MA1522 Tutorial 4

AY 24/25 Sem 1 — github/omgeta

Q1. (a) A is a line passing through $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and parallel to $\begin{bmatrix}1\\2\\3\end{bmatrix}$

(b) Let $x = 1 + t, y = 1 + 2t, z = 1 + 3t$

$$\begin{aligned} x + y - z &= (1 + t) + (1 + 2t) - (1 + 3t) \\ &= 1 \\ x - 2y + z &= (1 + t) - 2(1 + 2t) + (1 + 3t) \\ &= 0 \end{aligned}$$

$\therefore A = \{(x, y, z) | x + y - z = 1 \land x - 2y + z = 0\}$ ■

(c) $M = \begin{bmatrix}1 & 1 & -1\\1 & -2 & 1\\0 & 0 & 0\end{bmatrix}, \vec{b} = \begin{bmatrix}1\\0\\0\end{bmatrix}$

Q2. (a) (i)

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 \\ 3 \\ -7 \\ 3 \end{bmatrix} = 2\vec{u_1} - \vec{u_2} - \vec{u_3} \quad \blacksquare$$

(ii) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0\vec{u_1} + 0\vec{u_2} + 0\vec{u_3} \quad \blacksquare$

(iii)

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 5 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ cannot be expressed as a linear combination of $\vec{u_1}, \vec{u_2}, \vec{u_3} \quad \blacksquare$

(iv)

$$\begin{bmatrix} 2 & 3 & -1 & -4 \\ 1 & -1 & 0 & 6 \\ 0 & 5 & 2 & -13 \\ 3 & 2 & 1 & 4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -4 \\ 6 \\ -13 \\ 4 \end{bmatrix} = 3\vec{u_1} - 3\vec{u_2} + \vec{u_3} \quad \blacksquare$$

(b) Yes $\blacksquare$

Q3. (a) $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}5\\2\\3\end{bmatrix}$ satisfy $x - y - z = 0$

$\implies \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}5\\2\\3\end{bmatrix} \in V$

$\implies$ by closure over addition and multiplication in subspaces, $\text{Span}(S) \subseteq V$

$x - y - z \implies x = y + z$

$\implies \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}y+z\\y\\z\end{bmatrix} = y\begin{bmatrix}1\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\0\\1\end{bmatrix}$

$\implies V = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$

$$\begin{bmatrix}1 & 5 & 1 & 1\\1 & 2 & 1 & 0\\0 & 3 & 0 & 1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 1 & -2/3\\0 & 1 & 0 & 1/3\\0 & 0 & 0 & 0\end{bmatrix}$$

$\implies V \subseteq \text{Span}(S)$

$\therefore \text{Span}(S) = V \quad \blacksquare$

(b)

$$\begin{bmatrix}1 & 5 & 0\\1 & 2 & 0\\0 & 3 & 1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$\therefore \text{rank}(T) = 3 \implies \text{Span}(T) = \mathbb{R}^3 \quad \blacksquare$

Q4. (a)

$$\begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & 1 & 1\\0 & 0 & 1 & 1\\1 & 0 & 1 & 0\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$\therefore S$ spans $\mathbb{R}^4 \quad \blacksquare$

(b) Since S has only 3 vectors, S does not span $\mathbb{R}^4 \quad \blacksquare$

(c)

$$\begin{bmatrix}6 & 2 & 3 & 5 & 0\\4 & 0 & 2 & 6 & 4\\-2 & 0 & -1 & -3 & -2\\4 & 1 & 2 & 2 & -1\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 1/2 & 0 & -1/2\\0 & 1 & 0 & 0 & -1\\0 & 0 & 0 & 1 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$\therefore S$ does not span $\mathbb{R}^4 \quad \blacksquare$

(d)

$$\begin{bmatrix}1 & 1 & 0 & 2 & 1\\1 & 2 & 0 & 1 & 2\\0 & -1 & 1 & 2 & 3\\0 & 1 & 1 & 1 & 4\end{bmatrix} \xrightarrow{RREF} \begin{bmatrix}1 & 0 & 0 & 0 & 3\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 5\\0 & 0 & 0 & 1 & -1\end{bmatrix}$$

$\therefore S$ spans $\mathbb{R}^4 \quad \blacksquare$

Q5. (a)

$$\left[\begin{array}{ccc|cc} 2 & -1 & 0 & 1 & 0 \\ -2 & 1 & 0 & -1 & 1 \\ 0 & -1 & 9 & -5 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|cc} 1 & 0 & -\frac{9}{2} & 3 & 0 \\ 0 & 1 & -9 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\therefore \text{Span}\{\vec{v_1}, \vec{v_2}\} \not\subseteq \text{Span}\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ ■

$$\left[\begin{array}{cc|ccc} 1 & 0 & 2 & -1 & 0 \\ -1 & 1 & -2 & 1 & 0 \\ -5 & 1 & 0 & -1 & 9 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|ccc} 1 & 0 & \frac{1}{5} & -\frac{9}{5} & \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & \frac{9}{10} \end{array}\right]$$

$\therefore \text{Span}\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} \not\subseteq \text{Span}\{\vec{v_1}, \vec{v_2}\}$ ■

(b)

$$\left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \text{Span}\{\vec{v_1}, \vec{v_2}\} \subseteq \text{Span}\{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ ■

$$\left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \text{Span}\{\vec{u_1}, \vec{u_2}, \vec{u_3}\} \subseteq \text{Span}\{\vec{v_1}, \vec{v_2}\}$ ■

Q6. (a) $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ ■

(b) S is not a linear span/subspace since $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \in S \wedge -\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \notin S$ ■

(c) $S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 4/3 \\ 0 \\ -2/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ ■

(d)

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = a - c - d$$

$$\implies S = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : a - c - d = 0 \right\}$$

$$\implies S = \left\{ \begin{bmatrix} s+t \\ u \\ s \\ t \end{bmatrix} : s, t, u \in \mathbb{R} \right\}$$

$\therefore S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ ■

(e) $S = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ ■

(f) S is not a linear span/subspace since $\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \in S \wedge \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \notin S$ ■

(g)

$$\left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \left\{ \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

$$\therefore S = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \blacksquare$$