CS1231S Tutorial 8

AY 24/25 Sem 1 — github/omgeta

Q1. $g(n) = (-1)^n \left\lceil \frac{n}{2} \right\rceil$ ■

Q2. (a) **Direct Proof**

1. $b_1, b_2, \cdots$ is a sequence in which every element of B appears (Lemma 9.2)
2. Let $|C| = n$, then $C = \{c_1, c_2, \cdots, c_n\}$ (Definition of finite sets)
3. Then $c_1, c_2, \cdots, c_n, b_1, b_2, \cdots$ is a sequence in which every element of $B \cup C$ appears
4. $\therefore B \cup C$ is countable ■ (Lemma 9.2)

(b) **Direct Proof**

1. Since B is countably infinite set, $\exists$ bijection $f : \mathbb{Z}^+ \to B$
2. Let $C' = C \setminus B = \{c_1, c_2, \cdots, c_k\}$
3. Define $g : \mathbb{Z}^+ \to B \cup C$:

$$g(i) = \begin{cases} c_i & i <= k \\ f(i-k) & \text{otherwise} \end{cases}$$

4. Prove g is injective:
   4.1. Suppose $x_1, x_2 \in \mathbb{Z}^+$ s.t. $g(x_1) = g(x_2)$
   4.2. Case 1 ($x_1, x_2 \leq k$): $c_{x_1} = c_{x_2} \implies x_1 = x_2$ (Distinct values of C)
   4.3. Case 2 ($x_1, x_2 > k$): $f(x_1 - k) = f(x_2 - k) \implies x_1 = x_2$ (Injectivity of f)
   4.4. Case 3 (either x_1 or $x_2 \leq k$): WLOG, $x_1 \leq k \implies c_{x_1} = f(x_1 - k)$ but this is a contradiction with $B \cap C' = \phi$
   4.5. In both cases, $x_1 = x_2$
5. Prove g is surjective:
   5.1. Suppose $y \in B \cup C$
   5.2. Case 1 ($y \in B$): $\exists i, (g(i) = y)$ (Surjectivity of f)
   5.3. Case 2 ($y \notin B$): $\exists c_i = y \implies \exists i, g(i) = y$
   5.4. In both cases, $\exists i, (g(i) = y)$
6. $\therefore g : \mathbb{Z}^+ \to B \cup C$ is bijective (Definition of bijection)
7. $\therefore B \cup C$ is countable ■ (Definition of countably infinite)

Q3. (a) We cannot assume $A_{k+1} = \phi$, and must instead solve for the general case where A_{k+1} is any finite set ■

(b) Suppose $A_k = \{k\}$, then $\bigcup_{k=1}^{\infty} A_k = \mathbb{Z}^+$ which is infinite, disproving the statement ■

Q4. (a) **Proof by 1MI**

1. Let $P(n) \equiv \bigcup_{i=1}^{n} A_i$ is countable for $n \in \mathbb{Z}^+$
2. Basis step: $\bigcup_{i=1}^{1} A_1 = A_1$ which is given to be countable, therefore $P(1)$ is true
3. Assume $P(k)$ for some $k \in \mathbb{Z}^+$
4. Inductive step:
   4.1. $\bigcup_{i=1}^{k+1} = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1}$
   4.2. By induction hypothesis $\bigcup_{i=1}^{k} A_i$ is countable, and A_{k+1} is given to be countable, therefore their union is countable (Lemma 9.4)
   4.3. $P(k+1)$ is true
5. Therefore, $\bigcup_{i=1}^{n} A_i$ is countable for any $n \in \mathbb{Z}^+$ ■

(b) No, by Qn 3(b) ■

Q5. **Direct Proof**

1. Given $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable, we have bijection $f : \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$
2. There exists sequence $s_{i1}, s_{i2}, \cdots \in S_i$ in which every element of S_i appears (Lemma 9.2)
3. Hence, $\forall s_{ij} \in \bigcup_{i\in\mathbb{Z}^+}$, we have $(i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+$
4. Define sequence $c_1, c_2, \cdots$, s.t. $c_k = b_{ij}$ whenever $f(k) = (i, j)$
5. It suffices to show any element of $\bigcup_{i\in\mathbb{Z}^+} S_i$ appears in the sequence defined in line 4:
   - 5.1. Let $x \in \bigcup_{i\in\mathbb{Z}^+} S_i$
   - 5.2. $\exists i \in \mathbb{Z}^+(x \in S_i)$ (Definition of union)
   - 5.3. $\exists j \in \mathbb{Z}^+(x = b_{ij})$ (Line 3)
   - 5.4. $\exists k \in \mathbb{Z}^+(x = c_k)$ (Definition of sequence)
6. Therefore, $\bigcup_{i\in\mathbb{Z}^+}$ is countable ■ (Lemma 9.2)

Q6. **Direct Proof**

1. Take $B' \subseteq B$ s.t. B' is countably infinite (Proposition 9.3)
2. Since B is countably infinite set, $\exists$ bijection $f : \mathbb{Z}^+ \to B$
3. Let $C' = C \setminus B = \{c_1, c_2, \cdots, c_k\}$
4. $B' \cup C'$ is countable (Qn 2)
5. $\exists$ bijection $f : B' \cup C' \to B'$ (Definition of cardinality)
6. Define $g : B \cup C \to B$:

$$g(x) = \begin{cases} f(x) & x \in B' \cup C' \\ x & \text{otherwise} \end{cases} \quad ■$$

Q7. **Proof by Contradiction**

1. Suppose not, that is, $\mathcal{P}(A)$ is countable
   - 1.1. $\forall a \in A, \{a\} \in \mathcal{P}(A) \implies \mathcal{P}(A)$ is infinite
   - 1.2. $\exists$ sequence $a_1, a_2, \cdots \in A$ in which every element of A appears (Lemma 9.2)
   - 1.3. $\exists$ sequence $S_1, S_2, \cdots \in \mathcal{P}(A)$ in which every element of $\mathcal{P}(A)$ appears (Lemma 9.2)
   - 1.4. Construct $X = \{a_i : a_i \notin S_i\}$
   - 1.5. Note that $\forall S_i \in \mathcal{P}(A), X \neq S_i$
   - 1.6. But $X \in \mathcal{P}(A)$ which contradicts 1.3.
2. Hence, the supposition is false.
3. $\therefore \mathcal{P}(A)$ is uncountable ■ (Contradiction rule)

Q8. (a) **Direct Proof**

1. Suppose R is a reflexive relation on A for $|A| = n \in \mathbb{N}$
2. $A = \{a_1, a_2, \cdots, a_n\}$ (Definition of finite set)
3. $\forall a \in A, (a, a) \in R$ (Definition of reflexive relation)
4. Define $f : A \to R$. where $f(a) = (a, a), \forall a \in A$
5. $\forall x, y \in A, f(x) = f(y) \implies (x, x) = (y, y) \implies x = y$. Therefore, f is injective.
6. $|A| \leq |R|$ ■ (By pigeonhole principle)

(b) **Counterexample:** $A = \{a\}, R = \phi \implies |A| = 1 > 0 = |R|$ ■

(c) **Counterexample:** $A = \{a\}, R = \phi \implies |A| = 1 > 0 = |R|$ ■

Q9. **Proof by 2MI**

1. Let $P(n) \equiv Even(F_n) \leftrightarrow Even(F_{n+3}), \forall n \in \mathbb{N}$
2. Basis step: $F_0 = 0$ and $F_3 = 2$ are both even, therefore $P(0)$ is true
3. Assume $P(i)$ is true for $0 \leq i \leq k$:

$$Even(F_k) \leftrightarrow Even(F_{k+3})$$

4. Inductive step:
   - 4.1. $Even(F_{k+1}) \leftrightarrow Even(F_k + F_{k-1})$ (Definition of F)
   - 4.2. $\leftrightarrow (Even(F_k) \leftrightarrow Even(F_{k-1}))$ (Fact 1)
   - 4.3. $\leftrightarrow (Even(F_{k+3}) \leftrightarrow Even(F_{k+2}))$ (By inductive hypothesis)
   - 4.4. $\leftrightarrow Even(F_{k+3} + F_{k+2})$ (Fact 1)
   - 4.5. $\leftrightarrow Even(F_{k+4})$ (Definition of F)
   - 4.6. $P(k+1)$ is true
5. Therefore, $P(n)$ is true for all $n \in \mathbb{N}$ ■

Q10. (a) **Direct Proof**

1. Prove F1:
   - 1.1. Suppose $[x] \in X/\sim$
   - 1.2. Since g is a function defined on $\forall x \in X$, $f([x]) = g(x)$
   - 1.3. $\forall [x] \in X/\sim \exists g(x) \in Y (g(x) = f([x]))$
2. Prove F2:
   - 2.1. Suppose $[x] = [y]$ s.t. $f([x]) = f([y])$
   - 2.2. $x \sim y$ (Definition of equivalence class)
   - 2.3. $g(x) = g(y)$ (Definition of g)
   - 2.4. $\forall [x] \in X/\sim z_1, z_2 \in Y(([x], z_1) \in f \land ([x], z_2) \in f \rightarrow z_1 = z_2)$
3. Therefore, f is well-defined ■

(b) **Direct Proof**

1. Suppose $f([x]) = f([y])$
2. $g(x) = g(y)$ (Definition of f)
3. $x \sim y$ (Definition of g)
4. $[x] = [y]$ (Definition of equivalent classes)
5. f is injective ■

(c) $f : 4$ ■
$g : 4$ ■
$f^{-1} \circ g : 3$ ■