CS1231S Tutorial 2

AY 24/25 Sem 1 — github/omgeta

Q1. (a) **Original:** $\forall n \in \mathbb{Z}(6 \mid n \rightarrow 2 \mid n \wedge 3 \mid n)$ **true** $\blacksquare$
Converse: $\forall n \in \mathbb{Z}(2 \mid n \wedge 3 \mid n \rightarrow 6 \mid n)$ **true** $\blacksquare$
Inverse: $\forall n \in \mathbb{Z}(6 \nmid n \rightarrow 2 \nmid n \vee 3 \nmid n)$ **true** $\blacksquare$
Contrapositive: $\forall n \in \mathbb{Z}(2 \nmid n \vee 3 \nmid n \rightarrow 6 \nmid n)$ **true** $\blacksquare$

(b) **Original:** $\forall x(x \in \mathbb{Q} \rightarrow x \in \mathbb{Z})$ **false** $(3/2 \in \mathbb{Q} \wedge 3/2 \notin \mathbb{Z})$ $\blacksquare$
Converse: $\forall x(x \in \mathbb{Z} \rightarrow x \in \mathbb{Q})$ **true** $\blacksquare$
Inverse: $\forall x(x \notin \mathbb{Q} \rightarrow x \notin \mathbb{Z})$ **true** $\blacksquare$
Contrapositive: $\forall x(x \notin \mathbb{Z} \rightarrow x \notin \mathbb{Q})$ **false** $(3/2 \notin \mathbb{Z} \wedge 3/2 \in \mathbb{Q})$ $\blacksquare$

(c) **Original:** $\forall p, q \in \mathbb{Z}(Even(p) \wedge Even(q) \rightarrow Even(p+q))$ **true** $\blacksquare$
Converse: $\forall p, q \in \mathbb{Z}(Even(p+q) \rightarrow Even(p) \wedge Even(q))$ **false** $(p = 3, q = 5)$ $\blacksquare$
Inverse: $\forall p, q \in \mathbb{Z}({\sim}Even(p) \vee {\sim}Even(q) \rightarrow {\sim}Even(p+q))$ **false** $(p = 3, q = 5)$ $\blacksquare$
Contrapositive: $\forall p, q \in \mathbb{Z}({\sim}Even(p+q) \rightarrow {\sim}Even(p) \vee {\sim}Even(q))$ **true** $\blacksquare$

Q2. (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}(y > x)$ $\blacksquare$

(b) $\forall x, z \in \mathbb{R}, \exists y \in \mathbb{R}(x < z \rightarrow (x < z) \wedge (z < y))$ $\blacksquare$

Q3. (a) R is reflexive $\leftrightarrow \forall x \in A(xRx)$ $\blacksquare$

(b) R is symmetric $\leftrightarrow \forall x, y \in A(xRy \rightarrow yRx)$ $\blacksquare$

(c) R is transitive $\leftrightarrow \forall x, y, z \in A(xRy \wedge yRz \rightarrow xRz)$ $\blacksquare$

Q4. (a) **Disproof by Counterexample**
$3 \in \mathbb{Z} \wedge 2 \in \mathbb{Z}$ but $\frac{3}{2} \notin \mathbb{Z}$ $\blacksquare$

(b) **Direct Proof**
Let $x, y \in \mathbb{R}$, then by definition of rational numbers, $\exists a, b, c, d \in \mathbb{Z}$ where $b, d \neq 0$ such that:

$$x = \frac{a}{b}$$
$$y = \frac{c}{d}$$

Consider addition of x and y:

$$\begin{aligned} x + y &= \frac{a}{b} + \frac{c}{d} \\ &= \frac{ad + bc}{bd} \end{aligned}$$

Since integers are closed over addition and multiplication:

$$\begin{aligned} A &= ad + bc \in \mathbb{Z} \\ B &= bd \in \mathbb{Z} \end{aligned}$$

Since $b \neq 0, d \neq 0, B = bd \neq 0$.
Therefore, $\forall x, y \in \mathbb{R}, x + y = \frac{A}{B}$ is rational (by definition of rational numbers). Hence, rational numbers are closed under addition. $\blacksquare$

(c) **Disproof by Counterexample**
$\forall x \in \mathbb{R}, 0 \in \mathbb{R}, \frac{x}{0} \notin \mathbb{R}$. $\blacksquare$

Q5. (a) False. If $x < y$ then $x - y \notin B$ $\blacksquare$

(b) True. $\blacksquare$

(c) False. Predicate is only true if $(x, y) \in \{(1, 0), (3, 2), (5, 4), (7, 6)\}$ and not $\forall x \in A, \forall y \in B$ $\blacksquare$

(d) True. $\blacksquare$

(e) True. $\blacksquare$

(f) False. $\exists x \in A, \exists y \in B(x = y + 1)$ $\blacksquare$

(g) True. $\blacksquare$

(h) True. $\blacksquare$

(i) False. If $x \in 7, 11, 13$ predicate is false because the largest element $y \in B$ is 6. $\blacksquare$

(j) True. Take $y = 0$. $\blacksquare$

Q6. (a) False. There is no title read by all the female readers. Ms Emily has only read "Dream of the Red Chamber", "Da Vinci Code", "She: A History of Adventure" and "Black Beauty", all of which are not read by both of the other two females. $\blacksquare$

(b) False. Ms Dueet has not read any books in the Fantasy genre. $\blacksquare$

(c) True. Ms Dueet has read all books of the Mystery genre. $\blacksquare$

(d) True. Fantasy has none of its books read by Ms Dueet. $\blacksquare$

Q7. (a) Universal statements in each of the cases cannot be proven by a single example. $\blacksquare$

(b) There are 3 cases to consider: $x < 0, x = 0$ and $x > 0$. If $x < 0$, for example, $x = -1$, then $x^3 = -1 = x$; if $x = 0$, then $x^3 = 0 = x$; if $x > 0$, say $x = 1$, then $x^3 = -1 = x$. Therefore, in all cases, $x^3 = x$. $\blacksquare$

(c) The proof is false because it assumes that a single counterexample is enough to prove the falsity of a statement for all real values of x. $\blacksquare$

(d) Suppose $x^3 \neq x$ for all real numbers x. Let $x = 1$, then $x^3 = 1 = x$ which is a contradiction. Therefore, $\forall x \in \mathbb{R}(x^3 = x)$ $\blacksquare$

Q8. (a) No. In the case $r^2 \leq r$, our statement is vacuously true. $\blacksquare$

(b) 2.3. **Case 1:** $r > 0 \land r - 1 > 0$
This implies $r > 0$ and $r > 1$, which satisfies $r < 0 \lor r > 1$
2.4. **Case 2:** $r < 0 \land r - 1 < 0$
This implies $r < 0$ and $r < 1$, which satisfies $r < 0 \lor r > 1$
2.5. In both cases, the conclusion $r < 0 \lor r > 1$ is satisfied.

(c) By proving that a statement holds for an arbitrary element, we can conclude a universal statement. $\blacksquare$

Q9. (a) $\forall v \in V(W(v))$ $\blacksquare$

(b) $\forall v \in V(G(v) \to T(v))$ $\blacksquare$

(c) $\exists v \in V(T(v) \land G(v))$ $\blacksquare$

(d) $\forall v \in V(E(v) \to \sim W(v))$ $\blacksquare$

(e) $\exists v \in V(T(v) \land E(v)) \land \exists v \in V(T(v) \land \sim E(v))$ $\blacksquare$

Q10. (a) 3. Every black object is a square
2. If an object is square, then it is above all the grey objects
4. Every object that is above all the gray objects is above all the triangles.
1. If an object is above all the triangles, then it is above all the blue objects.
$\therefore$ If an object is black, then it is above all the blue object.

(b) 3. $\forall x(Black(x) \to Square(x))$
2. $\forall x, y(Square(x) \to (Gray(y) \to Above(x, y)))$
4. $\forall x, y, z((Gray(y) \to Above(x, y)) \to (Triangle(z) \to Above(x, z)))$
1. $\forall x, y, z((Triangle(y) \to (Above(x, y))) \to (Blue(z) \to Above(x, z)))$
$\therefore \forall x, y(Black(x) \to (Blue(y) \to Above(x, y)))$

Q11. **Proof by Contraposition.**

1. Let a, b be two positive integers

2. Assume the opposite of the conclusion, i.e., $a > n^{1/2} \wedge b > n^{1/2}$.

   2.1 $a \cdot b > n^{1/2} \cdot n^{1/2}$

   2.2 $a \cdot b > n$

   2.3 $a \cdot b \neq n$

3. Therefore, by contraposition, since $\forall a, b \in \mathbb{Z}^+, ((a > \sqrt{n}) \wedge (b > \sqrt{n})) \to n \neq a \cdot b$, then $\forall a, b \in \mathbb{Z}^+, n = ab \to ((a \leq \sqrt{n}) \vee (b \leq \sqrt{n}))$.

Proof by Contradiction.

1. Let a, b be two positive integers

2. Assume $n = ab$ and $a > n^{1/2} \wedge b > n^{1/2}$.

   2.1 $a \cdot b > n^{1/2} \cdot n^{1/2}$

   2.2 $a \cdot b > n$

   2.3 This is a contradiction to $n = ab$

3. Therefore, by contradiction, since $n = ab \to (a \leq \sqrt{n} \vee b \leq \sqrt{n})$