Lecture 8: Functions of Random Variables

MATH 697

Omidali Jazi

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McGill University

Goals for this Chapter

- Finding the Probability Distribution of Functions of Random Variables
- The Method of Distribution Functions
- Univariate Methods
  - The Transformation Method
- Multivariate Methods
  - The Method of Moment-Generating Function
- Data Generation Methods

Introduction

- To determine the probability distribution for a function of n random variables $Y_1, Y_2, ..., Y_n$, we must find the joint probability distribution for the random variables themselves. We generally assume that observations are obtained through random sampling.

- All quantities used to estimate population parameters or to make decisions about a population are functions of the n random observations that appear in a sample.

- Example: We draw a random sample of n samples $Y_1, Y_2, ..., Y_n$ (n copies of Y) from the population and employ the sample mean $\bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}$.

The Method of Distribution Functions

If Y has probability density function $f(y)$ and if U is some function of Y, then we can find $F_U(u) = P(U \leq u)$ directly by integrating $f(y)$ over the region for which $U \leq u$. The probability density function for U is found by differentiating $F_U(u)$.

The Method of Distribution Functions

Let U be a function of the random variables $Y_1, Y_2, ..., Y_n$.

1. Find the region $U = u$ in the $(y_1, y_2, ..., y_n)$ space.
2. Find the region $U \leq u$.
3. Find $F_U(u) = P(U \leq u)$ by integrating $f(y_1, y_2, ..., y_n)$ over the region $U \leq u$.
4. Find the density function $f_U(u)$ by differentiating $F_U(u)$. Thus, $f_U(u) = \frac{dF_U(u)}{du}$.

Example

A process for refining sugar yields up to 1 ton of pure sugar per day, but the actual amount produced, Y, is a random variable because of machine breakdowns and other slowdowns. Suppose that Y has density function given by $f(y) = 2y$ for $0 \leq y \leq 1$. The company is paid at the rate of 300$ per ton for the refined sugar, but it also has a fixed overhead cost of 100$ per day. Thus the daily profit, in hundreds of dollars, is $U = 3Y - 1$.

Solution

- Find the probability density function for U.
- Solution: Using the method of the distribution function:

$$F_U(u) = P(U \leq u) = P(3Y - 1 \leq u) = P(Y \leq \frac{u+1}{3})$$
$$= \int_0^{(u+1)/3} 2y dy = \left(\frac{u+1}{3}\right)^2, \qquad -1 \leq u \leq 2$$

Therefore, $f_U(y) = \frac{dF_U(u)}{du} = \frac{2(y+1)}{9}$ for $-1 \leq u \leq 2$.

Transformation Method

If we are given the density function of a random variable Y, the method of transformations results in a general expression for the density of $U = h(Y)$ for an increasing or decreasing function $h(y)$. Then if Y_1 and Y_2 have a bivariate distribution, we can use the univariate result explained earlier to find the joint density of Y_1 and $U = h(Y_1, Y_2)$. By integrating over y_1, we find the marginal probability density function of U which is our objective.

Univariate Transformation Method

- We previously discussed the problem of starting with a single random variable X, forming some function of X, such as X^2 or e^X, to obtain a new random variable $Y = h(X)$

Univerariate Transformation Method

Theorem 1

Let Y have probability density function $f_Y(y)$. If $h(y)$ is either increasing or decreasing for all y such that $f_Y(y) > 0$, then $U = h(Y)$ has density function

$$f_U(u) = f_Y[h^{-1}(u)] \left| \frac{dh^{-1}}{du} \right|$$

where $\frac{dh^{-1}}{du} = \frac{d[h^{-1}(u)]}{du}$

Increasing Case

We see from the graph that the set of points y such that $h(y) \leq u_1$ is precisely the same as the set of points y such that $y \leq h^{-1}(u_1)$.

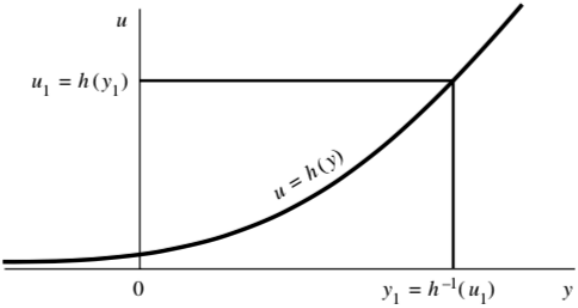
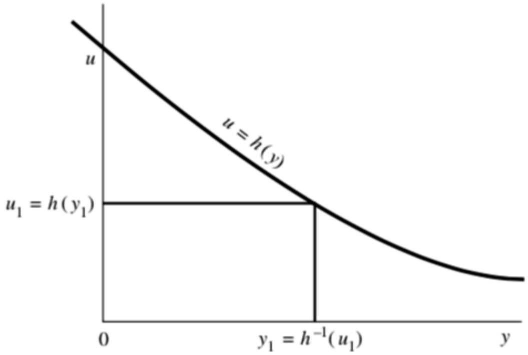


FIGURE 6.8 An increasing function

Decreasing Case

If $h(y)$ is a decreasing function of y, then $h^{-1}(u)$ is a decreasing function of u. That is, if $u_1 < u_2$, then $h^{-1}(u_1) = y_1 > y_2 = h^{-1}(u_2)$. Also, as in the graph, the set of points y such that $h(y) \leq u_1$ is the same as the set of points such that $y \geq h^{-1}(u_1)$.

FIGURE 6.9
A decreasing function



Example

Let Y have the Beta distribution with parameters $(\alpha, 1)$.

- What is the distribution of $U = -\log(Y)$
- Solution: Because U is a decreasing function. By the transormation method, since $f_Y(y) = \alpha y^{\alpha-1}$ and $h^{-1}(u) = e^{-u}$ where $0 < u < \infty$, we have

$$
\begin{aligned}
f_U(u) &= f_Y(h^{-1}(u))|\frac{dh^{-1}}{du}| = \alpha e^{-u(\alpha-1)} \times e^{-u} \\
&= \alpha e^{-\alpha u}
\end{aligned}
$$

That is, U is $Exp(\alpha^{-1})$

Example

- What is the distribution of $W = -2\alpha \log(Y)$
- Solution: Likewise, $h^{-1}(w) = e^{-w/2\alpha}$ and $0 < w < \infty$. Then

$$f_W(w) = f_Y(h^{-1}(w))|\frac{dh^{-1}}{dw}| = \frac{1}{2}e^{-w/2}$$

That is, $W \sim Exp(2) = Gamma(1, 2)$ or $W \sim \mathcal{X}^2_{(2)}$.

Multivariate Transformations

- Note that the transformation method can also be used in multivariate situations.
  - Consider a system having a component that can be replaced just once before the system itself expires.
  - X: lifetime of the original component
  - Y: lifetime of the replacement component
- Any of the following functions of X and Y may be of interest to an investigator:
  - The total lifetime $X + Y$
  - The ratio of lifetimes X/Y; e.g. if the ratio is 2 the original component lasted twice as long as its replacement
  - The ratio $X/(X + Y)$; the proportion of system lifetime during which the original component operated

The Method of Moment-Generating Functions

The moment-generating function method for finding the probability distribution of a function of random variables $Y_1, Y_2, ..., Y_n$ is based on the following uniqueness theorem.

Theorem 2

Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of random variables X and Y respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

Example

- Suppose that Y is a normally distributed random variable with mean μ and variance σ^2. Show that $Z = \frac{Y-\mu}{\sigma}$ has a standard normal distribution, a normal distribution with mean 0 and variance 1.
- Solution: Note that $m_Y(t) = e^{\mu t + \sigma^2 t^2/2}$. Hence

$$m_Z(t) = E(e^{tZ}) = E\left[e^{t\left(\frac{Y-\mu}{\sigma}\right)}\right] = e^{t^2/2}$$

That is, $Z \sim N(0,1)$

Useful Application

The method of moment-generating functions is often very useful for finding the distributions of sums of independent random variables.

Theorem 3

Let $Y_1, Y_2, ..., Y_n$ be independent random variables with moment-generating functions $m_{Y_1}(t), m_{Y_2}(t), ..., m_{Y_n}(t)$ respectively. If $U = Y_1 + ... + Y_n$, then

$$m_U(t) = m_{Y_1}(t) \times .... \times m_{Y_n}(t)$$

Example

- The number of customer arrivals at a checkout counter in a given interval of time possesses approximately a Poisson probability distribution. If Y_1 denotes the time until the first arrival, and Y_i denotes the time between the $(i-1)$th and ith arrival for $i = 2, .., n$. Then it can be shown that $Y_1, Y_2, ..., Y_n$ are independent Exponential random variables with the pdf $f_Y(y) = \frac{e^{-y/\theta}}{\theta}$ for $y > 0$.
- Find the probability density function for the waiting time from the opening of the counter until the nth customer arrives. (If $Y_1, Y_2, ...Y_n$ denote successive interarrival times, we want the density function of $U = Y_1 + Y_2 + \cdots + Y_n$.)

Remark

The method of moment-generating functions can be used to establish some useful results about the distributions of functions of normally distributed random variables.

Theorem 4

Let $Y_1, Y_2, ..., Y_n$ be independent normally distributed random variables with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, ..., n$, and let $a_1, a_2, ..., a_n$ be constants. If

$$U = \sum_{i=1}^{n} a_i Y_i = a_1 Y_1 + ... + a_n Y_n$$

Then U is a normally distrinuted random variable with $E(U) = \sum_{i=1}^{n} a_i \mu_i$ and $V(U) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$

Example

- Let $Y_1, ..., Y_n$ be normal random variables with mean μ_i and variance σ_i^2 respectively.
- Find the distribution of $\sum\limits_{i=1}^{n} Z_i^2$ where $Z_i = \frac{Y_i - \mu_i}{\sigma_i}$, that is standard normal random variables.

Generate Discrete Random Variables

- R can be used for generateing random variables from the Uniform distribution.
- Bernoulli: Simulate tossing a coin with probability of heads p.
- Let U be a Uniform(0,1) random variable. We can write Bernoulli random variable X as: $X = 1$ if $U \leq p$ and $X = 0$ if $U > p$.

R code

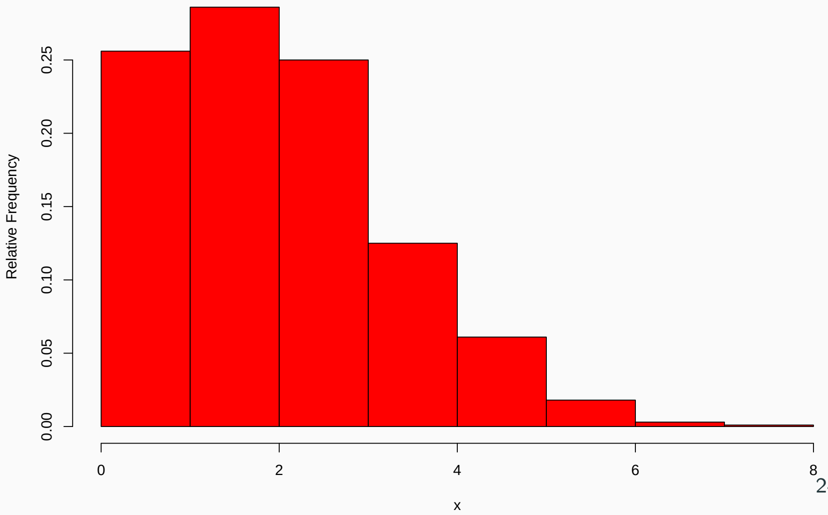
- Generate 5 samples from Bernoulli distribution with $p = 0.5$ (Toss a balanced coin 5 times).

```
p = 0.5;
U = runif(5,min = 0,max = 1);
X =ifelse(U<=p,1,0);
X
```

```
## [1] 1 1 0 0 1
```

Generate a Binomial(50, 0.25) random variable.

```
N=1000;p = 0.25;n = 10;X<-c(); for(i in 1:N){U = runif(n,min = 0,max = 1);
X = c(X,sum(U < p))}; # print(mean(X));
hist(X,freq=FALSE, xlab = "x",ylab ="Relative Frequency",main="",col="red")
```



Exercise

Write R programs to generate Geometric(p) and Negative Binomial (r,p) random variables. Produce the histograms of the generated data.

- Hint 1: If Y has $Exp(\beta)$ distribution, then $[Y]$ (the greatest integer value smaller than Y) has geometric distribution.
- Hint 2: If $Y_1, ..., Y_r$ independent random variables from the $Ge(p)$, then $\sum_{i=1}^{r} Y_i$ has the $NB(r, p)$.

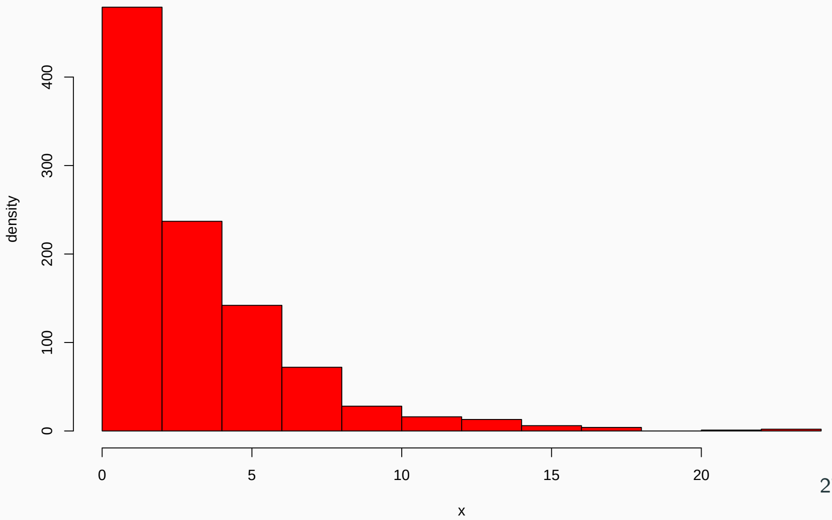
Distribution-Inverse Transformation Method

Theorem 5

Let U have the Uniform distribution in (0, 1) and F be a CDF which is strictly increasing. Also, consider a random variable X defined as $X = F^{-1}(U)$. Then X has distribution of F.

Example: Generate Exp($\beta = 3$)

```
N=1000;beta =3;n = 10;X<-c();
U = runif(N,min = 0,max = 1); X = -beta*log(1-U)
hist(X, xlab = "x",ylab ="density",main="",col="red")
```



Exercise

Generate random data from $Gamma(\alpha = 5, \beta = 3)$ and $\mathcal{X}^2_{r=6}$. Produce the histograms of the generated data.

- Hint: Generate samples from the $Exp(\beta)$ using an appropriate tranformation on $U(0,1)$. Now, if $Y_1, ..., Y_k$ are independent random variables from $Exp(\beta)$, then $\sum_{i=1}^{k} Y_i$ has $Gamma(k, \beta)$. Further, given $\beta = 2$, $\sum_{i=1}^{k} Y_i$ has $\mathcal{X}^2_{2k}$ (only when the degrees of freedom is an even number).
- Note: For Chi-square distribution with any arbitrary degrees of freedom, use $\sum_{i=1}^{n} Z_i^2 \sim \mathcal{X}^2_n$ where $Z_i \sim N(0,1)$

Poisson Distribution

- Use the algorithm for generating discrete random variables to obtain a Poisson random variable with parameter $\lambda = 2$.
- Assume N represents the number of events (arrivals) in $[0, t]$. If the interarrival times are distributed exponentially (with parameter λ) and independently, then the number of arrivals occurred in $[0, t]$, N, has Poisson distribution with parameter λt. Therefore, to solve this problem, we can repeat generating $Exp(\lambda)$ random variables while their sum is not larger than 1 ($t = 1$).

R code

```
Lambda = 2;
i = 0;
U = runif(1,min = 0,max = 1); Y =-(1/Lambda)*log(U);
sum = Y ;
while(sum < 1)
{U = runif(1,min = 0,max = 1); Y =-(1/Lambda)*log(U);
sum = sum + Y ;
i = i + 1; }
X=i
X
```

```
## [1] 3
```

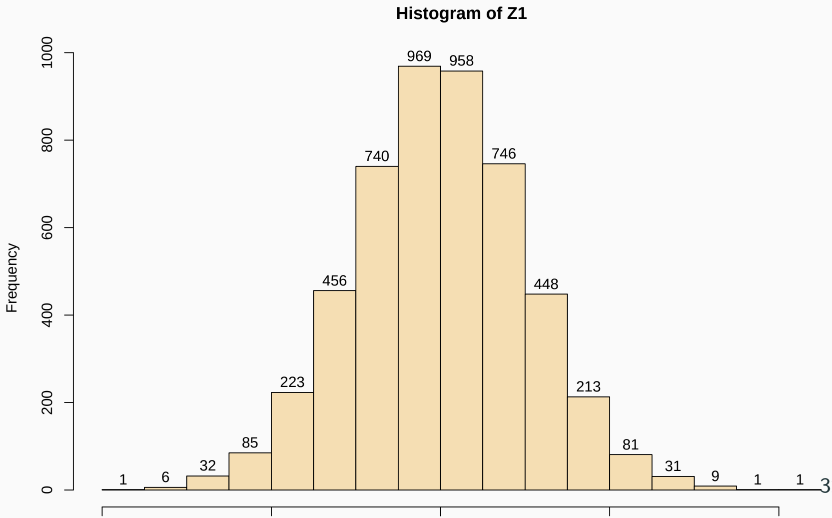
Generate Normal Samples from Uniform Distribution (Box-Muller transformation)

Theorem 6

We can generate a pair of independent normal variables (Z_1, Z_2) *by transforming a pair of independent* $U(0,1)$ *random variables* (U_1, U_2)*, as* $Z_1 = \sqrt{-2\log U_1}\cos(2\pi U_2)$ *and* $Z_2 = \sqrt{-2\log U_1}\sin(2\pi U_2)$.

R code: Generate 5000 pairs of normal RVs

```
n = 5000;U1 = runif(n,min = 0,max = 1);U2=runif(n,min = 0,max = 1);
Z1 = sqrt(-2 * log(U1)) * cos(2 * pi * U2);
Z2=sqrt(-2 * log(U1))*sin(2 * pi * U2);hist(Z1,col = "wheat", label = T)
```



What We Have Just Learned

- Finding the Probability Distribution of Functions of Random Variables
- The Method of Distribution Functions
- Univariate Methods
  - The Transformation Method
- Multivariate Methods
  - The Method of Moment-Generating Function
- Data Generation Methods