954:596 Regression & Time Series Analysis
Homework #1

Omikumar Bhavinkumar Makadia
om232@scarletmail.rutgers.edu

September 15, 2024

Problem 3. (25 points)

Let X and Y be two random variables and assume that the conditional expectation of Y given X is linear in X; i.e.,

$$E(Y|X) = \alpha + \beta X$$

1. (5 pts) Show that the model above can be equivalently expressed as

$$Y = \alpha + \beta X + \epsilon$$

where ϵ satisfies $E[\epsilon|X] = 0$.

2. (10 pts) Solve for the constants α and β in terms of $E(X)$, $E(Y)$, $\text{Cov}(X,Y)$, and $\text{Var}(X)$.

3. (10 pts) Now suppose that the linear model does not necessarily hold. So, we want to allow for

$$E[Y|X] \neq \alpha + \beta X$$

In terms of $E(X)$, $\text{Cov}(X,Y)$, and $\text{Var}(X)$, what are the values of α and β that minimize

$$E\{(Y - (\alpha + \beta X))^2\}?$$

Can you intuitively describe what we find by minimizing this expression?

Solution :

1:

We know the conditional expectation:

$$E(Y|X) = \alpha + \beta X \tag{1}$$

Error term is defined as :

$$\varepsilon = Y - E(Y|X) \tag{2}$$

Using (1) in (2)

$$\varepsilon = Y - (\alpha + \beta X)$$

$$\boxed{Y = \alpha + \beta X + \varepsilon \qquad (3)}$$

Using (3) we can derive

$$E[\epsilon|X] = E[Y - \alpha - \beta X|X]$$

$$E[\epsilon|X] = E[Y|X] - \alpha - \beta X$$

$$E[\epsilon|X] = \alpha + \beta X - \alpha - \beta X$$

$$\boxed{E[\epsilon|X] = 0.}$$

where ε satisfies $\text{E}(\varepsilon|X) = 0$

$$E[Y] = \alpha + \beta E[X] \tag{4}$$

2:

We will first multiply both sides of the equation $Y = \alpha + \beta X + \epsilon$ by X and take the expectation.

$$E[XY] = E[\alpha X + \beta X^2 + \epsilon X]$$

$$E[XY] = \alpha E[X] + \beta E[X^2] + E[\epsilon X]$$

Now we will use the law of iterated expectations

$$E[\epsilon X] = E[E[\epsilon X|X]]$$

$$E[\epsilon X] = E[XE[\epsilon|X]]$$

$$E[\epsilon X] = 0$$

$$E[XY] = \alpha E[X] + \beta E[X^2]$$

Now we will use the definition of covariance to solve for β

$$Cov(X, Y) = E[XY] - E[X]E[Y]$$

$$Cov(X, Y) = \alpha E[X] + \beta E[X^2] - E[X]E[Y]$$

Now using (4) we get,

$$Cov(X, Y) = \beta(E[X^2] - E[X]^2)$$

$$\boxed{\beta = \frac{\text{Cov}(X, Y)}{\text{Var}(X)}}$$

$$\boxed{\alpha = E[Y] - \frac{\text{Cov}(X, Y)}{\text{Var}(X)} E[X]}$$

3:

To find the β minimum value of the term we need to take the derivative first with respect to β and then equate it to 0

$$\frac{\partial}{\partial \beta} E\left\{(Y-\alpha-\beta X)^2\right\} = -2E\left[(Y-\alpha-\beta X)X\right]$$

$$E\left[(Y-\alpha-\beta X)X\right] = 0$$

$$E(YX) - \alpha E(X) - \beta E(X^2) = 0$$

$$\beta = \frac{E(YX) - \alpha E(X)}{E(X^2)}$$

$$\beta = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$$

To find the α minimum value of the term we need to take the derivative first with respect to α and then equate it to 0

$$\frac{\partial}{\partial \alpha} E\left\{(Y-\alpha-\beta X)^2\right\} = -2E\left[(Y-\alpha-\beta X)\right]$$

$$E(Y-\alpha-\beta X) = 0$$

$$\alpha = E(Y) - \beta E(X)$$

$$\boxed{\beta = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}}$$

$$\boxed{\alpha = E[Y] - \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} E[X]}$$

We conclude that the best linear approximation to the relationship is the line that best fits the data in terms of least square.