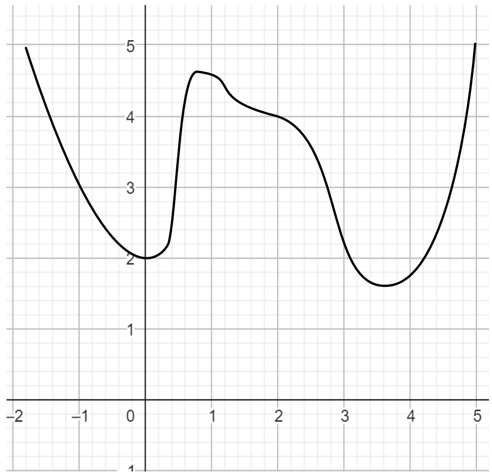
MA1521 Homework 3

AY 24/25 Sem 1 — github/omgeta

Q1.



Q2. $f'(x) = \sec^2 x + \sec x + \tan x$
$f(x)$ is not defined for $x = \frac{\pi}{2}, \frac{3\pi}{2}$ and $f'(x) > 0$ for all x.
$\therefore f$ is defined and $f'(x) > 0$ in the interval $(0, 2\pi)$ for $x \in (0, 2\pi) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ $\blacksquare$

Q3. By observing the graph,
Local maximum: $x = 1, 6$ $\blacksquare$
Local minimum: $x = -2, 2$ $\blacksquare$
Absolute maximum: $x = 6$ $\blacksquare$
Absolute minimum: $x = 2$ $\blacksquare$

Q4. (i) Suppose $y = \dfrac{x+1}{x^2+1}$, for $x \in [-3, 3]$.
First, find the first derivative y',

$$\begin{aligned} y' &= \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2} \\ &= \frac{-x^2 - 2x + 1}{(x^2+1)^2} \end{aligned}$$

Critical points are found at the points where $f'(x) = 0$

$$\begin{aligned} -x^2 - 2x + 1 &= 0 \\ x^2 + 2x - 1 &= 0 \\ x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{8}}{2} \\ &= -1 \pm \sqrt{2} \end{aligned}$$

When $x = -1 + \sqrt{2}, y = \frac{\sqrt{2}}{4 - 2\sqrt{2}} = \frac{\sqrt{2}+1}{2}$
When $x = -1 - \sqrt{2}, y = \frac{-\sqrt{2}}{4 + 2\sqrt{2}} = \frac{-\sqrt{2}+1}{2}$
$\therefore$ the critical points are $(-1 + \sqrt{2}, \dfrac{\sqrt{2}+1}{2}), (-1 - \sqrt{2}, \dfrac{-\sqrt{2}+1}{2})$ $\blacksquare$

(ii) When $x = (-1 + \sqrt{2})_-$, $y' > 0$ and when $x = (-1 + \sqrt{2})_+$, $y' < 0$.
When $x = (-1 - \sqrt{2})_-$, $y' < 0$ and when $x = (-1 - \sqrt{2})_+$, $y' > 0$.
$\therefore f$ is decreasing in $[-3, -1 - \sqrt{2}) \cup (-1 + \sqrt{2}, 3]$ and increasing in $(-1 - \sqrt{2}, -1 + \sqrt{2})$ $\blacksquare$

(iii) By the first derivative test, $y = \dfrac{\sqrt{2}+1}{2}$ is a local and absolute minimum, $y = \dfrac{-\sqrt{2}+1}{2}$ is a local and the absolute maximum $\blacksquare$.

Q5. Let C_g, C_s be the cost of installing the fiber-optic cable underground and undersea respectively in \$a. Suppose the cost per km of underground cable is \$$a$. The total cost C is given by:

$$\begin{aligned} C_g &= (13-x)\cdot 1 = (13-x) \\ C_s &= \sqrt{5^2+x^2}\cdot 1.4 = 1.4\sqrt{25+x^2} \\ C &= (13-x) + 1.4\sqrt{25+x^2} \end{aligned}$$

First find the rate of change of C w.r.t x:

$$\begin{aligned} \frac{dC}{dx} &= -1 + 1.4(\frac{1}{2})(25+x^2)^{-\frac{1}{2}}(2x) \\ &= \frac{1.4x}{\sqrt{25+x^2}} - 1 \end{aligned}$$

At critical points, $\frac{dC}{dx} = 0$:

$$\begin{aligned} \frac{1.4x}{\sqrt{25+x^2}} - 1 &= 0 \\ \frac{x}{\sqrt{25+x^2}} &= \frac{1}{1.4} \\ \frac{x}{\sqrt{25+x^2}} &= \frac{5}{7} \\ \frac{x^2}{25+x^2} &= \frac{25}{49} \\ 49x^2 &= 625 + 25x^2 \\ 24x^2 &= 625 \\ x^2 &= \frac{625}{24} \\ x &= \pm\frac{25}{\sqrt{24}} \\ x &= \frac{25}{\sqrt{24}} \qquad \text{(Distance } x \geq 0) \end{aligned}$$

By first derivative test, $\frac{dC}{dx}|_{x=\frac{25}{\sqrt{24}}-} < 0$ and $\frac{dC}{dx}|_{x=\frac{25}{\sqrt{24}}+} > 0$, implies that C is a local minimum when $x = \frac{25}{\sqrt{24}}$.
Since C at $x = \frac{25}{\sqrt{24}}$ is the only minima for $x \in [0, 13]$, it is also the absolute minimum cost.
$\therefore$ the distance between B and C if the total cost of installing the cable is to be minimized is $\frac{25}{\sqrt{24}} \approx 5.1$km. $\blacksquare$

Q6. (a) $\displaystyle\lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x}$

$$\begin{aligned} \lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x} &= \lim_{x\to\pi/2} \frac{-\cos x}{-2\sin 2x} && \text{(By L'Hopital's Rule)} \\ &= \lim_{x\to\pi/2} \frac{-\sin x}{4\cos 2x} && \text{(By L'Hopital's Rule)} \\ &= \frac{-1}{-4} \\ &= \frac{1}{4} \quad \blacksquare \end{aligned}$$

(b) $\lim_{x\to 0} \frac{\ln(\cos ax)}{\ln(\cos bx)}$

$$\begin{aligned}
\lim_{x\to 0} \frac{\ln(\cos ax)}{\ln(\cos bx)} &= \lim_{x\to 0} \frac{\frac{-a\sin ax}{\cos ax}}{\frac{-b\sin bx}{\cos bx}} && \text{(By L'Hopital's Rule)} \\
&= \lim_{x\to 0} \frac{a\tan ax}{b\tan bx} \\
&= \lim_{x\to 0} \frac{a^2 x}{b^2 x} && \text{(Small angle approximation)} \\
&= \frac{a^2}{b^2} \quad \blacksquare
\end{aligned}$$

(c) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$$\begin{aligned}
\ln x^{\frac{1}{1-x}} &= \frac{\ln x}{1-x} \\
\lim_{x\to 1} \ln x^{\frac{1}{1-x}} &= \lim_{x\to 1} \frac{\ln x}{1-x} \\
&= \lim_{x\to 1} \frac{\frac{1}{x}}{-1} && \text{(By L'Hopital's Rule)} \\
&= -1 \\
\therefore \lim_{x\to 1} x^{\frac{1}{1-x}} &= e^{-1} \quad \blacksquare
\end{aligned}$$

(d) $\lim_{x\to 0^+} x^{\sin x}$

$$\begin{aligned}
\ln x^{\sin x} &= \sin x \ln x \\
\lim_{x\to 0^+} \ln x^{\sin x} &= \lim_{x\to 0^+} \sin x \ln x \\
&= \lim_{x\to 0^+} x \ln x && \text{(By L'Hopital's Rule)} \\
&= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \\
&= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} && \text{(By L'Hopital's Rule)} \\
&= \lim_{x\to 0^+} -x \\
&= 0 \\
\therefore \lim_{x\to 0^+} x^{\sin x} &= e^0 \\
&= 1 \quad \blacksquare
\end{aligned}$$