

Lecture Notes week 3

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1 Lecture 1

$k^{-\gamma-1}$ or $k^{-\tau}$, with $\tau = \gamma + 1$ and $\gamma \in (1, 3)$

$P_k \approx \text{const} \cdot k^{-\gamma-1}$

$$\bar{F}_k = P(x \geq k) = \sum_{s=k}^{\infty} P_s \approx c \cdot k^{-\gamma}$$

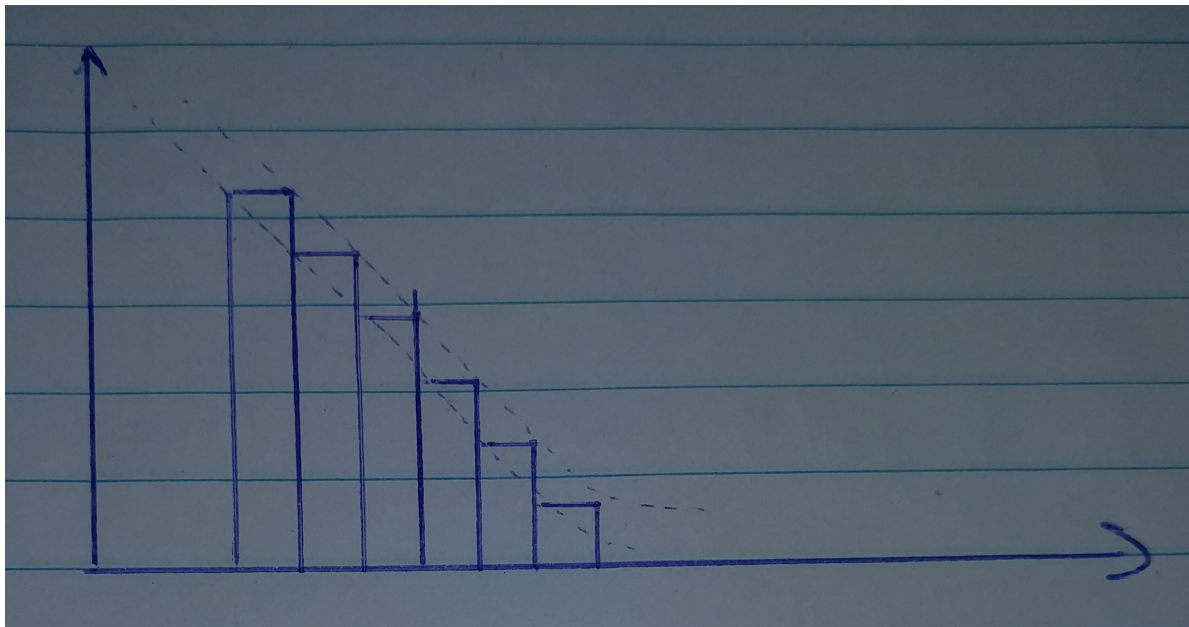


Figure 1: Visual representation of the formula on slide 4.

$$EX^2 = \sum_{k=k_0}^{\infty} k^2 \cdot c \cdot k^{-\gamma-1} = \sum_{k=k_0}^{\infty} c \cdot k^{1-\gamma} < \infty \iff \gamma > 2$$

X_1, X_2, X_3, \dots - Independent and identically distributed random variable.

LLN: $\frac{X_1 + X_2 + \dots + X_n}{n} \xrightarrow{a.s.} EX$ (the mean)

$$\sum_{i=1}^n X_i^P \approx n^{\frac{P}{j}}$$

1.1 Preferential Attachment (PA)

A new node connects to an existing node with probability proportional to the degree of the existing node, (the idea of "the rich get richer"). The degree of a new node m , we consider $m = 1$.

New node connects to an existing node with probability proportional to the degree $m = \#$ of links of a new node.

Definition 1. $P_{k,t}$ is the fraction of nodes with degree k at time t (after node t arrived).

$$\sum_{k=m}^{\infty} k \cdot P_{k,t} = \frac{1}{t} \cdot \sum_{k=m}^{\infty} k \cdot [\# \text{ of nodes with degree } k] = \frac{1}{t} \cdot [\text{total degree}] = \frac{2 \cdot m \cdot t}{t} = 2 \cdot m$$

The probability that a new node connects to a new node with degree k is:

$$\frac{k \cdot [\# \text{ of nodes with degree } k]}{\sum_{l=m}^{\infty} l \cdot [\# \text{ of nodes with degree } l]} \stackrel{(\text{divide by } t)}{=} \frac{k \cdot P_{k,t}}{\sum_{l=m}^{\infty} l \cdot P_{l,t}} = \frac{k \cdot P_{k,t}}{2 \cdot m}$$

Change in number of nodes with degree k at $t + 1$:

$$(t+1) \cdot P_{k,t+1} - t \cdot P_{k,t} = \frac{(k-1) \cdot P_{k-1,t}}{2 \cdot m} \cdot m - \frac{k \cdot P_{k,t}}{2 \cdot m} \cdot m, \text{ with } k > m$$

$$(t+1) \cdot P_{m,t+1} - t \cdot P_{m,t} = 1 - \frac{m \cdot P_{m,t}}{2 \cdot m} \cdot m$$

The questions we have to solve are:

Solve for $t \rightarrow \infty$

$$P_k = \frac{1}{2} \cdot (k-1) \cdot P_{k-1} - \frac{1}{2} \cdot k \cdot P_k, \text{ with } k > m$$

$$P_m = 1 - \frac{1}{2} \cdot m \cdot P_m$$

Solutions:

$$P_m \cdot \left(\frac{1}{2}m + 1\right) = 1 \implies P_m = \frac{2}{m+2}$$

$$P_k = \frac{(k-1)(k-2)(k-3)\dots m}{(k+2)(k+1)\dots m+3} = \frac{2 \cdot m \cdot (m+1)}{k \cdot (k+1) \cdot (k+2)} \approx c \cdot k^{-3}$$

Zipf's Law is for self-study.

1.2 Maximal degree

$D_1, D_2, D_3, \dots, D_n$ - Degrees

$$P(D > x) \approx c \cdot x^{-\gamma}$$

Maximal degree \implies Largest value \implies probability is $\frac{1}{n}$

$$P(D > d_{max}) \approx \frac{1}{n} \approx c \cdot (d_{max})^{-\gamma}$$

$$d_{max} = c' \cdot n^{\frac{1}{\gamma}}$$

j -th largest value $d^{(j)}$

$$P(D > d^{(j)}) \approx \frac{j}{n} \approx c \cdot (d^{(j)})^{-\gamma}$$

$$d^{(j)} \approx c'' \cdot n^{\frac{1}{\gamma}} \cdot j^{-\frac{1}{\gamma}}$$

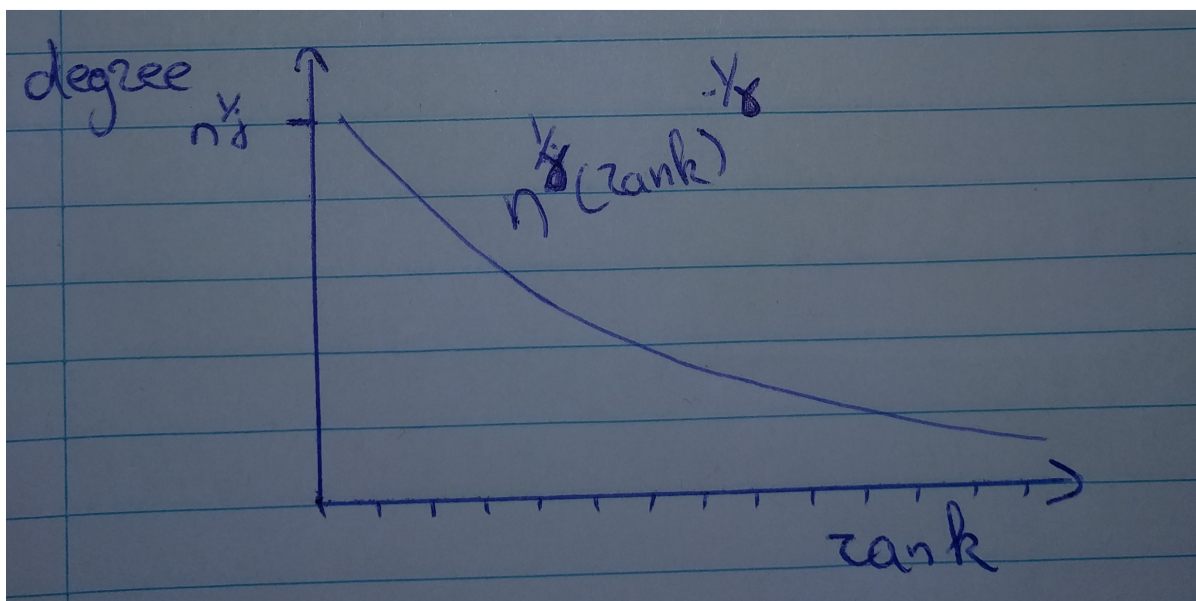
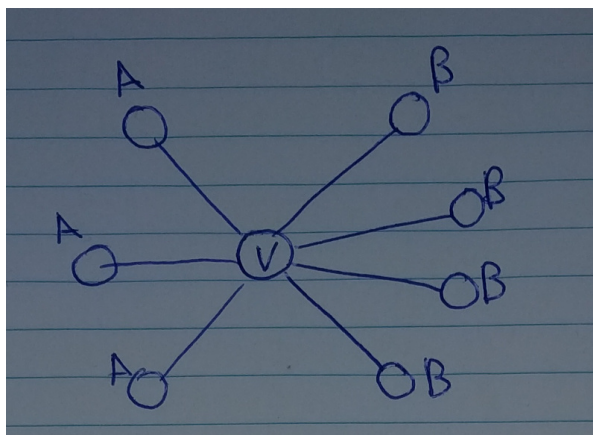


Figure 2: Nice picture with text.

2 Lecture 2 Innovation diffusion through a network



		w	
		A	B
v	A	(a, a)	$(0, 0)$
	B	$(0, 0)$	(b, b)

Definition 2. Fraction p of the neighbours of v adapted A.
 Fraction $1-p$ of the neighbours of v adapted B.

Adapting A: reward dpa
 Adapting B: reward $d(1-p)b$

Adapt A if $dpa > d(1-p)b \implies pa > (1-p)b \implies p(a+b) > b$

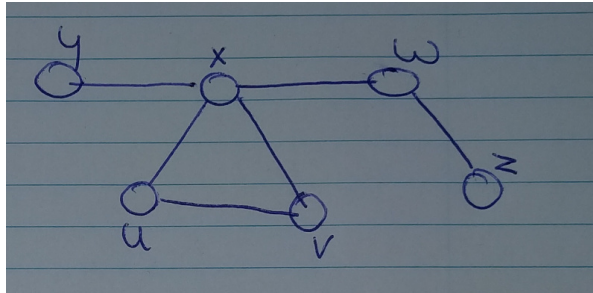
We can rewrite this as (this is important):

- $p > \frac{b}{a+b} \implies$ adapt A
- $p < \frac{b}{a+b} \implies$ adapt B
- $p = \frac{b}{a+b} \implies$ adapt A (This is something we decided)

Example: if $a = b \implies \frac{b}{a+b} = \frac{1}{2}$.

- Step 0: all B
- Step 1: $u \rightarrow A, v \rightarrow A$
- Step 2: $x \rightarrow A$

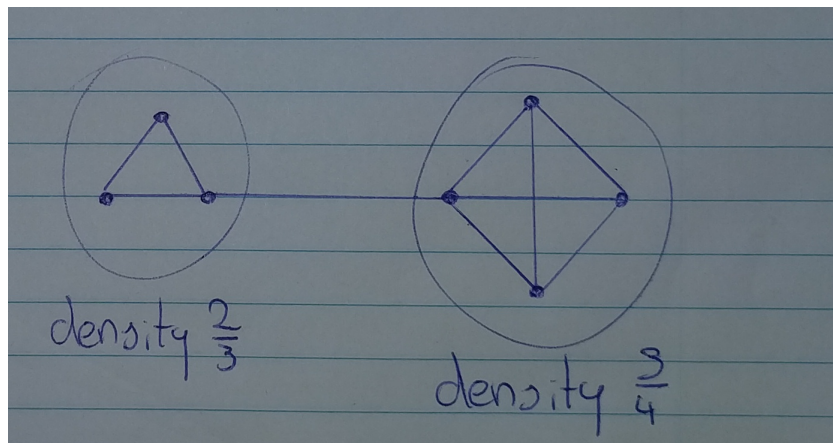
- Step 3: $y \rightarrow A, w \rightarrow A$
- Step 4: $z \rightarrow A$



The result is a cascade which results in the whole network being A.
If we were to add a node a to the neighbourhood of x, the cascading effect stops.

Example Clusters

Definition 3. Cluster of density $1-q$ is a set of nodes, each of which has at least $1-q$ fraction of their links in the cluster.



Claim: Denote $q = \frac{b}{a+b}$. Consider a set of initial adapters of A. The cascade is not complete \iff network contains a cluster of density $> 1 - q$ (without the initial adapters).

Cascading behaviour: $[fraction \geq q]$ adapted A \implies the node adapts A.

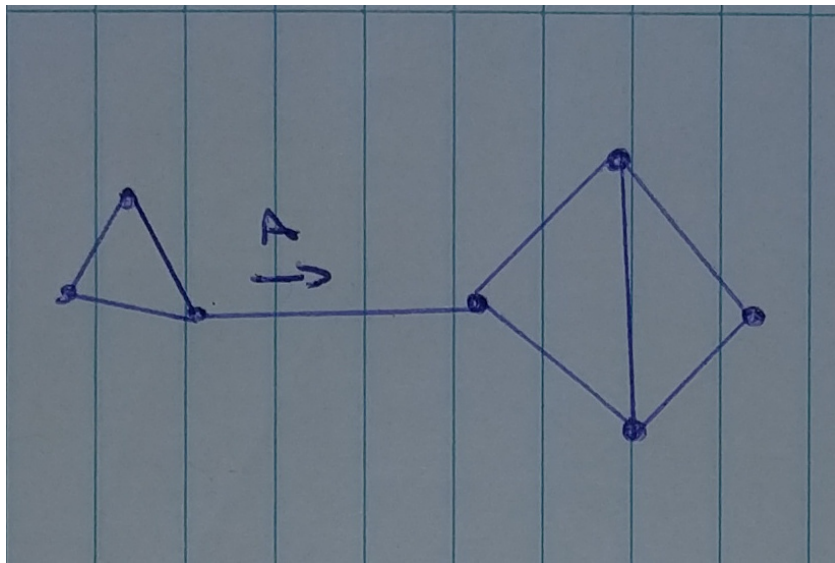
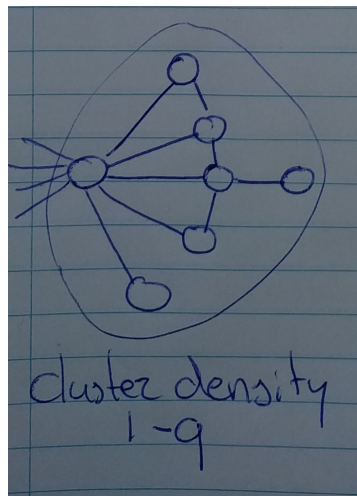


Figure 3: For this to happen, $q \leq \frac{1}{3}$. Think about q as a threshold. The lower the threshold, the easier to go over it

Example Hubs:



2.1 Knowledge and collective action (chapter 19.6)

I act only if at least k people act. I know only what my neighbours do. Solution: common knowledge.

How to choose initial adapters? Influence maximization 2003 Competitive technologies. (don't know what it means, but they are in my notes, so I kept them in)

2.2 Small World Phenomenon SWP

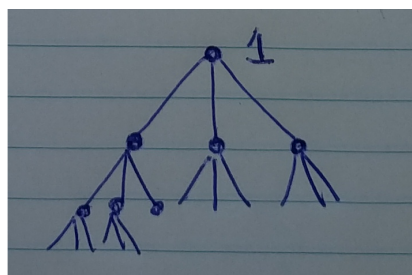
U_1, U_2 - 2 random nodes.

$d(U_1, U_2)$ - graph distance between U_1 and U_2 .

$d(U_1, U_2) < \infty$ (the path exists).

The SWP states $d(U_1, U_2) = O(\log n)$ with n = number of nodes. Let $h_n = d(U_1, U_2)$
 $\exists c > 0, P(H_n > c \cdot \log(n)) = O(1), n \rightarrow \infty$.

Explanation: regular tree with number of offspring d .

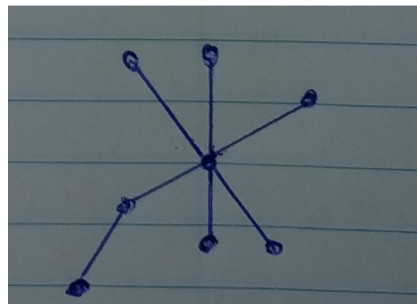


Number of nodes at distance 1: d
 Number of nodes at distance 2: d^2
 \vdots
 Number of nodes at distance k : $d^k \leftarrow \text{last generation}$

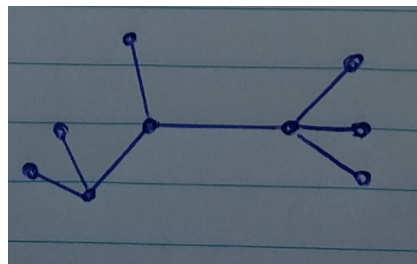
$$1 + d + d^2 + \dots + d^k = n$$

$$\frac{d^{k+1}-1}{d-1} \approx c \cdot d^k = n \text{ where } k = \log(n) - c$$

2.3 The role of Hubs



- Many short routes through the hubs
- Friendship Paradox



$$P(d_w = k) = \frac{k \cdot P_k}{\sum_{l=1}^w l \cdot P_l} = \frac{k \cdot P_l}{\text{avg. Degree}}$$

$P_k = c \cdot k^{-\gamma-1}$ - fraction of nodes with degree k .

$$E(d_w) = \sum_{k=1}^{\infty} k \cdot k \cdot P_k = \frac{1}{\text{avg. Degree}} \cdot \sum_{k=1}^{\infty} k^2 \cdot c \cdot k^{-\gamma-1} = \infty \text{ if } \gamma \in (1, 2)$$

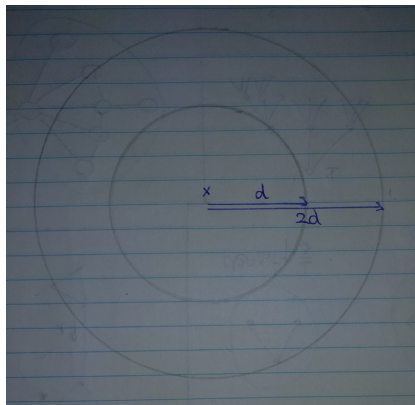
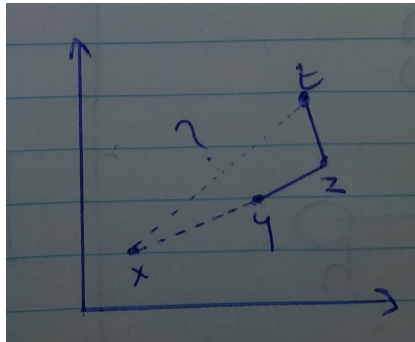
$$d(U_1, U_2) = O(\log \log n)$$

Decentralized-Search
 Migran's experiment

2.4 Spatial Graph

$$x = (x_1, x_2), y = (y_1, y_2)$$

$$P(x \longleftrightarrow y) = (d(x, y))^{-q}$$



The distances are short of $q = 2$, the number of nodes on distance $(d, 2d)$ from x is $\sim d^2$

$$[\pi \cdot (2d)^2 - \pi \cdot d^2] = c \cdot d^2$$

$$q > 2, P(x \longleftrightarrow y) = d^{-q}$$

$$\text{Number of neighbours on distance } (d, 2d) \approx c \cdot d^2 \cdot d^{-q}$$

$q > 2, c \cdot d^{2-q}$. If d increases, d^{2-q} decreases.

2.5 Some additional notes from the tutorial

$$\text{density} = \frac{\# \text{ of neighbours in cluster}}{\# \text{ of neighbours}}$$