PHYS 514 Final Project

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NEWTON

This chapter investigates star structures within the scope of Newtonian gravity. We will use both analytical and numerical methods for this cause.

Part a. Lane-Emden Equation

With the continuity equation and hydrostatic equilibrium conditions we can derive the Lane-Emden Equation.

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{Gm}{r^2} \tag{2}$$

where ρ and m is a function of r. Differentiating Eq.(2)

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) = \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} = -\frac{2}{\rho r}\frac{dP}{dr} - 4\pi G\rho \tag{3}$$

The mass gradient is replaced by the continuity equation. If we multiply both sides with r^2 and collect the derivatives on the left-hand side

$$r^2\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) + \frac{2r}{\rho}\frac{dP}{dr} = \frac{d}{dr}\left(r^2\frac{dP}{\rho\, dr}\right) = -4\pi G r^2\rho \tag{4}$$

The polytropic equation of state is expressed as

$$P = K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \tag{5}$$

where

$$\rho = \rho_c\theta^n \tag{6}$$

If we plug Eq.(5) in Eq.(4) we get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K\rho_c^{\frac{1}{n}}(n+1)\frac{d\theta}{dr}\right) = -4\pi G\rho_c\theta^n \tag{7}$$

Now, if we introduce $r = \alpha\xi$, where

$$\alpha^2 = (n+1)K\rho_c^{\frac{1}{n}-1}/4\pi G \tag{8}$$

If we write r and the derivatives in terms of ξ, we derive the Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{9}$$

A more detailed, handwritten derivation can be found in the repository. But the steps above pictures the main process nicely. The next step is solving Lane-Emden Equation analytically in mathematica. First we have to show that for $\theta(0) = 1$, solution behaves as a power expansion around $\xi = 0$. We use the following form in mathematica

$$2\frac{d\theta}{d\xi} + \xi\frac{d^2\theta}{d\xi^2} + \xi\theta^n = 0 \tag{10}$$

Using AsymptoticDSolveValue, we get the following solution at the center

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 \tag{11}$$

The implementation can be found on phys414.nb file which is included in the repository. For the given initial conditions and polytropic index $n = 1$, we use DSolve to analytically solve Lane-Emden Equation to get

$$\theta(\xi) = \frac{sin(\xi)}{\xi} \tag{12}$$

Now to find the total mass M, we go back to the continuity equation

$$\frac{dm}{dr} = 4\pi r^2\rho(r) \tag{13}$$

If we integrate up to R, where $R = \xi_s\alpha$

$$M = \int_0^R 4\pi r^2\rho(r)\, dr \tag{14}$$

And plug ξ instead of r

$$M = 4\pi\rho_c\alpha^3\int_0^{\xi_s}\xi^2\theta^n\, d\xi \tag{15}$$

$$M = \frac{-4\pi\rho_c R^3}{(\xi_s)^3}\int_0^{\xi_s}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi})d\xi \tag{16}$$

And we derive the expression for total mass M

$$M = 4\pi\rho_c R^3\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right)_{\xi=\xi_s} \tag{17}$$

Lastly, we need to show that if a group of stars share the same polytropic EOS, then their mass is a function of their radius. To do this, M and R can be related.

$$R = \xi_s\alpha = \xi_s\left[(n+1)K\rho_c^{\frac{1}{n}-1}/4\pi G\right]^{1/2} \tag{18}$$

Then we get the following expression for ρ_c

$$\frac{R^2}{\xi_s^2}\cdot\frac{4\pi G}{(n+1)K} = \rho_c^{-1+1/n} \implies \left(\frac{R^2}{\xi_s^2}\cdot\frac{4\pi G}{(n+1)K}\right)^{\frac{1}{\frac{1}{n}-1}} = \rho_c \tag{19}$$

Then we get the following expression for M

$$M = 4\pi\left(\frac{R^2}{\xi_s^2}\cdot\frac{4\pi G}{(n+1)K}\right)^{\frac{1}{\frac{1}{n}-1}} R^3\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right)_{\xi=\xi_s} \tag{20}$$

If we simplify a little bit we get

$$M = \left[-4\pi\left(\frac{4\pi G}{K(n+1)}\right)^{\frac{n}{1-n}}\xi_s^{\frac{-1+n}{1-n}}\theta'(\xi_s)\right]R^{\frac{3-n}{1-n}} \tag{21}$$

Which yields the constant of proportionality as

$$B = -4\pi\left(\frac{4\pi G}{K(n+1)}\right)^{\frac{n}{1-n}}\xi_s^{\frac{-1+n}{1-n}}\theta'(\xi_s) \tag{22}$$

Part b. White Dwarfs

The CSV file has been read with Pandas' $pd.read\_csv()$ method. The logarithm of the surface gravity can be easily converted to radius using basic Newtonian gravity with the following relation

$$R = \sqrt{\frac{GM}{g}} \tag{23}$$

With the conversion to MKS units we get the following plot for $M-R$ data

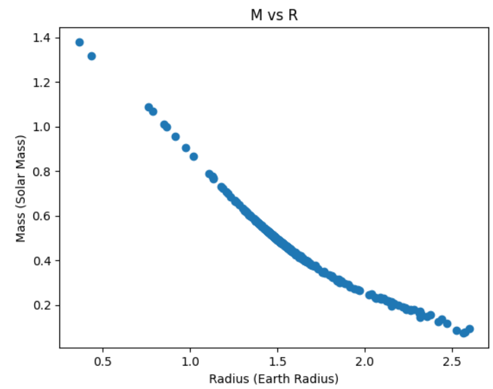

FIG. 1. M-R plot for White Dwarf Data

Part c. Low Mass White Dwarfs

EOS for a cold white dwarf is given by

$$P = C\left[x\left(2x^2-3\right)\left(x^2+1\right)^{1/2} + 3\sinh^{-1}x\right] \tag{24}$$

where

$$x = \left(\frac{\rho}{D}\right)^{1/q} \tag{25}$$

Solving Eq.(24) in Mathematica yields

$$P = \frac{8Cx^5}{5} - \frac{4Cx^7}{7} + \frac{Cx^9}{3} + \mathcal{O}(x^{11}) \tag{26}$$

For a low mass WD we can use $x << 1$ condition. The only surviving term in the expansion is the one with the largest degree. And by plugging Eq.(25) we get the following expression

$$P = \frac{8C}{5D^{5/q}}\rho^{1+\frac{5-q}{q}} \tag{27}$$

Hence we get the following relations for k_* and n_*

$$n_* = \frac{q}{5-q}, \quad K_* = \frac{8C}{5D^{5/q}} \tag{28}$$

Now we need to define a limit for low mass WDs. Based on scatter plot on Fig.1., I've chosen the limit to be 0.4 solar mass. For curve fitting we use the relation between M and R in Eq.(21). To linearize the relation between M and R, we take the logarithm of both sides

$$ln(M) = \frac{3-n}{1-n}ln(R) + ln(B) \tag{29}$$

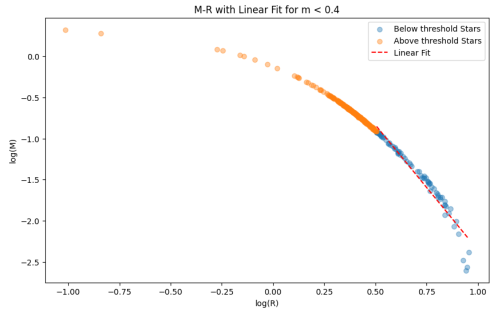

FIG. 2. Linear fit to the White Dwarf data. Blue dots represent low mass, while the orange ones represent high mass stars. Both axes are in logarithmic scale.

We use built-in Numpy polyfit() to fit a line to the data and get the slope and the intercept.

The polyfit() gives us the following slope

$$Slope = -3.040$$

Plugging the slope we have found into the $\frac{3-n_*}{1-n_*}$ as an integer we get $n_* = 1.5$ and $q = 3$. Now we should calculate K_* which requires the values of ξ and $\theta(\xi)$ for n = 1.5. Solving the Lane-Emden Equation numerically using built-in solve_ivp gives the following values

$$\xi = 3.654 \tag{30}$$

$$\theta' = -0.203 \tag{31}$$

$$k_* = 2.849.10^6 \tag{32}$$

Now we can use Eq.(17) to calculate the ρ_c to get Fig.3.

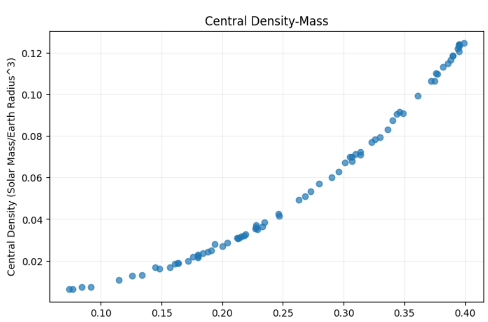

FIG. 3. Central density vs mass for low mass data.

Part e. Chandrasekhar Mass

To see the Chandrasekhar mass, we apply $x >> 1$ condition into Eq.(24). Series expansion of Eq.(24) with degeneracy around infinity in Mathematica gives the following expression

$$P = 2Cx^4 - 2Cx^2 + \left(-\frac{7C}{4} + \frac{3}{2}\ln(4) + 3\ln(x)\right) \tag{33}$$

For the $x >> 1$ condition the expression can be approximated as $P = 2Cx^4$ and if we plug Eq.(25) we get

$$P = \frac{2C}{D^{4/q}}\rho^{1+\frac{4-q}{q}} \tag{34}$$

It can easily be seen that for $n = 3$, we get $q = 3$ and the EOS becomes a polytrope. Plugging $n = 3$ into Eq.(22) gives the following expression for mass

$$M = \left(-4\pi\left(\frac{\pi G D^{4/3}}{2C}\right)^{-3/2}\right)\xi_n^2\theta'(\xi_n) \tag{35}$$

If we plug the theoretical expressions for C and D into equation above and simplify we get M_{ch} in terms of known constants

$$M = -\frac{\sqrt{3}\,\pi}{2}\left(\frac{\hbar\,c}{G}\right)^{\frac{3}{2}}\frac{1}{(m_u\,\mu_e)^2}\,\xi_n^2\,\theta'(\xi_n)\,. \tag{36}$$

EINSTEIN

Part a. Tolman-Oppenheimer-Volkoff equations

To solve Tolman–Oppenheimer–Volkoff (TOV) equation for different central density and central pressure values, I have used built-in solve _ivp as usual for ρ_c values ranging from 10^{-6} to 10^{-1} in logarithmic space. For 50 samples we get the curve in the figure below.

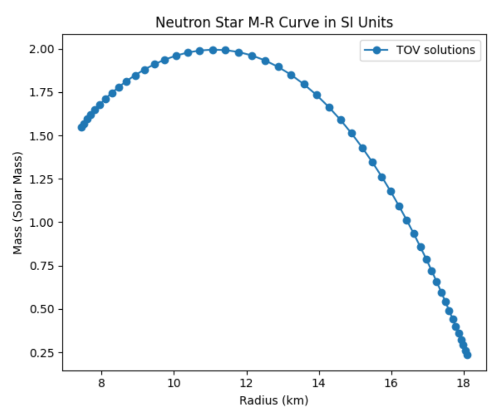

FIG. 4. M-R for ranging central density values.

Part b. Baryonic Mass

Baryonic mass is given by,

$$m'_P = 4\pi\left(1 - \frac{2m}{r}\right)^{-1/2} r^2\rho \tag{37}$$

We can just add Baryonic mass equation into the algorithm that solves previous part which solves TOV equations and get the M_p to calculate fractional binding energy according to

$$\Delta \equiv \frac{M_P - M}{M} \tag{38}$$

We get the figure below from this approach for the binding energy

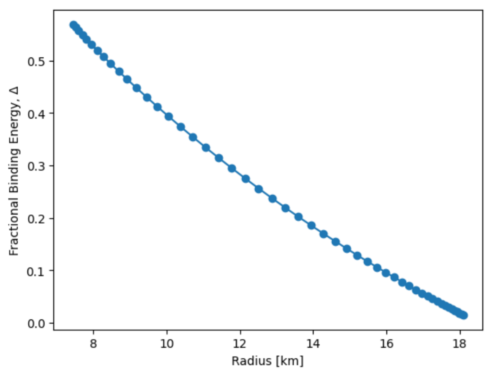

FIG. 5. Fractional Binding Energy vs. Radius

Part c. Stability

For Neutron stars the criterion for stability is

$$\frac{dM}{d\rho_c} > 0 \quad \rightarrow \quad \text{stable} \tag{39}$$

$$\frac{dM}{d\rho_c} < 0 \quad \rightarrow \quad \text{unstable} \tag{40}$$

We can find the relation between stability and $M - \rho_C$ by utilizing conditions above. Our ρ_c values are in range of 10^{-5} to 10^{-2} divided into 50 sample points. To easily separate the unstable ones from stable ones, we can find the maximum of the curve. We can infer the sign of the slope directly from the fact that the function is monotonically increasing up to the maximum and then monotonically decreasing afterward. So, the points after the maximum will be unstable, while the points before will be the stable ones.

The maximal NS mass allowed, in terms of solar mass is found to be

$$M_{max} = 1.995 \tag{41}$$

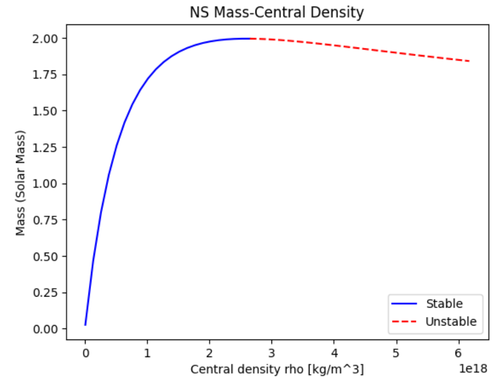

FIG. 6. Stable and unstable regions on Mass vs. Central Density plot

Part d. Allowed K Values

We need to find maximum allowed value of K_{ns}. To find the allowed K values based on the most massive NS, we solve TOV for each K value and get the maximum mass and we fit a CubicSpline to the pairs. Then, using fsolve, we find that the upper bound for K_{ns} is

$$K_{ns} = 115.03738042685654 \tag{42}$$

The results can be seen on Fig.5.

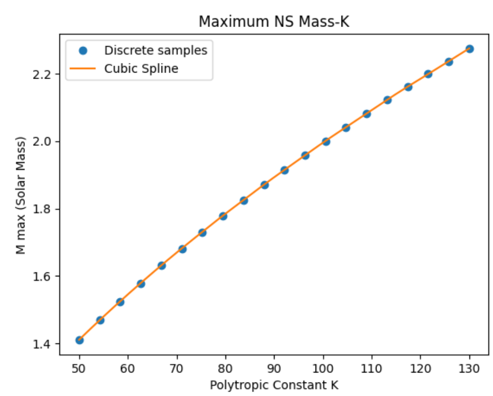

FIG. 7. Maximum mass for different values of $K_n s$

Part e. Mass Outside the Star

Since there is no matter outside the star, ν' becomes

$$\nu' = \frac{2M}{r(r-2M)} \quad (r > R) \tag{43}$$

We can solve this equation in Mathematica using DSolve to get this expression

$$v(r) = c - 2M\left[\frac{\ln(r)}{2M} - \frac{\ln(-2M+r)}{2M}\right] \tag{44}$$

$$v(r) = c + \ln\left[\frac{-2M+r}{r}\right] \tag{45}$$

And further simplifying

$$v(r) = c + \ln\left[\frac{1-2M}{r}\right] \tag{46}$$

For initial value $v(R)$ we get

$$C = v(R) - \ln\left[1 - \frac{2M}{R}\right] \tag{47}$$

Using Eq.(46) and Eq.(44) we derive the final expression

$$v(r > R) = \ln\left[1 - \frac{2M}{r}\right] - \ln\left[1 - \frac{2M}{R}\right] + v(R) \tag{48}$$