MA1521 Homework 1

AY 24/25 Sem 1 — github/omgeta

Q1. For each of the following functions, find all real values of x for which it is defined, i.e. the maximal domain of each function:

(a) $f(x) = \dfrac{81 - x^2}{(4 + x^2)(27 - x^3)(16 - x^4)}$

For $f(x)$ to be defined, denominator $(4 + x^2)(27 - x^3)(16 - x^4) \neq 0$.

$$\begin{aligned} 4 + x^2 = 0 &\implies \text{(no real solutions)} \\ 27 - x^3 = 0 &\implies x = 3 \\ 16 - x^4 = 0 &\implies x = \pm 2 \end{aligned}$$

Therefore, domain of $f(x)$ is:

$$\mathbb{R} \setminus \{-2, 2, 3\} \quad \blacksquare$$

(b) $g(x) = \sqrt{2 - \ln(x + 1)}$

For $g(x)$ to be defined, argument $(2 - \ln(x + 1))$ must be non-negative

$$\begin{aligned} 2 - \ln(x + 1) &\geq 0 \\ 2 &\geq \ln(x + 1) \\ e^2 &\geq x + 1 \\ e^2 - 1 &\geq x \end{aligned}$$

For $\ln(x + 1)$ to be defined, $x + 1 > 0 \implies x > -1$.

Therefore, domain of $g(x)$ is:

$$\{x \in \mathbb{R} : -1 < x \leq e^2 - 1\} \quad \blacksquare$$

(c) $h(x) = \dfrac{\ln(\sqrt{16 - 4x} + 1)}{\sqrt{\ln x} - 1}$

For $\ln(\sqrt{16 - 4x} + 1)$ to be defined

$$\begin{aligned} \sqrt{16 - 4x} + 1 &> 0 \\ 16 - 4x &\geq 0 \\ x &\leq 4 \end{aligned}$$

For $\sqrt{\ln x}$ to be defined, $x > 0$

For $h(x)$ to be defined, denominator must be non-zero

$$\begin{aligned} \sqrt{\ln x} - 1 &\neq 0 \\ \ln x &\neq 1 \\ x &\neq e \end{aligned}$$

Therefore, domain of $h(x)$ is:

$$0 < x \leq 4 \text{ and } x \neq e \quad \blacksquare$$

Q2. Let $f(x)$ be defined on $(-\infty, \infty)$ such that $f(x) = \begin{cases} 4 & x \leq -2 \\ x^2 - 1 & -2 < x \leq -1 \\ 0 & -1 < x \leq 1 \\ \dfrac{1}{x-1} & x > 1 \end{cases}$

Find all x such that f is not continuous at x

$$x = -2, 1 \quad \blacksquare$$

Q3. Let $f(x)$ be defined on $[0, 8]$ such that $f(x) = \begin{cases} p^{\frac{1}{3}}\sqrt{x} & 0 \leq x < 4 \\ 7 & x = 4 \\ q(x-2)^2 + 5 & 4 < x \leq 6 \\ \dfrac{2r}{x-5} & 6 < x \leq 8 \end{cases}$

It is given that f is continuous at $x = 4$ and $\lim_{x\to 6} f(x)$ exists. Find the values of p, q, r.

Since f is continuous at $x = 4$,

$$\begin{aligned} p^{\frac{1}{3}}\sqrt{4} &= 7 \\ p^{\frac{1}{3}} &= \frac{7}{2} \\ p &= \frac{343}{8} \quad \blacksquare \end{aligned}$$

$$\begin{aligned} q(4-2)^2 + 5 &= 7 \\ 4q &= 2 \\ q &= \frac{1}{2} \quad \blacksquare \end{aligned}$$

Since $\lim_{x\to 6} f(x)$ exists, when $x = 6$,

$$\begin{aligned} \frac{1}{2}(6-2)^2 + 5 &= \frac{2r}{6-5} \\ 8 + 5 &= 2r \\ r &= \frac{13}{2} \quad \blacksquare \end{aligned}$$

Q4. Evaluate each of the following limits if it exists:

(a) $\lim_{x\to 2} \dfrac{4 - x^2}{x^2 - 3x + 2}$

$$\begin{aligned} \lim_{x\to 2} \frac{4 - x^2}{x^2 - 3x + 2} &= \lim_{x\to 2} \frac{-(x-2)(x+2)}{(x-2)(x-1)} \\ &= \lim_{x\to 2} \frac{-(x+2)}{x-1} \\ &= \frac{-(2+2)}{2-1} \\ &= -4 \quad \blacksquare \end{aligned}$$

(b) $\lim_{x\to -2} \dfrac{4-x^2}{\sqrt{x^2-x-2}-\sqrt{2-x}}$

$$\begin{aligned}
\lim_{x\to -2} \frac{4-x^2}{\sqrt{x^2-x-2}-\sqrt{2-x}} &= \lim_{x\to -2} \frac{(4-x^2)(\sqrt{x^2-x-2}+\sqrt{2-x})}{(\sqrt{x^2-x-2})^2-(\sqrt{2-x})^2} \\
&= \lim_{x\to -2} \frac{(4-x^2)(\sqrt{x^2-x-2}+\sqrt{2-x})}{(x^2-x-2)-(2-x)} \\
&= \lim_{x\to -2} \frac{-(x^2-4)(\sqrt{x^2-x-2}+\sqrt{2-x})}{x^2-4} \\
&= \lim_{x\to -2} -(\sqrt{x^2-x-2}+\sqrt{2-x}) \\
&= -4 \quad \blacksquare
\end{aligned}$$

(c) $\lim_{x\to 2} \dfrac{x^3-8}{(x-2)^2}$

$$\lim_{x\to 2} \frac{x^3-8}{(x-2)^2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)^2} \qquad = \lim_{x\to 2} \frac{x^2+2x+4}{x-2} = \frac{12}{0}$$

Limit is $\pm\infty$ depending on LHS or RHS limit, therefore limit does not exist $\blacksquare$

Q5. Evaluate the following limits:

(a) $\lim_{x\to\infty} \sqrt{\dfrac{9x^{10}+3x-1}{(x^2+3x-5)^3(2x+5)^4}}$

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{\frac{9x^{10}+3x-1}{(x^2+3x-5)^3(2x+5)^4}} &= \lim_{x\to\infty} \sqrt{\frac{9x^{10}+\dots}{16x^{10}+\dots}} \\
&= \sqrt{\frac{9}{16}} \\
&= \frac{3}{4} \quad \blacksquare
\end{aligned}$$

(b) $\lim_{x\to -\infty} \dfrac{1}{x}\sqrt{\dfrac{9x^{10}+3x-1}{(x^2+3x-5)^3(2x+5)^2}}$

$$\begin{aligned}
\lim_{x\to -\infty} \frac{1}{x}\sqrt{\frac{9x^{10}+3x-1}{(x^2+3x-5)^3(2x+5)^2}} &= \lim_{x\to -\infty} \frac{1}{x}\sqrt{\frac{9x^{10}+\dots}{4x^8+\dots}} \\
&= \lim_{x\to -\infty} \sqrt{\frac{9x^{10}+\dots}{4x^{10}+\dots}} \\
&= -\sqrt{\frac{9}{4}} \\
&= -\frac{3}{2} \quad \blacksquare
\end{aligned}$$

(c) $\lim_{x\to -\infty} \dfrac{\sqrt{9x^{10}+3x-1}}{(1+2x)^2(x^2+x-1)}$

$$\begin{aligned}
\lim_{x\to -\infty} \frac{\sqrt{9x^{10}+3x-1}}{(1+2x)^2(x^2+x-1)} &= \lim_{x\to -\infty} \frac{3x^5+\dots}{4x^4+\dots} \\
&= -\infty \quad \blacksquare
\end{aligned}$$