# MA1522 Homework 1

AY 24/25 Sem 1 — github/omgeta

Q1. (a)

$$
\begin{aligned}
2x + 3y + 4z &= 400 \quad \blacksquare && \text{(i)}\\
1x + 2y + 1z &= 200 \quad \blacksquare && \text{(ii)}\\
2y + 4z &= 160 \quad \blacksquare && \text{(iii)}
\end{aligned}
$$

(b) Form and reduce the corresponding augmented matrix for the system of equations:

$$
\begin{aligned}
\begin{bmatrix} 2 & 3 & 4 & 400 \\ 1 & 2 & 1 & 200 \\ 0 & 2 & 4 & 160 \end{bmatrix}
&\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 1 & 200 \\ 2 & 3 & 4 & 400 \\ 0 & 2 & 4 & 160 \end{bmatrix}\\
&\xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 & 200 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & 4 & 160 \end{bmatrix}\\
&\xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 5 & 200 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & 4 & 160 \end{bmatrix}\\
&\xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 0 & 5 & 200 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 8 & 160 \end{bmatrix}\\
&\xrightarrow{\frac{1}{8}R_3} \begin{bmatrix} 1 & 0 & 5 & 200 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 20 \end{bmatrix}\\
&\xrightarrow{R_2 - 2R_3} \begin{bmatrix} 1 & 0 & 5 & 200 \\ 0 & -1 & 0 & -40 \\ 0 & 0 & 1 & 20 \end{bmatrix}\\
&\xrightarrow{-R_2} \begin{bmatrix} 1 & 0 & 5 & 200 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 20 \end{bmatrix}\\
&\xrightarrow{R_1 - 5R_3} \begin{bmatrix} 1 & 0 & 0 & 100 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 20 \end{bmatrix}
\end{aligned}
$$

Hence, we find $x = 100, y = 40, z = 20$. Therefore, pumps of type X pump 100 litres/sec, pumps of type Y pump 40 litres/sec, and pumps of type Z pump 20 litres/sec. $\blacksquare$

Q2. Reduce the corresponding augmented matrix:

$$\begin{bmatrix} a & 2 & a & (a+b) & (a-b) \\ a & 2 & a & a & (a-b) \\ 3 & 3 & -b & 3 & -b \\ (a+1) & 3 & (a+1) & (a+1) & (a-b+1) \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 0 & 0 & 0 & b & 0 \\ a & 2 & a & a & (a-b) \\ 3 & 3 & -b & 3 & -b \\ (a+1) & 3 & (a+1) & (a+1) & (a-b+1) \end{bmatrix}$$

$$\xrightarrow{R_4-R_2} \begin{bmatrix} 0 & 0 & 0 & b & 0 \\ a & 2 & a & a & (a-b) \\ 3 & 3 & -b & 3 & -b \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_3-3R_4} \begin{bmatrix} 0 & 0 & 0 & b & 0 \\ a & 2 & a & a & (a-b) \\ 0 & 0 & (-b-3) & 0 & (-b-3) \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2-aR_4} \begin{bmatrix} 0 & 0 & 0 & b & 0 \\ 0 & (2-a) & 0 & 0 & -b \\ 0 & 0 & (-b-3) & 0 & (-b-3) \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & (2-a) & 0 & 0 & -b \\ 0 & 0 & (-b-3) & 0 & (-b-3) \\ 0 & 0 & 0 & b & 0 \end{bmatrix}$$

(a) No solution: $a = 2 \wedge b \neq 0$. (Row 2 will have inconsistent equation $0 \neq 0$) ■

(b) Unique solution: $a \neq 2 \wedge b \neq -3 \wedge b \neq 0$. (RREF has pivot in every row) ■

$$\xrightarrow{\frac{1}{b}R_4} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & (2-a) & 0 & 0 & -b \\ 0 & 0 & (-b-3) & 0 & (-b-3) \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1-R_4} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & (2-a) & 0 & 0 & -b \\ 0 & 0 & (-b-3) & 0 & (-b-3) \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{-b-3}R_3} \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & (2-a) & 0 & 0 & -b \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1-3R_3} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & (2-a) & 0 & 0 & -b \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2-a}R_2} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{b}{2-a} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{b}{2-a} \\ 0 & 1 & 0 & 0 & -\frac{b}{2-a} \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence, the unique solution will be $x_1 = -\frac{b}{a-2}, x_2 = \frac{b}{a-2}, x_3 = 1, x_4 = 0$ ■

(c) Infinite solutions w/ 1 parameter: $a \neq 2 \wedge (b = -3 \vee b = 0)$. (There will be exactly one zero row $\implies$ there will be exactly 1 free variable) ■

Suppose, there are infinitely many solutions and $x_3 = 1, x_4 = 0$:
**Case 1:** $a \neq 2, b = -3$:

$$\xrightarrow{\frac{1}{2-a}R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2-a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 1 & 1 & 1-\frac{3}{2-a} \\ 0 & 1 & 0 & 0 & \frac{3}{2-a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{b}R_4} \begin{bmatrix} 1 & 0 & 1 & 1 & 1-\frac{3}{2-a} \\ 0 & 1 & 0 & 0 & \frac{3}{2-a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1-R_4} \begin{bmatrix} 1 & 0 & 1 & 0 & 1-\frac{3}{2-a} \\ 0 & 1 & 0 & 0 & \frac{3}{2-a} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then, $x_1 = 1 - \frac{3}{2-a} - x_3 = \frac{3}{a-2}$ ■

**Case 2:** $a \neq_2, b = 0$:

$$\xrightarrow{\frac{1}{2-a}R_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_4} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, $x_1 = 0 - x_4 = 0$ ■

(d) It is not possible to have infinite solutions w/ 3 parameters.
Firstly, we need to ensure the system is not inconsistent (when $a = 2, b \neq 0$).
Note that $x_1$ can never be free since it is not dependent on any variables $a, b$.
Then, we can set $x_2$ free (when $a = 2 \wedge b = 0$), $x_3$ free (when $b = -3$), or $x_4$ free (when $b = 0$).
However, since $a, b$ can only take one value at a time, we can at most satisfy 2 of the conditions simultaneously (when $a = 2 \wedge b = 0$), allowing at most 2 parameters $x_2, x_4$. ■

Q3. (a) Elementary row operations $A \xrightarrow{R_3+R_1} \xrightarrow{R_4-R_2} \xrightarrow{R_2+5R_1} U$ can also be expressed in terms with matrix multiplication of $A$ by elementary row matrices such as:

$$E_3E_2E_1A = U$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose there exists the LU factorisation, $A = LU$:

$$\begin{aligned} E_3E_2E_1LU &= U && \text{(Substitute } A = LU\text{)} \\ LU &= E_1^{-1}E_2^{-1}E_3^{-1}U && \text{(Definition of inverse } EE^{-1} = E^{-1}E = I\text{)} \\ L &= E_1^{-1}E_2^{-1}E_3^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -5 & 1 & 0 & 1 \end{bmatrix} \quad \blacksquare \end{aligned}$$

(b) To solve $A\vec{x} = LU\vec{x} = \begin{bmatrix} 2 \\ 8 \\ -2 \\ 5 \end{bmatrix}$, let $U\vec{x} = \vec{y}$ and solve $L\vec{y} = \begin{bmatrix} 2 \\ 8 \\ -2 \\ 5 \end{bmatrix}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -5 & 1 & 0 & 0 & 8 \\ -1 & 0 & 1 & 0 & -2 \\ -5 & 1 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 18 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\therefore \vec{y} = \begin{bmatrix} 2 \\ 18 \\ 0 \\ -2 \end{bmatrix}$$

Then solve $U\vec{x} = \begin{bmatrix} 2 \\ 18 \\ 0 \\ -2 \end{bmatrix}$:

$$\begin{bmatrix} -1 & 2 & -2 & 4 & 2 \\ 0 & 15 & -7 & 24 & 18 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & -3 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \quad \blacksquare$$

(c) By using properties of the determinant:

$$\begin{aligned} \det(A) &= \det(L) \cdot \det(U) && \text{(Lay T3.6 Multiplicative property)} \\ &= (1 \cdot 1 \cdot 1 \cdot 1) \cdot (-1 \cdot 15 \cdot 1 \cdot -3) && \text{(Lay T3.2 Determinant of triangular matrix)} \\ &= 1 \cdot 45 \\ &= 45 \quad \blacksquare \end{aligned}$$

Q4. (a) (i) By the Invertible Matrix Theorem, $A$ is invertible $\iff \det(A) \neq 0$, so we find the values of $a$ for which $\det(A) \neq 0$:

$$\begin{aligned}
\det(A) &\neq 0 \\
\begin{vmatrix} a & a & a \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} &\neq 0 \\
a\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - a\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} &\neq 0 && \text{(Lay T3.1 Cofactor expansion)} \\
a(1-0) - a(1-0) + a(1-0) &\neq 0 && \text{(Determinant of } 2\times 2 \text{ matrices)} \\
a &\neq 0
\end{aligned}$$

Therefore, for $A$ to be invertible, $a \neq 0$. ■

(ii) Suppose $C_{ij}$ is the $(i,j)$ cofactor of $A$, and $M_{ij}$ is the $(i,j)$ matrix minor of $A$ obtained by deletion of the $i$th row and $j$th column. $C_{ij}$ is given by:

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

First, find the cofactors of $A$, finding the determinant of each $M_{ij}$ by definition of determinant for $2 \times 2$ matrices:

$$\begin{aligned}
C_{11} &= +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, & C_{21} &= -\begin{vmatrix} a & a \\ 1 & 1 \end{vmatrix} = 0, & C_{31} &= +\begin{vmatrix} a & a \\ 1 & 0 \end{vmatrix} = -a \\
C_{12} &= -\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -1, & C_{22} &= +\begin{vmatrix} a & a \\ 0 & 1 \end{vmatrix} = a, & C_{32} &= -\begin{vmatrix} a & a \\ 1 & 0 \end{vmatrix} = a \\
C_{13} &= +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, & C_{23} &= -\begin{vmatrix} a & a \\ 0 & 1 \end{vmatrix} = -a, & C_{33} &= +\begin{vmatrix} a & a \\ 1 & 1 \end{vmatrix} = 0
\end{aligned}$$

Then, adj$(A)$ is then given by:

$$\begin{aligned}
\operatorname{adj}(A) &= (C_{ij})^T \\
&= \begin{bmatrix} 1 & 0 & -a \\ -1 & a & a \\ 1 & -a & 0 \end{bmatrix} \quad ■
\end{aligned}$$

(iii) Suppose $A$ is invertible:

$$\begin{aligned}
A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) && \text{(Lay T3.8 Adjoint Formula for Inverse)} \\
&= \frac{1}{a} \begin{bmatrix} 1 & 0 & -a \\ -1 & a & a \\ 1 & -a & 0 \end{bmatrix} \quad ■ && \text{(From (i) and (ii))}
\end{aligned}$$

(b) Suppose there is some matrix $cA$, where $c \in \mathbb{R}$:

$$\begin{aligned}
(cA)^{-1} &= \frac{1}{\det(cA)} \operatorname{adj}(cA) && \text{(Lay T3.8 Adjoint Formula for Inverse)} \\
\frac{1}{c} A^{-1} &= \frac{1}{\det(cA)} \operatorname{adj}(cA) && \text{(Chapter 2 Slide 87, } (aA)^{-1} = \tfrac{1}{a} A^{-1}\text{)} \\
\frac{1}{c} \cdot \frac{1}{\det(A)} \operatorname{adj}(A) &= \frac{1}{\det(cA)} \operatorname{adj}(cA) && \text{(Substituting in } A^{-1}\text{)} \\
&= \frac{1}{c^n \det(A)} \operatorname{adj}(cA) && \text{(Chapter 2 Slide 158, } \det(cA) = c^n \det(A)\text{)} \\
&= \frac{1}{c^n} \cdot \frac{1}{\det(A)} \operatorname{adj}(cA) \\
\operatorname{adj}(A) &= \frac{1}{c^{n-1}} \operatorname{adj}(cA) && \text{(Cancelling common terms)} \\
\therefore \operatorname{adj}(cA) &= c^{n-1} \operatorname{adj}(A)
\end{aligned}$$

Hence, for $\operatorname{adj}(A) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$, $\operatorname{adj}(3A)$ is given by:

$$\begin{aligned}
\operatorname{adj}(3A) &= 3^3 \cdot \operatorname{adj}(A) \\
&= 27 \cdot \operatorname{adj}(A) \\
&= \begin{bmatrix} 27 & 27 & 27 & 0 \\ 27 & 27 & 0 & 27 \\ 27 & 0 & 27 & 27 \\ 0 & 27 & 27 & 27 \end{bmatrix} \quad \blacksquare
\end{aligned}$$