CS1231S Tutorial 5

AY 24/25 Sem 1 — github/omgeta

Q1. **Disproof by Counterexample**

1. Suppose $a, b \in S$, $a = "s", b = "u"$
2. $len(a) = 1 = len(b) \wedge a \neq b$
3. $\exists a, b \in S((aRb \wedge bRa) \wedge (a \neq b))$
4. $\therefore R$ is not antisymmetric (Definition of antisymmetry)
5. $\therefore R$ is not a partial order ■ (Definition of partial order)

Q2. (a) False. $7 \mid 21 \implies 7 \preccurlyeq 21 \implies 21 \not\preccurlyeq *7$ (by antisymmetry) ■

(b) True. $2, 3$ are minimal elements. E.g. $\{2, 3, 5, 7, 21, 30, 84, 99\}$ ■

(c) True. $21 \preccurlyeq 84 \wedge 5$ is noncomparable to $21, 84$. E.g. $\{2, 3, 7, 21, 5, 30, 84, 99\}$ ■

(d) True. $30, 84, 99$ are maximal elements. E.g. $\{2, 3, 5, 7, 21, 99, 84, 30\}$ ■

Q3. For $A = \{11, 12, 13, 14, 15, 16\}, F_x = \{k \in \mathbb{Z}^+ : k \mid x\}$:

$$
\begin{aligned}
F_{11} = \{1, 11\} &\implies |F_{11}| = 2 \\
F_{12} = \{1, 2, 3, 4, 6, 12\} &\implies |F_{12}| = 6 \\
F_{13} = \{1, 13\} &\implies |F_{13}| = 2 \\
F_{14} = \{1, 2, 7, 14\} &\implies |F_{14}| = 4 \\
F_{15} = \{1, 3, 5, 15\} &\implies |F_{15}| = 4 \\
F_{16} = \{1, 2, 4, 8, 16\} &\implies |F_{16}| = 5
\end{aligned}
$$

Minimal elements are $11, 13$, largest and maximal element is 12 ■

Q4. All linearizations are:

$$
\begin{aligned}
&11 \preccurlyeq^* 13 \preccurlyeq^* 14 \preccurlyeq^* 15 \preccurlyeq^* 16 \preccurlyeq^* 12 && \text{(Given)} \\
&11 \preccurlyeq^* 13 \preccurlyeq^* 15 \preccurlyeq^* 14 \preccurlyeq^* 16 \preccurlyeq^* 12 \\
&13 \preccurlyeq^* 11 \preccurlyeq^* 14 \preccurlyeq^* 15 \preccurlyeq^* 16 \preccurlyeq^* 12 \\
&13 \preccurlyeq^* 11 \preccurlyeq^* 15 \preccurlyeq^* 14 \preccurlyeq^* 16 \preccurlyeq^* 12 && ■
\end{aligned}
$$

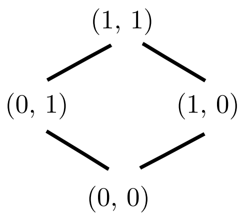
Q5. **Direct Proof**

1. Prove $\subseteq$ is reflexive:
   1.1. Let $S \in \mathcal{P}(A)$
   1.2. $S \subseteq S$ (Definition of subsets)
   1.3. $\therefore \forall S \in \mathcal{P}(A)(S \subseteq S)$ (Universal generalization)
   1.4. $\therefore \subseteq$ is reflexive (Definition of reflexivity)
2. Prove $\subseteq$ is antisymmetric:
   2.1. Let $S, T \in \mathcal{P}(A)$
   2.2. Suppose $S \subseteq T \land T \subseteq S$
   2.3. $S = T$ (Definition of set equality)
   2.4. $\therefore \forall S, T \in \mathcal{P}(A)(S \subseteq T \land T \subseteq S \rightarrow S = T)$ (Universal generalization)
   2.5. $\therefore \subseteq$ is antisymmetric (Definition of antisymmetry)
3. Prove $\subseteq$ is transitive:
   3.1. Let $S, T, U \in \mathcal{P}(A)$
   3.2. Suppose $S \subseteq T \land T \subseteq U$
   3.3. $\forall x(x \in S \rightarrow x \in T \land x \in T \rightarrow x \in U)$ (Definition of subset)
   3.4. $\forall x(x \in S \rightarrow x \in U)$ (Transitivity of implication)
   3.5. $S \subseteq U$ (Definition of subset)
   3.6. $\therefore \forall S, T, U \in \mathcal{P}(A)(S \subseteq T \land T \subseteq U \rightarrow S \subseteq U)$ (Universal generalization)
   3.7. $\therefore \subseteq$ is transitive (Definition of transitivity)
4. $\subseteq$ is reflexive, antisymmetric and transitive (Conjunction)
5. $\therefore \subseteq$ is a partial order $\blacksquare$ (Definition of partial order)

Q6. (a) **Direct Proof**

1. Prove R is reflexive:
   1.1. Let $(a, b) \in B \times B$
   1.2. $a \leq a \wedge b \leq b$
   1.3. $(a, b)R(a, b)$ (Definition of R)
   1.4. $\forall(a, b) \in B \times B((a, b)R(a, b))$ (Universal generalization)
   1.5. $\therefore R$ is reflexive (Definition of reflexive)
2. Prove R is antisymmetric:
   2.1. Let $(a, b), (c, d) \in B \times B$
   2.2. Suppose $(a, b)R(c, d) \wedge (c, d)R(a, b)$
   2.3. $(a \leq c \wedge c \leq a) \wedge (b \leq d \wedge d \leq b)$ (Definition of R)
   2.4. $a = c \wedge b = d$ (Definition of $\leq$)
   2.5. $\forall(a, b), (c, d) \in B \times B((a, b)R(c, d) \wedge (c, d)R(a, b) \rightarrow a = c \wedge b = d)$ (Universal generalization)
   2.6. R is antisymmetric (Definition of antisymmetry)
3. Prove R is transitive:
   3.1. Let $(a, b), (c, d), (e, f) \in B \times B$
   3.2. Suppose $(a, b)R(c, d) \wedge (c, d)R(e, f)$
   3.3. $a \leq c \leq e \wedge b \leq d \leq f$ (Definition of R)
   3.4. $a \leq e \wedge b \leq f$ (T18. Transitivity of $\leq$)
   3.5. $(a, b)R(e, f)$ (Definition of R)
   3.6. $\forall(a, b), (c, d), (e, f) \in B \times B((a, b)R(c, d) \wedge (c, d)R(e, f) \rightarrow (a, b)R(e, f))$(Universal generalization)
   3.7. $\therefore R$ is transitive (Definition of transitivity)
4. $\therefore R$ is reflexive, antisymmetric and transitive (Conjunction)
5. $\therefore R$ is a partial order ■ (Definition of partial order)

(b)



(c) Maximal and largest element is $(1, 1)$. Minimal and smallest element is $(0, 0)$ ■

(d) No. Counterexample: $(0, 1) \not R (1, 0) \wedge (1, 0) \not R (0, 1)$ ■

Q7. S is the reflexive closure of R

(a) **Direct Proof**

1. Prove S is reflexive:
   1.1. Let $x \in A$
   1.2. $x = x$
   1.3. xSx (Definition of S)
   1.4. $\therefore \forall x \in A(xSx)$ (Universal generalization)
   1.5. $\therefore S$ is reflexive ■ (Definition of reflexivity)

(b) **Direct Proof**

1. Prove $R \subseteq S$:
   1.1. Let $(x, y) \in R$
   1.2. xRy (Definition of R)
   1.3. xSy (Definition of S)
   1.4. $(x, y) \in S$
   1.5. $\therefore \forall (x, y) \in A \times A((x, y) \in R \to (x, y) \in S)$ (Universal generalization)
   1.6. $\therefore R \subseteq S$ ■ (Definition of subset)

(c) **Direct Proof**

1. Prove $S \subseteq S'$:
   1.1. Let $(x, y) \in S$
   1.2. $x = y \vee xRy$ (Definition of S)
   1.3. Case 1 ($x = y$): $xS'y$ (Reflexivity of S')
   1.4. Case 2 (xRy): $xS'y$ (Definition of S')
   1.5. In all cases, $xS'y$
   1.6. $(x, y) \in S'$
   1.7. $\therefore \forall (x, y) \in A \times A((x, y) \in S \to (x, y) \in S')$ (Universal generalization)
   1.8. $\therefore S \subseteq S'$ ■ (Definition of subset)

Q8. (a) $xRy \leftrightarrow x < y$ ■

(b) $xRy \leftrightarrow x \leq y$ ■

(c) DNE. ■

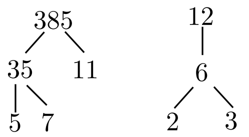
(d) $xRy \leftrightarrow xy \geq 0$ ■

Q9. (a) Comparable: $\{1, 1\}, \{1, 2\}, \{1, 4\}, \{1, 5\}, \{1, 10\}, \{1, 15\}, \{1, 20\}$
$\{2, 2\}, \{2, 4\}, \{2, 10\}, \{2, 20\}$
$\{4, 4\}, \{4, 20\}$
$\{5, 5\}, \{5, 10\}, \{5, 15\}, \{5, 20\}$
$\{10, 10\}, \{10, 20\}$
$\{15, 15\}$
$\{20, 20\}$ ■

(b) Compatible: $\{1, 1\}, \{1, 2\}, \{1, 4\}, \{1, 5\}, \{1, 10\}, \{1, 15\}, \{1, 20\}$
$\{2, 2\}, \{2, 4\}, \{2, 5\}, \{2, 10\}, \{2, 20\}$
$\{4, 4\}, \{4, 5\}, \{4, 10\}, \{4, 20\}$
$\{5, 5\}, \{5, 10\}, \{5, 15\}, \{5, 20\}$
$\{10, 10\}, \{10, 20\}$
$\{15, 15\}$
$\{20, 20\}$ ■

Q10. (a) Maximal chains: $\{\phi, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$ and $\{\phi, \{a\}, \{a, c\}, \{a, b, c\}, \{a, b, c, d\}\}$ ■

(b)



Maximal chains: $\{11, 385\}$ and $\{2, 6, 12\}$ ■

Q11. (a) True.

Direct Proof

1. Prove $\forall a, b \in A(a, b$ are comparable$\rightarrow a, b$ are compatible$)$
2. Suppose a, b are comparable
   - 2.1. $a \preccurlyeq b \vee b \preccurlyeq a$ (Definition of comparable)
   - 2.2. Case 1 ($a \preccurlyeq b$):
     - 2.2.1. $b \preccurlyeq b$ (Reflexivity of partial order)
     - 2.2.2. $\therefore \exists c = b \in A(a \preccurlyeq c \wedge b \preccurlyeq c)$ (Universal generalization)
     - 2.2.3. $\therefore a, b$ are compatible (Definition of compatible)
   - 2.3. Case 2 ($b \preccurlyeq a$):
     - 2.3.1. $a \preccurlyeq a$ (Reflexivity of partial order)
     - 2.3.2. $\therefore \exists c = a \in A(b \preccurlyeq c \wedge a \preccurlyeq c)$ (Universal generalization)
     - 2.3.3. $\therefore a, b$ are compatible (Definition of compatible)
   - 2.4. In all cases, a, b are compatible ■

(b) False. Counterexample from Q9: $\{4, 10\}$ is compatible $(4 \mid 20 \wedge 10 \mid 20)$ but not comparable $(4 \nmid 10 \wedge 10 \nmid 4)$ ■