Solution Q1

From $f(y|\theta) = \frac{1}{\theta}$ for $y = 1, ..., \theta$ and $\pi(\theta) = \frac{1}{5}$ for $\theta = 1, ..., 5$, the posterior function is obtained by

$$\pi(\theta|y=3) = \frac{\pi(3)P(y=3|\theta)}{m(3)} = \begin{cases} 20/47 & \theta = 3 \\ 15/47 & \theta = 4 \\ 12/47 & \theta = 5 \end{cases}$$

For example,

$$\pi(\theta=3|y=3) = \frac{\pi(3)P(y=3|\theta=3)}{m(3)} = \frac{1/3 \times 1/5}{\sum_{\theta=3}^{5} \frac{1}{5\theta}} = \frac{1/15}{1/5 \times (1/3 + 1/4 + 1/5)} = \frac{20}{47}$$

Now, let t be the number on the next bus that you happen to see in the town. Then

$$q(t|y,\theta) = \frac{1}{\theta} \quad \text{for} \quad t = 1, ..., \theta$$

or alternatively, $q(t|y,\theta) = I(t \le \theta)/\theta$, $t = 1, 2, ...$
The posterior predictive density of t then becomes

$$q(t|y) = \sum_{\theta} q(t|y,\theta)\pi(\theta|y) = \sum_{\theta=y}^{5} \frac{I(t \le \theta)}{\theta}\pi(\theta|y)$$

which leads to

$$q(t|y=3) = \begin{cases} \frac{1}{3} \times \frac{20}{47} + \frac{1}{4} \times \frac{15}{47} + \frac{1}{5} \times \frac{12}{47} = 0.2727 & t = 1 \\ \frac{1}{3} \times \frac{20}{47} + \frac{1}{4} \times \frac{15}{47} + \frac{1}{5} \times \frac{12}{47} = 0.2727 & t = 2 \\ \frac{1}{3} \times \frac{20}{47} + \frac{1}{4} \times \frac{15}{47} + \frac{1}{5} \times \frac{12}{47} = 0.2727 & t = 3 \\ \frac{1}{4} \times \frac{15}{47} + \frac{1}{5} \times \frac{12}{47} = 0.13085 & t = 4 \\ \frac{1}{5} \times \frac{12}{47} = 0.05106 & t = 5 \end{cases}$$

That is, for $y = 3$, we have

$$q(t|y=3) = \begin{cases} 0.2727 & t = 1, 2, 3 \\ 0.13085 & t = 4 \\ 0.05106 & t = 5 \end{cases}$$

(b) The probability that the next bus you see will have a number on it which is at least 4 equals

$$P(t \ge 4|y=3) = q(t=4|y=3) + q(t=5|y=3) = 0.13085 + 0.05106 = 0.182$$

(b) The expected value of the bus number you will next see is

$$E(T|y) = 1 \times 0.2727 + 2 \times 0.2727 + 3 \times 0.2727 + 4 \times 0.13085 + 5 \times 0.05106 = 2.4149$$

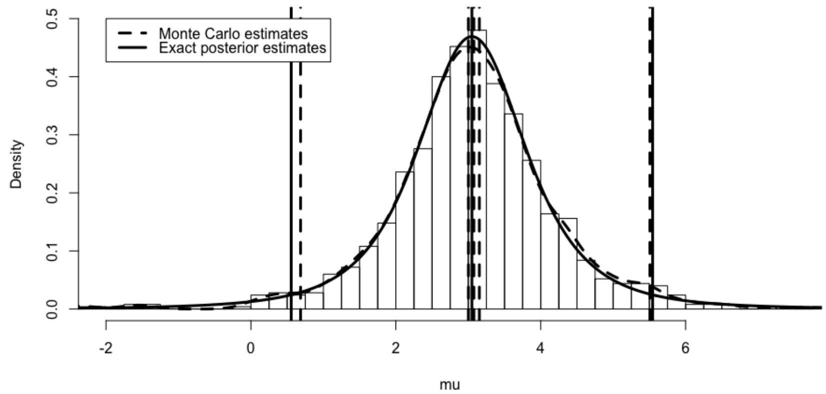

Figure 1: Histograms of the Generated Data from the Posterior Distribution.

Solution Q2

We generate $w_1, ..., w_J \sim t_{(n-1=3)}$ and then calculate $\mu_j = \bar{y} + \frac{S}{\sqrt{4}} w_j$ for $j = 1, ..., J = 1000$. We then use the sample $\mu_1, ..., \mu_J \sim \pi(\mu|y)$ for M.C inference about μ.

(a) We estimate μ's posterior mean $\hat{\mu} = E(\mu|y)$ by $\bar{\mu} = 3.077$ with $(3.001, 3.153)$ as the 95% M.C Confidence Interval for $\hat{\mu}$ from $\bar{y} \pm z_{0.025} \frac{S}{\sqrt{n}}$. The M.C estimate of μ's 95% CPDR is (0.685, 5.507) using the quantiles of the generated data at 0.025 and 0.975.

(b) We compare the above estimates with the true values: $\hat{\mu} = \bar{y} = 3.050$. 95% HPD interval for μ is $\bar{y} \pm t_{(n-1=3,0.025)} \frac{S}{\sqrt{4}} = (0.556, 5.544)$

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y=c(2.1, 3.2, 5.2, 1.7); n=length(y); ybar=mean(y); s=sd(y); s # 1.567
J=1000; set.seed(144); options(digits=4)
wv=rt(J,n-1); muv=ybar+s*wv/sqrt(n)
mubar=mean(muv); muci=mubar + c(-1,1)*qnorm(0.975)*sd(muv)/sqrt(J)
mucpdr=quantile(muv,c(0.025,0.975))
c(mubar,muci,mucpdr)  # 3.0770 3.0012 3.1528 0.6848 5.5069
muhat=ybar;  mucpdrtrue= ybar+(s/sqrt(n))*qt(c(0.025,0.975),n-1)
c(muhat,mucpdrtrue) # 3.050 0.556 5.544
hist(muv,prob=T,xlab="mu",xlim=c(-2,7.5), ylim=c(0,0.5),main="", breaks=seq(-20,20,0.25))
muvec=seq(-20,20,0.01); postvec=dt( (muvec-ybar)/(s/sqrt(n)),n-1)/(s/sqrt(n))
lines(muvec,postvec, lty=1,lwd=3)
lines(density(muv),lty=2,lwd=3)
abline(v=c(mubar,muci,mucpdr),lty=2,lwd=3)
abline(v=c(ybar, mucpdrtrue) , lty=1,lwd=3)
legend(-2,0.5,c("Monte Carlo estimates","Exact posterior estimates"),lty=c(2,1),lwd=c(3,3),bg="white")
```

Solution Q3

The density function is: $f(y|\mu,\lambda) = \frac{1}{\sqrt{2\pi/\lambda}} e^{-\frac{\lambda}{2}(y-\mu)^2}$ for $\mu \in \mathbb{R}$ and $\lambda > 0$. Taking the natural log function,

$$\log f(y|\mu,\lambda) = -\frac{1}{2}\log(\lambda) - \frac{\lambda}{2}(y-\mu)^2$$

It follows that the first partial derivative with respect to λ is $\frac{\partial \log f(y|\mu,\lambda)}{\partial \lambda} = \frac{1}{2\lambda} - \frac{1}{2}(y-\mu)^2$ and the second derivative with respect to λ is $\frac{\partial^2 \log f(y|\mu,\lambda)}{\partial \lambda^2} = -\frac{1}{2\lambda^2}$.

The Fisher information is then obtained by

$$I_{\mathbf{y}}(\lambda) = nI_{y_i}(\lambda) = nE\left[-\frac{\partial^2 \log f(y|\mu,\lambda)}{\partial \lambda^2}\right] = \frac{n}{2\lambda^2}$$

Therefore, the Jeffrey's prior becomes

$$\pi(\lambda) \propto \sqrt{I_{\mathbf{y}}(\lambda)} = \sqrt{\frac{n}{2\lambda^2}} \propto \frac{1}{\lambda}, \quad \lambda > 0$$

That is, the prior distribution is improper or non-informative.

Solution Q4

The null and alternative hypotheses are $H_0 : p_1 - p_2 = 0$ vs. $H_a : p_1 - p_2 > 0$, where p_1 and p_2 correspond to normal cell rates for cells treated with 0.6 and 0.7 respectively. concentrations of actinomycin D.

(a) Using the sample proportions 0.786 and .329, under H_0, the pooled estimate of p becomes

$$\hat{p}_{pooled} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2} = \frac{\hat{p}_1 + \hat{p}_2}{2} = \frac{0.786 + 0.329}{2} = 0.558$$

the test statistic is then obtained by

$$z = \frac{0.786 - 0.329}{\sqrt{0.558 \times 0.442 \times (2/70)}} = 5.444$$

Since the observed test statistic $z = 5.444$ is larger than z_α, it indicates that the rate of normal RNA synthesis is lower for cells exposed to the higher concentrations of actinomycin D at the level $\alpha = 0.05$.

(b) The p-value is $P(Z > 5.444) \approx 0$.

(c) Since the p-value is less than 0.05, we reject H_0 and conclude that the normal cell rate is lower for cells exposed to the higher actinomycin D concentration.

Solution Q5

(a) Define μ = mean trap weight. The sample statistics are $\bar{y} = 28.935$, $S = 9.507$. To test $H_0 : \mu = 30.31$ vs. $H_a : \mu < 30.31$, $t = -0.647$ with 19 degrees of freedom. With $\alpha = 0.05$, the critical value is $t_{0.05} = -1.729$. So we fail to reject H_0; that is we cannot conclude that the mean trap weight has decreased. Because $t = -0.647$ is Not smaller than $t_{0.05} = -1.729$. We conclude that there is not enough evidence to reject H_0, there is Not sufficient evidence to support the contention that the mean landings per trap has decreased since imposition of the Bahamian restrictions.
Alternatively, the p-value is obtained $P(T < -0.647) = 0.263$ where $T \sim t_{19}$ which leads to the same conclusion.

Solution Q6

(a) The most powerful test for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 = \sigma_1^2$, $\sigma_1^2 > \sigma_0^2$ is based on the Neyman-Perarson Lemma is

$$\frac{L(\sigma_0^2)}{L(\sigma_1^2)} = \left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum\limits_{i=1}^{n}(y_i-\mu)^2} < k$$

which is equivalent to $T = \sum\limits_{i=1}^{n}(y_i - \mu)^2 > c$ since $\sigma_1^2 > \sigma_0^2$. Therefore, we should reject if the statistic T is large. To find a rejection region of size α, note that $\frac{\sum_{i=1}^{n}(y_i-\mu)^2}{\sigma_0^2}$ has a chi-square distribution with n degrees of freedom. Thus, the most powerful test is equivalent to the chi-square test.

(b) Yes, the test is uniformly most powerful test since the RR is the same for any $\sigma_1^2 > \sigma_0^2$.

Solution Q7

(a) The power function is given by

$$1 - \beta = P(\text{Reject } H_0 \mid H_0 \text{ is false}) = P(Y > 0.5|\theta = \theta_a) = \int_{0.5}^{1} \theta_a y^{\theta_a - 1} dy = 1 - 0.5^{\theta_a}, \quad \theta_a > 0$$

(b) To test $H_0 : \theta = 1$ vs. $H_a : \theta = \theta_a$, $1 < \theta_a$, the likelihood ratio is : $\frac{L(1)}{L(\theta_a)} = \frac{1}{\theta_a y^{\theta_a - 1}} < k$.
which implies that

$$y > \left(\frac{1}{\theta_a k}\right)^{\frac{1}{\theta_a - 1}} = c$$

where c is chosen so that the test is of size α. This is given by

$$P(Y \geq c|\theta = 1) = \int_{c}^{1} dy = 1 - c = \alpha$$

so that $c = 1$–α. Since the RR does not depend on a specific $\theta_a > 1$, it is uniformly most powerful.

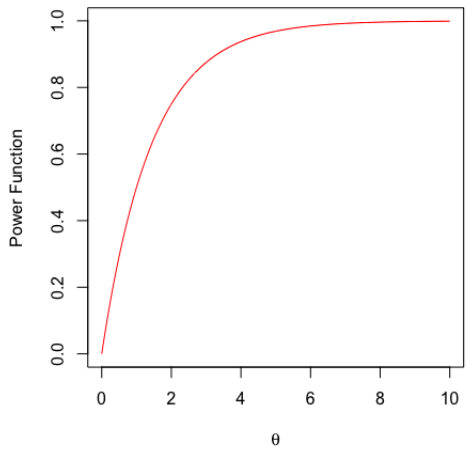

Figure 2: Power Function Curve for the Rejection Region $Y > 0.5$.

Solution Q8

(a) The hypothesis of interest is $H_0 : p_1 = p_2 = p_3 = p_4 = p$ versus $H_a : p_i \neq p_j$ for $i \neq j$.
The likelihood function is

$$L(p_1, p_2, p_3, p_4) = \prod_{i=1}^{4} \begin{pmatrix} 200 \\ y_i \end{pmatrix} {p_i}^{y_i}(1 - p_i)^{200-y_i} \quad for\ y_i = 0, ..., 200$$

Taking the natural log function, differentiating with respect to p (under H_0), the MLE of p becomes $\hat{p} = \frac{\sum_{i=1}^{4} y_i}{800}$. On the other hand, maximizing under Θ ($p_i \neq p_j$ for $i \neq j$), $\hat{p}_i = \frac{y_i}{200}$ for $i = 1, 2, 3, 4$. Then, the likelihood ratio becomes

$$\lambda = \frac{\prod_{i=1}^{4} \begin{pmatrix} 200 \\ y_i \end{pmatrix} \left(\frac{\sum_{i=1}^{4} y_i}{800}\right)^{y_i} \left(1 - \frac{\sum_{i=1}^{4} y_i}{800}\right)^{200-y_i}}{\prod_{i=1}^{4} \begin{pmatrix} 200 \\ y_i \end{pmatrix} \left(\frac{y_i}{200}\right)^{y_i} \left(1 - \frac{y_i}{200}\right)^{200-y_i}}$$

The sample sizes are large, by the asymptotic distribution of the likelihood ratio, $-2ln\lambda$ is approximately chi-square distribution with 3 ($\{r_0 = 1, r = 4\}$) degrees of freedom.
In the Table, $y_1 = 76$, $y_2 = 53$, $y_3 = 59$, and $y_4 = 48$ which gives $-2ln\lambda = 10.54$. Thus, we reject H_0 since $-2ln\lambda > \mathcal{X}^2_{(3,0.05)} = 7.81$. That is, the fractions of voters favouring candidate A are not the same in all four wards at the level $\alpha = 0.05$.