Solution Q1

Define the following event: $R =$\{Fish is red snapper\}. We are given that $P(R^C) = 0.77$. Hence,

(a) $P(R) = 1 - P(R^C) = 1 - 0.77 = 0.23$

(b) Let R_i denote the event that customer i is served red snapper. Then

$$\begin{aligned} P(\text{At least one customer is served red snapper}) &= 1 - P(\text{No customers are served red snapper}) \\ &= 1 - P(R_1^C \cap R_2^C \cap R_3^C \cap R_4^C \cap R_5^C) \end{aligned}$$

Now assuming that each customer is independent and has the same probability $P(R_i^C) = 0.77$, this simplifies to:

$$1 - P(R_1^C)P(R_2^C)P(R_3^C)P(R_4^C)P(R_5^C) = 1 - (0.77)^5 = 1 - 0.271 = 0.729$$

Remark: The assumption we make above is unreasonable since any given restaurant will get its fish from the same vendor and so the restaurant will either serve red snapper to its customers or not. It's unreasonable to assume that whether or not each customer in the same restaurant will be served red snapper are independent events (i.e. that R_1, R_2, R_3, R_4, R_5 are independent).

SolutionQ2

Define the following events:

H= \{NDE detects a hit\}

D= \{Defect exists\}

We are given that $P(H \mid D) = 0.97, P(H \mid D^C) = 0.005, P(D) = 1/100 = 0.01$. In order to find $P(D \mid H)$, we must first find $P(H)$ using the law of total probability:

$$\begin{aligned} P(H) &= P(H \cap D) + P(H \cap D^C) = P(H \mid D)P(D) + P(H \mid D^C)P(D^C) \\ &= (0.97)(0.01) + (0.005)(0.99) = 0.01465 \end{aligned}$$

Then, we have

$$P(D \mid H) = \frac{P(D \cap H)}{P(H)} = \frac{P(H \mid D)P(D)}{P(H)} = \frac{(0.97)(0.01)}{0.01465} = 0.6621$$

Note that we could have directly used Bayes' Rule to obtain this probability:

$$\begin{aligned} P(D \mid H) = \frac{P(D \cap H)}{P(H)} &= \frac{P(H \mid D)P(D)}{P(H \mid D)P(D) + P(H \mid D^C)P(D^C)} \\ &= \frac{(0.97)(0.01)}{(0.97)(0.01) + (0.005)(0.99)} = 0.6621 \end{aligned}$$

Solution Q3

Define the following events: $H = \{\text{person has HIV}\}$, $P = \{\text{positive test}\}$, and $N = \{\text{negative test}\}$.
We are given that $P(H) = 0.008, P(P \mid H) = 0.99, P(N \mid H^C) = 0.99$.

(a) Using Bayes' Rule, we have

$$\begin{aligned} P(H \mid P) &= \frac{P(H \cap P)}{P(P)} = \frac{P(H)P(P \mid H)}{P(H)P(P \mid H) + P(H^C)P(P \mid H^C)} \\ &= \frac{(0.008)(0.99)}{(0.008)(0.99) + (0.992)(0.01)} = \frac{0.00792}{0.00792 + 0.00992} \\ &= 0.44395 \end{aligned}$$

(b) In East Asia, $P(H) = 0.001$. Using Bayes' Rule, the probability is:

$$\begin{aligned} P(H \mid P) &= \frac{P(H \cap P)}{P(P)} = \frac{P(H)P(P \mid H)}{P(H)P(P \mid H) + P(H^C)P(P \mid H^C)} \\ &= \frac{(0.001)(0.99)}{(0.001)(0.99) + (0.999)(0.01)} = \frac{0.00099}{0.00099 + 0.00999} \\ &= 0.09016 \end{aligned}$$

(c) We are interested in the probability

$$P(H \mid P \text{ on first} \cap P \text{ on second})$$

Since the tests are independent, we have that:

$$P(P \text{ on first} \cap P \text{ on second} \mid H) = P(P \text{ on first} \mid H)P(P \text{ on second} \mid H)$$

Then, using Bayes's Rule, we have:

$$\begin{aligned} P(H \mid P \text{ on first} \cap P \text{ on second}) &= \frac{P(H \cap P \text{ on first} \cap P \text{ on second})}{P(P \text{ on first} \cap P \text{ on second})} \\ &= \frac{P(P \text{ on first} \cap P \text{ on second} \mid H)P(H)}{P(P \text{ on first} \cap P \text{ on second} \mid H)P(H) + P(P \text{ on first} \cap P \text{ on second} \mid H^C)P(H^C)} \\ &= \frac{P(P \text{ on first} \mid H)P(P \text{ on second} \mid H)P(H)}{P(P \text{ on first} \mid H)P(P \text{ on second} \mid H)P(H) + P(P \text{ on first} \mid H^C)P(P \text{ on second} \mid H^C)P(H^C)} \\ &= \frac{(0.99)(0.99)(0.008)}{(0.99)(0.99)(0.008) + (0.01)(0.01)(0.992)} \\ &= 0.987506297 \end{aligned}$$

(d) In East Asia, the probability is:

$$
\begin{aligned}
P(H \mid P \text{ on first} \cap P \text{ on second}) &= \frac{P(H \cap P \text{ on first} \cap P \text{ on second})}{P(P \text{ on first} \cap P \text{ on second})} \\
&= \frac{P(P \text{ on first} \cap P \text{ on second} \mid H)P(H)}{P(P \text{ on first} \cap P \text{ on second} \mid H)P(H) + P(P \text{ on first} \cap P \text{ on second} \mid H^C)P(H^C)} \\
&= \frac{P(P \text{ on first} \mid H)P(P \text{ on second} \mid H)P(H)}{P(P \text{ on first} \mid H)P(P \text{ on second} \mid H)P(H) + P(P \text{ on first} \mid H^C)P(P \text{ on second} \mid H^C)P(H^C)} \\
&= \frac{(0.99)(0.99)(0.001)}{(0.99)(0.99)(0.001) + (0.01)(0.01)(0.999)} \\
&= 0.9075
\end{aligned}
$$

Solution Q4

Define a sequence of events $\{B_1, B_2, ...\}$ as $B_1 = A_1$ and $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ for $n \geq 2$. Then, B_n for $n = 1, 2, ...$ are mutually exclusive. Using the third axiom of probability

$$P\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} P(B_n)$$

Furthermore, $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$ for every $n = 1, 2, ...$ while $P(B_i) \leq P(A_i)$. Hence,

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} P(B_i) \leq \sum_{i=1}^{n} P(A_i)$$

Taking limit as $n \to \infty$ from both sides, we have

$$P\left(\bigcup_{i=1}^{\infty} A_n\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

which completes the proof.

Solution Q5

There is a 25% chance the offspring of the parents will develop the disease. Then, Y = Number of offspring that develop the disease is binomial with $n = 3$ and $p = 0.25$.

(a) $P(Y = 3) = 0.25^3 = 0.015625$

(v) $P(Y = 1) = 3(.25)(.75^2) = 0.421875$

(c) Since the pregnancies are mutually independent, the probability is simply 25%.

Solution Q6

Let Y = Number of employees tested until three positives are found. Then, Y is negative binomial with $r = 3$ and $p = 0.4$. Therefore,

$$P(Y = 10) = \begin{pmatrix} 9 \\ 2 \end{pmatrix} 0.4^3 0.6^7 = 0.06$$

Solution Q7

Let Y = Number of treated seeds selected. Then, Y has the Hypergeometric distribution with $HG(10, 5, 5)$

(a) $P(Y = 4) = \dfrac{\begin{pmatrix} 5 \\ 4 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \end{pmatrix}}{\begin{pmatrix} 9 \\ 4 \end{pmatrix}} = 0.0238$

(b) $P(Y \leq 3) = 1 - P(Y = 4) = 1 - 0.0238 = 0.9762$

(c) This means at most 3 emerged from treated seeds which has the same probability as part b; that is 0.9762

Solution Q8

Using the Poisson approximation, $\lambda \approx np = 100(.03) = 3$. Hence

$$P(Y \geq 1) = 1–P(Y = 0) = 0.9524$$

Solution Q9

(a)

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2}y^2 & 0 \leq y < 1 \\ 2y - \frac{1}{2}y^2 - 1 & 1 \leq y < 2 \\ 1 & y \geq 2 \end{cases}$$

(b) $P(0.85 \leq Y \leq 1.15) = F_Y(1.15) - F_Y(0.85) = 0.2775$

(c) $E(Y) = \int_0^1 y \cdot y dy + \int_1^2 y \cdot (2 - y) dy = 1$. 10,000 gallons expected at \$2.10 per gallon gives an expected revenue of \$21,000.

Solution Q10

Let Y = be the measured resistance of a randomly selected wire.

(a) By standardizing the normal distribution, we have

$$P(0.12 \leq Y \leq 0.14) = P(\frac{0.12 - 0.13}{0.005} \leq Z \leq \frac{0.14 - 0.13}{0.005}) = P(-2 \leq Z \leq 2) = 0.9544$$

(b) Let X = Number of wires that do not meet specifications. Then, X is binomial with $n = 4$ and $p = 0.9544$. Thus, $P(X = 4) = 0.9544^4 = 0.8297$

Solution Q11

(a) By the definition of the expectation of a function of random variable, we have

$$E(e^{3Y/2}) = \int_{-\infty}^{0} e^{3y/2} e^{y} dy = \frac{2}{5} e^{5y/2} \Big|_{-\infty}^{0} = \frac{2}{5}$$

(b) By the definition of the Moment-generating function, we have

$$m(t) = E(e^{tY}) = \int_{-\infty}^{0} e^{ty} e^{y} dy = \frac{1}{t+1} \qquad t > -1$$

(c) By using the result of the Moment-generating function, $E(Y) = \frac{\partial}{\partial t}(t+1)^{-1}\Big|_{t=0} = -1$, and

$$E(Y^2) = \frac{\partial}{\partial t}[-(t+1)^{-2}]\Big|_{t=0} = 2$$

Hence, $V(Y) = 2 – (–1)^2 = 1$.