MA1522 Tutorial 9

AY 24/25 Sem 1 — github/omgeta

Q1. Let x_0, x_1, x_2 be the amount Jack, Jim, and John received respectively

(a) Conditions create the system of equations:

$$\begin{aligned} x_0 + 2x_1 &= 300 \\ x_1 + x_2 &= 300 \\ x_0 - 2x_2 &= 300 \end{aligned}$$

which is inconsistent ■

(b) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}, \vec{b} = \begin{pmatrix} 300 \\ 300 \\ 300 \end{pmatrix}$, solve $A^T A\vec{x} = A^T\vec{b}$:

$$[A^T A \mid A^T\vec{b}] = \begin{pmatrix} 2 & 2 & -2 & | & 600 \\ 2 & 5 & 1 & | & 900 \\ -2 & 1 & 5 & | & -300 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2 & | & 200 \\ 0 & 1 & 1 & | & 100 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

so a least square solution $\vec{x} = \begin{pmatrix} 200 \\ 100 \\ 0 \end{pmatrix}$ ■

Q2. (a) Suppose $Q = [q_1 \cdots q_n]$, where the columns of Q form an orthonormal set in $\mathbb{R}^m$, extend it to an orthonormal basis of $\mathbb{R}^m$ to form the orthogonal matrix $Q' = [q_1 \cdots q_m]$. Then let $R' = \begin{pmatrix} R \\ 0_{(m-n)\times m} \end{pmatrix}$ ■

(b) $Q = \begin{pmatrix} \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} \sqrt{3} & \sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{\sqrt{6}}{3} \\ 0 & 0 & 0 \end{pmatrix}$ ■

(c) Use $[QR] = qr(A, x)$ ■

Q3. (a) $p(X) = 0^{3\times 3}$ ■

(b) $\det(X) = x^3 - 4x^2 - x + 4 = p(x)$ ■

(c) $X^3 - 4X^2 - X + 4I = 0 \implies X(X^2 - 4X - I) = -4I \implies X^{-1} = -\frac{1}{4}(X^2 - 4X - I)$ ■

Q4. (a) $P = \begin{pmatrix} \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, P^{-1}AP = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ ■

(b) $P = \begin{pmatrix} -2 & \frac{5}{6} & 1 & -\frac{4}{5} \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ ■

(c) Algebraic multiplicity of eigenvalue $1 = 3$ but geometric multiplicity $= \dim(\text{Nul}(A - I)) = 1$ so there is no diagonalization ■

(d) $P = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ ■

(e) Characteristic equation does not factor into linear factors so there is no diagonalization ■

Q5. (a) $\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I)$ so their characteristic equations are the same $\blacksquare$

(b) If $A = PDP^{-1}$, then $A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = QDQ^{-1}$ since D is diagonal and we let $Q = P^{-1}$, so there is a diagonalization for A^T $\blacksquare$

(c) $A^k\vec{v} = A^{k-1}A\vec{v} = A^{k-1}\lambda\vec{v} = \cdots = \lambda^k\vec{v}$ so by definition λ^k is an eigenvalue for $\vec{v}$ in A^k $\blacksquare$

(d) $A\vec{v} = \lambda\vec{v} \implies \lambda^{-1}\vec{v} = A^{-1}\vec{v}$ so by definition and from (c), we have proven λ^k is an eigenvalue for $\vec{v}$ in A^k for negative k $\blacksquare$

(e) $A^k\vec{v} = \vec{0} \implies \lambda^k\vec{v} = \vec{0}$, and since $\vec{v} \neq 0$, $\lambda = 0$ $\blacksquare$

(f) If λ is the only eigenvalue, then for diagoanlisation $A = PDP^{-1}$, $D = \lambda I$ so then $A = P\lambda IP^{-1} = \lambda PIP^{-1} = \lambda I$ $\blacksquare$

(g) A is nilpotent $\implies$ 0 is the only eigenvalue, therefore D is the zero matrix and $A = P0P^{-1} = 0$ $\blacksquare$