MA1522 Tutorial 2

AY 24/25 Sem 1 — github/omgeta

Q1. (a) Suppose $B\vec{x} = \vec{0}$ has infinitely many solutions $\vec{u}$, then there are also infinitely many vectors $B\vec{u} = \vec{0}$ which when multiplied by matrix A satisfy the homogenous equation $AB\vec{u} = \vec{0}$.

Therefore if $B\vec{x} = \vec{0}$ has infinitely many solutions, then $AB\vec{x} = \vec{0}$ also has infinitely many solutions. ■

(b) Suppose $B\vec{x} = \vec{0}$ has only the trivial solution $\vec{x} = \vec{0}$. For example, let $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Consider two cases:

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AB\vec{x} = \vec{0}$ has only the trivial solution.

(ii) $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB\vec{x} = \vec{0}$ has infinitely many solutions.

Therefore, if $B\vec{x} = \vec{0}$ has only the trivial solution, it is not possible to know the number of solutions for $AB\vec{x} = \vec{0}$. ■

Q2. (a) To solve for X, reduce the augmented matrix $AX = I_3$:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_{14} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_{24} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_{34} \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Therefore, a possible solution is $X = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ■

(b) Solve $B^T Y^T = (YB)^T = I_3$ instead. Then $Y = (y_1, y_2, y_3)$ and $Y^T = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$\therefore y_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\therefore y_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\therefore y_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Therefore, a possible solution is $Y = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$ $\blacksquare$

Q3. (ai)

$$A = \begin{bmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & 12 & 2 \\ -2 & 1 & 3 & 1 \end{bmatrix}$$
$$\xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{bmatrix} = R \quad \blacksquare$$

(aii)

$$E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_1 + 2R_2)$$

$$E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_2 + 2R_1)$$

(aiii)

$$\begin{aligned} A &= E_1^{-1} E_2^{-1} R \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{bmatrix} \quad \blacksquare \end{aligned}$$

(bi)

$$A = \begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -3 & 4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$$

$$\xrightarrow[R_3+4R_1]{R_2-2R_1} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 10 & -7 \\ 0 & -10 & 7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{10}R_2} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -7/10 \\ 0 & -10 & 7 \end{bmatrix}$$

$$\xrightarrow[R_3+10R_2]{R_1+3R_2} \begin{bmatrix} 1 & 0 & 19/10 \\ 0 & 1 & -7/10 \\ 0 & 0 & 0 \end{bmatrix} = R \quad \blacksquare$$

(bii)

$$E_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (-R_1)$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_2 - 2R_1)$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_3 + 4R_1)$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (\tfrac{1}{10}R_2)$$

$$E_5 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_1 + 3R_2)$$

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_3 + 10R_2)$$

(biii)

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}R \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 19/10 \\ 0 & 1 & -7/10 \\ 0 & 0 & 0 \end{bmatrix} \quad \blacksquare \end{aligned}$$

(ci)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R \quad \blacksquare$$

(cii)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_2 - 2R_1)$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_3 - R_1)$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \blacksquare \qquad (R_2 \leftrightarrow R_3)$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (\tfrac{1}{3}R_2)$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_2 - R_1)$$

$$E_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \qquad (R_1 + R_2)$$

(ciii)

$$\begin{aligned} A &= E_1 E_2 E_3 E_4 E_5 E_6 R \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \blacksquare \end{aligned}$$

Q4. (a) By reducing the matrix into RREF

$$\begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We see that all of its columns are pivot columns. Therefore, it has an inverse. To find the inverse we augment the matrix with the identity matrix I_2, and reduce to RREF

$$\begin{bmatrix} -1 & 3 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2/7 & 3/7 \\ 3/7 & 1/7 \end{bmatrix}$$

Therefore, the inverse is the matrix $\begin{bmatrix} 2/7 & 3/7 \\ 3/7 & 1/7 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$ ■

(b) By reducing the matrix into RREF

$$\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 19/10 \\ 0 & 1 & -7/10 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that not all columns are pivot columns. Therefore, the inverse doesn't exist. ■

Q5. Reduce the matrix to REF

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \xrightarrow[R_3-aR_1]{R_2-aR_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{bmatrix}$$

$$\xrightarrow{R_3-(b+a)R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & c^2-a^2-(c-a)(b+a) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{bmatrix}$$

It can be seen that for the matrix to have 3 pivot columns, a, b, c must be distinct. ■

Q6. (a) If the inverse of $I - A$ is $I + A$, then $(I - A)(I + A) = I$, law of inverse matrix multiplication.

$$\begin{aligned} (I-A)(I+A) &= I^2 + IA - AI - A^2 \\ &= I + A - A - A^2 \\ &= I \quad ■ \end{aligned}$$

(b) Suppose we have the matrix $B = (I - A)^-1$ where $B = 1 + A + A^2$. To prove that B is an inverse for $I - A$ we must show that $(I - A)B = I$.

$$\begin{aligned} (I-A)(1+A+A^2) &= I + IA + IA^2 - A - A^2 - A^3 \\ &= I + A + A^2 - A - A^2 - A^3 \\ &= I \quad ■ \end{aligned}$$

(c) Suppose $A^n = 0$ and $B = (1 - A)^-1$ where $B = 1 + A + \ldots + A^{n-1}$. To show that for any nilpotent A, $I - A$ has an inverse, we must show that $(I - A)B = I$.

$$\begin{aligned} (I-A)(I+A+\ldots+A^{n-1}) &= (I+A+\ldots+A^{n-1}) - (A+A^2+A^n) \\ &= I - A^n \\ &= I \quad ■ \end{aligned}$$