Lecture Notes week 3

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Contents

1 Lecture 1

$k^{-\gamma-1}$ *or* $k^{-\tau}$, *with* $\tau = \gamma + 1$ *and* $\gamma \in (1, 3)$
$P_k \approx const \cdot k^{-\gamma-1}$
$\bar{F}_k = P(x \geq k) = \sum_{s=k}^{\infty} P_s \approx c \cdot k^{-\gamma}$

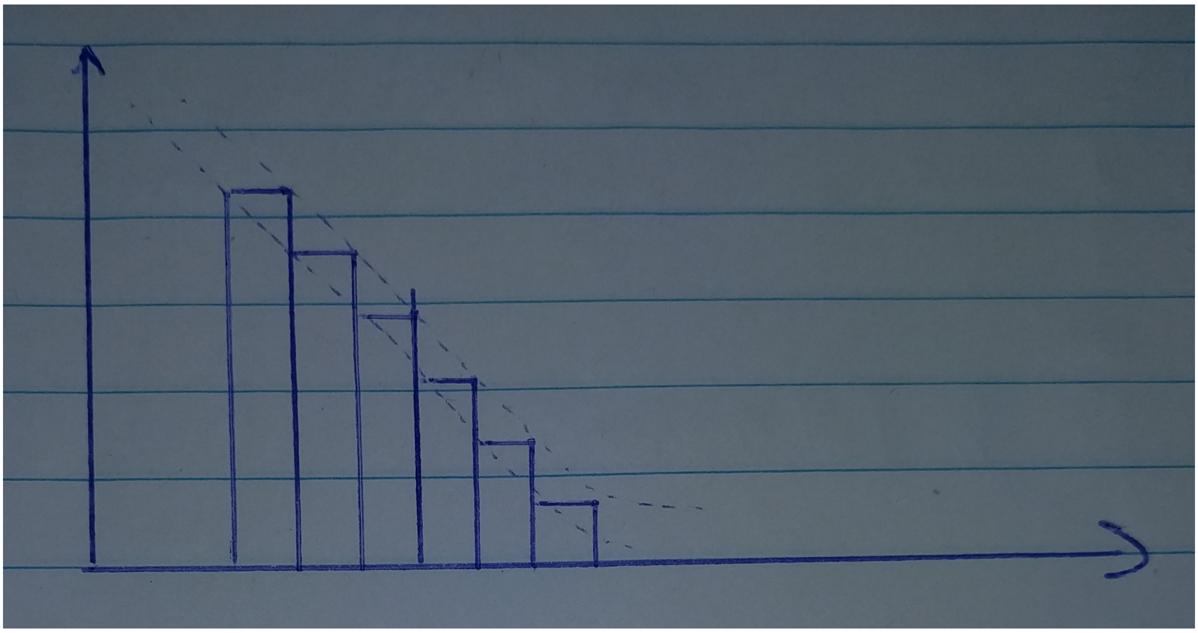


Figure 1: Visual representation of the formula on slide 4.

$EX^2 = \sum_{k=k_0}^{\infty} k^2 \cdot c \cdot k^{-\gamma-1} = \sum_{k=k_0}^{\infty} c \cdot k^{1-\gamma} < \infty \iff \gamma > 2$

$X_1, X_2, X_3 \ldots\ldots$ - Independent and identically distributed random variable.
LLN: $\frac{X_1+X_2+\ldots\ldots X_n}{n} \overrightarrow{a.s.} EX$ *(the mean)*

$$\sum_{i=1}^{n} X_i^P \approx n^{\frac{P}{j}}$$

1.1 Preferential Attachment (PA)

A new node connects to an existing node with probability proportional to the degree of the existing node, (the idea of of "the rich get richer"). The degree of a new node m, we coinsider $m = 1$.

New node connects to an existing node with probability proportional to the degree $m = \#$ of links of a new node.

Definition 1. *$P_{k,t}$ is the fraction of nodes with degree k at time t (after node t arrived).*

$$\sum_{k=m}^{\infty} k \cdot P_{k,t} = \frac{1}{t} \cdot \sum_{k=m}^{\infty} k \cdot [\# \; of \; nodes \; with \; degree \; k] = \frac{1}{t} \cdot [total \; degree] = \frac{2 \cdot m \cdot t}{t} = 2 \cdot m$$

The probability that a new node connects to a new node with degree k is:

$$\frac{k\cdot[\#\ of\ \ nodes\ \ with\ \ degree\ \ k]}{\sum\limits_{l=m}^{\infty} l\cdot[\#\ \ of\ \ nodes\ \ with\ \ degree\ \ k]} \overset{(divide\ \ by\ \ t)}{=} \frac{k\cdot P_{k,t}}{\sum\limits_{l=m}^{\infty} l\cdot P_{k,t}} = \frac{k\cdot P_{k,t}}{2\cdot m}$$

Change in number of nodes with degree k at $t+1$:

$(t+1)\cdot P_{k,t+1} - t\cdot P_{k,t} = \frac{(k-1)\cdot P_{k-1,t}}{2\cdot m}\cdot m - \frac{k\cdot P_{k,t}}{2\cdot m}\cdot m,\ with\ k > m$

$(t+1)\cdot P_{m,t+1} - t\cdot P_{m,t} = 1 - \frac{m\cdot P_{m,t}}{2\cdot m}\cdot m$

The questions we have to solve are:

Solve for $t \to \infty$

$P_k = \frac{1}{2}\cdot(k-1)\cdot P_{k-1} - \frac{1}{2}\cdot k\cdot P_k,\ with\ k > m$

$P_m = 1 - \frac{1}{2}\cdot m\cdot P_m$

Solutions:

$P_m \cdot (\frac{1}{2}m+1) = 1 \implies P_m = \frac{2}{m+2}$

$P_k = \frac{(k-1)(k-2)(k-3)......\cdot m}{(k+2)(k+1)......\cdot m+3} = \frac{2\cdot m\cdot(m+1)}{k\cdot(k+1)\cdot(k+2)} \approx c\cdot k^{-3}$

Zipf's Law is for self-study.

1.2 Maximal degree

$D_1, D_2, D_3, \ldots\ldots D_n$ - Degrees

$P(D > x) \approx c\cdot x^{-\gamma}$

Maximal degree $\implies$ Largest value $\implies$ probability is $\frac{1}{n}$

$P(D > d_{max}) \approx \frac{1}{n} \approx c\cdot(d_{max})^{-\gamma}$

$d_{max} = c'\cdot n^{\frac{1}{\gamma}}$

j-th largest value $d^{(j)}$

$P(D > d^{(j)}) \approx \frac{j}{n} \approx c\cdot(d^{(j)})^{-\gamma}$

$d^{(j)} \approx c" \cdot n^{\frac{1}{\gamma}}\cdot j^{-\frac{1}{\gamma}}$

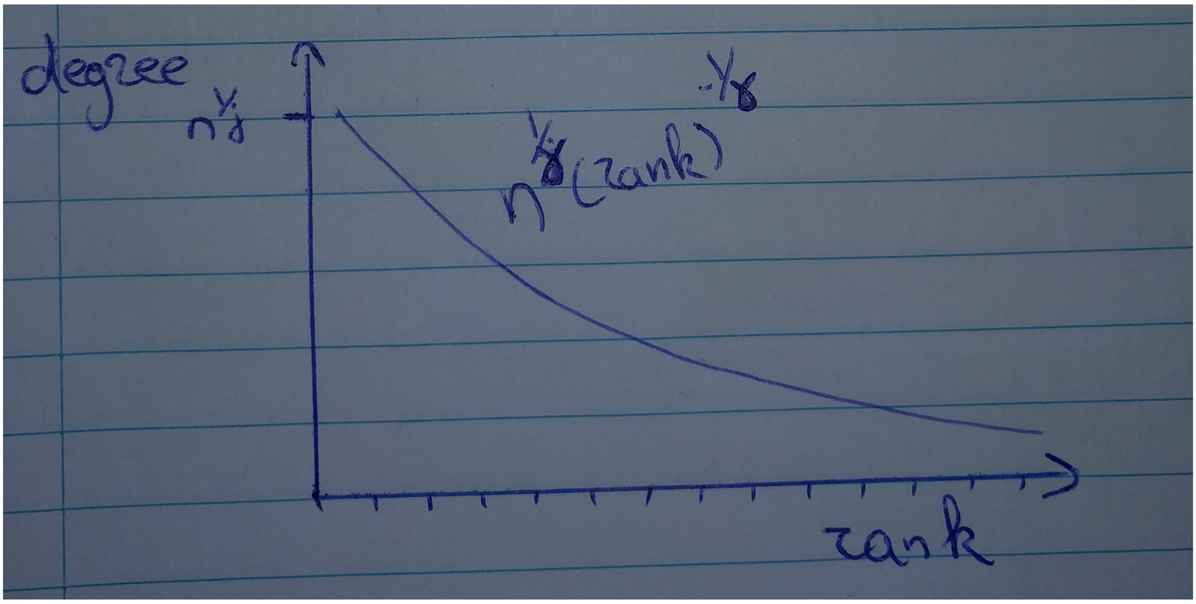
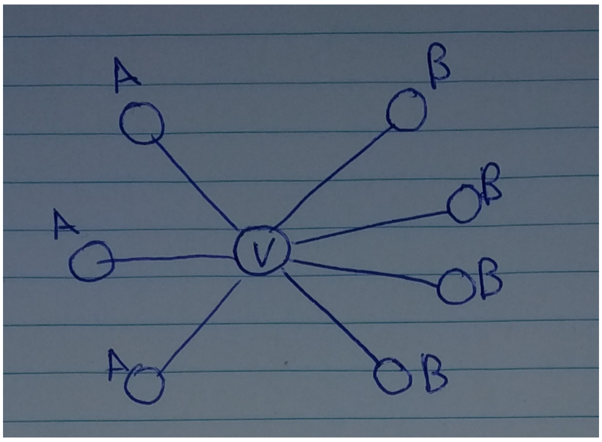


Figure 2: Nice picture with text.

2 Lecture 2 Innovation diffusion through a network



		w	
		A	B
v	A	(a, a)	(0, 0)
	B	(0, 0)	(b, b)

Definition 2. *Fraction p of the neighbours of v adapted A.*
Fraction 1-p of the neighbours of v adapted B.

Adapting A: reward dpa
Adapting B: reqard d(1-p)b

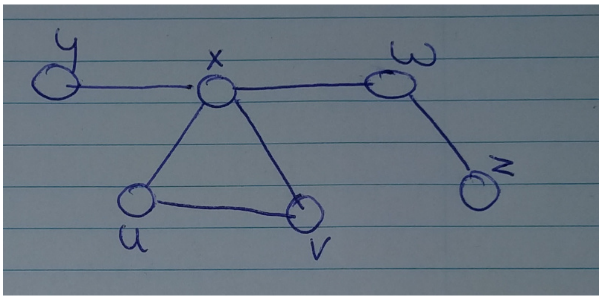
Adapt A if $dpa > d(1-p)b \implies pa > (1-p)b \implies p(a+b) > b$
We can rewrite this as (this is important):

- $p > \frac{b}{a+b} \implies$ adapt A
- $p < \frac{b}{a+b} \implies$ adapt B
- $p = \frac{b}{a+b} \implies$ adapt A (This is something we decided)

Example: if $a = b \implies \frac{b}{a+b} = \frac{1}{2}$.

- Step 0: all B
- Step 1: $u \to A$, $v \to A$
- Step 2: $x \to A$

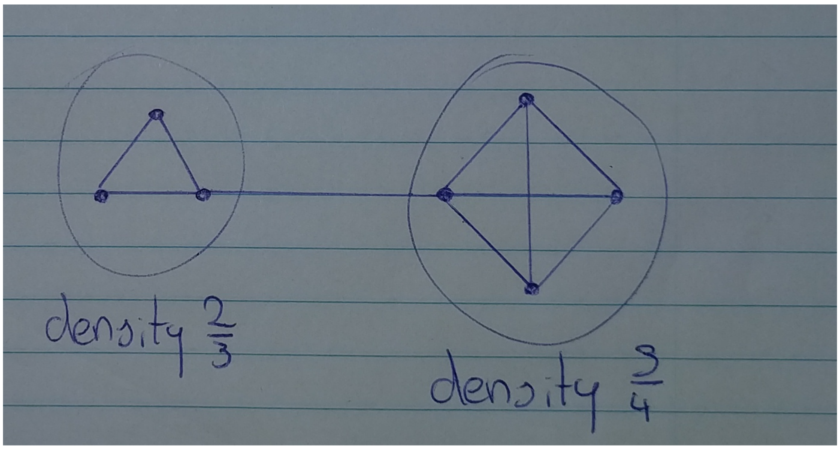
- Step 3: $y \to A$, $w \to A$
- Step 4: $z \to A$



The result is a cascade which results in the whole network being A.
If we were to add a node a to the neighbourhood of x, the cascading effect stops.

Example Clusters

Definition 3. *Cluster of density 1-q is a set of nodes, each of which has at least 1-q fraction of their links in the cluster.*



Claim: Denote $q = \frac{b}{a+b}$. Consider a set of inital adapters of A. The cascade is not complete $\iff$ network contains a cluster of density $> 1 - q$ (without the initial adapters).

Cascading behaviour: $[fraction \geq q]$ adapted A $\implies$ the node adapts A.

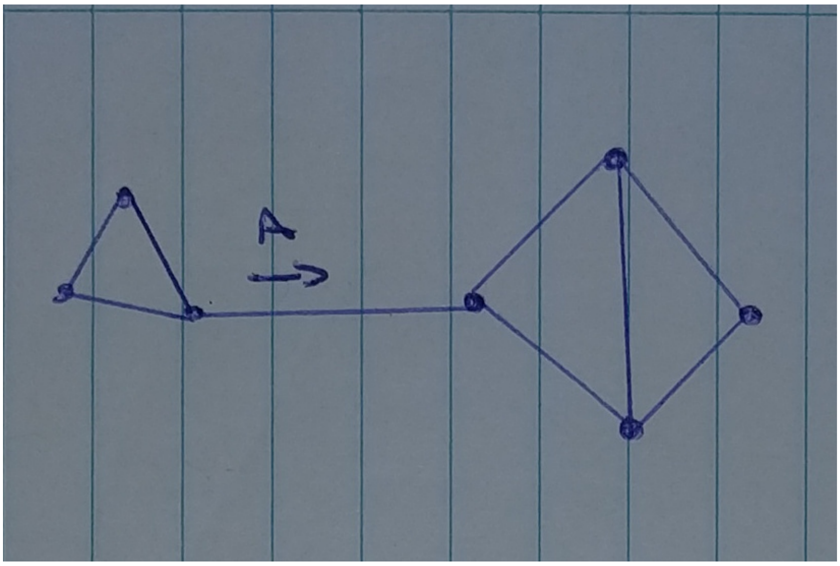
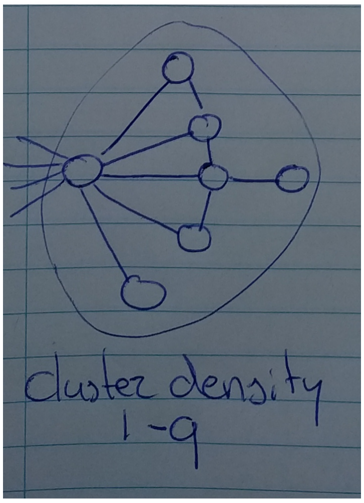


Figure 3: For this to happen, $q \leq \frac{1}{3}$. Think about q as a threshold. The lower the threshold, the easier to go over it

Example Hubs:



2.1 Knowledge and collective action (chapter 19.6)

I act only if at least k peiple act. I know only what my neighbours do. Solution: common knowledge.

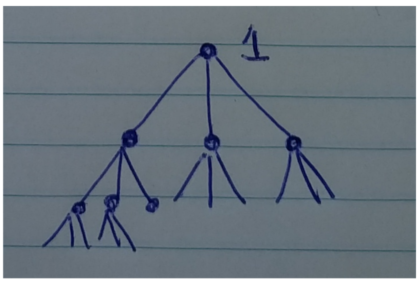
How to choose inital adapters? Influence maximization 2003 Competitive technologies. (dont know what is means, but they are in my notes, so I kept them in)

2.2 Small World Phenomenon SWP

U_1, U_2 - 2 random nodes.
$d(U_1, U_2)$ - graph distance between U_1 and U_2.
$d(U_1, U_2) < \infty$ (the path exists).

The SWP states $d(U_1, U_2) = O(logn)$ with n = number of nodes. Let $h_n = d(U_1, U_2)$
$\exists\ c > 0, P(H_n > c \cdot log(n)) = O(1),\ n \to \infty$.

Explanation: regular tree with number of offspring d.



Number of nodes at distance 1: d
Number of nodes at distance 2: d^2
$\vdots$
Number of nodes at distance k: $d^k \leftarrow lastgeneration$

$1 + d + d^2 + \cdots + d^k = n$
$\frac{d^{k+1}-1}{d-1} \approx c \cdot d^k = n$ where $k = log(n) - c$