CS3230 Tutorial 1

AY 25/26 Sem 1 — github/omgeta

Q1). (a) Prove $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \implies f(n) \in o(g(n))$

1. Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
2. $\forall \epsilon > 0, \exists n_0 > 0$ s.t. $\forall n \geq n_0,\ \frac{f(n)}{g(n)} < \epsilon$
3. $f(n) < \epsilon\, g(n)$
4. Let $c = \epsilon$, $f(n) < c\, g(n)$
5. By definition, $f(n) \in o(g(n))$

(b) Prove $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \in O(g(n))$

1. Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = k$ for some finite k
2. By definition of limit, $\forall \epsilon > 0, \exists n_0 > 0, \forall n \geq n_0,\ |\frac{f(n)}{g(n)} - k| < \epsilon$
3. $-\epsilon < \frac{f(n)}{g(n)} - k < \epsilon$
4. $\frac{f(n)}{g(n)} - k < \epsilon$
5. $f(n) < (k + \epsilon)g(n)$
6. $\therefore \exists c = k + \epsilon > 0, \exists n_0 > 0, \forall n \geq n_0,\ f(n) \leq cg(n)$
7. By definition, $f(n) \in O(g(n))$

(c) Prove $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0 \implies f(n) \in \Omega(g(n))$

1. Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$, then $\lim_{n\to\infty} \frac{f(n)}{g(n)} = k$ for some finite k
2. By definition of limit, $\forall \epsilon > 0, \exists n_0 > 0, \forall n \geq n_0,\ |\frac{f(n)}{g(n)} - k| < \epsilon$
3. $-\epsilon < \frac{f(n)}{g(n)} - k < \epsilon$
4. $-\epsilon < \frac{f(n)}{g(n)} - k$
5. $(k - \epsilon)g(n) < f(n)$
6. $\therefore \exists c = k - \epsilon > 0, \exists n_0 > 0, \forall n \geq n_0,\ cg(n) \leq f(n)$
7. By definition, $f(n) \in \Omega(g(n))$

(d) Prove $0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \in \Theta(g(n))$

1. By conjunction of (c) and (d)

(e) Prove $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \implies f(n) \in \omega(g(n))$

1. Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$
2. $\forall k \in \mathbb{R}, \exists n_0 > 0, \forall n \geq n_0,\ \frac{f(n)}{g(n)} > k$
3. $\forall k \in \mathbb{R}^+, \exists n_0 > 0, \forall n \geq n_0,\ f(n) > kg(n)$
4. By definition, $f(n) \in \omega(g(n))$

Q2). (a) Reflexivity: Let $c = 1$

(b) Transitivity: for O, Ω, Θ apply substitution; for o, ω use limit rules

(c) Symmetry: if $c_1 g(n) \leq f(n) \leq c_2 g(n)$, divide to get $\frac{f(n)}{c_2} \leq g(n) \leq \frac{f(n)}{c_1}$

(d) Complementarity: for O, Ω, if $f(n) \leq cg(n)$ then $g(n) \geq \frac{f(n)}{c}$ and vice versa; likewise o, ω

Q3). (a.) True; $3^{n+1} = 3 \cdot 3^n \leq 3 \cdot 3^n$, so let $c = 4$ and $n_0 = 1$

(b.) False; $4^n = 2^{2n} = 2^n \cdot 2^n$, so we have 2^n which is not a constant factor

(c.) True; $n - 1 \leq \lfloor n \rfloor \leq n \implies \log(n-1) \leq \log\lfloor n \rfloor \leq \log n \implies 2^{\log(n-1)} \leq 2^{\log\lfloor n \rfloor} \leq n \implies \frac{n}{2} \leq n - 1 \leq 2^{\log\lfloor n \rfloor} \leq n \implies 2^{\log n} \in \Theta(n)$

(d.) True; $(n+a)^i = \sum_{r=0}^{n} \binom{i}{r} n^{i-r} a^r \leq \sum_{r=0}^{i} \binom{i}{r} n^i a^r = n^i \sum_{r=0}^{i} \binom{i}{r} 1^{i-r} a^r = n^i (1+a)^i$, so $(n+a)^i \leq (1+a)^i \cdot n^i$ where $(1+a)^i$ is clearly a constant

Q4). Since $2^{\log_2 n} = n$

(a.) True $O(n)$; by reflexivity

(b.) True $\Omega(n)$; by reflexivity

(c.) False $\Theta(\sqrt{n})$; by previous two statements is $\Theta(n)$

(d.) False $\omega(n)$; let $c = 1$, then $cn \not< n$

Q5). $f_1, f_5 < f_4 < f_3 < f_2$