FDP2021 Special Physics Class 1, 2

AY 24/25 Sem 2 - 25/26 Sem 1 — github/omgeta

1. Vector Algebra

Differential Calculus

Gradient of scalar function f, ∇f, is a vector rate of change of f with maximum increase in the direction ∇f:

i. $\nabla(fg) = f(\nabla g) + g(\nabla f)$
ii. $\nabla(\vec{v}\cdot\vec{w}) = \vec{v}\times(\nabla\times\vec{w}) + \vec{w}\times(\nabla\times\vec{v}) + (\vec{v}\cdot\nabla)\vec{w} + (\vec{w}\cdot\nabla)\vec{v}$

Divergence of vector function $\vec{v}$, $\nabla\cdot\vec{v}$, is a scalar of how much $\vec{v}$ spreads out:

i. $\nabla\cdot(f\vec{v}) = f(\nabla\cdot\vec{v}) + \vec{v}\cdot(\nabla f)$
ii. $\nabla\cdot(\vec{v}\times\vec{w}) = \vec{w}\cdot(\nabla\times\vec{v}) - \vec{v}\cdot(\nabla\times\vec{w})$

Curl of vector function $\vec{v}$, $\nabla\times\vec{v}$, is a vector of how much $\vec{v}$ curls around:

i. $\nabla\times(f\vec{v}) = f(\nabla\times\vec{v}) - \vec{v}\times(\nabla f)$
ii. $\nabla\times(\vec{v}\times\vec{w}) = (\vec{w}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{w} + \vec{v}(\nabla\cdot\vec{w}) - \vec{w}(\nabla\cdot\vec{v})$

Laplacian of scalar function f, $\nabla^2 f = \nabla\cdot\nabla f$, is a scalar. Other second derivatives are:

i. $\nabla\cdot(\nabla\times\vec{v}) = 0$
ii. $\nabla\times(\nabla f) = \vec{0}$
iii. $\nabla\times(\nabla\times\vec{v}) = \nabla(\nabla\cdot\vec{v}) - \nabla^2\vec{v}$

Integral Calculus

Line Integral: $\int_a^b \vec{v}\cdot d\vec{\ell}$, for small $d\vec{\ell}$ along line

Surface Integral: $\int_{\mathcal{S}} \vec{v}\cdot d\vec{a}$, for small $d\vec{a}$ surface normal

Volume Integral: $\int_{\mathcal{V}} f\, d\tau$, for small $d\tau$ volume

Fundamental Theorems:

i. $\int_{\vec{v}}^{\vec{w}} (\nabla f)\cdot d\vec{\ell} = f(\vec{w}) - f(\vec{v})$ (Gradient)
ii. $\int (\nabla\cdot\vec{v})\, dV = \oint \vec{v}\cdot d\vec{a}$ (Divergence/ Gauss's)
iii. $\int (\nabla\times\vec{v})\cdot d\vec{S} = \oint \vec{v}\cdot d\vec{\ell}$ (Curl/ Stokes')

Coordinate Systems

Cartesian (x, y, z):

$$d\vec{\ell} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz, \quad d\tau = dx\,dy\,dz$$

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical (r, θ, ϕ) – origin radius r, z-angle θ, xy-angle ϕ:

$$d\vec{\ell} = \hat{r}\,dr + \hat{\theta}\,(r\,d\theta) + \hat{\phi}\,(r\sin\theta\,d\phi), \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$$

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\nabla\times\vec{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Cylindrical (s, ϕ, z) – z-radius s, xy-angle ϕ, height z:

$$d\vec{\ell} = \hat{s}\,ds + \hat{\phi}\,(s\,d\phi) + \hat{z}\,dz, \quad d\tau = s\,ds\,d\phi\,dz$$

$$\nabla f = \frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial\phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

$$\nabla\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s\, v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$$

$$\nabla\times\vec{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{s} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\phi} + \left[\frac{1}{s}\frac{\partial}{\partial s}(s\, v_\phi) - \frac{1}{s}\frac{\partial v_s}{\partial\phi}\right]\hat{z}$$

$$\nabla^2 f = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial f}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Standard Derivatives

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$\tan(g(x))$	$g'(x)\sec^2(g(x))$
$\sec(g(x))$	$g'(x)\sec(g(x))\tan(g(x))$
$\csc(g(x))$	$-g'(x)\csc(g(x))\cot(g(x))$
$\cot(g(x))$	$-g'(x)\csc^2(g(x))$
$\sin^{-1}(g(x))$	$\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\cos^{-1}(g(x))$	$-\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\tan^{-1}(g(x))$	$\frac{g'(x)}{1+g(x)^2}$
$\cot^{-1}(g(x))$	$-\frac{g'(x)}{1+g(x)^2}$
$\sec^{-1}(g(x))$	$\frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}}, \lvert g(x)\rvert > 1$
$\csc^{-1}(g(x))$	$-\frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}}, \lvert g(x)\rvert > 1$
a^x	$a^x\ln(a)$

Standard Integrals

$\mathbf{f(x)}$	$\mathbf{F(x) - C}$
$[f(x)]^n$, $n \neq -1$	$\frac{[f(x)]^{n+1}}{(n+1)f'(x)}$
$\tan(f(x))$	$\frac{1}{f'(x)}\ln\lvert\sec(f(x))\rvert$
$\sec(f(x))$	$\frac{1}{f'(x)}\ln\lvert\sec(f(x)) + \tan(f(x))\rvert$
$\csc(f(x))$	$-\frac{1}{f'(x)}\ln\lvert\csc(f(x)) + \cot(f(x))\rvert$
$\cot(f(x))$	$-\frac{1}{f'(x)}\ln\lvert\csc(f(x))\rvert$
$\sec^2(f(x))$	$\frac{1}{f'(x)}\tan(f(x))$
$\csc^2(f(x))$	$-\frac{1}{f'(x)}\cot(f(x))$
$\sec(f(x))\tan(f(x))$	$\frac{1}{f'(x)}\sec(f(x))$
$\csc(f(x))\cot(f(x))$	$-\frac{1}{f'(x)}\csc(f(x))$
$\frac{1}{a^2+[f(x)]^2}$	$\frac{1}{af'(x)}\tan^{-1}(\frac{f(x)}{a})$
$\frac{1}{\sqrt{a^2-[f(x)]^2}}$	$\frac{1}{f'(x)}\sin^{-1}(\frac{f(x)}{a})$
$-\frac{1}{\sqrt{a^2-[f(x)]^2}}$	$\frac{1}{f'(x)}\cos^{-1}(\frac{f(x)}{a})$
$\frac{1}{a^2-[f(x)]^2}$	$\frac{1}{2af'(x)}\ln\lvert\frac{f(x)+a}{f(x)-a}\rvert$
$\frac{1}{[f(x)]^2-a^2}$	$\frac{1}{2af'(x)}\ln\lvert\frac{f(x)-a}{f(x)+a}\rvert$
$\frac{1}{\sqrt{[f(x)]^2+a^2}}$	$\frac{1}{f'(x)}\ln\lvert f(x) + \sqrt{[f(x)]^2 + a^2}\rvert$
$\frac{1}{\sqrt{[f(x)]^2-a^2}}$	$\frac{1}{f'(x)}\ln\lvert f(x) + \sqrt{[f(x)]^2 - a^2}\rvert$
$\sqrt{a^2 - x^2}$	$\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(\frac{x}{a})$
$\sqrt{x^2 - a^2}$	$\frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\ln\lvert x + \sqrt{x^2 - a^2}\rvert$

2. Electrostatics

Electrostatics involves stationary source charges and their properties.

Coulomb's Law

Force $\vec{F}$ on test charge Q at $\vec{r}$ by source charge q at $\vec{r}'$ with separation vector $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$ is:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}^2} \hat{\mathscr{r}}$$

where $\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$ is permittivity of free space.

By Principle of Superposition, interactions between any two charges are unaffected by presence of any others; $\therefore$ Force on Q by point charges $q_1, \cdots, q_n$ at $\vec{r}_1, \cdots, \vec{r}_n$ is:

$$\vec{F} = \vec{F}_1 + \cdots + \vec{F}_n = \frac{Q}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i = Q\vec{E}$$

where $\vec{E}(\vec{r})$ is the electric field of source charges denoting force per unit charge exerted on a test charge at $\vec{r}$.

Line Charge: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}')}{\mathscr{r}^2} \hat{\mathscr{r}}\, d\ell'$

Surface Charge: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}')}{\mathscr{r}^2} \hat{\mathscr{r}}\, da'$

Volume Charge: $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{\mathscr{r}^2} \hat{\mathscr{r}}\, d\tau'$

Gauss's Law

Integral: $\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$ for enclosed charge Q_{enc}

Differential: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Symmetry

Spherical (total Q): $\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$ (out); $\vec{E} = \vec{0}$ (inside)

Cylindrical (line λ): $\vec{E}(\rho) = \frac{\lambda}{2\pi\epsilon_0 \rho} \hat{\rho}$

Planar (plane σ): $|\vec{E}| = \frac{\sigma}{2\epsilon_0}$ on each side, normal.

Potentials

Potential V at an electric field $\vec{E}$ is given by:

$$V(\vec{r}) \equiv -\int_{\infty}^{\vec{r}} \vec{E} \cdot d\vec{\ell} \quad \text{or} \quad \vec{E} = -\nabla V$$

and potential difference between $\vec{a}$ and $\vec{b}$ is:

$$V(\vec{b}) - V(\vec{a}) = -\int_a^b \vec{E} \cdot d\vec{\ell}$$

where principle of superposition applies, and in closed contour $\oint \vec{E} \cdot d\vec{\ell} = 0$ by conservative circulation.

Line Charge: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda(\vec{r}')}{\mathscr{r}} d\ell'$

Surface Charge: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\vec{r}')}{\mathscr{r}} da'$

Volume Charge: $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{\mathscr{r}} d\tau'$

Poisson's and Laplace's Equations

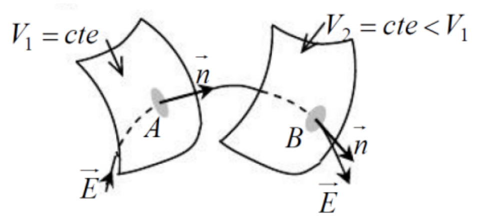
Poisson's Equation: $\nabla^2 V = \frac{-\rho}{\epsilon_0}$

Laplace's Equation: $\nabla^2 V = 0$ (for region of $\rho = 0$ charge)

Equipotential Surfaces

Equipotential surface is surface with constant potential V:

i. Field $\vec{E}$ follows direction of decreasing potentials
ii. Equipotential surfaces are (by definition of gradient), orthogonal to field lines
iii. In particular, a plane of antisymmetry is always an equipotential surface



Boundary Conditions

Electric field $\vec{E}$ is discontinuous over a surface charge σ by:

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \hat{n}$$

where $\hat{n}$ is the normal to the surface boundary, since

$$\vec{E}^{\perp}_{\text{above}} - \vec{E}^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0} \wedge \vec{E}^{\parallel}_{\text{above}} = \vec{E}^{\parallel}_{\text{below}}$$

Potential V is continuous over a boundary since:

$$V_{\text{above}} - V_{\text{below}} = -\int_a^b \vec{E} \cdot d\vec{\ell} \to 0 \text{ as } \ell \to 0$$

but gradient of V inherits discontinuity from $\vec{E} = -\nabla V$:

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{\sigma}{\epsilon_0} \hat{n}$$

Work and Energy

Work W taken to move test charge Q from $\vec{a}$ to $\vec{b}$ is:

$$W = \int_a^b \vec{F} \cdot d\vec{\ell} = -Q \int_a^b \vec{E} \cdot d\vec{\ell} = Q[V(\vec{b}) - V(\vec{a})]$$

or for system of point charges $q_1, \cdots, q_n$:

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$$

Volume Charge: $W = \frac{1}{2} \int \rho V \, d\tau = \frac{\epsilon_0}{2} \int_{\text{all space}} E^2 \, d\tau$

Capacitors

Capacitance C measured in farads (F) is coloumb-per-volt of electric charge stored, given by:

$$C = \frac{Q}{V}$$

Parallel-Plate Capacitor: $C = \frac{A\epsilon_0}{d}$ for area A, distance d

Electric Dipole

Electric dipoles consist of charges $\pm q$ separated by $\vec{d}$ characterised by dipole moment, $\vec{p}$, given by:

$$\vec{p} = q\vec{d}$$

Potential V_{dip} of a dipole is given by:

$$V_{\text{dip}}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\left|\vec{r} - \frac{\vec{d}}{2}\right|} - \frac{1}{\left|\vec{r} + \frac{\vec{d}}{2}\right|}\right) \underset{r \gg d}{\simeq} \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2}$$

Electric field $\vec{E}_{\text{dip}}$ of a dipole is given by:

$$\vec{E}_{\text{dip}}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\vec{r} - \frac{\vec{d}}{2}}{\left|\vec{r} - \frac{\vec{d}}{2}\right|^3} - \frac{\vec{r} + \frac{\vec{d}}{2}}{\left|\vec{r} + \frac{\vec{d}}{2}\right|^3}\right) \underset{r \gg d}{\simeq} \frac{3(\vec{p}\cdot\vec{r})\,\hat{r} - \vec{p}}{4\pi\epsilon_0 r^4}$$

Under External Fields

Torque $\vec{N}$ on a dipole $\vec{p}$ by uniform external field $\vec{E}$ is:

$$\vec{N} = \vec{p} \times \vec{E}$$

Force $\vec{F}$ on a dipole $\vec{p}$ by non-uniform external field $\vec{E}$ is:

$$\vec{F} = \nabla(\vec{p} \cdot \vec{E})$$

Energy U on a dipole $\vec{p}$ by external field $\vec{E}$ is:

$$U = -\vec{p} \cdot \vec{E}$$

Induced Dipole

Atoms can be polarized by external field $\vec{E}$ into a tiny dipole moment $\vec{p}$ given by:

$$\vec{p} = \alpha\vec{E}$$

where constant α is atomic polarizability of the atom.

Multipole Expansion

For general charge distributions, potential of a multipole is:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_0}{r} + \frac{\vec{p} \cdot \hat{r}}{r^2} + \frac{Q_{ij} r_i r_j}{r^3} + \cdots\right)$$

where q_0 is total charge, $\vec{p}$ is dipole moment, and Q_{ij} is quadrupole moment tensor

Examples

3. Magnetostatics

Magnetostatics involves steady currents and their properties.

Lorentz Force Law

Force $\vec{F}_{\text{mag}}$ on charge Q moving with velocity $\vec{v}$ in magnetic field $\vec{B}$ is:

$$\vec{F}_{\text{mag}} = Q(\vec{v} \times \vec{B})$$

Line Current: $\vec{F}_{\text{mag}} = \int \vec{I} \times \vec{B}\, d\ell = I \int d\vec{\ell} \times \vec{B}$, for $\vec{I} = \lambda \vec{v}$

Surface Current: $\vec{F}_{\text{mag}} = \int \vec{K} \times \vec{B}\, da, \quad \text{for } \vec{K} = \sigma \vec{v}$

Volume Current: $\vec{F}_{\text{mag}} = \int \vec{J} \times \vec{B}\, d\tau, \quad \text{for } \vec{J} = \rho \vec{v}$

Net force $\vec{F}$ on Q, in the presence of both electric and magnetic fields, is:

$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})]$$

Biot-Savart Law

Line Current: $\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{I}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\ell'$

Surface Current: $\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{K}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} da'$

Volume Current: $\vec{B}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{J}(\vec{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$

Ampère's Law

Differential: $\nabla \times \vec{B} = \mu_0 \vec{J}$

Integral: $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$ for enclosed current I_{enc}

Vector Potentials

Vector potential $\vec{A}$ at a magnetic field $\vec{B}$ is given by:

$$\vec{B} = \nabla \times \vec{A}$$

with the Coloumb gauge choice $\nabla \cdot \vec{A} = 0$

Line Current: $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{I}(\vec{r})}{\mathscr{r}} d\ell' = \dfrac{\mu_0}{4\pi I} \displaystyle\int \frac{1}{\mathscr{r}} d\vec{\ell}'$

Surface Current: $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{K}(\vec{r})}{\mathscr{r}} da'$

Surface Current: $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\vec{J}(\vec{r})}{\mathscr{r}} d\tau'$

Poisson's Equation

Poisson's Equation: $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

Boundary Conditions

Magnetic field $\vec{B}$ is discontinuous over a surface current density $\vec{K}$ by:

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0(\vec{K} \times \hat{n})$$

where $\hat{n}$ is the normal to the surface boundary, since

$$\vec{B}^{\perp}_{\text{above}} = \vec{B}^{\perp}_{\text{below}} \wedge \vec{B}^{\parallel}_{\text{above}} - \vec{B}^{\parallel}_{\text{below}} = \mu_0 \vec{K} \times \hat{n}$$

Vector potential $\vec{A}$ is continuous over a boundary since:

$$\vec{A}_{\text{above}} - \vec{A}_{\text{below}} = \int_a^b \vec{B} \cdot d\vec{S} \to 0 \text{ as } S \to 0$$

and $\nabla \cdot \vec{A} = 0$ and $\nabla \times \vec{A} = \vec{B}$ guarantees the normal and tangential components are continuous respectively but derivative of $\vec{A}$ inherits discontinuity from $\vec{B}$:

$$\frac{\partial \vec{A}_{\text{above}}}{\partial n} - \frac{\partial \vec{A}_{\text{below}}}{\partial n} = -\mu_0 \vec{K}$$

Magnetic Dipole

Magnetic dipoles consist of a current loop of I and area vector $\vec{a}$ characterised by dipole moment, $\vec{m}$, given by:

$$\vec{m} = I \int d\vec{a} = I\vec{a}$$

Vector potential $\vec{A}_{\text{dip}}$ of a dipole is given by:

$$\vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0(\vec{m} \times \vec{r})}{4\pi r^3}$$

Magnetic field $\vec{B}_{\text{dip}}$ of a dipole is given by:

$$\vec{B}_{\text{dip}}(\vec{r}) = \nabla \times \vec{A}_{\text{dip}}(\vec{r}) = \frac{\mu_0}{4\pi r^3}[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}]$$

Under External Fields

Torque $\vec{N}$ on a dipole $\vec{m}$ by uniform external field $\vec{B}$ is:

$$\vec{N} = \vec{m} \times \vec{B}$$

Force $\vec{F}$ on a dipole $\vec{m}$ by non-uniform external field $\vec{B}$ is:

$$\vec{F} = \nabla(\vec{m} \cdot \vec{B})$$

Energy U on a dipole $\vec{m}$ by external field $\vec{B}$ is:

$$U = -\vec{m} \cdot \vec{B}$$

Induced Dipole

Atoms can be magnetized by an external field $\vec{B}$ into a tiny dipole moment $\vec{m}$ given by:

$$\vec{m} = \chi \vec{B}$$

where constant χ is the magnetic susceptibility.

Multipole Expansion

For general current distributions, vector potential admits a multipole expansion:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{\vec{m} \times \hat{r}}{r^2} + \frac{Q^{(m)}_{ij} r_j}{r^3} + \cdots\right)$$

where $\vec{m}$ is dipole moment and $Q^{(m)}_{ij}$ is magnetic quadrupole tensor.

4. Electrodynamics

Maxwell's Equations