MA1521 Homework 11

AY 24/25 Sem 1 — github/omgeta

Q1. (a)

$$\begin{aligned}\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{1}{1+x^2+y^2} dydx &= \int_0^{\frac{\pi}{2}} \int_0^2 \frac{r}{1+r^2} drd\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} [\ln|1+r^2|]_0^2 d\theta \\ &= \frac{1}{2}\ln 5 \int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{1}{2}\ln 5 [\theta]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}\ln 5 \quad \blacksquare\end{aligned}$$

(b)

$$\begin{aligned}\int_0^1 \int_{)}^{1-x^2} e^{x^2+y^2} dydx &= \int_0^{\frac{\pi}{2}} \int_0^1 re^{r^2} drd\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} [e^{r^2}]_0^1 d\theta \\ &= \frac{1}{2}(e-1)\int_0^{\frac{\pi}{2}} d\theta \\ &= \frac{\pi}{4}(e-1) \quad \blacksquare\end{aligned}$$

Q2.

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi} \int_0^{\sqrt{2}} r|r\cos\theta - r\sin\theta| drd\theta \\ &= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sqrt{2}} r|r\cos\theta - r\sin\theta| drd\theta - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_0^{\sqrt{2}} r|r\cos\theta - r\sin\theta| drd\theta \\ &= \int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} (\cos\theta - \sin\theta)[\frac{1}{3}r^3]^{\sqrt{2}}0 - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\cos\theta - \sin\theta)[\frac{1}{3}r^3]^{\sqrt{2}}0 \\ &= \frac{2\sqrt{2}}{3}[\sin\theta + \cos\theta]_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} - \frac{2\sqrt{2}}{3}[\sin\theta + \cos\theta]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= \frac{2\sqrt{2}}{2}(2\sqrt{2}) - \frac{2\sqrt{2}}{2}(-2\sqrt{2}) \\ &= \frac{16}{3} \quad \blacksquare\end{aligned}$$

Q3. Suppose $f(x,y) = \sqrt{x^2+y^2}$, then:

$$\begin{aligned}f_x &= \frac{x}{\sqrt{x^2+y^2}} \\ f_y &= \frac{y}{\sqrt{x^2+y^2}}\end{aligned}$$

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\begin{aligned}
\text{Surface Area } &= \int\int_D \sqrt{f_x^2 + f_y^2 + 1}\, dA \\
&= \int\int_D \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}}\, dA \\
&= \int_{-1}^{2}\int_{x^2}^{x+2} \sqrt{2}\, dy dx \\
&= \sqrt{2}\int_{-1}^{2} [y]_{x^2}^{x+2} dx \\
&= \sqrt{2}\int_{-1}^{2} 2 + x - x^2 dx \\
&= \sqrt{2}[2x + \frac{x^2}{2} - \frac{x^3}{3}]_{-1}^{2} \\
&= \sqrt{2}(\frac{10}{3} + \frac{7}{6}) \\
&= \frac{9\sqrt{2}}{2} \quad \blacksquare
\end{aligned}
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Q4.

$$
\begin{aligned}
(1+y)y' + (3-4x)y^2 &= 0 \\
(1+y)y' &= (4x-3)y^2 \\
\int \frac{1+y}{y^2} dy &= \int 4x - 3 dx + C \\
\int \frac{1}{y^2} + \frac{1}{y} dy &= 2x^2 - 3x + C \\
\ln|y| - \frac{1}{y} &= 2x^2 - 3x + C \quad \blacksquare
\end{aligned}
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