MA1522 Tutorial 5

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \right\}$

$$\begin{bmatrix} 2 & 0 & 2 & 3 \\ -1 & 3 & 4 & 6 \\ 0 & 2 & 3 & 6 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 9/2 \\ 0 & 1 & 0 & 15/2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

S is not linearly independent. $\begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} = \frac{9}{2} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \frac{15}{2} \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ $\blacksquare$

(b) $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 0 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

S is linearly independent. $\blacksquare$

(c) $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

S is not linearly independent. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\blacksquare$

(d) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

S is linearly independent. $\blacksquare$

Q2. (a) S_1 is linearly independent. $\blacksquare$

(b) S_2 is is not linearly independent. $\vec{w} - \vec{u} = -(\vec{v} - \vec{w}) - (\vec{u} - \vec{v})$ $\blacksquare$

(c) S_3 is linearly independent. $\blacksquare$

(d) S_4 is linearly independent. $\blacksquare$

(e) S_5 is not linearly independent. $\vec{u} + \vec{v} + \vec{w} = -\frac{1}{2}(\vec{u} + \vec{v}) - \frac{1}{2}(\vec{v} + \vec{w}) - \frac{1}{2}(\vec{u} + \vec{w})$ $\blacksquare$

Q3. (a) $\vec{v} \in V = a\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + c\begin{bmatrix}0\\1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$

$$\begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & 1 & 0\\0 & 0 & 1 & 1\\0 & 1 & 0 & 1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\therefore \text{Basis } = \left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}\right\} \quad \blacksquare$$

(b)

$$\begin{bmatrix}1 & -1 & 0 & 1\\0 & 2 & 3 & -1\\-1 & 3 & 0 & 1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 0 & 1\\0 & 0 & 1 & -1\end{bmatrix}$$

$$\therefore \text{Basis } = \left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\0\end{bmatrix}\right\} \quad \blacksquare$$

(c)

$$\begin{bmatrix}1 & 0 & 1 & 1 & -1 & 0\\0 & 1 & 1 & 2 & 1 & 0\\1 & 1 & 2 & 1-2 & 0 & \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 1 & 0 & -2 & 0\\0 & 1 & 1 & 0 & -1 & 0\\0 & 0 & 0 & 1 & 1 & 0\end{bmatrix}$$

$$\therefore v \in V = a_3\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix} + a_5\begin{bmatrix}2\\1\\0\\-1\\1\end{bmatrix}$$

$$\therefore \text{Basis } = \left\{\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\\-1\\1\end{bmatrix}\right\} \quad \blacksquare$$

Q4. $\vec{u_1}, \vec{u_2}, \vec{u_3}$ form a bsis for $\mathbb{R}^3$ when the determinant formed by the corresponding matrix is non-zero:

$$\begin{vmatrix}a & -1 & 1\\1 & a & -1\\-1 & 1 & a\end{vmatrix} \neq 0$$

$$a\begin{vmatrix}a & -1\\1 & a\end{vmatrix} + \begin{vmatrix}1 & -1\\-1 & a\end{vmatrix} + \begin{vmatrix}1 & a\\-1 & 1\end{vmatrix} \neq 0$$

$$a(a^2+1) + (a-1) + (1+a) \neq 0$$

$$a^3 + 3a \neq 0$$

$$a(a^2+3) \neq 0$$

$$a \neq 0 \vee a^3 + 3 \neq 0 (\text{always true})$$

$$a \neq 0 \quad \blacksquare$$

Q5. (a) No, because any $\forall \vec{u} \in U, \vec{v} \in V, \vec{u} + \vec{v} \notin U \cup V$ $\blacksquare$

(b) Show $U + V$ as a span of its vectors.

$$U + V = \left\{ a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$
$$= \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

Find the number of linearly independent vectors in $U + V$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore U + V = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ with } \dim(U + V) = 3 \quad \blacksquare$$

(c) When $\vec{v} = \vec{0}$, $U + V = U$, and when $\vec{u} = \vec{0}$, $U + V = V$. $\therefore U + V$ contains U, V $\blacksquare$

(d) $\dim U = 2, \dim V = 2$ $\blacksquare$

(e) For $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in U \cap V$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \vec{x} = d \begin{bmatrix} 2 \\ -1 \\ -2 \\ 0 \end{bmatrix}, d \in \mathbb{R} \implies \dim(U \cap V) = 1 \quad \blacksquare$$

(f) $\dim(U + V) = 3 = 2 + 2 - 1 = \dim U + \dim V - \dim(U \cap V)$ $\blacksquare$