

PEVA Cheat Sheet 1

Lecture 1 General

Amdahl's law: $T(n) = (1 - \alpha) \cdot t + \alpha \cdot \frac{t}{n}$, $\lim_{n \rightarrow \infty} S(n) = \frac{1}{1 - \alpha}$

Kendall notation for queues:

arrivals | *service* | *servers* | *buffersize* | *population* | *scheduling*

arrival	Distribution of interarrival time
service	Distribution of service time
servers	Number of servers.
buffer size	Maximum number of customers in queueing station (including servers).
population	Number of customers in and outside the queueing station.
scheduling	Employed scheduling strategy.

Lecture 2 DTMCs

Limiting distribution: $\underline{v}(P - I) = \underline{0}$ and $\sum_j v_j = 1$

$$(P - I) = \begin{pmatrix} -0.1 & 0.1 \\ 0.4 & -0.4 \end{pmatrix} \Rightarrow \begin{matrix} -0.1v_0 + 0.4v_1 = 0 \\ 0.4v_0 - 0.4v_1 = 0 \end{matrix}$$

recurrent	A state is recurrent if we return to it with probability 1.
transient	A state is transient (or nonrecurrent) if there is a positive probability of not returning to this state.
positive recurrent	A state is positive recurrent (or recurrent non-null) if its mean recurrence time is finite.
null recurrent	A state is null recurrent if its mean recurrence time is infinite.
absorbing	A state i is absorbing if and only if $p_{i,i} = 1$.
period	The period d_i of state i is the greatest common divisor of all the values n for which $p_{i,i}(n) > 0$.
irreducible	A DTMC is called irreducible if every state can be reached from every other state in a finite number of steps. In an irreducible DTMC, all states have the same period.
Markov property	Future evolution (next state) only depends on the current state, not on the past history!
time-homogeneous	DTMCs are time-homogeneous: the matrix P does not change over time.

Lecture 3 CTMCs

For each state i , introduce rate q_i for an exponentially distributed residence time; mean residence time is $1/q_i$.

Compute steady-state of CTMC by: **1.** GBEs or "fluxin, fluxout", **2.** Determine steady-state of embedded DTMC and renormalize \underline{v} with $p_i = \left(\frac{v_i}{q_i}\right) \div \left(\sum_j \frac{v_j}{q_j}\right)$, **3.** Generator Matrix **Q** with

$q_{i,j} = q_i \cdot p_{i,j}$, $i \neq j$ and $-q_i$, $i = j$. With **Q**, solve $\underline{p} \cdot \mathbf{Q} = \underline{0}$.

Lecture 4 M|M|1 queues

FCFS	First come, first served.
RR	Round robin.
PS	Processor sharing.
SJN	Shortest job next.
LCFS	Last come, first served.
IS	Infinite server.
PRIQ	priority scheduling.
PASTA	Poisson Arrivals See Time Averages.

What is the expected number of jobs in the system in steady state?

1. compute steady-state probabilities using GBEs or something else,
2. use steady-state probabilities to compute expectation.

$$E[N] = \sum_{i=0}^n i \cdot p_i.$$

If server is infinite (like with a M|M|1 queue), the expected number of customer:

$$\text{In system, } E[N] = \frac{\rho}{1 - \rho}.$$

$$\text{In queue, } E[N_q] = \frac{\rho^2}{1 - \rho}.$$

$$\text{In server, } E[N_s] = \rho.$$

Little's Law helps to go from **system-oriented** measures to **user-oriented** measures.

Little's law for:

Full station	$E[N] = \lambda \cdot E[R]$, with $E[R]$ as the expected response time, average time each customer spends in system.
Queue only	$E[N_q] = \lambda \cdot E[W]$, with $E[W]$ as average waiting time.

Server only $\rho = \lambda \cdot E[S]$, with $E[S]$ as average service time.

For finite stations with one server:

Little's law	$E[N] = X \cdot E[R]$, with $X = \mu$ if overloaded
Loss prob $p_{\{m\}}$	probability that an arriving job has to leave because the buffer is full (PASTA)

Throughput $X = \lambda \cdot (1 - p_m) = \mu \cdot (1 - p_0)$, number of jobs served per time unit

$$\text{Utilization } X \cdot E[S] = 1 - p_0$$

For infinite stations with m server:

Utilization For each individual server $\rho = \frac{\lambda}{m \cdot \mu}$

Expected busy servers Number of busy servers $m \cdot \rho = \frac{\lambda}{\mu}$

Lecture 5 Simulation

Monte Carlo method: X_i and Y_i random variables, uniform on $[0,1]$ = random points in a unit square. Define J_i if $Y_i \leq X_i^2$ then 1 else 0. Estimate $\tilde{A} = \frac{1}{N} \sum_{i=1}^N J_i \approx \int_0^1 x^2 dx$.

The estimate \tilde{a} is a realization of the random variable \tilde{A} . Random variable \tilde{A} is called an estimator of a . \tilde{A} should be unbiased, so $E[\tilde{A}] = a$ and \tilde{A} should be consistent, so $\lim_{n \rightarrow \infty} var[\tilde{A}] = 0$

Different ways to classify simulation methods: **1.** Stochastic vs. deterministic: usage of random numbers, **2.** Discrete-event vs. continuous-event, **3.** Steady-state vs. transient and **4.** Time-based vs. event-based.

Time-based simulation: Also called synchronous simulation, Time proceeds in steps of size Δt , In each iteration all events are processed that happen in the interval $[t, t + \Delta t]$, System state is changed accordingly, Assumption: ordering of events in an interval is not important, events are independent and Δt has to be small.

Event-based simulation: Also called asynchronous simulation, Time 'jumps' from event to event, In each iteration: determine the

next event, set simulation time to its occurrence time, process the event and generate new events.

User-oriented measure: Estimate of average response time from n jobs = $\tilde{r} = \frac{1}{n} \sum_{i=1}^n (t_i^{(d)} - t_i^{(a)})$, with $t_i^{(a)}$ arrival time of i th job and $t_i^{(d)}$ departure time of i th job.

Mean values: We want to determine an approximation of $E[X]$ of a random variable X (for example, response time). Simulation is used to generate n samples, each of which is a realization of a random variable X_i . All X_i have the same distribution as X . The X_i are (should be) independent of each other. Random variable \tilde{X} is an estimator of $E[X]$: $\tilde{X} = \frac{1}{n} \cdot \sum_{i=1}^n X_i$ and hopefully \tilde{X} is unbiased ($E[\tilde{X}] = E[X]$) and consistent ($\lim_{n \rightarrow \infty} var[\tilde{X}] = 0$).

Confidence intervals: The X_i are (hopefully) independent and identically distributed, with mean $E[X]$ and some (unknown) variance $\sigma^2 = var[X]$. Central limit theorem says: \tilde{X} is approximately normal distributed with mean $E[N]$ and variance σ^2/n since it is the sum of independent variables X_i . σ^2 is not known either, but can be estimated by: $\tilde{\sigma}^2 = \frac{n}{n-1} \cdot \left(\sum_{i=1}^n X_i^2 - n\tilde{X}^2\right)$ which is the mean of the squares minus square of the mean. And

$$var[\tilde{X}] = \frac{\sigma^2}{n} \approx \frac{\tilde{\sigma}^2}{n}.$$

Std deviations Probability that a normally-distributed random variable deviates more than 1.645 standard deviations from the mean, is 10%.

$$\tilde{X} \in [E[X] - 1.645 \cdot \frac{\sigma}{\sqrt{n}}, E[X] + 1.645 \cdot \frac{\sigma}{\sqrt{n}}]$$

Random interval $[\tilde{X} - 1.645 \cdot \frac{\tilde{\sigma}}{\sqrt{n}}, \tilde{X} + 1.645 \cdot \frac{\tilde{\sigma}}{\sqrt{n}}]$ which with (approx.) 90% probability contains the true mean $E[X]$. This is called the 90% confidence interval.

68% Use σ instead of 1.645

95% Use $2 \cdot \sigma$ instead of 1.645

99% Use $3 \cdot \sigma$ instead of 1.645

Lecture 6 M|G|1 queues

$\sigma^2 = var[X] = E[X^2] - E[X]^2$ and normalized variance w.r.t. mean value $C_X^2 = \frac{var[X]}{E[X]^2}$

Deterministic	$X_{Det} \sim Det(d)$, $Pr\{X = d\} = 1$, $E[X_{Det}] = d$, $E[X_{Det}^2] = d^2$, $var[X_{Det}] = 0$ and $C_{X_{Det}}^2 = 0$
Uniform	$X_{Unif} \sim Unif(a, b)$, $E[X_{Unif}] = \frac{a+b}{2}$, $E[X_{Unif}^2] = \frac{b^3 - a^3}{3(b-a)}$, $var[X_{Unif}] = \frac{(b-a)^2}{12}$ and $C_{X_{Unif}}^2 = \frac{(b-a)^2}{3(a+b)^2}$
Exponential	$X_{Exp} \sim Exp(\lambda)$, $E[X_{Exp}] = 1/\lambda$, $E[X_{Exp}^2] = 2/\lambda^2$, $var[X_{Exp}] = 1/\lambda^2$ and $C_{X_{Exp}}^2 = 1$

Expected residual time: $E[R]$ of an arriving job: **1.**

$E[R_1] = \rho \cdot \frac{E[S^2]}{2E[S]}$ expected residual service time of the job in service (if any), **2.** $E[R_2] = E[N_q] \cdot E[S]$ expected service time of the job(s) already waiting and **3.** $E[R_3] = E[S]$ expected service time of the arriving job itself. Then $E[R] = E[S] + \frac{\lambda \cdot E[S^2]}{2(1-\rho)}$,

$$E[N] = \lambda E[R] = \lambda E[S] + \frac{\lambda^2 E[S^2]}{2(1-\rho)}, E[W] = \frac{\lambda E[S^2]}{2(1-\rho)} \text{ and } E[N_q] = \frac{\lambda^2 E[S^2]}{2(1-\rho)}$$