MA1522 Tutorial 7

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\blacksquare$

(b) Reduce the corresponding matrix:

$$\begin{pmatrix} 1 & 3 & -2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 0 \\ 0 & 0 & 5 & 10 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, the general solution is $s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ where $s, t \in \mathbb{R}$ $\blacksquare$

(c) This is equivalent to solving the system:

$$\begin{aligned} v_1 + 3v_2 - 2v_3 &= 0 \\ 2v_1 + 6v_2 - 5v_3 - 2v_4 &= 0 \\ 5v_3 + 10v_4 &= 0 \end{aligned}$$

From (b), choose $s = 1, t = 0$ then $\vec{v} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\blacksquare$

Q2. (a)

$$\begin{aligned} \vec{x} \cdot \vec{y} &= (\vec{v_1} - 2\vec{v_2} - 2\vec{v_3}) \cdot (2\vec{v_1} - 3\vec{v_2} + \vec{v}) \\ &= 2(\vec{v_1} \cdot \vec{v_1}) + 6(\vec{v_2} \cdot \vec{v_2}) - 2(\vec{v_3} \cdot \vec{v_3}) \\ &= 2 + 6 - 2 \\ &= 6 \quad \blacksquare \end{aligned}$$

(b)

$$\begin{aligned} ||\vec{x}|| &= \sqrt{\vec{x} \cdot \vec{x}} \\ &= \sqrt{(\vec{v_1} \cdot \vec{v_1}) + 4(\vec{v_2} + \vec{v_2}) + 4(\vec{v_3} + \vec{v_3})} \\ &= \sqrt{1 + 4 + 4} \\ &= 3 \quad \blacksquare \\ ||\vec{y}|| &= \sqrt{\vec{y} \cdot \vec{y}} \\ &= \sqrt{4(\vec{v_1} \cdot \vec{v_1}) + 9(\vec{v_2} + \vec{v_2}) + (\vec{v_3} + \vec{v_3})} \\ &= \sqrt{4 + 9 + 1} \\ &= \sqrt{14} \quad \blacksquare \end{aligned}$$

(c)

$$\begin{aligned} \theta &= \cos^{-1} \frac{6}{3\sqrt{14}} \\ &= 57.69^\circ \quad \blacksquare \end{aligned}$$

Q3. (a)

$$\vec{v_1}\cdot\vec{v_1} = 6 \quad \blacksquare$$
$$\vec{v_1}\cdot\vec{v_2} = 0 \quad \blacksquare$$
$$\vec{v_2}\cdot\vec{v_1} = 0 \quad \blacksquare$$
$$\vec{v_2}\cdot\vec{v_2} = 2 \quad \blacksquare$$

(b) $V^TV = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$ represents $\begin{pmatrix} \vec{v_1}\cdot\vec{v_1} & \vec{v_1}\cdot\vec{v_2} \\ \vec{v_2}\cdot\vec{v_1} & \vec{v_2}\cdot\vec{v_2} \end{pmatrix}$ $\blacksquare$

Q4. (a) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & -2 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, then $A^TA = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ which shows S is a set of orthongonal non-zero vectors which is automatically linearly independent $\blacksquare$

(b) Shown in (a)

(c) By Q1, $W^T = \text{Nul}(A^T)$ which is a subspace. To find W^T reduce A^T:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & -2 & -\frac{1}{4} \\ 0 & 1 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 2 & \frac{3}{4} \end{pmatrix}$$

Therefore, $\dim(W^T) = 2$ $\blacksquare$

(d) From (a), $T = \left\{ \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$ $\blacksquare$

(e) $\dfrac{\vec{v}\cdot\vec{w_1}}{||\vec{w_1}||^2}\vec{w_1} + \dfrac{\vec{v}\cdot\vec{w_2}}{||\vec{w_2}||^2}\vec{w_2} + \dfrac{\vec{v}\cdot\vec{w_3}}{||\vec{w_3}||^2}\vec{w_3} = \dfrac{1}{10}\begin{pmatrix} 10 \\ -1 \\ 12 \\ 3 \\ 6 \end{pmatrix}$ $\blacksquare$

(f)

$$A^T(\vec{v} - \vec{v_W}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $(\vec{v} - \vec{v_W}) \in W^{\perp}$

Q5. (a) Let $U = \begin{pmatrix} 1 & 1 & -1 & -2 \\ 2 & 1 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ -1 & 1 & -1 & 2 \end{pmatrix}$

$$U^TU = \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

Since U^TU is a scalar matrix, S is an orthogonal set with linearly independent vectors. Since $|S| = 4 = \dim(\mathbb{R}^4)$, $\mathrm{Span}(S) = \mathbb{R}^4$. Therefore, S is a basis for $\mathbb{R}^4$ ■

(b) No; because it is not possible to have a set of 5 linearly independent vectors in $\mathbb{R}^4$ ■

(c) From (a), $T = \left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{10}} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ ■

(d)

$$[\vec{v}]_S = (U^TU)^{-1}U^T\vec{V} = \begin{pmatrix} \frac{3}{10} \\ \frac{1}{2} \\ -1 \\ \frac{9}{10} \end{pmatrix} \quad ■$$

$$[\vec{v}]_T = \begin{pmatrix} \vec{v} \cdot u_1' \\ \vec{v} \cdot u_2' \\ \vec{v} \cdot u_3' \\ \vec{v} \cdot u_4' \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ 1 \\ -2 \\ \frac{9}{\sqrt{10}} \end{pmatrix} \quad ■$$