CS3230 Tutorial 2

AY 25/26 Sem 1 — github/omgeta

Q1). Give a tight asymptotic bound for $T(n) = 4 \cdot T(\frac{n}{4}) + \frac{n}{\log n}$

$$
\begin{aligned}
T(n) &= 4T(\frac{n}{4}) + \frac{n}{\log n} \\
\implies \frac{T(n)}{n} &= \frac{T(n/4)}{n/4} + \frac{1}{\log n} \\
\frac{T(n/4)}{n/4} &= \frac{T(n/4^2)}{n/4^2} + \frac{1}{\log n/4} \\
&\vdots \\
\frac{T(n/4^{\log_4(n)-1})}{n/4^{\log_4(n)-1}} &= \frac{T(n/4^{\log_4(n)})}{n/4^{\log_4(n)}} + \frac{1}{\log n/4^{\log_4(n)-1}} && (\text{when } \tfrac{n}{4^k} = 1, k = \log_4 n)
\end{aligned}
$$

Then by cancellation,

$$
\begin{aligned}
\frac{T(n)}{n} &= \frac{T(1)}{T(1)} + \frac{1}{\log n} + \cdots + \frac{1}{\log n/4^{\log_4(n)-1}} \\
&= \frac{T(1)}{T(1)} + \frac{1}{\log 4^i} + \cdots + \frac{1}{\log 4^1} && (\text{let } n = 4^i) \\
&= \frac{T(1)}{T(1)} + \frac{1}{\log 4}\left\{\frac{1}{i} + \cdots + \frac{1}{1}\right\} && (\text{let } n = 4^i) \\
&= \frac{T(1)}{T(1)} + \Theta(\log i) && (\text{harmonic sum}) \\
&= \frac{T(1)}{T(1)} + \Theta(\log\log n) \\
\therefore T(n) &\in \Theta(n \log\log n)
\end{aligned}
$$

Q2). $T(n) = 5T(\frac{n}{3}) + n$, $d = \log_3 5 = 1.46\ldots$ and $f(n) = n \in O(n^{\log_3 5 - \epsilon})$, so by case 1 $T(n) \in \Theta(n^{\log_3 5})$

Q3). $T(n) = 9T(\frac{n}{3}) + n^3$, $d = \log_3 9 = 2$ and $f(n) = n^3 \in \Omega(n^{2+\epsilon})$ ad for regularity $9(\frac{n}{3})^3 = \frac{1}{3}n^3 \leq \frac{1}{3}n^3 \land \frac{1}{3} < 1$, so by case 3 $T(n) \in \Theta(n^3)$

Q4). $T(n) = 16T(\frac{n}{4}) + n^2 \log n$, $d = \log_4 16 = 2$ and $f(n) = n^2 \log n \in \Theta(n^2 \log n)$, so by case 2 $T(n) \in \Theta(n^2 \log^2 n)$

Q5). Give a tight asymptotic bound for $T(n) = 4 \cdot T(\frac{n}{2}) + \sqrt{n}$.

1. Proof $T(n) \in O(n^2)$:
   1.1. Guess $T(n) \leq cn^2 - d\sqrt{n}$
   1.2. Base case: $T(0) = 0 \leq c \cdot 0^2 - d\sqrt{0}$
   1.3. Inductive step:
   $T(n) = 4T(\frac{n}{2}) + \sqrt{n} \leq 4(c\frac{n^2}{4} - d\frac{\sqrt{n}}{\sqrt{2}}) + \sqrt{n} = cn^2 - 2\sqrt{2}d\sqrt{n} + \sqrt{n} = cn^2 + (1 - 2\sqrt{2})d\sqrt{n}$,
   so choose $d < \frac{1}{1-2\sqrt{2}}$ and from base case $T(1) \leq q$, therefore
   $T(1) \leq c \times 1^2 - d \times 1 \leq q \implies c \leq q + d$
2. Proof $T(n) \in \Omega(n^2)$:
   2.1. Guess $T(n) \geq cn^2$
   2.2. Base case: $T(0) = 0 \geq c \cdot 0^2$
   2.3. Inductive step: $T(n) = 4T(\frac{n}{2}) + \sqrt{n} \geq 4 \cdot c(\frac{n}{2})^2 = cn^2$

Q6). $T(k, n) = T(\lceil \frac{k}{2} \rceil, n) + T(\lfloor \frac{k}{2} \rfloor, n) + kn = 2T(\frac{k}{2}, n) + \Theta(kn)$
By recursion tree, there are $\log k$ levels with $\Theta(kn)$ work at each level so $T(k, n) = \Theta(kn \log k)$

B1). $T(n) = \begin{cases} 1 & \text{, if } n \leq 4 \\ 8T(2^{\sqrt{\log n}}) & \text{, if } n > 4 \end{cases}$

Let $m = \log n, S(m) = T(2^m)$ to get rid of base 2

$$\begin{aligned} S(m) &= 8S(\sqrt{m}) \\ S(m^{\frac{1}{2}}) &= 8^2 S(m^{\frac{1}{4}}) \\ &\vdots \\ S(m^{\frac{1}{2^{k-1}}}) &= 8^k S(m^{\frac{1}{2^k}}) \end{aligned}$$

where we hit base case at $m^{\frac{1}{2^k}} \leq \log 4(2) \iff 2^k \geq \log m \iff k \geq \log\log m$

$$\begin{aligned} \therefore S(m) &\leq 8^{\log\log m} \\ &\leq 2^{3\log\log m} \\ &\leq (\log m)^3 \\ &\in \Theta((\log m)^3) \\ \therefore T(n) &= S(\log n) \in \Theta((\log\log n)^3) \end{aligned}$$

B2). $T(n) = \begin{cases} 1 & \text{, if } n = 1 \\ 5 + \frac{2}{n-1}\sum_{i=1}^{n-1} T(i) & \text{, if } n \in \{2,3,4,\cdots\} \end{cases}$

$$\begin{aligned} \sum_{i=1}^{n-1} T(i) - \sum_{i=1}^{n-2} T(i) &= T(n-1) \\ \frac{n-1}{2}(T(n)-5) - \frac{n-2}{2}(T(n-1)-5) &= T(n-1) \\ (n-1)T(n) - (n-2)T(n-1) &= 2T(n-1) + 5 \\ (n-1)T(n) &= nT(n-1) + 5 \\ \frac{T(n)}{n} &= \frac{T(n-1)}{n-1} + \frac{5}{n(n-1)} \\ &\vdots \\ &= 1 + 5\sum_{k=2}^{n}(\frac{1}{k-1} - \frac{1}{k}) \\ &= 1 + 5(1 - \frac{1}{n}) \\ &= 6 - \frac{5}{n} \\ T(n) &= 6n - 5 \\ &\in \Theta(n) \end{aligned}$$