MA1521 Homework 7

AY 24/25 Sem 1 — github/omgeta

Q1. (a) $\vec{w} = 3\hat{i} - \hat{j} + 4\hat{k} \quad \blacksquare$

(b) Suppose Q is a point on the plane:

$$\begin{aligned} 3(0) - 1(a) + 4(0) &= 0 \\ a &= -1 \quad \blacksquare \end{aligned}$$

(c) Find $\vec{QP}$:

$$\begin{aligned} \vec{QP} &= \vec{OP} - \vec{OQ} \\ &= \langle 2, 1, -3 \rangle - \langle 0, -1, 0 \rangle \\ &= \langle 2, 2, -3 \rangle \end{aligned}$$

Find the projection of $\vec{QP}$ onto the plane:

$$\begin{aligned} proj_{\mathcal{P}}\vec{QP} &= (\frac{\vec{QP} \cdot \vec{w}}{||\vec{w}||^2})\vec{w} \\ &= (\frac{2(3) + 2(-1) - 3(4)}{3^2 + 1^2 + 4^2})\vec{w} \\ &= -\frac{4}{13}(3\hat{i} - \hat{j} + 4\hat{k}) \quad \blacksquare \end{aligned}$$

(d) Since Q lies on the plane, the distance is given by:

$$\begin{aligned} ||proj_{\mathcal{P}}\vec{QP}|| &= || - \frac{4}{13}\langle 3, -1, 4 \rangle \\ &= \frac{4}{13}\sqrt{3^2 + 1^+ 4^2} \\ &= \frac{4\sqrt{26}}{13} \quad \blacksquare \end{aligned}$$

Q2. (a) Equation of the plane is given by:

$$\begin{aligned} -2(x-6)+5(y-3)+(z-2) &= 0 \\ -2x+12+5y-15+z-2 &= 0 \\ -2x+5y+z &= 5 \quad \blacksquare \end{aligned}$$

(b) Equation of the plane is given by:

$$\begin{aligned} 2(x-3)+4(y-0)+8(z-8) &= 0 \\ 2x-6+4y+8z-64 &= 0 \\ 2x+4y+8z &= 70 \quad \blacksquare \end{aligned}$$

(c) Line of intersection between planes is:

$$\begin{aligned} \vec{d_1} &= \langle 1,1,-1\rangle \times \langle 2,-1,3\rangle \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} \\ &= 2\hat{\imath} - 5\hat{\jmath} - 3\hat{k} \end{aligned}$$

Find a point on the line of intersection, let $z = 0$:

$$\begin{aligned} x+y &= 2 \\ 2x-y &= 1 \\ \implies x = y &= 1 \end{aligned}$$

Then we have a second vector on the plane:

$$\begin{aligned} \vec{d_2} &= \langle -1,2,1\rangle - \langle 1,1,0\rangle \\ &= \langle -2,1,1\rangle \end{aligned}$$

Find the normal of the plane:

$$\begin{aligned} \vec{n} &= \vec{d_1} \times \vec{d_2} \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -5 & -3 \\ -2 & 1 & 1 \end{vmatrix} \\ &= \langle 2,-4,8\rangle \end{aligned}$$

Equation of the plane is given by:

$$\begin{aligned} 2(x+1)-4(y-2)+8(z-1) &= 0 \\ 2x-4y+8z &= -2 \\ x-2y+4z &= -1 \quad \blacksquare \end{aligned}$$

Q3. Notice the equivalent form of Π_2:

$$\Pi_2 : 2x-3y+6z = -2$$

Then, the distance between the two planes is given by:

$$\begin{aligned} &\frac{|16-(-2)|}{\sqrt{2^2+3^2+6^2}} \\ &= \frac{18}{7} \quad \blacksquare \end{aligned}$$

Q4. Find two vectors on the plane:

$$\begin{aligned}\vec{AB} &= \langle 3,0,1\rangle - \langle 3,3,0\rangle \\ &= \langle 0,-3,1\rangle \\ \vec{AC} &= \langle 0,2,1\rangle - \langle 3,3,0\rangle \\ &= \langle -3,-1,1\rangle\end{aligned}$$

Find the normal to the plane:

$$\begin{aligned}\vec{n} = \vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -3 & 1 \\ -3 & -1 & 1 \end{vmatrix} \\ &= \langle -2,-3,-9\rangle\end{aligned}$$

Plane Π is given by:

$$\begin{aligned}-2(x-3)-3(y-3)-9(z-0) &= 0 \\ 2x+3y+9z &= 15 \quad \blacksquare\end{aligned}$$

Q5. Shortest distance from origin is given by:

$$\begin{aligned}&\frac{15}{\sqrt{2^2+3^2+9^2}} \\ &= \frac{15}{\sqrt{94}} \quad \blacksquare\end{aligned}$$

Q6. Line segment of $\vec{OD}$ is given by:

$$\vec{OD} : t\langle 4,2,1\rangle, \quad t \in \mathbb{R}$$

At the point of intersection:

$$\begin{aligned}2(4t)+3(2t)+9(t) &= 15 \\ 8t+6t+9t &= 15 \\ t &= \frac{15}{23}\end{aligned}$$

Therefore, the point of intersection is:

$$\begin{aligned}&\frac{15}{23}\langle 4,2,1\rangle \\ &= (\frac{60}{23}, \frac{30}{23}, \frac{15}{23}) \quad \blacksquare\end{aligned}$$

Q7. When the curves intersect:

$$\begin{aligned}r_1(t_1) &= r_2(t_2) \\ t_1\hat{i} + t_1^2\hat{j} + t_1^3\hat{k} &= (1+2t_2)\hat{i} + (1+6t_2)\hat{j} + (1+14t_2)\hat{k}\end{aligned}$$

which gives the system of equations:

$$t_1 = 1+2t_2, \quad t_1^2 = 1+6t_2, \quad t_1^3 = 1+14t_2$$

Solving these equations simultaneously gives the times when the curves are at the same point:

$$\begin{aligned}t_1 = 1 &\implies t_2 = 0 \\ t_1 = 2 &\implies t_2 = \frac{1}{2}\end{aligned}$$

(a) No; the intersections there are no intersections where $t_1 = t_2$ $\blacksquare$

(b) Yes; there are two points where the paths intersect at $t_1 = 1, t_2 = 0$ and $t_1 = 2, t_2 = \frac{1}{2}$ $\blacksquare$