CS1231S Tutorial 9

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $52^5 - 48^5 = 125,400,064$ $\blacksquare$

(b) $52^5 - 44^5 = 215,287,808$ $\blacksquare$

Q2. $5^5 - (5 \times 5 \times 3 \times 1 \times 1) = 3050$ $\blacksquare$

Q3.
1. Let $|P_n|$ denote number of permutations with integer n in the correct position
2. $|P_1| = |P_2| = |P_3| = (n-1)!$
3. $|P_1 \cap P_2| = |P_2 \cap P_3| = |P_1 \cap P_3| = (n-2)!$
4. $|P_1 \cap P_2 \cap P_3| = (n-3)!$
5. $|P_1 \cup P_2 \cup P_3| = 3(n-1)! - 3(n-2)! + (n+3)! = (n-3)!(3n^2 - 12n + 13)$ $\blacksquare$

Q4. $\Sigma_{i=1}^{n}(n - i + 1) = \Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\blacksquare$

Q5. (a) $7! \cdot \binom{8}{4} \cdot 4! = 8,467,200$ $\blacksquare$

(b) $5! - (2 \cdot 4!) = 72$ $\blacksquare$

(c) $(n-1)!$ $\blacksquare$

Q6. $(2 \times 3! \times 2! \times 2!) + (2 \times 3! \times 2!) = 72$ $\blacksquare$

Q7. (a) $\binom{7}{3}\binom{6}{2} + \binom{7}{4}\binom{6}{1} + \binom{7}{5} = 756$ $\blacksquare$

(b) $\dfrac{756}{\binom{13}{5}} = \dfrac{756}{1287} = 0.5874$ $\blacksquare$

Q8. (a) Constraint is: $x_1 + x_2 + x_3 + x_4 = 50$; so possible distributions are $\binom{53}{50} = 23426$ $\blacksquare$

(b) Constraint is: $x_1 + x_2 + x_3 + x_4 + x_5 = 50$; so possible distributions are $\binom{54}{50} = 316251$ $\blacksquare$

Q9.
1. Consider 25 subsquares of length $\frac{1}{5}$
2. Since $2 < \frac{51}{25}$, there exists a subsquare with 3 points (Generalised PGP)
3. Such a subsquare has diagonal $\sqrt{\frac{1}{5}^2 + \frac{1}{5}^2} = \frac{\sqrt{2}}{5} < \frac{2}{7}$ which is the diameter of a circle with radius $\frac{1}{7}$
4. Therefore, a subsquare with 3 points can be covered by a circle with radius $\frac{1}{7}$ $\blacksquare$

Q10.
1. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ be the set of 5 distinct non-negative integers, with relation R s.t. $xRy \leftrightarrow x \equiv y \bmod 4$
2. A/R has at most 4 equivalence classes $[0], [1], [2], [3]$ depending on the choice of A
3. In all cases, $|A| = 5 > |A/R| \implies \exists a_i, a_j \in A, (a_i R a_j)$ (Generalised PGP)
4. $a_i \equiv a_j \bmod 4$ (Definition of R)
5. $4 \mid (a_i - a_j)$ $\blacksquare$ (Definition of congruence)

Q11.
1. Let a_i denote games played for day i and $S_k = \Sigma_1^k a_k$
2. $1 \leq S_1 < S_2 < \cdots < S_{77} \leq 132$
3. $22 \leq S_1 + 21 < S_2 + 21 < \cdots < S_{77} + 21 \leq 153$
4. Note there are 154 elements in the sequence $S_1, \cdots, S_{77}, S_1 + 21, \cdots, S_{77} + 21$ but only 153 distinct integers in $[1, 153]$, two must be same (Generalised PGP)
5. Since $S_1, S_2, \cdots, S_{77}$ are distinct and $S_1 + 21, S_2 + 21, \cdots, S_{77} + 21$ are distinct $\exists S_i, S_j + 21$ s.t. $S_i = S_j + 21$
6. $\exists S_i - S_j = 21$ $\blacksquare$ (Basic algebra)