CS1231S Tutorial 7

AY 24/25 Sem 1 — github/omgeta

Q1. (a) Predicate $P(n)$ cannot be used as a binary operand ■

(b) We cannot assume equality to $P(k+1)$, we must show $P(k) \to P(k+1)$ ■

(c) If we assume $P(k)$ is true for all $k \in \mathbb{Z}^+$, then there is nothing to prove. ■

Q2. **Proof by 1MI**

1. Let $P(n) \equiv (1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2x+1)), \forall n \in \mathbb{Z}^+$
2. Basis step:
   2.1. $1^2 = \frac{1}{6}(2)(3)$, therefore $P(1)$ is true
3. Assume $P(k)$ is true for some $k \geq 1 \implies 1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$
4. Inductive step:
   4.1. $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$
   $= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$
   4.2. Therefore, $P(k+1)$ is true
5. Therefore, $P(n)$ is true for all $n \in \mathbb{Z}^+$ ■

Q3. **Proof by 1MI**

1. Let $P(n) \equiv (1 + nx \leq (1+x)^n), \forall n \in \mathbb{Z}^+, x \in \mathbb{Z}_{\geq -1}$
2. Basis step:
   2.1. $1 + x \leq (1+x)^1$, therefore $P(1)$ is true
3. Assume $P(k)$ is true for some $k \geq 1 \implies 1 + kx \leq (1+x)^k$
4. Inductive step:
   4.1. $1 + (k+1)x = 1 + kx + x \leq (1+x)^k + x \leq (1+x)^k + x(1+x)^k = (1+x)^{k+1}$
   4.2. Therefore, $P(k+1)$ is true
5. Therefore, $P(n)$ is true for all $n \in \mathbb{Z}^+$ ■

Q4. **Proof by 1MI**

1. Let $P(n) \equiv (2^{n+2} \mid (a^{2^n} - 1)), \forall n \in \mathbb{Z}^+, a$ is any odd integer
2. Basis step:
   2.1. $a^{2^1} - 1 = (a+1)(a-2)$ (Basic algebra)
   2.2. $= (2m+2)(2m) = 4(m+1)(m)$ (Definition of odd numbers)
   2.3. $= 4(2k)$ (Prod. of consecutive integers is even)
   2.4. $= 8k = k \cdot 2^3$
   2.5. $\therefore 2^3 \mid (a^{2^1} - 1)$ (Definition of divides)
   2.6. Therefore, $P(1)$ is true
3. Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$:
   3.1. $2^{k+2} \mid a^{2^n} - 1$ (Definition of $P(n)$)
   3.2. $\exists m \in \mathbb{Z}$, $m \cdot 2^{k+2} = a^{2^k} - 1$ (Definition of divides)
4. Inductive step:
   4.1. $a^{2^{k+1}} - 1 = (a^{2^k})^2 - 1 = (a^{2^k} - 1)(a^{2^k} + 1)$ (Basic algebra)
   4.2. $= m \cdot 2^{k+2} \cdot (a^{2^k} + 1)$ (By inductive hypothesis)
   4.3. $= m \cdot 2^{k+2} \cdot (m \cdot 2^{k+2} + 2)$ (By indutive hypothesis)
   4.4. $m \cdot 2^{k+3}(m \cdot 2^{k+1} + 1)$ (Basic algebra)
   4.5. Therefore, $P(k+1)$ is true
5. Therefore, $P(n)$ is true for all $n \in \mathbb{Z}^+$ ■

Q5. **Proof by 2MI**

1. Let $P(n) \equiv (n = 3x + 5y), \forall n \geq \mathbb{Z}_{\geq 8}, \exists x, y \in \mathbb{N}$
2. Basis step:
   2.1. $8 = 3(1) + 5(1)$, therefore $P(8)$ is true
   2.2. $9 = 3(3) + 5(0)$, therefore $P(9)$ is true
   2.3. $10 = 3(0) + 5(2)$, therefore $P(10)$ is true
3. Assume $P(i)$ is true for $8 \leq i \leq k$ for some k
4. Inductive step:
   4.1. $P(k-2)$ is true $\implies k - 2 = 3a + 5b$, for some $a, b \in \mathbb{Z}$
   4.2. $k + 1 = (k-2) + 3 = 3a + 5b + 3 = 3(a+1) + b$
   4.3. Therefore, $P(k+1)$ is true
5. Therefore, $P(n)$ is true for all $n \in \mathbb{Z}_{\geq 8}$ $\blacksquare$

Q6. **Proof by 2MI**

1. Let $P(n) \equiv (i_1 < i_2 < \cdots < i_l \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_l}), \forall n \in \mathbb{Z}^+ \exists l \in \mathbb{Z}^+ \exists i_1, i_2, \cdots, i_l \in \mathbb{N}$
2. Basis step: $1 = 2^0 \implies P(1)$ is true
3. Assume $P(i)$ is true for $1 \leq i \leq k$ for some k
4. Inductive step:
   4.1. Case 1 ($k+1$ is odd):
      4.1.1. $k + 1 = 2m + 1$, $m = \frac{k+1}{2} \in \mathbb{Z}$ (Definition of odd numbers)
      4.1.2. $m = 2^{i_1} + \cdots + 2^{i_l}$ (By inductive hypothesis)
      4.1.3. $k = 2(2^{i_1} + \cdots + 2^{i_l}) = 2^{i_1+1} + \cdots + 2^{i_l+1}$ where $i_1 + 1, i_2 + 1, \cdots i_l + 1 \geq 1$
      4.1.4. $k + 1 = 2^{i_1+1} + \cdots + 2^{i_l+1} + 2^0$
      4.1.5. Therefore, $P(k+1)$ is true
   4.2. Case 2 ($k+1$ is even):
      4.2.1. $k + 1 = 2m$, $m = \frac{k+1}{2} \in \mathbb{Z}$ (Definition of even numbers)
      4.2.2. $m = 2^{i_1} + \cdots + 2^{i_l}$ (By inductive hypothesis)
      4.2.3. $k + 1 = 2(2^{i_1} + \cdots + 2^{i_l}) = 2^{i_1+1} + \cdots + 2^{i_l+1}$
      4.2.4. Therefore, $P(k+1)$ is true
   4.3. In all cases, $P(k+1)$ is true
5. Therefore, $P(n)$ is true for all $n \in \mathbb{Z}^+$ $\blacksquare$

Q7. **Proof by 2MI**

1. Let $P(n) \equiv (a_n < 3^n), \forall n \in \mathbb{N}$
2. Basis step: $a_0 = 0 < 1 = 3^0$, therefore $P(0)$ is true
3. Assume $P(i)$ is true for $0 \leq i \leq k$ for some k
4. Inductive step:
   4.1. $a_{k+1} = a_k + a_{k-1} + a_{k-2} < 3^k + 3^{k-1} + 3^{k-2} < 3^k + 3^k + 3^k = 3^{k+1}$
   4.2. Therefore, $P(k+1)$ is true
5. Therefore, $P(n)$ is true for all $n \in \mathbb{N}$ $\blacksquare$

Q8. (a) $F(0+b) = F(b) = (F(1) \times F(b) + F(0) \times F(b-1))$, therefore $P(0,b)$ is true ■
$F(1+b) = F(b) + F(b-1) = (F(2) \times F(b) + F(1) \times F(b-1))$, therefore $P(1,b)$ is true ■

(b) 1. Assume $P(k-1,b) \wedge P(k,b)$ for some $k \in \mathbb{Z}^+$:

$$F(k-1+b) = F(k) \times F(b) + F(k-1) \times F(b-1)$$
$$F(k+b) = F(k+1) \times F(b) + F(k) \times F(b-1)$$

2. Inductive step:
   2.1. $F(k+1+b) = F(k+b) + F(k+b-1)$ (Definition of Fibonacci sequence)
   2.2. $= (F(k+1) \times F(b) + F(k) \times F(b-1)) + (F(k) \times F(b) + F(k-1) \times F(b-1))$
   2.3. $= F(b) \times (F(k+1) + F(k)) + F(b-1) \times (F(k) + F(k-1))$ (Distributive law)
   2.4. $= F(b) \times F(k+2) + F(b-1) \times F(k+1)$ (Definition of Fibonacci sequence)
   2.5. Therefore, $P(k+1,b)$ is true
3. Therefore, $P(n+1,b)$ is true for all $n \in \mathbb{Z}^+$ ■

Q9. **Proof by 1MI**

1. Basis step: $1 = 2^0 5^0 5^0$, therefore $P(1)$ is true
2. Assume $P(m)$ is true for some m, i.e. $\exists! i \exists! j \exists! k ((i,j,k \geq 0) \wedge m = 2^i 3^j 5^k)$
3. Inductive step:
   3.1. $2m = 2 \cdot 2^i 3^j 5^k = 2^{i+1} 3^k 5^k \implies P(2m)$ (By inductive hypothesis)
   3.2. $3m = 3 \cdot 2^i 3^j 5^k = 2^i 3^{k+1} 5^k \implies P(3m)$ (By inductive hypothesis)
   3.3. $5m = 5 \cdot 2^i 3^j 5^k = 2^i 3^k 5^{k+1} \implies P(5m)$ (By inductive hypothesis)
   3.4. Therefore $P(m) \rightarrow P(2m) \wedge P(3m) \wedge P(5m)$ (Conjunction)
4. Therefore, $\forall n \in H,\ P(n)$ ■ (Given 1MI rule)

Q10. $0, 15 \notin S$ and $6, 16, 36 \in S$ ■

Q11. (a) Yes; $C = (A \setminus B) \cup (B \setminus A) \in S$ ■

(b) No ■