Elementary Matrices

the three elementary row operations for matrices listed below were introduced.

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Definition of an Elementary Matrix

An $n \times n$ matrix is called an **elementary matrix** if it can be obtained from the identity matrix I_n by a single elementary row operation.

EXAMPLE

Which of the following matrices are elementary? For those that are, describe the corresponding elementary row operation.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

SOLUTION

(a) This matrix *is* elementary. It can be obtained by multiplying the second row of I_3 by 3.
(b) This matrix is *not* elementary because it is not square.
(c) This matrix is *not* elementary because it was obtained by multiplying the third row of I_3 by 0 (row multiplication must be by a *nonzero* constant).

(d) This matrix *is* elementary. It can be obtained by interchanging the second and third rows of I_3.
(e) This matrix *is* elementary. It can be obtained by multiplying the first row of I_2 by 2 and adding the result to the second row.
(f) This matrix is *not* elementary because two elementary row operations are required to obtain it from I_3.

A Method for Inverting Matrices

To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

$$[A:I] \rightarrow [A^{-1}:I]$$

Example:

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}.$$

$$A:I \rightarrow I:A^{-1}$$

By using elementary operations

$$\left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]..$$

$$\left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right] \qquad R_2 + R_1 \rightarrow R_2$$

$$\left[\begin{array}{cc:cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right] \qquad R_1 + (-4)R_2 \rightarrow R_1$$

$$\underset{A}{\left[\begin{array}{cc}1 & 4 \\ -1 & -3\end{array}\right.} \vdots \underset{I}{\left.\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right]} \rightarrow \underset{I}{\left[\begin{array}{cc}1 & 0 \\ 0 & 1\end{array}\right.} \vdots \underset{A^{-1}}{\left.\begin{array}{cc}-3 & -4 \\ 1 & 1\end{array}\right]}$$

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

SOLUTION

$$[A \vdots I] = \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right].$$

Now, using elementary row operations, rewrite this matrix in the form $[I \vdots A^{-1}]$, as follows.

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-1)R_1 \rightarrow R_2$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \qquad R_3 + (6)R_1 \rightarrow R_3$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \qquad R_3 + (4)R_2 \rightarrow R_3$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad (-1)R_3 \rightarrow R_3$$

$$\left[\begin{array}{rrr:rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_2 + R_3 \rightarrow R_2$$

$$\left[\begin{array}{rrr:rrr} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \qquad R_1 + R_2 \rightarrow R_1$$

The matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

Try confirming this by showing that $AA^{-1} = I = A^{-1}A$.

A Singular Matrix

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

SOLUTION

$$[A \; \vdots \; I] = \left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr:rrr} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \qquad R_2 + (-3)R_1 \rightarrow R_2$$

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right] \qquad R_3 + (2)R_1 \rightarrow R_3$$

Now, notice that adding the second row to the third row produces a row of zeros on the left side of the matrix.

$$\left[\begin{array}{ccc:ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \qquad R_3 + R_2 \rightarrow R_3$$

Because the "A portion" of the matrix has a row of zeros, you can conclude that it is not possible to rewrite the matrix $[A \vdots I]$ in the form $[I \vdots A^{-1}]$. This means that A has no inverse, or is noninvertible (or singular).

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We multiplied the third row by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

We added -2 times the second row to the first.

Thus, $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

If possible, find the inverse of each matrix.

(a) $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ (b) $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Properties of Inverses

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdot \cdot \cdot A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$

EXAMPLE

The Inverse of the Square of a Matrix

Compute A^{-2} in two different ways and show that the results are equal.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

SOLUTION One way to find A^{-2} is to find $(A^2)^{-1}$ by squaring the matrix A to obtain

$$A^2 = \begin{bmatrix} 3 & 5 \\ 10 & 18 \end{bmatrix}$$

and using the formula for the inverse of a 2×2 matrix to obtain

$$(A^2)^{-1} = \frac{1}{4}\begin{bmatrix} 18 & -5 \\ -10 & 3 \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

Another way to find A^{-2} is to find $(A^{-1})^2$ by finding A^{-1}

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$$

and then squaring this matrix to obtain

$$(A^{-1})^2 = \begin{bmatrix} \frac{9}{2} & -\frac{5}{4} \\ -\frac{5}{2} & \frac{3}{4} \end{bmatrix}.$$

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

EXAMPLE

Finding the Inverse of a Matrix Product

Find $(AB)^{-1}$ for the matrices

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$$

using the fact that A^{-1} and B^{-1} are represented by

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}.$$

SOLUTION

$$(AB)^{-1} = B^{-1}A^{-1} = \overset{B^{-1}}{\begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}} \overset{A^{-1}}{\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 8 & -5 & -2 \\ -8 & 4 & 3 \\ 5 & -2 & -\frac{7}{3} \end{bmatrix}.$$

Exercise: Find the inverse of each matrix , if it possible;

(a) $\begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$