MA1522 Tutorial 3

AY 24/25 Sem 1 — github/omgeta

Q1. If $A \in \mathbb{R}^{4\times 4}$ is obtained from I by the following sequence of elementary row operations:

$$I \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_1 - R_2} \xrightarrow{R_2 \leftrightarrow R_4} \xrightarrow{R_3 + 3R_1} A$$

Then A is also obtained from I by the following matrix multiplications:

$$A = E_4 E_3 E_2 E_1 I$$

Where the elementary matrices E_i are given by:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And their inverse are given by:

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$E_2^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then the inverse A^{-1} is obtained from I by the following matrix mulitiplications:

$$\begin{aligned} A^{-1} &= (E_4 E_3 E_2 E_1 I)^{-1} \\ &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} \quad \blacksquare \end{aligned}$$

Q2. (a) Find the LU factorisation for A:

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix} \xrightarrow{R_2+3R_1} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 8 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_3-4R_1} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 8 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2+3R_1} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare$$

Let $\vec{y} = U\vec{x}$ and solve $L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 \\ 4 & -1 & 1 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Then solve $U\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & -3 & 4 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 11/3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Therefore, $\vec{x} = \begin{bmatrix} -2/3 \\ 11/3 \\ 3 \end{bmatrix} \quad \blacksquare$

(b) Find the LU factorisation for A:

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \xrightarrow[R_2+\frac{1}{2}R_1]{R_2-3R_1} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3+2R_2} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \blacksquare$$

Let $\vec{y} = U\vec{x}$ and solve $L\vec{y} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1/2 & -2 & 1 & 17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 17 \end{bmatrix}$$

Then solve $U\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 17 \end{bmatrix}$

$$\begin{bmatrix} 2 & -4 & 4 & -2 & 0 \\ 0 & 3 & -5 & 3 & 0 \\ 0 & 0 & 0 & 5 & 17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -4/3 & 0 & -17/5 \\ 0 & 1 & -5/3 & 0 & -17/5 \\ 0 & 0 & 0 & 1 & 17/5 \end{bmatrix}$$

Therefore, $\vec{x} = \frac{17}{5}\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \frac{x_3}{3}\begin{bmatrix} -4 \\ -5 \\ 3 \\ 0 \end{bmatrix}$ ■

Q3. (a) Find an LU factorisation for A

$$A = \begin{bmatrix} 2 & -6 & 6 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -6 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -19 \\ 0 & -14 & 10 \\ 0 & 21 & 15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 30 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U \quad ■$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{bmatrix} \quad ■$$

(b) The MATLAB LU is the same. ■

Q4. First calculate the determinant by cofactor expansion:

$$\begin{aligned} \det(A) &= -x\begin{vmatrix} -x & 1 \\ -5 & 4-x \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 2 & 4-x \end{vmatrix} \\ &= -x[-x(4-x) - 1(-5)] - [0(4-x) - 1(2)] \\ &= -x(-4x + x^2 + 5) - (-2) \\ &= -x^3 + 4x^2 - 5x + 2 \quad ■ \end{aligned}$$

For A to be singular, $\det(A) = 0$:

$$\begin{aligned} -x^3 + 4x^2 - 5x + 2 &= 0 \\ (x-1)^2(x-2) &= 0 \\ x &= 1, 2 \quad ■ \end{aligned}$$

Q5. First, reduce the relevant matrices:

$$\begin{bmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{bmatrix} \xrightarrow{R_2 - xR_3} \begin{bmatrix} a+px & b+qx & c+rx \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{bmatrix}$$

$$\xrightarrow{R_1 - xR_2} \begin{bmatrix} a(1+x^3) & b(1+x^3) & c(1+x^3) \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{bmatrix} \xrightarrow{R_2 + x^2R_1} \begin{bmatrix} a & b & c \\ p & q & r \\ u+ax & v+bx & w+cx \end{bmatrix}$$

$$\xrightarrow{R_3 - xR_1} \begin{bmatrix} a & b & c \\ p & q & r \\ u & v & w \end{bmatrix}$$

Then, we can conclude:

$$\begin{vmatrix} a+px & b+qx & c+rx \\ p+ux & q+vx & r+wx \\ u+ax & v+bx & w+cx \end{vmatrix} = (1+x^3) \begin{vmatrix} a & b & c \\ p-ax^2 & q-bx^2 & r-cx^2 \\ u+ax & v+bx & w+cx \end{vmatrix}$$

$$= (1+x^3) \begin{vmatrix} a & b & c \\ p & q & r \\ u & v & w \end{vmatrix}$$

Q6. First, find $\det(A)$ and $\det(B)$

$$\begin{aligned} \det(A) &= 1 \begin{vmatrix} 2 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 2 \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \\ &= 2[1(1) - 2(1)] \\ &= -2 \\ \det(B) &= 1 \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \\ &= 3 \end{aligned}$$

(a) $\det(3A^T) = 3^4 \det(A) = -162$ ■

(b) $\det(3AB^{-1}) = 3^4 \frac{\det(A)}{\det(B)} = -54$ ■

(c) $\det(3A^T) = \frac{1}{\det(3B)} = \frac{1}{3^4 \det(B)} = \frac{1}{243}$ ■

Q7. Let $A = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 2 & -2 \\ 0 & 1 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Then, $\det(A) = 1\begin{vmatrix} 2 & -2 \\ 1 & 3 \end{vmatrix} = 8$. By Cramer's rule:

$$\begin{aligned}
x_1 &= \frac{\begin{vmatrix} 1 & 5 & 3 \\ 2 & 2 & -2 \\ 0 & 1 & 3 \end{vmatrix}}{8} \\
&= \frac{(6+2) - 2(15-3)}{8} \\
&= -2 \\
x_2 &= \frac{\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{vmatrix}}{8} \\
&= \frac{6}{8} \\
&= \frac{3}{4} \\
x_3 &= \frac{\begin{vmatrix} 1 & 5 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 0 \end{vmatrix}}{8} \\
&= \frac{-2}{8} \\
&= -\frac{1}{4}
\end{aligned}$$

Therefore, $\vec{x} = \begin{bmatrix} -2 \\ 3/4 \\ -1/4 \end{bmatrix}$ ■

Q8. First find the adjoint of A:

$$\begin{aligned}
\operatorname{adj}(A) &= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \\
&= \begin{bmatrix} 12 & 6 & -5 \\ 3 & 0 & -1 \\ -6 & -3 & 2 \end{bmatrix} \quad ■
\end{aligned}$$

Then find the determinant:

$$\begin{aligned}
\det(A) &= 1\begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} \\
&= 12 + 3(-1-4) \\
&= -3
\end{aligned}$$

Then the inverse A^{-1} is given by:

$$\begin{aligned}
A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) \\
&= -\frac{1}{3}\begin{bmatrix} 12 & 6 & -5 \\ 3 & 0 & -1 \\ -6 & -3 & 2 \end{bmatrix} \quad ■
\end{aligned}$$