MATH 141 — Solution practice problems 6.

Problem 1)

a) $$\int_0^{\frac{\pi}{2}} \sin^7\theta \cos^5\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^7\theta (\cos^2\theta)^2 \cos\theta \, d\theta = \int_0^1 u^7(1-u^2)^2 du$$

$\sin\theta = u \Rightarrow \cos\theta \, d\theta = du$

$$= \int_0^1 u^7(1+u^4-2u^2) du$$

$$= \frac{1}{8} + \frac{1}{12} - \frac{2}{10} = \frac{15+10-24}{120} = \frac{1}{120}$$

b) $$\int \frac{\sin^3(\sqrt{x})}{\sqrt{x}} dx = 2\int \frac{\sin^3(\sqrt{x})}{2\sqrt{x}} dx = 2\int \sin^3(u) du$$

$\sqrt{x} = u \Rightarrow \frac{dx}{2\sqrt{x}} = du$

$$= 2\int \sin^2 u \sin u \, du = 2\int (1-\cos^2 u) \sin u \, du$$

$\cos u = z \Rightarrow \sin u \, du = dz$

$$= 2\int (1-z^2) dz = 2(z - \frac{z^3}{3}) + c$$

$$= 2(\cos u - \frac{\cos^3 u}{3}) + c \quad = 2(\cos\sqrt{x} - \frac{\cos^3\sqrt{x}}{\sqrt{x}})$$

c) $$\int_0^{\pi/4} \cos^4(2t) dt = \int_0^{\pi/4} \left(\frac{1+\cos 4t}{2}\right)^2 dt = \frac{1}{4}\int_0^{\frac{\pi}{4}} (1 + 2\cos 4t + \cos^2 4t) dt$$

$$= \frac{1}{4}\left(\frac{\pi}{4} + \frac{2}{4}\sin 4t \Big|_0^{\pi/4} + \int_0^{\pi/4} \frac{1+\cos 8t}{2} dt \right)$$

$$= \frac{1}{4}\left[\frac{\pi}{4} + 0 + \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{16}\sin 8t \Big|_0^{\frac{\pi}{4}}\right)\right] = \frac{1}{4}\left[\frac{\pi}{4} + \frac{\pi}{8} + \frac{1}{32}\right]$$

Problem 2

a) $\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos u\, du}{4\sin^2 u\sqrt{4-4\sin^2 u}} = \int \frac{2\cos u\, du}{(4\sin^2 u)(\cancel{2\cos u})}$

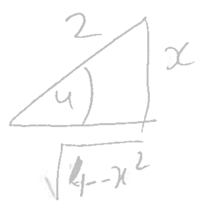
if $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$

$x = 2\sin u \Rightarrow dx = 2\cos u\, du$

$\sin u = \frac{x}{2}$

$= \int \frac{du}{4\sin^2 u} = \frac{1}{4}\int \csc^2 u\, du = -\frac{1}{4}\cot u + C$

$= -\frac{\sqrt{4-x^2}}{4x} + C$



$\rightarrow \cot u = \frac{\sqrt{4-x^2}}{x}$

b) $\int \frac{x^3\, dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3 u)(2\sec^2 u\, du)}{\sqrt{4\tan^2 u + 4}} = \frac{16}{2}\int \frac{\tan^3 u \sec^2 u}{|\sec u|}\, du$

$x = 2\tan u \Rightarrow dx = 2\sec^2 u\, du$

if $-\frac{\pi}{2} \le u \le \frac{\pi}{2} \rightarrow \sec u > 0$

$= 8\int \tan^3 u \sec u\, du$

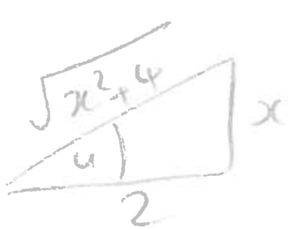
$= 8\int \tan^2 u \tan u \sec u\, du = 8\int (\sec^2 u - 1)\sec u \tan u\, du$

$\sec u = z \Rightarrow \sec u \tan u\, du = dz$

$= 8\int (z^2 - 1)\, dz$

$= 8\left(\frac{z^3}{3} - z\right) + C$

$= 8\left(\frac{\sec^3 u}{3} - \sec u\right) + C$



$\tan u = \frac{x}{2} \Rightarrow \sec u = \frac{\sqrt{x^2+4}}{2}$

$= 8\left(\frac{(x^2+4)^{3/2}}{24} - \frac{\sqrt{x^2+4}}{2}\right) + C$

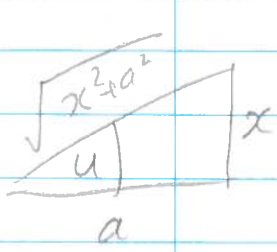
problem 3

Right hand side $\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2 u\, du}{\sqrt{a^2\tan^2 u + a^2}} = \int \frac{\sec^2 u\, du}{|\sec u|}$

$x = a\tan u$ if $\frac{-\pi}{2} \le u \le \frac{\pi}{2}$

$dx = a\sec^2 u\, du$

$= \int \sec u\, du = \ln|\sec u + \tan u| + C_1$



$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$

$= \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + C_1$

$= \ln\left|\sqrt{x^2+a^2}+x\right| - \ln|a| + C_1$

$= \ln(x + \sqrt{x^2+a^2}) + C$

problem (4)

a) $\int \frac{y}{(y+4)(2y-1)} dy = \int \frac{A}{y+4} dy + \int \frac{B}{2y-1} dy$

$A(2y-1) + B(y+4) = y \Rightarrow \begin{cases} 2A + B = 1 \Rightarrow \boxed{B = 1/9} \\ -A + 4B = 0 \Rightarrow \boxed{A = 4B} \end{cases}$

$\boxed{A = 4/9}$

$= \int \frac{4/9}{y+4} dy + \int \frac{1/9}{2y-1} dy$

$= \frac{4}{9} \ln|y+4| + \frac{1}{18} \ln\left|y - \frac{1}{2}\right| + C$

b) $\int_0^1 \frac{2}{2x^2+3x+1}\,dx$

First we find its indefinite integral:

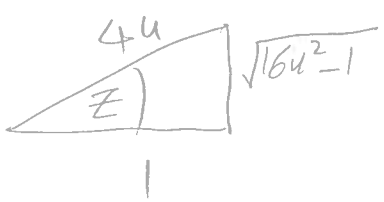
$$\int \frac{2}{2x^2+3x+1}\,dx = \int \frac{dx}{x^2+\frac{3}{2}x+\frac{1}{2}} = \int \frac{dx}{\left(x+\frac{3}{4}\right)^2-\frac{9}{16}+\frac{1}{2}}$$

$$= \int \frac{dx}{\left(x+\frac{3}{4}\right)^2-\frac{1}{16}} = \int \frac{du}{u^2-\frac{1}{16}} = \int \frac{\frac{1}{4}\sec z \tan z\,dz}{\frac{1}{16}\sec^2 z-\frac{1}{16}}$$

$$= 4\int \frac{\sec z \tan z}{\tan^2 z}\,dz = 4\int \frac{\sec z}{\tan z}\,dz$$

$x+\frac{3}{4} = u \Rightarrow dx = du$

$u = \frac{1}{4}\sec z \Rightarrow du = \frac{1}{4}\sec z \tan z\,dz$



$$= 4\int \frac{\frac{1}{\cos z}}{\frac{\sin z}{\cos z}}\,dz = 4\int \frac{1}{\sin z}\,dz = 4\int \csc z\,dz$$

$$= -4\ln|\csc z + \cot z| + C$$

$$= -4\ln\left|\frac{4u}{\sqrt{16u^2-1}} + \frac{1}{\sqrt{16u^2-1}}\right| + C$$

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx = -4\ln\left|\frac{4\left(x+\frac{3}{4}\right)}{\sqrt{16\left(x+\frac{3}{4}\right)^2-1}} + \frac{1}{\sqrt{16\left(x+\frac{3}{4}\right)^2-1}}\right| \Bigg|_0^1$$

$=$

problem 5)

a) $\int \frac{dx}{2\sqrt{x+3}+x} = \int \frac{2u\,du}{2u+u^2-3} = \int \frac{2u\,du}{(u+1)^2-1-3} = \int \frac{2u\,du}{(u+1)^2-4}$

$\sqrt{x+3} = u \Rightarrow x+3 = u^2$
$\Rightarrow dx = 2u\,du$

$= \int \frac{2u\,du}{(u+1-2)(u+1+2)} = \int \frac{2u\,du}{(u-1)(u+3)}$

$= \int \frac{A}{u-1}\,du + \int \frac{B\,du}{u+3}$

$\Rightarrow A(u+3)+B(u-1) = 2u \Rightarrow \begin{cases} A+B=2 \Rightarrow \boxed{A=1/2} \\ 3A-B=0 \Rightarrow B=3A \end{cases}$

$\boxed{B=3/2}$

$\int \frac{1/2}{u-1}\,du + \int \frac{3/2}{u+3}\,du = \frac{1}{2}\ln|u-1| + \frac{3}{2}\ln|u+3| + c$

$= \frac{1}{2}\ln\left|\sqrt{x+3}-1\right| + \frac{3}{2}\ln\left|\sqrt{x+3}+3\right| + C$

b) $\int_0^1 \frac{dx}{1+\sqrt[3]{x}} = \int_0^1 \frac{3u^2\,du}{1+u} = 3\int_0^1 \frac{\overbrace{u^2-1}^{(u-1)(u+1)}+1}{u+1}\,du = 3\left\{\int_0^1 \left(\frac{u^2-1}{u+1}+\frac{1}{u+1}\right)du\right\}$

$u = \sqrt[3]{x} \Rightarrow u^3 = x$
$3u^2\,du = dx$

$= 3\left\{\int_0^1 (u-1)\,du + \int_0^1 \frac{du}{u+1}\right\}$

$= 3\left\{\frac{1}{2} - 1 + \ln(u+1)\Big|_0^1\right\}$

$= 3\left\{-\frac{1}{2} + \ln 2\right\}$

problem 6

a) $\int \ln(x^2-x+2)\,dx = x\ln(x^2-x+2) - \int \frac{x(2x-1)\,dx}{x^2-x+2}$

$u = \ln(x^2-x+2)$, $dx = dv$ $\Rightarrow$ $du = \frac{2x-1}{x^2-x+2}\,dx$, $v = x$

$= x\ln(x^2-x+2) - \int \frac{2x^2-x-x+x-4+4}{x^2-x+2}\,dx$

$= x\ln(x^2-x+2) - \int \frac{\cancel{2x^2-2x+4}^{\,2}}{\cancel{x^2-x+2}}\,dx - \int \frac{x-4}{x^2-x+2}\,dx$

$= x\ln(x^2-x+2) - 2x - \frac{1}{2}\int \frac{2x-8}{x^2-x+2}\,dx$

$= x\ln(x^2-x+2) - 2x - \frac{1}{2}\int \frac{2x-1}{x^2-x+2} - \frac{1}{2}\int \frac{-7}{x^2-x+2}\,dx$

$= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln|x^2-x+2| + \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{7}{4}}$

$= \dots\dots$

problem 7) a) $\int \frac{dx}{x^2-2x} = \int \frac{dx}{x^2-2x+1-1} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$

$x-1 = u \Rightarrow dx = du$

$u = \sec z \Rightarrow du = \sec z \tan z\,dz$

$= \int \frac{\sec z \tan z\,dz}{\tan^2 z} = \int \csc z\,dz$

$= -\ln|\csc z + \cot z| + C = -\ln\left|\frac{x-1}{\sqrt{(x-1)^2+1}} + \frac{1}{\sqrt{(x-1)^2+1}}\right| +$

or use partial fraction method (easier)