MA1521 Homework 9

AY 24/25 Sem 1 — github/omgeta

Q1. Given $f(x, y) = xe^{-y}$:

$$\begin{aligned} f_x = e^{-y}, \quad f_y = -xe^{-y} \\ \implies \nabla f(x, y) = \langle e^{-y}, -xe^{-y} \rangle \end{aligned}$$

(a) (i) At $P(-2, 0)$ and $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$:

$$\begin{aligned} D_{\hat{u}} f(-2, 0) &= \nabla f(-2, 0) \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \\ &= \langle 1, 2 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle \\ &= \frac{3}{\sqrt{2}} \quad \blacksquare \end{aligned}$$

(ii) At $P(-2, 0)$ and $\vec{u} = \langle 3, 4 \rangle \implies \hat{u} = \frac{1}{5}\langle 3, 4 \rangle$:

$$\begin{aligned} D_{\hat{u}} f(-2, 0) &= \nabla f(-2, 0) \cdot \frac{1}{5}\langle 3, 4 \rangle \\ &= \langle 1, 2 \rangle \cdot \frac{1}{5}\langle 3, 4 \rangle \\ &= \frac{11}{5} \quad \blacksquare \end{aligned}$$

(b) f increases fastest at $P(-2, 0)$ with the unit gradient vector:

$$\begin{aligned} \frac{\nabla f(-2, 0)}{|\nabla f(-2, 0)|} &= \frac{\langle 1.2 \rangle}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\hat{i} + \frac{2}{\sqrt{5}}\hat{j} \quad \blacksquare \end{aligned}$$

Q2. Given $f(x, y, z) = xy + \sin(xyz)$:

$$\begin{aligned} f_x = y + yz\cos(xyz), \quad f_y = xz\cos(xyz), \quad f_z = xy\cos(xyz) \\ \implies \nabla f(x, y, z) = \langle y + yz\cos(xyz), xz\cos(xyz), xy\cos(xyz) \rangle \end{aligned}$$

(i) At $P(\frac{1}{2}, \frac{1}{3}, \pi)$ and $\hat{u} = \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$:

$$\begin{aligned} D_{\hat{u}} f(P) &= \nabla f(P) \cdot \hat{u} \\ &= \langle \frac{1}{3} + \frac{\pi\sqrt{3}}{6}, \frac{1}{2} + \frac{\pi\sqrt{3}}{4}, \frac{\sqrt{3}}{12} \rangle \cdot \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle \\ &= (\frac{1}{3\sqrt{3}} + \frac{\pi}{6}) + (-\frac{1}{2\sqrt{3}} - \frac{\pi}{4}) + \frac{1}{12} \\ &= (\frac{2}{6\sqrt{3}} - \frac{3}{6\sqrt{3}}) + (\frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{12}) \\ &= \frac{1}{12}(1 - \pi) - \frac{1}{6\sqrt{3}} \quad \blacksquare \end{aligned}$$

(ii) By linear approximation,

$$\begin{aligned} \Delta f &= D_{\hat{u}} f(P) \cdot 0.1 \\ &\approx -0.0275 \quad \blacksquare \end{aligned}$$

Q3. (i) Given $f(x,y) = \ln(x^2y) - xy - 2x + 2$, where $x > 0, y > 0$:

$$f_x = \frac{2}{x} - y - 2, \quad f_y = \frac{1}{y} - x$$
$$f_{xx} = -\frac{2}{x^2}, \quad f_{xy} = -1, \quad f_{yy} = -\frac{1}{y^2}$$

At critical points:

$$f_x = \frac{2}{x} - y - 2 = 0$$
$$f_y = \frac{1}{y} - x = 0$$

Solving simultaneously, $x = \frac{1}{2}, y = 2$, which by second derivative test:

$$\begin{aligned} D &= f_{xx}(\frac{1}{2},2)f_{yy}(\frac{1}{2},2) - (f_{xy}(\frac{1}{2},2))^2 \\ &= (-8)(-\frac{1}{4}) - (-1)^2 \\ &= 1 > 0 \end{aligned}$$

Therefore, $f(\frac{1}{2}, 2) = -\ln 2$ is a local maximum. ■

(ii) Given $g(x,y) = xy(1 - x - y)$:

$$f_x = y - 2xy - y^2, \qquad f_y = x - x^2 - 2xy$$
$$f_{xx} = -2y, \quad f_{xy} = 1 - 2x - 2y, \quad f_{yy} = -2x$$

At critical points:

$$f_x = y - 2xy - y^2 = 0$$
$$f_y = x - x^2 - 2xy = 0$$

Solving simultaneously, we get points $(0,0), (1,0), (0,1), (\frac{1}{3}, \frac{1}{3})$ which by second derivative test:

$$D(0,0) = -1 < 0$$
$$D(1,0) = -1 < 0$$
$$D(0,1) = -1 < 0$$
$$D(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3} > 0, f_{xx} = -\frac{2}{3} < 0$$

Therefore, $(0,0), (1,0), (0,1)$ are saddle points and $g(\frac{1}{3}, \frac{1}{3}) = \frac{1}{27}$ is a local maximum. ■

Q4. (a)

$$\begin{aligned}
\int\int_R x^3 + y^3 dA &= \int_0^b \int_0^a x^3 + y^3 dxdy \\
&= \int_0^b [\frac{1}{4}x^4 + xy^3]_0^a dy \\
&= \int_0^b \frac{1}{4}a^4 + ay^3 dy \\
&= [\frac{1}{4}a^4 y + \frac{1}{4}ay^3]_0^b \\
&= \frac{1}{4}a^4 b + \frac{1}{4}ab^4 \quad \blacksquare
\end{aligned}$$

(b)

$$\begin{aligned}
\int\int_R \frac{xy}{\sqrt{4-x^2}} dA &= \int_1^3 \int_0^2 \frac{xy}{\sqrt{4-x^2}} dxdy \\
&= \int_1^3 y \int_0^2 \frac{x}{\sqrt{4-x^2}} dxdy \\
&= \int_1^3 y \int_4^0 -\frac{1}{2} \cdot \frac{1}{\sqrt{u}} dudy \qquad (u = 4 - x^2,\ du = -2xdx) \\
&= \int_1^3 y[\sqrt{u}]_0^4 dy \\
&= \int_1^3 2ydy \\
&= [y^2]_1^3 \\
&= 8 \quad \blacksquare
\end{aligned}$$

Alternatively:

$$\begin{aligned}
\int\int_R \frac{xy}{\sqrt{4-x^2}} dA &= (\int_0^2 \frac{x}{\sqrt{4-x^2}} dx)(\int_1^3 ydy) \\
&= (\int_0^4 \frac{1}{2\sqrt{u}} du)[\frac{1}{2}y^2]_1^3 \qquad (u = 4 - x^2,\ du = -2xdx) \\
&= [\sqrt{u}]_0^4 \cdot 4 \\
&= 8 \quad \blacksquare
\end{aligned}$$

Q5.

$$\begin{aligned}
\int_0^3 \int_0^2 2 + (x-1)^2 + 4y^2 dydx &= \int_0^3 [2y + (x-1)^2 y + \frac{4}{3}y^3]_0^2 dx \\
&= \int_0^3 4 + 2(x-1)^2 + \frac{32}{3} dx \\
&= \int_0^3 2x^2 - 4x + \frac{50}{3} dx \\
&= [\frac{2}{3}x^3 - 2x^2 + \frac{50}{3}x]_0^3 \\
&= 50 \quad \blacksquare
\end{aligned}$$