Lecture 13: Point and Interval Estimation

MATH 697

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Goals for this Chapter

- The Bias and Mean Square Error of Point Estimators
- Some Common Unbiased Point Estimators
- Evaluating the Goodness of a Point Estimator
- Confidence Intervals
- Large-Sample Confidence Intervals
- Selecting the Sample Size
- Small-Sample Confidence Intervals for μ and $\mu_1 - \mu_2$
- Confidence Intervals for σ^2

Introduction

- The purpose of statistics is to use the information contained in a sample to make inferences about the population from which the sample is taken.
- Because populations are characterized by numerical descriptive measures called parameters, the objective of many statistical investigations is to estimate the value of one or more relevant parameters.

Examples

- A manufacturer of washing machines might be interested in estimating the proportion p of washers that can be expected to fail prior to the expiration of a 1-year guarantee time.
- We might wish to estimate the mean survival time (μ) of people over 65 with dementia.
- The standard deviation of the error of measurement (σ) of a new treatment.

Types of Estimate

- Suppose that we wish to estimate the average amount of mercury μ that a newly developed process can remove from 1 ounce of ore obtained at a geographic location.

- We could give our estimate in two distinct forms.
  - First, we could use a single number. This type of estimate is called a point estimate because a single value, or point, is given as the estimate of μ.
  - Second, in this estimation procedure, the two values that we give may be used to construct an interval intended to enclose the parameter of interest. Thus, the estimate is called an interval estimate.

Estimator

Definition 1

An estimator is a rule, often expressed as a formula, that tells how to calculate the value of an estimate based on the measurements contained in a sample.

- The sample mean $\bar{Y}_n$ is one possible point estimator of the population mean μ.
- The sample standard deviation $S = \sqrt{S^2} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2}$ is an estimator of the population standard deviation σ.

How to Evaluate an Estimator

- Some estimators are considered good, and others, bad. The management of a construction firm must define good and bad as they relate to the estimation of the cost of a job.
- How can we establish criteria of goodness to compare statistical estimators?
- The point is that we cannot evaluate the goodness of a point estimation procedure on the basis of the value of a single estimate; rather, we must observe the results when the estimation procedure is used many many times.

Unbiased Estimator

Definition 2

Let $\hat{\theta}$ be a point estimator for a parameter θ. Then $\hat{\theta}$ is an unbiased estimator if $E(\hat{\theta}) = \theta$. If $E(\hat{\theta}) \neq \theta$, $\hat{\theta}$ is said to be biased.

Definition 3

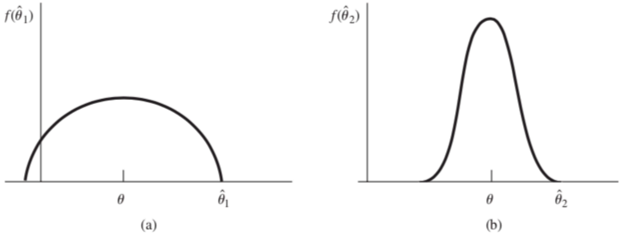
The bias of a point estimator $\hat{\theta}$ is given by $B(\hat{\theta}) = E(\hat{\theta}) - \theta$.

Prefereneces for Estimators

- We would prefer that our estimator have the type of distribution indicated in Figure because the smaller variance guarantees that in repeated sampling a higher fraction of values of $\hat{\theta}_2$ will be close to θ.
- Thus, in addition to preferring unbiasedness, we want the variance of the distribution of the estimator $V(\hat{\theta})$ to be as small as possible.
- Given two unbiased estimators of a parameter θ, and all other things being equal, we would select the estimator with the smaller variance.

Figure

FIGURE **8.3** Sampling distributions for two unbiased estimators: (a) estimator with large variation; (b) estimator with small variation



Example

Suppose that Y_1, Y_2, Y_3 denote a random sample from an $Exp(\theta)$ with density function $f(y) = \frac{1}{\theta}e^{-y/\theta}$ for $y > 0$. Consider the following estimators of θ:

$$\hat{\theta}_1 = Y_1 \qquad \hat{\theta}_2 = \frac{Y_1 + Y_2}{2}$$

$$\hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}, \qquad \hat{\theta}_4 = \min(Y_1, Y_2, Y_3), \qquad \hat{\theta}_5 = \bar{Y}$$

- Which estimators are unbiased ?
- solution: It easy to see that

$$E(\hat{\theta}_1) = E(\hat{\theta}_2) = E(\hat{\theta}_3) = E(\hat{\theta}_5) = \theta$$

Example 1

- For $\hat{\theta}_4$, we first find the ditribution of the estimator. Let $U = \min(Y_1, Y_2, Y_3)$

$$F_U(u) = P(U \leq u) = 1 - P(\min(Y_1, Y_2, Y_3) \geq u)$$
$$= 1 - [P(Y_1 \geq u)]^3 = 1 - e^{-3y/\theta}$$

  That is, $\min(Y_1, Y_2, Y_3) \sim Exp(\theta/3)$. Thus, $E(\hat{\theta}_4) = \theta/3$

- Among the unbiased estimators, which has the smallest variance?

- solution: We see that $V(\hat{\theta}_1) = \theta^2$, $V(\hat{\theta}_2) = \theta^2/2$, $V(\hat{\theta}_3) = 5\theta^2/9$ and $V(\hat{\theta}_5) = \theta^2/3$.

- Thus, $\hat{\theta}_5 = \bar{Y}$ has the minimum variance.

Example 2

Let $Y_1, ..., Y_n$ be random sample from $U(0, \theta)$. Consider two estimators $\hat{\theta}_1 = 2\bar{Y}$ and $\hat{\theta}_2 = \frac{n+1}{n} Y_{(n)}$ where $Y_{(n)} = \max(Y_1, ..., Y_n)$

- Show that both estimators are unbiased.

- Solution $E(2\bar{Y}) = 2E(\bar{Y}) = 2E(Y) = 2\frac{\theta}{2} = \theta$. Further, define $U_i = \frac{Y_i}{\theta}$ then $U_i \sim U(0, 1)$ and

$$F_{U_{(n)}}(u) = P(U_{(n)} \leq u) = P(\max(U_1, ..., U_n) \leq u)$$
$$= [F_{U_1}(u)]^n = u^n, \qquad for \;\; 0 < u < 1$$

That is, $U_{(n)} \sim Beta(n, 1)$ or $E(U_{(n)}) = E(\frac{Y_{(n)}}{\theta}) = \frac{n}{n+1}$

- Hence, $E(\frac{n+1}{n} Y_{(n)}) = \theta$.

Example

- Which estimator has smaller variance?
- Solution Firstly, $V(2\bar{Y}) = 4\frac{V(Y)}{n} = \frac{4\theta^2}{12n} = \frac{\theta^2}{3n}$.
- Secondly,

$$
\begin{aligned}
V(\frac{n+1}{n}Y_{(n)}) &= \frac{(n+1)^2}{n^2}\frac{n\theta^2}{(n+1)^2(n+2)} \\
&= \frac{\theta^2}{n(n+2)}
\end{aligned}
$$

- Since, for $n \geq 2$, $\frac{1}{n(n+2)} < \frac{1}{3n}$, $\hat{\theta}_2$ has smaller variance.

Mean Square Error

What if we have biased estimators, How can we evaluate them?

Definition 4

The mean square error of a point estimator $\hat{\theta}$ is defined as

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2]$$

The mean square error of an estimator $\hat{\theta}$, $MSE(\hat{\theta})$, is a function of both its variance and its bias. That is

$$MSE(\hat{\theta}) = V(\hat{\theta}) + [B(\hat{\theta})]^2$$

Example

Let $Y_1, ...Y_n$ have $N(\mu, 2)$.

- Show that $\bar{Y}$ is an unbiased estimator of μ
- Solution $\bar{Y}$ is always unbiased for the population mean (by linearity of the expecation) $E(\bar{Y}) = \mu$.
- Show $\bar{Y}^2$ is a biased estimator of μ^2.
- Solution: $E(\bar{Y}^2) = V(\bar{Y}) + E^2(\bar{Y}) = \frac{2}{n} + \mu^2$. Therefore, $B(\bar{Y}^2) = \frac{2}{n}$.

Example

- Find $MSE(\bar{Y}^2)$. Solution:

$$V\left[\bar{Y}^2\right] = V\left[\left(\frac{\bar{Y}-\mu+\mu}{\sqrt{2/n}}\right)^2 \frac{2}{n}\right]$$

$$= \frac{4}{n^2} V\left[Z^2 + \frac{2\mu}{\sqrt{2/n}} Z + \left(\frac{\mu}{\sqrt{2/n}}\right)^2\right]$$

where $Z = \frac{\bar{Y}-\mu}{\sqrt{2/n}}$. Becasue $Cov(Z, Z^2) = 0$ and $V(Z^2) = 2$ as $Z^2 \sim \mathcal{X}_1^2$, we have

$$V(\bar{Y}^2) = \frac{4}{n^2}\{2 + 2n\mu^2\}$$

- Thus, $MSE(\bar{Y}^2) = \frac{4}{n^2}\{3 + 2n\mu^2\}$

Some Common Unbiased Point Estimators

- It seems natural to use the sample mean $\bar{Y}$ to estimate the population mean μ and to use the sample proportion $\hat{p} = Y/n$ to estimate a binomial parameter p.
- If an inference is to be based on independent random samples of n_1 and n_2 observations selected from two different populations, how would we estimate the difference between means $(\mu_1 - \mu_2)$ or the difference in two binomial parameters, $(p_1 - p_2)$
- The standard deviation of the sampling distribution of the estimator $\hat{\theta}$, $\sigma_{\hat{\theta}} = \sqrt{\sigma^2_{\hat{\theta}}}$ is usually called the standard error of the estimator $\hat{\theta}$.

Table 8.1

All four point estimators are unbiased and that they possess the standard deviations shown in the Table.

Table 8.1 Expected values and standard errors of some common point estimators

Target Parameter θ	Sample Size(s)	Point Estimator $\hat{\theta}$	$E(\hat{\theta})$	Standard Error $\sigma_{\hat{\theta}}$
μ	n	$\overline{Y}$	μ	$\dfrac{\sigma}{\sqrt{n}}$
p	n	$\hat{p} = \dfrac{Y}{n}$	p	$\sqrt{\dfrac{pq}{n}}$
$\mu_1 - \mu_2$	n_1 and n_2	$\overline{Y}_1 - \overline{Y}_2$	$\mu_1 - \mu_2$	$\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}^{*\dagger}$
$p_1 - p_2$	n_1 and n_2	$\hat{p}_1 - \hat{p}_2$	$p_1 - p_2$	$\sqrt{\dfrac{p_1 q_1}{n_1} + \dfrac{p_2 q_2}{n_2}}^{\dagger}$

$^*\sigma_1^2$ and σ_2^2 are the variances of populations 1 and 2, respectively.

†The two samples are assumed to be independent.

Notes

- The expected values and standard errors for $\bar{Y}$ and $\bar{Y}_1 - \bar{Y}_2$ given in the Table 8.1 are valid regardless of the distribution of the population(s) from which the sample(s) is (are) taken.
- The expected value and the standard error for $\hat{p}$ is given only if Y is binomial distribution.
- All four estimators including $\hat{p}$ possess probability distributions that are approximately normal for large samples. (Note that if the underlying distribution of the population is normal, then $\bar{Y}$ and $\bar{Y}_1 - \bar{Y}_2$ are Exactly normal distribution)

Unbiased Estimator of Population Variance

Although unbiasedness is often a desirable property for a point estimator, not all estimators are unbiased.

- Let $Y_1, Y_2, ..., Y_n$ be a random sample with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$. Show that the following estimator is biased for σ^2.

$$S'^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

- However, this estimator is unbiased.

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

- Exercise

Evaluating the Goodness of a Point Estimator

- One way to measure the goodness of any point estimation procedure is in terms of the distances between the estimates that it generates and the target parameter, which is called the error of estimation.
- Naturally, we would like the error of estimation to be as small as possible

Error of Estimation

Definition 5

The error of estimation ϵ is the distance between an estimator and its target parameter. That is, $\epsilon = |\hat{\theta} - \theta|$

- The error of estimation is a random quantity as $\hat{\theta}$ is a random quantity.
- We cannot say how large or small it will be for a particular estimate. However, we can make probability statements about it.

$$P(|\hat{\theta} - \theta| < b) = P(\theta - b < \hat{\theta} < \theta + b)$$

- We can think of b as a probabilistic bound on the error of estimation.

Error of estimation

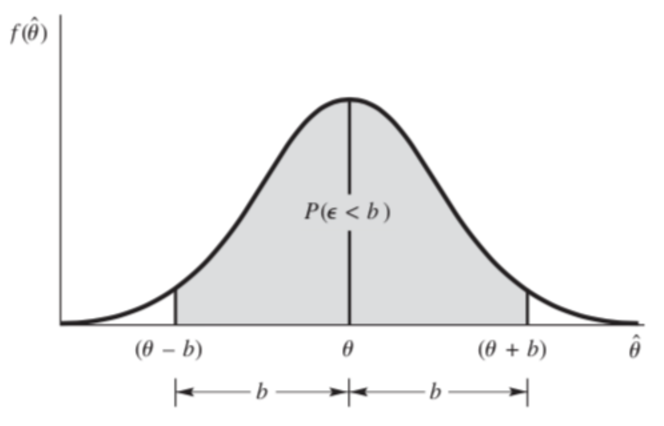
- This probability identifies the fraction of times, in repeated sampling, that the estimator $\hat{\theta}$ falls within b units of θ, the target parameter.

- Suppose that we want to find the value of b so that $P(\epsilon < b) = 0.90$. This is easy if we know the probability density function of $\hat{\theta}$. Then we seek a value b such that

$$\int_{\theta - b}^{\theta + b} f(\hat{\theta}) d\hat{\theta} = 0.90$$

- we know from Tchebysheff's theorem that ϵ will be less than $k\sigma_{\hat{\theta}}$ with probability at least $1 - 1/k^2$.

Error of estimation



Example

A sample of $n = 1000$ voters, randomly selected from a city, showed $y = 560$ in favor of candidate Jones.

- Estimate p, the fraction of voters in the population favoring a candidate, and place a 2-standard-error bound on the error of estimation.

- Solution: From Table 8.1

$$b = 2\sigma_{\hat{p}} = 2\sqrt{\frac{pq}{n}} \approx 2\sqrt{\frac{0.56 \times 0.44}{1000}} = 0.03$$

- The probability that the error of estimation is less than 0.03 is approximately 0.95. Consequently, we can be reasonably confident that our estimate, 0.56, is within 0.03 of the true value of p.

Confidence Intervals

- An interval estimator is a rule specifying the method for using the sample measurements to calculate two numbers that form the endpoints of the interval.
- Ideally, the resulting interval will have two properties:
  - First, it will contain the target parameter θ
  - Second, it will be relatively narrow.
- The objective is to find an interval estimator capable of generating narrow intervals that have a high probability of enclosing θ.

Interval Bounds

- Interval estimators are commonly called confidence intervals.
- The upper and lower endpoints of a confidence interval are called the upper and lower confidence limits, respectively.
- The probability that a (random) confidence interval will enclose θ (a fixed quantity) is called the confidence coefficient.

Two-sided Confidence Interval

- Suppose that $\hat{\theta}_L$ and $\hat{\theta}_U$ are the (random) lower and upper confidence limits, respectively, for a parameter θ. Then, if

$$P(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U) = 1 - \alpha$$

  the probability $(1 - \alpha)$ is the confidence coefficient.

- The resulting random interval defined by $[\hat{\theta}_L, \hat{\theta}_U]$ is called a two-sided confidence interval.

One-sided Confidence Interval

- It is also possible to form a one-sided confidence interval such that

$$P(\hat{\theta}_L \leq \theta) = 1 - \alpha$$

  Although only $\hat{\theta}_L$ is random in this case, the confidence interval is $[\hat{\theta}_L, \infty)$.

- Similarly, we could have an upper one-sided confidence interval such that

$$P(\theta \leq \hat{\theta}_U) = 1 - \alpha$$

  The implied confidence interval here is $(-\infty, \hat{\theta}_U]$

Pivotal Quantity Method

- One very useful method for finding confidence intervals is called the pivotal method.
- This method depends on finding a pivotal quantity that possesses two characteristics:
  - It is a function of the sample measurements and the unknown parameter θ, where θ is the only unknown quantity.
  - Its probability distribution does not depend on the parameter θ.

Example

Suppose that we are to obtain a single observation Y from an $Exp(\theta)$. Use Y to form a confidence interval for θ with confidence coefficient 0.90

- It is easy to show that $U = \frac{2Y}{\theta}$ has $Exp(2)$ which is Chi-square distribution with 2 degrees of freedom. (Recall: $Gamma(n, 2) = \mathcal{X}^2_{2n}$)
- Hence

$$0.90 = P(\mathcal{X}^2_{2,0.05} \leq \frac{2Y}{\theta} \leq \mathcal{X}^2_{2,0.95})$$

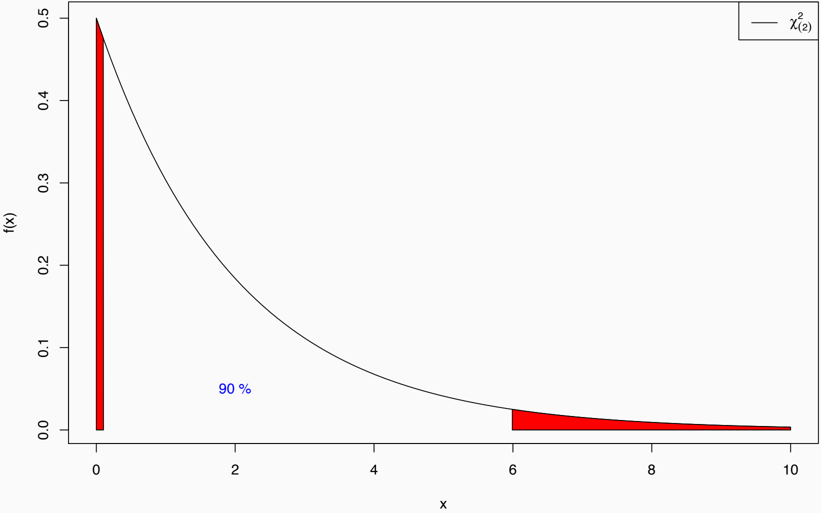
- That is

$$0.90 = P(\frac{2Y}{\mathcal{X}^2_{2,0.95}} \leq \theta \leq \frac{2Y}{\mathcal{X}^2_{2,0.05}})$$

R Code

```
a<-qchisq(0.05,df=2)  # or qgamma(0.05,shape = 1, scale = 2)
b<-qchisq(0.95,df=2)  # or qgamma(0.95,shape = 1, scale = 2)
cord.x <- c(0,seq(0,a,0.01),a)
curve(dgamma(x, shape = 1, scale = 2), from = 0, to = 10, ylab = "f(x)")
cord.x <- c(0,seq(0,a,0.01),a)
cord.y <- c(0,dgamma(seq(0,a,0.01),shape = 1, scale = 2),0)
polygon(cord.x,cord.y,col="red")
cord.x <- c(b,seq(b,10,0.001),10)
cord.y <- c(0,dgamma(seq(b,10,0.001),shape = 1, scale = 2),0)
polygon(cord.x,cord.y,col='red')
legend("topright",legend = expression(chi[(2)]^2),  col =  "black", lty = 1)
text(2,0.05, "90 %",  col =  "blue")
```

R Plot



Large-Sample Confidence Intervals

- As we indicated earlier, for large samples all these point estimators have approximately normal sampling distributions with standard errors as given in the Table 8.1.
- If the target parameter θ is μ, p, $\mu_1 - \mu_2$,or $p_1 - p_2$,then for large samples,

$$Z = \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}}$$

- Using the pivotal quantity methd

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1 - \alpha \tag{1}$$

Large-Sample Confidence Intervals

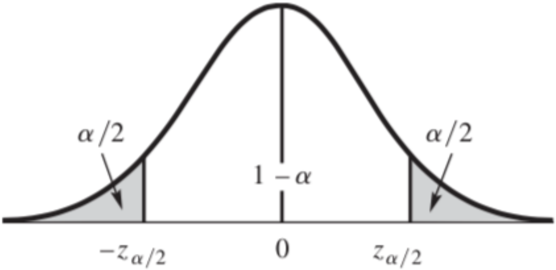
- Substituting for Z in the probability statement and solving it, we obtain

$$
\begin{aligned}
&P(-z_{\alpha/2} \leq \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}} \leq z_{\alpha/2}) \\
&= P(\hat{\theta} - z_{\alpha/2}\sigma_{\hat{\theta}} \leq \theta \leq \hat{\theta} + z_{\alpha/2}\sigma_{\hat{\theta}}) \\
&= 1 - \alpha
\end{aligned}
$$

- Thus, the endpoints for a $100(1-\alpha)\%$ confidence interval for θ are given by $(\hat{\theta} - z_{\alpha/2}\sigma_{\hat{\theta}}, \hat{\theta} + z_{\alpha/2}\sigma_{\hat{\theta}})$

Figure: Normal

FIGURE **8.7**
Location of $z_{\alpha/2}$ and $-z_{\alpha/2}$



Example

The shopping times of $n = 64$ randomly selected customers at a local supermarket were recorded. The average and variance of the 64 shopping times were 33 min and 256 min^2, respectively.

- Estimate μ, the true average shopping time per customer, with a confidence coefficient of $1 - \alpha = 0.90$.
- Solution: The sample is sufficiently large, using lerge sample size, a 90% confidence interval for μ is

$$\bar{Y} \pm z_{\alpha/2}\sigma_{\bar{Y}} = \bar{Y} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \approx \bar{Y} \pm z_{0.05}\frac{s}{\sqrt{n}}$$

$$= 33 \pm 1.645\frac{\sqrt{256}}{\sqrt{64}} = (29.71, 36.29)$$

Selecting the Sample Size

- The sampling procedure (experimental design) affects the quantity of information per measurement. This, together with the sample size n controls the total amount of relevant information in a sample.
- We would like to know how accurate the experimenter wishes the estimate to be.
- We can indicate the desired accuracy by specifying a bound on the error of estimation.

Example

- suppose that we wish to estimate the average daily yield μ of a chemical and we wish the error of estimation to be less than 5 tons with probability 0.95.
- Followig the CLT, approximately 95% of the sample means will lie within $2\sigma_{\bar{Y}}$.
- Thus, $\frac{2\sigma}{\sqrt{n}} = 5$ or $n = \frac{4\sigma^2}{25}$
- Lacking an exact value for σ, we can use the range which is approximately equal to 4σ.

Example

- Let the range of the daily yield is approximately 84. Then $n = \frac{4\times(21)^2}{25} \approx 71$
- Conclusion: Using a sample size $n = 71$, we can be reasonably certain (with confidence coefficient approximately equal to 0.95) that our estimate will lie within 5 tons of the true average daily yield.

General Case

- The method of choosing the sample sizes for all the large-sample estimation procedures outlined in Table 8.1 is analogous to that just described. The experimenter must specify a desired bound on the error of estimation and an associated confidence level $1 - \alpha$.
- For example, if the parameter is θ and the desired bound is B, we equate

$$B = z_{\alpha/2}\sigma_{\hat{\theta}}$$

Exercise

The reaction of an individual to a stimulus in a psychological experiment may take one of two forms, A or B. If an experimenter wishes to estimate the probability p that a person will react in manner A, how many people must be included in the experiment? Assume that the experimenter will be satisfied if the error of estimation is less than 0.04 with probability equal to 0.90. Assume also that he expects p to lie somewhere in the neighborhood of 0.6.

Small-Sample C.I for μ and $\mu_1 - \mu_2$

- Assume that the experimenter's sample has been randomly selected from a normal population.
- The intervals are appropriate for samples of any size, and the confidence coefficients of the intervals are close to the specified values even when the population is not normal, as long as the departure from normality is not excessive.
- We would like to construct a confidence interval for the population mean when $V(Y_i) = \sigma^2$ is unknown and the sample size is too small to permit us to to apply the large-sample techniques of the previous section

t Distribution

Theorem 6

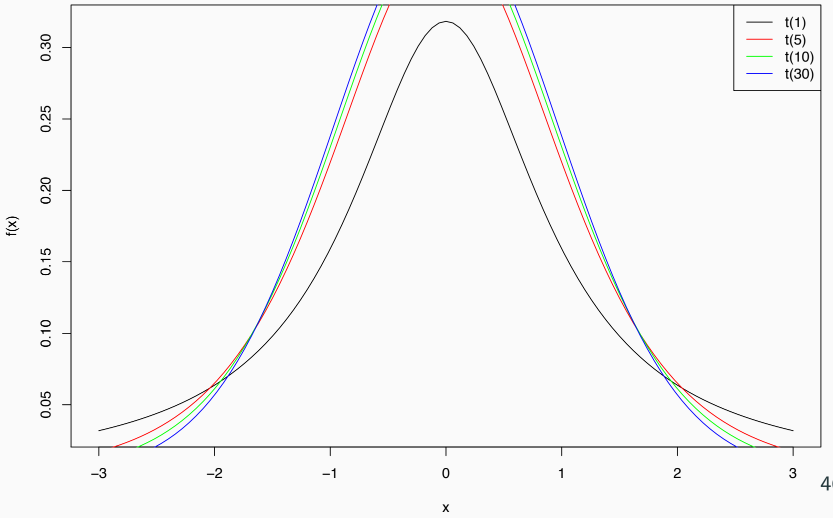
Let X_1 and X_2 be independent random vriables, from standard normal distribution and Chi-square distribution with r degrees of freedom. Then, the random variable

$$T = \frac{X_1}{\sqrt{\frac{X_2}{r}}}$$

has the t distribution with r degrees of freedom denoted by $t_{(r)}$.

t Distribution Curves

```
curve(dt(x, df=30), add = TRUE, col = "blue")
legend("topright",legend = c("t(1)", "t(5)", "t(10)","t(30)"),
       col = c("black","red","green","blue"), lty = 1)
```



Small-Sample Confidence Interval for μ

- The theorem implies that $T = \frac{\bar{Y}-\mu}{\frac{S}{\sqrt{n}}}$ has the $t_{(n-1)}$ distribution and it can be used as the pivotal quantity to construct the C.I for μ.

- Using the value of $t_{(n-1),\alpha/2}$ in the t Table, we have

$$P(-t_{(n-1),\alpha/2} \leq T \leq t_{(n-1),\alpha/2}) = 1 - \alpha$$

- After some calculation, we see that

$$P(\bar{Y} - t_{(n-1),\alpha/2}\frac{S}{\sqrt{n}} \leq \mu \leq \bar{Y} + t_{(n-1),\alpha/2}\frac{S}{\sqrt{n}}) = 1 - \alpha$$

Example

A manufacturer of gunpowder has developed a new powder, which was tested in eight shells. The resulting muzzle velocities, in feet per second, were as follows: 3005, 2925, 2935, 2965, 2995, 3005, 2937, 2905.

- Find a 95% confidence interval for the true average velocity μ for shells of this type. Assume that muzzle velocities are approximately normally distributed.
- Solution: By $t_{(n-1),\alpha/2} = t_{7,0.025} = 2.365$, we obtain $(\bar{y} \pm t_{7,0.025}\frac{S}{\sqrt{n}}) = (2959 \pm 32.7)$.

Remark

The quantity $(n-1)$ is referred to as "degrees of freedom" because the deviations $(y_i - \bar{y})$ must sum to zero, and so only $(n-1)$ of them are "free" to vary. A sample of size n provides only $(n-1)$ independent pieces of information about variability, that is, about σ. This is particularly clear if we consider the case $n = 1$; a sample of size 1 provides some information about μ, but no information about σ, and so no information about sampling error. It makes sense, then, that when $n = 1$, we cannot use t distribution to calculate a confidence interval; the sample standard deviation does not exist and there is no critical value with $df = 0$.

Remark: Confidence Intervals and Randomness

In what sense can we be "confident" in a confidence interval? To answer this question, let us assume that we are dealing with a random sample from a normal population. Consider, for instance, a 95% confidence interval. One way to interpret the confidence level (95%) is to refer to the meta-study of repeated samples from the same population. If a 95% confidence interval for μ is constructed for each sample, then 95% of the confidence intervals will contain μ. Of course, the observed data in an experiment comprise only one of the possible samples; we can hope "confidently" that this sample is one of the lucky 95%, but we will never know.

Small-Sample Confidence Interval for $\mu_1 - \mu_2$

- If $\bar{Y}_1$ and $\bar{Y}_2$ are the respective sample means obtained by samples n_1 and n_2 from two independent populations with normal distribution and common unknown variance σ^2.

- The small-sample confidence interval for $(\mu_1 - \mu_2)$ is developed by using the following pivotal quantity.

$$Z = \frac{\bar{Y}_1 - \bar{Y}_2 - (\mu_1 - \mu_2)}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- Because σ^2 is unknown, we need to find an estimator of the common variance σ^2 so that we can construct a quantity with a t distribution.

Small-Sample Confidence Interval for $\mu_1 - \mu_2$

- The usual unbiased estimator of the common variance σ^2 is obtained by pooling the sample data to obtain the pooled estimator

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

where, S_i^2 is the sample variance from sample i, $i = 1, 2$.

- Furthermore, we can show that

$$W = \frac{(n_1 + n_2 - 2)S_p^2}{\sigma^2} = \frac{(n_1 - 1)S_1^2}{\sigma^2} + \frac{(n_2 - 1)S_2^2}{\sigma^2}$$

is the sum of two independent $\mathcal{X}^2$-distributed random variables with $(n_1 - 1)$ and $(n_2 - 1)$ degrees of freedom respectively. That is, $W \sim \mathcal{X}^2_{(n_1+n_2-2)}$

Small-Sample Confidence Interval for $\mu_1 - \mu_2$

- It follows that the below quantity has t distribution with $(n_1 + n_2 - 2)$ degrees of freedom.

$$T = \frac{Z}{\sqrt{\frac{W}{n_1+n_2-2}}} = \frac{\bar{Y}_1 - \bar{Y}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- Thus, we see that the confidence interval for $\mu_1 - \mu_2$ is

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{(n_1+n_2-2),\alpha/2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

- where $t_{(n_1+n_2-2),\alpha/2}$ is determined from the t Table.

Example

- Bulimia study: The "fear of negative evaluation" (FNE) scores for 11 students known to suffer from the eating disorder bulimia and 14 students with normal eating habits (the higher the score, the greater the fear of a negative evaluation.)
  Bulimic students: 21, 13, 10, 20, 25, 19, 16, 21, 24, 13, 14
  Normal students: 13, 6, 16, 13, 8, 19, 23, 18, 11, 19, 7, 10, 15, 20
- Construct a 95% confidence interval for the difference between the population means of the FNE scores for bulimic and normal students.
- What assumptions are required for the C.I to be statistically valid? Are these assumptions reasonably satisfied?

Confidence Interval for σ^2

- The population variance σ^2 quantifies the amount of variability in the population.

- Assume that we have a random sample $Y_1, Y_2, ..., Y_n$ from a normal distribution with mean μ and variance σ^2, both unknown. We know

$$W = \frac{(n-1)S^2}{\sigma^2}$$

  has $\mathcal{X}^2_{(n-1)}$ distribution.

- We can then proceed by the pivotal method to find the upper and lower bounds

$$P\left(\mathcal{X}^2_L \leq \frac{(n-1)S^2}{\sigma^2} \leq \mathcal{X}^2_U\right) = 1 - \alpha$$

Confidence Interval for σ^2

- A $100(1-\alpha)$% Confidence Interval for σ^2

$$\left(\frac{(n-1)S^2}{\mathcal{X}^2_{(n-1),\alpha/2}}, \frac{(n-1)S^2}{\mathcal{X}^2_{(n-1),1-\alpha/2}}\right)$$

- Note that the confidence intervals are very sensitive to the assumption that the underlying population is normally distributed. Consequently, the actual confidence coefficient associated with the interval estimation procedure can differ markedly from the nominal value when this condition does not hold.

What We Have Just Learned

- The Bias and Mean Square Error of Point Estimators
- Some Common Unbiased Point Estimators
- Evaluating the Goodness of a Point Estimator
- Confidence Intervals
- Large-Sample Confidence Intervals
- Selecting the Sample Size
- Small-Sample Confidence Intervals for μ and $\mu_1 - \mu_2$
- Confidence Intervals for σ^2