MA1521 Homework 4

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $\displaystyle\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^3} ds$

$$\begin{aligned}
\int_1^{\sqrt{2}} \frac{s^2+\sqrt{s}}{s^3} ds &= \int_1^{\sqrt{2}} s^{-1} + s^{-5/2} ds \\
&= [\ln|s| - \frac{2}{3}s^{-3/2}]_1^{\sqrt{2}} \\
&= \ln\sqrt{2} - \frac{2}{3}\sqrt{2}^{-3/2} - \ln 1 + \frac{2}{3} \\
&= \ln\sqrt{2} - \frac{2}{3\cdot 2^{3/4}} + \frac{2}{3} \\
&= \ln\sqrt{2} - \frac{2^{1/4}}{3} + \frac{2}{3} \quad \blacksquare
\end{aligned}$$

(b) $\displaystyle\int_0^1 \frac{1}{(x+1)(x+2)(x+3)} dx$

$$\begin{aligned}
\text{Let } \frac{1}{(x+1)(x+2)(x+3)} &= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \\
1 &= A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)
\end{aligned}$$

When $x = -1, 2A = 1 \implies A = \frac{1}{2}$
When $x = -2, -B = 1 \implies B = -1$
When $x = -3, 2C = 1 \implies C = \frac{1}{2}$

$$\begin{aligned}
\therefore \int_0^1 \frac{1}{(x+1)(x+2)(x+3)} dx &= \int_0^1 \frac{1}{2(x+1)} - \frac{1}{x+2} + \frac{1}{2(x+3)} dx \\
&= [\frac{1}{2}\ln|x+1| - \ln|x+2| + \frac{1}{2}\ln|x+3|]_0^1 \\
&= (\frac{1}{2}\ln 2 - \ln 3 + \frac{1}{2}\ln 4) - (\frac{1}{2}\ln 1 - \ln 2 + \frac{1}{2}\ln 3) \\
&= (\frac{1}{2}\ln 2 - \ln 3 + \ln 2) - (-\ln 2 + \frac{1}{2}\ln 3) \\
&= \frac{5}{2}\ln 2 - \frac{3}{2}\ln 3 \quad \blacksquare
\end{aligned}$$

(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{2}(\cos x + |\cos x|) dx$

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} \frac{1}{2}(\cos x + |\cos x|) dx &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(\cos x + \cos x) dx \qquad (\cos x > 0, -\tfrac{\pi}{2} \le x \le \tfrac{\pi}{2}) \\
&= \int_{-\pi/2}^{\pi/2} \cos x dx \\
&= [\sin x]_{-\pi/2}^{\pi/2} dx \\
&= \sin\frac{\pi}{2} - \sin(-\frac{\pi}{2}) \\
&= 1 - (-1) \\
&= 2 \quad \blacksquare
\end{aligned}$$

(d) $\int_0^{\pi} \sin^2(1+\frac{\theta}{2})d\theta$

$$\begin{aligned}
\int_0^{\pi} \sin^2(1+\frac{\theta}{2})d\theta &= \int_0^{\pi} \frac{1-\cos(2+\theta)}{2}d\theta \\
&= \frac{1}{2}\int_0^{\pi} 1-\cos(2+\theta)d\theta \\
&= \frac{1}{2}[x-\sin(2+\theta)]_0^{\pi} \\
&= \frac{1}{2}[(\pi-\sin(2+\pi))-(0-\sin 2)] \\
&= \frac{1}{2}[\pi+\sin 2+\sin 2] \\
&= \frac{\pi}{2}+\sin 2 \quad \blacksquare
\end{aligned}$$

Q2. (a) $y = \int_0^{\sqrt{x}} e^t dt$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\int_0^{\sqrt{x}} e^t dt \\
&= e^{\sqrt{x}} \cdot \frac{d}{dx}\sqrt{x} && \text{(By FTC)} \\
&= e^{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}} \\
&= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \blacksquare
\end{aligned}$$

(b) $y = \int_0^{x^2} \cos\sqrt{t}dt$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\int_0^{x^2} \cos\sqrt{t}dt \\
&= \cos\sqrt{x^2} \cdot \frac{d}{dx}x^2 && \text{(By FTC)} \\
&= \cos x \cdot 2x \\
&= 2x\cos x \quad \blacksquare
\end{aligned}$$

(c) $\int_0^{\sin x} \frac{1}{\sqrt{1-t^2}}dt$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\int_0^{\sin x} \frac{1}{\sqrt{1-t^2}}dt \\
&= \frac{1}{\sqrt{1-\sin^2 x}} \cdot \frac{d}{dx}\sin x && \text{(By FTC)} \\
&= \frac{1}{\sqrt{\cos^2 x}} \cdot \cos x \\
&= 1 \quad \blacksquare
\end{aligned}$$

Q3. (a) $\int x^{1/2} \sin(x^{3/2}+1)dx$

$$\int x^{1/2} \sin(x^{3/2}+1)dx = \int 2u^2 \sin(u^3+1)du \qquad (\text{Sub } u = x^{1/2} \implies dx = 2u \cdot du)$$
$$= -\frac{2}{3}\cos(u^3+1) + C$$
$$= -\frac{2}{3}\cos(x^{3/2}+1) + C \quad \blacksquare$$

(b) $\int \frac{\sqrt{\ln(2x)}}{2x}dx$

$$\int \frac{\sqrt{\ln(2x)}}{2x}dx = \int \frac{\sqrt{u}}{e^u} \cdot \frac{e^u}{2}du \qquad (\text{Sub } u = \ln 2x \implies dx = \tfrac{e^u}{2}du)$$
$$= \int \frac{\sqrt{u}}{2}du$$
$$= \frac{1}{2} \cdot \frac{2}{3}u^{3/2} + C$$
$$= \frac{1}{3}u^{3/2} + C$$
$$= \frac{(\ln(2x))^{3/2}}{3} + C \quad \blacksquare$$

(c) $\int \frac{18\tan^2 x \sec^2 x}{(2+\tan^3 x)}dx$

$$\int \frac{18\tan^2 x \sec^2 x}{(2+\tan^3 x)}dx = \int \frac{6}{2+u}du \qquad (\text{Sub } u = \tan^3 x \implies \tfrac{du}{dx} = 3\tan^2 x \sec^2 x)$$
$$= 6\ln|2+u| + C$$
$$= 6\ln|2+\tan^3 x| + C \quad \blacksquare$$

(d) $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}d\theta$

$$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\cos^3\sqrt{\theta}}d\theta = \int \frac{\sin u}{u\cos^3 u} \cdot 2udu \qquad (\text{Sub } u = \sqrt{\theta} \implies d\theta = 2u \cdot du)$$
$$= 2\int \frac{\sin u}{\cos^3 u}du$$
$$= 2\int \frac{\sin u}{v^3} \cdot (-\sin u)dv \qquad (\text{Sub } v = \cos u \implies dv = -\sin u)$$
$$= -2\int \frac{1}{v^3}dv$$
$$= -2 \cdot \frac{-1}{2} \cdot \frac{1}{v^2} + C$$
$$= \frac{1}{cos^2 u} + C$$
$$= \sec^2\sqrt{\theta} + C \quad \blacksquare$$

Q4. (a) $\int x \sin(\frac{x}{2}) dx$

Let $u = x \implies \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \sin\frac{x}{2} \implies v = -2\cos\frac{x}{2}$

$$\begin{aligned}\int x\sin(\frac{x}{2})dx &= -2x\cos\frac{x}{2} + 2\int \cos\frac{x}{2}dx \qquad \text{(Integrate by parts)}\\ &= -2x\cos\frac{x}{2} + 4\sin\frac{x}{2} + C\\ &= -2(x\cos\frac{x}{2} - 2\sin\frac{x}{2}) + C \quad \blacksquare\end{aligned}$$

(b) $\int t^2 e^{4t} dt$

Let $u = t^2 \implies \frac{du}{dx} = 2t$

Let $\frac{dv}{dx} = e^{4t} \implies v = \frac{1}{4}e^{4t}$

$$\int t^2 e^{4t} dt = \frac{t^2}{4}e^{4t} - \frac{1}{2}\int te^{4t}dt \qquad \text{(Integrate by parts)}$$

Let $u = t \implies \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{4t} \implies v = \frac{1}{4}e^{4t}$

$$\begin{aligned}&= \frac{t^2}{4}e^{4t} - \frac{1}{2}[\frac{t}{4}e^{4t} - \frac{1}{4}\int e^4 tdt] \qquad \text{(Integrate by parts)}\\ &= \frac{t^2}{4}e^{4t} - \frac{1}{2}[\frac{t}{4}e^{4t} - \frac{1}{16}e^{4t} + D]\\ &= \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C\\ &= (\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32})e^{4t} + C \quad \blacksquare\end{aligned}$$

(c) $\int e^{-y}\cos y dy$

Let $u = e^{-y} \implies \frac{du}{dy} = -e^{-y}$

Let $\frac{dv}{dy} = \cos y \implies v = \sin y$

$$\int e^{-y}\cos y dy = e^{-y}\sin y + \int e^{-y}\sin y dy \qquad \text{(Integrate by parts)}$$

Let $u = e^{-y} \implies \frac{du}{dy} = -e^{-y}$

Let $\frac{dv}{dy} = \sin y \implies v = -\cos y$

$$\begin{aligned}\int e^{-y}\cos y dy &= e^{-y}\sin y + [-e^{-y}\cos y - \int e^{-y}\cos y dy] \qquad \text{(Integrate by parts)}\\ 2\int e^{-y}\cos y dy &= e^{-y}\sin y - e^{-y}\cos y\\ \int e^{-y}\cos y dy &= \frac{e^{-y}}{2}(\sin y - \cos y) + C \quad \blacksquare\end{aligned}$$

Q5. (a) $\displaystyle\int_0^1 \frac{1}{(x-1)^{4/5}}dx$

Let $u = x - 1 \implies du = dx$

$$\begin{aligned}\int_0^1 \frac{1}{(x-1)^{4/5}} &= \int_{-1}^0 \frac{1}{u^{4/5}}du && \text{(Sub } u = x-1 \implies du = dx\text{)}\\ &= [5u^{\frac{1}{5}}]_{-1}^0 \\ &= 0 - (-5) \\ &= 5 \quad \blacksquare\end{aligned}$$

(b) $\displaystyle\int_1^\infty \frac{\ln x}{x^3}dx$

Let $u = \ln x \implies \dfrac{du}{dx} = \dfrac{1}{x}dx$

Let $\dfrac{dv}{dx} = \dfrac{1}{x^3} \implies v = -\dfrac{1}{2x^2}$

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^3}dx &= [-\frac{\ln x}{2x^2}]_1^\infty + \frac{1}{2}\int \frac{1}{x^3}dx && \text{(Integrate by parts)}\\ &= (0-0) + \frac{1}{2}[-\frac{1}{2x^2}]_1^\infty \\ &= \frac{1}{2}(0 + \frac{1}{2}) \\ &= \frac{1}{4} \quad \blacksquare\end{aligned}$$