Moderation

Fundamentals:

- Moderation refers to a change in the relationship between an independent variable and a dependent variable, depending on the level of a third variable, termed the moderator variable. Moderating effects are also referred to as interaction and conditioning effects.

    * For two continuous variables, moderation means that the slope of the relationship between the independent and dependent variable varies (i.e., increases or decreases) according to the level of the moderator variable.

    * For a continuous independent variable and a categorical moderator variable, moderation means that the slope of the relationship between the independent and dependent variable differs across the groups represented by the categorical moderator variable.

    * For a categorical independent variable and a continuous moderator variable, moderation means that the differences between the group means represented by the levels of the categorical independent variable differ according to the level of the moderator variable.

    * For two categorical variables, moderation means that the difference between the group means for the categorical independent variable differ depending on group membership on the moderator variable.

- When the predictor and moderator variables are continuous, a single product is needed to capture the moderating effect. When one variable is continuous and the other is categorical, the required number of product terms is $g - 1$, where g equals the number of groups represented by the categorical variable. When both variables are categorical, the required number of product terms is $(g_1 - 1)(g_2 - 1)$, where g_1 and g_2 are the number of groups represented by the two categorical variables.

- Interactions can range up to the kth order, where k represents the number of variables on the right side of the equation. The kth order interaction is tested using the kth order product term, controlling for the first-order terms and all products formed by dropping one or more terms from the kth order product. Thus, a three-way interaction is testing using $X_1X_2X_3$ controlling for X_1, X_2, X_3, X_1X_2, X_1X_3, and X_2X_3. Analogously, a four-way interaction is testing using $X_1X_2X_3X_4$ controlling for X_1, X_2, X_3, X_4, X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_4, $X_1X_2X_3$, $X_1X_2X_4$, $X_1X_3X_4$, and $X_2X_3X_4$.
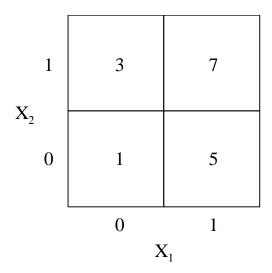
- Moderation is represented not by a product term (or set of product terms) itself, but rather by a product term from which all lower-order terms constituting the product have been partialed.

- A curvilinear effect is a special case of moderation in which the relationship between an independent and dependent variable varies according to the level of the independent variable itself. For example, a U-shaped relationship between X and Y indicates a negative slope at low levels of X, a flat slope at moderate levels of X, and a positive relationship at high levels of X.

- Moderator variables should be distinguished from mediator variables, in that the latter transmit the effect of the independent variable on the dependent variable. For example, X_2 is a mediator of the effect of X_1 on Y if X_1 influences X_2, which in turn influences Y. In contrast, X_2 is a moderator of the effect of X_1 on Y if the relationship between X_1 and Y varies according to the level of X_2. Although these processes are fundamentally different, the terms mediator and moderator are often confused in the literature.
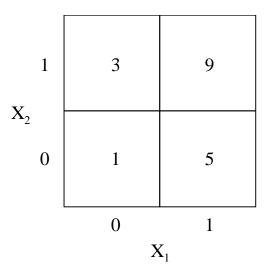
Methods of Analysis:

- Moderation can be tested by forming product terms using the variables that constitute the interaction and estimating the increment in variance explained by the product terms after the lower-order terms they contain have been controlled. However, the form and interpretation of these terms can vary, depending on whether the constituent variables are continuous or categorical and on the number of levels of the categorical variable.

- Two Categorical Variables: For two dichotomous variables (i.e., categorical variables with only two levels), moderation is estimated as follows:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_1X_2 + e$$

If X_1 and X_2 are dummy coded (i.e., 0, 1), then the product X_1X_2 will equal 1 when both X_1 and X_2 are 1 and will equal zero otherwise. Thus, X_1X_2 can be viewed as a third dummy variable that is coded 1 when X_1 and X_2 are both 1. If X_1 and X_2 are arranged in a 2 x 2 table, the coefficient on X_1X_2 (i.e., b_3) can be interpreted as the change in the mean differences of Y predicted by X_1 as one moves from $X_2 = 0$ to $X_2 = 1$. This interpretation is illustrated by the following table:

X_2	$X_1 = 0$	$X_1 = 1$
1	3	7
0	1	5

The entries in this table represent the mean of Y for different combinations of X_1 and X_2. The difference in the mean of Y for $X_1 = 0$ and $X_1 = 1$ is $5 - 1 = 4$ for $X_2 = 0$ and $7 - 3 = 4$ for $X_2 = 1$, both of which equal 4. Thus, there is no interaction between X_1 and X_2. Compare this to the following table:
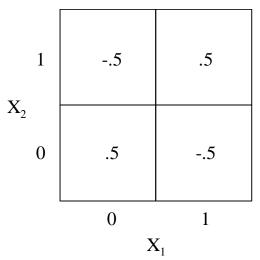
X_2	$X_1 = 0$	$X_1 = 1$
1	3	9
0	1	5

In this case, the mean difference in Y for $X_1 = 0$ and $X_1 = 1$ is $5 - 1 = 4$ for $X_2 = 0$ but is $9 - 3 = 6$ for $X_2 = 1$. This represents an interaction between X_1 and X_2, because the effect of X_1 varies, depending on the level of X_2.

In the above table, the mean of Y when X_1 and X_2 are both 1 is higher than in the table where X_1 and X_2 do not interact. It makes intuitive sense that this type of interaction could be captured by a dummy variable that contrasts the cell where X_1 and X_2 are both 1 against the other three cells, as represented by the $X_1 X_2$ product. However, due to the partialing that occurs in regression analysis, the $X_1 X_2$ product will capture an interaction between X_1 and X_2 of any form.

This principle is illustrated by estimating the means in the above table with X_1 and X_2 but not X_1X_2 as predictors. Assuming each cell contains the same number of observations, the effect for X_1 is the average of the mean differences in Y for $X_1 = 0$ and $X_1 = 1$ across levels of X_2. The difference was 4 when $X_2 = 0$ and 6 when $X_2 = 1$, and thus the average is 5. Under these conditions, the estimated means using X_1 and X_2 as predictors are:
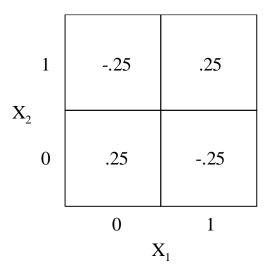
		0	1
	1	3.5	8.5
X_2			
	0	.5	5.5

$$X_1$$

An effect of 5 for X_1 is one unit too large when $X_2 = 0$ and is one unit too small when $X_2 = 1$. These discrepancies produce the following pattern of mean residuals:

		0	1
	1	-.5	.5
X_2			
	0	.5	-.5

$$X_1$$

Likewise, the effect for X_2 is the average of the mean differences in Y for $X_2 = 0$ and $X_2 = 1$ across levels of X_1. The difference was 2 when $X_1 = 0$ and 4 when $X_1 = 1$, and therefore the average is 3. Thus, the effect of X_2 is one unit too large when $X_1 = 0$ and one unit too small when $X_1 = 1$. Again, these discrepancies are consistent with the pattern of mean residuals shown above.

Although the pattern of mean residuals shown above does not correspond to the contrast represented by the X_1X_2 product, it corresponds to the partialed X_1X_2 product. This can be seen by residualizing X_1X_2 on X_1 and X_2, which yields the following values for X_1X_2 for the four combinations of X_1 and X_2:

	$X_1 = 0$	$X_1 = 1$
$X_2 = 1$	-.25	.25
$X_2 = 0$	.25	-.25

The residual values for X_1X_2 are half of the residual values for Y. Hence, after controlling for X_1 and X_2, the X_1X_2 product is perfectly correlated with the mean residuals for Y and therefore will explain the variance in the means of Y not explained by X_1 and X_2. The pattern of mean residuals for Y is the same for any interaction between X_1 and X_2, and therefore the partialed X_1X_2 product will always capture the interaction between X_1 and X_2.

For categorical variables with more than two levels, an interaction is represented by all pairwise products between the dichotomous variables used to represent the two categorical variables. For example, assume that both categorical variables represent three groups, and that two groups for the first variable are represented by the dummy variables X_1 and X_2, and two groups for the second variable are represented by the dummy variables X_3 and X_4. An equation containing their interaction would be as follows:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4$$

$$+ b_5X_1X_3 + b_6X_1X_4 + b_7X_2X_3 + b_8X_2X_4 + e$$

The four product terms X_1X_3, X_1X_4, X_2X_3, and X_2X_4 represent the interaction between the two original categorical variables. In general, these terms should be tested and interpreted as a set. If a significant increment in R^2 is yielded by these terms, an interaction is indicated, and the nine means associated with the 3x3 classification of the two categorical variables can be compared to determine the form of the interaction.

- <u>One Continuous and One Categorical Variable</u>: The interaction between a continuous and dichotomous variable is estimated as follows:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_1X_2 + e$$

If X_2 is dichotomous and framed as the moderator variable, then b_3 can be interpreted as the difference in the slope of Y on X_1 for the two groups represented by X_2. This can be seen by treating X_2 as a dummy variable and substituting either 0 or 1 for X_2 in the preceding equation. For example, if $X_2 = 0$, this equation becomes:

$$Y = b_0 + b_1X_1 + e$$

Thus, b_1 represents the slope of Y on X_1 for the omitted group (i.e., the group for which $X_2 = 0$). If $X_2 = 1$, the equation becomes:

$$Y = (b_0 + b_2) + (b_1 + b_3)X_1 + e$$

These expressions give simple slopes relating X_1 to Y when X_2 equals 0 and 1, respectively. Note that, if $b_3 = 0$, the slope of Y on X_1 is the same for both groups (i.e., for $X_2 = 0$ and $X_2 = 1$), indicating no interaction between X_1 and X_2.

If a categorical moderator variable has more than two levels, its interaction with a continuous variable is represented by the products of the continuous variable with the $g - 1$ dichotomous variables used to represent the categorical variable. For example, if the continuous variable is represented by X_1 and the categorical variable has three levels and is represented by the dummy variables X_2 and X_3, the equation containing their interaction is as follows:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_1X_2 + b_5X_1X_3 + e$$

The equation for the omitted group is found by substituting 0 for X_2 and X_3:

$$Y = b_0 + b_1X_1 + e$$

The equation for the group represented by X_2 (i.e., when $X_2 = 1$) is:

$$Y = (b_0 + b_2) + (b_1 + b_4)X_1 + e$$

The equation for the group represented by X_3 (i.e., when $X_3 = 1$) is:

$$Y = (b_0 + b_3) + (b_1 + b_5)X_1 + e$$

Thus, the difference between the slope for the omitted group and the group represented by X_2 is reflected by b_4, and the difference between the slope for the omitted group and the group represented by X_3 is reflected by b_5. A joint (i.e., omnibus) test of whether the slopes for the three groups differ from one another is performed by simultaneously testing the constraints $b_4 = 0$ and $b_5 = 0$. The slopes for the groups represented by X_2 and X_3 can be compared by testing the difference between b_4 and b_5.

- Two Continuous Variables: When the independent and moderator variables are continuous, their interaction is estimated as follows:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_1X_2 + e$$

The change in slope of Y on X_1, given a unit change in X_2, is represented by b_3. Note that the interpretation of b_3 is symmetric, in that it also represents the change in slope of Y on X_2, given a unit change in X_1.

The preceding equation can be rewritten to explicitly show how the slope and intercept of the equation relating X_1 to Y varies according to X_2:

$$Y = (b_0 + b_2X_2) + (b_1 + b_3X_2)X_1 + e$$

This equation represents the simple slope relating X_1 to Y at a given level of X_2 as the compound coefficient $b_1 + b_3X_2$. Note that the slope of Y on X_1 changes by b_3 for a unit change in X_2. This equation also shows that b_1 represents the slope of Y on X_1 when $X_2 = 0$.

The same expression for the slope of Y on X_1 is obtained by taking the partial derivative of Y with respect to X_1 in the preceding equation:

$$\delta Y/\delta X_1 = b_1 + b_3X_2$$

This again shows that the slope of Y on X_1 is not a constant, but instead varies according to the level of X_2, and that the slope of Y on X_1 at the point $X_2 = 0$ is represented by b_1.

Higher-Order Interactions:

- Higher-order interactions can be tested by extending the procedures described previously. For example, the interaction between three continuous variables X_1, X_2, and X_3 can be tested with the following equation:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + b_4X_1X_2 + b_5X_1X_3 + b_6X_2X_3 + b_7X_1X_2X_3 + e$$

In this equation, the three-way interaction is represented by the partialed triple product $X_1X_2X_3$. Thus, b_7 can be interpreted as the change in the interaction between any two variables (e.g., X_1 and X_2) according to the level of the third variable (e.g., X_3). Note that the equation for the three-way interaction contains the three first-order terms and all interactions terms of lower order (i.e., the three two-way interactions). This procedure can be extended to interactions of higher order.

Testing Simple Slopes:

-    In moderated regression, simple slopes refer to the relationship between the independent variable and the dependent variable at selected levels of the moderator variable. To illustrate, reconsider a regression equation in which the relationship between X_1 and Y is moderated by the dummy variable X_2:

$$Y = b_0 + b_1X_1 + b_2X_2 + b_3X_1X_2 + e$$

When $X_2 = 0$, this equation reduces to:

$$Y = b_0 + b_1X_1 + e$$

When $X_2 = 1$, the equation becomes:

$$Y = (b_0 + b_2) + (b_1 + b_3)X_1 + e$$

In these two expressions, the coefficients on X_1 are simple slopes relating X_1 to Y at $X_2 = 0$ and $X_2 = 1$. When $X_2 = 0$, the slope relating X_1 to Y is simply b_1, and this coefficient can be tested in the usual manner. When $X_2 = 1$, the slope relating X_1 to Y is given by the expression $b_1 + b_3$, which can be tested using procedures for testing linear combinations of regression coefficients. For instance, a t-test can be conducted for $b_1 + b_3$ by dividing this sum by its standard error, which can be obtained by taking the square root of the variance of $b_1 + b_3$. This variance can be derived using the squares of the standard errors of b_1 and b_3 to represent their variances and by computing the covariance between b_1 and b_3 as the product of their correlation with their standard errors. The resulting expression can be written as follows:

$$V(b_1 + b_3) = V(b_1) + V(b_3) + 2C(b_1,b_3)$$

Dividing $b_1 + b_3$ by the square root of this expression yields a t-test for the simple slope $b_1 + b_3$.

When X_2 is continuous, the simple slope relating X_1 to Y at a given level of X_2 can be obtained from the following expression, which is a simple algebraic manipulation of the conventional moderated regression equation:

$$Y = (b_0 + b_2X_2) + (b_1 + b_3X_2)X_1 + e$$

The simple slope relating X_1 to Y at a level of X_2 is the compound coefficient $b_1 + b_3X_2$. A t-test can be conducted by dividing this expression by an estimate of its standard error, which is obtained by taking the square root of the variance of $b_1 + b_3X_2$. The variance of this expression can be written as:

$$V(b_1 + b_3X_2) = V(b_1) + X_2^2V(b_3) + 2X_2C(b_1,b_3)$$

Note that, in this equation, X_2 is a value that represents a point of interest on the X_2 scale. Such points could include the mean of X_2, values one standard deviation above and below the mean of X_2, or other values chosen for theoretical or practical reasons.

Another method for testing simple slopes is to center the moderator variable at the value at which the simple slope is to be tested, recompute the product term, reestimate the regression equation using the independent variable, the rescaled moderator variable, and the recomputed product term, and test the coefficient on the independent variable. This approach draws from the principle that the coefficient on the independent variable equals the slope of the independent variable when the moderator variable equals zero. Thus, shifting the zero point of the moderator variable to a value that corresponds to the desired simple slope means that the coefficient on the independent variable represents the simple slope of interest. This procedure can be repeated by centering the moderator variable at values that correspond to different simple slopes, recomputing the product term, and reestimating the regression equation.

Simple slopes can also be tested using other procedures for testing linear combinations of regression coefficients. These procedures include matrix approaches in general linear modeling programs and methods that involve imposing the null hypothesis that a simple slope equals zero on a regression equation and testing the reduction in R^2 relative to the original moderated regression equation in which this constraint is relaxed.

<u>Scaling Issues</u>:

- In a regression equation using X_1, X_2, and X_1X_2 as predictors, adding or subtracting a constant to X_1 changes the coefficient on X_2, and vice-versa, provided that the coefficient on X_1X_2 is nonzero. Such transformations are permissible for interval measures, which are widely used in the social sciences, because such measures have no true origin (i.e., zero point). The effects of scaling X_1 and X_2 can be seen with the following variables:

$$X_1^* = X_1 + c$$

$$X_2^* = X_2 + d$$

Using X_1^*, X_2^*, and their product in a regression equation predicting Y yields:

$$Y = b_0^* + b_1^* X_1^* + b_2^* X_2^* + b_3^* X_1^* X_2^* + e$$

Substitution yields:

$$Y = b_0^* + b_1^*(X_1 + c) + b_2^*(X_2 + d) + b_3^*(X_1 + c)(X_2 + d) + e$$

$$= (b_0^* + cb_1^* + db_2^* + cdb_3^*) + (b_1^* + db_3^*)X_1 + (b_2^* + cb_3^*)X_2 + b_3^* X_1 X_2 + e$$

Comparing the preceding equation to an equation containing X_1 and X_2 in their original form yields the following equalities:

$$b_0 = b_0^* + cb_1^* + db_2^* + cdb_3^*$$

$$b_1 = b_1^* + db_3^*$$

$$b_2 = b_2^* + cb_3^*$$

$$b_3 = b_3^*$$

Thus, although adding the constants c and d to X_1 and X_2 does not influence the coefficient on X_1X_2 (i.e., b_3), it changes the coefficient on X_1 by a factor of db_3^* and the coefficient on X_2 by a factor of cb_3^* (the change in the intercept, i.e., cb_1^* + db_2^* + cdb_3^*, is usually irrelevant in tests of the joint effects of X_1 and X_2 on Y). Because c and d are arbitrary constants, some investigators consider the coefficients on X_1 and X_2 equally arbitrary, because values of c and d can be selected to make these coefficients take on virtually any value, provided b_3 is nonzero.

This apparent dilemma can be resolved by recalling that b_1 represents the slope of Y on X_1 where $X_2 = 0$. Selecting different values of d shifts the zero point of X_2 and effectively selects different points in the distribution of X_2 to evaluate the slope of Y on X_1. For example, to test the slope of Y on X_1 at the mean of X_2, d would be set equal to the negative of the mean of X_2. This strategy is used to test the slope of Y on X_1 at other values of X_2, such as one standard deviation above or below the mean of X_2. Although the slope of Y on X_1 could be forced to take on any arbitrary value based on the selection of d, it makes no sense to select values that extend beyond the range of X_2, because this would require extrapolating beyond the bounds of the data.

When X_1 and X_2 are multiplied by arbitrary constants, the coefficients on X_1, X_2, and X_1X_2 will change. However, the standard error of the coefficient on X_1X_2 (i.e., b_3) will change accordingly, such that tests of significance for b_3 are unaffected. The interpretations of b_1 and b_2 will also remain the same, representing the slope for X_1 at $X_2 = 0$ and the slope for X_2 for $X_1 = 0$, respectively.

Multicollinearity:

-   The product X_1X_2 is often highly correlated with X_1 and X_2, as when both X_1 and X_2 take on only positive values. This phenomenon represents non-essential ill conditioning and can be reduced by rescaling X_1 and X_2 such that they are centered near zero (this is often accomplished by centering X_1 and X_2 at their means). Note that rescaling X_1 and X_2 has no effect on tests of b_3, but it changes the interpretation of b_1 and b_2, such that they represent the slopes of Y on X_1 and X_2 at the rescaled zero points of X_2 and X_1, respectively. Rescaling X_1 and X_2 might facilitate interpretation, but there is little reason to rescale them to reduce multicollinearity, because substantive interpretation is unaffected by the scaling of X_1 and X_2.

Reliability:

-   The reliability of a product term is often less than that of either variable constituting the product. If X_1 and X_2 are orthogonal, the reliability of X_1X_2 is the product of the reliabilities of X_1 and X_2, which is necessarily lower than the reliability of either X_1 or X_2. If X_1 and X_2 are positively correlated, the reliability of X_1X_2 will be somewhat higher than the product of the reliabilities of X_1 and X_2. Measurement error drastically reduces the power to detect moderating effects.