MA1522 Tutorial 6

AY 24/25 Sem 1 — github/omgeta

Q1. (a) Reduce the corresponding matrix P_S:

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\text{rank}(P_S) = 3 \implies S$ is a basis for $\mathbb{R}^3$ $\blacksquare$

(b) Use the transition matrix P_S^{-1} to change $\vec{w}$ from the standard basis to S:

$$\begin{aligned} [\vec{w}]_S &= P_S^{-1}\vec{w} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ -1 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1/7 \\ 5/7 \end{pmatrix} \quad \blacksquare \end{aligned}$$

(c) Reduce the matrix $[P_S|P_T]$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 2 & 2 & -1 & 5 & 3 & 2 \\ -1 & 1 & 3 & 4 & 7 & 4 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{array}\right)$$

Therefore, $P_{S\leftarrow T} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 3 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ $\blacksquare$

(d) Find the inverse of $P_{S\leftarrow T}$:

$$P_{T\leftarrow S} = P_{S\leftarrow T}^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \end{pmatrix} \quad \blacksquare$$

(e) Use the required transition matrix to find $[\vec{w}]_T$:

$$\begin{aligned} [\vec{w}]_T &= P_{T\leftarrow S}[\vec{w}]_S \\ &= \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} & \frac{5}{8} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{7} \\ \frac{5}{7} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{7} \\ -\frac{1}{7} \\ \frac{5}{14} \end{pmatrix} \quad \blacksquare \end{aligned}$$

Q2. (a) $v_1, v_2, v_3 \in V \implies \text{Span}(T) \subseteq V$ and $|T| = 3 = \dim(V)$. Consider also:

$$\begin{aligned} c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} &= 0 \\ c_1(\vec{u_1} + \vec{u_2} + \vec{u_3}) + c_2(\vec{u_2} + \vec{u_3}) + c_3(\vec{u_2} - \vec{u_3}) &= 0 \\ c_1\vec{u_1} + (c_1 + c_2 + c_3)\vec{u_2} + (c_1 + c_2 - c_3)\vec{u} &= 0 \end{aligned}$$

which has only the trivial solution $c_1 = c_2 = c_3 = 0 \implies T$ is linearly independent. Therefore, T is a basis for V ■

(b) Transition matrix from S to T is $P_{S\leftarrow T}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ ■

Q3. (a) Check if $\vec{b} \in \text{Col}(A)$ by reducing the matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

This is an inconsistent equation, therefore $\vec{b} \notin \text{Col}(A)$ ■

(b) Check if $\vec{b}^T \in \text{Col}(A^T)$ by reducing the matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 9 & 3 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right) \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Therefore, $\vec{b} \in \text{Row}(A)$ and $\vec{b} = 1(1, 9, 1) - 3(-1, 3, 1) + 1(1, 1, 1)$ ■

(c) Reduce matrix A:

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\text{Col}(A) = \mathbb{R}^4$. By invertible matrix theorem, $\text{Row}(A) = \mathbb{R}^4$ ■

Q4. (a) Reduce A:

$$\begin{pmatrix} 1 & 2 & 5 & 3 \\ 1 & -4 & -1 & -9 \\ -1 & 0 & -3 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(i) Basis for $\text{Row}(A) = \{(1,0,3,-1),(0,1,1,2)\}$ ■

(ii) Basis for $\text{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ ■

(iii) Basis for $\text{Nul}(A) = \left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ ■

(iv) $\text{rank}(A) + \text{nullity}(A) = 2 + 2 = 4 =$ columns of A, so rank-nullity is verified ■

(v) $\text{rank}(A) = 2 < \min\{4,5\}$, therefore A is not full rank ■

(b) Reduce A:

$$\begin{pmatrix} 1 & 3 & 7 \\ 2 & 1 & 8 \\ 3 & -5 & -1 \\ 2 & -2 & 2 \\ 1 & 1 & 5 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(i) Basis for $\text{Row}(A) = \{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\}$ ■

(ii) Basis for $\text{Col}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ -1 \\ 2 \\ 5 \end{pmatrix} \right\}$ ■

(iii) Basis for $\text{Nul}(A) = \phi$ ■

(iv) $\text{rank}(A) + \text{nullity}(A) = 3 + 0 = 3 =$ columns of A, so rank-nullity is verified ■

(v) $\text{rank}(A) = 3 = \min\{3,5\}$, therefore A is full rank ■

Q5. Reduce the matrix formed by the columns of W^T:

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 1 & 15 & 8 & 6 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 6 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Basis for $W = \left\{ \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ ■

(b) $\dim(W) = 3$ ■

(c) Basis for $\mathbb{R}^5 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ ■

Q6. Reduce the matrix formed by the vectors in S to find linear independence:

$$\begin{pmatrix} 1 & 2 & -1 & 0 & 3 \\ 0 & -1 & 3 & 1 & -1 \\ 1 & 0 & 5 & 2 & 1 \\ 3 & 1 & 12 & 5 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 5 & 2 & 1 \\ 0 & 1 & -3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore, $S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ ■