MA1522 Homework 3

AY 24/25 Sem 1 — github/omgeta

Q1. (a) Using the information provided:

$$P = \begin{pmatrix} 0 & 0.4 & 0.4 \\ 0.3 & 0 & 0.6 \\ 0.7 & 0.6 & 0 \end{pmatrix} \quad \blacksquare$$

(b) Diagonalize stochastic matrix P:

$$P = \begin{pmatrix} 0 & \frac{8}{11} & 2 \\ -1 & \frac{9}{11} & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 0 & \frac{8}{11} & 2 \\ -1 & \frac{9}{11} & -3 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

Then P^n is given by:

$$P^n = \begin{pmatrix} 0 & \frac{8}{11} & 2 \\ -1 & \frac{9}{11} & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{3}{5}^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & -\frac{2}{5}^n \end{pmatrix} \begin{pmatrix} 0 & \frac{8}{11} & 2 \\ -1 & \frac{9}{11} & -3 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

Hence, we have steady state vector:

$$\begin{aligned}
P^n x_0 = P^n \begin{pmatrix} a \\ b \\ c \end{pmatrix} \xrightarrow[n\to\infty]{} x_\infty &= \begin{pmatrix} 0 & \frac{8}{11} & 2 \\ -1 & \frac{9}{11} & -3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{8}{11} & 2 \\ -1 & \frac{9}{11} & -3 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= \frac{1}{28} \begin{pmatrix} 8 & 8 & 8 \\ 9 & 9 & 9 \\ 11 & 11 & 11 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\
&= \frac{1}{28} \begin{pmatrix} 8\,a + 8\,b + 8\,c \\ 9\,a + 9\,b + 9\,c \\ 11\,a + 11\,b + 11\,c \end{pmatrix} \\
&= \frac{1}{28} \begin{pmatrix} 8 \\ 9 \\ 11 \end{pmatrix}
\end{aligned}$$

Therefore, we see from the steady state vector that in the long run: Ah Meng will visit gym C the most with probability $\frac{11}{28}$ and gym A the least with probability $\frac{8}{28}$. $\blacksquare$

Q2. (a) Suppose $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, then by column-vector multiplication for the given equation:

$$\begin{aligned} a_n &= m_1 \cdot a_{n-1} + m_2 \cdot a_n \implies m_1 = 0 \text{ and } m_2 = 1 \\ a_{n+1} &= m_3 \cdot a_{n-1} + m_4 \cdot a_n \implies m_3 = 1 \text{ and } m_4 = 1 \qquad \text{(Given recurrence relation)} \end{aligned}$$

Therefore $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ ■

(b) Using repeated applications of the given relation with matrix M:

$$\begin{aligned} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} &= M \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} \\ \implies \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} &= M \begin{pmatrix} a_{20} \\ a_{21} \end{pmatrix} \\ &= M \left(M \begin{pmatrix} a_{19} \\ a_{20} \end{pmatrix} \right) \qquad \vdots \\ &= M^{21} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ &= \begin{pmatrix} 10946 \\ 17711 \end{pmatrix} \end{aligned}$$

Therefore, $a_{22} = 17711$ ■

(c) Use the characteristic equation to find the eigenvalues of M:

$$\begin{aligned} \det(M - \lambda I) &= 0 \\ \det \begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} &= 0 \\ -\lambda(1-\lambda) - 1 &= 0 \\ \lambda^2 - \lambda - 1 &= 0 \\ \lambda &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

When $\lambda = \frac{1+\sqrt{5}}{2}$ for eigenvector $\vec{v_1}$, $\begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \vec{v_1} = \vec{0}$:

$$\begin{pmatrix} -\frac{1+\sqrt{5}}{2} & 1 & 0 \\ 1 & \frac{1-\sqrt{5}}{2} & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \vec{v_1} = s \begin{pmatrix} -1+\sqrt{5} \\ 2 \end{pmatrix}, s \in \mathbb{R}$$

When $\lambda = \frac{1-\sqrt{5}}{2}$ for eigenvector $\vec{v_2}$, $\begin{pmatrix} -\frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \vec{v_2} = \vec{0}$:

$$\begin{pmatrix} -\frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & \frac{1+\sqrt{5}}{2} & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \vec{v_2} = t \begin{pmatrix} -1-\sqrt{5} \\ 2 \end{pmatrix}, t \in \mathbb{R}$$

Then we can find the diagonalization:

$$M = \begin{pmatrix} -1+\sqrt{5} & -1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} -1+\sqrt{5} & -1-\sqrt{5} \\ 2 & 2 \end{pmatrix}^{-1} \quad ■$$

(d) Using powers of the diagonalization of M:

$$
\begin{aligned}
\begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} &= \begin{pmatrix} -1+\sqrt{5} & -1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^{n-1} & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^{n-1} \end{pmatrix} \begin{pmatrix} -1+\sqrt{5} & -1-\sqrt{5} \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -1+\sqrt{5} & -1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^{n-1} & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^{n-1} \end{pmatrix} \frac{1}{4\sqrt{5}} \begin{pmatrix} 2 & 1+\sqrt{5} \\ -2 & -1+\sqrt{5} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{4\sqrt{5}} \begin{pmatrix} -1+\sqrt{5} & -1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} (\frac{1+\sqrt{5}}{2})^{n-1} & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^{n-1} \end{pmatrix} \begin{pmatrix} 1+\sqrt{5} \\ -1+\sqrt{5} \end{pmatrix} \\
&= \frac{1}{4\sqrt{5}} \begin{pmatrix} -1+\sqrt{5} & -1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \frac{(1+\sqrt{5})^n}{2^{n-1}} \\ -\frac{(1-\sqrt{5})^n}{2^{n-1}} \end{pmatrix} \\
\therefore a_n &= \frac{1}{4\sqrt{5}} \left(2\frac{(1+\sqrt{5})^n}{2^{n-1}} - 2\frac{(1-\sqrt{5})}{2^{n-1}} \right) \\
&= \frac{1}{\sqrt{5}} \left(\frac{(1+\sqrt{5})^n}{2^n} - \frac{(1-\sqrt{5})}{2^n} \right) \\
&= \frac{1}{\sqrt{5}} \left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n \right) \quad \blacksquare
\end{aligned}
$$

(e)

Q3. (a) By Invertible Matrix Theorem, if A has a zero row then 0 must be an eigenvalue $\implies$ 0 is the last singular value σ_3 ■

(b) Compare eigenvalues of A^TA with known eigenvalues:

$$\det(A^TA - \lambda I) = 0$$

$$\det\begin{pmatrix} 16+b^2-\lambda & 4a & 24 \\ 4a & a^2-\lambda & 6a \\ 24 & 6a & 36-\lambda \end{pmatrix} = 0$$

$$\lambda^3 - \lambda^2(a^2+b^2+52) + \lambda(a^2b^2+36b^2) = 0$$

$$(\lambda-72)(\lambda-20)\lambda = 0 \qquad \text{(Known)}$$

$$\lambda^3 - 92\lambda^2 + 1440\lambda = 0$$

Which gives us two equations:

$$a^2+b^2+52 = 92 \qquad (1)$$

$$a^2b^2+36b^2 = 1440 \qquad (2)$$

Solving simultaneously and only including $a, b > 0$:

$$a = 2, b = 6 \quad ■$$

(c) Find the corresponding eigenvector for each singular value:

$$\lambda_1 = (6\sqrt{2})^2 \implies \vec{v_1} = \begin{pmatrix} 4/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\lambda_1 = (2\sqrt{5})^2 \implies \vec{v_2} = \begin{pmatrix} -5/6 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\lambda_1 = 0^2 \implies \vec{v_3} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

Then form V from the normalized unit eigenvectors:

$$V = \begin{pmatrix} 4/\sqrt{26} & -5/\sqrt{65} & 0 \\ 1/\sqrt{26} & 2/\sqrt{65} & -3/\sqrt{10} \\ 3/\sqrt{26} & 6/\sqrt{65} & 1/\sqrt{10} \end{pmatrix} \quad ■$$

(d) For each $\vec{v_i}$ column unit eigenvector of V, find the corresponding left singular vector:

$$\sigma_1 = 6\sqrt{2} \implies u_1 = \frac{1}{6\sqrt{2}} A \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$\sigma_2 = 2\sqrt{5} \implies u_2 = \frac{1}{2\sqrt{5}} A \frac{1}{\sqrt{65}} \begin{pmatrix} -5 \\ 2 \\ 6 \end{pmatrix} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$$

$$\sigma_3 = 0 \implies u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \text{(By choosing orthogonal vector)}$$

Therefore $U = \begin{pmatrix} 3/\sqrt{13} & 2/\sqrt{13} & 0 \\ 2/\sqrt{13} & -3/\sqrt{13} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ■

Q4. Check if the input vectors are linearly independent by reducing matrix U which has columns as the input vectors:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\implies U \text{ is linearly independent}$$

$$\implies U \text{ is invertible}$$

Suppose A is the standard matrix for T, and V is the output vectors, then by the relation of A, U, V:

$$\begin{aligned} V &= AU \\ A &= VU^{-1} \\ &= \begin{pmatrix} 0 & -2 & 0 & -4 \\ 0 & 1 & 0 & 1 \\ 3 & 3 & 3 & 2 \\ 3 & 6 & 3 & 11 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1 & -3 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 \\ 4 & 7 & -1 & 4 \end{pmatrix} \end{aligned}$$

Then by using MATLAB, the corresponding eigenvector for eigenvalue $\lambda = 2$ is:

$$\vec{u} = \begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \quad \blacksquare$$