MA1521 Homework 2

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $\lim_{x\to 0} \frac{4x\sin(3x)}{\tan^2(4x)}$

$$\begin{aligned}\lim_{x\to 0} \frac{4x\sin(3x)}{\tan^2(4x)} &= \lim_{x\to 0} \frac{4x\cdot 3x}{(4x)^2} && \text{(Small angle approximation)}\\ &= \lim_{x\to 0}\frac{3x}{4x}\\ &= \lim_{x\to 0}\frac{3}{4}\\ &= \frac{3}{4} \quad \blacksquare\end{aligned}$$

(b) $\lim_{x\to 3}(\frac{\tan(2\ln(x-2))}{3\ln(x-2)})^3$

$$\begin{aligned}\lim_{x\to 3}(\frac{\tan(2\ln(x-2))}{3\ln(x-2)})^3 &= \lim_{x\to 3}(\frac{2\ln(x-2)}{3\ln(x-2)})^3 && \text{(Small angle approximation)}\\ &= \lim_{x\to 3}(\frac{2}{3})^3\\ &= \lim_{x\to 3}\frac{8}{27}\\ &= \frac{8}{27} \quad \blacksquare\end{aligned}$$

(c) $\lim_{x\to 1}\frac{x^2-5x+4}{\tan(x^2-x)}$

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-5x+4}{\tan(x^2-x)} &= \lim_{x\to 1}\frac{x^2-5x+4}{x^2-x} && \text{(Small angle approximation)}\\ &= \lim_{x\to 1}\frac{(x-1)(x-4)}{x(x-1)}\\ &= \lim_{x\to 1}\frac{x-4}{x}\\ &= \frac{1-4}{1}\\ &= -3 \quad \blacksquare\end{aligned}$$

Q2. (a) $y = \frac{ax+b}{cx+d}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}\\ &= \frac{a(cx+d)-c(ax+b)}{(cx+d)^2}\\ &= \frac{acx+ad-acx-bc}{(cx+d)^2}\\ &= \frac{ad-bc}{(cx+d)^2} \quad \blacksquare\end{aligned}$$

(b) $y = \sin^n(x)\cos(mx)$

$$\begin{aligned}\frac{dy}{dx} &= \sin^n(x)\frac{d}{dx}(\cos(mx)) + \cos(mx)\frac{d}{dx}(\sin^n(x))\\ &= \sin^n(x)(-m\sin(mx)) + \cos(mx)(n\sin^{n-1}(x)\cos(x))\\ &= -m\sin^n(x)\sin(mx) + n\sin^{n-1}(x)\cos(x)\cos(mx) \quad \blacksquare\end{aligned}$$

(c) $y = e^{x+x^2+sin(x^3)}$

$$\begin{aligned}\frac{dy}{dx} &= e^{x+x^2+sin(x^3)}\frac{d}{dx}(x + x^2 + sin(x^3)) \\ &= e^{x+x^2+sin(x^3)}(1 + 2x + 3x^2\cos(x^3)) \quad \blacksquare\end{aligned}$$

(d) $y = x^3 - 4(x^2 + e^2 + \ln(x)) + 3(x + \pi)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3) - 4\frac{d}{dx}(x^2 + e^2 + \ln(x)) + 3\frac{d}{dx}(x + \pi) \\ &= 3x^2 - 4(2x + \frac{1}{x}) + 3(1) \\ &= 3x^2 - 8x - \frac{4}{x} + 3 \quad \blacksquare\end{aligned}$$

(e) $y = (\frac{\sin\theta}{\cos\theta - 1})^2$

$$\begin{aligned}\frac{dy}{d\theta} &= 2(\frac{\sin\theta}{\cos\theta - 1})\cdot(\frac{(\cos\theta - 1)\cos\theta - \sin\theta(-\sin\theta)}{(\cos\theta - 1)^2}) \\ &= 2(\frac{\sin\theta}{\cos\theta - 1})\cdot(\frac{\cos^2\theta - \cos\theta + \sin^2\theta}{(\cos\theta - 1)^2}) \\ &= 2(\frac{\sin\theta}{\cos\theta - 1})\cdot(\frac{1 - \cos\theta}{(\cos\theta - 1)^2}) \\ &= 2(\frac{\sin\theta(1 - \cos\theta)}{(\cos\theta - 1)^3}) \\ &= \frac{-2\sin\theta}{(\cos\theta - 1)^2} \quad \blacksquare\end{aligned}$$

(f) $y = t\tan(2\sqrt{t}) + 7$

$$\begin{aligned}\frac{dy}{dt} &= t\frac{d}{dt}(\tan(2\sqrt{t})) + \tan(2\sqrt{t})\frac{d}{dt}(t) + \frac{d}{dt}(7) \\ &= \sqrt{t}\sec^2(2\sqrt{t}) + \tan(2\sqrt{t}) \quad \blacksquare\end{aligned}$$

(g) $r = \sin(\theta + \sqrt{\theta + 1})$

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{d}{d\theta}(\theta + (\theta + 1)^{\frac{1}{2}})\cos(\theta + \sqrt{\theta + 1}) \\ &= (1 + \frac{1}{2}(\theta + 1)^{-\frac{1}{2}})\cos(\theta + \sqrt{\theta + 1}) \\ &= (1 + \frac{1}{2\sqrt{\theta + 1}})\cos(\theta + \sqrt{\theta + 1}) \quad \blacksquare\end{aligned}$$

(h) $s = \frac{4}{\cos x} + \frac{1}{\tan x}$

$$\begin{aligned}\frac{ds}{dx} &= 4\frac{d}{dx}(\cos x)^{-1} + \frac{d}{dx}(\tan x)^{-1} \\ &= 4(-1)(-\sin x)(\cos x)^{-2} - \sec^2 x(\tan x)^{-2} \\ &= 4\frac{\sin x}{(\cos x)^2} - (\frac{1}{(\cos x)^2})(\frac{(\cos x)^2}{(\sin x)^2}) \\ &= 4\tan x\sec x - \csc^2 x \quad \blacksquare\end{aligned}$$

(i) $r = \cos^{-1}(x^2 - 1)$

$$\begin{aligned}\frac{dr}{dx} &= \frac{-\frac{d}{dx}(x^2-1)}{\sqrt{1-(x^2-1)^2}}\\ &= \frac{-2x}{\sqrt{1-(x^2-1)^2}} \quad \blacksquare\end{aligned}$$

(j) $s = \tan^{-1}(e^x + 2\sqrt{x})$

$$\begin{aligned}\frac{ds}{dx} &= \frac{\frac{d}{dx}(e^x+2\sqrt{x})}{1+(e^x+2\sqrt{x})^2}\\ &= \frac{e^x + x^{-\frac{1}{2}}}{1+(e^x+2\sqrt{x})^2} \quad \blacksquare\end{aligned}$$

Q3. (a) Let V, h be the volume and height of the cylindrical coffeepot respectively. Let t be time (in minutes).
Since the radius of the coffeepot is $\frac{15}{2} = 7.5$cm,

$$\begin{aligned}V &= \pi(7.5)^2 \cdot h\\ &= 56.25\pi \cdot h\\ \implies h &= \frac{V}{56.25\pi}\\ \implies \frac{dh}{dV} &= \frac{1}{56.25\pi}\end{aligned}$$

Since the coffee is entering the coffeepot at 10cm^3/min,

$$\frac{dV}{dt} = 10$$

Using chain rule, the speed of the level in the pot rising, or $\frac{dh}{dt}$, is given by:

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt}\\ &= \frac{1}{56.25\pi} \cdot 10\\ &= \frac{8}{45\pi}\text{cm/min} \quad \blacksquare\end{aligned}$$

(b) Let V, r, h be the volume, radius and height of the coffee in the cone respectively.
Comparing ratios,

$$\begin{aligned}\frac{\text{Radius of Coffee}}{\text{Height of Coffee}} &= \frac{\text{Radius of Cone}}{\text{Height of Cone}}\\ \frac{r}{h} &= \frac{7.5}{15}\\ r &= \frac{h}{2}\end{aligned}$$

Substituting $r = \frac{h}{2}$ into the equation for V,

$$\begin{aligned}V &= \frac{1}{3} \cdot \pi(\frac{h}{2})^2 \cdot h\\ &= \frac{h^3\pi}{12}\\ \implies \frac{dV}{dh} &= \frac{h^2\pi}{4}\end{aligned}$$

Using chain rule, the rate of change in the height of the cone when $h = 5$cm is given by:

$$\begin{aligned}\frac{dh}{dt} &= \frac{dV}{dt} \div \frac{dV}{dh}\\ &= -10 \div \frac{5^2\pi}{4}\\ &= -\frac{8}{5\pi}\text{cm/min}\end{aligned}$$

Therefore, level in the cone is falling at $\frac{8}{5\pi}$cm/min when the coffee is at depth 5cm. ■

Q4. (a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$$\begin{aligned}\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -(\frac{x}{y})^{-\frac{1}{3}}\\ &= -(\frac{y}{x})^{\frac{1}{3}}\\ &= -(\frac{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}}{x})^{\frac{1}{3}}\\ &= -\frac{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{1}{2}}}{x^{\frac{1}{3}}}\\ &= -\sqrt{\frac{a^{\frac{2}{3}} - x^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\\ &= -\sqrt{(\frac{a}{x})^{\frac{2}{3}} - 1} \quad \blacksquare\\ \frac{d^2y}{dx^2} &= -\frac{1}{2}[(\frac{a}{x})^{\frac{2}{3}} - 1]^{-\frac{1}{2}} \cdot \frac{d}{dx}((\frac{a}{x})^{\frac{2}{3}} - 1)\\ &= -\frac{1}{2}[(\frac{a}{x})^{\frac{2}{3}} - 1]^{-\frac{1}{2}}(\frac{2}{3})(\frac{a}{x})^{-\frac{1}{3}}(-\frac{a}{x^2})\\ &= \frac{1}{3}[(\frac{a}{x})^{\frac{2}{3}} - 1]^{-\frac{1}{2}}(\frac{a^{\frac{2}{3}}}{x^{\frac{5}{3}}})\\ &= \frac{a^{\frac{2}{3}}}{3x^{\frac{5}{3}}\sqrt{(\frac{a}{x})^{\frac{2}{3}} - 1}}\\ &= \frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}}\sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}} \quad \blacksquare\end{aligned}$$

(b) $y = (\sin x)^{\sin x}$

$$\begin{aligned}\ln y &= \ln(\sin x)^{\sin x} = \sin x \ln(\sin x)\\ \frac{1}{y}\frac{dy}{dx} &= \sin x(\frac{\cos x}{\sin x}) + \ln(\sin x)\cos x\\ &= \cos x(1 + \ln(\sin x))\\ \frac{dy}{dx} &= y\cos x(1 + \ln(\sin x))\\ &= (\sin x)^{\sin x}(1 + \ln(\sin x))\cos x \quad \blacksquare\\ \frac{d^2y}{dx^2} &= \cos x(1 + \ln\sin x)\frac{d}{dx}((\sin x)^{\sin x}) + (\sin x)^{\sin x}[\cos x(\frac{\cos x}{\sin x}) - (1 + \ln\sin x)\sin x]\\ &= (\sin x)^{\sin x}[(1 + \ln\sin x)^2\cos^2 x + \frac{\cos^2 x}{\sin x} - (1 + \ln\sin x)\sin x] \quad \blacksquare\end{aligned}$$

(c) $x = a\cos t, y = a\sin t$

$$\begin{aligned}
\frac{dx}{dt} &= -a\sin t \\
\frac{dy}{dt} &= a\cos t \\
\frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} && \text{(By chain rule)} \\
&= \frac{a\cos t}{-a\sin t} \\
&= -\cot t \quad \blacksquare \\
\frac{d}{dt}(\frac{dy}{dx}) &= \csc^2 t \\
\frac{d^2y}{dx^2} &= \frac{d}{dt}(\frac{dy}{dx}) \div \frac{dx}{dt} \\
&= \frac{\csc^2 t}{-a\sin t} \\
&= -\frac{1}{a\sin^3 t} \quad \blacksquare
\end{aligned}$$