Exercise 11

We consider the functions:

$$f(x) = (x-1)^2 e^{-x}, \quad g(x) = \frac{3}{2}(x-1)^2$$

The plane is equipped with an orthonormal coordinate system (O, \vec{i}, \vec{j}) . The origin O is placed 5cm from the left edge of the page, and the graphic unit is 3cm.

1. Curve \mathcal{P} representing g(x)

The function $g(x) = \frac{3}{2}(x-1)^2$ is a parabola opening upward, with vertex at (1,0) and a vertical stretching factor of $\frac{3}{2}$.

2. Study of f(x)

a) Limit at $-\infty$:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} (x - 1)^2 e^{-x}$$

As $x \to -\infty$, $(x-1)^2 \to +\infty$, and $e^{-x} \to +\infty$, so:

$$\lim_{x \to -\infty} f(x) = +\infty$$

b) Limit at $+\infty$:

We rewrite:

$$f(x) = (x-1)^2 e^{-x} = x^2 e^{-x} - 2xe^{-x} + e^{-x}$$

Each term tends to 0 as $x \to +\infty$, so:

$$\lim_{x \to +\infty} f(x) = 0$$

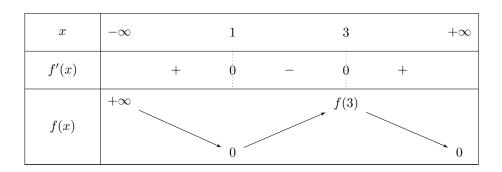
Graphical interpretation: the curve C rises sharply to the left and tends toward the x-axis to the right.

c) Derivative and variations:

$$f'(x) = \frac{d}{dx}[(x-1)^2 e^{-x}] = (x-1)^2 (-e^{-x}) + 2(x-1)e^{-x} = (x-1)(3-x)e^{-x}$$

$$f'(x) = (x-1)(3-x)e^{-x}$$

Sign table:



$$f(3) = (3-1)^2 e^{-3} = 4e^{-3}$$

d) Tangent at point A of abscissa 0:

$$f(0) = (0-1)^2 e^0 = 1, \quad f'(0) = (-1)(3)(1) = -3$$

So the equation of the tangent T at A(0,1) is:

$$y - 1 = -3(x - 0) \Rightarrow y = -3x + 1$$

3. Intersection points and comparison

a) Points of intersection f(x) = g(x):

$$(x-1)^2(e^{-x}-\frac{3}{2})=0 \Rightarrow x=1$$
 or $e^{-x}=\frac{3}{2} \Rightarrow x=-\ln\frac{3}{2}$

So the curves intersect at:

$$x = 1$$
 and $x = -\ln\frac{3}{2}$

b) Sign of f(x) - g(x):

We analyze:

$$f(x) - g(x) = (x - 1)^{2} (e^{-x} - \frac{3}{2})$$

- For $x<-\ln\frac{3}{2},\,e^{-x}>\frac{3}{2}\Rightarrow f(x)>g(x)$ - For $-\ln\frac{3}{2}< x<1$ and $x>1,\,f(x)< g(x)$

Graphical interpretation: curve C is above P before $x = -\ln \frac{3}{2}$, and below it elsewhere.

4. Sketching the curves

Draw \mathcal{C} and \mathcal{P} on the same coordinate plane, respecting: - Intersections at x=1 and $x=-\ln\frac{3}{2}$ - Tangent at A(0,1) with slope -3

5. Integral and primitive

a) Show that $F(x) = -e^{-x}(x^2 + 1)$ is a primitive of f(x):

Differentiate F:

$$F'(x) = e^{-x}(x^2 + 1) - 2xe^{-x} = (x^2 - 2x + 1)e^{-x} = (x - 1)^2e^{-x} = f(x)$$

b) Exact and approximate value of the integral *I*:

$$I = \int_0^1 f(x) \, dx = F(1) - F(0) = \left[-e^{-1}(1^2 + 1) \right] - \left[-e^{0}(0^2 + 1) \right] = -2e^{-1} + 1$$

Using $e^{-1} \approx 0.368$:

$$I \approx 1 - 2 \cdot 0.368 = 1 - 0.736 = 0.264 \Rightarrow I \approx 0.26$$

Geometric interpretation: I represents the (signed) area between the curve C and the x-axis from x = 0 to x = 1.

2