Practice problems9 _ Solution

problem1

1) (a) $x = 1+\ln(t)$ $\qquad y = t^2+2$ $\qquad (1,3)$, $x \to 1$, $y \to 3$

$x=1 \Rightarrow 1+\ln(t)=1 \Rightarrow \boxed{t=1}$

without eliminating parameter

slope of the tangent line $m = \dfrac{dy}{dx} = \left.\dfrac{dy/dt}{dx/dt}\right|_{t=1} = \left.\dfrac{2t}{1/t}\right|_{t=1} = \left.2t^2\right|_{t=1} = 2$

$y-3 = 2(x-1) \Rightarrow \boxed{y = 2x+1}$

First eliminate the parameter $\Rightarrow x = 1+\ln t \Rightarrow t = e^{x-1}$

$y = t^2+2 = (e^{x-1})^2+2 = e^{2(x-1)}+2$

$\boxed{y = e^{2(x-1)}+2}$

$m = y'|_{x=1} = \left.2e^{2(x-1)}\right|_{x=1} = 2 \Rightarrow \boxed{y-3 = 2(x-1)}$

problem 2

1) $x = a\cos^3\theta$, $y = a\sin^3\theta$ $\to a\sin^3\theta = a\cos^3\theta$ $\qquad 0 \le \theta \le \frac{\pi}{4}$

$A = \int y\,dx = \int_0^{\pi/4} a\sin^3\theta\,(-3a\sin\theta\cos^2\theta)\,d\theta = -3a^2\int_0^{\pi/4}(\sin^4\theta - \sin^6\theta)\,d\theta$

$dx = -3a\sin\theta\cos^2\theta$ $\qquad$ Use half angle formula to solve it.

(2) $x = t^2 - 2t$ $\quad y = \sqrt{t}$ $\quad$ y-axis

The curve intersects the y-axis ($x=0$) when

$$t^2 - 2t = 0 \Rightarrow t(t-2) = 0 \begin{cases} t = 0 \\ t = 2 \end{cases}$$

$$A = \int_0^2 (\underset{\downarrow \atop t^2-2t}{x_R} - \underset{\downarrow \atop x=0}{x_L})\,dy = \int_0^2 (t^2 - 2t)\,\frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^2 \sqrt{t}\,(t-2)\,dt$$

$$dy = \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\left(\frac{t^{5/2}}{5/2} - 2\,\frac{t^{3/2}}{3/2} \right)_0^2$$

$$= \frac{1}{2}\left(\frac{2}{5}\,2^{5/2} - \frac{4}{3}\,2^{3/2} \right)$$

problem 3

① $x = 1+3t^2 \qquad y = 4+2t^3 \qquad 0 \le t \le 1$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{[6t]^2 + [6t^2]^2}\, dt$$

$$= \int_0^1 \sqrt{(6t)^2(1+t^2)}\, dt \overset{t>0}{\uparrow}{=} \int_0^1 6t\sqrt{1+t^2}\, dt$$

$$= \frac{6}{2}\int_1^2 \sqrt{u}\, du = 3\left(\frac{u^{3/2}}{3/2}\right)_1^2 = 2(2^{3/2}-1)$$

$1+t^2 = u \Rightarrow 2t\,dt = du$

② $x = e^t + e^{-t} \qquad y = 5-2t \qquad 0 \le t \le 3$

$$L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt$$

$$= \int_0^3 \sqrt{e^{2t} + e^{-2t} - 2 + 4}\, dt = \int_0^3 \sqrt{e^{2t} + e^{-2t} + 2}\, dt$$

$$= \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt = \int_0^3 |\underbrace{e^t + e^{-t}}_{\text{always positive}}|\, dt = \int_0^3 (e^t + e^{-t})\, dt$$

$$= e^t - e^{-t}\Big|_0^3 = (e^3 - e^{-3}) - (1-1) = e^3 - e^{-3}$$

problem 4

① $x = t^3$, $y = t^2$ $\quad 0 \le t \le 1$ $\quad$ x-axis

$$\int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi t^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt$$

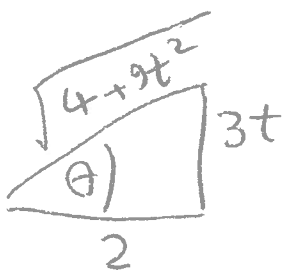
$$= \int_0^1 2\pi t^2 \sqrt{9t^4 + 4t^2}\, dt = \int_0^1 2\pi t^3 \sqrt{9t^2 + 4}\, dt$$

$t = \frac{2}{3}\tan\theta$

$dt = \frac{2}{3}\sec^2\theta \, d\theta$

$$\Rightarrow \int 2\pi t^3 \sqrt{9t^2+4}\, dt$$

$$= \int 2\pi \frac{8}{27}\tan^3\theta \sqrt{4\tan^2\theta + 4}\ \frac{2}{3}\sec^2\theta \, d\theta$$



$$= \int \frac{64\pi}{81} \tan^3\theta \sec^3\theta \, d\theta$$

$$= \frac{64\pi}{81} \int (\sec^2\theta - 1)\sec^2\theta \sec\theta \tan\theta \, d\theta$$

$\sec(\theta) = \dfrac{\sqrt{4+9t^2}}{2}$

$$= \frac{64\pi}{81} \int (u^2 - 1) u^2 \, du = \frac{64\pi}{81}\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$

$$= \frac{64\pi}{81}\left(\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}\right) + C$$

$$\int_0^1 2\pi t^3 \sqrt{9t^2+4}\, dt = \frac{64\pi}{81}\left(\frac{\left(\frac{\sqrt{4+9t}}{2}\right)^5}{5} - \frac{\left(\frac{\sqrt{4+9t}}{2}\right)^3}{3}\right)\Bigg|_0^1$$

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③ $x = 3t^2$ $\quad y = 2t^3$ $\quad 0 \le t \le 5$

y-axis

$$A = \int 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^5 2\pi (3t^2) \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$= - - - - - .$