CAS CS 538. Solutions to Problem Set 5

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Problem 1. (50 points)
In Discussion 5 we introduced the Signal symmetric ratchet and we showed that even if a subsequent key is leaked to the adversary, it is hard to recover the previous key. In this problem, we will prove something stronger: even if a subseuqent key is leaked (and the previous key has been deleted), past encryptions are still semantically secure.

Specifically, let $\mathcal{E} = (\mathsf{Enc}, \mathsf{Dec})$ be a semantically secure cipher over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$. Let $G : \mathcal{S} \to \mathcal{S} \times \mathcal{K}$ be a secure PRG such that $G : \{0,1\}^\ell \to \{0,1\}^{2\ell}$. Define the symmetric ratchet scheme as $\mathsf{Enc}^1(s, m) = \mathsf{Enc}(G(s)_2, m)$ and $\mathsf{Dec}^1(s, c) = \mathsf{Dec}(G(s)_2, c)$. Here $G(s)_2$ denotes the second part of the output of $G(s)$, i.e. the key part. We refer you to Figure 1 of Discussion 5 for a diagram of the symmetric ratchet.

At time step n, participants will advance the state of the ratchet by computing $(s_n, k_n) \twoheadleftarrow G(s_{n-1})$. Then, the past is erased by deleting the old seed s_{n-1} from memory. The new ciphertext is computed as $c_n = \mathsf{Enc}(k_n, m_n)$.

Prove that $\mathcal{E}^1$ is *semantically secure* for the previous encryption c_{n-1} even if a subsequent seed and key (s_n, k_n) is leaked to the adversary.

Getting started. To get started, you should consider an adversary $\mathcal{A}$ playing a new game called the SSKeyLeak game against $\mathcal{E}^1$. The SSKeyLeak is the same as the Semantic Security Game (Boneh + Shoup Attack Game 2.1) except that in addition to a ciphertext c_{n-1}, the adversary *also* receives a leaked seed and key (s_n, k_n) where s_n, k_n are pseudorandom values produced by G^n with an initial random seed.

In particular, say $\mathcal{A}$ outputs two messages m_0, m_1. Let Game 0 be when $c_{n-1} = \mathsf{Enc}^1(k_{n-1}, m_0)$ and Game 1 be when $c_{n-1} = \mathsf{Enc}(k_{n-1}, m_1)$. $\mathcal{A}$ receives the tuple (c_{n-1}, s_n, k_n) and outputs a bit $\hat{b}$. Let W_b be the event that $\hat{b} = 1$ in Game b of the SSKeyLeak game. Then we define the key-leakage advantage as $\text{SSKeyLeakadv}[\mathcal{A}, \mathcal{E}^1] = |\Pr[W_0] - \Pr[W_1]|$.

Building a hybrid argument. Use a hybrid argument to prove that $\mathcal{E}^1$ is semantically secure against leaked keys by showing that $\text{SSKeyLeakadv}[\mathcal{A}, \mathcal{E}^1]$ is negligible.

Hint: you will have to introduce two hybrid games in between Game 0 and Game 1 of SSKeyLeak game. Then you will give three reductions with wrapper adversaries $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$. In no particular order, two of these reductions yield a PRG distinguisher for G^n and one of them yields an SS adversary for $\mathcal{E}$.

Solution. The whole idea behind how I'll prove this by essentially splitting up the whole process into a bunch of hybrid arguments, and showing how the sum of all of the reductions end up being negligible.

Let $G^n : \{0,1\}^\ell \to \{0,1\}^{(n+1)\ell}$ be the n-wise sequential composition of G, outputting $(k_1, s_1, \ldots, k_n, s_n)$. Also assume that G is a secure PRG, thereby also meaning that G^n is also secure.

The Hybrid Games

Game 0: The initial real SSKeyLeak game w/ bit $b = 0$.

1. Challenger samples $s_0 \leftarrow \{0,1\}^\ell$.
2. Calculates $(k_1, s_1, \ldots, k_n, s_n) \twoheadleftarrow G^n(s_0)$.
3. $\mathcal{A}$ then outputs m_0, m_1.
4. Challenger computes $c_{n-1} = \mathsf{Enc}(k_{n-1}, m_0)$.
5. Challenger then gives (c_{n-1}, k_n, s_n) to $\mathcal{A}$.
6. $\mathcal{A}$ returns $\hat{b}$.

Let $\Pr[W_0]$ denote the probability of $\mathcal{A}$ outputting 1.

Hybrid 1: Same as Game 0, but but now the keys and seeds are replaced w/ rnd uniform values.

1. Challenger samples $(k_1, \ldots, k_n, s_n) \leftarrow \{0,1\}^{(n+1)\ell}$ uniformly at random.
2. $\mathcal{A}$ outputs m_0, m_1.
3. Challenger then Computes $c_{n-1} = \mathsf{Enc}(k_{n-1}, m_0)$.
4. Challenger will then give (c_{n-1}, k_n, s_n) to $\mathcal{A}$.
5. $\mathcal{A}$ finlly returns $\hat{b}$.

$\Pr[H_1]$ will be the probability of $\mathcal{A}$ outputting 1.

Hybrid 2: Hybrid 1, but now the msg is m_1.

1. Challenger samples $(k_1, \ldots, k_n, s_n) \leftarrow \{0,1\}^{(n+1)\ell}$ uniformly at random.
2. $\mathcal{A}$ outputs m_0, m_1.
3. Challenger calcualtes $c_{n-1} = \mathsf{Enc}(k_{n-1}, m_1)$.
4. Challenger then gives (c_{n-1}, k_n, s_n) to $\mathcal{A}$.
5. $\mathcal{A}$ returns $\hat{b}$.

Let $\Pr[H_2]$ be the chance of $\mathcal{A}$ outputting 1.

Game 1: Finally circling back to the SSKeyLeak game, but now w/ bit $b = 1$.

1. Challenger samples $s_0 \leftarrow \{0,1\}^\ell$.
2. Compute $(k_1, s_1, \ldots, k_n, s_n) \twoheadleftarrow G^n(s_0)$.
3. $\mathcal{A}$ outputs m_0, m_1.
4. Challenger then calculates $c_{n-1} = \mathsf{Enc}(k_{n-1}, m_1)$.
5. Give that value, (c_{n-1}, k_n, s_n), to $\mathcal{A}$.
6. $\mathcal{A}$ returns $\hat{b}$.

Let $\Pr[W_1]$ be the probability of $\mathcal{A}$ outputting 1.

By the triangle inequality, the advantage is then bounded by the following expression:

$$|\Pr[W_0] - \Pr[W_1]| \leqslant |\Pr[W_0] - \Pr[H_1]| + |\Pr[H_1] - \Pr[H_2]| + |\Pr[H_2] - \Pr[W_1]| \, . \tag{1}$$

Reducing and Analyzing

Game 0 $\to$ **Game** H_1 Let $\mathcal{B}_1$ be a new adversay against G^n.

- $\mathcal{B}_1$ will recieve a challenge string $r \in \{0,1\}^{(n+1)\ell}$ from G^n.
- $\mathcal{B}_1$ will then parse r as $(k_1, \ldots, k_n, s_n)$.
- $\mathcal{B}_1$ will run $\mathcal{A}$ to get m_0, m_1.
- $\mathcal{B}_1$ then computs $c_{n-1} = \mathsf{Enc}(k_{n-1}, m_0)$.
- Finally, $\mathcal{B}_1$ gives (c_{n-1}, k_n, s_n) to $\mathcal{A}$ and its response.

Analyzing this:

- If r is outputted by $G^n(s_0)$, then this is just **Game 0**. So then $\Pr[\mathcal{B}_1 \to 1 \mid \text{Real}] = \Pr[W_0]$.
- In the same manner, if r is uniform random, then this is just **Hybrid 1**. Thus, $\Pr[\mathcal{B}_1 \to 1 \mid \text{Random}] = \Pr[H_1]$.

$|\Pr[W_0] - \Pr[H_1]| = \text{PRG}\mathsf{adv}[\mathcal{B}_1, G^n]$, which is negligible.

Game H_1 $\to$ **Game** H_2 $\mathcal{B}_2$ will now be the adversay against the semantic security of $\mathcal{E}$.

- $\mathcal{B}_2$ will run $\mathcal{A}$ to get m_0, m_1.
- $\mathcal{B}_2$ then sends m_0, m_1 to the SS challenger for $\mathcal{E}$.
- $\mathcal{B}_2$ will then get a ciphertext, c^*, from the challenger (which is just $\mathsf{Enc}(k^*, m_b)$ for a rnd key k^*).
- $\mathcal{B}_2$ then samples $(k_n, s_n) \leftarrow \{0,1\}^{2\ell}$ uniformly at random.
- And then $\mathcal{B}_2$ will give (c^*, k_n, s_n) to $\mathcal{A}$, and output its response.

Looking at this:

- The key for the SS game, c^*, is rnd and independant of the leak. This is also the same distribution as k_{n-1} back in **Hybrid 1** and **Hybrid 2**.
- So then, if the SS challenger chooses m_0, then this is just **Hybrid 1**.
- If instead the SS challenger chooses m_1, then it is just **Hybrid 2**.

$|\Pr[H_1] - \Pr[H_2]| = \text{SS}\mathsf{adv}[\mathcal{B}_2, \mathcal{E}]$, which is negligible.

Game $H_2 \rightarrow$ **Game 1** $\mathcal{B}_3$ will now be challenging the PRG security of G^n.

- $\mathcal{B}_3$ gets a challenge string $r \in \{0,1\}^{(n+1)\ell}$ from G^n.
- $\mathcal{B}_3$ will parse r as $(k_1, \ldots, k_n, s_n)$.
- $\mathcal{B}_3$ then runs $\mathcal{A}$ to gets m_0, m_1.
- $\mathcal{B}_3$ computes $c_{n-1} = \mathsf{Enc}(k_{n-1}, m_1)$.
- And finally, $\mathcal{B}_3$ gives (c_{n-1}, k_n, s_n) to $\mathcal{A}$ and outputs its response.

Looking at this game:

- If r is outputted by $G^n(s_0)$, then this is just **Game 1**. This basically means that $\Pr[\mathcal{B}_3 \rightarrow 1 \mid \text{Real}] = \Pr[W_1]$.
- If instead r is uniform random, then this is just **Hybrid 2**, meaning that $\Pr[\mathcal{B}_3 \rightarrow 1 \mid \text{Random}] = \Pr[H_2]$.

$|\Pr[H_2] - \Pr[W_1]| = \text{PRGadv}[\mathcal{B}_3, G^n]$, which is negligible.

End

Combining the bounds, we get:

$$\text{SSKeyLeakadv}[\mathcal{A}, \mathcal{E}^1] \leqslant \text{PRGadv}[\mathcal{B}_1, G^n] + \text{SSadv}[\mathcal{B}_2, \mathcal{E}] + \text{PRGadv}[\mathcal{B}_3, G^n]. \tag{2}$$

I've already assumed, in the beginning, that G^n was secure, making the PRG advantages negligible. Also, since $\mathcal{E}$ is semantically secure, then that also means that SS advantage is negligible. And since the sum of negligibles is negligble too, then $\therefore$ $\text{SSKeyLeakadv}[\mathcal{A}, \mathcal{E}^1]$ is negligible.

Problem 2. (50 points at 10 each) Let $p > 2$ be an odd prime. For this problem, suppose the order of g is $p-1$ (which is even). Such a g is called a *generator* of $\mathbb{Z}_p^*$. Such a g exists for every p (we won't prove this fact; see, for example, Section 7.5 of Victor Shoup's book at http://www.shoup.net/ntb/ntb-v2.pdf for the existence proof and Section 11.1 for how to sample it efficiently).

(a) Let $a = g^x$ for some $x \in \mathbb{Z}_{p-1}$ (the exponent works modulo $p-1$ due to Fermat's little theorem). We can talk about the exponent x being even since $p-1$ is even, and therefore $x + k(p-1)$ has the same evenness as x for any $k \in \mathbb{Z}$.

Clearly if x is even, then a has a square root $g^{x/2}$ modulo p. Now show the converse: if a has a square root modulo p, then x is even. *(Hint: you can rewrite any element $b \in \mathbb{Z}_p^*$ as $b = g^y$ for some $y \in \mathbb{Z}_{p-1}$.)*

Solution. I can't really explain the way I'll solve this proof without just restating the proof, but I will say that this soltuion basically involves just doing a bunch of substitution, "technically".

Firstly, assume that a has a sqrt modulo p. Now let $r \in \mathbb{Z}_p^*$ be such that $r^2 \equiv_p a$.

Since g is a generator of $\mathbb{Z}_p^*$, then every element in $\mathbb{Z}_p^*$ can be represented as a power of g. Therefore, there exists some $y \in \mathbb{Z}_{p-1}$ such that $r = g^y$.

Substituting this into the square root equation:

$$a \equiv_p r^2 \equiv_p (g^y)^2 \equiv_p g^{2y}. \tag{3}$$

Also, since we know that $a = g^x$, then that also means:

$$g^x \equiv_p g^{2y}. \tag{4}$$

Since g has multiplicative order $p-1$ in $\mathbb{Z}_p^*$, then we have:

$$x \equiv 2y \pmod{p-1}. \tag{5}$$

This ultimately means that there exists some integer k such that:

$$x = 2y + k(p-1). \tag{6}$$

Now observe:

- $2y$ is even (it is a multiple of 2).
- $p-1$ is even (since $p > 2$ is an odd prime).
- Therefore $k(p-1)$ is even for any int k.

And since x is the sum of two even numbers, x must be even. QED.

It'd be nice to check if a value is a square modulo p without having to explicitly know what power of g it is. Luckily we have the following test: a is a square iff $a^{(p-1)/2} \equiv_p 1$.

(b) Prove the forward direction: if a is square then $a^{(p-1)/2} \equiv_p 1$.

Solution. Like again, these are one of those convoluted math-substitution problems, where I just gotta go back to the rules to solve it. But I think this particular question does rely on Fermat's little theorem.

Being by assuming that a is a square modulo p. Theb by its definition, there exists some $r \in \mathbb{Z}_p^*$ such that $r^2 \equiv_p a$.

Like I said before, every element can be written as a power of g. Thus, $\exists y \in \mathbb{Z}_{p-1}, r = g^y$.

Now by subbing this into the equation for a, we have:

$$a \equiv_p r^2 \equiv_p (g^y)^2 \equiv_p g^{2y}. \tag{7}$$

Calcuating $a^{(p-1)/2}$:

$$a^{(p-1)/2} \equiv_p (g^{2y})^{(p-1)/2} \equiv_p g^{2y \cdot \frac{p-1}{2}} \equiv_p g^{y(p-1)}. \tag{8}$$

And now, by Fermat's Little Theorem, since $g \in \mathbb{Z}_p^*$, then it must be the case that $g^{p-1} \equiv_p 1$. Therefore:

$$g^{y(p-1)} \equiv_p (g^{p-1})^y \equiv_p 1^y \equiv_p 1. \tag{9}$$

Thus, I have shwon that if a is a square, then $a^{(p-1)/2} \equiv_p 1$.

(c) Now prove the converse: if $a^{(p-1)/2} \equiv_p 1$, then a is square. *(Hint: Assume not and find a contradiction. Begin by writing a as g^x.)*

Solution. The solution to this problem will be, like the hint says, doing via a contradiction. Basically, assume that $a^{(p-1)/2} \pmod 1$, and eventually it'll become clear that this can't be.

Assume that $a^{(p-1)/2} \equiv_p 1$ but that$a$ is **not** a square modulo p.

Because of the property of g, we can write $a = g^x$ for some $x \in \mathbb{Z}_{p-1}$.

By (a), we alreay know that that a is a square IFF x is even. Since I assumed a is not a square, then x must be odd. Therefore, I can write $x = 2k+1$ for some int k.

Now calculate $a^{(p-1)/2}$:

$$a^{(p-1)/2} \equiv_p (g^x)^{(p-1)/2} \equiv_p g^{x \cdot \frac{p-1}{2}} \equiv_p g^{(2k+1) \cdot \frac{p-1}{2}}. \tag{10}$$

Expanding the exponent:

$$g^{(2k+1) \cdot \frac{p-1}{2}} \equiv_p g^{k(p-1) + \frac{p-1}{2}} \equiv_p g^{k(p-1)} \cdot g^{(p-1)/2}. \tag{11}$$

And now, by Fermat's Little Theorem:

$$a^{(p-1)/2} \equiv_p 1 \cdot g^{(p-1)/2} \equiv_p g^{(p-1)/2}. \tag{12}$$

However, since g has multiplicative order $p-1$ in $\mathbb{Z}_p^*$, then it is also the case that $g^m \equiv_p 1$ IFF $(p-1) \mid m$. And since $0 < (p-1)/2 < p-1$, then we get:

$$g^{(p-1)/2} \not\equiv_p 1. \tag{13}$$

This contradicts the initial assumption that $a^{(p-1)/2} \equiv_p 1$. Therefore, it must be false.

Thus, if $a^{(p-1)/2} \equiv_p 1$, then a is a square modulo p.

We have thus shown that exactly half the values in $\mathbb{Z}_p^*$ have square roots, and we know how to identify them: by raising to $(p-1)/2$. Note also that values that do have square roots have at least two of them: if $r \in \mathbb{Z}_p^*$ is a square root of $a \in \mathbb{Z}_p^*$, then so is $-r$, because $(-r)^2 = (-1) \cdot r \cdot (-1) \cdot r = (-1)(-1)r^2 = 1 \cdot r^2 = r^2$. The two square roots are different, because $p - r \neq r$ as p is odd. Therefore, each square cannot have more than two square roots by a simple counting argument: if some square had more than two square roots, there wouldn't be enough square roots for all the $(p-1)/2$ squares, because just two square roots per square already takes up all the $p-1$ possible square root values in $\mathbb{Z}_p^*$. Thus, each square has exactly two square roots.

(d) Show that if $(g^x)^2 \equiv_p a$, then $\left(g^{x+(p-1)/2}\right)^2 \equiv_p a$, as well. Show that these two square roots are distinct: that is, show that $g^x \not\equiv_p g^{x+(p-1)/2}$.

Solution. In order to prove this, I will essentially have to do 2 things. Firstly, I'll nee to verify the 2nd root. And then, I'll need to prove its distinctness.

I'll firstly prove that $g^{x+(p-1)/2}$ is also a square root of a. By computing the square of this value:

$$\left(g^{x+(p-1)/2}\right)^2 \equiv_p g^{2(x+(p-1)/2)} \equiv_p g^{2x+(p-1)}. \tag{14}$$

By the laws of exponents:

$$g^{2x+(p-1)} \equiv_p g^{2x} \cdot g^{p-1} \equiv_p (g^x)^2 \cdot g^{p-1}. \tag{15}$$

And by Fermat's Little Theorem, since $g \in \mathbb{Z}_p^*$, then $g^{p-1} \equiv_p 1$. Therefore:

$$(g^x)^2 \cdot g^{p-1} \equiv_p (g^x)^2 \cdot 1 \equiv_p a. \tag{16}$$

Thus, $\left(g^{x+(p-1)/2}\right)^2 \equiv_p a$.

Now, I'll show how these two square roots are distinct. Assume, for the sake of contradiction, that they are in fact the same:

$$g^x \equiv_p g^{x+(p-1)/2}. \tag{17}$$

Now, since $g^x \in \mathbb{Z}_p^*$, it has a multiplicative inverse modulo p. By multiplying both sides by $(g^x)^{-1}$, we get:

$$1 \equiv_p g^{(p-1)/2}. \tag{18}$$

But remember how g is a generator of $\mathbb{Z}_p^*$ with multiplicative order $p-1$. THen by the defintion of its order, $g^k \equiv_p 1$ IFF $(p-1) \mid k$. And since $p > 2$, we know that $0 < (p-1)/2 < p-1$, meaning that

$(p-1)$ does not divide $(p-1)/2$. Thusly:

$$g^{(p-1)/2} \not\equiv_p 1. \tag{19}$$

And this is where the contradiction occurs. That then means, ultimately, that the two square roots must be distinct:

$$g^x \not\equiv_p g^{x+(p-1)/2}. \tag{20}$$

We know from the paragraph above that a has only two square roots. But we have three values that all when squared give us a: g^x, $g^{x+(p-1)/2}$, and $-g^x$. Thus, it must be that $g^{x+(p-1)/2} \equiv_p -g^x$.

(e) Given that $g^{x+(p-1)/2} \equiv_p -g^x$, show that that $g^{(p-1)/2} \equiv_p -1$. Now show that for any $b \in \mathbb{Z}_p^*$ that is a non-square, $b^{(p-1)/2} \equiv_p -1$.

This refines our previous test for squares from parts (b) and (c): to test if something is a square, you raise it to $(p-1)/2$ modulo p and check if the result is 1 or -1. By the way, the value of $a^{(p-1)/2} \,\%\, p$ is called the Legendre symbol of a and is often written as $\left(\frac{a}{p}\right)$. The Legendre symbol can be generalized for composite p and this generalization is called the Jacobi symbol (we will not cover it here).

Solution. Now in order to prove this, this also involves 2 parts. The first is showing how $g^{(p-1)/2} \equiv_p -1$. Then, I'll have to show how $b^{(p-1)/2} \equiv_p -1$ for non-squares specifically.

Getting $g^{(p-1)/2} \equiv_p -1$

Ok so we start off knowing that $g^{x+(p-1)/2} \equiv_p -g^x$. Then, by the laws of exponents, I can rewrite the left-hand side to be the following:

$$g^{x+(p-1)/2} \equiv_p g^x \cdot g^{(p-1)/2}. \tag{21}$$

Now subbing this into the given relation:

$$g^x \cdot g^{(p-1)/2} \equiv_p -g^x. \tag{22}$$

And by the properties of g, it is a fact that $g^x \in \mathbb{Z}_p^*$, which means g^x has a multiplicative inverse modulo p. So then, by multiplying both sides by $(g^x)^{-1}$:

$$g^{(p-1)/2} \equiv_p -1. \tag{23}$$

$b^{(p-1)/2} \equiv_p -1$ for any non-square b.

Let $b \in \mathbb{Z}_p^*$ be a non-square. Now by the properties of g, we can write $b = g^y$ for some $y \in \mathbb{Z}_{p-1}$.

By (a), I alr know that b is a square IFF y is even. And since b is a non-square, y must be odd. So I'll just write $y = 2k+1$ for some int k.

Now determine $b^{(p-1)/2}$:

$$b^{(p-1)/2} \equiv_p (g^y)^{(p-1)/2} \equiv_p g^{y \cdot \frac{p-1}{2}} \equiv_p g^{(2k+1)\cdot\frac{p-1}{2}}. \tag{24}$$

Expand the exponent:

$$g^{(2k+1)\cdot\frac{p-1}{2}} \equiv_p g^{k(p-1)+\frac{p-1}{2}} \equiv_p g^{k(p-1)} \cdot g^{(p-1)/2}. \tag{25}$$

And by Fermat's Little Theorem, we get $g^{p-1} \equiv_p 1$:

$$g^{k(p-1)} \equiv_p (g^{p-1})^k \equiv_p 1^k \equiv_p 1. \tag{26}$$

Subbing this back:

$$b^{(p-1)/2} \equiv_p 1 \cdot g^{(p-1)/2} \equiv_p g^{(p-1)/2}. \tag{27}$$

And from the first part, we showed that $g^{(p-1)/2} \equiv_p -1$. So then therefore:

$$b^{(p-1)/2} \equiv_p -1. \tag{28}$$

This proves how, for any $a \in \mathbb{Z}_p^*$,

- If a is a square, then $a^{(p-1)/2} \equiv_p 1$ (from (b)).
- And how, if a is a non-square, then $a^{(p-1)/2} \equiv_p -1$.