MA1522 Homework 2

AY 24/25 Sem 1 — github/omgeta

Q1. (a) Check V by substituting $x_1 = 1, x_2 = 2, x_3 = 0, x_4 = 1$ into the equation of V:

$$2(1) + 1(2) - 3(1) = 1 \neq 0$$
$$\implies \text{the vector does not satisfy the equation for } V$$
$$\implies \begin{bmatrix}1\\2\\0\\1\end{bmatrix} \notin V$$

Check U by RREF:

$$\left[\begin{array}{cccc|c} 3 & 1 & 7 & 1 & 1 \\ 3 & 3 & 1 & 4 & 2 \\ -3 & -1 & -3 & -3 & 0 \\ 3 & 2 & 5 & 2 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\implies \text{inconsistent equation in the last row}$$
$$\implies \text{the vector is not in the column space}$$
$$\implies \begin{bmatrix}1\\2\\0\\1\end{bmatrix} \notin U$$

Therefore, $\begin{bmatrix}1\\2\\0\\1\end{bmatrix} \notin V \wedge \begin{bmatrix}1\\2\\0\\1\end{bmatrix} \notin U \quad \blacksquare$

(b) From the RREF of the matrix formed by the spanning set of U in (a), we can see the 4th column vector is a linear combination of the 1st and 3rd. By removing either the 1st, 3rd, or 4th vector, we can get a linearly independent set which also spans U.

Therefore, a possible basis is $\left\{ \begin{bmatrix}3\\3\\-3\\3\end{bmatrix}, \begin{bmatrix}1\\3\\-1\\2\end{bmatrix}, \begin{bmatrix}7\\1\\-3\\5\end{bmatrix} \right\} \quad \blacksquare$

(c) Suppose there is a linear equation for $U : a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = \vec{b}$, since $\vec{0} \in U$, the following equations must hold:

$$3a_1 + 3a_2 - 3a_3 + 3a_4 = 0$$
$$1a_1 + 3a_2 - 1a_3 + 2a_4 = 0$$
$$7a_1 + 1a_2 - 3a_3 + 5a_4 = 0$$

Reduce the corresponding matrix:

$$\left[\begin{array}{cccc|c} 3 & 3 & -3 & 3 & 0 \\ 1 & 3 & -1 & 2 & 0 \\ 7 & 1 & -3 & 5 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3/4 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 1/4 & 0 \end{array}\right]$$
$$\implies \begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} = s \begin{bmatrix}3\\2\\1\\-4\end{bmatrix}, s \in \mathbb{R}$$

Therefore, a linear equation is $3x_1 + 2x_2 + x_3 - 4x_4 = 0 \quad \blacksquare$

(d) Firstly, check if T is linearly independent:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies T \text{ is linearly independent}$$

Secondly, check if $\text{Span}(T) \subseteq V$:

$$2(1) + 1(1) - 3(1) = 0$$
$$2(2) + 1(-1) - 3(1) = 0$$
$$\implies \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \in V$$

$$\implies \text{Span}(T) \subseteq V, \text{ by closure over addition and multiplication}$$

Thirdly, check if $V \subseteq \text{Span}(T)$:

$$\vec{v} \in V = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_4 - 2x_1 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= s \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad s, t, u \in \mathbb{R}$$

$$\therefore V = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & 3 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 1 & 0 & -2/3 & 1/3 & 4/3 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\implies V \subseteq \text{Span}(T)$$

Since $Span(T) \subseteq V$ and $V \subseteq \text{Span}(T)$, then $\text{Span}(T) = V$, by the definition of set equality. Therefore, $\text{Span}(T) = V$ and T is linearly independent $\implies T$ is a basis for V ■

(e) For a vector $\vec{v} \in U \cap V$:

$$2x_1 + x_2 + 0x_3 - 3x_4 = 0$$
$$3x_1 + 2x_2 + 1x_3 - 4x_3 = 0$$

Reduce the corresponding matrix:

$$\left[\begin{array}{cccc|c} 2 & 1 & 0 & -3 & 0 \\ 3 & 2 & 1 & -4 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{array}\right]$$

$$\therefore \vec{v} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Therefore, a basis for $U \cap V$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ ■

Q2. Yes.
Includes zero vector: when $s_1 = -1, s_2 = 1, s_3 = 2, \vec{v} = \vec{0}$
Closure over addition:

$$\vec{u} + \vec{v} = \begin{bmatrix} 4 \\ -3 \\ 9 \\ 1 \end{bmatrix} + s_1 \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} + s_2 \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix} + s_3 \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ 9 \\ 1 \end{bmatrix} + t_1 \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix}, \text{ where } \vec{u}, \vec{v} \in V$$

$$= \begin{bmatrix} 4 \\ -3 \\ 9 \\ 1 \end{bmatrix} + (s_1 + t_1 + 1) \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} + (s_2 + t_2 - 1) \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix} + (s_3 + t_3 - 2) \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -3 \\ 9 \\ 1 \end{bmatrix} + s_1' \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} + s_2' \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix} + s_3' \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix} \qquad (\mathbb{R} \text{ closed over addition})$$

$$\implies u + v \in V$$

Closure over multiplication:

$$c \cdot \vec{v} = c \begin{bmatrix} 4 \\ -3 \\ 9 \\ 1 \end{bmatrix} + cs_1 \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} + cs_2 \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix} + cs_3 \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix}, \text{ where } c \in \mathbb{R}, \vec{v} \in V$$

$$= \begin{bmatrix} 4 \\ -3 \\ 9 \\ 1 \end{bmatrix} + (cs_1 + c) \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} + (cs_2 - c) \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix} + (cs_3 - 2c) \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -3 \\ 9 \\ 1 \end{bmatrix} + s_1' \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix} + s_2' \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix} + s_3' \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix} \qquad (\mathbb{R} \text{ closed over addition/multiplication})$$

$$\implies c\vec{v} \in V$$

To check if the spanning set is a basis, check for linearly independence:

$$\begin{bmatrix} 6 & 6 & -2 \\ 6 & 3 & 3 \\ 8 & 7 & -4 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\implies \text{all the vectors are linearly independent}$$

Therefore, a basis for V is $\left\{ \begin{bmatrix} 6 \\ 6 \\ 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -4 \\ 0 \end{bmatrix} \right\}$ $\blacksquare$

Q3. (a) Reduce matrix A:

$$\begin{bmatrix} 2 & 4 & 3 \\ 9 & 6 & 3 \\ -1 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = \dim(\text{Col}(A)) =$ number of pivot columns of A $= 3$ ■

(b) Yes; Suppose $B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$, then since $BA = I_3 \iff A^T B^T = I_3$, to solve, we reduce the system $[A^T \mid I_3]$:

$$\left[\begin{array}{cccc|ccc} 2 & 9 & -1 & 1 & 1 & 0 & 0 \\ 4 & 6 & 3 & 1 & 0 & 1 & 0 \\ 3 & 3 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|ccc} 1 & 0 & 0 & -1/3 & -5/9 & 13/9 & -11/9 \\ 0 & 1 & 0 & 2/9 & 7/27 & -11/27 & 10/27 \\ 0 & 0 & 1 & 1/3 & 2/9 & -7/9 & 8/9 \end{array}\right]$$

Therefore, $B = \begin{bmatrix} -5/9 + s/3 & 7/27 - 2s/9 & 2/9 - s/3 & s \\ 13/9 + t/3 & -11/27 - 2t/9 & -7/9 - t/3 & t \\ -11/9 + u/3 & 10/27 - 2u/9 & 8/9 - u/3 & u \end{bmatrix}, s, t, u \in \mathbb{R}$ ■

(c) No; $\text{rank}(A) \neq$ number of rows $= 4 \implies A$ has no right inverse. ■(Math Cafe 7, Slide 30)

(d) No; $\text{Nul}(A) \perp \text{Row}(A) \implies \text{Nul}(A^T) \perp \text{Col}(A) \implies \forall v \in \text{Nul}(A^T),\ \vec{v} \notin \text{Col}(A)$ ■

Alternatively, suppose nonzero $\vec{v} \in \text{Nul}(A^T)$, then $A^T\vec{v} = \vec{0}$:

$$\left[\begin{array}{cccc|c} 2 & 9 & -1 & 1 & 0 \\ 4 & 6 & 3 & 1 & 0 \\ 3 & 3 & 4 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & 2/9 & 0 \\ 0 & 0 & 1 & -1/3 & 0 \end{array}\right]$$

$$\therefore \vec{v} = s \begin{bmatrix} 3 \\ -2 \\ 3 \\ -9 \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

However, $A\vec{x} = \vec{v}$ will not be consistent as shown below:

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & 3 \\ 9 & 6 & 3 & -2 \\ -1 & 3 & 4 & 3 \\ 1 & 1 & 1 & -9 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore, there exists no nonzero vector $\vec{v} \in \text{Nul}(A^T) \land A\vec{x} = \vec{v}$ ■

Q4. (a) Find the transition matrix from S to T:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & -3 & 6 & 0 & 3 & 0 \\ -1 & 3 & -1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore $[w]_T = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \dfrac{x_1 + x_2}{2} - x_3 \\ \dfrac{x_1 - x_2}{2} + x_3 \\ x_3 \end{bmatrix}$ ■

(b) Find the transition matrix from B to the standard basis, using P_S, the transition matrix from S to the standard basis:

$$\begin{aligned} P_S P &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 2 \\ -3 & 3 & 3 \\ 6 & -3 & 6 \\ 1 & 0 & 2 \end{bmatrix} \end{aligned}$$

Therefore, a basis for B is $\left\{ \begin{bmatrix} 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \\ 2 \end{bmatrix} \right\}$ ■