(G))$, where $V(G)$ is a nonempty set, $E(G)$ is a set disjoint from $V(G)$, and $I(G)$is an incident relation that associates with each element of $E(G)$, an unordered pair of elements(same or distinct)of $V(G)$.

Elements of$V(G)$ are called **vertices(vertex)**, nodes or points and elements of $E(G)$ are called **edges** or **lines** of G. $V(G)$ and $E(G)$ are the **vertex set** and

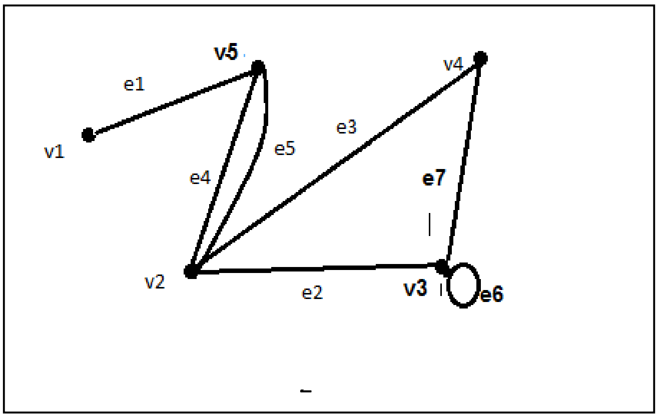
Edge set of G, respectively.

If, for the edges of $G, I_G(e) = \{u, v\}$, we write $I_G(e) = uv$, or simply $e = uv$.

Diagrammatic Representation of a graph:

Each graph can be represented by a diagram in the plane. In this diagram each vertex of the graph is represented by a point, with distinct vertices being represented by distinct point . Each edge is represented by a simple Jordan arc joining two(not necessary distinct) vertices.

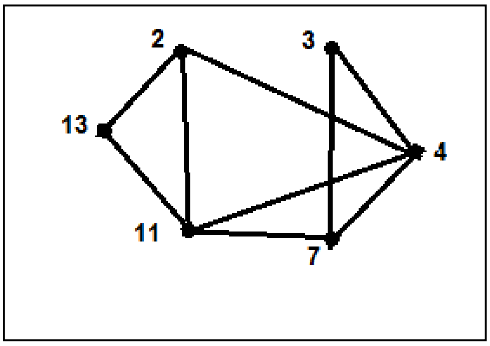
Example: Let $V(G) = \{v_1, v_2, \dots, v_5\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_{6,}e_{7,}\}$ and I_G is given by$e_1 = v_1v_5, e_2 = v_2v_3, e_3 = v_2v_4, e_4 = v_2v_5, e_5 = v_2v_5, e_6 = v_3v_{3,}, e_7 = v_3v_4$.



Example:

Let $S = \{2,3,4,7,11,13\}$. Draw the graph G whose vertex set is S and such that$ij \in E(G),\ for\ i,j \in S\ if\ i+j \in S\ or\ |i-j| \in S, i \neq j$.

Solution



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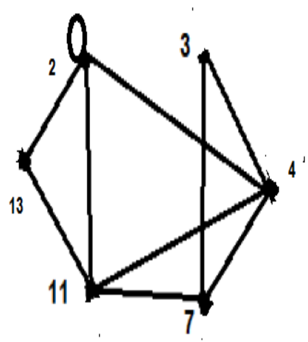
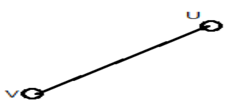
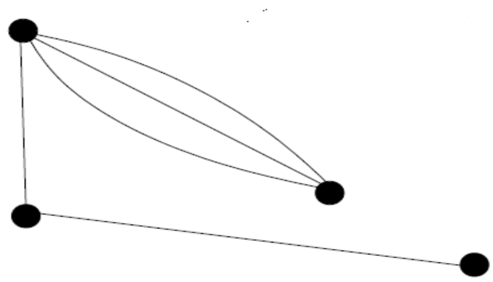


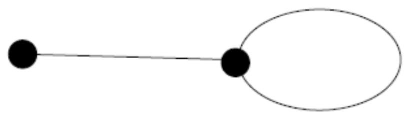
Definition: If $e = uv$, then the vertices u and v are the end vertices of the edge e, we say, e is **incident** with vertices u and v.



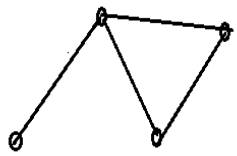
Definition: A set of two or more edges of a graph G is called a set of **multiple** edges or **parallel** edges if they have the same ends.



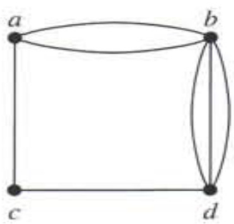
Definition: An edge for which the two ends are the same is called a **loop** at the common vertex.



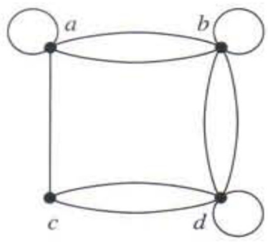
Definition: A graph G is called **simple** if it has no loops and no parallel edges.



Definition: A graph G is called multi graph if it has no loops.



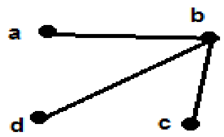
Definition: A graph G is called a pseudo if it has loops and parallel edges.



2.2 Adjacent vertices:

Definition: Two vertices v and u are **adjacent** if and only if there is an edge e start with v and endswith$u$.

Example:



Vertex a is adjacent to vertex b, vertex b is adjacent to the vertices c and d but vertices a, c, d are nonadjacent to each others, also, we can say vertices a, c, d are adjacent to the vertex b.

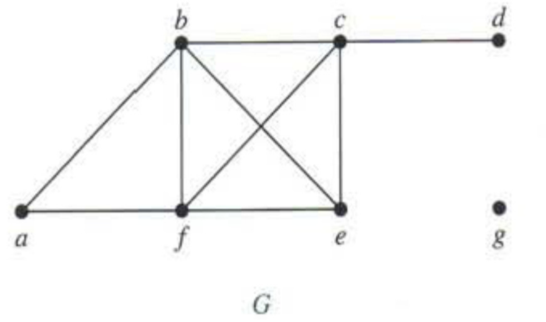
Degree of a vertex

Definition: The degree of a vertex in a graph G is the number of edges incident with it, and is denoted by$\deg(v)$.

Isolated vertex: A vertex of degree zero is called an isolated vertex.

Pendant vertex: A vertex of degree one is called pendant vertex

Example: Find the degree of each vertex of the following graph.

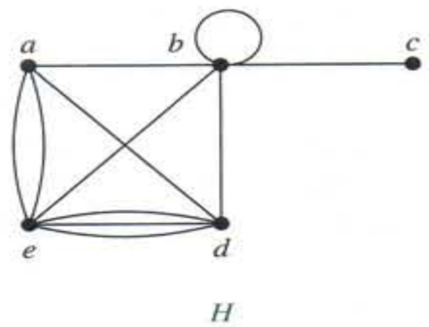


$\deg(a) = 2, \deg(b) = 4$, $\deg(c) = 4, \deg(d) = 1, \deg(e) = 3, \deg(f) = 4,$

and $\deg(g) = 0$. So, in the given graph the vertex g is an isolated vertex while the vertex d is a pendant vertex.

Note: If there is a loop we can count it twice.

Example: Find the degrees of the graph H.



$\deg(a) = 4$, $\deg(b) = 6, \deg(c) = 1, \deg(d) = 5$ and $\deg(e) = 6$.

2.3 Order and size of a graph

Definition: Let $G = (V, E)$. The order of G, denoted by $|V(G)|$ $or |V|$, is the number of vertices and the size of G, denoted by $|E(G)|$ $or|E|$ is the number of edges . So, if $|V| = n$ and $|E| = q$, then the order of G is n and is size is q and we called such a graph is a $(n, q) graph$.

So, the order and size of the graph given in the previous example are 5 and 11 respectively.

The First Theorem of Graph Theory(The hand shaking theorem)

The first theorem of graph theory was due to Leanhard Euler(1707-1783). This theorem connects the degrees of the vertices and the number of edges.

Theorem: *Let G be a graph of order n and size q. Then* $\sum_{v\in G} deg(v) = 2q$.

Proof: Each edge, since it has two ends vertices contributes precisely 2 to the sum of the degrees, i.e., when the sum of the vertices are summed, each edge is counted twice.

Example: How many edges are there in a graph with 10 vertices each of degree 6?

Solution: By the first theorem of graph theory, $\sum \deg(v) = 2q$

Therefore 10.6=2q $\Rightarrow$ $q = 30$ $edges$.

<u>Example:</u> How many vertices are there in a graph with 30 edges such that each vertex is of degree 6?

<u>Solution:</u> Let n be the number of the vertices of such a graph.

By the first theorem of graph theory, $\sum \deg(v) = 2q$,

So, $6n = 30$,$\Rightarrow n = 10$.

<u>Example</u>: Let G be a graph of order10 and size 30, such that all its vertices have the same degree. Find the degree of each vertex.

Solution: **Since all the vertices have the same degree, Let r_be the degree of each vertex.**

Therefore 10 $r = 2.30 \Rightarrow r = 6$.

Coro rally: **If G is a graph of order n and size q and each vertex of G is of degree r. Then** $nr = 2q$.

Example: **A certain graph G has order 14 and size 27. The degree of each vertex of G is 3,4 or 5. There are 6 vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5?**

Solution: **Let x be the number of vertices of G of degree 3, since order of G is 14 and 6 vertices of degree 5, then there are $(8 - x)$ vertices of degree 4.**

$$\therefore 4.6 + 3.x + 5(8 - x) = 2.27$$

$$or \;\; 24 + 3x + 40 - 5x = 54$$

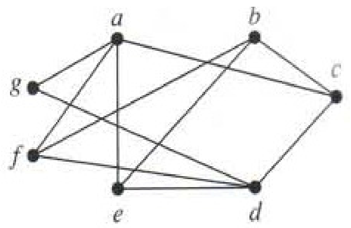
$$\Rightarrow \; x = 5.$$

And so, there are 5 vertices of degree 3 and there are $8 - 5 = 3$ vertices of degree 5.

Odd and even vertices

Definition: A vertex in a graph G is called **odd** or **even** depending on whether its degree is odd or even.

Example: **Find the odd and the even vertices of the following graph.**



<u>Solution:</u> The odd vertices are $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}$ and the even are $\boldsymbol{a}, \boldsymbol{d}$ and $\boldsymbol{g}$.

<u>Corollary</u>: *In any graph G, there is an even number of odd vertices.*

Proof: Let W $(W \neq \emptyset)$ be the set of odd vertices and U be the set of even vertices of G**.** Then for each $u \in U, \deg(u)$ is even and so, $\sum_{u\in U} \deg(u)$ being sum of even integers is even. Therefore

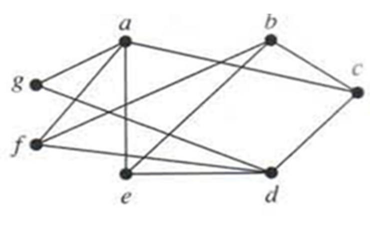
$$\sum_{\boldsymbol{u}\in\boldsymbol{U}} \boldsymbol{deg}(\boldsymbol{u}) + \sum_{\boldsymbol{w}\in\boldsymbol{W}} \boldsymbol{deg}(\boldsymbol{w}) = \sum_{\boldsymbol{v}\in\boldsymbol{V}} \boldsymbol{deg}(\boldsymbol{v}) = \boldsymbol{2q}$$

$\therefore \sum_{\boldsymbol{w}\in\boldsymbol{W}} \boldsymbol{deg}(\boldsymbol{w}) = \boldsymbol{2q} - \sum_{\boldsymbol{u}\in\boldsymbol{U}} \boldsymbol{deg}(\boldsymbol{u})$ iseven being the difference of two even numbers. As all the terms in $\sum_{w\in W} \deg(w)$ are odd and their sum is even. Then there must be an even number of terms.

Maximum and minimum degree

The maximum degree of a vertex of a graph G is denoted by $\Delta(G)$ and the minimum is denoted by $\delta(G)$.

<u>Example :</u> Find the maximum and minimum degree for the following graph.



<u>Solution:</u> $\Delta(\boldsymbol{G}) = \boldsymbol{4}$ **and** $\boldsymbol{\delta}(\boldsymbol{G}) = \boldsymbol{2}$.

Theorem: *For any simple graph G of order n, $\Delta(G) \leq n-1$.*

Proof: Let $v \in V(G)$. Since G is simple, then no multiple edges nor loops are allowed. Thus v can adjacent to at most $n-1$ vertices in G.

Thus $\Delta(G) \leq n-1$.

Corollary: *For any simple graph G of order n, $0 \leq \delta(G) \leq \Delta(G) \leq n-1$.*

Theorem: *Let G be a simple graph of order n and size q. Let $\Delta(G)$ and $\delta(G)$ be the maximum degree and minimum degree of G, respectively.*

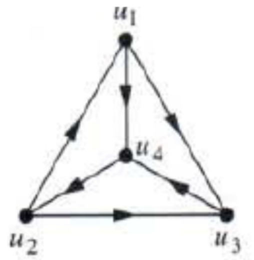
Then $\delta(G) \leq \frac{2q}{n} \leq \Delta(G)$.

Directed Graphs

Definition: A directed graph (or digraph) $G = (V, E)$ consists of a nonempty set of vertices V and a set of direct edges E. Each edge is a associated with an ordered pair of vertices.

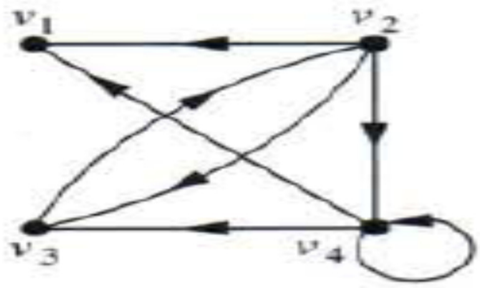
The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Simple directed graph: When a directed graph has no loops and has no multiple



edges it is called a simple directed graph.

Directed multigraph: When a directed graph has multiple edges it is called the directed multi graph.



Definition: When uv is an edge in the directed graph, u, is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called initial vertex of the edge $e = uv$ and v is called the terminal vertex of the edge $e = uv$. For the loop they are the same.

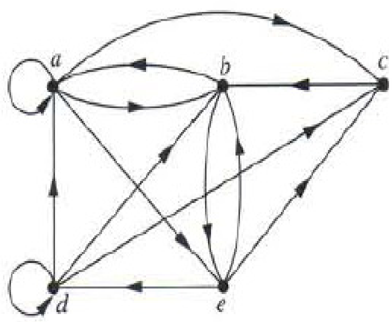
Degree of a vertex in in a directed graph:

The **in-degree** of a vertex v, denoted by $deg^-(v)$, is the number of edges with $v$as their terminal vertex. The **out-degree** of a vertex v, denoted by$deg^+(v)$, is the number of edges with $v$as their initial vertex.

Note that a loop at a vertex contributes one to both the in-degree and the out-degree of this vertex.

Example: Find the in-degree and the out-degree of the following digraph.

Solution:



$$deg^-(a) = 3$$

$$deg^-(b) = 4$$

$$deg^-(c) = 3$$

$$deg^-(d) = 2$$

$$deg^{-}(e) = 2.$$

$$deg^{+}(a) = 4$$

$$deg^{+}(b) = 2$$

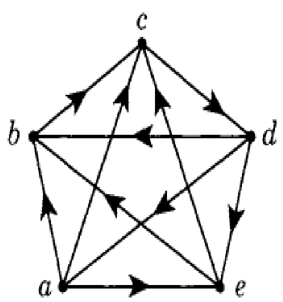
$$deg^{+}(c) = 1$$

$$deg^{+}(d) = 4$$

$$deg^{+}(e) = 3.$$

Theorem: *Let* $G = (V, E)$ *be a directed graph. Then* $\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$.

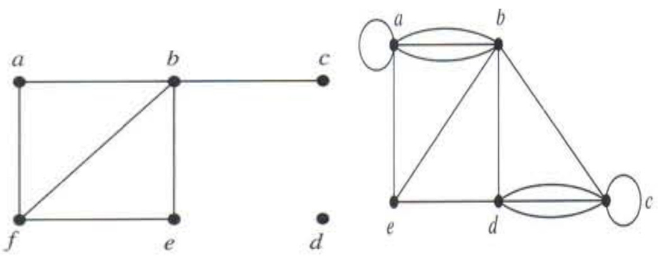
PROOF:
Since indeg(v_i) denotes the number of edges terminating at v_i, each is counted exactly once in $\sum_{i=1}^{n}$ indeg(v_i). Similarly, $\sum_{i=1}^{n}$ outdeg$(v_i) = e$. ■



2.5 Exercises

Let $S = \{-6, -3, 0, 3, 6\}$. Draw the graph G whose vertex set is S and such that $ij \in E(G), for\ i, j \in S\ if\ i + j \in S\ or\ |i - j| \in S$.

1. Find the number of vertices, the number of edges, and the degree of each vertex in the given graphs. Identify all isolated and pendent vertices.



2. Find the sum of the degrees of the vertices of each graph in Exercises 2.

and verify that it equals twice the number of edges in the graph.

3. Can a simple graph exist with 15 vertices each of degree five?
4. Give an example of the following or explain why no such example exists:

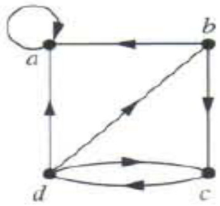
(a) a graph of order 7 whose vertices have degrees 1,1,1,2,2,3,3.

(b) a graph of order 7 whose vertices have degrees 1,2,2,2,3,3,7

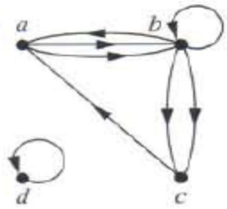
(c) a graph of order 4 whose vertices have degrees 1,3,3,3.

5. The degree of each vertex of a certain graph of order 12 and size 31 is either 4 or 6. How many vertices of degree 4 are there?
6. Show that the maximum number of edges in a simple graph with n vertices is (n(n-1))/2.
7. In Exercises 7-9 determine the number of vertices and edges and find the in –degree and out- degree of each vertex for the given directed multi graph.

7.



8.



9.