CS1231S Tutorial 4

AY 24/25 Sem 1 — github/omgeta

Q1.

$$
\begin{aligned}
R = \{&(2,2),(2,4),(2,6),(2,8),(2,10),(2,12),(2,14),\\
&(3,6),(3,12),\\
&(5,10),\\
&(7,14),\\
&\} \quad \blacksquare\\
R^{-1} = \{&(14,2),(12,2),(10,2),(8,2),(6,2),(4,2),(2,2),\\
&(12,3),(6,3),\\
&(10,5),\\
&(14,7)\} \quad \blacksquare
\end{aligned}
$$

Q2.

1. Prove R is symmetric $\rightarrow \forall x, y \in A(xRy \leftrightarrow yRx)$:
   1.1. Suppose R is symmetric, i.e. $\forall x, y \in A(xRy \rightarrow yRx)$
   1.2. Then, $\forall y, x \in A(yRx \rightarrow xRy)$ (By supposition 1.1)
   1.3. $\therefore \forall x, y \in A(xRy \leftrightarrow yRx)$ (Definition of iff)
2. Prove $\forall x, y \in A(xRy \leftrightarrow yRx) \rightarrow R = R^{-1}$:
   2.1. Suppose $\forall x, y \in A(xRy \leftrightarrow yRx)$
   2.2. Suppose $(x, y) \in R$:
      2.2.1. $\leftrightarrow xRy$ (Definition of R)
      2.2.2. $\leftrightarrow yRx$ (By supposition 2.1)
      2.2.3. $\leftrightarrow xR^{-1}y$ (Definition of R^{-1})
      2.2.4. $\leftrightarrow (x, y) \in R^{-1}$
   2.3. $R = R^{-1}$
3. Prove $R = R^{-1} \rightarrow R$ is symmetric:
   3.1. Suppose $R = R^{-1}$ and $(x, y) \in R$
   3.2. $(y, x) \in R^{-1}$ (Definition of R^{-1})
   3.3. $(y, x) \in R$ (By supposition 3.1)
   3.4. $\therefore \forall x, y \in A((x, y) \in R \rightarrow (y, x) \in R)$ (Universal generalization)
   3.5. $\therefore \forall x, y \in A(xRy \rightarrow yRx)$ (Definition of relation)
4. Hence, R is symmetric $\leftrightarrow \forall x, y \in A(xRy \leftrightarrow yRx) \leftrightarrow R = R^{-1}$ $\blacksquare$ (Definition of iff)

Q3. (a) Q is reflexive. ■
Q is not symmetric. Counterexample: $(1, 2) \in Q \wedge (2, 1) \notin Q$ ■
Q is transitive. ■
$\therefore Q$ is not an equivalence relation. ■

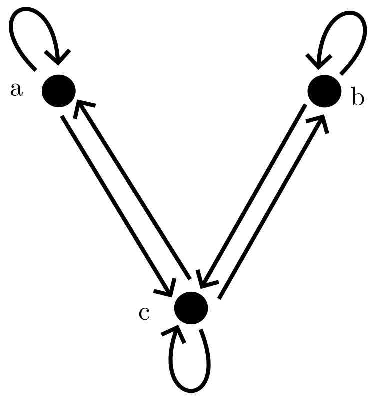
(b) E is reflexive. ■
E is symmetric. ■
E is transitive. ■
$\therefore E$ is an equivalence relation. ■

(c) R is reflexive. ■
R is symmetric. ■
R is not transitive. Counterexample: $(1, 0) \in R \wedge (0, -1) \in R \wedge (1, -1) \notin R$ ■
$\therefore R$ is not an equivalence relation. ■

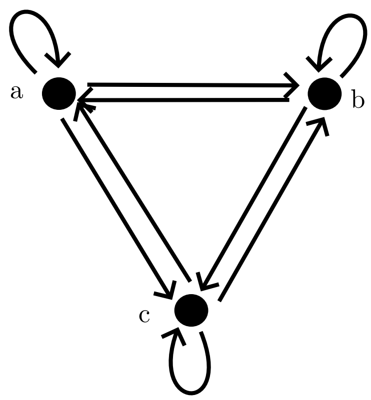
(d) S is not reflexive. Counterexample: $(0, 0) \notin S$ ■
S is symmetric. ■
S is transitive. ■
$\therefore R$ is not an equivalence relation. ■

(e) T is reflexive. ■
T is symmetric. ■
T is not transitive. Counterexample: $(2, 1) \in T \wedge (1, -1) \in T \wedge (2, -1) \notin T$ ■
$\therefore T$ is not an equivalence relation. ■

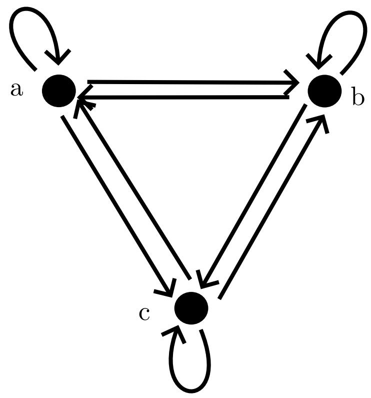
Q4. (a) $R \circ R$ is not transitive. Counterexample: $(a, c) \in R \circ R \land (c, b) \in R \circ R \land (a, c) \notin R \circ R$ ■



(b) $R \circ R \circ R$ is transitive. ■



(c) $(R \circ R) \cup R$ is transitive. ■



Q5. (a) True. $\blacksquare$

(b) True. $\blacksquare$

1. Suppose $(x, y) \in R$:
   1.1. $(y, y) \in R$ (Reflexivity of R)
   1.2. $(x, y) \in R \circ R$ (Definition of composition)
2. $\forall (x, y) \in R((x, y) \in R \circ R)$ (Universal generalization)
3. $R \subseteq R \circ R$

(c) True. $\blacksquare$

1. Suppose $(x, y) \in R \circ R$:
   1.1. $\exists z(xRz \wedge zRy)$ (Definition of $R \circ R$)
   1.2. $(x, z) \in R \wedge (z, y) \in R$ (Definition of R)
   1.3. $(x, y) \in R$ (Transitivity of R)
2. $\forall (x, y) \in R \circ R((x, y) \in R)$ (Universal generalization)
3. $R \circ R \subseteq R$

(d) True. $\blacksquare$

Q6. From Q5, $R \subseteq R \circ R \wedge R \circ R \subseteq R$, therefore by definition of set equality, $R = R \circ R$. This means:

$$
\begin{aligned}
&R \circ R \circ R \circ R \circ R \circ R \circ R \\
&= (R) \circ (R) \circ (R) \circ R \\
&= (R) \circ (R) \\
&= R \quad \blacksquare
\end{aligned}
$$

Q7. $T \circ (S \circ R)$

$= \{(a, d) \in A \times D : \exists c \in C((a, c) \in S \circ R \wedge (c, d) \in T)\}$ (Definition of composition)

$= \{(a, d) \in A \times D : \exists c \in C((\exists b \in B((a, b) \in R \wedge (b, c) \in S)) \wedge (c, d) \in T)\}$ (Definition of $S \circ R$)

$= \{(a, d) \in A \times D : \exists b \in B \exists c \in C((a, b) \in R \wedge (b, c) \in S \wedge (c, d) \in T)\}$ (Distributive law)

$= \{(a, d) \in A \times D : \exists b \in B((a, b) \in R \wedge (\exists c \in C((b, c) \in S \wedge (c, d) \in T)))\}$ (Distributive law)

$= \{(a, d) \in A \times D : \exists b \in B((a, b) \in R \wedge (b, d) \in T \circ S)\}$ (Definition of $T \circ S$)

$= (T \circ S) \circ R \quad \blacksquare$ (Definition of composition)

Q8. $[(1, 1)] = \{(1, 1)\} \quad \blacksquare$

$[(4, 3)] = \{(4, 3), (3, 4), (6, 2), (2, 6), (12, 1), (1, 12)\} \quad \blacksquare$

Q9. (a) $S^{-1} = \{(n, m) \in \mathbb{Z}^2 : (m, n) \in S\}$ (Definition of inverse relation)

$= \{(n, m) \in \mathbb{Z}^2 : m^3 + n^3 \text{ is even}\}$ (Definition of S)

$= \{(m, n) \in \mathbb{Z}^2 : m^3 + n^3 \text{ is even}\}$ (F1. Commutativity of addition)

$= S \quad \blacksquare$

(b)
1. Prove $S \circ S \subseteq S$
   1.1. Suppose $(x, z) \in S \circ S$:
   1.2. $\exists y(x^3 + y^3 \text{ is even } \wedge y^3 + z^3 \text{is even})$ (Definition of composition)
   1.3. $x^3 + 2y^3 + z^3$ is even
   1.4. Since $2y^3$ is even, $x^3 + z^3$ is even
   1.5. $(x, z) \in S$ (Definition of S)
2. Prove $S \subseteq S \circ S$
   2.1. Suppose $(x, z) \in S$:
   2.2. $(x, x) \in S$ ($x^3 + x^3$ is even)
   2.3. $(x, z) \in S \circ S$ (Definition of composition)
3. $\therefore S \circ S = S$

(c) $S \circ S^{-1} = S \circ S$ (By 9a)

$= S \quad \blacksquare$ (By 9b)

Q10. (a)

1. Prove $\sim$ is reflexive:
   1.1. Suppose $a \in \mathbb{Z} \setminus \{0\}$
   1.2. $a \cdot a = a^2 > 0$ (T21. $a \neq 0 \to a^2 > 0$)
   1.3. $\therefore \forall a \in \mathbb{Z} \setminus \{0\}, a \sim a$ (Universal generalization)
   1.4. $\therefore \sim$ is reflexive (Definition of reflexivity)
2. Prove $\sim$ is symmetric:
   2.1. Suppose $a \sim b$, then $ab > 0$ (Definition of $\sim$)
   2.2. $ba = ab > 0$ (F1. $\forall a, b \in \mathbb{R}, ab = ba$)
   2.3. $b \sim a$ (Definition of $\sim$)
   2.4. $\therefore \forall a, b \in \mathbb{Z}(a \sim b \to b \sim a)$ (Universal generalization)
   2.5. $\therefore \sim$ is symmetric (Definition of symmetry)
3. Prove $\sim$ is transitive:
   3.1. Suppose $a \sim b \wedge b \sim c$, then $ab > 0 \wedge bc > 0$ (Definition of $\sim$)
   3.2. $(ab)(bc) = ab^2c > 0$ (T25. $ab > 0 \leftrightarrow (a > 0 \wedge b > 0) \vee (a < 0 \wedge b < 0)$)
   3.3. $b^2 > 0$ (T21. $a \neq 0 \to a^2 > 0$)
   3.4. $\therefore ac > 0$ (T25. $ab > 0 \leftrightarrow (a > 0 \wedge b > 0) \vee (a < 0 \wedge b < 0)$)
   3.5. $a \sim c$ (Definition of $\sim$)
   3.6. $\forall a, b, c \in \mathbb{Z}(((a \sim b) \wedge (b \sim c)) \to a \sim c)$ (Universal generalization)
   3.7. $\therefore \sim$ is transitive (Definition of transitivity)
4. Hence, $\sim$ is reflexive, symmetric and transitive.
5. $\therefore \sim$ is an equivalence relation. $\blacksquare$

(b) $(\mathbb{Z} \setminus \{0\})/\sim$
$= \{\mathbb{Z}^+, \mathbb{Z}^-\}$ $\blacksquare$