MA1521 Homework 5

AY 24/25 Sem 1 — github/omgeta

Q1. (a)

$$\begin{aligned}
\text{Area} &= \int_{-\pi/3}^{\pi/3} \frac{1}{2}\sec^2 x + 4\sin^2 x dx \\
&= \int_{-\pi/3}^{\pi/3} \frac{1}{2}\sec^2 x + 2 - 2\cos 2x dx \\
&= [\frac{1}{2}\tan x + 2x - \sin 2x]_{-\pi/3}^{\pi/3} \\
&= (\frac{1}{2}\tan\frac{\pi}{3} + \frac{2\pi}{3} - \sin\frac{2\pi}{3}) - (\frac{1}{2}\tan -\frac{\pi}{3} - \frac{2\pi}{3} - \sin -\frac{2\pi}{3}) \\
&= (\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}) - (-\frac{\sqrt{3}}{2} - \frac{2\pi}{3} + \frac{\sqrt{3}}{2}) \\
&= \frac{4\pi}{3} \quad \blacksquare
\end{aligned}$$

(b) Express $y = \frac{1}{4}x^2$ in terms of y: $x = \pm 2\sqrt{y}$
Express $y = x$ in terms of y: $x = y$

$$\begin{aligned}
\text{Area} &= \int_0^1 2\sqrt{y} - y dy \\
&= [\frac{4}{3}y^{\frac{3}{2}} - \frac{y^2}{2}]_0^1 \\
&= (\frac{4}{3} - \frac{1}{2}) - (0 - 0) \\
&= \frac{5}{6} \quad \blacksquare
\end{aligned}$$

(c) Graphs of $y = 4 - x^2$ and $y = 2 - x$ intersect at:

$$\begin{aligned}
4 - x^2 &= 2 - x \\
x^2 - x - 2 &= 0 \\
(x-2)(x+1) &= 0 \\
x &= -1, 2
\end{aligned}$$

Then, find the area:

$$\begin{aligned}
\text{Area} &= \int_{-2}^{-1} (2-x) - (4-x^2) dx + \int_{-1}^{2} (4-x^2) - (2-x) dx \\
&= \int_{-2}^{-1} x^2 - x - 2 dx + \int_{-1}^{2} 2 + x - x^2 dx \\
&= [\frac{x^3}{3} - \frac{x^2}{2} - 2x]_{-2}^{-1} + [2x + \frac{x^2}{2} - \frac{x^3}{3}]_{-1}^{2} \\
&= (-\frac{1}{3} - \frac{1}{2} + 2) - (-\frac{8}{3} - 2 + 4) + (4 + 2 - \frac{8}{3}) - (-2 + \frac{1}{2} + \frac{1}{3}) \\
&= \frac{19}{3} \quad \blacksquare
\end{aligned}$$

Q2. (a) First, find the derivative:

$$\begin{aligned} y &= \ln(\sec x) \\ \frac{dy}{dx} &= \frac{\sec x \tan x}{\sec x} \\ &= \tan x \end{aligned}$$

Then, find the arc length:

$$\begin{aligned} \text{Arc length} &= \int_0^{\pi/4} \sqrt{1+\tan^2 x} dx \\ &= \int_0^{\pi/4} \sec x dx \\ &= [\ln(\sec x + \tan x)]_0^{\pi/4} \\ &= \ln(\sqrt{2}+1) - \ln(1+0) \\ &= \ln(\sqrt{2}+1) \quad \blacksquare \end{aligned}$$

(b) First, find the derivative:

$$\begin{aligned} x &= \frac{1}{2}e^{2y} + \frac{1}{8}e^{-2y} \\ \frac{dx}{dy} &= e^{2y} - \frac{1}{4}e^{-2y} \end{aligned}$$

Then, find the arc length:

$$\begin{aligned} \text{Arc length} &= \int_{\ln 2}^{\ln 3} \sqrt{1+(e^{2y} - \frac{1}{4}e^{-2y})^2} dy \\ &= \int_{\ln 2}^{\ln 3} \sqrt{1+e^{4y} + \frac{1}{16}e^{-4y} - \frac{1}{2}} dy \\ &= \int_{\ln 2}^{\ln 3} \sqrt{e^{4y} + \frac{1}{2} + \frac{1}{16}e^{-4y}} dy \\ &= \int_{\ln 2}^{\ln 3} \sqrt{(e^{2y} + \frac{1}{4}e^{-2y})^2} dy \\ &= \int_{\ln 2}^{\ln 3} e^{2y} + \frac{1}{4}e^{-2y} dy \\ &= [\frac{e^{2y}}{2} - \frac{1}{8}e^{-2y}]_{\ln 2}^{\ln 3} \\ &= (\frac{e^{2\ln 3}}{2} - \frac{e^{-2\ln 3}}{8}) - (\frac{e^{2\ln 2}}{2} - \frac{e^{-2\ln 2}}{8}) \\ &= (\frac{3^2}{2} - \frac{3^{-2}}{8}) - (\frac{2^2}{2} - \frac{2^{-2}}{8}) \\ &= \frac{725}{288} \quad \blacksquare \end{aligned}$$

Q3.

$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 x^2 dx + \pi \int_1^2 4x^2 - 4x + 1 dx \\
&= \pi[\frac{x^3}{3}]_0^1 + \pi[\frac{4}{3}x^3 - 2x^2 + x]_1^2 \\
&= \pi(\frac{1}{3}) + \pi(\frac{14}{3} - \frac{1}{3}) \\
&= \frac{14\pi}{3} \quad \blacksquare
\end{aligned}$$

Q4. Curves $x = y^2 + 1$ and $x = 3$ intersect at:

$$\begin{aligned}
y^2 + 1 &= 3 \\
y^2 &= 2 \\
y &= \pm\sqrt{2}
\end{aligned}$$

Then, find the volume:

$$\begin{aligned}
\text{Volume} &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (3 - (y^2 + 1))^2 dy \\
&= \pi \int_{-\sqrt{2}}^{\sqrt{2}} (2 - y^2)^2 dy \\
&= \pi \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 4y^2 + y^4 dy \\
&= \pi[4y - \frac{4}{3}y^3 + \frac{1}{5}y^5]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \pi[(4\sqrt{2} - \frac{4}{3}(2\sqrt{2}) + \frac{1}{5}(4\sqrt{2})) \\
&\quad - (-4\sqrt{2} + \frac{4}{3}(2\sqrt{2}) - \frac{1}{5}(4\sqrt{2}))] \\
&= \pi(8\sqrt{2} - \frac{16}{3}\sqrt{2} + \frac{8}{5}\sqrt{2}) \\
&= \frac{64}{15}\sqrt{2}\pi \quad \blacksquare
\end{aligned}$$

Q5. Curves $y = \sqrt{x-1}$ and $y = (x-1)^2$ intersect at:

$$\begin{aligned}
(x-1)^2 &= \sqrt{x-1} \\
(x-1)^4 &= x-1 \\
u^4 &= u && \text{(Sub } u = x-1\text{)} \\
u(u^3-1) &= 0 \\
u &= 0, 1 \\
\therefore x &= 1, 2
\end{aligned}$$

Then, find the volume:

$$\begin{aligned}
\text{Volume} &= 2\pi \int_1^2 x|\sqrt{x-1} - (x-1)^2| dx \\
&= 2\pi(\int_1^2 x\sqrt{x-1}dx - \int_1^2 x(x-1)^2 dx) \\
&= 2\pi(\int_0^1 (u+1)\sqrt{u}du - \int_1^2 x^3 - 2x^2 + xdx) && \text{(Sub } u = x-1\text{)} \\
&= 2\pi(\int_0^1 u^{3/2} + u^{1/2}du - [\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}]_1^2) \\
&= 2\pi([\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}]_0^1 - (\frac{2}{3} - \frac{1}{12})) \\
&= 2\pi[(\frac{16}{15}) - (\frac{7}{12})] \\
&= 2\pi(\frac{29}{60}) \\
&= \frac{29}{30}\pi \quad \blacksquare
\end{aligned}$$