ST2334 Tutorial 5

AY 25/26 Sem 1 — github/omgeta

Short Form Questions

Q1. $V(Z) = V(-2X + 4Y - 3) = 2^2V(X) + 4^2V(Y) + 2(-2)(4)Cov(X,Y) = 68 - 16Cov(X,Y)$
Therefore, (a) is true since independence implies $Cov(X,Y) = 0$, and (c) due to the formulation

Q2. (a), (b), (c)

Q3. (a); $P(\text{no red}) = (0.98)^n < 0.5 \implies n > \frac{\ln 0.98}{\ln 0.5} \implies n = 35$

Long Form Questions

Q1. (i) $P(X_1 < 1; X_2 < 1) = \int_0^1 \int_0^1 x_1x_2 \, dx_1dx_2 = \frac{1}{4}$

(ii) $f_{X_1} = \int_0^1 x_1x_2 \, dx_2 = \frac{x_1}{2}$, $f_{X_2} = \int_0^2 x_1x_2 \, dx_1 = 2x_2$. Since $f = f_{X_1}f_{X_2}$, independent.

(iii) Since independent, $E[g(X_1, X_2)] = E[X_1] + E[X_2] = \int_0^2 x_1\frac{x_1}{2} \, dx_1 + \int_0^1 x_2 \cdot 2x_2 \, dx_2 = 2$

Q2. (i) Yes

(ii) $E(Y \mid X = 2) = \frac{0.1}{0.4}(1) + \frac{0.2}{0.4}(3) + \frac{0.1}{0.4}(5) = 3$

(iii) $E(X \mid Y = 3) = \frac{0.2}{0.5}(2) + \frac{0.3}{0.5}(4) = 3.2$

(iv) $E(2X - 3Y) = 2E(X) - 3E(Y) = 2(3.2) - 3(3) = -2.6$

(v) $E(XY) = E(X)E(Y) = 3.2 \cdot 3 = 9.6$

(vi) $V(X) = E(X^2) - E(X)^2 = 4(0.4) + 16(0.6) - 3.2^2 = 0.96$

(vii) $V(Y) = E(Y^2) - E(Y)^2 = 1(0.25) + 9(0.5) + 25(0.25) - 3^2 = 2$

Q3. (i) $f_X = \int_0^1 \frac{3}{2}(x^2 + y^2) \, dy = \frac{3}{2}x^2 + \frac{1}{2}$, $f_Y = \frac{3}{2}y^2 + \frac{1}{2}$ which is not product so not independent

(ii) $E(X) = E(Y) = \int_0^1 xf_X(x) \, dx = \frac{5}{8}$ and $E(X^2) = E(Y^2) = \frac{7}{15} \implies V(X) = V(Y) = \frac{73}{960}$

(iii) $E[XY] = \int_0^1 \int_0^1 xy\frac{3}{2}(x^2 + y^2) \, dxdy = \frac{3}{8} \implies Cov(X,Y) = \frac{3}{8} - (\frac{5}{8})^2 = -\frac{1}{64}$

(iv) $E(X + Y) = \frac{5}{4}, V(X + Y) = \frac{73}{960} + \frac{73}{960} + 2(-\frac{1}{64}) = \frac{29}{240}$

Q4. $X \sim Bin(20, 0.3)$

(i) $P(X \geq 10) = \sum^2 0_{k=10} \binom{20}{k} 0.3^k 0.7^{20-k} \approx 0.0480$

(ii) $P(X \leq 4) = \sum_{k=10}^{4} \binom{20}{k} 0.3^k 0.7^{20-k} \approx 0.2375$

(iii) $P(X = 5) = \binom{20}{5} 0.3^5 0.7^{15} \approx 0.1789$

Q5. $X \sim Poisson(\lambda = np = 10000 \cdot 0.001 = 10)$

(i) $P(X = 6, 7, 8) \approx e^{-10}(\frac{10^6}{6!} + \frac{10^7}{7!} + \frac{10^8}{8!}) \approx 0.2657$

(ii) $E(X) = 10, Var(X) = 10(0.999) = 9.99$

Q6. $X \sim NB(2, 0.5)$

(i) $P(\text{7th child is 2nd son}) = \binom{6}{1} 0.5^2 0.5^5 = 0.0469$

(ii) $E(X) = \frac{2}{0.5} = 4$