CS1231S Tutorial 6

AY 24/25 Sem 1 — github/omgeta

Q1. (a) **Direct Proof**

1. Let $x, y \in \mathbb{N} \wedge xR_1y$
2. Case 1 ($x = 0$):
   2.1. $x^2 = 0 = y^2$ (Definition of R_1)
   2.2. $y = 0$ (T11)
3. Case 2 ($x \neq 0$)
   3.1. Suppose $x = n \in \mathbb{N}^+$
   3.2. $x^2 = n^2 = y^2$ (Definition of R_1)
   3.3. $y = n \in \mathbb{N} \vee y = -n \notin \mathbb{N}$ (Basic algebra)
   3.4. $y = n$ (Elimination)
4. $\therefore \forall x \in \mathbb{N}, \exists! y \in \mathbb{N}((x, y) \in R_1)$
5. $\therefore R_1$ is a function ■ (Definition of function)

(b) **Disproof by Counterexample**

1. Let $x = 2$
2. $1|x \wedge 2|x \implies y = 1 \vee y = 2$ (Definition of R_2)
3. $\exists x, y_1, y_2 \in \mathbb{N}((x, y_1) \in R_2 \wedge (x, y_2) \in R_2 \wedge y_1 \neq y_2)$
4. $\therefore R_2$ is not a function ■ (Definition of function)

(c) **Direct Proof**

1. Suppose $x = n \in \mathbb{N}$, and $\exists y_1, y_2 \in \mathbb{N}$ s.t. $xR_3y_1 \wedge xR_3y_2$
2. $y_1 = n + 1$ (Definition of R_3)
3. $y_2 = n + 1$ (Definition of R_3)
4. $\therefore y_1 = y_2$ (Substitute 2 into 3)
5. $\therefore \forall x, y_1, y_2 \in \mathbb{N}(((x, y_1) \in R_3 \wedge (x, y_2) \in R_3) \rightarrow y_1 = y_2)$
6. $\therefore R_3$ is a function ■ (Definition of function)

Q2. (a) **Direct Proof**

1. Let $s_1, s_2 \in S$ s.t. $C(s_1) = C(s_2)$
2. $as_1 = as_2$ (Definition of C)
3. Let n be length of as_1, as_2 (Definition of string equality)
4. Thus, s_1, s_2 are of same length $n - 1$
5. Let $s_1 = a_1a_2 \dots a_{n-1}$
6. Let $s_2 = b_1b_2 \dots b_{n-1}$
7. $s_1 = s_2$ (Definition of string equality)
8. $\therefore \forall s_1, s_2 \in S(C(s_1) = C(s_2) \rightarrow s_1 = s_2)$
9. $\therefore C$ is injective ■ (Definition of injectivity)

(b) **Proof by Contradiction**

1. Let $y = b$
2. Assume s is any string s.t. $C(s) = b$
   2.1. $as = b$
   2.2. By definition of string equality $len(as) = lenb(b) = 1$
   2.3. $\therefore s = \varepsilon$
   2.4. $\therefore a = b$
   2.5. This is a contradiction
3. $\therefore C$ is not surjective ■ (Definition of surjectivity)

Q3. (a) $len(suu) = 3$ ■

(b) $len(\{\varepsilon, ss, uu, ssss\}) = \{0, 2, 4\}$ ■

(c) $len^{-1}(\{3\}) = \{sss, ssu, sus, uss, suu, usu, uus, uuu\}$ ■

(d) **Disproof by Counterexample**

1. Let $a_1 = sss \neq uuu = a_2$
2. $len(a_1) = 3 = len(a_2)$ (Definition of len)
3. $\exists a_1, a_2 \in A^*(len(a_1) = len(a_2) \wedge a_1 \neq a_2)$
4. $\therefore len$ is not injective (Definition of injectivity)
5. $\therefore len$ is not bijective (Definition of bijectivity)
6. $\therefore len^{-1}$ does not exist (Definition of inverse)

Q4. **Direct Proof**

1. Prove $f^{-1} \circ g^{-1}$ is a left inverse:
   1.1. $(f^{-1} \circ g^{-1}) \circ (g \circ f)$
   1.2. $= f^{-1} \circ (g^{-1} \circ g) \circ f$ (Associativity of functions)
   1.3. $= f^{-1} \circ (id_B) \circ f$ (Definition of inverse)
   1.4. $= f^{-1} \circ f$ (Definition of identity)
   1.5. $= id_A$ (Definition of inverse)
   1.6. $\therefore f^{-1} \circ g^{-1}$ is a left inverse (Definition of left inverse)
2. Prove $f^{-1} \circ g^{-1}$ is a right inverse:
   2.1. $(g \circ f) \circ (f^{-1} \circ g^{-1})$
   2.2. $= g \circ (f \circ f^{-1}) \circ g^{-1}$ (Associativity of functions)
   2.3. $= g \circ (id_B) \circ g^{-1}$ (Definition of inverse)
   2.4. $= g \circ g^{-1}$ (Definition of identity)
   2.5. $= id_C$ (Definition of inverse)
   2.6. $\therefore f^{-1} \circ g^{-1}$ is a right inverse (Definition of right inverse)
3. $\therefore f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$ (Definition of inverse)
4. $\therefore f^{-1} \circ g^{-1} = (g \circ f)^{-1}$ $\blacksquare$

Q5. (a)

1. Prove f is injective:
   1.1. Let $x_1, x_2 \in \mathbb{Q}$ s.t. $f(x_1) = f(x_2)$
   1.2. $12x_1 + 31 = 12x_2 + 31$ (Definition of f)
   1.3. $x_1 = x_2$ (Basic algebra)
   1.4. $\forall x_1, x_2 \in \mathbb{Q}(f(x_1) = f(x_2) \rightarrow x_1 = x_2)$ (Universal generalization)
   1.5. f is injective $\blacksquare$ (Definition of injectivity)
2. Prove f is surjective:
   2.1. Let $y = f(x) \in \mathbb{Q}$
   2.2. $y = 12x + 31$ (Definition of f)
   2.3. $x = \frac{y-31}{12} \in \mathbb{Q}$ (Closure of rationals over subtraction and division)
   2.4. $\forall y \in \mathbb{Q} \exists x \in \mathbb{Q}(y = f(x))$
   2.5. f is surjective $\blacksquare$ (Definition of surjectivity)

$f^{-1} = \frac{y-31}{12}$ $\blacksquare$

(b)

1. Prove g is not injective:
   1.1. $g(false, false) = g(false, true) = g(true, true) = false$
   1.2. $(false, false) \neq (false, true) \neq (true, true)$
   1.3. $\therefore \exists x_t rue, x_2 \in Bool^2(g(x_t rue) = g(x_2) \wedge x_t rue \neq x_2)$
   1.4. $\therefore g$ is not injective $\blacksquare$ (Definition of injective)
2. Prove g is surjective:
   2.1. Let $y \in Bool$
   2.2. Case true ($y = false$):
      2.2.1. $g(false, false) = g(false, true) = g(true, true) = false$
      2.2.2. $\exists x \in Bool^2(false = g(x))$
   2.3. Case 2 ($y = true$):
      2.3.1. $g(true, false) = true$
      2.3.2. $\exists x \in Bool^2(true = g(x))$
   2.4. $\therefore \forall y \in Bool, \exists x \in Bool^2(y = g(x))$
   2.5. g is surjective $\blacksquare$ (Definition of surjectivity)

(c)

1. Prove h is not injective:
   1.1. $h(false, true) = h(1, 0) = (0, 1)$
   1.2. $(false, true) \neq (1, 0)$
   1.3. $\therefore \exists x_t rue, x_2 \in Bool^2(h(x_1) = h(x_2) \wedge x_1 \neq x_2)$
   1.4. $\therefore h$ is not injective $\blacksquare$ (Definition of injective)
2. Prove h is not surjective:
   2.1. Suppose $y = (true, false)$
   2.2. $\therefore \exists y \in Bool^2, \forall x \in Bool^2(y \neq h(x))$
   2.3. h is not surjective $\blacksquare$ (Definition of surjectivity)

(d)

1. Prove k is injective:
   1.1. Let $x_1, x_2 \in \mathbb{Z}$ s.t. $k(x_1) = k(x_2)$
   1.2. Case 1 (x_1 and x_2 are even):
      1.2.1. $x_1 = x_2$ (Definition of k)
   1.3. Case 2 (x_1 and x_2 are odd):
      1.3.1. $2x_1 - 1 = 2x_2 - 1$ (Definition of k)
      1.3.2. $x_1 = x_2$ (Basic algebra)
   1.4. $\forall x_1, x_2 \in \mathbb{Z}(k(x_1) = k(x_2) \to x_1 = x_2)$
   1.5. k is injective ■ (Definition of injectivity)
2. Prove k is not surjective:
   2.1. Let $y = 3$
   2.2. Assume $x \in \mathbb{Z}$ s.t. $k(x) = 3$
      2.2.1. x is odd
      2.2.2. $3 = 2x - 1$ (Definition of k)
      2.2.3. $x = 2$
      2.2.4. This contradicts 2.2.1
   2.3. $\exists y \in \mathbb{Z} \forall x \in \mathbb{Z}(y \neq k(x))$
   2.4. k is not surjective ■ (Definition of surjectivity)

Q6. (a) f, k ■

(b) f, g ■

(c) (i) False. ■

(ii) False. ■

Q7. **Proof by Contraposition**

1. Suppose f is not injective
2. $\exists x_1, x_2 \in B$ s.t. $f(x_1) = f(x_2) \wedge x_1 \neq x_2$ (Definition of not injective)
3. $g(f(x_1)) = g(f(x_2))$
4. $(g \circ f)(x_1) = (g \circ f)(x_2)$ (Definition of composition)
5. $(g \circ f)(x_1) = (g \circ f)(x_2) \wedge x_1 \neq x_2$ (Conjunction)
6. $g \circ f$ is not injective (Definition of not injective)
7. f is not injective $\to g \circ f$ is not injective
8. $\equiv g \circ f$ is injective $\to f$ is injective ■

Q8. Order of $g : 2$ ■
Order of $h : 2$ ■
Order of $g \circ h : 3$ ■
Order of $h \circ g : 3$ ■

Q9. (a)

1. Prove $X \subseteq f^{-1}(f(X))$
   1.1. Suppose $x \in X$
   1.2. $f(x) \in f(X)$ (Definition of image)
   1.3. $x \in f^{-1}(f(X))$ (Definition of preimage)
   1.4. $\forall x(x \in X \to x \in f^{-1}(f(x)))$ (Universal generalisation)
   1.5. $X \subseteq f^{-1}(f(X))$ $\blacksquare$ (Definition of subset)
2. Prove $f^{-1}(f(X)) \not\subseteq X$:
   2.1. Suppose $f : \{a, b\} \to \{c\}$
   2.2. Let $X = \{a\}$
   2.3. $f(X) = \{c\}$ (Definition of setwise image)
   2.4. $f^{-1}(f(X)) = f^{-1}(\{c\}) = \{a, b\} \not\subseteq X$ $\blacksquare$ (Definition of setwise preimage)

(b)

1. Prove $Y \not\subseteq f(f^{-1}(Y))$:
   1.1. Suppose $f : \{a\} \to \{b, c\}, f(a) = b$
   1.2. Let $Y = \{c\}$
   1.3. $f^{-1}(Y) = \{\}$ (Definition of setwise image)
   1.4. $f(f^{-1}(f(Y))) = f(f^{-1}(\{c\})) = \{\} \not\subseteq Y$ $\blacksquare$ (Definition of setwise preimage)
2. Prove $f(f^{-1}(Y)) \subseteq Y$:
   2.1. Let $y \in f(f^{-1}(Y))$
   2.2. $f^{-1}(y) \in f(Y)$ (Definition of preimage)
   2.3. $y \in Y$ (Definition of image)
   2.4. $\forall y(y \in ff^{-1}(Y) \to y \in Y)$ (Universal generalisation)
   2.5. $f(f^{-1}(Y)) \subseteq Y$ $\blacksquare$ (Definition of subset)

Q10. (a) **Direct Proof**

1. Let $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Q}/\!\sim$ s.t. $[x_1] = [x_2]$ and $[y_1] = [y_2]$
2. $x_1 - x_2 \in k \in \mathbb{Z} \wedge y_1 - y_2 = l \in \mathbb{Z}$ (Definition of $\sim$)
3. Consider, $(x_1 + y_1) - (x_2 + y_2)$
4. $= (x_1 - x_2) + (y_1 - y_2)$
5. $= k + l \in \mathbb{Z}$
6. $\therefore x_1 + y_1 \sim x_2 + y_2$ (Definition of $\sim$)
7. $\therefore +$ is well-defined for $\sim$ $\blacksquare$

(b) **Disproof by Counterexample**

1. Notice $[\frac{1}{2}] \sim [-\frac{1}{2}]$
2. Consider, $[\frac{1}{2}] \cdot [\frac{1}{2}] = [\frac{1}{4}]$
3. and $[\frac{1}{2}] \cdot [-\frac{1}{2}] = [-\frac{1}{4}]$
4. However, $[-\frac{1}{4}] \not\sim [-\frac{1}{4}]$
5. $\therefore \cdot$ is not well-defined for $\sim$ $\blacksquare$

Q11. $+ : (\mathbb{Q}/\!\sim, \mathbb{Q}/\!\sim) \to \mathbb{Q}/\!\sim$ $\blacksquare$