PENROSE TILINGS

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Written in XeLaTeX

v0: 02/08/2020

An article on https://ethanarnold.uk

QUASIPERIODICITY

INTRODUCTION

In 1982, the discovery of a solid with a diffraction pattern displaying five-fold symmetry represented the start of a paradigm shift for the study of crystallography. The view, at the time, was that such a diffraction pattern was impossible. Following the traditional view of a crystal being a *periodic* entity over all of space, there are only a finite possible number of rotational symmetries that could be displayed in the diffraction pattern: dictated by the crystallographic restriction theorem, it was expected that only two-fold, three-fold, four-fold and six-fold symmetries were possible - so clearly, five-fold patterns were not allowed, following this rule.

What was the explanation for such an unexpected observation? The description of a crystal being periodic no longer held up to experiment. Soon after, the solids displaying this phenomena were affectionately named "quasicrystals", to reflect their most unusual behaviour. This document focusses on generating the "lattice points" for a quasicrystalline surface.

WHAT IS A QUASICRYSTAL?

The attentive reader will have noticed that the very nature of a quasicrystal has yet to be discussed. With quasicrystals not having the periodic structure most familiar to you, understanding what they are is vital to proceed in generating their lattice points. To put it most simply, quasicrystals are any substance with a long-range order, but no periodicity; this is ambiguous, and indeed is intended to be so: there is no set structure for quasicrystals.

Penrose Tilings

It is often the case that references to the structure of quasicrystals will lead to a discussion of Penrose tilings. Why is this? We only just said quasicrystals have no set structure, so why are Penrose tilings so important? Taking a phenomenological approach to this study of quasicrystals, we essentially need to test various structures until something works well. Penrose tilings have been shown to fit many common quasicrystals, with a good degree of accuracy! Thus it is clear that it is rewarding to understanding how to construct a Penrose tiling. This will be the focus of this article.

CONSTRUCTING PENROSE TILINGS

There are many known methods for constructing Penrose tilings. Continuing with the context of quasicrystals, it is only sensible that we attempt to construct a Penrose tiling that matches well with many different known quasicrystals; this is an easy task: the most well known construction, arising from the tesselation of two specific rhombi, allows the plane to be tiled in a manner than generates the five fold symmetry so frequently seen.

Constituent Shapes

How do we generate a construction using these rhombi? First, we need to establish the exact geometry of the shapes used. We use two distinct rhombi; the first of which has an internal ingle of $\pi/5$, and the second of which has an internal angle of $3\pi/10$. All of the sides on both shapes are of equal length. This length is arbitrary.

Decomposition

The most straightforward, and computationally viable, method of finding the analogous points to the "lattice points" of a conventional crystal in a Penrose tiling is to use the method of decomposition. What is this method? It makes sense to summarise it before approaching it in full. Simply put, it iteratively breaks down the shapes making up the plane into smaller components. The decomposition each shape takes depends on which rhombus it is based on. This is relatively straightforward - the easist trap to fall into is to overcomplicate the process.

We will now proceed with describing the procedure used to decompose the plane into a Penrose tile. As always, the dominating problem we may come across is the boundary condition. What tiling do we start with? It should be clear to the reader that at all stages, the tiling we have must be further decomposable; otherwise our progress will stagnate, and the process will fail. Thus we require to start with a set of points that conform to *any* section of a complete Penrose tiling. The typical starting shape used is one of the triangles in the middle of figure 1, as this as the most basic shape we can form (it is formed from cutting the rhombus into two triangles, along the appropriate line of symmetry).

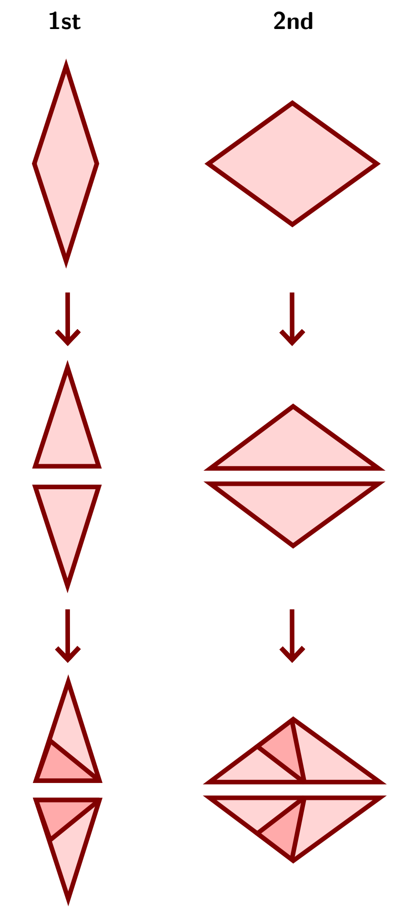

Figure 1 | The process for decomposing a region of the plane into a Penrose tiling. For each occurance of the slim triangle (with a $\pi/5$ angle at the top vertex), we decompose it into the configuration on the bottom line. The next step is to decompose each of the fatter triangles (with a $3\pi/10$ angle at the top vertex) into the arrangement on the correponding shapes on the bottom line. This process is iterated, in its entirety, an infinite number of times.

Once the boundary condition has been established, it is possible to begin decomposing the plane. Throughout this description, use the figure 1 to assist with your interpretation of the process. There are two steps: we first decompose the triangles from the slim rhombus into one triangle from the thick rhombus, and one triangle from the slim rhombus. The triangles from the thick rhombus are decomposed in a fairly similar way - into two thick rhombus triangles, and one slim rhombus triangle. This description is unfortunately quite verbose. We again recommend that you study the figure, to better understand the process.

With the description above (best captured by the diagram), we have all the information we need to construct a Penrose tiling. No high powered maths is necessary! More mathematically inclined methods exist, but they are less suitable for computer programs. We briefly discuss them below, but will not go into detail.

Other Methods

Aside from using decomposition, there are various other methods that can be used to tile the plane. It is conceptually simpler to construct a Penrose tiling using the appropriate "matching rule". This simply fits the pieces together, like a jigsaw, completely tiling the plane. However, this is not easy: a computer can not easily tell how to put it together. There are alternatives; we could use methods such as the "duel grid method" to construct a Penrose tiling, but these are computationally heavy. This summary, also seen with other methods, is a brief explanation of why the decomposition method, being computationally simple, is what we opt for.

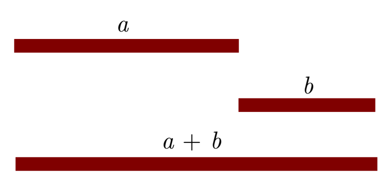

Figure 2 | The golden ratio is defined by the ratio of lengths. Two lengths are said to be in the golden ratio if the ratio of their lengths is equal to the ratio of their sum to the larger length.

THE GOLDEN RATIO

Two lengths are said to be in the golden ratio if the ratio of their lengths is equal to the ratio of their sum to the larger length. Thus, from our diagram, we can quantify these statements. If we set the two lengths be a and b, where a is longer than b, then we require that

$$\frac{a}{b} = \frac{a+b}{a}.$$

Solving for the ratio a/b suggests:

$$\frac{a}{b} = \frac{1+\sqrt{5}}{2},$$

which we set as the golden ratio ϕ. We encourage the reader to find expressions for ϕ^2, ϕ^{-1} and $\sqrt{\phi}$. How can these can be written as a linear combination of the golden ratio ϕ?

TASK

There are a number of steps used to generate a Penrose tiling using a computer program. We will break these down in this tutorial, to make the objective of generating a Penrose tiling from scratch less daunting.

1. Prior to doing any geometric calculations, it is wise to consider the behaviour of the cosine and sine functions for the relevant angles in the geometry that will arise.
   (a) Calculate $\cos 2\pi/5$ as a linear expression of the golden ratio.
   (b) Calculate $\sin 2\pi/5$ as a linear expression of the golden ratio.
   (c) Calculate $\cos \pi/5$ as a linear expression of the golden ratio.
   (d) Calculate $\sin \pi/5$ as a linear expression of the golden ratio.
2. Consider the rhombi in figure 1. Suppose that, at the start of each decomposition, the side length of a rhombus is x.
   (a) For each rhombus, assign position vectors to each of the vertices of the triangles the rhombus can be decomposed into.
   (b) Find the lengths of the sides of all of the triangles in the decomposed shapes.
   (c) Find the position vectors of the "new" vertices (*i.e.* the vertices of the triangles in the decomposed shape that are *not* in the original shape).
3. The majority of the work has now been done. What remains is to transform the process above into a functioning iterative process. How do we do this? We need to be careful not to "double count" any points (or worse...), as this would ruin future tasks. Therefore the procedure we adopt is to define two lists: one of which contains the lattice points, and the other containing a list of triangles in a queue to be decomposed. Thus our task is simple:
   (a) Define two lists. These should both be lists of lists. The first of these should be the positions of all of the "lattice point" type coordinates. The latter should be a list of triangles that are awaiting decomposition. Each of the triangles in this list should be an ordered trio of coordinates, along with a label of whether the triangle is associated with a "thick" or "slim" rhombus.
   (b) Define two functions. The first of these should take the position vectors of three vertices (in a specific order - think about how to consistently generate the same pattern), corresponding to a "slim" triangle, and perform a very specific task. It should generate a list of the "new" triangles produced, each having a label of whether they correspond to "thick" or "slim" rhombi. An equivalent function should be produced for a "thick" triangle.
   (c) The final stage in the production of a Penrose tiling is to iterate

through the triangles in the plane, producing a ever more detailed tiling. Write a script that loops through 5 (or more) iterations of decomposing the plane.

4. Plot a small circle on each of the lattice points you have calculated, that well illustrates the pattern of the tiling. Save the figure as a lossless file. This is the least important part of the tutorial - do not think of it as the "finished product"; it shows you what your arrangement of lattice points looks like.

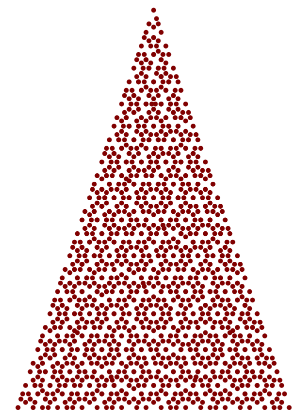

Figure 3 | An example of what your generated figure should look like.

SUMMARY

In this article, we have introduced the concept of a quasicrystal, and how to represent them with a Penrose tiling. We constructed a Penrose tiling using the decomposition method.