Complex Analysis Cheat Sheet

Guide

- Unless explicitly stated, G is a domain (open and connected).
- $Hol\left(G\right)$ is the set of holomorphic functions on G, $Mer\left(G\right)$ and $Har\left(G\right)$ are the sets of meromorphic (holomorphic, save for countable isolated singularities) and harmonic functions on G.
- $C\left(G\right)$ is the set of continuous functions on G.

Theorems

Topology

1. The set S is closed $\iff$ S^c is open
2. Let $G\in\mathbb{C}$ be an open set. The following definitions for *connectivity* are equivalent:
   (a) G cannot be decomposed into two disjoint open sets: if $X\subseteq G$ is open and $X\setminus G$ is open, then either $X=G$ or $X=\emptyset$.
   (b) Let $a,b\in G$. So there exitsts a polygonal curve that starts at a and ends at b.
   (c) For each *locally constant* $f:G\to\mathbb{C}$, f is necessarily *globally constant*.
   (d) Every continuous $f:G\to\mathbb{R}$ satisfies the intermediate value property: $\exists\alpha,\beta.f\left(\alpha\right)=s,f\left(\beta\right)=t\Rightarrow\left[s,t\right]\subseteq\mathrm{Img}\left(f\right)$
3. $\gamma:\left[a,b\right]\to\mathbb{C}$ is cont. and does not vanish. Then $\exists\psi:\left[a,b\right]\to\mathbb{C}$ such that $e^{\psi(t)}=\gamma\left(t\right)$. It is uniqe up to $\pm2\pi i$.
4. $\mathrm{ind}_{\gamma}\left(z_0\right)=\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-z_0}$
5. $\gamma\subset G$ is closed and piecewise C^1, $f\in Hol\left(\gamma\right)$, them $\int_{\gamma}\frac{f'}{f}dz=2\pi i\cdot\mathrm{ind}_{f\circ\gamma}\left(0\right)$
6. $f\in Hol\left(G\right);\gamma_0,\gamma_1:\left[\alpha,\beta\right]\to G$. Then $\gamma_0\underset{G}{\sim}\gamma_1\Rightarrow\int_{\gamma_0}f=\int_{\gamma_1}f$
7. Jordan: if γ is simple, it divides $\mathbb{C}$ into two connected components.
8. If G is simply connected:
   (a) γ is closed, $f\in Hol\left(G\right)$. Then $\int_{\gamma}f=0$
   (b) $f\in Hol\left(G\right)$ has a primitive, $F\left(z\right)=\int_{z\to z_0}f$
   (c) γ is closed, $z_0\notin G$, then $\mathrm{ind}_{\gamma}\left(z_0\right)=0$
   (d) $f\in Hol\left(G\right)$ and does not vanish, then it has a branch of $\log f,\sqrt{f}$ there.
9. $f\in Hol\left(G\right),\gamma\subset G$ is closed. Then: G is simply connected $\iff$ $\overline{\mathbb{C}}\setminus G$ is connected $\iff$ $\forall z_0\notin G$, is closed them $\mathrm{ind}_{\gamma}\left(z_0\right)=0$ $\iff$ $\int_{\gamma}f=0$ $\iff$ if f does not vanish on G there exists $\log f\in Hol\left(G\right)$.
10. $G_1,G_2\subset\mathbb{C}$, $f:G_1\to G_2$ is continuous, 1-1 and onto with a continuous f^{-1}, then $G_1$is simply connected $\iff$ G_2 is simply connected.
11. **Schwarz's Lemma:**$Hol\left(\mathbb{D}\right)\ni f:\mathbb{D}\to\mathbb{D}$ with $f\left(0\right)=0$. Then $\left|f\left(z\right)\right|\leq\left|z\right|$ and only reaches equality when $f\left(z\right)=\lambda z,\left|\lambda\right|=1$
12. $Hol\left(\mathbb{D}\right)\ni f:\mathbb{D}\to\mathbb{D}$, 1-1 and onto with $f\left(0\right)=0$. Then there exists $\theta\in\left[0,2\pi\right]$ such that $f\left(z\right)=e^{i\theta}z$
13. $f:\mathbb{D}\to\mathbb{D}$ is conformal on $\mathbb{D}$. Then there exist $\left|a\right|<1,\theta\in\left[0,2\pi\right]$ such that $f\left(z\right)=\frac{z-a}{1-\bar{a}z}$
14. **Riemann's thm:** Let $G\subset\mathbb{C}$, $G\neq\mathbb{C}$ and is simply connected. Then there exists a unique holomorphic transform $f:G\to\mathbb{D}$, 1-1 and onto, with $f\left(a\right)=0,\mathbb{R}\ni f'\left(a\right)>0$.
15. **Pick's Lemma :** $f:\mathbb{D}\to\mathbb{D}$ is holomorphic, so $\left|f'\left(z\right)\right|\leq\frac{1-\left|f(z)\right|^2}{1-\left|z\right|^2}$, and $\left|\frac{f(z)-f(w)}{1-\overline{f(z)}f(w)}\right|\leq\left|\frac{z-w}{1-z\bar{w}}\right|$

Differentiation

1. If f is differentiable at z_0, g is differentiable at $w_0=f\left(z_0\right)$, then $\frac{d}{dt}g\left(f\left(z\right)\right)\big|_{z=z_0}=g'\left(w_0\right)f\left(z_0\right)$.
2. f is differentiable, at x_0 $\iff$ it verifies the Cauchy-Riemann equations.

Curves

1. Chain rule for curves: suppose γ is differentiable at t_0 , f is holomorphic at $\gamma\left(t_0\right)$. So $\left(f\circ g\right)'=f'\left(\gamma\left(t_0\right)\right)\dot{\gamma}\left(t_0\right)$

Holomorphic Functions

1. Suppose $f:G\to\mathbb{C}$ is holomorphic. Then f is constant $\iff$ $\forall z\in G:\ f'\left(z\right)=0$
2. If $f:G\to\mathbb{C}$ is holomorphic and real, then f is constant.
3. Suppose $f:G\to\mathbb{C}$ is holomorphic such that $\left|f\right|$ is constant in G, then f is globally constant.
4. Suppose that $f:G\to\mathbb{C}$ is holomorphic and $f'\left(t\right)\neq0,\forall z\in G$. Then f is a conformal transformation, such that $\measuredangle\left(f\left(\gamma_1\right),f\left(\gamma_2\right)\right)=\measuredangle\left(\gamma_1,\gamma_2\right)$.
5. Suppose $f:\mathbb{C}\to\mathbb{C}$ is an $\mathbb{R}$-linear transformation. then $\forall z\in\mathbb{C}\,f\left(z\right)=az+b\bar{z}$, such that $a=\frac{\partial f}{\partial z}\left(0\right);b=\frac{\partial f}{\partial\bar{z}}\left(0\right)$.
6. Suppose $f:G\to\mathbb{C}$ is $\mathbb{R}$-differentiable and conformal. Then f is holomorphic.

Harmonic Functions

1. $f\in Hol\left(G\right)$. If f has continuous second partial derivatives, then f is harmonic (meaning $\mathrm{Im}\left(f\right),\mathrm{Re}\left(f\right)$ are both harmonic),
2. $u:G\to\mathbb{C}$ is harmonic, then $\frac{\partial u}{\partial z}\in Hol\left(G\right)$.
3. Let $u:G\to\mathbb{R}$ be harmonic. Then:
   (a) If $G=\mathbb{C}$, there always exists a harmonic conjugate v.
   (b) For each G, it is single up to $\pm c$.

Möbius Transformations ($\varphi\left(z\right)=\frac{az+b}{cz+d}$, C is a circle)

1. For each $A\in GL_2$, $h_A\left(z\right)$ transformations circles to circles.
2. If $A,B\in GL_2$, then $h_A\circ h_B=h_{AB}$. Therefore:
   (a) $\forall\lambda\neq0$, $h_{\lambda A}=h_A$.
   (b) $h_{\lambda d\bar{d}}=h_I$.
3. φ has at most 2 fixed points. If h_A has 3 fixed points, then it is h_I.
4. $z_1,z_2,z_3,z_4\in\mathbb{C}$ are distinct. Then $\left[z_1,z_2,z_3,z_4\right]=\left[\varphi\left(z_1\right),\varphi\left(z_2\right),\varphi\left(z_3\right),\varphi\left(z_4\right)\right]$.
   Therefore, if $w_1,w_2,w_3,w_4\in\mathbb{C}$ are distinct, there exists φ such that $\forall_{1\leq j\leq4}Tz_j=w_j$
5. Suppose φ operates on $C_1\longmapsto C_2$. Then every $z,\left[z\right]^*_{C_1}\mapsto z,\left[z\right]^*_{C_2}$.
6. z_2,z_3,z_4 are distinct and on C. Then $\left[\left[z_1\right]^*_C,z_2,z_3,z_4\right]=\overline{\left[z_1,z_2,z_3,z_4\right]}$.
7. $z_j,1\leq j\leq4$ are all on the same circle if $\left(z_1,z_2,z_3,z_4\right)\in\mathbb{R}$.

Functions

1. Let $G\subseteq\mathbb{C}$, u_1,u_2 be branches of log. Then $u_1-u_2\equiv\mathrm{const}$.
2. Let $G\subseteq\mathbb{C}$, $l:G\to\mathbb{C}$ be a branch of log. Then $l\in Hol\left(G\right)$ and $l'\left(z\right)=\frac{1}{z}$.
3. Let $f\left(:G\to\mathbb{C}\right)\in Hol\left(G\right)$ and suppose that u is a branch of $\log\left(f\right)$. Then u is holomorphic and $\forall z\in G.u'\left(z\right)=\frac{f'(z)}{f(z)}$.
4. Let $f\left(:G\to\mathbb{C}\right)\in Hol\left(G\right)$, $z_0\in G,f\left(z_0\right)\neq0$. Then $\exists\delta>0$ such that $\mathbb{D}\left(z_0,\delta\right)$ contains a branch of $log\left(f\right)$.

Series

1. If a series converges absolutely to a, it is invariant to any permutation of its order of summation.
2. **Weierstrass' M-Test:** let $\left\{u_n\left(z\right)\right\}_{n=0}^{\infty}$ be a series of functions in G, and $\left\{M_n\right\}_{n=0}^{\infty}$ a series of positive numbers such that:
   (a) $\sup_{z\in G}\left|u_n\right|<M_n$
   (b) $\sum_{n=0}^{\infty}M_n<+\infty$

   Then $\sum_{n=0}^{\infty}u_n\left(z\right)$ uniformly and absolutely in G.
3. **D'alembert:** For $\sum_{n=0}^{\infty}c_nz^n$, $R=\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$; **Cauchy-Hadamard:** $R^{-1}=\lim_{n\to\infty}\sup\sqrt[n]{\left|a_n\right|}$
4. Suppose $f\left(z\right)=\sum_{n=0}^{\infty}c_n\left(z-z_0\right)^n$ is a power series with convergence radius $R\in\left[0,\infty\right]$.
   (a) The series converges normally at $\mathbb{D}\left(z_0,R\right)$.
   (b) If $\left|z-z_0\right|>R$ the series diverges.
   (c) $f\left(z\right)$ is holomorphic in $\mathbb{D}\left(z_0,R\right)$ and its derivative is $f'\left(z\right)=\sum_{n=1}^{\infty}c_nn\left(z-z_0\right)^{n-1}$. Its convergence radius is the same as in $f\left(z\right)$.

   Therefore:
   (a) $f\left(z\right)=\sum_{n=0}^{\infty}c_n\left(z-z_0\right)^n$ is C^{∞} on $\mathbb{D}\left(z_0,R\right)$, and $f^{(k)}\left(z\right)=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}c_n\left(z-z_0\right)^{n-k}$. Specifically for $z=z_0$: $\frac{f^{(k)}(z_0)}{k!}=c_k$.
   (b) Suppose WLOG that $z_0=0$. f has a primitive function with the same radius of convergence: $G\left(z\right)=\sum_{n=0}^{\infty}c_n\frac{z^{n+1}}{n+1}$.
5. **Tauber:** Suppose $n\left|c_n\right|\overset{n\to\infty}{\to}0$, and $f\left(z\right)=\sum c_nz^n$ has a limit $\lim_{\mathbb{R}\ni n\to1}f\left(x\right)=L$, so $\sum c_n=L$.
6. **Abel:** Suppose $\sum c_n$ converges. Then if $z_n\overset{n\to\infty}{\to}1$ and $\sup\frac{\left|z_n-1\right|}{1-\left|z_n\right|}<\infty$, then $\lim_{n\to\infty}f\left(z_n\right)=\sum c_n$.

Integrals

1. **Newton-Leibnitz Formula:** Suppose $f:G\to\mathbb{C}$ is Holomorphic, $\gamma:\left[a,b\right]\to G$ is piecewise differentiable. Then $\int_{\gamma}f'\left(z\right)dz=f\left(\gamma\left(b\right)\right)-f\left(\gamma\left(a\right)\right)$
   (a) $\int_{\gamma}f'$ depends only on the edges of γ
   (b) If the curve closed, the integral is 0.
2. Suppose $f:\left[a,b\right]\to\mathbb{C}$, then $\int_a^bf\left(t\right)dt=\int_a^b\mathrm{Re}f\left(t\right)dt+i\int_a^b\mathrm{Im}f\left(t\right)dt$.
3. Suppose γ_2 is a reparametrization of $\gamma_1$and f is defined on the image of γ_1, then $\int_{\gamma_1}f=\int_{\gamma_2}f$
4. Suppose $f_n:G\to\mathbb{C}$ is a sequence of continuous functions such that $f_n\rightrightarrows f$ on G. Let $\gamma:\left[a,b\right]\to\mathbb{C}$ be piecewise C^1. Then $\int_{\gamma}f_n\to\int_{\gamma}f$.
5. $\left|\int_{\gamma}f\right|\leq\mathrm{Length}\left(\gamma\right)\cdot\sup_{t\in[a,b]}\left|f\left(\gamma\left(t\right)\right)\right|$
6. Let $G\subset\mathbb{C}$, $f:G\to\mathbb{C}$ be continuous and $\gamma:\left[a,b\right]$ be piecewise C^1. Let $\varepsilon>0$. So there exists $\delta>0$ such that for every partition $\pi=\left\{a=t_0<\ldots<t_N=b\right\}$, such that $\lambda\left(\pi\right)<\delta$:
   (a) $\left|\int_{\gamma}f-\sum_{i=0}^{N-1}f\left(z_i\right)\left(z_{i+1}-z_i\right)\right|<\varepsilon$, $z_i=\gamma\left(t_i\right)$.
   (b) Let γ_{π} be the polygonal curve connecting the vertices in $f\left\{\pi\right\}$, then $\left|\int_{\gamma_{\pi}}-\int_{\gamma}\right|<\varepsilon$
7. **Goursat:** Let $G\subset\mathbb{C}$, $f:G\to\mathbb{C}$ be holomorphic. Suppose T is a triangle in G, with counterclockwise orientation. Then $\int_{\gamma}f=0$.
8. If $G\subset\mathbb{C}$ is convex and $f:G\to\mathbb{C}$ is holomorphic, then f has a primitive in G.
9. If $G\subset\mathbb{C}$ is convex and $f:G\to\mathbb{C}$ is holomorphic, and suppose γ is closed and piecewise continuous. Then $\int_{\gamma}f=0$.
10. **Cauchy's formula** for a disk: See Section.

11. Intermediate Value Theorem: suppose f is holomorphic inside a circle containing $\overline{D} \subset \mathbb{C}$ about z_0, then $\forall z \in D$: $f(z) = \frac{1}{2\pi i}\int_0^{2\pi} f\left(z+e^{it}\right)dt$. Therefore, any holomorphic function is locally a Power Series.
12. The Cauchy integral is holomorphic outside the curve on which it is defined, and is C^∞ there.
13. **Power Series Coefficients:** Suppose f is holomorphic on $\overline{D(z_0,R)}$. So $\forall z \in D(z_0,R), f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, and:
    (a) $a_n = \frac{n!}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(z)dz}{(z-z_0)^{n+1}}$
    (b) $f^{(n)}(z_1) = \frac{n!}{2\pi i}\int_{\partial D(z_0,R)} \frac{f(z)dz}{(z-z_1)^{n+1}}$
14. **Morera (Convex):** Suppose G is convex, then (see 20)
15. **Liouville:** If f is entire (harmonic/holomorphic) and bounded, it is fixed.
16. **Fundamental thm of Algebra:** $P \in \mathbb{C}[z]$ is a nonconstant polynomial, then P has a root.
17. **General Cauchy thm+formula:** See Section
18. There Exists no nonconstant holomorphic function for the following cases:
    (a) $f : \mathbb{C} \to \mathbb{D}$ (but any domain that isn't $\mathbb{C}$ is applicable)
    (b) $f : \mathbb{CP}^1 \to \mathbb{C}$
    (c) $f : \mathbb{C} \to \mathbb{C}$ with two linearly independent cycles.
19. Maximum Modulus Principle: G is bounded with a regular contour, $f \in Hol(G) \cap C(\overline{G})$. So the strict maximum is achieved on the boundary. Otherwise, the function is constant.
20. **Morera (General):** Suppose f is continuous in G and for every triangle $T \subset G$: $\int_{\partial T} f(z)\,dz = 0$. Then f is holomorphic in G.
21. Suppose $I \subset G$ is a closed contour. $f \in Hol(G \backslash I) \cap C(G)$, Then $f \in Hol(G)$.

Laurent Series

1. A nonconstant holomorphic function has a finite nullset.
2. Suppose f is holomorphic around z_0. If f has a zero of order m there, then $f(z) = (z-z_0)^m g(z)$, where $Hol(G) \ni g(z) = \begin{cases} (z-z_0)^{-m} f(z) & z = z_0 \\ a_m & z \neq z_0 \end{cases}$
3. The nullset for f does not contain an accumulation point in G.
4. Suppose $f, g \in Hol(G)$ and $f = g$ on $A \subset G$. If A has an accumulation point, $f \equiv g$. Therefore functions in $\mathbb{R}$ have a single analytic continuation.
5. **Weierstraß convergence thm:** Suppose $f_n : G \to \mathbb{C}$ is a sequence of holomorphic functions converging uniformly to a limit function f.
    (a) $f \in Hol(G)$.
    (b) For every k, $f_n^{(k)} \to f^{(k)}$ locally uniformly.
6. **Laurent Coefficients:** Suppose $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in the annulus $A = \{R_1 < |z| < R_2\}$. Then for every $r \in (R_1, R_2)$, $a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z^{n+1}}dz$. Therefore:
    (a) Laurent series can be reconstructed from one circle.
    (b) Two laurent series which agree on one circle are equivalent.
7. **Laurent's thm:** $f \in Hol(A)$ for $A = \{R_1 < |z| < R_2\}$. Then there exist $\{a_n\}_{n=-\infty}^{\infty}$ such that $f = \sum_{n=-\infty}^{\infty} a_n z^n$

Isolated Singularities

1. **Riemann's Criterion :** f is bounded in $D \backslash \{z_0\}$, then z_0 is *removable.* If $z_0 = \infty$. it also needs a bounded neigborhood.
2. **Pole Criterion:** $f \in Hol(D(z_0,r) \setminus \{z_0\})$. Then z_0 is a *pole* $\iff \lim_{z \to z_0} |f(z)| = \infty$.
3. **Casorati-Weierstraß:** f has an *essential* singularity at $z_0 \iff \forall \varepsilon > 0$, $f\{D(z_0,\varepsilon) \setminus \{z_0\}\}$ is dense in $\mathbb{C}$.
4. **Picard's Thm:** If f has an *essential* singualrity, then $\#\{\mathbb{C} \backslash f\{D(z_0,\varepsilon) \setminus \{z_0\}\}\} \le 1$.
5. $f \in Hol(D(z_0,r) \setminus \{z_0\})$. Then $\forall 0 < \varepsilon < r$, $\int_{|z-z_0|=\varepsilon} f(z)\,dz = 2\pi i \cdot \mathrm{res}_{z_0} f$.
6. Argument principle:$f \in Mer(G) \cap C(\overline{G}) \Rightarrow \int_{\partial G} \frac{f'}{f} = 2\pi i \sum (Z_G - P_G)$. ∂G Must not vanish.
7. **Rouché's thm:** $f, g \in Hol(G) \cap C(\overline{G})$, $\forall z \in \partial G$ $|f(z) - g(z)| < |f(z)|$. Then $Z_f = Z_g$ in G.
8. Open Mapping thm:$f \in Hol(G)$ and nonconstant. Then f maps open sets to open sets.
9. Inverse function thm: $f : G \to \mathbb{C}$ is holomorphic and 1-1. Then f^{-1} is holomorphic and $\left(f^{-1}\right)'(z) = \frac{1}{f'(f^{-1}(z))}$.
10. Change of variables: $\int_{g(\gamma)} f(z)\,dz = \int_\gamma f(g(w))\,g'(w)\,dw$
11. Residue at ∞: If $f \in Hol(\{z \mid |z| > R\})$ then $\forall r > R$, $-\int_{|z|=r} f(z)\,dz = 2\pi i \cdot \mathrm{res}_\infty(f)$
12. The sum of residues on $\mathbb{C}$ is 0.
13. $f \in Hol(\mathbb{C})$ with a pole at $\infty \Rightarrow f$ is a polynomial.
14. $f \in Mer(\mathbb{C})$ with a pole/removable sg. at $\infty \Rightarrow f$ is rational.
15. **Local Mapping thm:** $f \in Hol(G)$ and nonconstant. If $z_0 \in G, w_0 = f(z_0)$ with multiplicity m. Then for every sufficiently small $\delta > 0$, there exists $\varepsilon > 0$ such that every $|w - w_0| < \varepsilon$ has m preimages in $|z - z_0| < \delta$.

Cauchy Theorems:

1. General thm: Suppose G is bounded with a regular contour, $f : \overline{G} \to \mathbb{C}$ is continuous and holomorphic in G. Then $\int_{\partial G} f(z)\,dz = 0$
2. Isolated Singularities: $G \backslash \{a_1, \ldots, a_n\}$, $\int_{\partial G} f = 2\pi i \sum \mathrm{res}_{a_k} f$
3. Residue thm: Γ is a contour which doesn't run through any singularities. Then $\int_\Gamma f = 2\pi i \sum \mathrm{ind}_\Gamma(a_k)\,\mathrm{res}_{a_k} f$
4. Using Winding Numbers: $\gamma \subset G$ is closed, $z_0 \in G \backslash \gamma$. Then $\mathrm{ind}_\gamma(z_0) f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - z_0 dz}$
5. Disk formula: if $D \subseteq \mathbb{C}$ is open and f is holomorphic inside a disk containing $\bar{D}$, then $\forall z_0 \in D$, $f(z_0) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{z - z_0}$
6. General formula: For said G and $Hol(G) \cap C(\overline{G}) \supset f : G \to \mathbb{C}$: $\frac{1}{2\pi i}\int_{\partial G} \frac{f(\zeta)}{\zeta - z}d\zeta = \begin{cases} 0 & z \notin G \\ f(z) & z \in G \end{cases}$

Hyperbolic Geometry

1. $Aut(\mathbb{D}) \ni f : \mathbb{D} \to \mathbb{D}, \gamma : [a,b] \to \mathbb{D}$ piecewise C^1. Then $L_H(\gamma) = L_H(f \circ \gamma)$.
2. If said f is not an autmorphism, then $L_H[f \circ \gamma] \le L_H[\gamma]$
3. There exists a uniqe Geodesic between two points, and it is contained in a circle orthogonal to $\partial\mathbb{D}$.

Equations

Arithmetic

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2};\ \overline{zw} = \overline{zw}\ \ |z|\,|w| = |zw|$$

Residues

- For $\sum a_n z^n$ about p, $\mathrm{res}_p f = a_{-1}$
- g, h are holomorphic in an open set with z_0, and suppose h has a simple pole. Then $\mathrm{res}_{z_0} \frac{g}{h} = \frac{g(z_0)}{h'(z_0)}$.
- If f is holomorphic in an open set with z_0 and it has a zero of order m there, then $\mathrm{res}_{z_0} \frac{f'}{f} = m$
- if c is a pole of order n:

$$\mathrm{res}_c f = \frac{1}{(n-1)!}\lim_{z \to c} \frac{d^{n-1}}{dz^{n-1}}\left((z-c)^n f(z)\right)$$

Residues at ∞

- If $\lim_{z\to\infty} f(z) = 0$,then $\mathrm{res}_\infty f = \lim_{z\to\infty} z \cdot f(z)$
- $\mathrm{res}_\infty f = \mathrm{res}_0\left(-\frac{1}{w^2} f\left(\frac{1}{w}\right)\right)$
- $\mathrm{res}_{z_0} f \circ g = \mathrm{res}_{z_0}\left(f(g(z_0))\,g'(z_0)\right)$

Laurent Series

- $r = \overline{\lim}_{n\to\infty} |a_{-n}|^{1/n}$
- $\frac{1}{R} = \overline{\lim}_{n\to\infty} |a_n|^{1/n}$

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