MA1522 Tutorial 10

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $A = \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.1 & 0.6 & 0.3 \\ 0.5 & 0.2 & 0.3 \end{pmatrix}$ which has each column adding up to 1 and all values between 0 and 1 $\blacksquare$

(b) $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{10} & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$ so:

$$\begin{aligned} x_3 = A^3 x_0 &= \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1^3 & 0 & 0 \\ 0 & -\frac{1}{10}^3 & 0 \\ 0 & 0 & \frac{2}{5}^3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 35 \\ 31.2 \\ 33.8 \end{pmatrix} \quad \blacksquare \end{aligned}$$

(c) $A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{10} & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$ which is the same result as in (b)

(d) As $n \to \infty$, $D^n = \begin{pmatrix} 1^n & 0 & 0 \\ 0 & -\frac{1}{10}^n & 0 \\ 0 & 0 & \frac{2}{5}^n \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$:

$$\begin{aligned} x_\infty &= \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 33.3 \\ 33.3 \\ 33.3 \end{pmatrix} \quad \blacksquare \end{aligned}$$

so in the long run all ants will be equally distributed $\blacksquare$

(e) As $n \to \infty$, $D^n = \begin{pmatrix} 1^n & 0 & 0 \\ 0 & -\frac{1}{10}^n & 0 \\ 0 & 0 & \frac{2}{5}^n \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$:

$$\begin{aligned} x_\infty &= \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & -5 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \\ \alpha+\beta+\gamma \end{pmatrix} \quad \blacksquare \end{aligned}$$

which is always an equilibrium vector $\blacksquare$

Q2. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$, so

$B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$ $\blacksquare$

Q3. (a) $P = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ ■

(b) $P = \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{5}}{5} & -\frac{2\sqrt{5}}{15} \\ -\frac{2}{3} & \frac{\sqrt{5}}{5} & \frac{4\sqrt{5}}{15} \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} \end{pmatrix}$ ■

Q4. (a) $P = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$ and $P^T A P = D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ ■

(b) Result is the same ■

Q5. (a) $A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & -\frac{2}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} & \frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$ ■

(b) Since A is the transpose of the matrix in (a),

$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{6} & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ ■

(c) $A = \begin{pmatrix} -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \\ -\frac{\sqrt{2}\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}$ ■

Q6. (a) $A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & -\frac{4}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{4}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & \frac{4}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} & -\frac{4}{5} \end{pmatrix}$ ■

(b) Result is the same ■