practice problems 7 : Solution

problem 1

(a) $$\int_{-\infty}^{0} \frac{dx}{3-4x} = \lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{3-4x}$$

$u = 3-4x$
$du = -4\,dx$

First Solve indefinite integral

$$\int \frac{dx}{3-4x} = \int \frac{-du/4}{u} = -\frac{1}{4}\int \frac{du}{u} = -\frac{1}{4}\ln|u| + c$$
$$= -\frac{1}{4}\ln|3-4x| + c$$

So,

$$\lim_{t\to-\infty} \int_{t}^{0} \frac{dx}{3-4x} = \lim_{t\to-\infty} -\frac{1}{4}\ln|3-4x|\Big|_{t}^{0}$$
$$= -\frac{1}{4}\ln|3| + \lim_{t\to-\infty} \frac{1}{4}\ln|3-4t|$$
$$= \infty$$

The integral is divergent.

(b) $$\int_{1}^{\infty} \frac{dx}{(2x+1)^3} = \lim_{t\to\infty} \int_{1}^{t} \frac{dx}{(2x+1)^3}$$

$U = 2x+1 \Rightarrow du = 2dx$, First solve indefinite integral

$$\int \frac{du/2}{u^3} = \frac{1}{2}\int \frac{du}{u^3} = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + c = -\frac{1}{4}(2x+1)^{-2} + c$$

So, $$\lim_{t\to\infty} -\frac{1}{4}(2x+1)^{-2}\Big|_{1}^{t} = \frac{1}{36} - \frac{1}{4}\lim_{t\to\infty}\frac{1}{(2t+1)^2} = \frac{1}{36}$$

integral $\Rightarrow$ Convergent

Problem 2

Not testable

However, in case you're interested

(a) $\int_0^{\infty} \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^{\infty} \frac{x}{x^3+1}\,dx$

when $x>0$ (Since the integral is from $(0,\infty)$)

$$\frac{x}{x^3+1} < \frac{x}{x^3} = \frac{1}{x^2}$$

Assume, $f(x) = \frac{x}{x^3+1}$ and $g(x) = \frac{1}{x^2}$

Since $\frac{x}{x^3+1} < \frac{1}{x^2}$ and $\int_1^{\infty} \frac{dx}{x^2}$ is Convergent (show it)

So, $\int_1^{\infty} \frac{x}{x^3+1}\,dx$ is Convergent

and $\int_0^1 \frac{x}{x^3+1}\,dx$ is a proper integral and Finite

So, $\int_0^{\infty} \frac{x}{x^3+1}\,dx$ is Convergent

Problem 3

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{c}{3x+1}\right)dx = \lim_{t\to\infty}\int_0^t \left(\frac{x}{x^2+1} - \frac{c}{3x+1}\right)dx$$

First, we solve indefinite integral using v-substitution

$$\int \left(\frac{x}{x^2+1} - \frac{c}{3x+1}\right)dx = \int\left(\frac{1}{2}\,\frac{2x}{x^2+1} - \frac{c}{3}\,\frac{3}{3x+1}\right)dx$$

$u = x^2+1 \Rightarrow du = 2x\,dx$

$v = 3x+1 \Rightarrow dv = 3d$

$$= \frac{1}{2}\ln|u| - \frac{c}{3}\ln|v| + \text{Constant}$$

$$= \frac{1}{2}\ln|x^2+1| - \frac{c}{3}\ln|3x+1| + \text{Constant}$$

so,

$$\lim_{t\to\infty}\int_0^t \left(\frac{x}{x^2+1} - \frac{c}{3x+1}\right)dx = \lim_{t\to\infty}\left(\frac{1}{2}\ln|x^2+1| - \frac{c}{3}\ln|3x+1|\right)\Big|_0^t$$

$$= \lim_{t\to\infty}\left(\ln\sqrt{t^2+1} - \ln(3t+1)^{c/3}\right) = \lim_{t\to\infty}\ln\frac{\sqrt{t^2+1}}{(3t+1)^{c/3}}$$

since ln function is continuous $= \ln \lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{c/3}}$

For convergency

the lim should be exist and positive (since domain ln is) $\Longrightarrow$ $c = 3$

In this case, $\int_0^\infty \left(\frac{x}{x^2+1} - \frac{3}{3x+1}\right)dx = \ln\lim_{t\to\infty}\frac{\sqrt{t^2+1}}{3t+1}$

$$= \ln\lim_{t\to\infty}\frac{t}{3t} = \ln\frac{1}{3} = -\ln 3$$

problem 4:

(a) $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x^p} = \lim_{t\to 0^+} \frac{x^{-p+1}}{-p+1}\Big|_t^1$

$= \frac{1}{1-p} - \lim_{t\to\infty} \frac{t^{-p+1}}{-p+1} \longrightarrow p\neq 1$

the integral is convergent if the limit exists

So, $-p+1<0 \Longrightarrow p>1$

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b) $\int_e^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^p}$

first, we solve indefinite integral

$\ln x = u \Rightarrow \frac{dx}{x} = du$

$\int \frac{dx}{x(\ln x)^p} = \int \frac{du}{u^p} = \frac{u^{-p+1}}{-p+1} + C = \frac{(\ln x)^{1-p}}{1-p}$

So, $\lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty} \frac{(\ln x)^{1-p}}{1-p}\Big|_e^t$

$= \lim_{t\to\infty} \frac{(\ln t)^{1-p}}{1-p} - \frac{1}{1-p} \xrightarrow[\text{Convergency}]{\text{for}} 1-p<0$

$p\neq 1$ $\qquad p>1$

problem 5

1) $y = 1+6x^{3/2}$ $\quad 0 \le x \le 1 \longrightarrow y' = 9x^{1/2}$

the exact length $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

Leibnitz notation

$$L = \int_0^1 \sqrt{1+(9x^{1/2})^2}\,dx = \int_0^1 \sqrt{1+81x}\,dx$$

$u = 1+81x \Rightarrow du = 81dx$ $\quad = \int_1^{\sqrt{82}} \sqrt{u}\,\frac{du}{81}$

$$= \frac{1}{81}\frac{u^{3/2}}{3/2}\Bigg|_1^{\sqrt{82}} = \frac{2(81)^{3/4}}{243}$$

2) $y^2 = 4(x+4)^3 \xrightarrow{y>0} y = 2(x+4)^{3/2}$

$$y' = 2\left(\frac{3}{2}\right)(x+4)^{1/2}$$

$$L = \int_0^2 \sqrt{1+9(x+4)}\,dx = \int_{37}^{55} \sqrt{u}\,\frac{du}{9}$$

$u = 1+9(x+4)$

$du = 9dx$

$$= \frac{1}{9}\left(\frac{u^{3/2}}{3/2}\right)_{37}^{55}$$

$$= \frac{2}{27}\left(55^{3/2} - 37^{3/2}\right)$$

Problem 6:

1) $y = x^2/2$ $\quad P(-1, \frac{1}{2})$, $Q(1, \frac{1}{2})$

$y' = x \Rightarrow L = \int_{-1}^{1} \sqrt{1+x^2}\, dx \underset{\downarrow}{=} 2\int_0^1 \sqrt{1+x^2}\, dx$

because It is even function

$x = \tan z$
$dx = \sec^2 z\, dz$

$= 2\int_0^{\pi/4} \sqrt{1+\tan^2 z}\, \sec^2 z\, dz$

if $0 < z < \frac{\pi}{4} \Rightarrow \sec z > 0$

$= 2\int_0^{\pi/4} \underbrace{\sec z}_{u}\, \underbrace{\sec^2 z\, dz}_{dv}$

integration by part

$= 2\left[\sec z \tan z \Big|_0^{\pi/4} - \int_0^{\pi/4} \sec z \tan^2 z\, dz \right]$

$\sec z = u \Rightarrow \sec z \tan z\, dz = du$
$\sec^2 z\, dz = dv \Rightarrow \tan z = v$

$= 2\left[\sqrt{2} - \int_0^{\pi/4} \sec z(\sec^2 z - 1)\, dz \right]$

$= 2\left[\sqrt{2} - \int_0^{\pi/4} \sec^3 z\, dz + \int_0^{\pi/4} \sec z\, dz \right]$

$\Rightarrow 2\int_0^{\pi/4} \sec^3 z\, dz = 2\sqrt{2} - 2\int_0^{\pi/4} \sec^3 z\, dz + 2\int_0^{\pi/4} \sec z\, dz$

Rearrange the equation $\Rightarrow \underbrace{2\int_0^{\pi/4} \sec^3 z\, dz + 2\int_0^{\pi/4} \sec^3 z\, dz}_{4\int_0^{\pi/4} \sec^3 z\, dz} = 2\sqrt{2} + 2\int_0^{\pi/4} \sec z\, dz$

$2\int_0^1 \sqrt{1+x^2}\, dx = 2\int_0^{\pi/4} \sec^3 z\, dz = \sqrt{2} + \int_0^{\pi/4} \sec z\, dz$

$= \sqrt{2} + \ln\left| \sec z + \tan z \right| \Big|_0^{\pi/4} = \sqrt{2} + \ln|1+\sqrt{2}|$