MA1521 Calculus for Computing

AY 24/25 Sem 1 — github/omgeta

1. Limits

Limit of a function $f(x)$ is given by:

i. $\lim_{x \to c^-} f(x) = L$ ($f(x) \to L$ from left)

ii. $\lim_{x \to c^+} f(x) = L$ ($f(x) \to L$ from right)

iii. $\lim_{x \to c} f(x) = L$ ($f(x) \to L$ from both)

Function $f(x)$ is continuous at $x = c$ if and only if it is differentiable at c or $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = f(c)$.

Laws of Limits:

i. $\lim_{x \to c}(f(x) \pm g(x)) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$

ii. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$

iii. $\lim_{x \to c}(f(x)g(x)) = (\lim_{x \to c} f(x))(\lim_{x \to c} g(x))$

iv. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

v. g is continuous at $x = b \wedge \lim_{x \to c} f(x) = b$
$\implies \lim_{x \to c} g(f(x)) = g(b) = g(\lim_{x \to c} f(x))$

Squeeze Theorem:

$$g(x) \leq f(x) \leq h(x) \wedge \lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L \implies \lim_{x \to c} f(x) = L$$

Intermediate Value Theorem:

f is continuous on $[a, b] \wedge k$ is between $f(a)$ and $f(b)$
$\implies f(c) = k$ for some $c \in [a, b]$

Trignometric Identities:

$$\lim_{x \to c} g(x) = 0$$
$$\implies \lim_{x \to c} \frac{g(x)}{\sin(g(x))} = \lim_{x \to c} \frac{\sin(g(x))}{g(x)} = 1$$
$$\implies \lim_{x \to c} \frac{g(x)}{\tan(g(x))} = \lim_{x \to c} \frac{\tan(g(x))}{g(x)} = 1$$

L'Hôpital's Rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty} \implies \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

2. Differentiation

Derivative of a function f at $x = x_0$ is given by:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Derivative of a parametric function in t is given by:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}, \quad \frac{dy^2}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}) \div \frac{dx}{dt}$$

Critical points at $x = c$ of function f are non-endpoints where $f'(c)$ is 0 or does not exist.

First Derivative Test:

i. $f'(c^-) > 0 \wedge f'(c^+) < 0$ (Local maximum)

ii. $f'(c^-) < 0 \wedge f'(c^+) > 0$ (Local minimum)

iii. Otherwise (Point of inflection)

Second Derivative Test:

i. $f''(c) < 0$ (Local maximum)

ii. $f''(c) > 0$ (Local minimum)

Rolle's Theorem:

f continuous on $[a, b]$, differentiable on $(a, b) \wedge f(a) = f(b)$
$\implies f'(c) = 0$ for some $c \in [a, b]$

Mean Value Theorem:

f is continuous on $[a, b] \wedge f$ is differentiable on (a, b)
$\implies f'(c) = 0$ for some $c \in [a, b]$

Standard Derivatives

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$\tan(g(x))$	$g'(x)\sec^2(g(x))$
$\sec(g(x))$	$g'(x)\sec(g(x))\tan(g(x))$
$\text{cosec}(g(x))$	$-g'(x)\,\text{cosec}(g(x))\cot(g(x))$
$\cot(g(x))$	$-g'(x)\,\text{cosec}^2(g(x))$
$\sin^{-1}(g(x))$	$\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\cos^{-1}(g(x))$	$-\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\tan^{-1}(g(x))$	$\frac{g'(x)}{1+g(x)^2}$
$\cot^{-1}(g(x))$	$-\frac{g'(x)}{1+g(x)^2}$
$\sec^{-1}(g(x))$	$\frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}}, \lvert g(x)\rvert > 1$
$\text{cosec}^{-1}(g(x))$	$-\frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}}, \lvert g(x)\rvert > 1$
a^x	$a^x \ln(a)$

3. Integration

Definite integrals of function f have Riemann Sum:

$$\int_a^b f(x)dx = \lim_{n \to \infty} \Sigma_{i=1}^n \frac{b-a}{n} f\left(a + (b-a)\frac{i}{n}\right)$$

Integration by substitution involves choosing $u = g(x)$ and replacing all original variables, limits and dx.

Integration by parts for $\int f(x)g(x)dx$ involves choosing u and $\frac{dv}{dx}$ (u by LIATE) so $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Volume of revolution about same axis, in a disk:

$$V = \pi \int_a^b [f(x)]^2 dx, \quad V = \pi \int_c^d [g(y)]^2 dy$$

Volume of revolution about diff. axis, in a cylindrical shell:

$$V = 2\pi \int_a^b x\lvert f(x)\rvert dx, \quad V = 2\pi \int_c^d y\lvert g(y)\rvert dy$$

Arc length of a curve measured along x or y:

$$l = \int_a^b \sqrt{1 + [f'(x)]^2} dx, \quad l = \int_c^d \sqrt{1 + [g'(y)]^2} dy$$

Standard Integrals

$\mathbf{f(x)}$	$\mathbf{F(x) - C}$
$[f(x)]^n, n \neq -1$	$\frac{[f(x)]^{n+1}}{(n+1)f'(x)}$
$\tan(f(x))$	$\frac{1}{f'(x)} \ln\lvert\sec(f(x))\rvert$
$\sec(f(x))$	$\frac{1}{f'(x)} \ln\lvert\sec(f(x)) + \tan(f(x))\rvert$
$\text{cosec}(f(x))$	$-\frac{1}{f'(x)} \ln\lvert\text{cosec}(f(x)) + \cot(f(x))\rvert$
$\cot(f(x))$	$-\frac{1}{f'(x)} \ln\lvert\text{cosec}(f(x))\rvert$
$\sec^2(f(x))$	$\frac{1}{f'(x)} \tan(f(x))$
$\text{cosec}^2(f(x))$	$-\frac{1}{f'(x)} \cot(f(x))$
$\sec(f(x))\tan(f(x))$	$\frac{1}{f'(x)} \sec(f(x))$
$\text{cosec}(f(x))\cot(f(x))$	$-\frac{1}{f'(x)} \text{cosec}(f(x))$
$\frac{1}{a^2+[f(x)]^2}$	$\frac{1}{af'(x)} \tan^{-1}(\frac{f(x)}{a})$
$\frac{1}{\sqrt{a^2-[f(x)]^2}}$	$\frac{1}{f'(x)} \sin^{-1}(\frac{f(x)}{a})$
$-\frac{1}{\sqrt{a^2-[f(x)]^2}}$	$\frac{1}{f'(x)} \cos^{-1}(\frac{f(x)}{a})$
$\frac{1}{a^2-[f(x)]^2}$	$\frac{1}{2af'(x)} \ln\lvert\frac{f(x)+a}{f(x)-a}\rvert$
$\frac{1}{[f(x)]^2-a^2}$	$\frac{1}{2af'(x)} \ln\lvert\frac{f(x)-a}{f(x)+a}\rvert$
$\frac{1}{\sqrt{[f(x)]^2+a^2}}$	$\frac{1}{f'(x)} \ln\lvert f(x) + \sqrt{[f(x)]^2 + a^2}\rvert$
$\frac{1}{\sqrt{[f(x)]^2-a^2}}$	$\frac{1}{f'(x)} \ln\lvert f(x) + \sqrt{[f(x)]^2 - a^2}\rvert$
$\sqrt{a^2 - x^2}$	$\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(\frac{x}{a})$
$\sqrt{x^2 - a^2}$	$\frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\ln\lvert x + \sqrt{x^2 - a^2}\rvert$

4. Sequences and Series

n^{th} **Term:** $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

Integral Test for $a_n = f(n)$, where f is continuous, positive, decreasing for $x \geq 1$:
$$\int_1^{\infty} f(x) \text{ converges} \iff \sum_{n=1}^{\infty} a_n \text{ converges}$$

p**-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\iff p > 1$

Comparison Test for $0 \leq a_n \leq b_n$:
$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$
$$\sum_{n=1}^{\infty} a_n \text{ diverges} \implies \sum_{n=1}^{\infty} b_n \text{ diverges}$$

Ratio/Root Test:
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \text{ or } \lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

i. $0 \leq L < 1$ (Absolute Convergence)
ii. $L > 1$ (Divergence)
iii. $L = 1$ (Inconclusive)

Alternating Series Test for terms $a_n = (-1)^n b_n$ or $a_n = (-1)^{n-1} b_n$, where b_n is decreasing:
$$\lim_{n\to\infty} b_n = 0 \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

Radius of Convergence $R = \frac{1}{L}$ about $x = a$ for power series $b_n = c_n(x-a)^n$ is interval for absolute convergence:
$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L \text{ or } \lim_{n\to\infty} \sqrt[n]{|c_n|} = L$$

Functions with power series representation for $R > 0$ have:

i. $f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$
ii. $\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

Taylor Series for a function with power series representation is:
$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$
with MacLaurin Series at $x = 0$:
$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$

5. Vectors

Projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by:
$$\text{proj}_{\mathbf{a}}\, \mathbf{b} = \text{comp}_{\mathbf{a}}\, \mathbf{b} \times \hat{\mathbf{a}} = (\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}}$$

Perpendicular distance from position vector $\mathbf{b}$ to $\mathbf{a}$ is given by:
$$\|\mathbf{b} \times \hat{\mathbf{a}}\|$$

Projection of $\mathbf{b}$ onto plane $\Pi : \mathbf{r} \cdot \mathbf{n} = D$:
$$\text{proj}_{\Pi}\, \mathbf{b} = \mathbf{b} - \text{proj}_{\hat{\mathbf{n}}}\, \mathbf{b} = \mathbf{b} - (\mathbf{b} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$$
$$\|\text{proj}_{\Pi}\, \mathbf{b}\| = \|\mathbf{b} \times \hat{\mathbf{n}}\|$$

Perpendicular distance from position vector $\mathbf{b}$ to plane $\mathbf{r} \cdot \mathbf{n} = D$:
$$\frac{|D - \mathbf{b} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

Dot and Cross product are given by:
$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$
$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$$
$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Vector-valued Functions

Derivative of $r(t) = \langle f(t), g(t), h(t)\rangle$ at $t = a$ is given by:
$$r'(a) = \langle f'(a), g'(a), h'(a)\rangle$$

Arc length of a path measured along t:
$$l = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

6. Multivariate Calculus

Derivative of $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$ is given by:
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Derivative of $z = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$ is given by:
$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Normal vector to the tangent plane for $z = f(x, y)$ at (x_0, y_0) is given by:
$$\langle f_x(a, b), f_y(a, b), -1\rangle$$

Derivative of z in $F(x, y, z) = 0$ is given by:
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Normal vector to the tangent plane for level surface of $F(x, y, z)$ at (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0)$

Directional derivative of f at $P = (x_0, y_0)$ in direction of unit vector $\hat{\mathbf{u}}$ is given by:
$$D_{\hat{\mathbf{u}}} f(P) = \nabla f(x_0, y_0) \cdot \hat{\mathbf{u}}$$
where $\nabla f = \langle f_x, f_y\rangle$ is the gradient vector and rate of change is optimized at:

$\|\nabla f(P)\|$ in direction $\nabla f(P)$ (Max.)
$-\|\nabla f(P)\|$ in direction $-\nabla f(P)$ (Min.)

Gradient vector $\nabla f(x_0, y_0) \neq 0$ is normal to level curve $f(x, y) = k$ at (x_0, y_0)

Critical points at (a, b) of function f are non-endpoints where $f_x(a, b) = f_y(a, b) = 0$ or a partial derivative does not exist.

Second Derivative Test:
$$D = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

i. $D > 0 \wedge f_{xx}(a, b) < 0$ (Local max.)
ii. $D > 0 \wedge f_{xx}(a, b) > 0$ (Local min.)
iii. $D < 0$ (Saddle point)
iv. $D = 0$ (Inconclusive)

7. Double Integrals

Double Integral $\iint_R f(x, y)dA$ over rectangular region $R = [a, b] \times [c, d]$:
$$\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$
with special case $f(x, y) = g(x)h(y)$:
$$\int_a^b g(x)dx + \int_c^d h(y)dy$$

Area of general plane region D: $\iint_D dA$

Surface area: $\iint_D \sqrt{f_x^2 + f_y^2 + 1} dA$

Polar coordinates: $\iint_D f(r\cos\theta, r\sin\theta) r d\theta dr$

8. ODEs

Separable ODEs, reducing if necessary by $v = \frac{y}{x}$ or $u = ax + by$:
$$\frac{dy}{dx} = f(x)g(y) \Rightarrow \int \frac{1}{g(y)}dy = \int f(x)dx + C$$

Linear ODEs using $I(x) = e^{\int P(x)dx}$:
$$\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow yI(x) = \int Q(x)I(x)dx$$

Bernoulli equation using $u = y^{1-n}$:
$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$
$$\implies \frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

Appendix

$$\sec^2 x - 1 = \tan^2 x$$
$$\text{cosec}^2 x - 1 = \cot^2 x$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\sin P + \sin Q = 2\sin\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$
$$\sin P - \sin Q = 2\cos\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$
$$\cos P + \cos Q = 2\cos\tfrac{1}{2}(P+Q)\cos\tfrac{1}{2}(P-Q)$$
$$\cos P - \cos Q = -2\sin\tfrac{1}{2}(P+Q)\sin\tfrac{1}{2}(P-Q)$$