This time: constraint satisfaction

- Constraint Satisfaction Problems (CSP)
- Backtracking search for CSPs
- Local search for CSPs

Constraint satisfaction problems

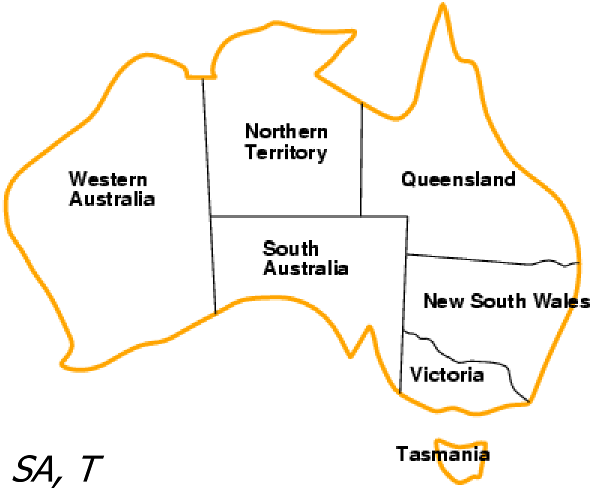
Standard search problem:

- state is a “black box” – any data structure that supports successor function, heuristic function, and goal test

CSP:

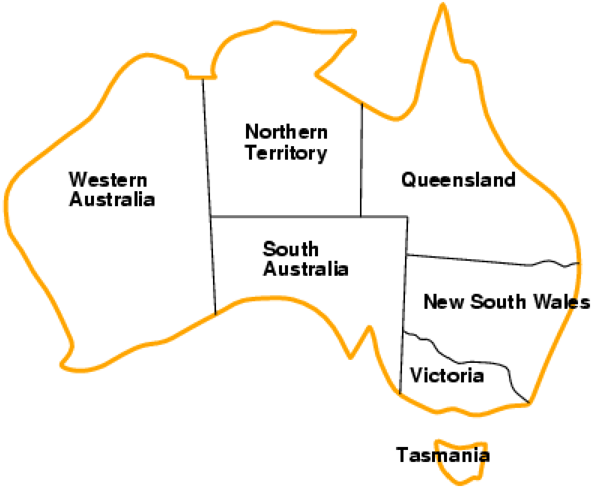
- state is defined by variables Xi with values from domains Di
- goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a formal representation language
- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: map coloring problem



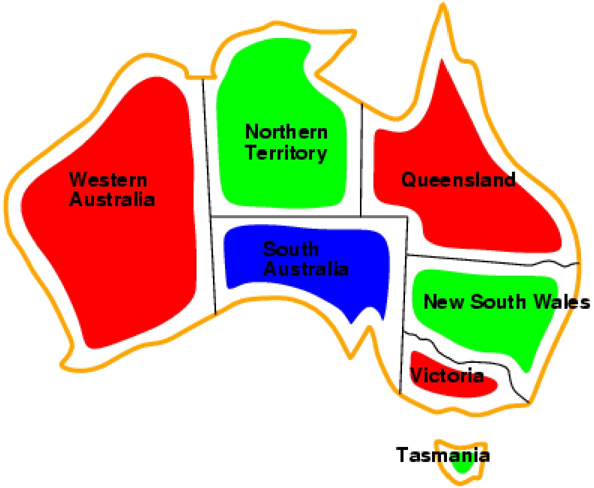
- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: D_i = {red, green, blue}   (one for each variable)
- Constraints: $C_i = <scope, rel>$ where *scope* is a tuple of variables and *rel* is the relation over the values of these variables
  - E.g., here, adjacent regions must have different colors
  - e.g., WA ≠ NT, or (WA,NT) in {(red,green), (red,blue), (green,red), (green,blue), (blue,red), (blue,green)}

Example: map coloring problem



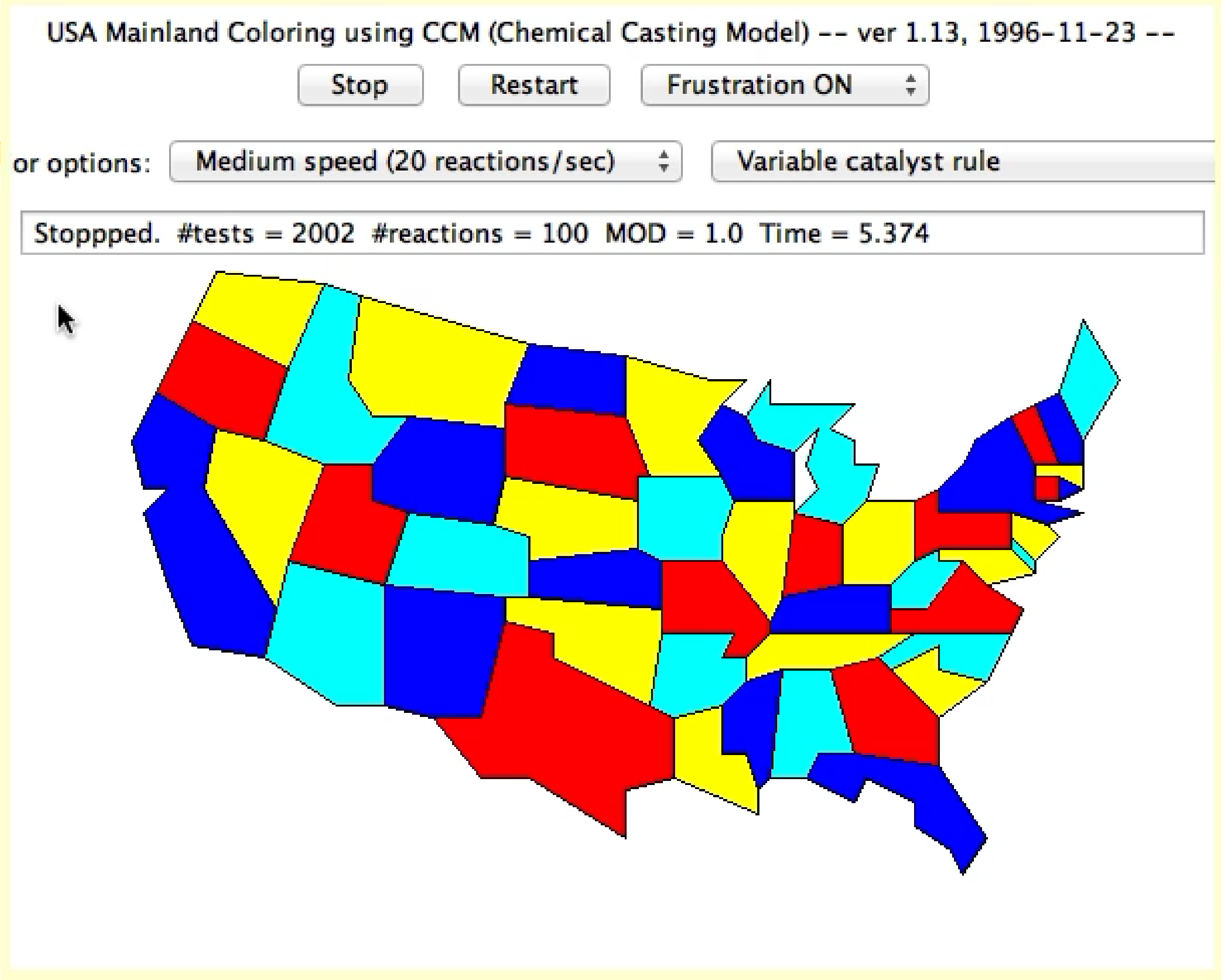
- Assignment: values are given to some or all variables
- Consistent (legal) assignment: assigned values do not violate any constraint
- Complete assignment: every variable is assigned
- Solution to a CSP: a consistent and complete assignment

Example: map coloring problem



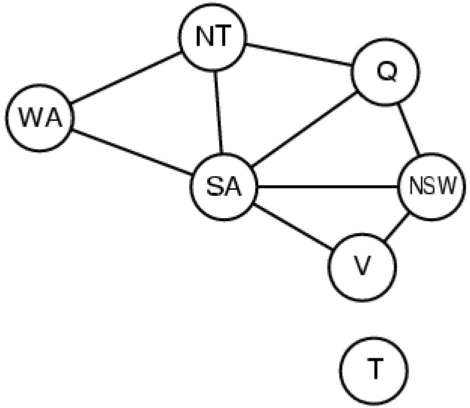
- Solutions are complete and consistent assignments,
- e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

Demo


USA Mainland Coloring using CCM (Chemical Casting Model) -- ver 1.13, 1996-11-23 --
Stop
Restart
Frustration ON
or options:
Medium speed (20 reactions/sec)
Variable catalyst rule
Stoppped. #tests = 2002 #reactions = 100 MOD = 1.0 Time = 5.374

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints



Varieties of CSPs

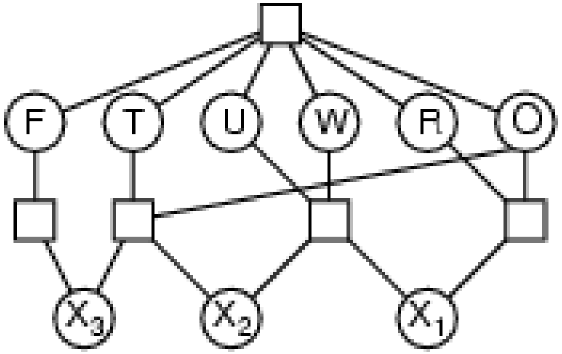
- Discrete variables
  - finite domains:
    - n variables, domain size $d \rightarrow O(d^n)$ complete assignments
    - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- Continuous variables
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order (sometimes called global) constraints involve 3 or more variables,
  - e.g., cryptarithmetic column constraints

Example: cryptarithmetic

$$\begin{array}{cccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



- Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
- Domains: $\{0,1,2,3,4,5,6,7,8,9\}$
- Constraints:
  - *Alldiff (F,T,U,W,R,O)*
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F,\ T \neq 0,\ F \neq 0$

Constraint hypergraph
Circles: nodes for variable
Squares: hypernodes for n-ary constraints

Real-world CSPs

- Assignment problems
  - e.g., who teaches what class
- Timetabling problems
  - e.g., which class is offered when and where?
- Transportation scheduling
- Factory scheduling
- Notice that many real-world problems involve real-valued variables

Example: sudoku

			2	6		7		1
6	8			7			9	
1	9				4	5		
8	2		1				4	
		4	6		2	9		
	5				3		2	8
		9	3				7	4
	4			5			3	6
7		3		1	8			

?

Variables: each square (81 variables)
Domains: [1 .. 9]
Constraints: each column, each row, and each of the nine 3×3 sub-grids that compose the grid contain all of the digits from 1 to 9

Example: sudoku

			2	6		7		1
6	8			7			9	
1	9				4	5		
8	2		1				4	
		4	6		2	9		
	5				3		2	8
		9	3				7	4
	4			5			3	6
7		3		1	8			

4	3	5	2	6	9	7	8	1
6	8	2	5	7	1	4	9	3
1	9	7	8	3	4	5	6	2
8	2	6	1	9	5	3	4	7
3	7	4	6	8	2	9	1	5
9	5	1	7	4	3	6	2	8
5	1	9	3	2	6	8	7	4
2	4	8	9	5	7	1	3	6
7	6	3	4	1	8	2	5	9

Variables: each square (81 variables)
Domains: [1 .. 9]
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Formulation as a search problem

Let's start with the straightforward approach, then fix it

States are defined by the values assigned so far

- Initial state: the empty assignment { }
- Successor function: assign a value to an unassigned variable that does not conflict with current assignment
  → fail if no legal assignments

- Goal test: the current assignment is complete

1. This is the same for all CSPs
2. Every solution appears at depth *n* with *n* variables → use depth-first search
3. Path is irrelevant, so can be discarded
4. $b = (n - \ell)d$ at depth ℓ, hence $n! \cdot d^n$ leaves

Backtracking search

- Variable assignments are commutative, i.e.,
  [WA = red then NT = green] same as [NT = green then WA = red]

- Only need to consider assignments to a single variable at each node
  → b = d and there are d^n leaves

- Depth-first search for CSPs with single-variable assignments is called backtracking search

- Backtracking search is the basic uninformed algorithm for CSPs

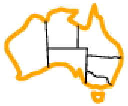
- Can solve *n*-queens for $n \approx 25$

Backtracking search (note: textbook has a slightly more complex version)

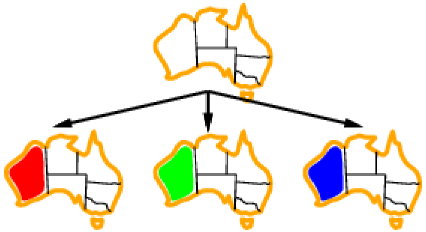
```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
    return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment according to Constraints[csp] then
            add { var = value } to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failue then return result
            remove { var = value } from assignment
    return failure
```

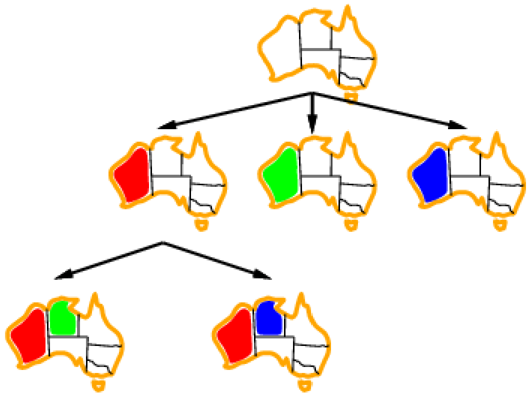
Backtracking example



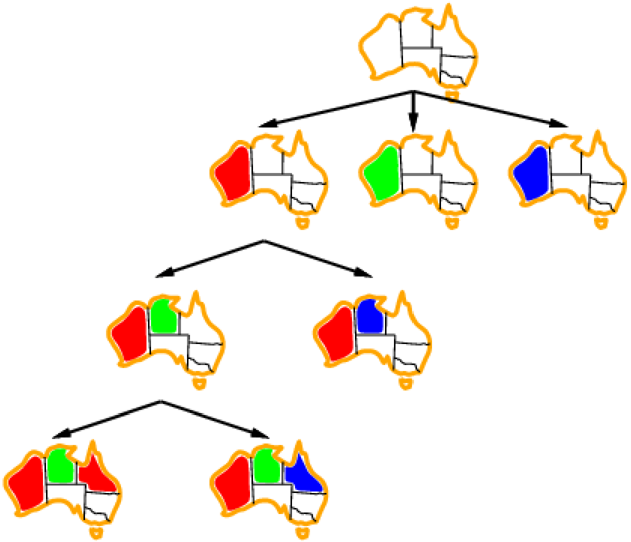
Backtracking example



Backtracking example



Backtracking example



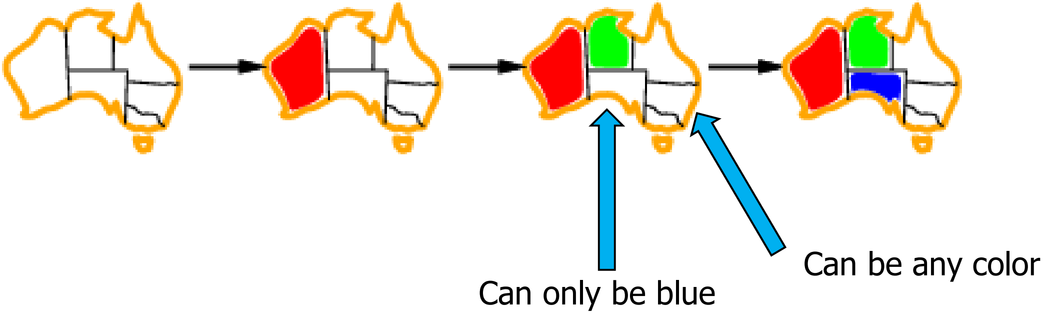
Improving backtracking efficiency

- General-purpose methods can give huge gains in speed (like using heuristics in informed search):
    - Which variable should be assigned next?
    - In what order should its values be tried?
    - Can we detect inevitable failure early?

Most constrained variable

- Most constrained variable:

  choose the variable with the fewest legal values

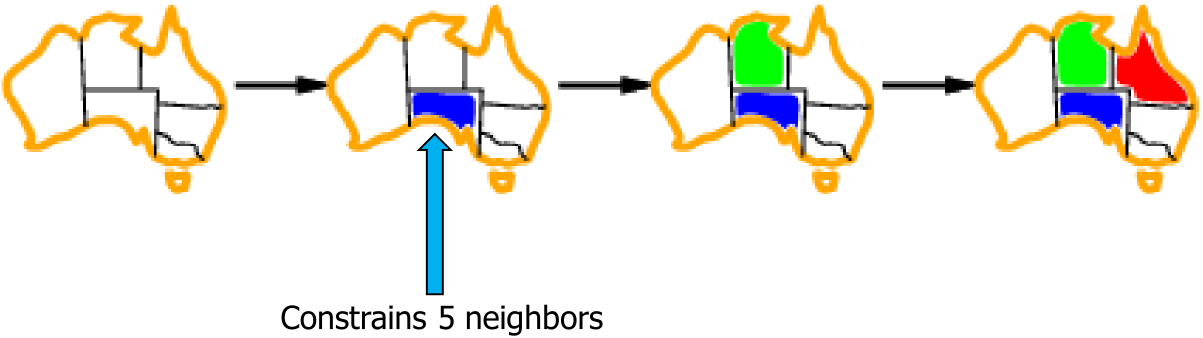


- a.k.a. minimum remaining values (MRV) heuristic

Most constraining variable

- Tie-breaker among most constrained variables
- Most constraining variable:

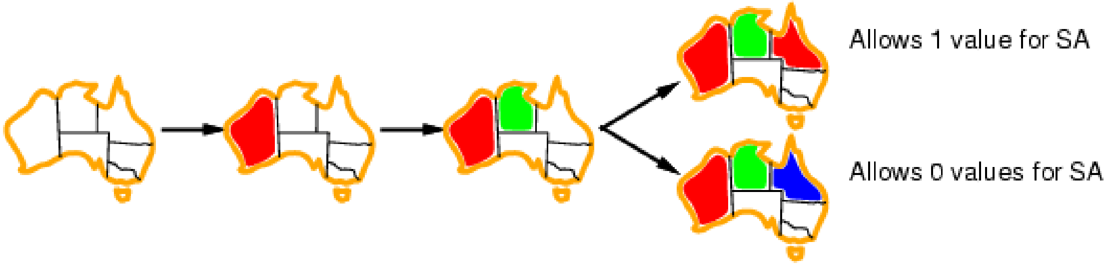
  - choose the variable with the most constraints on remaining variables



  - also known as the **degree heuristic**

Least constraining value

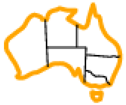
- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables

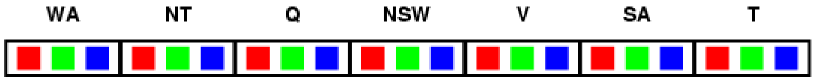


- Combining these heuristics makes 1000 queens feasible

Forward checking

- Idea:
  - Keep track of remaining legal values for unassigned variables (inference step)
  - Terminate search when any variable has no legal values





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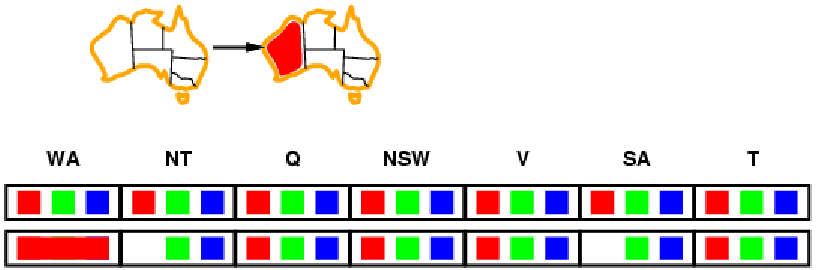

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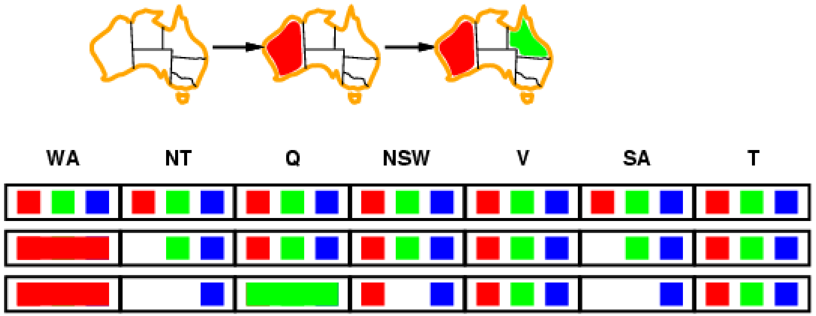

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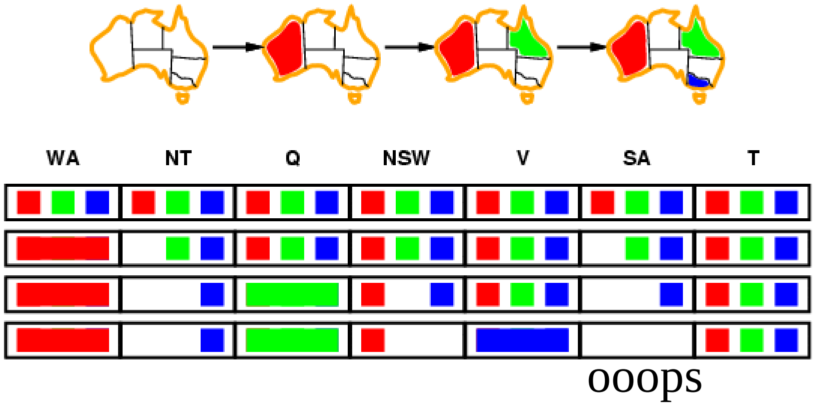
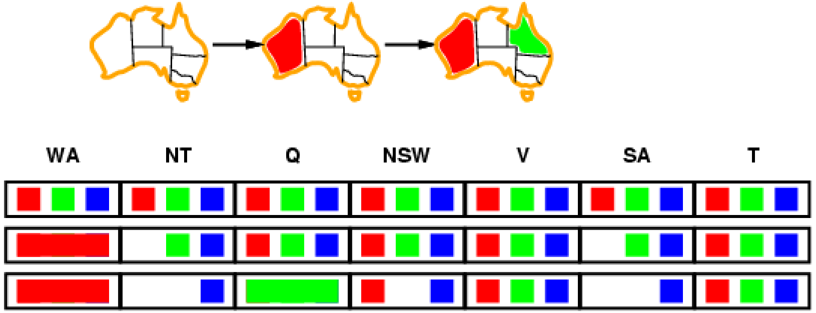


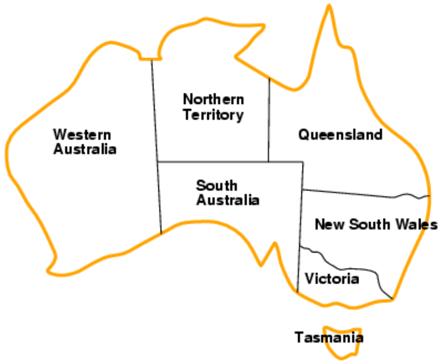
Constraint propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!

- Constraint propagation repeatedly enforces constraints locally



Node and Arc consistency

A single variable is node-consistent if all the values in its domain satisfy the variable's unary constraints

A variable is arc-consistent if every value in its domain satisfies the binary constraints

- i.e., Xi arc-consistent with Xj if for every value in Di there exists a value in Dj that satisfies the binary constraints on arc (Xi, Xj)

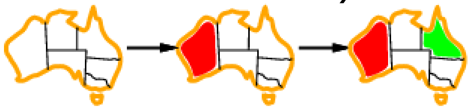
A network is arc-consistent if every variable is arc-consistent with every other variable.

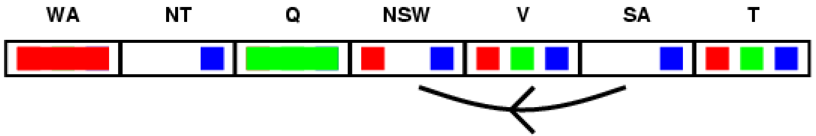
Arc-consistency algorithms: reduce domains of some variables to achieve network arc-consistency.

Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* → *Y* is consistent iff

for every value *x* of *X* there is some allowed *y*

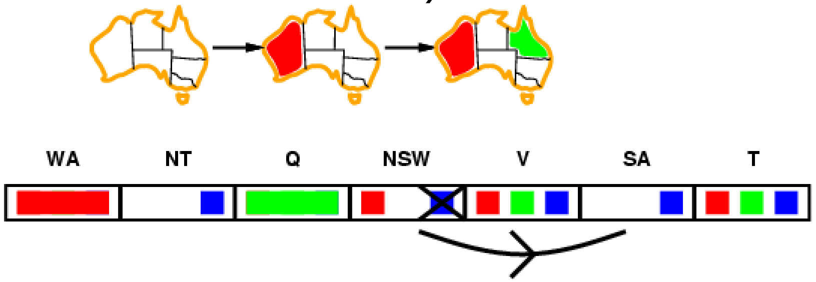




Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

for every value x of X there is some allowed y



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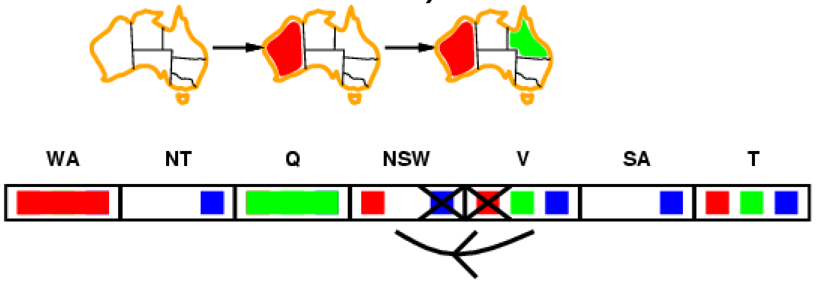

- If X loses a value, neighbors of X need to be rechecked

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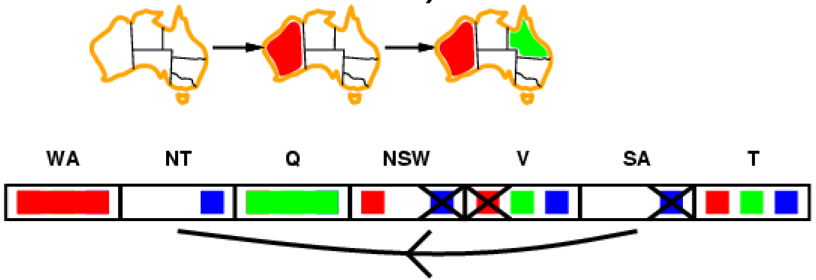

- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- After running AC-3, either every arc is arc-consistent or some variable has empty domain, indicating the CSP cannot be solved.
- Can be run as a preprocessor or after each assignment

Arc consistency algorithm AC-3

- Start with a queue that contains all arcs

- Pop one arc (Xi, Xj) and make Xi arc-consistent with respect to Xj
  - If Di was not changed, continue to next arc,
  - Otherwise, Di was revised (domain was reduced), so need to check all arcs connected to Xi again: add all connected arcs (Xk, Xi) to the queue. (this is because the reduction in Di may yield further reductions in Dk)
  - If Di is revised to empty, then the CSP problem has no solution.

Arc consistency algorithm AC-3

```
function AC-3( csp) returns the CSP, possibly with reduced domains
    inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
    local variables: queue, a queue of arcs, initially all the arcs in csp

    while queue is not empty do
        (Xi, Xj) ← REMOVE-FIRST(queue)
        if RM-INCONSISTENT-VALUES(Xi, Xj) then
            for each Xk in NEIGHBORS[Xi] do
                add (Xk, Xi) to queue
```

```
function RM-INCONSISTENT-VALUES( Xi, Xj) returns true iff remove a value
    removed ← false
    for each x in DOMAIN[Xi] do
        if no value y in DOMAIN[Xj] allows (x,y) to satisfy constraint(Xi, Xj)
            then delete x from DOMAIN[Xi];   removed ← true
    return removed
```

- Time complexity: ? (n variables, d values)

Arc consistency algorithm AC-3

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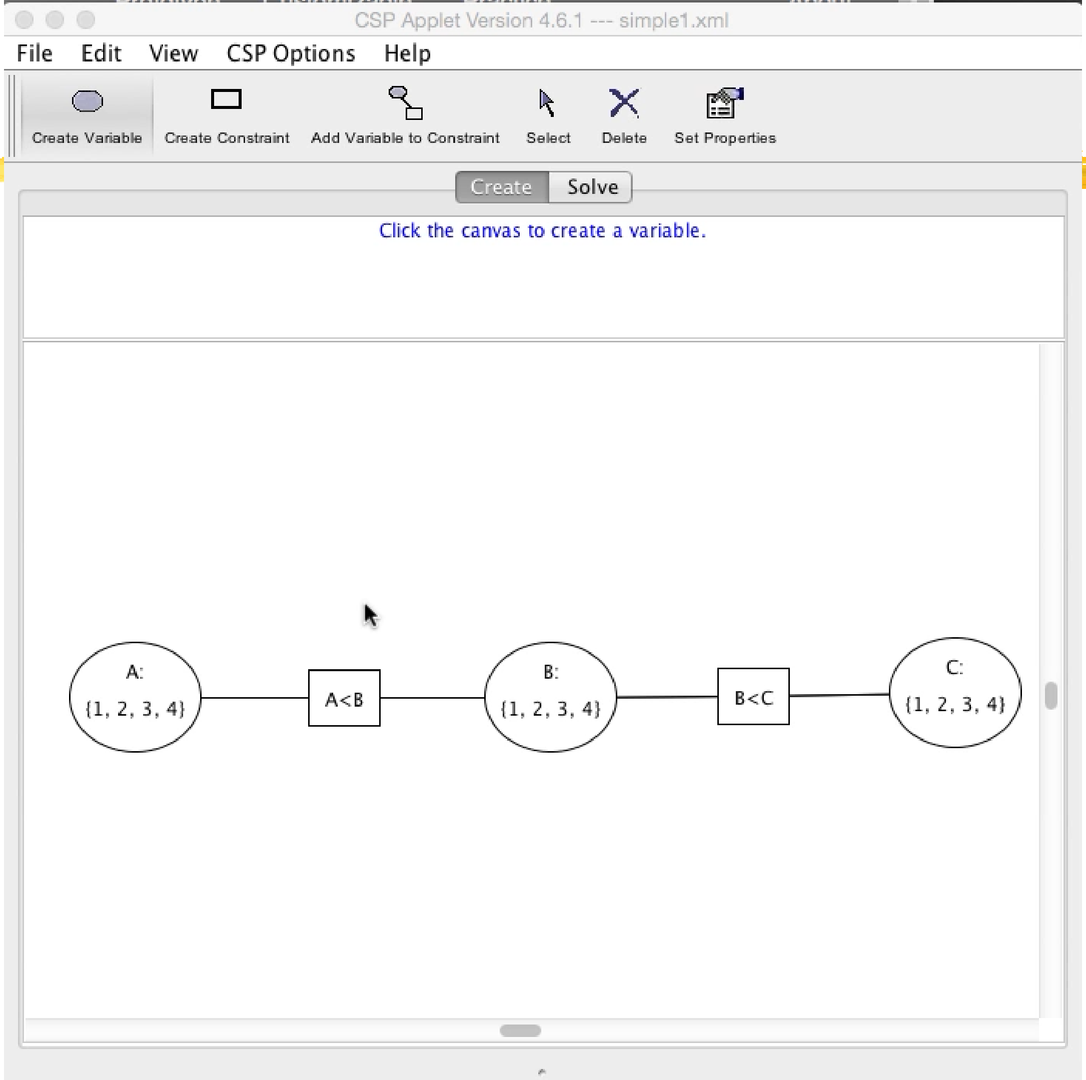
   return removed
```

- Time complexity: $O(n^2d^3)$ (n variables, d values)
- (each arc can be queued only d times, n^2 arcs (at most), checking one arc is $O(d^2)$)

Demo



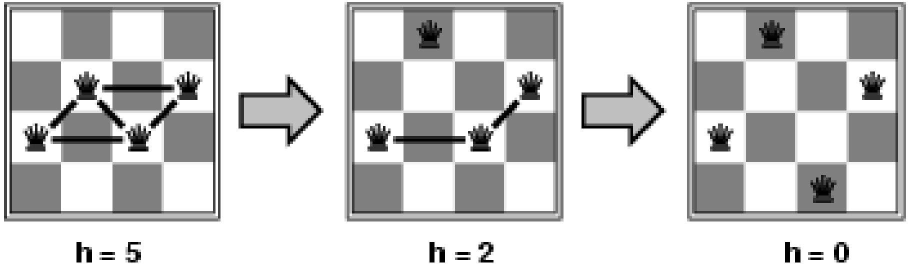
CSP Applet Version 4.6.1 --- simple1.xml
File
Edit
View
CSP Options
Help
Create Variable
Create Constraint
Add Variable to Constraint
Select
Delete
Set Properties
Create
Solve
Click the canvas to create a variable.
A:
{1, 2, 3, 4}
A<B
B:
{1, 2, 3, 4}
B<C
C:
{1, 2, 3, 4}

Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: $h(n)$ = number of attacks



- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)

Min-conflicts algorithm

```
function MIN-CONFLICTS(csp, max_steps) returns a solution or failure
  inputs: csp, a constraint satisfaction problem
          max_steps, the number of steps allowed before giving up

  current ← an initial complete assignment for csp
  for i = 1 to max_steps do
     if current is a solution for csp then return current
     var ← a randomly chosen conflicted variable from csp.VARIABLES
     value ← the value v for var that minimizes CONFLICTS(var, v, current, csp)
     set var = value in current
  return failure
```

Figure 6.8 The MIN-CONFLICTS algorithm for solving CSPs by local search. The initial state may be chosen randomly or by a greedy assignment process that chooses a minimal-conflict value for each variable in turn. The CONFLICTS function counts the number of constraints violated by a particular value, given the rest of the current assignment.

Example: N-Queens

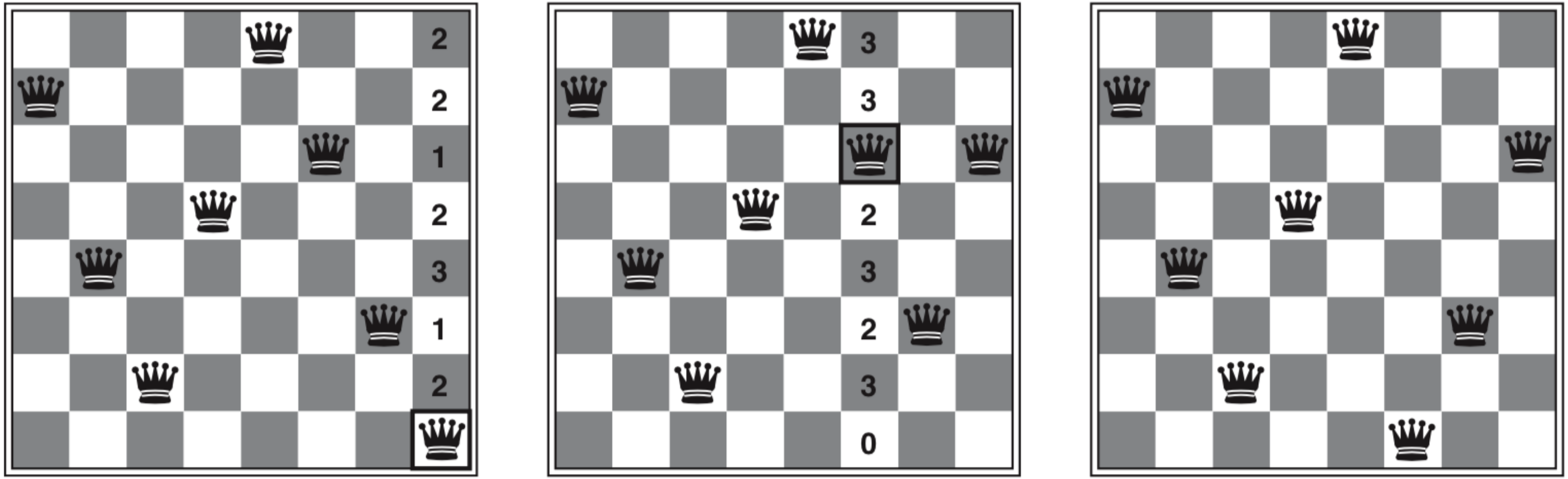


Figure 6.9 A two-step solution using min-conflicts for an 8-queens problem. At each stage, a queen is chosen for reassignment in its column. The number of conflicts (in this case, the number of attacking queens) is shown in each square. The algorithm moves the queen to the min-conflicts square, breaking ties randomly.

Summary

- CSPs are a special kind of problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice