

Lecture Notes week 3

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1 Lecture 1

1.1 Bayes' Rule

Given events A and B :

Definition 1. The Conditional Probability $P(A|B) = \frac{P(A \cap B)}{P(B)}$, with $P(B) > 0$.

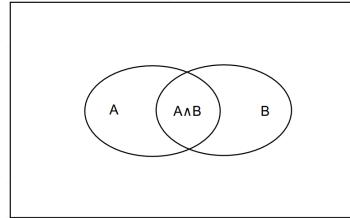


Figure 1: Visual representation of Bayes' Rule

Example: There are 40 Math students of which 15 are girls and there are 50 Computer Science students of which 10 are girls.

Let $A = \text{randomly chosen student is a girl}$ and $B = \text{randomly chosen student is a math student}$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{15}{90}}{\frac{40}{90}} = \frac{3}{8}$$

And from Bayes' Rule follows $\Rightarrow P(A \cap B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A)$, $P(A) > 0$.
Thus we can rewrite it as:

Definition 2. The Conditional Probability $P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)}$.

1.1.1 Full Probability Formula

Definition 3. $P(A) = P(B) \cdot P(A|B) + P(\bar{B}) \cdot P(A|\bar{B})$

$$\text{Example: } P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A)} = \frac{\frac{40}{90} \cdot \frac{3}{8}}{\frac{25}{90}} = \frac{3}{8}$$

1.2 A Herding Experiment

Envelope 1 contains 8 red and 4 blue domino pieces (R) and envelope 2 contains 4 red and 8 blue domino pieces (B).

$$P(R) = P(B) = \frac{1}{2}$$

The first person that draws either a red or blue piece.

$$P(R|(\text{blue})\text{blue}) = \frac{P(\text{blue}|R) \cdot P(R)}{P(\text{blue})}.$$

$$P(\text{blue}) = P(R) \cdot P(\text{blue}|R) + P(B) \cdot P(\text{blue}|B) = \frac{1}{2} \cdot \frac{4}{12} + \frac{1}{2} \cdot \frac{8}{12} = \frac{1}{2}$$

$$P(R|\text{blue}) = \frac{\frac{4}{12} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{1}{3} < \frac{1}{2} \text{ and } P(B|\text{blue}) = \frac{\frac{8}{12} \cdot \frac{1}{2}}{\frac{1}{2}} = \frac{2}{3} > \frac{1}{2}$$

Now lets look at when a second person draws.

Definition 4. $D_i = \{\text{blue}\}$ or $\{\text{red}\}$ – what person i says they saw

Definition 5. $E_i = \{\text{blue}\}$ or $\{\text{red}\}$ – what person i saw

Lets say that the first person says what he sees ($D_1 = E_1$)

$$P(B|\text{blue, blue}) = \frac{P(B) \cdot P(\text{blue, blue}|B)}{P(\text{blue, blue})}$$

$$P(\text{blue, blue}) = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 + \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{5}{18}$$

$$\text{Thus } P(B|\text{blue, blue}) = \frac{\frac{1}{2} \cdot \left(\frac{2}{3}\right)^2}{\frac{5}{18}} = \frac{4}{5} > \frac{1}{2}$$

Conclusion: If $D_1 = \{\text{blue}\}$ and $E_2 = \{\text{blue}\} \implies D_2 = \{\text{blue}\}$

$$P(R) = P(B) = \frac{1}{2}$$

$$P(B|\text{blue, red}) = \frac{P(B) \cdot P(\text{blue, red}|B)}{P(\text{blue, red})} = \frac{\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{2}} = \frac{1}{2}$$

$$D_1 = \{\text{blue}\}, E_2 = \{\text{red}\} \implies D_2 = \{\text{red}\}$$

If your opinions differ (blue, red or red, blue) then you are back at the initial situation. Thus the third person will be equivalent to the first person.

Let $D_1 = D_2 = \{\text{blue}\}$. If $E_3 = \{\text{blue}\} \implies D_3 = \{\text{blue}\}$ If $E_3 = \{\text{red}\}$, then $P(R|\text{blue, blue, red}) = \frac{P(R) \cdot P(\text{blue, blue, red}|R)}{P(\text{blue, blue, red})} = \frac{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}} = \frac{2}{6} = \frac{1}{3} < \frac{1}{2}$. Thus when $E_3 = \{\text{red}\} \implies D_3 = \{\text{blue}\}$. Now the cascade has started, which also means 4th person \equiv 3rd person.

$$P(B \text{ cascade}|R) = \frac{1}{3} \cdot \frac{1}{3} + \left(\frac{1}{3} \cdot \frac{2}{3}\right) \cdot 2 \cdot \frac{1}{3} \cdot \frac{1}{3} + \left(\frac{1}{3} \cdot \frac{2}{3} \cdot 2\right)^2 \cdot \frac{1}{3} \cdot \frac{1}{3} + \dots = \frac{1}{9} \cdot \frac{1}{1-\frac{4}{9}} = \frac{1}{5} \equiv 20\%$$

1.3 General Cascade Model

Example: Adopting a product

1) Let B be bad and G be good. We estimate $P(B) = 1 - p$ and $P(G) = p$.

2) And let v_g be if we guess G right and v_b if we guess B right.

$v_g \cdot p + v_b \cdot (1 - p) = 0$ - Expected Reward. Rewritten it looks like $v_b + (v_g - v_b) \cdot p$.

It is accepted if $P(G|signal) > p$ and rejected if $P(G|signal) < p$.

3) low(L) and high(H)

	B	G
L	q	1-q
H	1-q	q

Table 1: With $q > \frac{1}{2}$

$$P(L|B) = q > \frac{1}{2}, P(H|B) = 1 - q. P(L|G) = 1 - q, P(H|G) = q > \frac{1}{2}.$$

$$\text{First person } P(G|H) = \frac{P(H|G) \cdot P(G)}{P(H)} = \frac{p \cdot q}{p \cdot q + (1-p) \cdot (1-q)} > \frac{p \cdot q}{p \cdot q + (1-p) \cdot q} = \frac{p \cdot q}{q} = p$$

$$\text{Thus } E_1 = \{H\} \implies D_1 = \{G\}$$

Let $a + b$ - person and S - signal a times H and b times L.

$$P(G|S) = \frac{P(S|G) \cdot P(G)}{P(S)} = \frac{p \cdot q^a \cdot (1-q)^b}{p \cdot q^a \cdot (1-q)^b + (1-p) \cdot (1-q)^a \cdot q^b} > \frac{p \cdot q^a \cdot (1-q)^b}{p \cdot q^a \cdot (1-q)^b + (1-p) \cdot q^a \cdot (1-q)^b} = p$$

This is when $a < b \iff (1-q)^a \cdot q^b > q^a \cdot (1-q)^b$ (because $q > 1-q$).

Thus when $a < b$, $D = \{B\}$

If $a > b \iff (1-q)^a \cdot q^b < q^a \cdot (1-q)^b$. This would result in $P(G|S) > p \implies D = \{G\}$

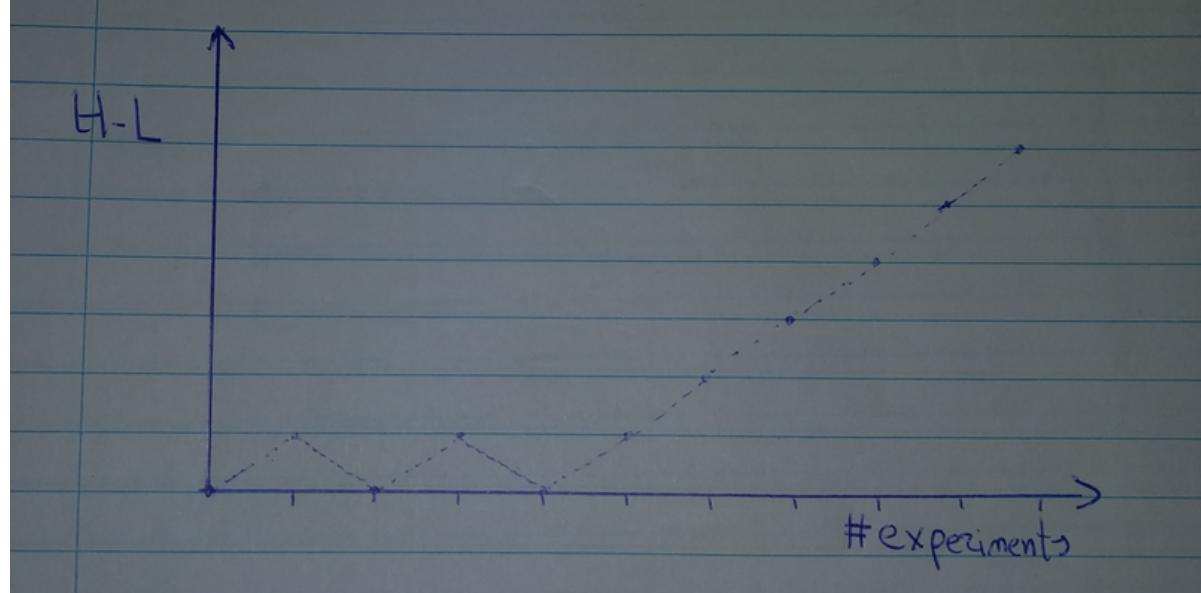


Figure 2: Graph showing cascade with up = H and down = L

2 Lecture 2 Network Effects

Definition 6. *Externality: welfare affected by actions of others without a "contract" (mutual agreement).*

Definition 7. *Interest: reservation price (how much you are willing to pay).*

Definition 8. *Consumers: numbers between 0 and 1.*

When $x < y$, x has a higher reservation price than y .

Definition 9. $r(x)$: *reservation price where $0 < x < 1$.*

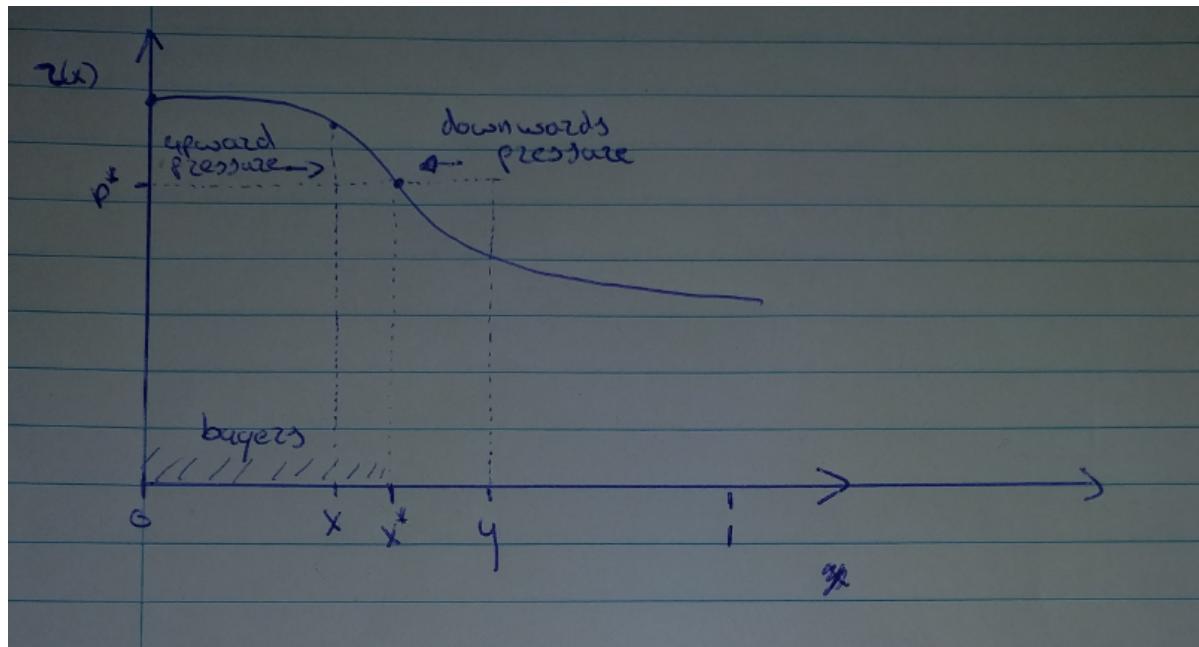


Figure 3: Reservation price

Definition 10. p : *market price. $p \geq r(0)$ means nobody buys it and $p \leq r(1)$ means everybody buys it.*

Definition 11. p^* : *equilibrium market price. x^* is chosen such that $r(x^*) = p^*$ - equilibrium quality.*

$x < x^* \implies r(x) > p^* \implies x$ wants to buy.

$y > x^* \implies r(y) < p^* \implies y$ regrets it (downwards implies regret).

Definition 12. z : *fraction of the population that uses the product*

Definition 13. $r(x)$: *reservation price of $x \in [0, 1]$*

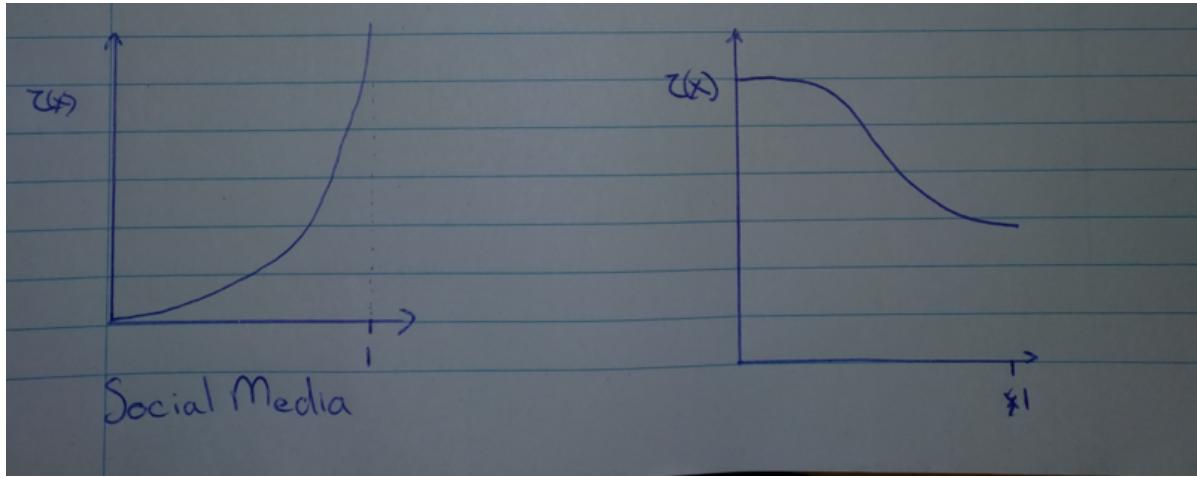


Figure 4: Example for social media

The price x wants to pay $p^* = r(x) \cdot f(z)$ (the "new" reservation price). $f(z)$ is increasing in z .

$f(0) = 0$ (if nobody uses the product) $\Rightarrow p(0) = r(0) \cdot f(0) = 0$

So the customer expects fraction z and buys the product if $r(x) \cdot f(z) \geq p^*$ where p^* is the market price.

2.1 Self-fulfilling expectation

If all customers agree that fraction f will buy the product and behave based on that, then the fraction of people who buys the product will be z .

How it works:

$z = 0 : f(0) = 0 \Rightarrow \forall x \text{ holds } r(x) \cdot f(0) = 0 \Rightarrow z = 0$

$x' > 0$ buys the product ($z > 0$). $r(x') \cdot f(z) > p^*$, $r(x)$ is decreasing fraction.

$x < x' \Rightarrow r(x) > r(x') \Rightarrow r(x) \cdot f(z) > r(x') \cdot f(z) > p^* \Rightarrow x \text{ buys.}$

Customer z has lowest intrinsic value $r(z)$ on $[0, z]$.

If $p^* = f(z) \cdot r(z) \Rightarrow$ self-fulfilling expectation.

Suppose $p^* = f(z) \cdot r(x)$ for some $x \Rightarrow$ fraction x will but \Rightarrow actual value for x is $f(x) \cdot r(x)$. $x < y \Rightarrow f(x) \cdot r(x) < p^* \Rightarrow x \text{ paid too much!}$

If $p^* = r(z) \cdot f(z)$ ($\exists z$ (there is always such a z)) $\Rightarrow z \text{ buys} \Rightarrow x < z \text{ also buys} \Rightarrow$ fraction z will have it.

z is the fraction of the population that uses the product. $r(x)$ is the intrinsic price of $x \in [0, 1]$. The price x wants to pay $p(x) = r(x) \cdot f(z)$, in other words the reservation price. $p^* = r(z) \cdot f(z)$ is the equilibrium. $f(0) = 0 \Rightarrow p^* = r(0) \cdot f(0) = 0$, in other words equilibrium.

Example: $r(x) = 1 - x^2$, $f(z) = z^2$

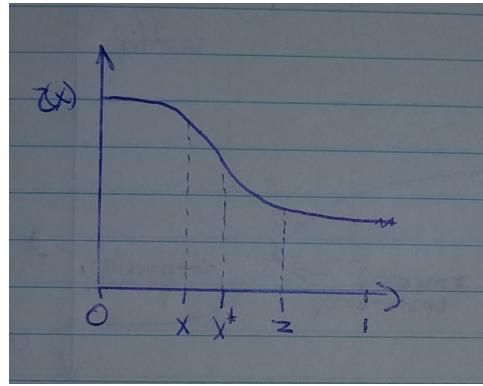


Figure 5: Reservation Price

$$\begin{aligned}
 p^* &= r(z) = (1 - z^2) \cdot z^2 \\
 p^* > \frac{3}{16} &\implies \text{no solution} \\
 p^* < \frac{3}{16} &\implies r(z) \cdot f(z) \text{ has 2 solutions.}
 \end{aligned}$$

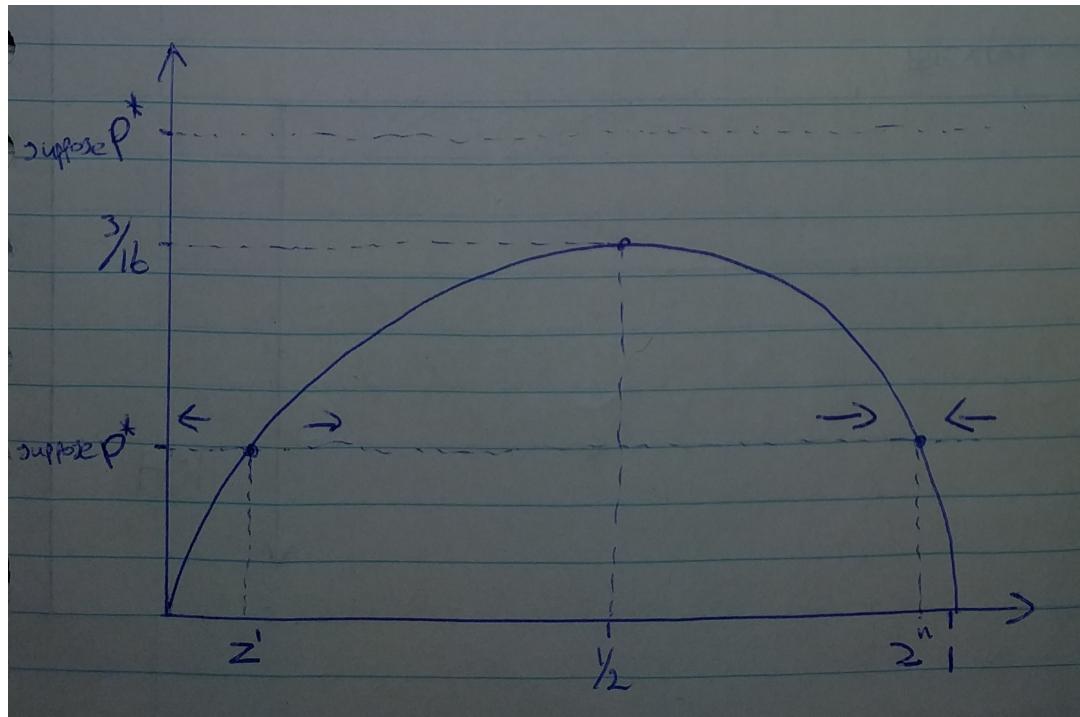
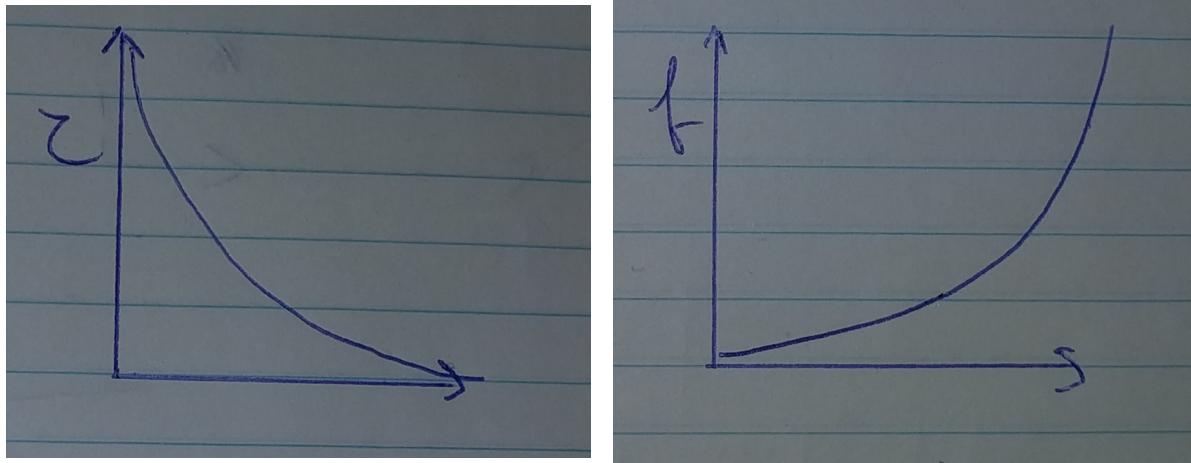


Figure 6: p^*

We will look at different cases:

1. $z = 0$: This is a stable equilibrium.
2. $0 < z < z'$: $r(z) \cdot f(z) < p^* \implies$ if they have bought it, they regret it (downward pressure).



3. $z' < z < z'': r(z) \cdot f(z) > p^* \implies$ more people want the product (upward pressure).

4. $z > z'': r(z) \cdot f(z) < p^* \implies$ downwards pressure.

Conclusion: 0 and z'' are stable equilibrium and z' is a tipping point.

2.2 A Dynamic View

x wants to purchase, thus $r(x) \cdot f(z) \geq p^*$. We define \hat{z} as a solution for $r(\hat{z}) \cdot f(z) = p^*$. This can be rewritten as $r(\hat{z}) = \frac{p^*}{f(z)}$. It holds that when $r(0) \geq \frac{p^*}{f(z)} \iff$ solution exists. We can also rewrite the formula to $\hat{z} = r^{-1}(\frac{p^*}{f(z)}) = g(z)$, (\hat{z} is the fraction of people that will buy given z and p).

Example: $r(x) = 1 - x^2$, $f(z) = z^2$, $\hat{z} = \sqrt{1 - \frac{p^*}{z^2}}$.

If $z > \sqrt{p^*} \implies \hat{z}$ is defined and if $z \leq \sqrt{p^*} \implies \hat{z} = 0$.

$$q = r(x) = 1 - x^2 \implies x^2 = 1 - q$$

$x = \sqrt{1 - q} = r^{-1}(q)$ Remember that in these cases we have always assumed that $f(0) = 0$.

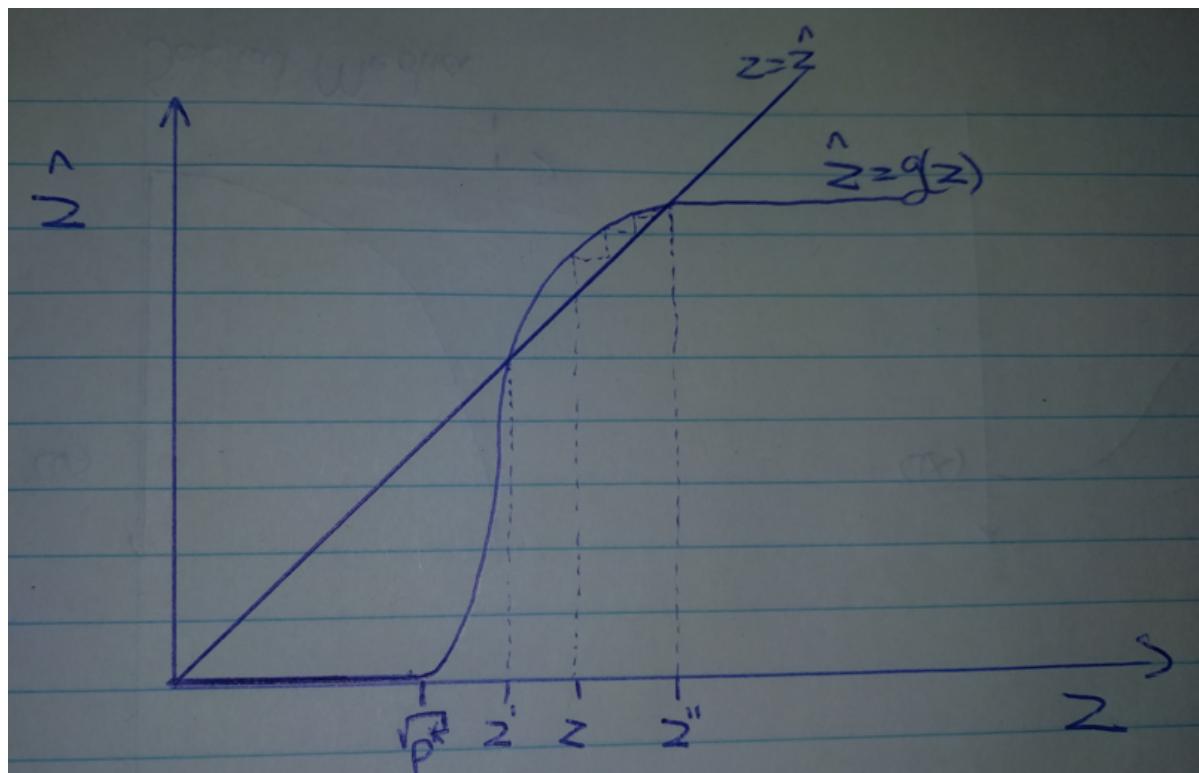


Figure 7: \hat{z} plotted against z