CS1231S Tutorial 3

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $\mathcal{P}(\{a,b,c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$ ■

(b) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$ ■

Q2. (a) $A \cup B = \{x \in \mathbb{R} : -2 \le x < 3\} = [-2, 3)$ ■

(b) $A \cap B = \{x \in \mathbb{R} : -1 < x \le 1\} = (-1, 1]$ ■

(c) $\overline{A} = \{x \in \mathbb{R} : x < -2 \vee x > 1\} = (-\infty, -2) \cup (1, \infty)$ ■

(d) $\overline{A} \cap \overline{B} = \{x \in \mathbb{R} : x < -2 \vee x > 3\} = (-\infty, -2) \cup (3, \infty)$ ■

(e) $A \setminus B = \{x \in \mathbb{R} : -2 \le x \le -1\} = [-2, -1]$ ■

Q3. (a) True. If $A \cap B = \varnothing$, e.g. let $A = \{1\}, B = \{2\}$ ■

(b) True. If $A \cap B \neq \varnothing$, e.g. let $A = \{2\}, B = \{2\}$ ■

(c) False. $\forall A, B,\ \varnothing \in \mathcal{P}(A \times B)$ but $\varnothing \notin A \times \mathcal{P}(B)$ ■

Q4. 1. Prove $A \subseteq B$:

1.1 Suppose $a \in A$, then $a = 2n + 1, n \in \mathbb{Z}$.

1.2 Let $m = n + 3$, $m \in \mathbb{Z}$ by closure of integers over addition.

1.3 $a = 2n + 1 = 2(n + 3) - 5 = 2m - 5 \in B$.

1.4 $\therefore \forall a \in A, a \in B$ (Universal generalization)

1.5 $\therefore A \subseteq B$ (Definition of subsets)

2. Prove $B \subseteq A$:

2.1 Suppose $b \in B$, then $a = 2n - 5, n \in \mathbb{Z}$.

2.2 Let $m = n - 3$, $m \in \mathbb{Z}$ by closure of integers over addition.

2.3 $b = 2n - 5 = 2(n - 3) + 1 = 2m + 1 \in A$.

2.4 $\therefore \forall b \in B, b \in A$ (Universal generalization)

2.5 $\therefore B \subseteq A$ (Definition of subsets)

3. $\therefore A \subseteq B \wedge B \subseteq A$ (Conjunction)

4. $\therefore A = B$ ■ (Definition of set equality)

Q5. Let A, B, C be sets. To prove $A \cap (B \setminus C) = (A \cap B) \setminus C$:

1. Prove $A \cap (B \setminus C) \subseteq (A \cap B) \setminus C$:

   1.1 $A \cap (B \setminus C) = \{x : x \in A \cap (B \setminus C)\}$

   1.2 $= \{x : (x \in A) \wedge (x \in B \setminus C)\}$ (Definition of set difference)

   1.3 $= \{x : (x \in A) \wedge ((x \in B) \wedge (x \notin C))\}$ (Definition of set difference)

   1.4 $= \{x : ((x \in A) \wedge (x \in B)) \wedge (x \notin C)\}$ (Associative law)

   1.5 $= \{x : (x \in A \cap B) \wedge (x \notin C)\}$ (Definition of intersection)

   1.6 $= \{x : x \in (A \cap B) \setminus C\}$ (Definition of set difference)

   1.7 $= (A \cap B) \setminus C$

2. We must show $\forall x, x \in (A \cap B) \setminus C \rightarrow x \in A \cap (B \setminus C)$

   1.1 Let $x \in (A \cap B) \setminus C$

   1.2 $x \in (A \cap B) \wedge x \notin C$ (Definition of set difference)

   1.3 $x \notin C$ (Specialisation)

   1.4 $x \in (A \cap B)$ (Specialisation)

   1.5 $x \in A \wedge x \in B$ (Definition of set intersection)

   1.6 $x \in A$ (Specialisation)

   1.7 $x \in B$ (Specialisation)

   1.8 $x \in B \wedge x \notin C$ (Conjunction)

   1.9 $x \in (B \setminus C)$ (Definition of set difference)

   1.10 $x \in A \wedge x \in (B \setminus C)$ (Conjunction)

   1.11 $x \in A \cap (B \setminus C)$ (Definition of set intersection)

3. $\therefore (A \cap (B \setminus C) \subseteq (A \cap B) \setminus C) \wedge ((A \cap B) \setminus C \subseteq A \cap (B \setminus C))$ (Conjunction)

4. $\therefore A \cap (B \setminus C) = (A \cap B) \setminus C$ (Definition of set equality)

Therefore, $\forall A, B, C, A \cap (B \setminus C) = (A \cap B) \setminus C$ ■

Q6. Let A, B, C be sets.

1. $A \setminus (B \setminus C)$
2. $= A \setminus (B \cap \overline{C})$ (Set difference law)
3. $= A \cap \overline{(B \cap \overline{C})}$ (Set difference law)
4. $= A \cap (\overline{B} \cup C)$ (DeMorgan's law)
5. $= (A \cap \overline{B}) \cup (A \cap C)$ (Distributive law)
6. $= (A \setminus B) \cup (A \cap C)$ (Set difference law)

Therefore, $\forall A, B, C, A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$ ■

Q7. (a) $A \oplus B = \{1, 9\}$ ■

(b) Let A, B be sets, and U is the universal set.

1. $A \oplus B$
2. $= (A \setminus B) \cup (B \setminus A)$ (Definition of XOR)
3. $= (A \cap \overline{B}) \cup (B \setminus A)$ (Set difference law)
4. $= (A \cap \overline{B}) \cup (B \cap \overline{A})$ (Set difference law)
5. $= ((A \cap \overline{B}) \cup B) \cap ((A \cap \overline{B}) \cup \overline{A})$ (Distributive law)
6. $= ((A \cup B) \cap (\overline{B} \cup B)) \cap ((A \cup \overline{A}) \cap (\overline{B} \cup \overline{A}))$ (Distributive law)
7. $= ((A \cup B) \cap U) \cap (U \cap (\overline{B} \cup \overline{A}))$ (Complement law)
8. $= (A \cup B) \cap (\overline{B} \cup \overline{A})$ (Idempotent law)
9. $= (A \cup B) \cap (\overline{A} \cup \overline{B})$ (Commutative law)
10. $= (A \cup B) \cap \overline{(A \cap B)}$ (DeMorgan's law)
11. $= (A \cup B) \setminus (A \cap B)$ (Set difference law)

Therefore, $\forall A, B, A \oplus B = (A \cup B) \setminus (A \cap B)$ ■

Q8. Let A, B be sets. To prove $A \subseteq B \leftrightarrow A \cup B = B$

1. Prove $A \subseteq B \rightarrow A \cup B = B$

    1.1 Suppose $A \subseteq B$

    1.2 Prove $A \cup B \subseteq B$

    1.2.1 Let $x \in A \cup B$

    1.2.2 $x \in A \vee x \in B$ (Definition of union)

    1.2.3 Case 1: $x \in A \implies x \in B$ (By 1.1)

    1.2.4 Case 2: $x \in B$

    1.2.5 In both cases, $x \in B$

    1.2.6 $\therefore A \cup B \subseteq B$ (Definition of subset)

    1.3 Prove $B \subseteq A \cup B$

    1.3.1 Let $x \in B$

    1.3.2 $x \in A \vee x \in B$ (Generalisation)

    1.3.3 $x \in (A \cup B)$ (Definition of union)

    1.3.4 $\therefore B \subseteq A \cup B$ (Definition of subset)

    1.4 $(A \cup B \subseteq B) \wedge (B \subseteq A \cup B)$ (Conjunction)

    1.5 $A \cup B = B$ (Definition of set equality)

2. Prove $A \cup B = B \rightarrow A \subseteq B$

    2.1 Suppose $A \cup B = B$

    2.2 Let $x \in A$

    2.3 $x \in A \vee x \in B$ (Generalisation)

    2.4 $x \in A \cup B$ (Definition of union)

    2.5 $x \in B$ (By 2.1)

    2.6 $\therefore A \subseteq B$ (Definition of subset)

3. $(A \subseteq B \rightarrow A \cup B = B) \wedge (A \cup B = B \rightarrow A \subseteq B)$ (Conjunction)

4. $A \subseteq B \leftrightarrow A \cup B = B$ (Definition of iff)

Q9. (a) Step 4 is an incorrect application of distribution over disjunction. $\blacksquare$

(b) 1. Prove $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$

1.1 Suppose $x \in (A \setminus B) \cup (B \setminus A)$

1.2 $x \in (A \setminus B) \vee x \in (B \setminus A)$ (Definition of union)

1.3 $x \in (A \cap \overline{B}) \vee x \in (B \cap \overline{A})$ (Set difference law)

1.4 Case 1: $x \in (A \cap \overline{B}) \implies x \in A \wedge x \notin B$ (Definition of intersection)

1.5 Case 2: $x \in (B \cap \overline{A}) \implies x \in B \wedge x \notin A$ (Definition of intersection)

1.6 In either case, $x \in A \cup B$ (Definition of union)

1.7 In either case, $x \notin A \cap B$ (Definition of intersection)

1.8 $x \in (A \cup B) \wedge x \in \overline{(A \cap B)}$ (Conjunction)

1.9 $x \in (A \cup B) \cap \overline{(A \cap B)}$ (Definition of intersection)

1.10 $x \in (A \cup B) \setminus (A \cap B)$ (Set difference law)

1.11 $\therefore (A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$

2. Prove $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$

2.1 Suppose $x \in (A \cup B) \setminus (A \cap B)$

2.2 $x \in (A \cup B) \cap \overline{(A \cap B)}$ (Set difference law)

2.3 $x \in (A \cup B) \wedge x \in \overline{(A \cap B)}$ (Definition of intersection)

2.4 $(x \in A \vee x \in B) \wedge x \in \overline{(A \cap B)}$ (Definition of union)

2.5 Case 1: $x \in A \wedge x \notin B \Rightarrow x \in A \setminus B$ (Definition of set difference)

2.6 Case 2: $x \in B \wedge x \notin A \Rightarrow x \in B \setminus A$ (Definition of set difference)

2.7 In either case, $x \in (A \setminus B) \cup (B \setminus A)$ (Definition of union)

2.8 $\therefore (A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$

3. Therefore, $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ $\blacksquare$ (Definition of set equality)

Q10. For $\{G, H, R, S\}$ to be a partition of $HSWW$, the sets G, H, R, S must be mutually disjoint and non-empty. Symbolically, if $HOUSE_i \in (G, H, R, S)$, when $n \neq k$,
$HOUSE_n \cap HOUSE_k = \varnothing \wedge HOUSE_n \neq \varnothing$. $\blacksquare$

Q11. (a) A_{-2}

$$\begin{aligned} A_{-2} &= \{\} \quad \blacksquare \\ &= \{x \in \mathbb{Z} : -2 \leq x \leq -4\} \quad \blacksquare \\ &= [-2, -4] \quad \blacksquare \end{aligned}$$

(b) $\bigcup_{i=3}^{5} A_i$

$$\begin{aligned} \bigcup_{i=3}^{5} A_i &= \{3, 4, \ldots, 10\} \quad \blacksquare \\ &= \{x \in \mathbb{Z} : 3 \leq x \leq 10\} \quad \blacksquare \\ &= [3, 10] \quad \blacksquare \end{aligned}$$

(c) $\bigcap_{i=3}^{5}$

$$\begin{aligned} \bigcap_{i=3}^{5} &= \{5, 6\} \quad \blacksquare \\ &= x \in \mathbb{Z} : 5 \leq x \leq 6 \quad \blacksquare \\ &= [5, 6] \quad \blacksquare \end{aligned}$$

Q12. (a) $\bigcup_{i=1}^{4} V_i$

$$
\begin{aligned}
\bigcup_{i=1}^{4} V_i &= V_1 \cup V_2 \cup V_3 \cup V_4 \\
&= [-\frac{1}{1}, \frac{1}{1}] \cup [-\frac{1}{2}, \frac{1}{2}] \cup [-\frac{1}{3}, \frac{1}{3}] \cup [-\frac{1}{4}, \frac{1}{4}] \\
&= [-\frac{1}{1}, \frac{1}{1}] \\
&= [-1, 1] \quad \blacksquare
\end{aligned}
$$

(b) $\bigcap_{i=1}^{4} V_i$

$$
\begin{aligned}
\bigcap_{i=1}^{4} V_i &= V_1 \cap V_2 \cap V_3 \cap V_4 \\
&= [-\frac{1}{1}, \frac{1}{1}] \cap [-\frac{1}{2}, \frac{1}{2}] \cap [-\frac{1}{3}, \frac{1}{3}] \cap [-\frac{1}{4}, \frac{1}{4}] \\
&= [-\frac{1}{4}, \frac{1}{4}] \\
&= [-\frac{1}{4}, \frac{1}{4}] \quad \blacksquare
\end{aligned}
$$

(c) $\bigcup_{i=1}^{n} V_i$

$$
\begin{aligned}
\bigcup_{i=1}^{n} V_i &= V_1 \cup \ldots \cup V_n \\
&= [-\frac{1}{1}, \frac{1}{1}] \cup \ldots \cup [-\frac{1}{n}, \frac{1}{n}] \\
&= [-\frac{1}{1}, \frac{1}{1}] \\
&= [-1, 1] \quad \blacksquare
\end{aligned}
$$

(d) $\bigcap_{i=1}^{n} V_i$

$$
\begin{aligned}
\bigcap_{i=1}^{n} V_i &= V_1 \cap \ldots \cap V_n \\
&= [-\frac{1}{1}, \frac{1}{1}] \cap \ldots \cap [-\frac{1}{n}, \frac{1}{n}] \\
&= [-\frac{1}{n}, \frac{1}{n}] \\
&= [-\frac{1}{n}, \frac{1}{n}] \quad \blacksquare
\end{aligned}
$$

(e) No, because $\forall V_n, 0 \in V_n$. $\blacksquare$