MA1521 Homework 6

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $\displaystyle\sum_{n=1}^{\infty} \cos^2 \frac{1}{n}$

$$\lim_{n\to\infty} \cos^2 \frac{1}{n} = 1$$

$\therefore$ By nth term test, $\sum_{n=1}^{\infty} \cos^2 \frac{1}{n}$ diverges. $\blacksquare$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{r+1}}$

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\ln x)^{r+1}} dx &= \int_{\ln 2}^{\infty} \frac{1}{u^{r+1}} du \qquad \text{(Sub } u = \ln x \implies dx = xdu) \\ &= -\frac{1}{r}[u^{-r}]_{\ln 2}^{\infty} \\ &= -\frac{1}{r}[0 - (\ln 2)^{-r}] \\ &= \frac{1}{r(\ln 2)^r}\end{aligned}$$

$\therefore$ By integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{r+1}}$ is convergent. $\blacksquare$

(c) $\displaystyle\sum_{n=1}^{\infty} \sin^{2n}(\frac{1}{\sqrt{n}})$

$$\begin{aligned}\sin^{2n}(\frac{1}{\sqrt{n}}) &\approx (\frac{1}{\sqrt{n}})^{2n} \qquad \text{(For large } n) \\ &= \frac{1}{n^2}\end{aligned}$$

Since $0 \leq \sin^{2n} \frac{1}{\sqrt{n}} \leq \frac{1}{n^2}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, then by comparison test, $\displaystyle\sum_{n=1}^{\infty} \sin^{2n} \frac{1}{\sqrt{n}}$ is convergent. $\blacksquare$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{c}{\sqrt{d+n^2}}$

$$\begin{aligned}\frac{d}{dn}(\frac{c}{\sqrt{d+n^2}}) &= -\frac{c \cdot n}{(d+n^2)^{3/2}} \\ &< 0, \text{ for all } n \geq 1 \\ \lim_{n\to\infty} \frac{c}{\sqrt{d+n^2}} &= \lim_{n\to\infty} \frac{c}{n} \\ &= 0\end{aligned}$$

$\therefore$ by alternating series test, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{c}{\sqrt{d+n^2}}$ is convergent. $\blacksquare$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{3+\sin n}{n^3}$

$$
\begin{aligned}
-1 \leq \sin n \leq 1 \\
2 \leq 3+\sin n \leq 4 \\
\frac{2}{n^3} \leq \frac{3+\sin n}{n^3} \leq \frac{4}{n^3}
\end{aligned}
$$

Since both $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{4}{n^3}$ are convergent p-series, then by comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{3+\sin n}{n^3}$ is convergent. ■

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{1+3n}(n+1)}{n^2 5^{1+n}}$

$$
\begin{aligned}
\lim_{n\to\infty} \left|\frac{\frac{2^{1+3n+3}(n+2)}{(n+1)^2 5^{2+n}}}{\frac{2^{1+3n}(n+1)}{n^2 5^{1+n}}}\right| &= \lim_{n\to\infty} \left|\frac{2^3 \cdot n^2(n+2)}{5(n+1)^3}\right| \\
&= \lim_{n\to\infty} \left|\frac{2^3(n^3+\ldots)}{5(n^3+\ldots)}\right| \\
&= \frac{8}{5} \\
&> 1
\end{aligned}
$$

$\therefore$ by ratio test, $\displaystyle\sum_{n=1}^{\infty} \frac{2^{1+3n}(n+1)}{n^2 5^{1+n}}$ is divergent. ■

Q2. (a) $\sum_{n=1}^{\infty}(-1)^n \frac{(2x+3)^n}{n}$

For the power series to be convergent:

$$\begin{aligned}
\lim_{n\to\infty} |\frac{(\frac{(-1)^{n+1}(2x+3)^{n+1}}{n+1})}{\frac{(-1)^n(2x+3)^n}{n}}| &< 1 \\
\lim_{n\to\infty} |-(2x+3)\cdot\frac{n}{n+1}| &< 1 \\
|-(2x+3)| &< 1 \\
|2x+3| &< 1 \\
-1 < 2x+3 &< 1 \\
-2 < x &< -1
\end{aligned}$$

At $x=-1$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(2x+3)^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

which is the divergent harmonic series

At $x=-2$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(2x+3)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

which is the convergent alternating harmonic series

$\therefore$ radius of convergence is $\frac{1}{2}$ and interval of convergence is $(-2,-1]$ ■

(b) $\sum_{n=1}^{\infty}(nx)^{n/5}$

By ratio test:

$$\begin{aligned}
\lim_{n\to\infty}|\frac{((n+1)x)^{n+1/5}}{(nx)^{n/5}}| &= \lim_{n\to\infty}|\frac{(n+1)^{n+1/5}}{n^{n/5}}\cdot x^{1/5}| \\
&= \lim_{n\to\infty}|(\frac{n+1}{n})^{n/5}\cdot(n+1)^{1/5}\cdot x^{1/5}| \\
&= \lim_{n\to\infty}|e^{1/5}\cdot(n+1)^{1/5}\cdot x^{1/5}| \\
&= \infty
\end{aligned}$$

$\therefore$ radius of convergence is 0 ■

Q3. $\sum_{n=1}^{\infty} a_n(-1)^n x^{2n}$

For the power series to be convergent:

$$\lim_{n\to\infty} |\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}| < 1$$
$$\frac{1}{5}|x|^2 < 1$$
$$|x|^2 < 5$$
$$|x| < \sqrt{5}$$

$\therefore$ radius of convergence is $\sqrt{5}$ ■

Q4. (a) $\frac{x}{1-x}$ at $x = 0$

$$\begin{aligned}\frac{x}{1-x} &= x \cdot \frac{1}{1-x} \\ &= x \cdot \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^{n+1} \quad \blacksquare\end{aligned}$$

(b) $\frac{1}{x^2}$ at $x = 1$

$$\begin{aligned}f(x) &= \frac{1}{x^2} \\ f'(x) &= -\frac{2}{x^3} \\ f''(x) &= \frac{6}{x^4} \\ f'''(x) &= -\frac{24}{x^5} \\ \frac{1}{x^2} &= 1 + (-\frac{2}{(1)^3})(\frac{1}{1!})(x-1) + (\frac{6}{1^4})(\frac{1}{2!})(x-1)^2 + \dots \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots \\ &= \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n \quad \blacksquare\end{aligned}$$

(c) $\frac{x}{1+x}$ at $x = -2$

$$\begin{aligned}f(x) &= \frac{x}{1+x} \\ f'(x) &= \frac{1}{(1+x)^2} \\ f''(x) &= \frac{-2}{(1+x)^3} \\ \frac{x}{1+x} &= 2(\frac{1}{0!})(x+2)^0 + 1(\frac{1}{1!})(x+2)^1 + (2)(\frac{1}{2!})(x+2)^2 + \dots \\ &= 2 + \sum_{n=1}^{\infty}(x+2)^n \quad \blacksquare\end{aligned}$$

Q5. Given $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$:

$$\begin{aligned}
xe^x &= x\sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} \\
\int_0^1 xe^x dx &= [\sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)n!}]_0^1 \\
[xe^x - e^x]_0^1 &= \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} \\
1 &= S \\
\therefore S &= 1 \quad \blacksquare
\end{aligned}$$