

Sheaf semantics of termination-insensitive noninterference

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Abstract

We propose a new *sheaf semantics* for secure information flow over a space of abstract behaviors, based on synthetic domain theory: security classes are open/closed partitions, types are sheaves, and redaction of sensitive information corresponds to restricting a sheaf to a closed subspace. Our security-aware computational model satisfies termination-insensitive noninterference automatically, and therefore constitutes an intrinsic alternative to state of the art extrinsic/relational models of noninterference. Our semantics is the latest application of Sterling and Harper’s recent re-interpretation of *phase distinctions* and noninterference in programming languages in terms of Artin gluing and topos-theoretic open/closed modalities. Prior applications include parametricity for ML modules, the proof of normalization for cubical type theory by Sterling and Angiuli, and the cost-aware logical framework of Niu *et al.* In this paper we employ the phase distinction perspective *twice*: first to reconstruct the syntax and semantics of secure information flow as a lattice of phase distinctions between “higher” and “lower” security, and second to verify the computational adequacy of our sheaf semantics with respect to a version of Abadi *et al.*’s *dependency core calculus* to which we have added a construct for declassifying termination channels.

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1 Introduction

Security-typed languages restrict the ways that classified information can flow from high-security to low-security clients. Abadi *et al.* [1] pioneered the use of *idempotent monads* to



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deliver this restriction in their *dependency core calculus* (DCC), parameterized in a poset of security levels \mathcal{P} . Covariantly in security levels $l \in \mathcal{P}$, a family of type operations $T_l A$ satisfying the rules of an idempotent monad are added to the language; the idea is then that sensitive data can be hidden underneath T_l and unlocked only by a client with a type that can be equipped with a T_l -algebra structure, *i.e.* a $\langle l \rangle$ -sealed type in our terminology.¹ For instance, a high-security client can read a medium-security bit:

$$\begin{aligned} f &: T_M \text{bool} \rightarrow T_H \text{bool} \\ f\ u &= x \leftarrow u; \text{seal}_H(\text{not } x) \end{aligned}$$

There is however no corresponding program of type $T_H \text{bool} \rightarrow T_M \text{bool}$, because the type $T_M \text{bool}$ of medium-security booleans is not $\langle H \rangle$ -sealed, *i.e.* it cannot be equipped with the structure of a T_H -algebra. In fact, up to observational equivalence it is possible to state a *noninterference result* that fully characterizes such programs:

► **Proposition (Noninterference).** *For any closed function $\cdot \vdash f : T_H \text{bool} \rightarrow T_M \text{bool}$, there exists a closed $\cdot \vdash b : T_M \text{bool}$ such that $f \simeq \lambda_ . b$.*

Intuitively the noninterference result above follows because you cannot “escape” the monad, but to prove such a result rigorously a model construction is needed. Today the state of the art is to employ a *relational model* in the sense of Reynolds in which a type is interpreted as a binary relation on some domain, and a term is interpreted by a relation-preserving function. Our contribution is to introduce an *intrinsic* and *non-relational semantics* of noninterference presenting several advantages that we will argue for, inspired by the recent modal reconstruction of *phase distinctions* by Sterling and Harper [62].

1.1 Termination-insensitivity and the meaning of “observation”

► **Notation 1.** We will write LA for the lifting monad that Abadi *et al.* notate A_\perp .

When we speak of noninterference up to observational equivalence, much weight is carried by the choice of what, in fact, counts as an observation. In a functional language with general recursion, it is conventional to say that an observation is given by a computation of *unit* type — which necessarily either diverges or converges with the unique return value $()$. Under this notion of observation, noninterference up to observations takes a very strong character:

Termination-sensitive noninterference. For a closed partial function $\cdot \vdash f : T_H \text{bool} \rightarrow L(T_M \text{bool})$, either $f \simeq \lambda_ . \perp$ or there exists $\cdot \vdash b : T_M \text{bool}$ such that $f \simeq \lambda_ . b$.

If on the other hand we restrict observations to only terminating computations of type *bool*, we evince a more relaxed *termination-insensitive* version of noninterference that allows leakage through the termination channel but *not* through the “return channel”:

Termination-insensitive noninterference. For a closed partial function $\cdot \vdash f : T_H \text{bool} \rightarrow L(T_M \text{bool})$, given any closed u, v on which f terminates, we have $f u \simeq f v$.

¹ We use the term “sealing” for what Abadi *et al.* [1] call “protection”; to avoid confusion, we impose a uniform terminology to encompass both our work and that of *op. cit.* A final notational deviation on our part is that we will distinguish a security level $l \in \mathcal{P}$ from the corresponding syntactical entity $\langle l \rangle$.

1.2 Relational vs. intrinsic semantics

To verify the noninterference property for the dependency core calculus, Abadi *et al.* [1] define a *relational semantics* that starts from an insecure model of computation (domain theory *qua* dcpos) and restricts it by means of binary relations indexed in security levels that express the indistinguishability of sensitive bits to low-security clients. The indistinguishability relations are required to be preserved by all functions, ensuring the security properties of the model. The relational approach has an extrinsic flavor, being characterized by the *post hoc* imposition of order (noninterference) on an inherently disordered computational model. We contrast the extrinsic relational semantics of *op. cit.* with an *intrinsic* denotational semantics in which the underlying computational model has security concerns “built-in” from the start.

1.3 Our contribution: intrinsic semantics of noninterference

The main contribution of our paper is to develop an *intrinsic semantics* in the sense of Section 1.2, in which termination-insensitive noninterference (Section 1.1) is not bolted on but rather arises directly from the underlying computational model. To summarize our approach, instead of controlling the security properties of ordinary dcpos using a \mathcal{P} -indexed logical relation, we take semantics in a category of \mathcal{P} -indexed dcpos, *i.e.* sheaves of dcpos on a space \mathbf{P} in which each security level $l \in \mathcal{P}$ corresponds to an open/closed partition. Employing the viewpoint of Sterling and Harper [62], each of these partitions induces a *phase distinction* between data visible below security level l (open) and data that is hidden (closed), leading to a novel account of the sealing monad T_l as restriction to a closed subspace.

Our intrinsic semantics has several advantages over the relational approach. Firstly, termination-insensitive noninterference arises directly from our computational model. Secondly, our model of secure information flow contributes to the consolidation and unification of ideas in programming languages by treating general recursion and security typing as instances of two orthogonal and well-established notions, namely *axiomatic & synthetic domain theory* and *phase distinctions/Artin gluing* respectively. Termination-insensitivity then arises from the non-trivial interaction between these orthogonal layers.

In particular, our computational model employs the ideas of axiomatic domain theory in the sense of Fiore [15], embedding into a sheaf model of synthetic domain theory [19, 14, 17, 18, 16, 20, 40]. Hence the interpretation of the PCF fragment of DCC is interpreted exactly as in the standard Plotkin semantics of general recursion in categories of partial maps, in contrast to the relational model of Abadi *et al.* Lastly, the view of security levels as phase distinctions per Sterling and Harper [62] advances a uniform perspective on noninterference scenarios that has already proved fruitful for resolving several problems in programming languages:

1. A generalized abstraction theorem for ML modules with strong sums [62].
2. Normalization and decidability of type checking for cubical type theory [61, 59] and multi-modal type theory [23]; guarded canonicity for guarded dependent type theory [24].
3. The design and metatheory of the **calf** logical framework [41] for simultaneously verifying the correctness and complexity of functional programs.

The final benefit of the phase distinction perspective is that logical relations arguments can be re-cast as imposing an *additional* orthogonal phase distinction between *syntax* and *logic/specification*, an insight originally due to Peter Freyd in his analysis of the existence and disjunction properties in terms of Artin gluing [21]. We employ this insight in the present paper to develop a uniform treatment of our denotational semantics and its computational adequacy in terms of phase distinctions.

2 Background: relational semantics of noninterference

To establish noninterference for the dependency core calculus, Abadi *et al.* [1] define a relational model of their monadic language in which each type A is interpreted as a dcpo $|A|$ equipped with a family of admissible binary relations R_l^A indexed in security levels $l \in \mathcal{P}$. In the relational semantics, a term $\Gamma \vdash M : A$ is interpreted as a continuous function $|M| : |\Gamma| \rightarrow |A|$ such that for all $l \in \mathcal{P}$, if $\gamma R_l^\Gamma \gamma'$ then $|M|\gamma R_l^A |M|\gamma'$.

► **Remark 2.** Two elements $u, v \in A$ such that $u R_l^A v$ have been called *equivalent* in subsequent literature, but this terminology may lead to confusion as there is nothing forcing the relation to be transitive, nor even symmetric nor reflexive.

The essence of the relational model is to impose *relations* between elements that should not be distinguishable by a certain security class; a type like `bool` or `string` whose relation is totally discrete, then, allows any security class to distinguish all distinct elements. Non-discrete types enter the picture through the sealing modality T_l :

$$|T_l A| = |A| \quad u R_k^{T_l A} v \iff \begin{cases} u R_k^A v & \text{if } l \sqsubseteq k \\ \top & \text{otherwise} \end{cases}$$

Under this interpretation, the denotation of a function $T_H \text{bool} \rightarrow T_M \text{bool}$ must be a constant function, as $u R_H^{\text{bool}} v$ if and only iff $u = v$. By proving computational adequacy for this denotational semantics, one obtains the analogous *syntactic* noninterference result up to observational equivalence.

Generalization and representation of relational semantics. The relations imposed on each type give rise to a form of cohesion in the sense of Lawvere [37], where elements that are related are thought of as “stuck together”. Then noninterference arises from the behavior of maps from a relatively codiscrete space into a relatively discrete space, as pointed out by Kavvos [34] in his *tour de force* generalization of the relational account of noninterference in terms of axiomatic cohesion. Another way to understand the relational account is by *representation*, as attempted by Tse and Zdancewic [66] and executed by Bowman and Ahmed [10]: one may embed DCC into a polymorphic lambda calculus in which the security abstraction is implemented by *actual* type abstraction.

Adapting the relational semantics for termination-insensitivity

In the relational semantics of the dependency core calculus, the termination-sensitive version of noninterference is achieved by interpreting the *lift* of a type in the following way:

$$|A_\perp| = |A|_\perp \quad u R_l^{A_\perp} v \iff (u, v \downarrow \wedge u R_l^A v) \vee (u = v = \perp)$$

To adapt the relational semantics for termination-insensitivity, Abadi *et al.* change the interpretation of lifts to identify *all* elements with the bottom element:

$$|A_\perp| = |A|_\perp \quad u R_l^{A_\perp} v \iff (u, v \downarrow \wedge u R_l^A v) \vee (u = \perp) \vee (v = \perp)$$

That all data is “indistinguishable” from the non-terminating computation means that the indistinguishability relation cannot be both transitive and non-trivial, a somewhat surprising state of affairs that leads to our critique of relational semantics for information flow below and motivates our new perspective based on the analogy between *phase distinctions* in programming languages and *open/closed partitions* in topological spaces [62].

Critique of relational semantics for information flow

From our perspective there are several problems with the relational semantics of Abadi *et al.* [1] that, while not fatal on their own, inspire us to search for an alternative perspective.

Failure of monotonicity. First of all, within the context of the relational semantics it would be appropriate to say that an object A is $\langle l \rangle$ -sealed when the relation R_l^A is indiscrete (*i.e.* the total relation that identifies all elements). But in the semantics of Abadi *et al.*, it is not necessarily the case that a $\langle l \rangle$ -sealed object is $\langle k \rangle$ -sealed when $k \sqsubseteq l$. It is true that objects that are *definable* in the dependency core calculus are better behaved, but in proper denotational semantics one is not concerned with the image of an interpretation function but rather with the entire category.

Failure of transitivity. A more significant and harder to resolve problem is the fact that the indistinguishability relation R_l^A assigned to each type cannot be construed as an equivalence relation — despite the fact that in real life, indistinguishability is indeed reflexive, symmetric, and transitive. As we have pointed out, the adaptation of DCC’s relational semantics for termination-insensitivity is evidently incompatible with using (total or partial) equivalence relations to model indistinguishability, as transitivity would ensure that no two elements of A_\perp can be distinguished from another.

Where is the dominance? Conventionally the denotational semantics for a language with general recursion begins by choosing a category of “predomains” and then identifying a notion of *partial map* between them that evinces a *dominance* [15, 52]. It is unclear in what sense the DCC’s relational semantics reflects this hard-won arrangement; as we have seen, the adaptation of the relational semantics for termination-insensitivity further increases the distance from ordinary domain-theoretic semantics.

Perspective. Abadi *et al.*’s relational semantics is based on imposing secure information flow properties on an existing insecure model of partial computation, but this is quite distinct from an *intrinsic denotational semantics* for secure information flow — which would necessarily entail new notions of predomain and partial map that are sensitive to security from the start. In this paper we report on such an intrinsic semantics for secure information flow in which termination-insensitive noninterference arises inexorably from the chosen dominance.

3 Central ideas of this paper

In this section, we dive a little deeper into several of the main concepts that substantiate the contributions of this paper. We begin by fixing a poset \mathcal{P} of security levels closed under finite meets, for example $\mathcal{P} = \{\mathsf{L} \sqsubseteq \mathsf{M} \sqsubseteq \mathsf{H} \sqsubseteq \top\}$. The purpose of including a security level even higher than H will become apparent when we explain the meaning of the sealing monad T_l .

► **Notation 3.** Given a space \mathbf{X} and an open set $U \in \mathcal{O}_{\mathbf{X}}$, we will write $\mathbf{X}_{/U}$ for the open subspace spanned by U and $\mathbf{X}_{\bullet U}$ for the corresponding complementary closed subspace. We also will write $\mathcal{S}_{\mathbf{X}}$ for the category of sheaves on the space \mathbf{X} .

3.1 A space of abstract behaviors and security policies

We begin by transforming the security poset \mathcal{P} into a topological space \mathbf{P} of “abstract behaviors” whose algebra of open sets $\mathcal{O}_{\mathbf{P}}$ can be thought of as a lattice of *security policies* that govern whether a given behavior is permitted.

► **Definition 4.** An *abstract behavior* is a filter on the poset \mathcal{P} , *i.e.* a monotone subset $x \subseteq \mathcal{P}$ such that $\bigwedge_{i < \omega} l_i \in x$ if and only if each $l_i \in x$.

► **Definition 5.** A *security policy* is a lower set in \mathcal{P} , i.e. an antitone subset $U \subseteq \mathcal{P}$. We will write $U \Vdash x$ to mean U permits the behavior x , i.e. the subset $x \cap U$ is inhabited.

An abstract behavior x denotes the set of security levels $l \in \mathcal{P}$ at which it is permitted; a security policy U denotes the set of security levels *above which* some behavior is permitted.

► **Construction 6.** We define \mathbf{P} to be the topological space whose points are abstract behaviors, and whose open sets are of the form $\{x \mid U \Vdash x\}$ for some security policy U .²

We have a meet-preserving embedding of posets $\langle - \rangle : \mathcal{P} \hookrightarrow \mathcal{O}_{\mathbf{P}}$ that exhibits $\mathcal{O}_{\mathbf{P}}$ as the free completion of \mathcal{P} under joins, or equivalently the free frame on the meet semi-lattice \mathcal{P} .

► **Intuition 7** (Open and closed subspaces). Each security level $l \in \mathcal{P}$ represents a security policy $\langle l \rangle \in \mathcal{O}_{\mathbf{P}}$ whose corresponding open subspace $\mathbf{P}_{/\langle l \rangle}$ is spanned by the behaviors *permitted* at security levels l and above. Conversely the complementary closed subspace $\mathbf{P}_{\bullet \langle l \rangle} = \mathbf{P} \setminus \mathbf{P}_{/\langle l \rangle}$ is spanned by behaviors that are *forbidden* at security level l and below.

3.2 Sheaves on the space of abstract behaviors

Our intention is to interpret each type of a dependency core calculus as a *sheaf* on the space \mathbf{P} of abstract behaviors. To see why this interpretation is plausible as a basis for secure information flow, we note that a sheaf on \mathbf{P} is the same thing as a presheaf on the poset \mathcal{P} , i.e. a family of sets $(A_l)_{l \in \mathcal{P}}$ indexed contravariantly in \mathcal{P} in the sense that for $k \sqsubseteq l$ there is a chosen restriction function $A_l \rightarrow A_k$ satisfying two laws. Hence a sheaf on \mathbf{P} determines (1) for each security level $l \in \mathcal{P}$ a choice of what data is visible under the security policy $\langle l \rangle$, and (2) a way to *redact* data as it passes under a more restrictive security policy $\langle k \rangle \subseteq \langle l \rangle$.

3.3 Transparency and sealing from open and closed subspaces

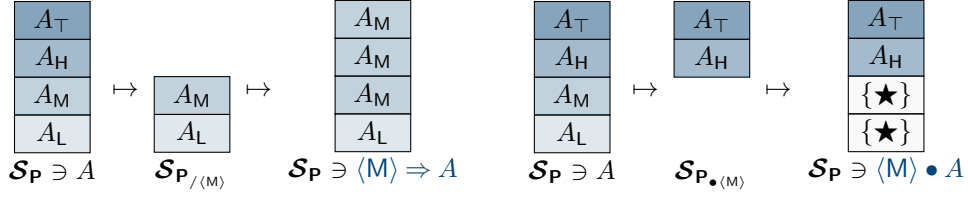
For any subspace $\mathbf{Q} \subseteq \mathbf{P}$, a sheaf $A \in \mathcal{S}_{\mathbf{P}}$ can be restricted to \mathbf{Q} , and then extended again to \mathbf{P} . This composite operation gives rise to an *idempotent monad* on $\mathcal{S}_{\mathbf{P}}$ that has the effect of purging any data from $A \in \mathcal{S}_{\mathbf{P}}$ that cannot be seen from the perspective of \mathbf{Q} . The idempotent monads corresponding to the open and closed subspaces induced by a security level $l \in \mathcal{P}$ are named and notated as follows:

1. The *transparency monad* $A \mapsto (\langle l \rangle \Rightarrow A)$ replaces A with whatever part of it can be viewed under the policy $\langle l \rangle$. The transparency monad is the function space $A^{\langle l \rangle}$, recalling that an open set of \mathbf{P} is the same as a subterminal sheaf. When the unit is an isomorphism at A , we say that A is *$\langle l \rangle$ -transparent*.
2. The *sealing monad* $A \mapsto (\langle l \rangle \bullet A)$ removes from A whatever part of it can be viewed under the policy $\langle l \rangle$. The sealing monad can be constructed as the pushout $\langle l \rangle \sqcup_{\langle l \rangle \times A} A$. When the unit is an isomorphism at A , we say that A is *$\langle l \rangle$ -sealed*.

The transparency and sealing monads interact in two special ways, which can be made apparent by appealing to the visualization of their behavior that we present in Figure 1.

1. The $\langle l \rangle$ -transparent part of a $\langle l \rangle$ -sealed sheaf is trivial, i.e. we have $(\langle l \rangle \Rightarrow (\langle l \rangle \bullet A)) \cong \{\star\}$.
2. Any sheaf $A \in \mathcal{S}_{\mathbf{P}}$ can be reconstructed as the fiber product $(\langle l \rangle \Rightarrow A) \times_{\langle l \rangle \bullet (\langle l \rangle \Rightarrow A)} \langle l \rangle \bullet A$.

² Those familiar with the point-free topology of topoi [30, 67, 3] will recognize that \mathbf{P} is more simply described as the presheaf topos \mathcal{P} : viewed as a space, it is the dcpo completion of \mathcal{P}^{op} , and as a frame it is the free cocompletion of \mathcal{P} . The definition of $U \Vdash x$ then presents a computation of the *stalk* U_x of the subterminal sheaf $U \in \mathcal{S}_{\mathbf{P}}$ at the behavior $x \in \mathbf{P}$.



■ **Figure 1** The transparency and sealing monads for $M \in \mathcal{P}$ on a sheaf $A \in \mathcal{S}_{\mathcal{P}}$ visualized.

The first property above immediately gives rise to a form of noninterference, which justifies our intent to interpret DCC's sealing monad as $T_l A = \langle l \rangle \bullet A$.

► **Observation 8** (Noninterference). *Any map $\langle l \rangle \bullet A \rightarrow \text{bool}$ is constant.*

Proof. We may verify that the boolean sheaf bool is $\langle l \rangle$ -transparent for all $l \in \mathcal{P}$. ◀

Our sealing monad above is well-known to the type-and-topos-theoretic community as the *closed modality* [51, 53, 4] corresponding to the open set $\langle l \rangle \in \mathcal{O}_{\mathcal{P}}$. In the context of (total) dependent type theory, our sealing monad has excellent properties not shared by those of Abadi *et al.* [1], such as justifying *dependent* elimination rules and commuting with identity types. In contrast to the *classified sets* of Kavvos [34] which cannot form a topos, our account of information flow is compatible with the full internal language of a topos.

3.4 Recursion and termination-insensitivity via sheaves of domains

To incorporate recursion into our sheaf semantics of information flow, in this section we consider *internal dcpos* in $\mathcal{S}_{\mathcal{P}}$, *i.e.* sheaves of dcpos. Later in the technical development of our paper, we work in the axiomatic setting of synthetic domain theory, but all the necessary intuitions can also be understood concretely in terms of dcpos. Domain theory internal to $\mathcal{S}_{\mathcal{P}}$ works very similarly to classical domain theory, but it must be developed without appealing to the law of the excluded middle or the axiom of choice as these do not hold in $\mathcal{S}_{\mathcal{P}}$ except for a particularly degenerate security poset. De Jong and Escardó [12] explain how to set up the basics of domain theory in a suitably constructive manner, which we will not review.

The sheaf-theoretic domain semantics sketched above leads immediately to a new and simplified account of termination-insensitivity. It is instructive to consider whether there is an analogue to Observation 8 for partial continuous functions $\langle l \rangle \bullet A \rightarrow \mathbb{L} \text{ bool}$. It is not the case that $\mathbb{L} \text{ bool}$ is $\langle l \rangle$ -transparent for all $l \in \mathcal{P}$, so it would not follow that any continuous map $\langle l \rangle \bullet A \rightarrow \mathbb{L} \text{ bool}$ is constant. A partial function always extends to a total function on a restricted domain, however, so we may immediately conclude the following:

► **Observation 9** (Termination-insensitive noninterference). *For any continuous map $f : \langle l \rangle \bullet A \rightarrow \mathbb{L} \text{ bool}$ and elements $u, v : \langle l \rangle \bullet A$ with fu and fv defined, we have $fu = fv$.*

This is the sense in which termination-insensitive noninterference arises automatically from the combination of domain theory with sheaf semantics for information flow.

4 Refined dependency core calculus

We now embark on the technical development of this paper, beginning with a call-by-push-value (cbpv) style [38] refinement of the dependency core calculus over a poset \mathcal{P} of security

levels. We will work informally in the logical framework of locally Cartesian closed categories *à la* Gratzer and Sterling [26]; we will write \mathcal{T} for the free locally Cartesian closed category generated by all the constants and equations specified herein.

4.1 The basic language

We have value types $A : \mathbf{tp}^+$ and computation types $X : \mathbf{tp}^\ominus$; because our presentation of cbpv does not include stacks, we will not include a separate syntactic category for computations but instead access them through thunking. The sorts of value and computation types and their adjoint connectives are specified below:

$$\mathbf{tp}^+, \mathbf{tp}^\ominus : \mathbf{Sort} \quad \mathbf{tm} : \mathbf{tp}^+ \rightarrow \mathbf{Sort} \quad \mathbf{U} : \mathbf{tp}^\ominus \rightarrow \mathbf{tp}^+ \quad \mathbf{F} : \mathbf{tp}^+ \rightarrow \mathbf{tp}^\ominus$$

We let A, B, C range over \mathbf{tp}^+ and X, Y, Z over \mathbf{tp}^\ominus . We will often write A instead of $\mathbf{tm} A$ when it causes no ambiguity. Free computation types are specified as follows:

$$\begin{aligned} \mathbf{ret} : A \rightarrow \mathbf{U}FA & & \mathbf{bind}(\mathbf{ret} u) f &\equiv_{\mathbf{U}X} f u \\ \mathbf{bind} : \mathbf{U}FA \rightarrow (A \rightarrow \mathbf{U}X) \rightarrow \mathbf{U}X & & \mathbf{bind} u \mathbf{ret} &\equiv_{\mathbf{U}FA} u \\ & & \mathbf{bind}(\mathbf{bind} u f) g &\equiv_{\mathbf{U}X} \mathbf{bind} u (\lambda x. \mathbf{bind}(f x) g) \end{aligned}$$

We support general recursion in computation types:

$$\mathbf{fix} : (\mathbf{U}X \rightarrow \mathbf{U}X) \rightarrow \mathbf{U}X \quad \mathbf{fix} f \equiv f(\mathbf{fix} f)$$

We close the universe $X : \mathbf{tp}^\ominus \vdash \mathbf{tm} \mathbf{U}X$ of computation types and thunked computations under all function types $\mathbf{tm} A \rightarrow \mathbf{tm} \mathbf{U}X$ by adding a new computation type constant \mathbf{fn} equipped with a universal property like so:

$$\mathbf{fn} : \mathbf{tp}^+ \rightarrow \mathbf{tp}^\ominus \rightarrow \mathbf{tp}^\ominus \quad \mathbf{fn}.\mathbf{tm} : (A \rightarrow \mathbf{U}X) \cong \mathbf{U}(\mathbf{fn} A X)$$

We will treat this isomorphism implicitly in our informal notation, writing $\lambda x. u(x)$ for both meta-level and object-level function terms. Finite product types are specified likewise:

$$\begin{aligned} \mathbf{prod} : \mathbf{tp}^+ \rightarrow \mathbf{tp}^+ \rightarrow \mathbf{tp}^+ & & \mathbf{unit} : \mathbf{tp}^+ \\ \mathbf{prod}.\mathbf{tm} : A \times B \cong \mathbf{prod} A B & & \mathbf{unit}.\mathbf{tm} : \mathbf{1} \cong \mathbf{unit} \end{aligned}$$

Sum types must be treated specially because we do not intend them to be coproducts in the logical framework: they should have a universal property for types, not for sorts.

$$\begin{aligned} \mathbf{sum} : \mathbf{tp}^+ \rightarrow \mathbf{tp}^+ \rightarrow \mathbf{tp}^+ & & \mathbf{case} : \mathbf{sum} A B \rightarrow (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow C \\ \mathbf{inl} : A \rightarrow \mathbf{sum} A B & & \mathbf{case}(\mathbf{inl} u) f g \equiv_C f u \\ \mathbf{inr} : B \rightarrow \mathbf{sum} A B & & \mathbf{case}(\mathbf{inr} v) f g \equiv_C g v \\ & & \mathbf{case} u (\lambda x. f(\mathbf{inl} x)) (\lambda x. f(\mathbf{inr} x)) \equiv_C f u \end{aligned}$$

4.2 The sealing modality and declassification

For each $l \in \mathcal{P}$, we add an *abstract* proof irrelevant proposition $\langle l \rangle : \mathbf{Prop}$ to the language; this proposition represents the condition that the “client” has a lower security clearance than l . This “redaction” is implemented by isolating the types that are *sealed* at $\langle l \rangle$, *i.e.* those that become singletons in the presence of $\langle l \rangle$:

$$\begin{aligned} \langle l \rangle : \mathbf{Prop} & & \mathbf{sealed}_{\langle l \rangle} : \mathbf{tp}^+ \rightarrow \mathbf{Prop} \\ \langle k \rangle \rightarrow \langle l \rangle & & \mathbf{sealed}_{\langle l \rangle} A := \langle l \rangle \rightarrow \{x : A \mid \forall y : A. x \equiv_A y\} \\ \langle k \rangle \rightarrow \langle l \rangle \rightarrow \langle k \wedge l \rangle & & \end{aligned}$$

We will write $\text{tp}_{\bullet\langle l \rangle}^+ \subseteq \text{tp}^+$ for the subtype spanned by value types A for which $\text{sealed}_{\langle l \rangle} A$ holds. As in Section 3.3, we will write \star for the unique element of an $\langle l \rangle$ -sealed type in the presence of $u : \langle l \rangle$. Next we add the sealing modality itself:

$$\begin{aligned} \text{T}_l : \text{tp}^+ &\rightarrow \text{tp}_{\bullet\langle l \rangle}^+ & \text{unseal}_l : \{B : \text{tp}_{\bullet\langle l \rangle}^+\} &\rightarrow \text{T}_l A \rightarrow (A \rightarrow B) \rightarrow B \\ \text{seal}_l : A &\rightarrow \text{T}_l A & \text{unseal}_l (\text{seal}_l u) f &\equiv_B f u \\ & & \text{unseal}_l u (\lambda x. f (\text{seal}_l x)) &\equiv_B f u \end{aligned}$$

Finally a construct for declassifying the termination channel of a sealed computation:

$$\text{tdcl}_{\langle l \rangle} : \{A : \text{tp}_{\bullet\langle l \rangle}^+\} \rightarrow \text{T}_l \text{UFA} \rightarrow \text{UFA} \quad \text{tdcl}_{\langle l \rangle} (\text{seal}_l (\text{ret } u)) \equiv_{\text{UFA}} \text{ret } u$$

► **Remark 10.** The $\langle l \rangle$ propositions play a purely book-keeping role, facilitating verification of program equivalences in the same sense as the ghost variables of Owicki and Gries [43].

5 Denotational semantics in synthetic domain theory

We will define our denotational semantics for information flow and termination-insensitive noninterference in a category of domains indexed in \mathcal{P} . To give a model of the theory presented in Section 4 means to define a locally Cartesian closed functor $\mathcal{T} \rightarrow \mathcal{E}$ where \mathcal{E} is locally Cartesian closed. Unfortunately no category of domains can be locally Cartesian closed, but we can *embed* categories of domains in a locally Cartesian closed category by following the methodology of *synthetic domain theory* [19, 14, 17, 18, 16, 20, 40].³

5.1 A topos for information flow logic

Recall that \mathcal{P} is a poset of security levels closed under finite meets. The presheaf topos \mathbf{P} defined by the identification $\mathcal{S}_{\mathbf{P}} = [\mathcal{P}^{\text{op}}, \text{Set}]$ contains propositions $y_{\mathcal{P}}l$ corresponding to every security level $l \in \mathcal{P}$, and is closed under both sealing and transparency modalities $y_{\mathcal{P}}l \Rightarrow E, y_{\mathcal{P}}l \bullet E$ in the sense of Section 3.3; in more traditional parlance, these are the *open* and *closed* modalities corresponding to the proposition $y_{\mathcal{P}}l$ [51]. It is possible to give a denotational semantics for a *total* fragment of our language in $\mathcal{S}_{\mathbf{P}}$, but to interpret recursion we need some kind of domain theory. We therefore define a topos model of synthetic domain theory that lies over \mathbf{P} and hence incorporates the information flow modalities seamlessly.

5.2 Synthetic domain theory over the information flow topos

We will now work abstractly with a Grothendieck topos \mathbf{C} equipped with a dominance $\Sigma \in \mathcal{S}_{\mathbf{C}}$, called the *Sierpiński space*, satisfying several axioms that give rise to a reflective subcategory of objects that behave like predomains. We leave the construction of \mathbf{C} to our technical appendix, where it is built by adapting the recipe of Fiore and Plotkin [17].

► **Definition 11** (Rosolini [52]). A *dominion* on a category \mathcal{E} is a stable class of monos closed under identity and composition. Given a dominion \mathcal{M} such that \mathcal{E} has finite limits, a *dominance* for \mathcal{M} is a classifier $\top : \mathbf{1}_{\mathcal{E}} \rightarrow \Sigma$ for the elements of \mathcal{M} in the sense that every $U \rightarrow A \in \mathcal{M}$ gives rise to a unique map $\chi_U : A \rightarrow \Sigma$ such that $U \cong \chi_U^* \top$.

³ In particular we focus on the style of synthetic domain theory based on Grothendieck topoi and well-complete objects. There is another very productive strain of synthetic domain theory based on realizability and replete objects that has different properties [28, 44, 65, 45, 46, 47, 48, 49].

If \mathcal{E} is locally cartesian closed, we may form the *partial element classifier* monad $L : \mathcal{E} \rightarrow \mathcal{E}$ for a dominance Σ , setting $LE = \sum_{\phi: \Sigma} \phi \Rightarrow E$; given $e \in LE$, we will write $e \downarrow \in \Sigma$ for the termination support $\pi_1 e$ of e . We are particularly interested in the case where L has a final coalgebra $\bar{\omega} \cong L\bar{\omega}$ and an initial algebra $L\omega \cong \omega$. When \mathcal{E} is the category of sets, ω is just the natural numbers object \mathbb{N} and $\bar{\omega}$ is \mathbb{N}_∞ , the natural numbers with an infinite point adjoined. In general, one should think of ω as the “figure shape” of a formal ω -chain $\omega \rightarrow E$ that takes into account the data of the dominance; then $\bar{\omega}$ is the figure shape of a formal ω -chain equipped with its supremum, given by evaluation at the infinite point $\infty \in \bar{\omega}$. There is a canonical inclusion $\iota : \omega \rightarrow \bar{\omega}$ witnessing the *incidence relation* between a chain equipped with its supremum and the underlying chain.

► **Axiom SDT-1.** Σ has *finite joins* $\bigvee_{i < n} \phi_i$ that are preserved by the inclusion $\Sigma \subseteq \Omega$. We will write \perp for the empty join and $\phi \vee \psi$ for binary joins.

► **Definition 12** (Complete types). In the internal language of \mathcal{E} , a type E is called *complete* when it is internally orthogonal to the comparison map $\omega \rightarrow \bar{\omega}$. In the internal language, this says that for any formal chain $e : \omega \rightarrow E$ there exists a unique figure $\hat{e} : \bar{\omega} \rightarrow E$ such that $\hat{e} \circ \iota = e$. In this scenario, we write $\bigsqcup_{i \in \omega} e_i$ for the evaluation $\hat{e} \infty$.

► **Axiom SDT-2.** The initial lift algebra ω is the colimit of the following ω -chain of maps:

$$\emptyset \xrightarrow{!} L\emptyset \xrightarrow{L!} L^2\emptyset \xrightarrow{L^2!} \dots$$

► **Definition 13.** A type E is called a *predomain* when LE is complete.

► **Axiom SDT-3.** The dominance Σ is a predomain.

The category of predomains is complete, cocomplete, closed under lifting, exponentials, and powerdomains, and is a reflective exponential ideal in $\mathcal{S}_{\mathcal{C}}$ — thus better behaved than any classical category of predomains. The predomains with L -algebra structure serve as an appropriate notion of *domain* in which arbitrary fixed points can be interpreted by taking the supremum of formal ω -chains of approximations $f^n \perp$; in addition to “term-level” recursion, we may also interpret recursive types. We impose two additional axioms for information flow:

► **Axiom SDT-4.** The topos \mathcal{C} is equipped with a geometric morphism $p_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{P}$ such that the induced functor $p_{\mathcal{C}}^* y_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{O}_{\mathcal{C}}$ is fully faithful and is valued in Σ -propositions. We will write $\langle l \rangle$ for each $p_{\mathcal{C}}^* y_{\mathcal{P}} l$.

Axiom SDT-4 ensures that our domain theory include computations whose termination behavior depends on the observer’s security level. The following **Axiom SDT-5** is applied to the semantic noninterference property.

► **Axiom SDT-5.** Any constant object $\mathbf{C}^*[n] \in \mathcal{S}_{\mathcal{C}}$ for $[n]$ a finite set is an $\langle l \rangle$ -transparent predomain for any $l \in \mathcal{P}$.

The category $\mathcal{S}_{\mathcal{C}}$ is closed under as many topos-theoretic universes [63] as there are Grothendieck universes in the ambient set theory. For any such universe \mathbf{U}_i , there is a subuniverse $\mathbf{Predom}_i \subseteq \mathbf{U}_i$ spanned by *predomains*; we note that being a predomain is a property and not a structure. The object \mathbf{Predom}_i can exist because being a predomain is a local property that can be expressed in the internal logic. In fact, the predomains can be seen to be not only a reflective subcategory but also a reflective *subfibration* as they are obtained by the internal localization at a class of maps [55]; therefore the reflection can be internalized as a connective $\mathbf{U}_i \rightarrow \mathbf{Predom}_i$ implemented as a quotient-inductive type [54]. We may define the corresponding universe of domains \mathbf{Dom}_i to be the collection of predomains in \mathbf{Predom}_i equipped with L -algebra structures. We hereafter suppress universe levels.

5.3 The stabilizer of a predomain and its action

In this section, we work internally to the synthetic domain theory of \mathcal{S}_C ; first we recall the definition of an *action* for a commutative monoid.

► **Definition 14.** *Let $(M, 0, +)$ be a monoid object in the category of predomains; an ***M-action structure*** on a predomain A is given by a function $\parallel_A : M \times A \rightarrow A$ satisfying the identities $0 \parallel_A a = a$ and $m \parallel_A n \parallel_A a = (m + n) \parallel_A a$.*

Write Σ^\vee for the additive monoid structure of the Sierpiński domain, with addition given by Σ -join $\phi \vee \psi$ and the unit given by the non-terminating computation \perp . Our terminology below is inspired by stabilizer subgroups in algebra.

► **Definition 15** (The stabilizer of a predomain). *Given a predomain A , we define the ***stabilizer*** of A to be the submonoid $\text{Stab}_{\Sigma^\vee} A \subseteq \Sigma^\vee$ spanned by $\phi : \Sigma^\vee$ such that A is ϕ -sealed, i.e. the projection map $A \times \phi \rightarrow \phi$ is an isomorphism.*

► **Remark 16.** We can substantiate the analogy between Definition 15 and stabilizer subgroups in algebra. Up to coherence issues that could be solved using higher categories, any category \mathcal{P} of predomains closed under subterminals and pushouts can be structured with a monoid action over Σ^\vee ; the action $\parallel_{\mathcal{P}} : \Sigma^\vee \times \mathcal{P} \rightarrow \mathcal{P}$ takes A to the ϕ -sealed object $\phi \parallel_{\mathcal{P}} A := \phi \bullet A$. Up to isomorphism, the identities for a Σ^\vee -action can be seen to be satisfied. Then we say that the stabilizer of a predomain $A \in \mathcal{P}$ is the submonoid $\text{Stab}_{\Sigma^\vee} A \subseteq \Sigma^\vee$ consisting of propositions ϕ such that $\phi \parallel_{\mathcal{P}} A \cong A$.

► **Lemma 17.** *For any predomain A , we may define a canonical $\text{Stab}_{\Sigma^\vee} A$ -action on $\mathsf{L}A$:*

$$\begin{aligned} \parallel_{\mathsf{L}A} : \text{Stab}_{\Sigma^\vee} A \times \mathsf{L}A &\rightarrow \mathsf{L}A \\ \phi \parallel_{\mathsf{L}A} a &= (\phi \vee a \downarrow, [\phi \hookrightarrow \star, a \downarrow \hookrightarrow a]) \end{aligned}$$

The stabilizer action described in Lemma 17 will be used to implement declassification of termination channels in our denotational semantics.

► **Lemma 18.** *The stabilizer action preserves terminating computations in the sense that $\phi \parallel_{\mathsf{L}A} u = u$ for $\phi : \text{Stab}_{\Sigma^\vee} A$ and terminating $u : \mathsf{L}A$.*

Proof. We observe that $\phi \vee \top = \top$, hence for terminating a we have $\phi \parallel_{\mathsf{L}A} a = a$. ◀

5.4 The denotational semantics

We now define an algebra for the theory \mathcal{T} in \mathcal{S}_C ; the initial prefix of this algebra is standard:

$$\begin{array}{ll} \llbracket \text{tp}^+ \rrbracket = \mathbf{Predom} & \llbracket \text{prod} \rrbracket A B = A \times B \\ \llbracket \text{tp}^\ominus \rrbracket = \mathbf{Dom} & \llbracket \text{prod.tm} \rrbracket = \langle \text{canonical} \rangle \\ \llbracket \mathsf{U} \rrbracket X = X & \llbracket \text{unit} \rrbracket = \mathbf{1}_{\mathbf{Predom}} \\ \llbracket \mathsf{F} \rrbracket A = \mathsf{L}A & \llbracket \text{unit.tm} \rrbracket = \langle \text{canonical} \rangle \\ \llbracket \text{ret} \rrbracket a = a & \llbracket \text{sum} \rrbracket A B = A + B \\ \llbracket \text{bind} \rrbracket m f = f^\# m & \llbracket \text{inl} \rrbracket a = \text{inl } a \\ \llbracket \text{fix} \rrbracket f = \text{fix } f & \llbracket \text{inr} \rrbracket a = \text{inr } a \\ \llbracket \text{fn} \rrbracket A X = A \Rightarrow X & \llbracket \text{case} \rrbracket u f g = \begin{cases} f(x) & \text{if } u = \text{inl } x \\ g(x) & \text{if } u = \text{inr } x \end{cases} \\ \llbracket \text{fn.tm} \rrbracket = \langle \text{canonical} \rangle & \end{array}$$

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Note that the coproduct $A + B$ above is computed in the category of predomains⁴ and need not be preserved by the embedding into $\mathcal{S}_{\mathbf{C}}$. We next add the security levels and the sealing modality, interpreted as the pushout of predomains $\langle l \rangle \bullet A$, again computed in the category of predomains. We define the unsealing operator for $B : \llbracket \mathbf{tp}_{\bullet \langle l \rangle}^+ \rrbracket$ using the universal property of the pushout.

$$\begin{aligned} \llbracket \langle l \rangle \rrbracket &= \langle l \rangle = p_{\mathbf{C}}^* y_{\mathcal{P}} l & \llbracket \text{unseal}_l \rrbracket u f &= \\ \llbracket \mathbf{T}_l \rrbracket A &= \langle l \rangle \bullet A & \begin{cases} f x & \text{if } u = \eta_{\bullet \langle l \rangle} x \\ \star & \text{if } u = \star \end{cases} \\ \llbracket \text{seal}_l \rrbracket a &= \eta_{\bullet \langle l \rangle} a \end{aligned}$$

► **Observation 19.** *Morphisms $\langle l \rangle \bullet A \rightarrow B$ are in bijective correspondence with morphisms $A \rightarrow B$ that restricts to a weakly constant function under $\langle l \rangle$.*

We may now interpret the termination declassification operation. Fixing a sealed type $A : \llbracket \mathbf{tp}_{\bullet \langle l \rangle}^+ \rrbracket$, we must define the dotted lift below using the universal property of the pushout and the action of the stabilizer of A on $\mathbf{L}A$, noting that $\langle l \rangle \in \mathbf{Stab}_{\Sigma^v} A$ by assumption:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathbf{L}A \\ \eta_{\bullet \langle l \rangle} \circ \eta_A \downarrow & \nearrow \llbracket \text{tdcl}_{\langle l \rangle} \rrbracket & \\ \langle l \rangle \bullet \mathbf{L}A & & \end{array} \quad \llbracket \text{tdcl}_{\langle l \rangle} \rrbracket u = \begin{cases} \langle l \rangle \parallel_{\mathbf{L}A} x & \text{if } u = \eta_{\bullet \langle l \rangle} x \\ \langle l \rangle \parallel_{\mathbf{L}A} \perp & \text{if } u = \star \end{cases}$$

To see that the above is well-defined, we observe that under $\langle l \rangle$ both branches return the (unique) computation whose termination support is $\langle l \rangle$. With this definition, the required computation rule holds by virtue of Lemma 18.

5.5 Noninterference in the denotational semantics

► **Definition 20.** *A function $u : A \rightarrow B$ is called **weakly constant** [35] if for all $x, y : A$ we have $u x = u y$. A partial function $u : A \rightarrow \mathbf{L}B$ is called **partially constant** if for all $x, y : A$ such that $u x \downarrow \wedge u y \downarrow$, we have $u x = u y$.*

For the following, let $l \in \mathcal{P}$ be a security level.

► **Lemma 21.** *Let A be a $\langle l \rangle$ -sealed predomain and let B be a $\langle l \rangle$ -transparent predomain; then (1) any function $A \rightarrow B$ is weakly constant, and (2) any partial function $A \rightarrow \mathbf{L}B$ is partially constant.*

The following lemma follows from **Axiom** SDT-5.

► **Lemma 22.** *The predomain $\llbracket \mathbf{bool} \rrbracket$ is $\langle l \rangle$ -transparent.*

In order for Lemma 21 to have any import as far as the equational theory is concerned, we must establish computational adequacy. This is the topic of Section 6.

⁴ Any reflective subcategory of a cocomplete category is cocomplete: first compute the colimit in the outer category, and then apply the reflection.

6 Adequacy of the denotational semantics

We must argue that the denotational semantics agrees with the theory as far as convergence and return values is concerned. We do so using a Plotkin-style logical relations argument, phrased in the language of Synthetic Tait Computability [59, 62, 61].

6.1 Synthetic Tait computability of formal approximation

In this section we will work abstractly with a Grothendieck topos \mathbf{G} satisfying several axioms that will make it support a Kripke logical relation for adequacy.

► **Notation 23.** For each universe $\mathbf{U} \in \mathcal{S}_{\mathbf{G}}$ there is a type $\mathcal{T}\text{-Alg}_{\mathbf{U}}$ of internal \mathcal{T} -algebras whose type components are valued in \mathbf{U} . $\mathcal{T}\text{-Alg}_{\mathbf{U}}$ is a dependent record containing a field for every constant in the signature by which we generated \mathcal{T} . Assuming enough universes, functors $\mathcal{T} \rightarrow \mathcal{S}_{\mathbf{G}}/E$ correspond up to isomorphism to morphisms $E \rightarrow \mathcal{T}\text{-Alg}_{\mathbf{U}}$. This is the relationship between the internal language and the *functorial semantics* à la Lawvere [36].

► **Axiom STC-1.** *There are two disjoint propositions $\mathsf{T}, \mathsf{C} \in \mathcal{O}_{\mathbf{G}}$ such that $\mathsf{T} \wedge \mathsf{C} = \perp$. We will refer to these as the *syntactic* and *computational phases* respectively. We will write $\mathsf{B} = \mathsf{T} \vee \mathsf{C}$ for the disjoint union of the two phases.*

► **Axiom STC-2.** *Within the syntactic phase, there exists a \mathcal{T} -algebra $\mathcal{A}^{\mathsf{T}} : \mathcal{T}\text{-Alg}_{\mathbf{U}/\mathsf{T}}$ such that the corresponding functor $\mathcal{T} \rightarrow \mathcal{S}_{\mathbf{G}}/\mathsf{T}$ is fully faithful.*

► **Axiom STC-3.** *Within the computational phase, the axioms of \mathcal{P} -indexed synthetic domain theory (Axioms SDT-1–SDT-5) are satisfied.*

As a consequence of Axiom STC-3, we have a *computational* \mathcal{T} -algebra $\mathcal{A}^{\mathsf{C}} : \mathcal{T}\text{-Alg}_{\mathbf{U}/\mathsf{C}}$ given by the constructions of Section 5.4. Gluing together the two models $\mathcal{A}^{\mathsf{T}}, \mathcal{A}^{\mathsf{C}}$ we see that $\mathbf{G}_{/\mathsf{B}}$ supports a model $\mathcal{A}^{\mathsf{B}} = [\mathsf{T} \hookrightarrow \mathcal{A}^{\mathsf{T}}, \mathsf{C} \hookrightarrow \mathcal{A}^{\mathsf{C}}]$ of \mathcal{T} . The final Axiom STC-4 above is needed in the approximation structure of $\mathbf{tdcl}_{\langle l \rangle}$.

► **Axiom STC-4.** *For each $l \in \mathcal{P}$ we have $\mathcal{A}^{\mathsf{C}}.\langle l \rangle \leq \mathsf{B} \bullet \mathcal{A}^{\mathsf{T}}.\langle l \rangle$.*

► **Theorem 24.** *There exists a topos \mathbf{G} satisfying Axioms STC-1–STC-4 containing open subtopoi $\mathbf{G}_{/\mathsf{T}} = \widehat{\mathcal{T}}$ and $\mathbf{G}_{/\mathsf{C}} = \mathbf{C}$ such that the complementary closed subtopos is $\mathbf{G}_{\bullet\mathsf{B}} = \mathbf{P}$.*

Proof. We may construct a topos using a variant of the Artin gluing construction of Sterling and Harper [62], which we detail in our technical appendix. ◀

By Axioms STC-1 and STC-2, any such topos \mathbf{G} supports a model of the *synthetic Tait computability* of Sterling and Harper [62, 59]. In the internal language of $\mathcal{S}_{\mathbf{G}}$, the phase B induces a pair of complementary transparency/open and sealing/closed modalities that can be used to synthetically construct formal approximation relations in the sense of Plotkin between computational objects and syntactical objects. Viewing an object $E \in \mathcal{S}_{\mathbf{G}}$ as a family $x : \mathsf{C} \Rightarrow E, x' : \mathsf{T} \Rightarrow E \vdash \{E \mid \mathsf{C} \hookrightarrow x, \mathsf{T} \hookrightarrow x'\}$ of B -sealed types over the B -transparent type $(\mathsf{B} \Rightarrow E) \cong ((\mathsf{C} \Rightarrow E) \times (\mathsf{T} \Rightarrow E))$, we may think of E as a *proof-relevant* formal approximation relation between its computational and syntactic parts, which we might term a “formal approximation structure”.

► **Notation 25** (Extension types). We recall *extension types* from Riehl and Shulman [50]. Given a proposition $\phi : \Omega$ and a partial element $e : \phi \Rightarrow E$, we will write $\{E \mid \phi \hookrightarrow e\}$ for the collection of elements of E that restrict to e under ϕ , i.e. the subobject $\{x : E \mid \phi \Rightarrow (x = e)\} \rightarrowtail E$. Note that $\{E \mid \phi \hookrightarrow e\}$ is always ϕ -sealed, since it becomes the singleton type $\{e\}$ under ϕ .

Each universe \mathbf{U} of $\mathcal{S}_{\mathbf{G}}$ satisfies a remarkable *strictification* property with respect to any proposition $\phi : \Omega$ that allows one to construct codes for dependent sums of families of ϕ -sealed types over a ϕ -transparent type in such a way that they restrict *exactly* to the ϕ -transparent part under ϕ . This refinement of dependent sums is called a *strict glue type*:⁵

$$\begin{array}{c} \text{STRICT GLUE TYPES} \\ \hline A : \phi \Rightarrow \mathbf{U} \quad B : ((z : \phi) \Rightarrow Az) \rightarrow \mathbf{U} \quad \forall x. \text{isSealed}_{\phi}(B, x) \\ \hline (x : A) \ltimes_{\phi} Bx : \{\mathbf{U} \mid z : \phi \hookrightarrow Az\} \\ \text{glue}_{\phi} : \{((x : (z : \phi) \Rightarrow Az) \times Bx) \cong (x : A) \ltimes_{\phi} Bx \mid \phi \hookrightarrow \pi_1\} \end{array}$$

► **Notation 26** (Strict glue types). We impose two notations assuming A, B as above. Given $a : (z : \phi) \Rightarrow Az$ and $b : B a$, we write $\text{glue}[b \mid \phi \hookrightarrow a]$ for $\text{glue}_{\phi}(a, b)$. Given $g : (x : A) \ltimes_{\phi} Bx$, we write $\text{unglue}_{\phi} g : Bg$ for the element $\pi_2(\text{glue}_{\phi}^{-1} g)$.

Using the strict glue types it is possible to define very strict universes $\mathbf{U}_{/\phi}$, $\mathbf{U}_{\bullet\phi}$ of ϕ -transparent and ϕ -sealed types respectively which are themselves ϕ -transparent and ϕ -sealed in the next universe as in §3.6 of Sterling’s dissertation [59]:

$$\begin{array}{ll} \mathbf{U}_{/\phi} : \{\mathbf{V} \mid \phi \hookrightarrow \mathbf{U}\} & (\phi \Rightarrow -) : \{\mathbf{U} \rightarrow \mathbf{U}_{/\phi} \mid \phi \hookrightarrow \lambda A. A\} \\ \mathbf{U}_{\bullet\phi} : \{\mathbf{V} \mid \phi \hookrightarrow \mathbf{1}\} & (\phi \bullet -) : \mathbf{U} \rightarrow \mathbf{U}_{\bullet\phi} \end{array}$$

The transparent subuniverse $\mathbf{U}_{/\phi}$ is canonically isomorphic to $\phi \Rightarrow \mathbf{U}$; given an element $A : \mathbf{U}_{/\phi}$, elements of A are the same as partial elements $(z : \phi) \rightarrow Az$ under the former identification. In our notation, we suppress these identifications *as well as* the introduction and elimination form for these ϕ -partial elements. The sealed subuniverse $\mathbf{U}_{\bullet\phi}$ is canonically isomorphic to $\{\mathbf{U} \mid \phi \hookrightarrow \mathbf{1}\}$. The modal universes $\mathbf{U}_{/\phi}$, $\mathbf{U}_{\bullet\phi}$ serve as (*weak*) *generic objects* in the sense of Jacobs [29] for the full subfibrations of $\text{cod}_{\mathcal{S}_{\mathbf{G}}} : \mathcal{S}_{\mathbf{G}}^{\rightarrow} \rightarrow \mathcal{S}_{\mathbf{G}}$ spanned by fiberwise \mathbf{U} -small ϕ -transparent and ϕ -sealed families of types respectively.

► **Notation 27.** Let E be a type in $\mathcal{S}_{\mathbf{G}}$ and fix elements $e : C \Rightarrow E$ and $e' : T \Rightarrow E$ of the computational and syntactical parts of E respectively; we will write $e \triangleleft_E e'$, pronounced “ e formally approximates e' ”, for the extension type $\{E \mid C \hookrightarrow e, T \hookrightarrow e'\}$.

This is the connection between synthetic Tait computability and analytic logical relations; the open parts of an object correspond to the *subjects* of a logical relation and the closed parts of an object correspond to the evidence of that relation.

► **Definition 28** (Formal approximation relations). *A type E is called a **formal approximation relation** when for any B -point $e : B \Rightarrow E$, the extension type $\{E \mid B \hookrightarrow e\}$ is a proposition, i.e. any two elements of $e \triangleleft_E e$ are equal.*

We will write $\text{Rel}_{\mathbf{U}} \subseteq \mathbf{U}$ for the subuniverse of formal approximation relations.

► **Definition 29** (Admissible formal approximation relations). *Let E be a formal approximation relation such that $C \Rightarrow E$ is a predomain equipped with an \mathbf{L} -algebra structure. We say that E is **admissible** at $x : T \Rightarrow E$ when the subobject $\{E \mid T \hookrightarrow x\} \subseteq C \Rightarrow E$ is admissible in the sense of synthetic domain theory, i.e. contains \perp and is closed under formal suprema of formal ω -chains. We say that E is admissible when it is admissible at every such x .*

⁵ In presheaves, the universes of Hofmann and Streicher [27, 63] satisfy this property directly; for sheaves, there is an alternative transfinite construction of universes enjoying this property [25]. Our presentation in terms of transparency and sealing is an equivalent reformulation of the strictness property identified by several authors in the context of the semantics of homotopy type theory [33, 64, 56, 11, 42, 8, 57, 5].

► **Lemma 30** (Scott induction). *Let X be a formal approximation relation such that $C \Rightarrow X$ is a domain. Let $f : X \rightarrow X$ be an endofunction on X and let $x : T \Rightarrow X$ be a syntactical fixed point of f in the sense that $T \Rightarrow (x = f x)$; if X is admissible at x , then we have $\text{fix } f \triangleleft_X x$.*

Our goal can be rephrased now in the internal language; choosing a universe $\mathbf{V} \supset \mathbf{U}$, we wish to define a suitable \mathbf{V} -valued algebra $\mathcal{A} \in \mathcal{T}\text{-Alg}_{\mathbf{V}}$ that restricts under B to \mathcal{A}^B , i.e. an element $\mathcal{A} \in \{\mathcal{T}\text{-Alg}_{\mathbf{V}} \mid B \hookrightarrow \mathcal{A}^B\}$. This can be done quite elegantly in the internal language of $\mathcal{S}_{\mathbf{G}}$, i.e. the *synthetic Tait computability of formal approximation structures*. The high-level structure of our model construction is summarized as follows:

We interpret value types as *formal approximation structures* over a syntactic value type and a predomain; we interpret computation types as *admissible formal approximation relations* between a syntactic computation type and a domain.

To make this precise, we will define $\mathcal{A}.\text{tp}^+ \in \{\mathbf{V} \mid B \hookrightarrow \mathcal{A}^B.\text{tp}^+\}$ as the collection of types that restrict to an element of $\mathcal{A}^T.\text{tp}^+$ in the syntactic phase and to an element of $\mathcal{A}^C.\text{tp}^+ = \mathbf{Predom}$ in the computational phase. This is achieved using strict gluing:

$$\mathcal{A}.\text{tp}^+ = (A : \mathcal{A}^B.\text{tp}^+) \ltimes_B \{\mathbf{U} \mid B \hookrightarrow \mathcal{A}^B.\text{tm } A\} \quad \mathcal{A}.\text{tm} = \text{unglue}_B$$

The above is well-defined because $\mathcal{A}^B.\text{tp}^+$ is B -transparent and $\{\mathbf{U} \mid B \hookrightarrow \mathcal{A}^B.\text{tm } A\}$ is B -sealed. We also have $T \Rightarrow \mathcal{A}.\text{tp}^+ = \mathcal{A}^T.\text{tp}^+$ and $C \Rightarrow \mathcal{A}.\text{tp}^+ = \mathbf{Predom}$. Next we define the formal approximation structure of computation types:

$$\mathcal{A}.\text{tp}^\ominus = (X : \mathcal{A}^B.\text{tp}^\ominus) \ltimes_B \{X' : \{\text{Rel}_{\mathbf{U}} \mid B \hookrightarrow \mathcal{A}^B.\text{tm } (\mathcal{A}^B.\text{U } X)\} \mid X' \text{ is admissible}\}$$

To see that the above is well-defined, we must check that the family component of the gluing is pointwise B -sealed, which follows because the property of being admissible is B -sealed. To see that this is the case, we observe that it is obviously T -sealed and also (less obviously) C -sealed: under C , X' restricts to the “total” predicate on X which is always admissible. To define the thunking connective, we simply forget that a given admissible approximation relation was admissible: $\mathcal{A}.\text{U } X = \text{glue} [\text{unglue}_B X \mid B \hookrightarrow \mathcal{A}^B.\text{U } X]$. To interpret free computation types, we proceed in two steps; first we define the formal approximation relation as an element of $\text{Rel}_{\mathbf{U}}$ and then we glue it onto syntax and semantics.

$$\begin{aligned} [F] A &= (u : \mathcal{A}^B.(UF) A) \ltimes_B (C \Rightarrow u \downarrow) \Rightarrow B \bullet \exists a : A.B \Rightarrow u = \mathcal{A}^B.\text{ret } a \\ \mathcal{A}.F A &= \text{glue} [[F] A \mid B \hookrightarrow \mathcal{A}^B.F] \end{aligned}$$

In simpler language, we have $u \triangleleft_{[F] A} v$ if and only if v terminates syntactically whenever u terminates such that the value of u formally approximates the value of v . This is the standard clause for lifting in an adequacy proof, phrased in synthetic Tait computability. ; the use of the sealing modality is an artifact of synthetic Tait computability, ensuring that the relation is pointwise B -sealed. The `ret`, `bind` operations are easily shown to preserve the formal approximation relations. The construction of formal approximation structures for product and function spaces is likewise trivial. Using Scott induction (Lemma 30) we can show that fixed points also lie in the formal approximation relations; we elide the details. Next we deal with the information flow constructs, starting by interpreting each security policy $\mathcal{A}.\langle l \rangle$ as $\mathcal{A}^B.\langle l \rangle$. The sealing modality is interpreted below:

$$\begin{aligned} [T_l] A &= (u : \mathcal{A}^B.T_l A) \ltimes_B B \bullet \mathcal{A}.\langle l \rangle \bullet \{a : A \mid B \Rightarrow u = \mathcal{A}^B.\text{seal}_l a\} \\ \mathcal{A}.T_l A &= \text{glue} [[T_l] A \mid B \hookrightarrow \mathcal{A}^B.T_l] \end{aligned}$$

► **Theorem 31** (Fundamental theorem of logical relations). *The preceding constructions arrange into an algebra $\mathcal{A} \in \{\mathcal{T}\text{-Alg}_{\mathbf{V}} \mid B \hookrightarrow \mathcal{A}^B\}$.*

6.2 Adequacy and syntactic noninterference results

The following definitions and results in this section are global rather than internal. We may immediately read off from the logical relation of Section 6.1 a few important properties relating value terms and their denotations. The results of this section depend heavily on the assumption that the functor $\mathcal{T} \hookrightarrow \mathcal{S}_{\mathbf{G}}/\mathbf{T}$ is fully faithful (Axiom STC-2).

► **Theorem 32** (Value adequacy). *For any closed values $u, v : \mathbf{1}_{\mathcal{T}} \rightarrow \text{bool}$, we have $\llbracket u \rrbracket = \llbracket v \rrbracket$ if and only if $u \equiv_{\text{bool}} v$; moreover we have either $u \equiv_{\text{bool}} \text{tt}$ or $u \equiv_{\text{bool}} \text{ff}$.*

Let $u : \mathbf{1}_{\mathcal{T}} \rightarrow \text{UFA}$ be a closed computation.

► **Definition 33** (Convergence and divergence). *We say that u **converges** when there exists $a : \mathbf{1}_{\mathcal{T}} \rightarrow A$ such that $u = \text{ret } a$. Conversely, we say that u **diverges** when there does not exist such an a . We will write $u \Downarrow$ to mean that u converges, and $u \Uparrow$ to mean that u diverges.*

► **Theorem 34** (Computational adequacy). *The computation u converges iff $\llbracket u \rrbracket \Downarrow = \top$.*

► **Theorem 35** (Termination-insensitive noninterference). *Let A be a syntactic type such that $\text{sealed}_{\langle l \rangle} A$ holds; fix a term $c : A \rightarrow \text{UF bool}$. Then for all $x, y : \mathbf{1}_{\mathcal{T}} \rightarrow A$ such that $c x \Downarrow$ and $c y \Downarrow$, we have $c x \equiv_{\text{UF bool}} c y$.*

We give an example of a program whose termination behavior hinges on a classified bit to demonstrate that our noninterference result is non-trivial.

► **Example 36.** There exists an $\langle l \rangle$ -sealed type A and a term $c : A \rightarrow \text{UF unit}$ such that for some $x, y : \mathbf{1}_{\mathcal{T}} \rightarrow A$ we have $c x \Downarrow$ and yet $c y \Uparrow$.

Proof. Choose $A := \top_l \text{ bool}$ and consider the following terms:

$$\begin{aligned} \top &:= \text{ret } () & \perp &:= \text{fix } (\lambda z. z) & x &:= \text{seal}_l \text{tt} & y &:= \text{seal}_l \text{ff} \\ c &:= \lambda u. \text{tdcl}_{\langle l \rangle} (\text{unseal}_l u (\lambda b. \text{seal}_l (\text{if } b \top \perp))) \end{aligned}$$

We then have $c x \equiv_{\text{UF unit}} \top$ and therefore $c x \Downarrow$. On the other hand, we have $c y \equiv_{\text{UF unit}} \text{tdcl}_{\langle l \rangle} (\text{seal}_l \perp)$; executing the denotational semantics, we have $\llbracket c y \rrbracket \Downarrow = \langle l \rangle$. From the full and faithfulness assumption of Axiom SDT-4, we know that $\langle l \rangle$ is not globally equal to \top ; hence we conclude from Theorem 34 that $c y \Uparrow$. ◀

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$$\begin{array}{lll}
\text{tp}^+, \text{tp}^\ominus : \mathbf{Sort} & \text{ret} : A \rightarrow \mathbf{UFA} & \text{fn} : \text{tp}^+ \rightarrow \text{tp}^\ominus \rightarrow \text{tp}^\ominus \\
\text{tm} : \text{tp}^+ \rightarrow \mathbf{Sort} & \text{bind} : \mathbf{UFA} \rightarrow (A \rightarrow \mathbf{UX}) \rightarrow \mathbf{UX} & \text{fn.tn} : (A \rightarrow \mathbf{UX}) \cong \mathbf{U}(\text{fn } A \text{ } X) \\
\mathbf{U} : \text{tp}^\ominus \rightarrow \text{tp}^+ & \text{bind}(\text{ret } u) f \equiv_{\mathbf{UX}} f u & \text{prod} : \text{tp}^+ \rightarrow \text{tp}^+ \rightarrow \text{tp}^+ \\
\mathbf{F} : \text{tp}^+ \rightarrow \text{tp}^\ominus & \text{bind } u \text{ ret} \equiv_{\mathbf{UFA}} u & \text{prod.tn} : A \times B \cong \text{prod } A \text{ } B \\
& \text{bind}(\text{bind } u f) g \equiv_{\mathbf{UX}} \text{bind } u (\lambda x. \text{bind}(f x) g) & \text{unit} : \text{tp}^+ \\
& & \text{unit.tn} : \mathbf{1} \cong \text{unit}
\end{array}$$

$$\begin{array}{ll}
\text{sum} : \text{tp}^+ \rightarrow \text{tp}^+ \rightarrow \text{tp}^+ & \langle l \rangle : \mathbf{Prop} \\
\text{inl} : A \rightarrow \text{sum } A \text{ } B & \langle k \rangle \rightarrow \langle l \rangle \quad (k \leq l \in \mathcal{P}) \\
\text{inr} : B \rightarrow \text{sum } A \text{ } B & \langle k \rangle \rightarrow \langle l \rangle \rightarrow \langle k \wedge l \rangle \\
\text{case} : \text{sum } A \text{ } B \rightarrow (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow C & \text{sealed}_{\langle l \rangle} : \text{tp}^+ \rightarrow \mathbf{Prop} \\
\text{case}(\text{inl } u) f g \equiv_C f u & \text{sealed}_{\langle l \rangle} A := \langle l \rangle \rightarrow \{x : A \mid \forall y : A. x \equiv_A y\} \\
\text{case}(\text{inr } v) f g \equiv_C g v & \text{tp}_{\bullet, \langle l \rangle}^+ := \{A : \text{tp}^+ \mid \text{sealed}_{\langle l \rangle} A\} \\
\text{case } u (\lambda x. f(\text{inl } x)) (\lambda x. f(\text{inr } x)) \equiv_C f u & \\
\\
\mathbf{T}_l : \text{tp}^+ \rightarrow \text{tp}_{\bullet, \langle l \rangle}^+ & \text{fix} : (\mathbf{UX} \rightarrow \mathbf{UX}) \rightarrow \mathbf{UX} \\
\text{seal}_l : A \rightarrow \mathbf{T}_l A & \text{fix } f \equiv f(\text{fix } f) \\
\\
\text{unseal}_l : \{B : \text{tp}_{\bullet, \langle l \rangle}^+\} \rightarrow \mathbf{T}_l A \rightarrow (A \rightarrow B) \rightarrow B & \text{tdcl}_{\langle l \rangle} : \{A : \text{tp}_{\bullet, \langle l \rangle}^+\} \rightarrow \mathbf{T}_l \mathbf{UFA} \rightarrow \mathbf{UFA} \\
\text{unseal}_l(\text{seal}_l u) f \equiv_B f u & \text{tdcl}_{\langle l \rangle}(\text{seal}_l(\text{ret } u)) \equiv_{\mathbf{UFA}} \text{ret } u \\
\text{unseal}_l u (\lambda x. f(\text{seal}_l x)) \equiv_B f u &
\end{array}$$

■ **Figure 2** The LF presentation of the theory \mathcal{T} .

How to read the technical appendix

We divide our technical appendix into two parts. Section A provides proofs of the main results of this paper that depend only on the axiomatics we have imposed for synthetic domain theory and synthetic Tait computability. To verify that these axioms are substantiated, we construct concrete models of synthetic domain theory and synthetic Tait computability in Section B.

A Synthetic results

A.1 Denotational semantics in synthetic domain theory

In this section we assume all the axioms of synthetic domain theory.

► **Lemma 37.** *The predomains form a **reflective exponential ideal** of \mathcal{S}_C .*

Proof. In a locally presentable category, such as any Grothendieck topos, any orthogonal subcategory is reflective. ◀

A.1.1 Denotational noninterference

► **Lemma 21.** *Let A be a $\langle l \rangle$ -sealed predomain and let B be a $\langle l \rangle$ -transparent predomain; then (1) any function $A \rightarrow B$ is weakly constant, and (2) any partial function $A \rightarrow \mathbf{LB}$ is partially constant.*

Proof. Fix $\langle l \rangle$ -sealed A and $\langle l \rangle$ -transparent B . For (1), we fix a function $u : A \rightarrow B$ to check that for any $x, y : A$ we have $u x = u y$. Because B is $\langle l \rangle$ -transparent, we may assume $\langle l \rangle = \top$; but then $x = y$ and hence $u x = u y$. For (2), we fix a partial function $u : A \rightarrow B_\perp$ along with $x, y : A$ such that u is defined on both x and y ; we must check that $u x = u y : B$. Because B is $\langle l \rangle$ -transparent, we may again assume $\langle l \rangle = \top$; but A is $\langle l \rangle$ -sealed and hence under this assumption, we have $x = y$ and therefore $u x = u y$. ◀

► **Lemma 22.** *The predomain $\llbracket \text{bool} \rrbracket$ is $\langle l \rangle$ -transparent.*

Proof. $\llbracket \text{bool} \rrbracket$ is the binary coproduct $\mathbf{2}_{\text{Predom}_{\mathbf{C}}}$ of singleton predomains, which just the constant sheaf $\mathbf{2}_{\mathcal{S}_{\mathbf{C}}} \cong \mathbf{C}^* \{0, 1\}$; this is transparent by **Axiom** SDT-5. ◀

A.2 Adequacy via synthetic Tait computability

► **Lemma 30** (Scott induction). *Let X be a formal approximation relation such that $\mathbf{C} \Rightarrow X$ is a domain. Let $f : X \rightarrow X$ be an endofunction on X and let $x : \mathbf{T} \Rightarrow X$ be a syntactical fixed point of f in the sense that $\mathbf{T} \Rightarrow (x = f x)$; if X is admissible at x , then we have $\text{fix } f \triangleleft_X x$.*

Proof. We employ the fact that admissible approximation relations are closed under bottom and formal suprema of ω -chains. ◀

A.2.1 Universe of value types

$$\begin{aligned} \mathcal{A}.\text{tp}^+ &: \{ \mathbf{V} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\text{tp}^+ \} \\ \mathcal{A}.\text{tp}^+ &= (A : \mathcal{A}^{\mathbf{B}}.\text{tp}^+) \ltimes_{\mathbf{B}} \{ \mathbf{U} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\text{tm } A \} \\ \mathcal{A}.\text{tm} &: \{ \mathcal{A}.\text{tp}^+ \rightarrow \mathbf{V} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\text{tm} \} \\ \mathcal{A}.\text{tm} &= \text{unglue}_{\mathbf{B}} \end{aligned}$$

A.2.2 Universe of computation types

$$\begin{aligned} \mathcal{A}.\text{tp}^{\ominus} &: \{ \mathbf{V} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\text{tp}^{\ominus} \} \\ \mathcal{A}.\text{tp}^{\ominus} &= (X : \mathcal{A}^{\mathbf{B}}.\text{tp}^{\ominus}) \ltimes_{\mathbf{B}} \{ X' : \{ \text{Rel}_{\mathbf{U}} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\text{tm } (\mathcal{A}^{\mathbf{B}}.\mathbf{U} X) \} \mid X' \text{ is admissible} \} \\ \mathcal{A}.\mathbf{U} &: \{ \mathcal{A}.\text{tp}^{\ominus} \rightarrow \mathcal{A}.\text{tp}^+ \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\mathbf{U} \} \\ \mathcal{A}.\mathbf{U} X &= \text{glue} [\text{unglue}_{\mathbf{B}} X \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\mathbf{U} X] \end{aligned}$$

A.2.3 Free computation types

First we define the underlying formal approximation relation.

$$\begin{aligned} [\mathbf{F}] &: \{ \mathcal{A}.\text{tp}^+ \rightarrow \text{Rel}_{\mathbf{U}} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.(\text{tm} \circ \mathbf{U} \circ \mathbf{F}) \} \\ [\mathbf{F}] A &= (u : \mathcal{A}^{\mathbf{B}}.(\mathbf{U} \mathbf{F}) A) \ltimes_{\mathbf{B}} (\mathbf{C} \Rightarrow u \downarrow) \Rightarrow \mathbf{B} \bullet \exists a : A.\mathbf{B} \Rightarrow (u = \mathcal{A}^{\mathbf{B}}.\text{ret } a) \end{aligned}$$

► **Computation 38.** It is useful to characterize the relation in more familiar notation:

$$u \triangleleft_{[\mathbf{F}]} A x \iff u \downarrow \Rightarrow \mathbf{B} \bullet \exists u', x'. \mathbf{C} \Rightarrow (u = \eta_A u') \wedge \mathbf{T} \Rightarrow (x = \mathcal{A}^{\mathbf{T}}.\text{ret } x') \wedge u' \triangleleft_A x'$$

► **Lemma 39.** *Each $[\mathbf{F}] A$ is admissible.*

Proof. We fix a syntactic point $x : \mathcal{A}^{\mathbf{T}}.\mathbf{U} \mathbf{F} A$ to check that each $\{[\mathbf{F}] A \mid \mathbf{T} \hookrightarrow x\} \subseteq \mathbf{L} A$ is an admissible subdomain.

1. It is immediate that $\perp_{\mathbf{L} A} \triangleleft_{[\mathbf{F}]} A x$, since $\perp_{\mathbf{L} A} \downarrow = \perp$.
2. Next we fix an ω -chain of formal approximations $u_i \triangleleft_{[\mathbf{F}]} A x$ to check that $\bigsqcup_i u_i \triangleleft_{[\mathbf{F}]} A x$. Because $\bigsqcup_i u_i \downarrow = \bigsqcup_i u_i \downarrow$, we assume that $u_i \downarrow$ for some i to check that away from \mathbf{B} there exists an element $a : A$ such that x restricts under \mathbf{B} to $\mathcal{A}^{\mathbf{T}}.\text{ret } a$ and under \mathbf{C} we have $\bigsqcup_i u_i = \eta_A a$. Because $u_i \downarrow$ and therefore the value of u_i is the same as the value of $\bigsqcup_i u_i$, we are done. ◀

By Lemma 39 we are justified in specifying the code for the free computation type:

$$\begin{aligned} \mathcal{A}.\mathbf{F} &: \{ \mathcal{A}.\text{tp}^+ \rightarrow \mathcal{A}.\text{tp}^{\ominus} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\mathbf{F} \} \\ \mathcal{A}.\mathbf{F} A &= \text{glue} [[\mathbf{F}] A \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}.\mathbf{F}] \end{aligned}$$

► **Lemma 40.** *If $u \triangleleft_A u'$ then $\eta_A u \triangleleft_{[A] A} \mathcal{A}^T.\text{ret } u'$*

Proof. Immediate. ◀

► **Lemma 41.** *If $u \triangleleft_{[F] A} u'$ and for all $a \triangleleft_A a'$ we have $ha \triangleleft_{A.UX} h'a'$ then we have $h^\sharp u \triangleleft_{A.UX} \mathcal{A}^T.\text{bind } u' h'$.*

Proof. Assume $h^\sharp u \downarrow$, i.e. there exists a such that $u = \eta_A a$ and $ha \downarrow$. Therefore there exists a' with $u' = \mathcal{A}^T.\text{ret } a'$ and $a \triangleleft_A a'$. In this case $h^\sharp u = ha$ and $\mathcal{A}^T.\text{bind } u' h' = h'a'$, and we have $ha \triangleleft_{A.UX} h'a'$ by assumption. ◀

A.2.4 Function types

First we define the underlying formal approximation relation.

$$[\text{fn}] A X = (u : \mathcal{A}^B.U (\mathcal{A}^B.\text{fn } A X)) \ltimes_B \{A \rightarrow \mathcal{A}.UX \mid B \hookrightarrow \mathcal{A}^B.\text{fn}.\text{tm}^{-1} u\}$$

That the above is admissible follows immediately from the definition of formal suprema in function spaces.

$$\begin{aligned} \mathcal{A}.\text{fn} &: \{\mathcal{A}.\text{tp}^+ \rightarrow \mathcal{A}.\text{tp}^\ominus \rightarrow \mathcal{A}.\text{tp}^\ominus \mid B \hookrightarrow \mathcal{A}^B.\text{fn}\} \\ \mathcal{A}.\text{fn } A X &= \text{glue } [[\text{fn}] A X \mid B \hookrightarrow \mathcal{A}^B.\text{fn}] \end{aligned}$$

The canonical isomorphism $\mathcal{A}.\text{fn}.\text{tm}$ is immediate:

$$\begin{aligned} \mathcal{A}.\text{fn}.\text{tm} &: \{(A \rightarrow \mathcal{A}.UX) \cong \mathcal{A}.U (\mathcal{A}.\text{fn } A X) \mid B \hookrightarrow \mathcal{A}^B.\text{fn}.\text{tm}\} \\ \mathcal{A}.\text{fn}.\text{tm } u &= \text{glue } [u \mid B \hookrightarrow \mathcal{A}^B.\text{fn}.\text{tm } u] \end{aligned}$$

A.2.5 Product types

First we define the underlying formal approximation structure.

$$[\text{prod}] A B = (u : \mathcal{A}^B.\text{prod } A B) \ltimes_B \{A \times B \mid B \hookrightarrow \mathcal{A}^B.\text{prod}.\text{tm}^{-1} u\}$$

The rest is defined as follows:

$$\begin{aligned} \mathcal{A}.\text{prod} &: \{\mathcal{A}.\text{tp}^+ \rightarrow \mathcal{A}.\text{tp}^+ \rightarrow \mathcal{A}.\text{tp}^+ \mid B \hookrightarrow \mathcal{A}^B.\text{prod}\} \\ \mathcal{A}.\text{prod } A B &= \text{glue } [[\text{prod}] A B \mid B \hookrightarrow \mathcal{A}^B.\text{prod}] \\ \mathcal{A}.\text{prod}.\text{tm} &: \{(A \times B) \cong \mathcal{A}.\text{prod } A B \mid B \hookrightarrow \mathcal{A}^B.\text{prod}.\text{tm}\} \\ \mathcal{A}.\text{prod}.\text{tm } u &= \text{glue } [u \mid B \hookrightarrow \mathcal{A}^B.\text{prod}.\text{tm } u] \end{aligned}$$

A.2.6 Sum types

First we define the underlying formal approximation structure.

$$[\text{sum}] A B = (u : \mathcal{A}^B.\text{sum } A B) \ltimes_{B \bullet} (\{x : A \mid B \Rightarrow u = \mathcal{A}^B.\text{inl } x\} + \{x : B \mid B \Rightarrow u = \mathcal{A}^B.\text{inr } x\})$$

The type code and its constructors are defined as follows:

$$\begin{aligned} \mathcal{A}.\text{sum} &: \{\mathcal{A}.\text{tp}^+ \rightarrow \mathcal{A}.\text{tp}^+ \rightarrow \mathcal{A}.\text{tp}^+ \mid B \hookrightarrow \mathcal{A}^B.\text{sum}\} \\ \mathcal{A}.\text{sum } A B &= \text{glue } [[\text{sum}] A B \mid B \hookrightarrow \mathcal{A}^B.\text{sum}] \\ \mathcal{A}.\text{inl} &: \{A \rightarrow \mathcal{A}.\text{sum } A B \mid B \hookrightarrow \mathcal{A}^B.\text{inl}\} \\ \mathcal{A}.\text{inl } a &= \text{glue } [\text{inl } a \mid B \hookrightarrow \mathcal{A}^B.\text{inl } a] \\ \mathcal{A}.\text{inr} &: \{B \rightarrow \mathcal{A}.\text{sum } A B \mid B \hookrightarrow \mathcal{A}^B.\text{inr}\} \\ \mathcal{A}.\text{inr } b &= \text{glue } [\text{inr } b \mid B \hookrightarrow \mathcal{A}^B.\text{inr } b] \end{aligned}$$

The case statement is defined as follows:

$$\begin{aligned} \mathcal{A}.\text{case} &: \{ \mathcal{A}.\text{sum } A B \rightarrow (A \rightarrow C) \rightarrow (B \rightarrow C) \rightarrow C \mid B \hookrightarrow \mathcal{A}^B.\text{case} \} \\ \mathcal{A}.\text{case } u g h &= \begin{cases} B \hookrightarrow \mathcal{A}^B.\text{case } u g h \\ \text{unglue}_B u = \eta_{\bullet B} \text{inl } x \hookrightarrow g x \\ \text{unglue}_B u = \eta_{\bullet B} \text{inr } x \hookrightarrow h x \end{cases} \end{aligned}$$

The well-definedness of the case split above follows from the computation rules for $\mathcal{A}^B.\text{case}$.

A.2.7 The sealing modality

We interpret each $\mathcal{A}.\langle l \rangle$ as $\mathcal{A}^B.\langle l \rangle = \mathcal{A}^C.\langle l \rangle \wedge \mathcal{A}^T.\langle l \rangle$.

$$\begin{aligned} [\mathsf{T}_l] &: \{ \mathcal{A}.\text{tp}^+ \rightarrow \mathsf{U} \mid B \hookrightarrow \mathcal{A}^B.(\text{tm} \circ \mathsf{T}_l) \} \\ [\mathsf{T}_l] A &= (u : \mathcal{A}^B.\mathsf{T}_l A) \ltimes_B (B \vee \mathcal{A}.\langle l \rangle) \bullet \{ a : A \mid B \Rightarrow (u = \mathcal{A}^B.\text{seal}_l a) \} \\ \mathcal{A}.\mathsf{T}_l &: \{ \mathcal{A}.\text{tp}^+ \rightarrow \mathcal{A}.\text{tp}^+ \mid B \hookrightarrow \mathcal{A}^B.\mathsf{T}_l \} \\ \mathcal{A}.\mathsf{T}_l A &= \text{glue } [[\mathsf{T}_l] A \mid B \hookrightarrow \mathcal{A}^B.\mathsf{T}_l] \end{aligned}$$

By definition, $\mathcal{A}.\mathsf{T}_l$ is $\mathcal{A}.\langle l \rangle$ -sealed; we define its introduction and elimination form next.

$$\begin{aligned} \mathcal{A}.\text{seal}_l &: \{ A \rightarrow \mathcal{A}.\mathsf{T}_l A \mid B \hookrightarrow \mathcal{A}^B.\text{seal}_l \} \\ \mathcal{A}.\text{seal}_l a &= \text{glue } [\eta_{\bullet(B \vee \mathcal{A}.\langle l \rangle)} a \mid B \hookrightarrow \mathcal{A}^B.\text{seal}_l a] \\ \mathcal{A}.\text{unseal}_l &: \{ \{ B : \mathcal{A}.\text{tp}_{\bullet \langle l \rangle} \} \rightarrow \mathcal{A}.\mathsf{T}_l A \rightarrow (A \rightarrow B) \rightarrow B \mid B \hookrightarrow \mathcal{A}^B.\text{unseal}_l \} \\ \mathcal{A}.\text{unseal}_l \{ B \} u h &= \begin{cases} B \hookrightarrow \mathcal{A}^B.\text{unseal}_l \{ B \} u h \\ \mathcal{A}.\langle l \rangle \hookrightarrow \star \\ \text{unglue}_B u = \eta_{\bullet(B \vee \mathcal{A}.\langle l \rangle)} x \hookrightarrow h x \end{cases} \end{aligned}$$

A.2.8 Fixed points and declassification

► **Lemma 42.** *If for all $u \triangleleft_{\mathcal{A}.\text{UX}} u'$ we have $hu \triangleleft_{\mathcal{A}.\text{UX}} h'u'$, then $\text{fix } h \triangleleft_{\mathcal{A}.\text{UX}} \mathcal{A}^T.\text{fix } h'$.*

Proof. By Scott induction (Lemma 30). ◀

► **Lemma 43.** *For any $\mathcal{A}.\langle l \rangle$ -sealed value type A , if $u \triangleleft_{\mathcal{A}.\mathsf{T}_l \text{UF} A} u'$ then $\mathcal{A}^C.\text{tdcl}_{\langle l \rangle} u \triangleleft_{\mathcal{A}.\text{UF} A} \mathcal{A}^T.\text{tdcl}_{\langle l \rangle} u'$.*

Proof. We consider the proof of $u \triangleleft_{\mathcal{A}.\mathsf{T}_l \circ \text{U} \circ \text{F}} A u'$ and proceed by cases:

1. Suppose we have $\mathcal{A}.\langle l \rangle = \top$; this case is trivial, because both sides terminate with the unique element of A under $\mathcal{A}.\langle l \rangle$.
2. Suppose we have some $v \triangleleft_{[\mathcal{A}.\text{F}] A} v'$ such that $u = \eta_{\bullet \mathcal{A}^C.\langle l \rangle} v$ and $u' = \mathcal{A}^T.\text{seal}_l v'$. In this case, $\mathcal{A}^C.\text{tdcl}_{\langle l \rangle} u = \mathcal{A}^C.\langle l \rangle \parallel v$. To check $(\mathcal{A}^C.\langle l \rangle \parallel v) \triangleleft_{[\mathcal{A}.\text{F}] A} \mathcal{A}^T.\text{tdcl}_{\langle l \rangle} u$, we may assume that $(\mathcal{A}^C.\langle l \rangle \parallel v) \downarrow$ which is the Σ -join $\mathcal{A}^C.\langle l \rangle \vee v \downarrow$; we proceed by cases:
 - a. Suppose that $\mathcal{A}^C.\langle l \rangle = \top$. Because we are proving a B-sealed proposition, **Axiom STC-4** allows us to assume that $\mathcal{A}^T.\langle l \rangle = \top$. Under this assumption we have $\mathcal{A}.\langle l \rangle = \top$, a case that we have already discharged above.
 - b. If $v \downarrow$, then by assumption we have a syntactic point $w' : \top \Rightarrow A$ such that $v \triangleleft_A w'$ and $v' = \mathcal{A}^T.\text{ret } w'$. Therefore $\mathcal{A}^T.\text{tdcl}_{\langle l \rangle} u' = \mathcal{A}^T.(\text{tdcl}_{\langle l \rangle} \circ \text{seal}_l \circ \text{ret}) w' = \mathcal{A}^T.\text{ret } w'$; on the other side, we have $\mathcal{A}^C.\text{tdcl}_{\langle l \rangle} u = \mathcal{A}^C.\text{tdcl}_{\langle l \rangle} \eta_{\bullet \langle l \rangle} v = v$ by Lemma 18 using our assumption that $v \downarrow$. Therefore our goal follows from our assumption $v \triangleleft_A w'$. ◀

A.2.9 Fundamental Theorem

► **Theorem 31** (Fundamental theorem of logical relations). *The preceding constructions arrange into an algebra $\mathcal{A} \in \{\mathcal{T}\text{-}\mathbf{Alg}_{\mathbf{V}} \mid \mathbf{B} \hookrightarrow \mathcal{A}^{\mathbf{B}}\}$.*

Proof. We have provided components for all the generating constants of \mathcal{T} , verifying that all the formal approximation relations assigned to computation type connectives are admissible. ◀

B Analytic results: verifying the axioms

B.1 Model of synthetic domain theory

In this section we give an explicit construction of the SDT topos **C**. Our plan is as follows:

1. We first develop a concrete version of domain theory incorporating information flow logic based on \mathcal{P} -indexed dcpos (Section B.1.1).
2. Finally in Sections B.1.2 and B.1.4 we adapt the methods of Fiore and Plotkin [17] to obtain a sheaf topos model of synthetic domain theory satisfying all the axioms we have imposed.

B.1.1 Dcpo over the information flow topos

We begin by developing a theory of domains that is *stratified* over \mathcal{P} ; by this we mean domain theory internal to presheaf category $\mathcal{S}\hat{\mathcal{P}} = [\mathcal{P}^{\text{op}}, \mathbf{Set}]$. Such a stratified domain theory is a specialization of the work of De Jong and Escardó [12] on constructive domain theory in homotopy type theory (noting that *op. cit.* made no use of homotopical axioms that do not hold for a 1-topos). We recall the basic definitions here, but do not belabor them. Everything in this section should be read *internally* to $\mathcal{S}\hat{\mathcal{P}}$; standard techniques can be used to translate such statements to ordinary mathematics, as explained by Mac Lane and Moerdijk [39] as well as Awodey *et al.* [6].

► **Definition 44.** A **directed-complete partial order** or **dcpo** is a partial order A that is closed under suprema of directed subsets $S \subseteq A$. A continuous function of dcpos is one that preserves these directed suprema.

► **Definition 45.** A **pointed dcpo** or **dcppo** is a dcpo that possesses a bottom element, i.e. an element \perp such that $\perp \leq u$ for all $u : A$. A strict morphism between dcpo's is a continuous morphism between the underlying dcpos that preserves the bottom element.

► **Lemma 46.** The forgetful functor from the category of dcpo's and strict maps to the category of dcpos and continuous maps has a left adjoint \mathbf{L} called the “lift”. The underlying set of $\mathbf{L}A$ is given by the **partial element classifier** in $\mathcal{S}\hat{\mathcal{P}}$, i.e. we have $|\mathbf{L}A| \cong (\phi : \Omega) \times \phi \Rightarrow |A|$.

► **Notation 47.** Given an element $u : \mathbf{L}A$, we will write $u \downarrow : \Omega$ for the **support** of u . Because the unit map $\eta_A : A \rightarrow \mathbf{L}A$ is a monomorphism, our notation will silently identify elements $u : A$ with their images $(\top, \lambda_{\perp}.u) : \mathbf{L}A$. We will write $\perp : \mathbf{L}A$ for the element $(\perp, \lambda_{\perp}.\perp)$;

We specify that $u \leq_{\mathbf{L}A} v$ if and only if whenever $u = \eta_A u'$, there exists v' such that $v = \eta_A v'$ and $u' \leq_A v'$.

► **Notation 48.** Given a morphism $f : A \rightarrow X$ of dcpos where X is a dcpo, we will write $f^\# : \mathbf{L}A \rightarrow X$ for the unique extension of f as a strict map between dcpo's.

We have a *Sierpiński space* $\Sigma = \mathbf{L1}$ whose carrier is the subobject classifier Ω itself. It is not difficult to see that the projection map $(-\downarrow) : \mathbf{L}A \rightarrow \Sigma$ is both continuous and strict. In addition to the usual suspects of domain theory, we also have *new* objects that come from the stratification over \mathcal{P} . In particular, for every $l \in \mathcal{P}$ we have a new element $y_{\mathcal{P}}l : \Sigma$. Every proposition $\phi : \Sigma$ can be thought of (somewhat degenerately) as a dcpo; there is exactly one partial order on ϕ , and as soon as we have a directed subset of ϕ we also have $\phi = \top$, since directed subsets are stipulated to be non-empty.

B.1.2 Algebraic compactness results

We have a fibered category of dcpos over $\mathcal{S}_{\mathcal{P}}$; restricting to the full internal subcategory of $\mathcal{S}_{\mathcal{P}}$ spanned by some universe $\mathbf{U} \in \mathcal{S}_{\mathcal{P}}$ in the sense of Hofmann and Streicher [27, 63] and considering the fiber over the terminal object $\mathbf{1}_{\mathcal{S}_{\mathcal{P}}}$, we obtain a small category $\mathbf{dcpo}_{\mathcal{P}}$ of small(er) \mathcal{P} -indexed dcpos and dcppos.

The (internal) lifting monad restricts to an ordinary lifting monad $\mathbf{L} : \mathbf{dcpo}_{\mathcal{P}} \rightarrow \mathbf{dcpo}_{\mathcal{P}}$. We will write $\mathbf{dcpo}_{\mathcal{P}}$ for the Eilenberg–Moore category of \mathbf{L} and $\mathbf{pdcpo}_{\mathcal{P}}$ for the Kleisli category of \mathbf{L} . Here is our glossary:

1. The category $\mathbf{dcpo}_{\mathcal{P}}$ consists of “predomains” A, B and continuous maps $A \rightarrow B$.
2. The category $\mathbf{dcpo}_{\mathcal{P}}$ consists of “domains” X, Y and *strict* continuous maps $A \multimap B$.
3. The category $\mathbf{pdcpo}_{\mathcal{P}}$ consists of predomains A, B and *partial maps* $A \multimap B$; equivalently, strict continuous maps $\mathbf{L}A \multimap \mathbf{L}B$.

► **Remark 49.** Each of $\mathbf{dcpo}_{\mathcal{P}}, \mathbf{dcpo}_{\mathcal{P}}, \mathbf{pdcpo}_{\mathcal{P}}$ is $\mathbf{dcpo}_{\mathcal{P}}$ -enriched in a canonical way.

In classical domain theory every \mathbf{L} -algebra/dcpo/domain is free so we may identify $\mathbf{dcpo}_{\mathcal{P}}$ with $\mathbf{pdcpo}_{\mathcal{P}}$; this does *not* hold in the domain theory indexed over \mathcal{P} . The reason is that lifting in this setting does much more than adding a single point; for instance, $\Sigma = \mathbf{L1}$ contains strictly more distinct global points than there are elements of \mathcal{P} .

B.1.2.1 Preliminaries on embedding-projection pairs

► **Definition 50.** In an poset-enriched category \mathcal{E} , we define an *embedding* $U \hookrightarrow A$ to be a monomorphism $\epsilon : U \rightarrow A$ that has a right adjoint, called its *projection* $\pi : A \multimap U$. We refer to the pair $\epsilon \dashv \pi$ as an *embedding-projection pair* or *ep-pair*.

► **Observation 51.** Note that any (ϵ, π) is an embedding-projection pair if and only if $\pi \circ \epsilon = \text{id}_U$ and $\epsilon \circ \pi \leq \text{id}_A$.

► **Definition 52.** In a poset-enriched category \mathcal{E} , an *e-initial* object is defined to be an initial object $\emptyset \in \mathcal{E}$ such that every morphism $\emptyset \rightarrow E$ is an embedding. Dually a *p-terminal* object is defined to be a terminal object $\mathbf{1} \in \mathcal{E}$ such that every morphism $E \rightarrow \mathbf{1}$ is a projection. An object $\mathbf{0} \in \mathcal{E}$ is called an *ep-zero object* when it is both e-initial and p-terminal.

► **Notation 53.** For a poset-enriched category \mathcal{E} , we will write $\mathcal{E}^e \subseteq \mathcal{E}$ for the wide subcategory spanned by embeddings and $\mathcal{E}^p \subseteq \mathcal{E}$ for the wide subcategory spanned by projections.

It is not difficult to verify that $(\mathcal{E}^e)^{\text{op}} = \mathcal{E}^p$ and vice versa [58].

B.1.2.2 The inductive fixed point object in $\mathbf{dcpo}_{\mathbf{P}}$

In some cases, a category \mathcal{C} equipped with a dominance Σ and its lifting monad will give rise to a model of intuitionistic linear–non-linear logic, *i.e.* a monoidal adjunction between a monoidal closed category and a cartesian closed category. In this case, the latter is \mathcal{C} and the corresponding monoidal closed category is the Eilenberg–Moore category $\mathcal{C}^{\mathbf{L}}$. Fiore and Plotkin [17] show that when these data are equipped with an *inductive fixed point object*, \mathcal{C} can be (fully faithfully) extended to a model of synthetic domain theory.

The closed structure of $\mathcal{C}^{\mathbf{L}}$ is not in question Kock [?], but it remains unclear whether the corresponding monoidal structure (tensor products) can be exhibited in a constructive setting:⁶ tensor products *do* exist, for instance, when \mathbf{L} preserves reflexive coequalizers, but we do not expect this to be the case for our model. Therefore, we will use the ideas of Fiore and Plotkin [17] but not their exact results.

► **Observation 54.** *The category $\mathbf{dcpo}_{\mathbf{P}}$ is cartesian closed and has an initial object. The universal family of our dominance is the open inclusion $\top : \mathbf{1}_{\mathbf{dcpo}_{\mathbf{P}}} \multimap \Sigma$.*

► **Lemma 55.** *The final coalgebra $\bar{\omega}$ for the lift monad in $\mathbf{dcpo}_{\mathbf{P}}$ is an *inductive fixed point object*, *i.e.* it is the colimit of the following chain:*

$$\emptyset \xrightarrow{!} \mathbf{L}\emptyset \xrightarrow{\mathbf{L}!} \mathbf{L}^2\emptyset \xrightarrow{\mathbf{L}^2!} \dots$$

Proof. The final coalgebra $\bar{\omega} \cong \mathbf{L}\bar{\omega}$ in $\mathbf{dcpo}_{\mathbf{P}}$ can be computed as the limit of the following ω^{op} -indexed diagram in the category $\mathbf{dcpo}_{\mathbf{P}}^{\mathbf{P}}$ of projections [58]:

$$\mathbf{1} \xleftarrow{\pi_1 = !} \mathbf{L}\mathbf{1} \xleftarrow{\pi_2 = \mathbf{L}\pi_1} \mathbf{L}^2\mathbf{1} \xleftarrow{\pi_3 = \mathbf{L}\pi_2} \dots$$

The embeddings $\epsilon_i \dashv \pi_i$ corresponding to each of the projections above can be computed recursively, defining $\epsilon_1 = \perp$ and $\epsilon_{n+1} = \mathbf{L}\epsilon_n$. By the limit-colimit coincidence for diagrams of embedding-projection pairs in $\mathbf{dcpo}_{\mathbf{P}}$, verified in a topos-valid way by De Jong and Escardó [12], we see that $\bar{\omega}$ is *also* the colimit in $\mathbf{dcpo}_{\mathbf{P}}^{\mathbf{P}}$ of the following diagram of embeddings:

$$\mathbf{1} \xrightarrow{\epsilon_1 = \perp} \mathbf{L}\mathbf{1} \xrightarrow{\epsilon_2 = \mathbf{L}\epsilon_1} \mathbf{L}^2\mathbf{1} \xrightarrow{\epsilon_3 = \mathbf{L}\epsilon_2} \dots$$

Because $\mathbf{1} = \mathbf{L}\emptyset$, we may rewrite our diagram like so:

$$\mathbf{L}\emptyset \xrightarrow{\epsilon_1 = \perp} \mathbf{L}^2\emptyset \xrightarrow{\epsilon_2 = \mathbf{L}\epsilon_1} \mathbf{L}^3\emptyset \xrightarrow{\epsilon_3 = \mathbf{L}\epsilon_2} \dots$$

The above is not only colimiting in $\mathbf{dcpo}_{\mathbf{P}}^{\mathbf{P}}$ but also in $\mathbf{dcpo}_{\mathbf{P}}$ [58]. Moreover it clearly remains colimiting when a further map $\emptyset \multimap \mathbf{L}\emptyset$ is adjoined to the left, so we are done. ◀

B.1.2.3 Algebraic compactness

The following result is easily adapted from Fiore’s dissertation [15].

► **Fact 56.** *Let \mathcal{E} be $\mathbf{dcpo}_{\mathbf{P}}$ -enriched; if \mathcal{E} has an ep-zero object and \mathcal{E}^e is closed under colimits of ω -chains, then \mathcal{E} is $\mathbf{dcpo}_{\mathbf{P}}$ -algebraically compact.*

⁶ In a previous version of this appendix, we mistakenly claimed that the existence of tensor products was obvious.

By $\mathbf{dcpo}_{\mathbf{P}}$ -algebraic compactness, we mean that every $\mathbf{dcpo}_{\mathbf{P}}$ -enriched endofunctor $F : \mathcal{E} \rightarrow \mathcal{E}$ has a *free algebra*, *i.e.* an object that simultaneously carries F 's initial algebra and final coalgebra.

We leave to the reader the following easily verified fact.

► **Fact 57.** *The Kleisli category $\mathbf{pdcpo}_{\mathbf{P}} = (\mathbf{dcpo}_{\mathbf{P}})_{\mathbf{L}}$ has an ep-zero object.*

► **Lemma 58.** *The category of partial embeddings $\mathbf{pdcpo}_{\mathbf{P}}^{\mathbf{e}}$ is closed under colimits of ω -chains.*

Proof. By Theorem 5.3.14 of Fiore's dissertation [15] and the fact that the lift monad is $\mathbf{dcpo}_{\mathbf{P}}$ -enriched and therefore preserves ep-chains, it suffices to recall that $\mathbf{dcpo}_{\mathbf{P}}^{\mathbf{e}}$ is closed under colimits of ω -chains [12]. ◀

► **Remark 59.** For intuition, Sterling [60] has also verified an explicit (Grothendieck) topos-valid computation of the colimit of an arbitrary diagram $A_{\bullet} : \mathcal{J} \rightarrow \mathbf{pdcpo}_{\mathbf{P}}^{\mathbf{e}}$ where \mathcal{J} is a small filtered poset, generalizing the claim of Jones and Plotkin [31, 32]:

$$A_{\infty} := \left\{ \sigma : \prod_{i \in \mathcal{J}} \mathbf{L}A_i \mid (\exists i \in \mathcal{J}. \sigma_i \downarrow) \wedge \forall i \leq j \in \mathcal{J}. \pi_{i \leq j} \sigma_j = \sigma_i \right\}$$

► **Corollary 60.** *The category of dcpos and partial maps $\mathbf{pdcpo}_{\mathbf{P}}$ is $\mathbf{dcpo}_{\mathbf{P}}$ -algebraically compact.*

B.1.3 Dominances and orthogonality

Let \mathcal{E} be a locally cartesian closed category equipped with a dominance Σ . Let \mathcal{M} be a class of monomorphisms in \mathcal{E} ; we define \mathcal{M}_{\times} be the smallest class of monomorphisms in \mathcal{E} stable under products and containing \mathcal{M} , and we define \mathcal{M}_{Σ} be the smallest class of monomorphisms in \mathcal{E} stable under pullback along Σ -monomorphisms and containing \mathcal{M}_{Σ} .

► **Notation 61.** We will write $m \perp f$ to mean that f is externally orthogonal to m ; we will write $\mathcal{M} \perp f$ to mean that f is that $m \perp f$ for every $m \in \mathcal{M}$.

► **Fact 62.** *An object $E \in \mathcal{E}$ is internally orthogonal to \mathcal{M} if and only if $\mathcal{M}_{\times} \perp E$.*

► **Lemma 63.** *If $\mathcal{M}_{\times} \perp \Sigma$ and $\mathcal{M}_{\Sigma} \perp E$, then $\mathcal{M}_{\times} \perp \mathbf{L}E$.*

Proof. Fix $I \twoheadrightarrow J \in \mathcal{M}_{\times}$ and a lifting problem of the following form:

$$\begin{array}{ccc} I & \xrightarrow{a} & \mathbf{L}E \\ \downarrow & \nearrow \gamma & \\ J & & \end{array}$$

The map $a : I \rightarrow \mathbf{L}E$ corresponds to a unique total map $\tilde{a} : I_a \rightarrow E$ defined on a Scott-open subset $I_a \subseteq I$; because $\mathcal{M}_{\times} \perp \Sigma$, we may compute the support of the desired map

$J \dashrightarrow \mathbb{L}E$ by solving another lifting problem:

$$\begin{array}{ccc}
 I_a & \xrightarrow{\tilde{a}} & E \\
 \downarrow & \lrcorner & \downarrow \\
 I & \xrightarrow{a} & \mathbb{L}E \\
 \downarrow & \nearrow \phi & \downarrow \\
 J & & \Sigma
 \end{array}
 \quad
 \begin{array}{ccc}
 J_a & \xrightarrow{\quad} & \mathbf{1} \\
 \downarrow & \lrcorner & \downarrow \\
 J & \xrightarrow{\phi} & \Sigma
 \end{array}$$

Using the pullback lemma we deduce that the left-hand square below is cartesian:

$$\begin{array}{ccccc}
 I_a & \xrightarrow{\quad} & J_a & \xrightarrow{\quad} & \mathbf{1} \\
 \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 I & \xrightarrow{\quad} & J & \xrightarrow{\phi} & \Sigma \\
 & \searrow a & \nearrow \mathbb{L}! & & \\
 & & \mathbb{L}E & &
 \end{array}$$

Therefore $I_a \succrightarrow J_a \in \mathcal{M}_\Sigma$ and so we may solve the following lifting problem in E , and use the universal property of the partial map classifier to obtain the desired lift:

$$\begin{array}{ccc}
 I_a & \xrightarrow{\tilde{a}} & E \\
 \downarrow & \nearrow \tilde{b} & \downarrow \\
 J_a & & \mathbb{L}E \\
 \downarrow & \nearrow \tilde{c} & \downarrow \\
 J & &
 \end{array}$$

The following lemma appears as Proposition 3.2(2) in Fiore and Plotkin [17], but is left unproved in *op. cit.*

► **Lemma 64.** *The inclusion of any orthogonal subcategory creates limits.*

Proof. Let \mathcal{N} be a class of monomorphisms and let $\mathcal{E}_{\mathcal{N}^\perp}$ be the full subcategory of \mathcal{E} spanned by objects E such that $\mathcal{N} \perp E$. Let $E_\bullet : \mathcal{J} \rightarrow \mathcal{E}_{\mathcal{N}^\perp}$ be a diagram that has a limit in \mathcal{E} ; then we will argue that $\mathcal{N} \perp \varprojlim_{\mathcal{J}} E_\bullet$. Fixing $U \succrightarrow V \in \mathcal{N}$, consider any lifting problem of the following kind in \mathcal{E} :

$$\begin{array}{ccc}
 U & \xrightarrow{\quad} & \varprojlim_{\mathcal{J}} E_\bullet \\
 \downarrow & \nearrow \gamma & \downarrow \\
 V & &
 \end{array}$$

By the universal property of the limit, a solution to the lifting problem above in \mathcal{E} is uniquely determined by a solution to the following lifting problem in $[\mathcal{J}, \mathcal{E}]$.

$$\begin{array}{ccc} \{U\} & \longrightarrow & E_{\bullet} \\ \downarrow & \nearrow \gamma & \\ \{V\} & & \end{array}$$

Such a lifting problem can be solved pointwise, recalling that each E_i is orthogonal to \mathcal{N} :

$$\begin{array}{ccc} U & \longrightarrow & E_i \\ \downarrow & \nearrow \exists! & \\ V & & \end{array}$$

Naturality of the induced cone $\{V\} \rightarrow E_{\bullet}$ follows from the uniqueness of lifts. \blacktriangleleft

► **Lemma 65.** *Let $\mathcal{N} \in \{\mathcal{M}_{\times}, \mathcal{M}_{\Sigma}, \mathcal{M}\}$. If $\mathcal{N} \perp \Sigma$ and $\mathcal{N} \perp \mathbb{L}E$ then $\mathcal{N} \perp E$;*

Proof. By Lemma 64, since E is the following pullback:

$$\begin{array}{ccc} E & \longrightarrow & \mathbf{1}_{\mathcal{E}} \\ \downarrow \lrcorner & & \downarrow \\ \mathbb{L}E & \longrightarrow & \Sigma \end{array}$$

► **Lemma 66.** *If $\mathcal{M}_{\times} \perp \mathbb{L}E$, then also $\mathcal{M}_{\Sigma} \perp \mathbb{L}E$.*

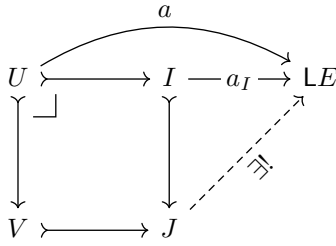
Proof. Fix $I \rightarrowtail J \in \mathcal{M}_{\times}$ and let $V \rightarrowtail J$ be a Σ -monomorphism; consider a lifting problem of the following form:

$$\begin{array}{ccccc} I & \longleftarrow & U & \xrightarrow{a} & \mathbb{L}E \\ \downarrow & & \downarrow \lrcorner & \nearrow \gamma & \\ J & \longleftarrow & V & & \end{array}$$

We may factor $a : U \rightarrow \mathbb{L}E$ through I as follows:

$$\begin{array}{ccc} U_a & \xrightarrow{\tilde{a}} & E \\ \downarrow & & \downarrow \\ U & \xrightarrow{a} & \mathbb{L}E \\ \downarrow & & \parallel \\ I & \xrightarrow{a_I} & \mathbb{L}E \end{array}$$

Therefore we may solve the following lifting problem:



► **Lemma 67.** *Let \mathcal{E} be locally presentable. If $\mathcal{M}_\times \perp \Sigma$, then the subcategory of \mathcal{E} spanned by E such that $\mathcal{M}_\times \perp LE$ is reflective.*

Proof. In a locally presentable category, any orthogonal subcategory is reflective. We therefore argue that $\mathcal{M}_\times \perp LE$ if and only if $\mathcal{M}_\Sigma \perp E$.

1. Suppose that $\mathcal{M}_\Sigma \perp E$; then Lemma 63 applies as we have also assumed $\mathcal{M}_\times \perp \Sigma$.
2. Suppose that $\mathcal{M}_\times \perp LE$; by Lemma 65 it suffices to check that $\mathcal{M}_\Sigma \perp \Sigma$ and $\mathcal{M}_\Sigma \perp LE$ which both follow from our assumptions via Lemma 66.

B.1.3.1 Definition of cartesian coverages and sheaves

► **Definition 68.** *Let \mathcal{C} be a category and let $u \in \mathcal{C}$; a sink on u is defined to be a collection of morphisms into u , i.e. an element of $\mathbf{Fam}_{\text{Set}}(\mathcal{C}_{/u})$.*

When \mathcal{C} has pullbacks, we may consider the base change of a sink $\underline{u} = (u_i \rightarrow u \mid i \in I)$ along a map $v \rightarrow u$, namely the family $v^*\underline{u} = (v \times_u u_i \rightarrow v \mid i \in I)$.

► **Definition 69.** *Let \mathcal{C} be a category with pullbacks. Then a cartesian coverage on \mathcal{C} is given by an assignment to each object $u \in \mathcal{C}$ a collection $K(u) \subseteq \mathbf{Fam}_{\text{Set}}(\mathcal{C}_{/u})$ of sinks on u that is stable under base change: if $\underline{u} \in K(u)$ then for any $v \rightarrow u$, the sink $v^*\underline{u}$ lies in $K(v)$.*

We refer to an element of $K(u)$ as a *covering sink* on u .

► **Definition 70.** *Let \mathcal{C} be category with pullbacks and let \underline{u} be a sink on $u \in \mathcal{C}$. For a presheaf $F \in \mathcal{S}\hat{\mathcal{C}}$, a matching family for \underline{u} is given by an assignment of elements $x_i \in F(u_i)$ to each $u_i \rightarrow u$ in \underline{u} , such that for all $u_i, u_j \in \underline{u}$ we have $u_{ij}^*x_i = u_{ij}^*x_j \in F(u_{ij})$ where u_{ij} is the fiber product $u_i \times_u u_j \in \mathcal{C}_{/u}$.*

► **Definition 71.** *Let \mathcal{C} be a category with pullbacks and let K be a coverage on \mathcal{C} . A presheaf $F \in \mathcal{S}\hat{\mathcal{C}}$ is defined to be a sheaf relative to K when for any covering sink $\underline{u} \in K(u)$, there is a bijection between matching families for \underline{u} in F and elements of $F(u)$.*

B.1.4 A sheaf model of synthetic domain theory

Taking the Kleisli model of axiomatic domain theory from the previous section, we may now adapt the conservative extension theorem of Fiore and Plotkin [17] to embed it into a sheaf model of synthetic domain theory.

► **Notation 72.** In this section, we will write \mathcal{C} for dcpop .

B.1.4.1 Descent properties of dpcos

We begin by showing that two classes of colimit in \mathcal{C} enjoy a useful descent property, which will enable us to embed \mathcal{C} into a topos such that the resulting Yoneda embedding preserves these colimits. Monomorphisms of dpcos are not generally well-behaved. But there is a dominion of monomorphisms in \mathcal{C} that we shall call *Scott-open immersions* that has a number of very useful properties.

► **Definition 73.** A *Scott-open immersion* $U \hookrightarrow A$ of dpcos is any map that arises by pullback from the dominance $\top : \mathbf{1}_{\mathcal{C}} \hookrightarrow \Sigma$. (In the category of dpcos, we shall use this special arrow to denote open immersions.)

Every Scott-open immersion $j : U \hookrightarrow A$ factors uniquely through an isomorphism $U \cong \text{Im } j$ and a subdcpo inclusion $\text{Im } j \subseteq A$ such that the order on $\text{Im } j$ is the restriction of the order on A . We will refer to subdcpo inclusions that arise from such a factorization as *Scott-open subdcpo inclusions*; of course, every Scott-open subdcpo inclusion is trivially a Scott-open immersion. Therefore we need not pay much attention to the difference between Scott-open immersions and Scott-open subdcpo inclusions.

► **Lemma 74.** The Scott-open subdcpo inclusions are closed under finite unions.

Proof. We mean that given a finite family of subobjects $(U_i \in \mathbf{Sub}_A \mid i \in I)$ that are each Scott-open, the union $\bigvee_{i \in I} U_i$ exists and moreover is Scott-open. This is not difficult to see, as we may compute $\bigvee_i U_i$ using the operation $\bigvee : \Sigma^I \rightarrow \Sigma$. Let $\phi_i : A \rightarrow \Sigma$ be the characteristic maps of each U_i ; then we have $\bigvee_i \phi_i : A \rightarrow \Sigma$ which will serve as the characteristic map for $\bigvee_i U_i \hookrightarrow A$. ◀

► **Lemma 75.** Finite unions of Scott-open subobjects in \mathcal{C} are *stable* under pullback.

Proof. We consider the partition of a dcpo $X \in \mathcal{C}$ as the union of a finite family of Scott-open subdcpo inclusions $U_i \hookrightarrow X$ indexed in some finite set I . We must check the union cocone is preserved by pulling back along some $k : Y \rightarrow X$, i.e. we need to reconstruct Y as the union of the Scott-open subdcpo inclusions k^*U_i . That $X = \bigvee_i U_i$ means that the following square is cartesian:

$$\begin{array}{ccccc} X & \longrightarrow & \cdots & \longrightarrow & \mathbf{1}_{\mathcal{C}} \\ \parallel \lrcorner & & \downarrow \lrcorner & & \downarrow \top \\ X & \xrightarrow{\phi_{\bullet}} & \Sigma^I & \xrightarrow{(\vee)} & \Sigma \end{array}$$

Pulling back further along $k : Y \rightarrow X$, we have:

$$\begin{array}{ccccccc} Y & \longrightarrow & X & \longrightarrow & \cdots & \longrightarrow & \mathbf{1}_{\mathcal{C}} \\ \parallel \lrcorner & & \parallel \lrcorner & & \downarrow \lrcorner & & \downarrow \top \\ Y & \xrightarrow{k} & X & \xrightarrow{\phi_{\bullet}} & \Sigma^I & \xrightarrow{(\vee)} & \Sigma \end{array}$$

But the characteristic map $k; \phi_{\bullet}; \vee$ is equal to $(\lambda i.k; \phi_i); \vee$. Therefore Y is the union of k^*U_i because $k; \phi_i$ is the characteristic map for k^*U_i . ◀

► **Lemma 76.** *Let D be a directed subset of a dcpo $A \in \mathcal{C}$ and let $U \subseteq A$ be a Scott-open subdcpo of A such that $U \cap D$ is inhabited; then $U \cap D$ is also directed.*

Proof. Fixing $x, y \in U \cap D$ we must find some $z \in U \cap D$ such that $x, y \leq z$. Because D is directed there does exist $z \in D$ such that $x, y \leq z$. As U is a Σ -subset, there exists a unique Scott continuous characteristic map $\phi_U : A \rightarrow \Sigma$ encoding U . A continuous map is also monotone, so from $x \leq z$ we conclude $\phi_U x \leq \phi_U z$. Therefore $z \in U$ follows from $x \in U$. ◀

► **Lemma 77.** *Finite unions of Scott-open subobjects in \mathcal{C} are **effective** in the sense of Barr [7] and Garner and Lack [22].*

Proof. Let $U_i \hookrightarrow X$ be a finite family of Scott-open immersions in \mathcal{C} indexed in $i \in I$, such that each U_i is classified by $\phi_i : X \rightarrow \Sigma$ respectively. We recall that any Scott-open immersion is an order-embedding in the sense each U_i has the order induced by restricting that of X to points satisfying ϕ_i . We recall that $|\Sigma|$ is the subobject classifier of \mathcal{S}_P and $\vee : \Sigma^I \rightarrow \Sigma$ is tracked in \mathcal{S}_P by the actual topos disjunction. Therefore the underlying sheaf of points of the union $\bigvee_i U_i$ can be computed in \mathcal{S}_P as the *wide* (effective) pushout of all the projections $p_j : |\bigwedge_i U_i| = \bigwedge_i |U_i| \rightarrow |U_j|$, as any topos has effective unions of small arity. The effectivity property for wide pushouts of monomorphisms in a topos means that $|\bigwedge_i U_i|$ is the wide pullback of all the $|U_j| \rightarrow |\bigvee_i U_i|$.

Then $\bigvee_i U_i$ is the Scott-open subdcpo of X spanned by $|\bigvee_i U_i|$ with order induced from X . We note that $\bigwedge_i U_i$ remains the wide pullback in \mathcal{C} of all the $U_j \hookrightarrow \bigvee_i U_i$ because pullbacks of dcpo are computed as in \mathcal{S}_P . It remains to show that the $\bigvee_i U_i$ is *also* the wide pushout in \mathcal{C} of all the $\bigwedge_i U_i \hookrightarrow U_j$. To this end, fix any cocone $h_\bullet : U_\bullet \rightarrow \{Y\}$ in which Y is an arbitrary object of \mathcal{C} ; at the level of points, there is a unique function $h_\infty : |\bigvee_i U_i| \rightarrow |Y|$ making the following diagram commute in $[I, \mathcal{S}_P]$:

$$\begin{array}{ccc} U_\bullet & \longrightarrow & \{|\bigvee_i U_i|\} \\ & \searrow \scriptstyle |h_\bullet| & \downarrow \scriptstyle \exists! h_\infty \\ & & \{|Y|\} \end{array}$$

Therefore it remains to show that $h_\infty : |\bigvee_i U_i| \rightarrow |Y|$ is Scott continuous and hence tracks a morphism $\bigvee_i U_i \rightarrow Y$ in \mathcal{C} . We fix a directed subset $D \subseteq \bigvee_i U_i$ to check that $h_\infty(\bigsqcup D) = \bigsqcup_{x \in D} h_\infty x$. Note that $\bigsqcup D$ must lie in U_k for some k ; observe that $\bigsqcup D$ is also the least upper bound of the subset $D \cap U_k \subseteq U_k$ which remains directed by Lemma 76. Because h_k is assumed Scott continuous, we have $h_\infty(\bigsqcup D) = h_k(\bigsqcup D) = \bigsqcup_{x \in D \cap U_k} h_k x$; but this is equal to $\bigsqcup_{x \in D} h_\infty x$ because any element of D is bounded by an element of U , namely $\bigsqcup D$ itself. ◀

► **Lemma 78.** *Coproduct injections in \mathcal{C} are Scott-open immersions.*

Proof. Let I be a set and let A_i be a family of dcpo indexed in $i \in I$. For $k \in I$ we must verify that the injection $A_k \hookrightarrow \coprod_i A_i$ is in fact a Scott-open immersion. We do so by constructing the unique characteristic map $\phi_k : \coprod_i A_i \rightarrow \Sigma$ making the following square

cartesian:

$$\begin{array}{ccc}
 A_k & \xrightarrow{\quad} & 1_{\mathcal{C}} \\
 \downarrow \lrcorner & & \downarrow \\
 \coprod_i A_i & \dashrightarrow & \Sigma
 \end{array}$$

Of course, a map $\coprod_i A_i \rightarrow \Sigma$ is the same as for each $i \in I$, a map $A_i \rightarrow \Sigma$. When $i = k$ we return $\top \in \Sigma$ and otherwise we return $\perp \in \Sigma$. \blacktriangleleft

► **Corollary 79.** *Any coproduct in \mathcal{C} is the union of a family of Scott-open immersions.*

B.1.4.2 A coverage on the category of dcpos

We will impose a coverage on \mathcal{C} that will ensure an embedding into a Grothendieck topos that preserves finite unions of Scott-open subobjects (and thus finite coproducts). For any $A \in \mathcal{C}$ we let $K(A)$ be the set of sinks given by finitely indexed families Scott-open immersions $(U_i \hookrightarrow A \mid i \in I)$ such that $A \cong \bigvee_{i \in I} U_i$.

► **Lemma 80.** *The assignment K of sets of sinks to objects of \mathcal{C} is a cartesian coverage.*

Proof. By Lemma 75. \blacktriangleleft

► **Lemma 81.** *The cartesian coverage K is subcanonical.*

Proof. Let $A \in \mathcal{C}$ be a dcpo; we must check that $y_{\mathcal{C}} A$ is a sheaf for K . This follows from the fact that unions of Scott-open subobjects are effective (Lemma 77). Indeed, considering only the binary case for simplicity, we must check the collection of pairs of morphisms $a_U : U \rightarrow A$ and $a_V : V \rightarrow A$ that agree on the intersection $U \wedge V$ is bijective with the collection of morphisms $U \vee V \rightarrow A$; this follows immediately by effectivity which guarantees that the following square is both cartesian and cocartesian:

$$\begin{array}{ccc}
 U \wedge V & \hookrightarrow & U \\
 \downarrow \lrcorner & & \downarrow \\
 V & \hookrightarrow & U \vee V
 \end{array}$$

► **Construction 82.** We will write $\mathbf{C} \hookrightarrow \widehat{\mathcal{C}}$ for the subtopos determined by localizing $\mathcal{S}_{\widehat{\mathcal{C}}}$ at the coverage K .

Because the localization is subcanonical, the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{S}_{\widehat{\mathcal{C}}}$ factors through a fully faithful functor $y_{\mathbf{C}} : \mathcal{C} \hookrightarrow \mathcal{S}_{\mathbf{C}}$:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{y_{\mathcal{C}}} & \mathcal{S}_{\widehat{\mathcal{C}}} \\
 y_{\mathbf{C}} \downarrow & \nearrow & \\
 \mathcal{S}_{\mathbf{C}} & &
 \end{array}$$

► **Fact 83.** *The subcategory $\mathcal{S}_{\mathbf{C}} \subseteq \mathcal{S}_{\widehat{\mathcal{C}}}$ is spanned precisely by the presheaves that send finite unions of Scott-open subobjects in \mathcal{C} to pullbacks in \mathbf{Set} .*

B.1.4.3 Conservative extension and basic axioms of SDT

An analogue to the conservative extension result (Theorem 3.4) of Fiore and Plotkin [17] can be adapted to our setting to construct a model of synthetic domain theory in $\mathcal{S}_{\mathbf{C}}$ satisfying **Axioms** SDT-1–SDT-3 extending the development of Section B.1.2 (we verify the remaining **Axioms** SDT-4 and SDT-5 in Section B.1.4.4). The difference between our setting and *op. cit.* is that the latter localize with the 0-ary extensive topology, whereas we do so with a more sophisticated topology (Construction 82); an important consequence is that our lifting monad on \mathbf{dcpos} preserves only filtered colimits, whereas that of *op. cit.* preserves all connected colimits.

Writing $\mathbf{Predom}_{\mathbf{C}} \subseteq \mathcal{S}_{\mathbf{C}}$ for the full subcategory spanned by predomains, the Yoneda embedding $\mathcal{C} \hookrightarrow \mathcal{S}_{\mathbf{C}}$ factors through $\mathbf{Predom}_{\mathbf{C}}$ and preserves limits, exponentials, the final lift-coalgebra $\bar{\omega}$, finite coproducts, and the dominance Σ ; moreover, the lifting monad is extended by each embedding in the sense that the following squares commute up to isomorphism:

$$\begin{array}{ccccc} \mathcal{C} & \hookrightarrow & \mathbf{Predom}_{\mathbf{C}} & \hookrightarrow & \mathcal{S}_{\mathbf{C}} \\ \downarrow L & & \downarrow L & & \downarrow L \\ \mathcal{C} & \hookrightarrow & \mathbf{Predom}_{\mathbf{C}} & \hookrightarrow & \mathcal{S}_{\mathbf{C}} \end{array}$$

► **Axiom SDT-1.** Σ has *finite joins* $\bigvee_{i < n} \phi_i$ that are preserved by the inclusion $\Sigma \subseteq \Omega$. We will write \perp for the empty join and $\phi \vee \psi$ for binary joins.

Proof. Finite joins of Σ -subobjects are preserved by the Yoneda embedding by virtue of the localization that we have imposed on $\mathcal{S}_{\mathbf{C}}$ in Construction 82. ◀

► **Computation 84.** The original lifting monad on \mathcal{C} lifts into $\mathcal{S}_{\widehat{\mathcal{C}}}$ by Yoneda extension in the following way:

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{L} & \mathcal{C} & \xrightarrow{ye} & \mathcal{S}_{\widehat{\mathcal{C}}} \\ \downarrow ye & & & \nearrow L_! & \\ \mathcal{S}_{\widehat{\mathcal{C}}} & & & & \end{array}$$

$L_!$ above is the classifier for Σ -partial maps in $\mathcal{S}_{\widehat{\mathcal{C}}}$, as can be seen by adapting Corollary 1.5 of [17]. Moreover $L_!$ restricts to the lifting monad on $\mathcal{S}_{\mathbf{C}}$ in the sense that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} \mathcal{S}_{\mathbf{C}} & \xrightarrow{e_*} & \mathcal{S}_{\widehat{\mathcal{C}}} \\ \downarrow L & & \downarrow L_! \\ \mathcal{S}_{\mathbf{C}} & \xrightarrow{e_*} & \mathcal{S}_{\widehat{\mathcal{C}}} \end{array}$$

This can be seen by considering the explicit computation of the partial map classifier for a given dominance, using the following facts:

1. The dominance Σ is representable, and hence a sheaf because our localization was subcanonical.
2. Sheaves comprise a Σ -closed exponential ideal in every slice.

The following lemma abstracts the argument of Theorem 1.7(2) of Fiore and Plotkin [17].

► **Lemma 85.** *The lifting functor $L : \mathcal{S}_{\mathcal{C}} \rightarrow \mathcal{S}_{\mathcal{C}}$ preserves any shape of colimit that is preserved by the direct image functor $e_* : \mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{S}_{\widehat{\mathcal{C}}}$.*

Proof. Let \mathcal{J} be a category such that colimits of shape \mathcal{J} are preserved by e_* ; letting $D_{\bullet} : \mathcal{J} \rightarrow \mathcal{S}_{\mathcal{C}}$ be a diagram, we compute the colimit of LD_{\bullet} .

$$\begin{aligned}
& \text{Hom}_{\mathcal{S}_{\mathcal{C}}}(X, \varinjlim_{\mathcal{J}} LD_{\bullet}) \\
& \cong \text{Hom}_{\mathcal{S}_{\widehat{\mathcal{C}}}}(e_*X, e_*\varinjlim_{\mathcal{J}} LD_{\bullet}) && e \text{ is embedding} \\
& \cong \text{Hom}_{\mathcal{S}_{\widehat{\mathcal{C}}}}(e_*X, \varinjlim_{\mathcal{J}} e_*LD_{\bullet}) && \text{by assumption} \\
& \cong \text{Hom}_{\mathcal{S}_{\widehat{\mathcal{C}}}}(e_*X, \varinjlim_{\mathcal{J}} L!e_*D_{\bullet}) && \text{Computation 84} \\
& \cong \text{Hom}_{\mathcal{S}_{\widehat{\mathcal{C}}}}(e_*X, L!\varinjlim_{\mathcal{J}} e_*D_{\bullet}) && \text{Yoneda extension is cocontinuous} \\
& \cong \text{Hom}_{\mathcal{S}_{\widehat{\mathcal{C}}}}(e_*X, L!e_*\varinjlim_{\mathcal{J}} D_{\bullet}) && \text{by assumption} \\
& \cong \text{Hom}_{\mathcal{S}_{\widehat{\mathcal{C}}}}(e_*X, e_*L\varinjlim_{\mathcal{J}} D_{\bullet}) && \text{Computation 84} \\
& \cong \text{Hom}_{\mathcal{S}_{\mathcal{C}}}(X, L\varinjlim_{\mathcal{J}} D_{\bullet}) && e \text{ is embedding} \quad \blacktriangleleft
\end{aligned}$$

► **Lemma 86.** *Write $e : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ for the embedding of topoi that exhibits $\mathcal{S}_{\mathcal{C}}$ as the sheaf subcategory spanned by presheaves that send the colimits specified in Construction 82 to limits. Then the direct image functor $e_* : \mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{S}_{\widehat{\mathcal{C}}}$ preserves ω -filtered colimits.*

Proof. We consider an ω -filtered category \mathcal{J} and a diagram $D_{\bullet} : \mathcal{J} \rightarrow \mathcal{S}_{\mathcal{C}}$; it is enough to check that the colimit of $e_*D_{\bullet} : \mathcal{J} \rightarrow \mathcal{S}_{\widehat{\mathcal{C}}}$ is a sheaf. We shall write $Q = \varinjlim_{i \in \mathcal{J}} e_*D_i$ for this colimit. We must verify that finite unions of Scott-open subobjects are sent to limits in \mathbf{Set} . To that end, fix a finite partition $a \cong \bigvee_k a_k$ of a dcpo a into Scott-open subdcpos $a_k \subseteq a \in \mathcal{C}$. We must check that $Q(a)$ is isomorphic to $\bigwedge_k Q(a_k)$. We compute:

$$\begin{aligned}
Q(a) & \cong Q(\bigvee_k a_k) \\
& \cong \varinjlim_i e_*D_i(\bigvee_k a_k) && \text{colimits pointwise in } \mathcal{S}_{\widehat{\mathcal{C}}} \\
& \cong \varinjlim_i \bigwedge_k e_*D_i(a_k) && \text{each } e_*D_i \text{ is a sheaf} \\
& \cong \bigwedge_k \varinjlim_i e_*D_i(a_k) && \omega\text{-filtered colimits commute with finite limits} \\
& \cong \bigwedge_k Q(a_k) && \text{colimits pointwise in } \mathcal{S}_{\widehat{\mathcal{C}}} \quad \blacktriangleleft
\end{aligned}$$

► **Lemma 87.** *The functor $L : \mathcal{S}_{\mathcal{C}} \rightarrow \mathcal{S}_{\mathcal{C}}$ preserves ω -filtered colimits.*

Proof. By Lemma 85 and Lemma 86. ◀

► **Remark 88.** Fiore and Plotkin [17] show that in their setting the lift functor preserves *connected* colimits; this is stronger than preserving filtered colimits, and does not hold in our scenario. The reason is that they impose only the 0-extensive topology on \mathcal{C} , hence the only product cone that must be considered in their version of Lemma 86 is the empty one. Because we have imposed a different topology, we must consider a broader class of limit cones; although it is indeed the case that the empty product is preserved by connected colimits, we cannot hope for the same more generally [9].

► **Axiom SDT-2.** The initial lift algebra ω is the colimit of the following ω -chain of maps:

$$\emptyset \xrightarrow{!} L\emptyset \xrightarrow{L!} L^2\emptyset \xrightarrow{L^2!} \dots$$

Proof. By the classic result of Adámek [2], it suffices to observe that L is ω -cocontinuous by virtue of Lemma 87, hence its initial algebra is the colimit of the standard ω -chain above. ◀

► **Lemma 89.** Every representable object in $\mathcal{S}_{\mathcal{C}}$ is complete.

Proof. Fixing a dcpo $d \in \mathcal{C}$, we must argue that $y_{\mathcal{C}}d$ is complete. This follows from the limit-colimit coincidence in \mathcal{C} recalling that $\bar{\omega} = y_{\mathcal{C}}\bar{\omega}$ is both the initial L -algebra and final L -coalgebra from the perspective of \mathcal{C} . Therefore we are solving lifting problems of the following form, recalling that both lifting and initial objects are preserved by $y_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{S}_{\mathcal{C}}$:

$$\begin{array}{ccc} I \times \varinjlim_n y_{\mathcal{C}} L^n \emptyset_{\mathcal{C}} & \longrightarrow & y_{\mathcal{C}} d \\ \downarrow & \nearrow \gamma & \\ I \times y_{\mathcal{C}} \varinjlim_n L^n \emptyset_{\mathcal{C}} & & \end{array}$$

Applying the fully faithful direct image functor $e_* : \mathcal{S}_{\mathcal{C}} \hookrightarrow \mathcal{S}_{\widehat{\mathcal{C}}}$, recalling that it commutes with filtered colimits (Lemma 86), we may rewrite our lifting problem to one in $\mathcal{S}_{\widehat{\mathcal{C}}}$:

$$\begin{array}{ccc} e_* I \times \varinjlim_n y_{\mathcal{C}} L^n \emptyset_{\mathcal{C}} & \longrightarrow & y_{\mathcal{C}} d \\ \downarrow & \nearrow \gamma & \\ e_* I \times y_{\mathcal{C}} \varinjlim_n L^n \emptyset_{\mathcal{C}} & & \end{array}$$

Because $e_* I$ is canonically a colimit of representables $y_{\mathcal{C}}e$ that parameterize an element of I , we may rewrite the lifting problem once more using the fact that the product functor $(e \times -)$ in both $\mathcal{S}_{\widehat{\mathcal{C}}}$ and \mathcal{C} is cocontinuous as both are cartesian closed:

$$\begin{array}{ccc} \varinjlim_n y_{\mathcal{C}}(e \times L^n \emptyset_{\mathcal{C}}) & \longrightarrow & y_{\mathcal{C}} d \\ \downarrow & \nearrow \gamma & \\ y_{\mathcal{C}}(\varinjlim_n e \times L^n \emptyset_{\mathcal{C}}) & & \end{array}$$

By the universal property of the colimit in $\mathcal{S}_{\widehat{\mathcal{C}}}$, the upper map is determined by a cocone $y_{\mathcal{C}}(e \times L^n \emptyset_{\mathcal{C}}) \rightarrow \{y_{\mathcal{C}}d\}$ in $\mathcal{S}_{\widehat{\mathcal{C}}}$, but by the Yoneda lemma this is identical to a cocone $e \times L^n \emptyset_{\mathcal{C}} \rightarrow \{d\}$ in \mathcal{C} . Such a cocone in \mathcal{C} uniquely determines the dotted map, via the Yoneda lemma and the universal property of the colimit in \mathcal{C} . ◀

► **Corollary 90.** Every representable object in $\mathcal{S}_{\mathcal{C}}$ is a predomain.

Proof. Representables are closed under lifting, hence the result follows by Lemma 89. ◀

► **Axiom SDT-3.** The dominance Σ is a predomain.

Proof. By Corollary 90. ◀

B.1.4.4 Axioms for information flow and declassification

We have a geometric morphism $p_C : \mathbf{C} \rightarrow \mathbf{P}$; viewing \mathbf{P} as the classifying topos for the “security level theory” \mathcal{P} , it suffices to exhibit a finite meet preserving functor $\mathcal{P} \rightarrow \mathcal{S}_C$. We will send each $l \in \mathcal{P}$ to the subterminal $y_{\mathcal{C}\mathcal{P}}l$.

► **Axiom SDT-4.** *The topos \mathbf{C} is equipped with a geometric morphism $p_C : \mathbf{C} \rightarrow \mathbf{P}$ such that the induced functor $p_C^* y_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{O}_C$ is fully faithful and is valued in Σ -propositions. We will write $\langle l \rangle$ for each $p_C^* y_{\mathcal{P}}l$.*

Proof. We consider the Yoneda extension of the following fully faithful and finitely continuous composite functor $\mathcal{P} \hookrightarrow \mathcal{S}_C$, noting that the image of $y_{\mathcal{P}}$ lies in \mathcal{C} :

$$\begin{array}{ccccc}
 \mathcal{P} & \xrightarrow{y_{\mathcal{P}}} & \mathcal{C} & \xrightarrow{y_C} & \mathcal{S}_C \\
 \downarrow y_{\mathcal{P}} & & & \nearrow p_C^* & \\
 \mathcal{S}_{\mathbf{P}} & & & &
 \end{array}$$

By Diaconescu’s theorem [13], the cocontinuous extension indicated above is also finitely continuous, hence the inverse image part of a morphism of topoi $p_C : \mathbf{C} \rightarrow \mathbf{P}$. It is evident that $p_C^* y_{\mathcal{P}} : \mathcal{P} \hookrightarrow \mathcal{O}_C$ is valued in Σ -propositions. ◀

► **Computation 91.** The direct image functor $(p_C)_* : \mathcal{S}_C \rightarrow \mathcal{S}_{\mathbf{P}}$ is given by precomposition with $y_{\mathcal{P}} : \mathcal{P} \hookrightarrow \mathcal{C}$.

Proof. This is computed by adjointness, fixing $E \in \mathcal{S}_C$.

$$\begin{aligned}
 ((p_C)_* E)l &\cong \text{Hom}_{\mathcal{S}_{\mathbf{P}}}(y_{\mathcal{P}}l, (p_C)_* E) \\
 &\cong \text{Hom}_{\mathcal{S}_C}(p_C^* y_{\mathcal{P}}l, E) \\
 &\cong \text{Hom}_{\mathcal{S}_C}(y_C y_{\mathcal{P}}l, E) \\
 &\cong E(y_{\mathcal{P}}l)
 \end{aligned}$$

► **Lemma 92.** *The direct image functor $(p_C)_* : \mathcal{S}_C \rightarrow \mathcal{S}_{\mathbf{P}}$ extends the the forgetful functor $|-| : \mathcal{C} \rightarrow \mathcal{S}_{\mathbf{P}}$ sending a small dcpo to its underlying presheaf of points, in the sense that the following diagram commutes up to isomorphism:*

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{|-|} & \mathcal{S}_{\mathbf{P}} \\
 \downarrow y_C & \nearrow (p_C)_* & \\
 \mathcal{S}_C & &
 \end{array}$$

Proof. We fix a small predomain $A \in \mathcal{C}$ to compute $(p_C)_* y_C A$:

$$\begin{aligned}
 ((p_C)_* y_C A)(k) &\cong (y_C A)(\langle k \rangle) && \text{Computation 91} \\
 &\cong \text{Hom}_{\mathcal{C}}(\langle k \rangle, A) && \text{Yoneda lemma} \\
 &\cong \text{Hom}_{\mathcal{S}_{\mathbf{P}}}(y_{\mathcal{P}}k, |A|) && \text{subterminal dcpos are discrete} \\
 &\cong |A|(k) && \text{Yoneda lemma}
 \end{aligned}$$

► **Axiom SDT-5.** Any constant object $\mathbf{C}^*[n] \in \mathcal{S}_{\mathbf{C}}$ for $[n]$ a finite set is an $\langle l \rangle$ -transparent predomain for any $l \in \mathcal{P}$.

Proof. That $\mathbf{C}^*[n]$ is a predomain follows from the fact that a finite coproduct of the form $\coprod_{i < n} \mathbf{1}_{\mathcal{S}_{\mathbf{C}}}$ is represented by the dcpo $\{i \mid i < n\} \in \mathcal{C} \subseteq \mathcal{S}_{\mathbf{P}}$, and all representables are predomains. That it is $\langle l \rangle$ -transparent follows from the discreteness of the representing dcpo as a constant presheaf in $\mathcal{S}_{\mathbf{P}}$. ◀

B.2 Model of synthetic Tait computability

B.2.1 \mathbf{P} as a classifying topos

Because \mathcal{P} is finitely complete, it can be thought of as a *finite limit theory* — the essentially algebraic theory of security levels.

► **Remark 93.** In concrete applications we also expect \mathcal{P} to have joins, but we are not paying attention to these; while this may appear strange, the joins of a security lattice are also ignored by Abadi *et al.* [1], which can be seen from the fact that a type that is both l -protected and k -protected nonetheless need not be $l \vee k$ -protected in the Dependency Core Calculus. Furthermore, it is not the case in *op. cit.* that every type is \perp -protected when \mathcal{P} has a bottom element. We simply reproduce this mismatch between security levels and the security typing, though it would be interesting to remedy it in future work.

Any finitely continuous functor $\mathcal{P} \rightarrow \mathcal{E}$ can be thought of as a *model* of the theory \mathcal{P} . The collection of all topos models of \mathcal{P} is concentrated in the topos $\mathbf{P} = \widehat{\mathcal{P}}$: by Diaconescu’s theorem [13], a model $L : \mathcal{P} \rightarrow \mathcal{S}_{\mathbf{X}}$ corresponds essentially uniquely to a morphism of topoi $[L] : \mathbf{X} \rightarrow \mathbf{P}$ whose inverse image functor is obtained by Yoneda extension:

$$\begin{array}{ccc}
 \mathcal{P} & \xrightarrow{L} & \mathcal{S}_{\mathbf{X}} \\
 \downarrow y_{\mathcal{P}} & \searrow [L]^* & \\
 \mathcal{S}_{\mathbf{P}} & &
 \end{array} \tag{1}$$

The direct image functor $[L]_* : \mathcal{S}_{\mathbf{X}} \rightarrow \mathcal{S}_{\mathbf{P}}$ can be computed by adjointness:

$$[L]_* X \cong \text{Hom}_{\mathcal{S}_{\mathbf{P}}}(y_{\mathcal{P}} -, [L]_* X) \quad \text{Yoneda} \tag{2}$$

$$\cong \text{Hom}_{\mathcal{S}_{\mathbf{X}}}([L]^* y_{\mathcal{P}} -, X) \quad [L]^* \dashv [L]_* \tag{3}$$

$$\cong \text{Hom}_{\mathcal{S}_{\mathbf{X}}}(L -, X) \quad \text{Diagram 1} \tag{4}$$

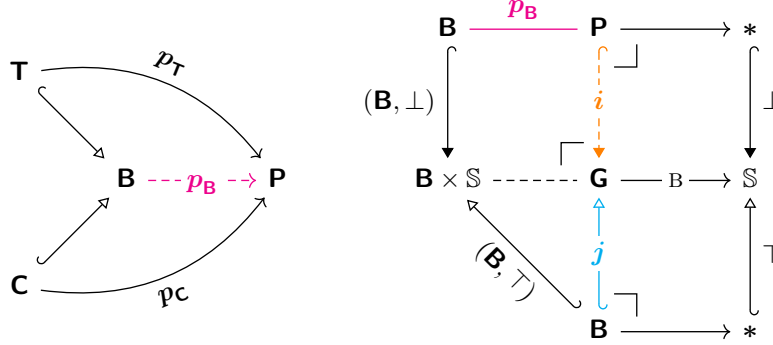
To see that $[L]^* \dashv [L]_*$ we use the fact that $[L]^*$ is cocontinuous.

B.2.2 The formal approximation topos

► **Notation 94** (Syntactic topos). We will write $\mathbf{T} = \widehat{\mathcal{T}}$ for the presheaf topos defined by the identification $\mathcal{S}_{\mathbf{T}} = [\mathcal{T}^{\text{op}}, \text{Set}]$.

In **Axiom SDT-4** we have assumed a morphism of topoi $p_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{P}$; viewing \mathbf{P} as the classifying topos for the “theory” of security levels \mathcal{P} , this morphism is the *characteristic map* for the model of \mathcal{P} in $\mathcal{S}_{\mathbf{C}}$ in which $\langle l \rangle = p_{\mathbf{C}}^* y_{\mathcal{P}} l$. Because the syntactic category \mathcal{T} of our programming language also contains security levels, we likewise obtain a morphism of topoi $p_{\mathbf{T}} : \mathbf{T} \rightarrow \mathbf{P}$ such that $y_{\mathcal{T}} \langle l \rangle = p_{\mathbf{T}}^* y_{\mathcal{P}} l$.

Writing \mathbf{B} for the coproduct topos $\mathbf{T} + \mathbf{C}$, we may combine these maps together into a single map $p_{\mathbf{B}} : \mathbf{B} \rightarrow \mathbf{P}$ along which glue to obtain the relative Sierpiński cone \mathbf{G} :



► **Intuition 95.** The purpose of constructing \mathbf{G} is as follows: in the category $\mathcal{S}_{\mathbf{G}}$ we may define a Kripke logical relation between the syntax and the denotational semantics, with the Kripke variation taken in \mathcal{P} . Indeed, we will see that every object of $\mathcal{S}_{\mathbf{G}}$ is a kind of “formal approximation relation” between something syntactical and something domain-theoretic.

In the construction above, we recover \mathbf{B} as an *open subtopos* of \mathbf{G} and \mathbf{P} as a *closed subtopos*; the inclusions of these subspaces are given by open and closed immersions j, i respectively. Such an open-closed partition corresponds essentially uniquely to a characteristic map that we name $\mathbf{B} : \mathbf{G} \rightarrow \mathbb{S}$; it is well-known in topos theory that the open \mathbf{B} may be identified with a subterminal object $\mathbf{B} \in \mathcal{O}_{\mathbf{G}}$ such that $\mathcal{S}_{\mathbf{B}} \simeq \mathcal{S}_{\mathbf{G}/\mathbf{B}}$ and under this identification, the inverse image functor $j^* : \mathcal{S}_{\mathbf{G}} \rightarrow \mathcal{S}_{\mathbf{B}}$ is the base change functor $\mathbf{B}^* : \mathcal{S}_{\mathbf{G}} \rightarrow \mathcal{S}_{\mathbf{G}/\mathbf{B}}$.

► **Axiom STC-1.** *There are two disjoint propositions $\mathbf{T}, \mathbf{C} \in \mathcal{O}_{\mathbf{G}}$ such that $\mathbf{T} \wedge \mathbf{C} = \perp$. We will refer to these as the *syntactic* and *computational phases* respectively. We will write $\mathbf{B} = \mathbf{T} \vee \mathbf{C}$ for the disjoint union of the two phases.*

Proof. The coproduct injections $\mathbf{T}, \mathbf{C} \hookrightarrow \mathbf{B}$ are (cl)open immersions and therefore induce disjoint opens $\mathbf{T}, \mathbf{C} : \mathbf{G} \rightarrow \mathbb{S}$ such that $\mathbf{B} = \mathbf{T} \vee \mathbf{C}$ and $\mathbf{T} \wedge \mathbf{C} = \perp$. ◀

Consequently we may likewise identify $\mathcal{S}_{\mathbf{T}}, \mathcal{S}_{\mathbf{C}}$ with $\mathcal{S}_{\mathbf{G}/\mathbf{T}}, \mathcal{S}_{\mathbf{G}/\mathbf{C}}$ respectively.

► **Remark 96 (Logical Relations as Types).** The configuration of the glued topos \mathbf{G} and its open and closed subtopoi is in essence a generalization of the topos model of parametricity structures by Sterling and Harper [62]. In *op. cit.* the role of \mathbf{P} was played by the Sierpiński topos \mathbb{S} , and two copies of the syntactic topos were taken as (disjoint) open subspaces whereas here we have taken the coproduct of the syntactic and computational topoi. To understand the difference between the two constructions, consider that Sterling and Harper [62] were targeting a single phase distinction between static and dynamic code, whereas we are concerned with a spectrum of phase distinctions corresponding to the security poset \mathcal{P} ; whereas \mathbb{S} is the classifying topos of a single phase distinction, \mathbf{P} is the classifying topos for such an array of phase distinctions. Finally, whereas *op. cit.* were capturing a binary logical relation between a language and itself, we are considering a heterogeneous logical relation between a language and its denotational semantics.

► **Axiom STC-2.** *Within the syntactic phase, there exists a \mathcal{T} -algebra $\mathcal{A}^{\mathbf{T}} : \mathcal{T}\text{-Alg}_{\mathbf{U}/\mathbf{T}}$ such that the corresponding functor $\mathcal{T} \rightarrow \mathcal{S}_{\mathbf{G}/\mathbf{T}}$ is fully faithful.*

Proof. We may define \mathcal{A}^T to be the internal algebra corresponding to the (fully faithful) functor $\mathcal{T} \hookrightarrow \mathcal{S}_{\mathbf{G}}$ obtained by restricting the direct image of the open immersion $\mathbf{T} \hookrightarrow \mathbf{G}$ along the Yoneda embedding $\mathcal{T} \hookrightarrow \mathcal{S}_{\mathbf{T}}$. This does indeed give rise to a model of \mathcal{T} because both the Yoneda embedding as well as the direct image of an open immersion are locally cartesian closed functors. ◀

► **Axiom STC-3.** *Within the computational phase, the axioms of \mathcal{P} -indexed synthetic domain theory (Axioms SDT-1–SDT-5) are satisfied.*

Proof. This follows immediately by definition, since we have ensured that the open subtopos corresponding to the computational phase \mathbf{C} is \mathbf{C} itself, *i.e.* $\mathbf{G}_{/\mathbf{C}} \simeq \mathbf{C}$. ◀

► **Axiom STC-4.** *For each $l \in \mathcal{P}$ we have $\mathcal{A}^{\mathbf{C}}.\langle l \rangle \leq \mathbf{B} \bullet \mathcal{A}^T.\langle l \rangle$.*

Proof. Working externally, we note that $\mathcal{A}^{\mathbf{C}}.\langle l \rangle = j_* \text{inr}_* p_{\mathbf{C}}^* y_{\mathcal{P}} l$ and $\mathcal{A}^T.\langle l \rangle = j_* \text{inl}_* p_{\mathbf{T}}^* y_{\mathcal{P}} l$; to establish our goal, it suffices to check that $i^* j_* \text{inr}_* p_{\mathbf{C}}^* y_{\mathcal{P}} l \leq i^* j_* \text{inr}_* p_{\mathbf{T}}^* y_{\mathcal{P}} l$. We compute the left-hand side using Computation 91 and Lemma 92:

$$i^* j_* \text{inr}_* p_{\mathbf{C}}^* y_{\mathcal{P}} l \cong (p_{\mathbf{B}})_* \text{inr}_* p_{\mathbf{C}}^* y_{\mathcal{P}} l \cong (p_{\mathbf{C}})_* p_{\mathbf{C}}^* y_{\mathcal{P}} l \cong (p_{\mathbf{C}})_* y_{\mathbf{C}} y_{\mathcal{P}} l \cong |y_{\mathcal{P}} l| \cong y_{\mathcal{P}} l \quad \blacktriangleleft$$