Exercise 11

We consider the functions:

$$f(x) = (x-1)^2 e^{-x}, \quad g(x) = \tfrac{3}{2}(x-1)^2$$

The plane is equipped with an orthonormal coordinate system $(O, \vec{i}, \vec{j})$. The origin O is placed 5cm from the left edge of the page, and the graphic unit is 3cm.

1. Curve $\mathcal{P}$ representing $g(x)$

The function $g(x) = \frac{3}{2}(x-1)^2$ is a parabola opening upward, with vertex at $(1,0)$ and a vertical stretching factor of $\frac{3}{2}$.

2. Study of $f(x)$

a) **Limit at $-\infty$:**

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x-1)^2 e^{-x}$$

As $x \to -\infty$, $(x-1)^2 \to +\infty$, and $e^{-x} \to +\infty$, so:

$$\boxed{\lim_{x \to -\infty} f(x) = +\infty}$$

b) **Limit at $+\infty$:**

We rewrite:

$$f(x) = (x-1)^2 e^{-x} = x^2 e^{-x} - 2x e^{-x} + e^{-x}$$

Each term tends to 0 as $x \to +\infty$, so:

$$\boxed{\lim_{x \to +\infty} f(x) = 0}$$

Graphical interpretation: the curve $\mathcal{C}$ rises sharply to the left and tends toward the x-axis to the right.

c) **Derivative and variations:**

$$f'(x) = \frac{d}{dx}[(x-1)^2 e^{-x}] = (x-1)^2(-e^{-x}) + 2(x-1)e^{-x} = (x-1)(3-x)e^{-x}$$

$$\boxed{f'(x) = (x-1)(3-x)e^{-x}}$$

Sign table:

x	$-\infty$		1		3		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$f(3)$	$\searrow$	0

$$f(3) = (3-1)^2 e^{-3} = 4e^{-3}$$

d) **Tangent at point A of abscissa 0:**

$$f(0) = (0-1)^2 e^0 = 1, \quad f'(0) = (-1)(3)(1) = -3$$

So the equation of the tangent T at $A(0,1)$ is:

$$y - 1 = -3(x-0) \Rightarrow \boxed{y = -3x + 1}$$

3. Intersection points and comparison

a) **Points of intersection $f(x) = g(x)$:**

$$(x-1)^2(e^{-x} - \tfrac{3}{2}) = 0 \Rightarrow x = 1 \quad \text{or} \quad e^{-x} = \tfrac{3}{2} \Rightarrow x = -\ln \tfrac{3}{2}$$

So the curves intersect at:

$$x = 1 \quad \text{and} \quad x = -\ln \tfrac{3}{2}$$

b) **Sign of $f(x) - g(x)$:**

We analyze:

$$f(x) - g(x) = (x-1)^2(e^{-x} - \tfrac{3}{2})$$

- For $x < -\ln \frac{3}{2}$, $e^{-x} > \frac{3}{2} \Rightarrow f(x) > g(x)$ - For $-\ln \frac{3}{2} < x < 1$ and $x > 1$, $f(x) < g(x)$

Graphical interpretation: curve $\mathcal{C}$ is above $\mathcal{P}$ before $x = -\ln \frac{3}{2}$, and below it elsewhere.

4. Sketching the curves

Draw $\mathcal{C}$ and $\mathcal{P}$ on the same coordinate plane, respecting: - Intersections at $x = 1$ and $x = -\ln \frac{3}{2}$ - Tangent at $A(0,1)$ with slope -3

5. Integral and primitive

a) **Show that $F(x) = -e^{-x}(x^2+1)$ is a primitive of $f(x)$:**

Differentiate F:

$$F'(x) = e^{-x}(x^2+1) - 2xe^{-x} = (x^2 - 2x + 1)e^{-x} = (x-1)^2 e^{-x} = f(x)$$

b) **Exact and approximate value of the integral I:**

$$I = \int_0^1 f(x)\,dx = F(1) - F(0) = [-e^{-1}(1^2+1)] - [-e^0(0^2+1)] = -2e^{-1} + 1$$

Using $e^{-1} \approx 0.368$:

$$I \approx 1 - 2 \cdot 0.368 = 1 - 0.736 = 0.264 \Rightarrow \boxed{I \approx 0.26}$$

Geometric interpretation: I represents the (signed) area between the curve $\mathcal{C}$ and the x-axis from $x = 0$ to $x = 1$.