Solution to Exercise 1 – Mathematics Test 2 (2020)

Exercise 1

1) Let n be a natural number greater than 1. Let $\omega = e^{2i\pi/n}$. Consider the sum:

$$S = 1 + \omega + \omega^2 + \cdots + \omega^{n-1}$$

This is a geometric series of n terms with ratio $\omega \neq 1$. So we use the formula:

$$S = \frac{1 - \omega^n}{1 - \omega}$$

But since $\omega^n = e^{2i\pi} = 1$, it follows that:

$$S = \frac{1 - 1}{1 - \omega} = 0$$

Therefore:

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

2) Let $\omega = e^{2i\pi/5}$. Then we know:

$$\omega^k = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$$

So the real part of the sum $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ is:

$$\mathrm{Re}(1+\omega+\omega^2+\omega^3+\omega^4) = 1+\cos\left(\frac{2\pi}{5}\right)+\cos\left(\frac{4\pi}{5}\right)+\cos\left(\frac{6\pi}{5}\right)+\cos\left(\frac{8\pi}{5}\right)$$

Therefore:

$$1 + \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) = 0$$

3) We use the trigonometric identity:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

a) For $\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right)$:

$$= 2\cos(\pi)\cos\left(\frac{3\pi}{5}\right) = -2\cos\left(\frac{3\pi}{5}\right)$$

But we know:

$$\cos\left(\frac{3\pi}{5}\right) = 1 - 2\cos^2\left(\frac{\pi}{5}\right)$$

So:

$$-2\cos\left(\frac{3\pi}{5}\right) = -2(1 - 2\cos^2\left(\frac{\pi}{5}\right)) = 4\cos^2\left(\frac{\pi}{5}\right) - 2$$

b) For $\cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right)$:

$$= 2\cos(\pi)\cos\left(\frac{\pi}{5}\right) = -2\cos\left(\frac{\pi}{5}\right)$$

And since $\cos\left(\frac{2\pi}{5}\right) = \cos\left(\frac{\pi}{5}\right)$ due to symmetry:

$$\cos\left(\frac{4\pi}{5}\right) + \cos\left(\frac{6\pi}{5}\right) = -2\cos\left(\frac{2\pi}{5}\right)$$

4) From above we know:

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{8\pi}{5}\right) = 4\cos^2\left(\frac{\pi}{5}\right) - 2$$

Using previous results, we can solve for $\cos\left(\frac{\pi}{5}\right)$. It is known that:

$$\cos\left(\frac{\pi}{5}\right) = \frac{\sqrt{5}+1}{4}$$

5) Linearize $\cos^3 x$ and deduce the exact value of $\cos\left(\frac{3\pi}{5}\right)$
We use the identity:

$$\cos^3 x = \frac{3\cos x + \cos(3x)}{4}$$

Substitute $x = \frac{\pi}{5}$:

$$\cos^3\left(\frac{\pi}{5}\right) = \frac{3\cos\left(\frac{\pi}{5}\right) + \cos\left(\frac{3\pi}{5}\right)}{4}$$

Then solving for $\cos\left(\frac{3\pi}{5}\right)$:

$$\cos\left(\frac{3\pi}{5}\right) = 4\cos^3\left(\frac{\pi}{5}\right) - 3\cos\left(\frac{\pi}{5}\right)$$

Now substitute $\cos\left(\frac{\pi}{5}\right) = \dfrac{\sqrt{5}+1}{4}$ to get the exact value.