

PEVA Cheat Sheet 2

Lecture 6 M|G|1 queues

Comparison of different queues	
M M 1	$E[W_M] = \frac{\rho E[S]}{1-\rho}$
M D 1	$E[W_D] = \frac{1}{2} E[W_M]$
M E_2 1	$E[W_{E_2}] = \frac{3}{4} E[W_M]$
Batch arrivals: $E[H] = \sum_{k=0}^{\infty} k \cdot h_k$	
Arrival	Poisson arrival process with rate λ .
Packets	An arrival consists of H mini-packets.
E[H]	$E[H] = \sum_{k=0}^{\infty} k \cdot h_k$
2nd factorial moment	$E[H^2] - E[H]^2 = E[H^2 - H] = E[H(H-1)] = \sum_{k=0}^{\infty} k \cdot (k-1) \cdot h_k$
Service time	service time distribution for a mini-packet with $E[S]$ and $E[S^2]$
Utilisation	$\rho = \lambda E[H] \cdot E[S]$
E[W]	$E[W] = \frac{\lambda E[H] \cdot E[S^2]}{2(1-\rho)} + \frac{E[H(H-1)]E[S]}{2E[H](1-\rho)}$
E[W_unsplit]	$E[W_{unsplit}] = \frac{\lambda E[S^2]}{2(1-\rho)} = \frac{\lambda 2 \cdot E[S]^2}{2(1-\rho)}$
Priority PRIO: With PRIO every class has an λ_r and $E[S_r]$.	
Non-Preemptive	a job in service is always first finished before a new job is put into service
remaining service time	$E[T_P] = \frac{1}{2} \sum_{k=1}^P \lambda_k E[S_k^2]$
Cobham	$E[W_r] = \frac{E[T_P]}{(1-\sigma_r)(1-\sigma_{r-1})}$, with $\sigma_0 = 0$
Sigma r	$\sigma_r = \sum_{k=1}^r \rho_k$
Conservation law	$\rho E[W] = \sum_{r=1}^P \rho_r E[W_r]$
Preemptive	a job in service is stopped being served as soon as a higher priority job arrives.
Scheduling	preemptive resume (PRS), preemptive repeat identical (PRI) and preemptive repeat different (PRD).
For class 1 job	$E[W_1] = \frac{\lambda_1 E[S_1^2]}{2(1-\sigma_r)}$
E[W_r]	$E[W_r] = \frac{\sigma_{r-1} E[S_r]}{1-\sigma_{r-1}} + \frac{E[T_r]}{(1-\sigma_r)(1-\sigma_{r-1})}$
Shorest Job next (SJN):	
Utilisation	$\rho_t = \lambda_t t = \lambda s f_s(s) dt \cdot t$
Cumulative utilisation	$\beta_t = \int_0^t \lambda s f_s(s) ds$
Total utilisation	$\beta_{\infty} = \lambda \int_0^{\infty} s f_s(s) ds = \lambda E[S] = \rho$
Remaining service time	$E[T_{\infty}] = \frac{1}{2} \lambda E[S^2]$
Waiting time	$E[W_t] = \frac{\lambda E[S^2]}{2(1-\beta_t)^2}$
Waiting time	$E[W_{SJN}] = \int_0^{\infty} E[W_t] f_s(t) dt$
Round Robin (RR): an arriving job needs an integer number of service rounds, each of constant length Q (quantum). The number of quanta G an arriving job needs follows a (discrete) geometric distribution. For an arriving job consisting of 1 quantum only, the mean response time is simple: $E[R_{G=1}] = \rho \frac{Q}{2} + E[N_q]Q + Q$	
RR to the limit (processor Sharing (PS)) $Q \rightarrow 0$: with response time $E[R_{PS}] = \frac{t}{1-\rho}$, stretch factor $(1-\rho)^{-1}$ and virtual average waiting time for jobs of length t $E[W_{PS}(t)] = E[R_{PS}] - t = \frac{\rho t}{1-\rho}$. Unconditional virtual average waiting time $E[W_{PS}] = \frac{\rho E[S]}{1-\rho}$	

Lecture 7 Open queueing networks

Open queueing networks: Infinite source/sink (node 0), arrivals from source, departures to sink, number of customers in QN is not constant, stability if number of arrivals equals number of departures.

Feed-Forward QNs: Nodes can be ordered in such a way that whenever customers flow from node i to node j , we have $i < j$. Acyclic customer/packet/job flow. FF QN must be open.

Burke's Theorem: The departure process from a stable single server M|M|1 queue with arrival rate λ and service rate μ is a Poisson process with rate λ .

Consequences of Burke's Theorem: $E[N_i] = \frac{\rho_i}{1-\rho_i}$ with $\rho_i = \frac{\lambda_i}{\mu_i}$.

Steady-state probs: $Pr\{\underline{N} = \underline{n}\} = \frac{1}{G} \prod_{i=1}^M \rho_i^{n_i}$ with

$$G = (\prod_{i=1}^M (1 - \rho_i))^{-1}.$$

Feedback Jackson QNs: Steady-state probs

$Pr\{\underline{N} = \underline{n}\} = \frac{1}{G} \prod_{i=1}^M \rho_i^{n_i}$ with $G = (\prod_{i=1}^M (1 - \rho_i))^{-1}$. $E[R]$?

Calculate $E[N]$ which is the sum of $E[N_i] = \frac{\rho_i}{1-\rho_i}$ for all queues and

use Little's Law $E[R] = \frac{E[N]}{\lambda}$

Lecture 8 Closed queueing networks

Gordon Newell QNs: always stable.

Visit count $V_j = \sum_{i=1}^M V_i r_{i,j}$, Solavble if one of the Vs is given
Service Demand $D_i = V_i \cdot E[S_i]$. Highest service demand is bottleneck

If we increase K, then $X(K) \rightarrow \frac{1}{D_b}$ and $\rho_b \rightarrow 1$. For non-bottleneck

stations, $\rho_i \rightarrow \frac{D_i}{D_b} < 1$

Gordon And Newell product-form: $I(M, K)$ is every possible combination of $\underline{n} = (n_1, n_2, \dots, n_M)$. $P(\underline{N} = \underline{n}) = \frac{1}{G(M, K)} \prod_{i=1}^M D_i^{n_i}$ with normalisation constant $G(M, K) = \sum_{n \in I(M, K)} \prod_{i=1}^M D_i^{n_i}$. Size of $I(M, K) = \binom{M+K-1}{M-1}$

MVA:

$E[\hat{R}_i(k)]$	$(E[N_i(k-1)] + 1)D_i$
$E[\hat{R}_i(k)]$	$\sum E[\hat{R}_i(K)]$
$X(K)$	$\frac{K}{E[\hat{R}(k)]}$
$E[N_i(k)]$	$X(k)E[\hat{R}_i(k)]$
For IS node	$E[R_i(K)] = E[S_i]$ and $E[\hat{R}_i(K)] = D_i$

k	0	1	2	3
$E[\hat{R}_1(k)]$	—			
$E[\hat{R}_2(k)]$	—			
$E[\hat{R}_3(k)]$	—			
$X(k)$	—			
$E[N_1(k)]$	0			
$E[N_2(k)]$	0			
$E[N_3(k)]$	0			

Asumptotic bounds:

X(K) upperbound $X(K) \leq \min\{\frac{K}{E[Z]+D_{\Sigma}}, \frac{1}{D_+}\}$

E[R] lower bound $E[\hat{R}(K)] \geq \max\{E[Z] + D_{\Sigma}, KD_+\}$

Saturation point $K^* = \frac{D_{\Sigma} + E[Z]}{D_+}$. The integer part of K^* can be interpreted as the maximum number of customers that could be accommodated without any queueing when all the service times are of deterministic length. Stated differently, if the number of customers is larger than K^* we are sure that queueing effects in the network contribute to the response times.

Bard-Schweiter Approximation: MVA with $E[\hat{R}_i(K)] \approx (\frac{K-1}{K} E[N_i(K)] + 1)D_i$ and as a first guess

$$E[N_i](k) = \frac{K}{M}.$$

Balanced Queueing Networks: Assume all station have the same service demands and $E[N_i(K)] = \frac{K}{M}$.

$E[\hat{R}_i(K)]$	$\frac{D(K+M-1)}{M}$
$E[\hat{R}_i(K)]$	$D(K+M-1)$
$X(K)$	$\frac{K}{D(K+M-1)}$
$\rho_i(K)$	$\frac{K}{K+M-1}$
Simple bounds	

$$\frac{K}{D_+(K+M-1)} \leq X(K) \leq \frac{K}{D_-(K+M-1)}$$

$$D_-(K+M-1) \leq E[\hat{R}(K)] \leq D_+(K+M-1)$$

Tighter bounds

Performance best if $D = \bar{D} = \frac{D_{\Sigma}}{M_K}$

Throughput $X(K) \leq \min\{\frac{K}{D(K+M-1)}, \frac{1}{D_+}\}$

$$\max\{\bar{D}(K+M-1), KD_+\} \leq E[\hat{R}(K)] \leq D_+(K-1) + D_{\Sigma}$$

Lecture 9 Random number generation

Inversion method: Given some $F_y(y)$, solve $z = F_y(y)$. Here $Z \sim U(0, 1)$.

If density function $f_y(y)$ is given, calculate $F_y(y)$. For example, if $f_y(y) = \frac{2}{3}x^2$ for $-1 \leq x \leq 1$ and zero for everything else, then

$$F_y(y) = \int_{-\infty}^x f_y(u) du = \int_{-1}^x \frac{2}{3}u^2 du = [\frac{1}{2}x^3]_{-1}^x = \frac{1}{2}x^3 + \frac{1}{2}.$$

Rejection Method: Given the desity function with finite domain $[a, b]$ and finite image $[0, c]$, you can get approximate random numbers. Generate two uniformly distributed numbers u_1 and u_2 on $[0, 1]$. Derive random numbers $x = a + (b-a)u_1$ and $y = cu_2$. If $(y < f_X(x))$, then accept x , else generate new u_1 and u_2 . This is fairly efficient when area under $f_X(x)$ is close to $c(b-a)$.

Erreta

Lecture 2: Limiting distribution of DTMC exists if it is irreducible, aperiodic and has positive recurrent states.

Lecture 5: CI interval 90%: 1.645, 95%: 1.96, 99%: 2.575.