MATH 141 – practice problems 11 – Solution

Problem 1

(a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1-0.2^n) = 1 - \lim_{n\to\infty} 0.2^n = 1$ Convergent

(b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^{n+2}}{5^n} = 3^2 \lim_{n\to\infty} \frac{3^n}{5^n} = 3^2 \lim_{n\to\infty}\left(\frac{3}{5}\right)^n = 0$ Convergent

(c) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1} n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{(-1)^{n+1} n}{n} = \pm 1$ divergent

d) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n/2)$ DNE (Not unique) divergent

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problem 2

(a) $a_n = (-2)^{n+1} \rightarrow$
$a_1 = (-2)^2 = 4$
$a_2 = (-2)^3 = -8$
$a_3 = (-2)^4 = 16$
$\vdots$
Not monotonic

(b) $a_n = \frac{1}{2n+1} \rightarrow$
$a_1 = \frac{1}{3}$
$a_2 = \frac{1}{5}$
$a_3 = \frac{1}{7}$
$\vdots$
decreasing

or differentiate $f(n) = \frac{1}{2n+1} \Rightarrow f'(n) = -2(2n+1)^{-2} = \frac{-2}{(2n+1)^2} < 0 \Rightarrow a_n$ is decreasing

(c) $a_n = \frac{2n-3}{3n+4} \Rightarrow f'(n) = \frac{2(3n+4) - 3(2n-3)}{(3n+4)^2} = \frac{17}{(3n+4)^2} > 0$

a_n is increasing

problem3)

(a) $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{2}{k^2-1} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{2}{(k-1)(k+1)}$

$= \lim_{n\to\infty} \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k+1} \right) = \lim_{n\to\infty} \left\{ (1-\cancel{\tfrac{1}{3}}) + (\tfrac{1}{2} - \cancel{\tfrac{1}{4}}) + (\cancel{\tfrac{1}{3}} - \tfrac{1}{5}) + \cdots + (\cancel{\tfrac{1}{n-1}} - \tfrac{1}{n+1}) \right\}$

$= \lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{n+1} \right) = 3/2$ Convergent

(c) $\sum_{n=2}^{\infty} \frac{2}{n^3-n} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{2}{k^3-k} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{2}{k(k^2-1)} = \lim_{n\to\infty} \sum_{k=2}^{n} \frac{2}{k(k+1)(k-1)}$

$= \lim_{n\to\infty} \sum_{k=2}^{n} \left(\frac{A}{k-1} + \frac{B}{k} + \frac{C}{k+1} \right) = \lim_{n\to\infty} \sum_{k=2}^{n} \left\{ \frac{1}{k-1} - \frac{2}{k} + \frac{1}{k+1} \right\}$

$Ak(k+1) + B(k-1)(k+1) + Ck(k-1) = 2$

$$\begin{array}{l|l} 2 & A+B+C=0 \quad \mid 2A+B=0 \Rightarrow \boxed{A=1} \\ 1 & A-C=0 \quad \boxed{C=1} \\ 0 & -B=2 \Rightarrow B=-2 \end{array}$$

$= \lim_{n\to\infty} \sum_{k=2}^{n} \left\{ \left(\frac{1}{k-1} - \frac{1}{k} \right) + \left(\frac{1}{k+1} - \frac{1}{k} \right) \right\}$

$= \lim_{n\to\infty} \left\{ (1-\cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{3}} - \tfrac{1}{2}) + (\tfrac{1}{2} - \cancel{\tfrac{1}{3}}) + (\tfrac{1}{4} - \cancel{\tfrac{1}{3}}) + (\cancel{\tfrac{1}{3}} - \tfrac{1}{4}) + (\tfrac{1}{n-1} - \tfrac{1}{n}) + (\tfrac{1}{n+1} - \tfrac{1}{n}) \right\}$

$= \lim_{n\to\infty} \left\{ (1-\tfrac{1}{2}) + (-\tfrac{1}{n}) + (\tfrac{1}{n+1}) \right\}$

$= 1/2$

problem 4

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}} \Rightarrow \int_1^{\infty} \frac{dx}{\sqrt[5]{x}} = \lim_{t\to\infty} \int_1^t x^{-1/5}\,dx = \lim_{t\to\infty} \left(\frac{x^{4/5}}{4/5} \right)\Big|_1^t$

$= \frac{5}{4} \lim_{t\to\infty} (t^{4/5} - 1) = \infty$

it is divergent (or use the power test $p = \frac{1}{5} < 1$)

(c) $\sum_{n=1}^{\infty} n^2 e^{-n^3} \Rightarrow \int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3}\,dx$

$= -\frac{1}{3} \lim_{t\to\infty} \left(e^{-x^3} \Big|_1^t \right) = -\frac{1}{3} \lim_{t\to\infty} (e^{-t^3} - e^{-1})$

$= e^{-1}/3$

It is Convergent

d) $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$\Rightarrow \int_1^{\infty} \frac{dx}{x\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^{3/2}} = \lim_{t\to\infty} \left(\frac{x^{-1/2}}{-1/2} \right)\Big|_1^t$

$= -2 \lim_{t\to\infty} (t^{-1/2} - 1) = 2$

It is Convergent (or use the power test $p = 3/2 > 1$)