AI for *Robotics*

(*Just Bayes Filter and a bit more*)

Guest Lecture

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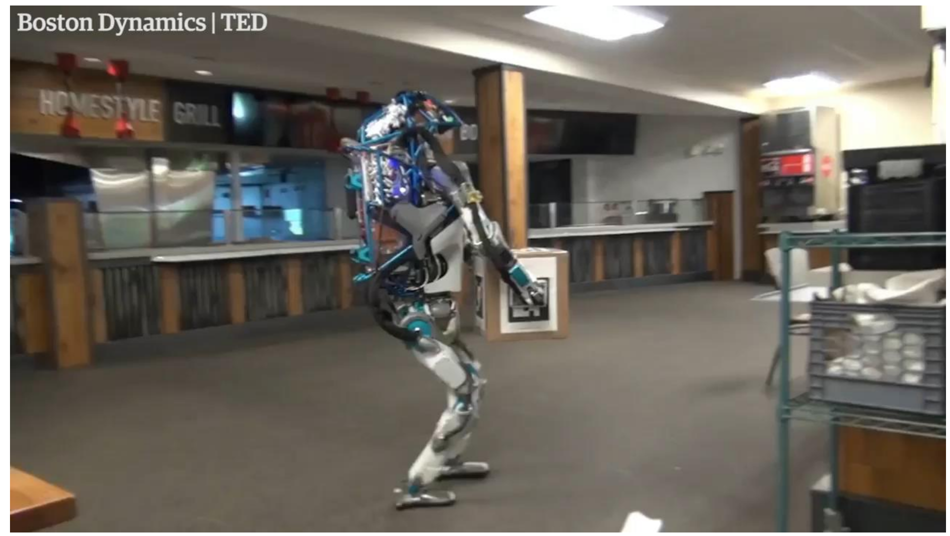
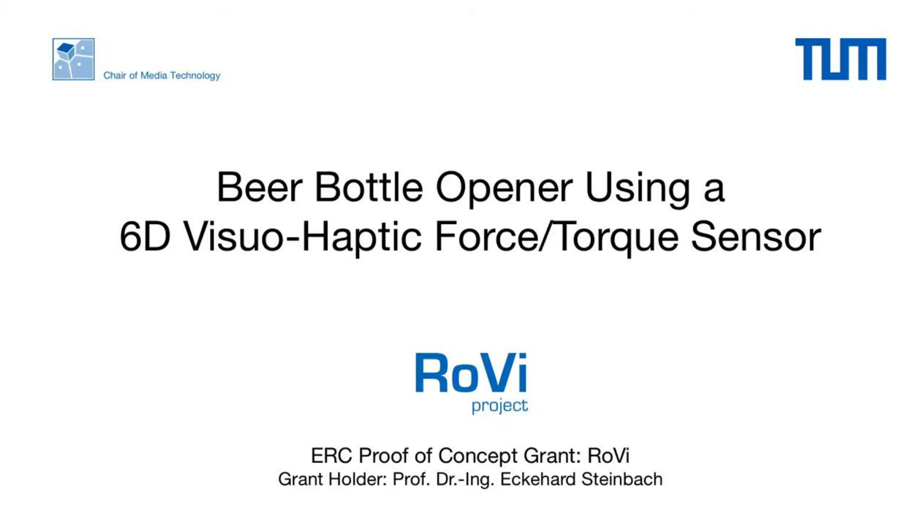
ICAROS

What is a Robot?

What is a Robot?

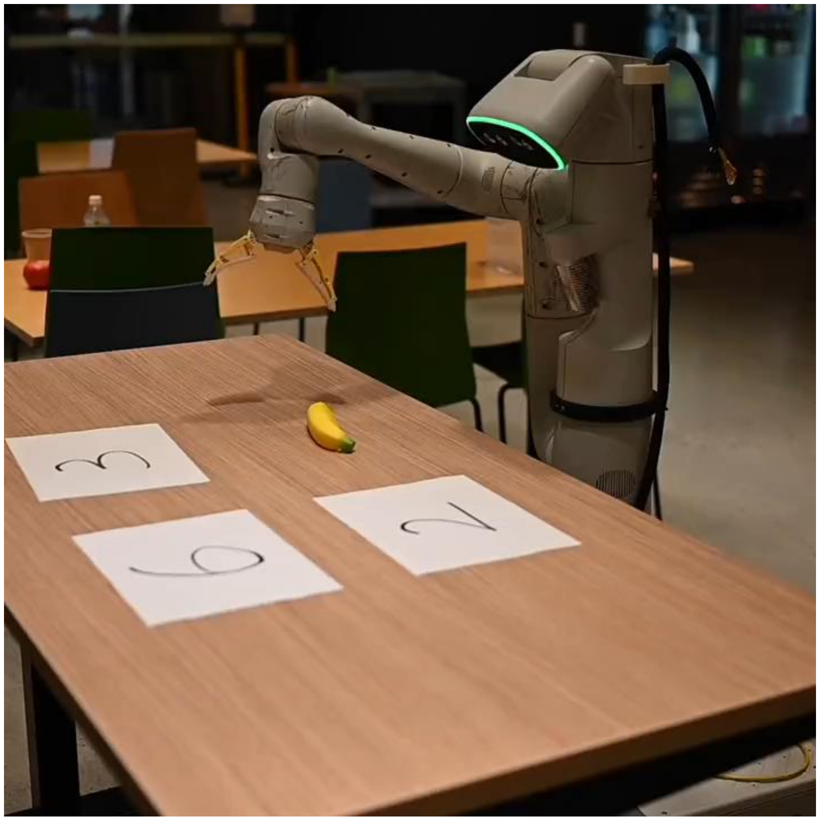


What is a Robot?

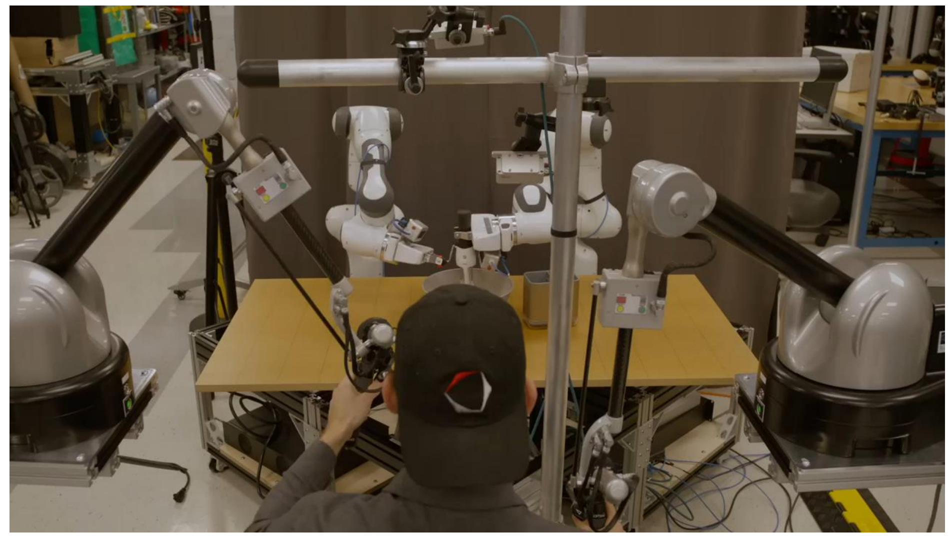


What is a Robot?

Robotics with LLM



Robotics with Learning from Demonstration



What is a Robot?

- Is robot just a shell of computer vision and natural language?
- Will robotics be solved by CV and NLP?
- Generalist vs Specialist
- Data-driven vs physics-based
- What is your thought on future of robotics?

What is a Robot?

Do it yourself!

What is a Robot?

A robot is an *autonomous* system
which exists in the *physical* world,
can *sense* its environment,
and *acts* in it
to *achieve some goals*.

au·ton·o·mous *adjective* \ȯ-ˈtä-nə-məs\

: existing or acting separately from other things or people

: having the power or right to govern itself

Full Definition of AUTONOMOUS

Cite! g+1 Like

1 : of, relating to, or marked by autonomy

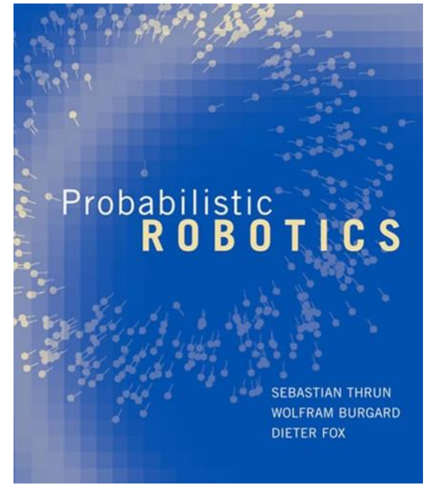
2 a : having the right or power of self-government

b : undertaken or carried on without outside control **:** SELF-CONTAINED <an *autonomous* school system>

Sensing and *Estimation* in Robotics

Contents adapted from Prof. Gaurav Sukhatme's lecture on *Fundamentals of Robotics* and Book *"Probabilistic Robotics"*

Probabilistic Robotics



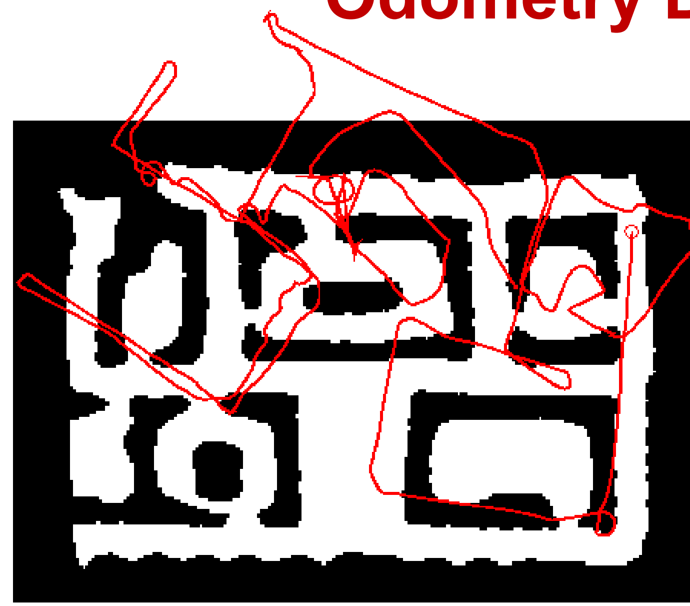
Key idea: Explicit representation of uncertainty using the calculus of probability theory

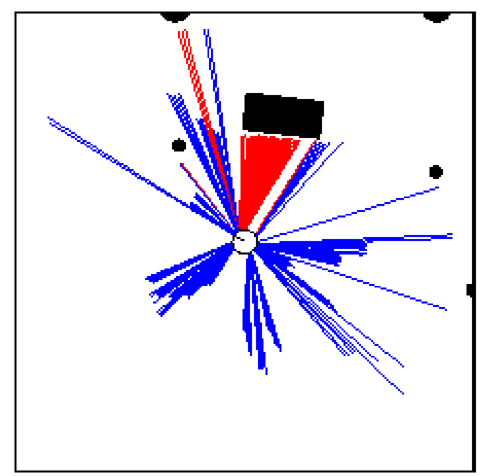
- Perception = state estimation
- Action = utility optimization

Uncertainty is Inherent/Fundamental

- Uncertainty arises from four major factors:
  - Environment is stochastic, unpredictable
  - Robot's actions are stochastic
  - Sensors are limited and noisy
  - Models are inaccurate, incomplete

Odometry Data





Range Data

Advantages and Pitfalls

- Can accommodate inaccurate models
- Can accommodate imperfect sensors
- Robust in real-world applications
- Best known approach to many hard robotic problems
- Computationally demanding
- False assumptions
- Approximate

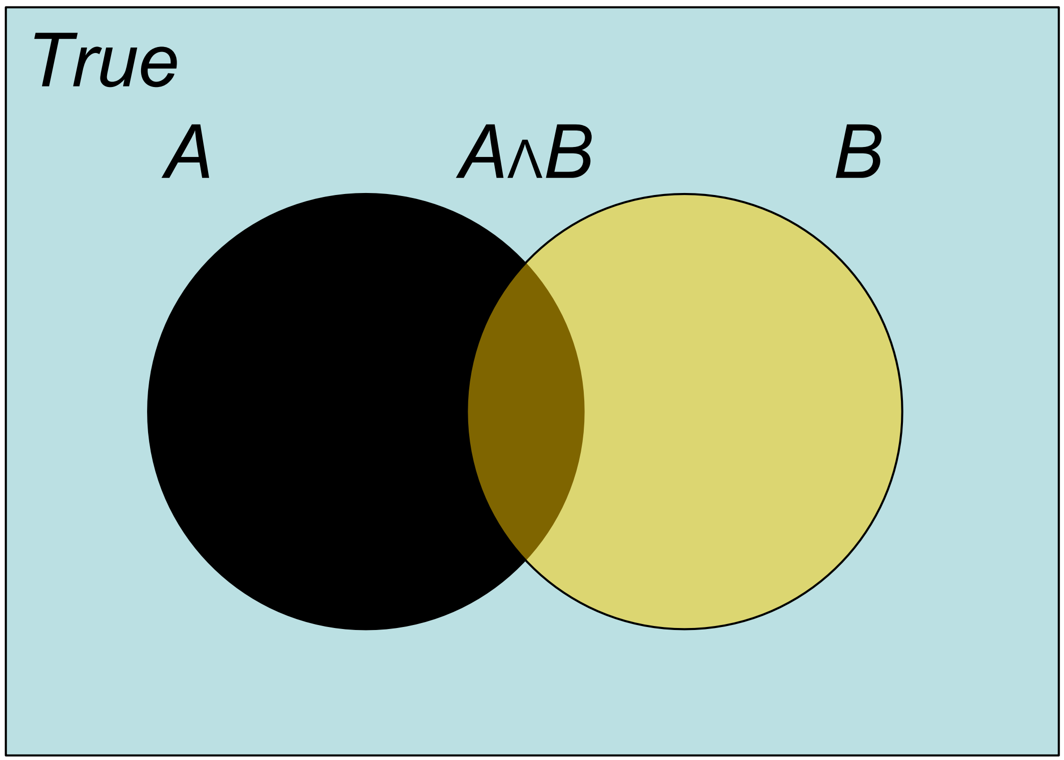
Axioms of Probability Theory

Pr(*A*) denotes probability that proposition *A* is true.

- $0 \leq \Pr(A) \leq 1$
- $\Pr(True) = 1$, $\Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

A Closer Look at Axiom 3

$$\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$$



Using the Axioms

$$
\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(\mathit{True}) &= \Pr(A) + \Pr(\neg A) - \Pr(\mathit{False}) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}
$$

Discrete Random Variables

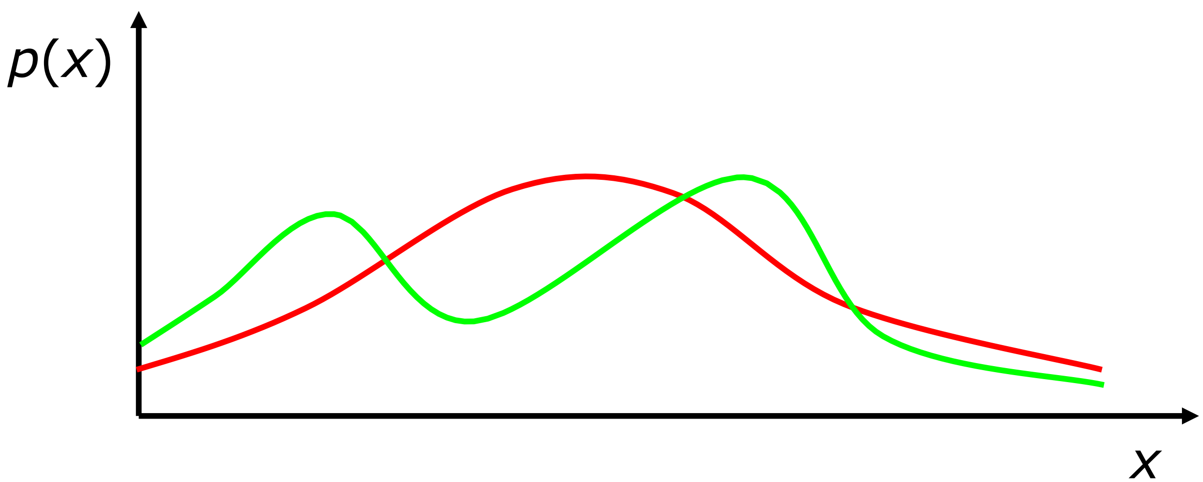
- X denotes a random variable.
- X can take on a finite number of values in $\{x_1, x_2, \ldots, x_n\}$.
- $P(X=x_i)$, or $P(x_i)$, is the probability that the random variable X takes on value x_i.
- $P(\cdot)$ is called probability mass function.
- E.g. $P(\text{Room}) = \{0.7, 0.2, 0.08, 0.02\}$

Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$, or $p(x)$, is a probability density function.

$$\Pr(x \in [a,b]) = \int_a^b p(x)\,dx$$

- E.g.



Joint and Conditional Probability

- $P(X = x \text{ and } Y = y) = P(x, y)$

- If X and Y are independent then

  $$P(x, y) = P(x)\, P(y)$$

- $P(x \mid y)$ is the probability of x given y

  $$P(x \mid y) = P(x, y) \,/\, P(y)$$

  $$P(x, y) = P(x \mid y)\, P(y)$$

- If X and Y are independent then

  $$P(x \mid y) = P(x)$$

Law of Total Probability

Discrete case

$$\sum_x P(x) = 1$$

Marginalization:

$$P(x) = \sum_y P(x, y)$$

Conditioning:

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$\int p(x)dx = 1$$

$$p(x) = \int p(x, y)dy$$

$$p(x) = \int p(x \mid y)p(y)dy$$

Reverend Thomas Bayes, FRS (1702-1761)

Clergyman and mathematicians first used probability inductively and established a mathematical basis for probability inference

Bayes Formula

$$P(x,y) = P(x|y)P(y) = P(y|x)P(x)$$

$$\Downarrow$$

$$\boxed{P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \frac{\text{likehood prior}}{\text{evidence}}}$$

Normalization

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)} = \eta\, P(y|x)P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y|x)P(x)}$$

Algorithm:

$$\forall x: \mathrm{aux}_{x|y} = P(y|x)P(x)$$

$$\eta = \frac{1}{\sum_x \mathrm{aux}_{x|y}}$$

$$\forall x: P(x|y) = \eta\ \mathrm{aux}_{x|y}$$

Conditioning

- Total probability:

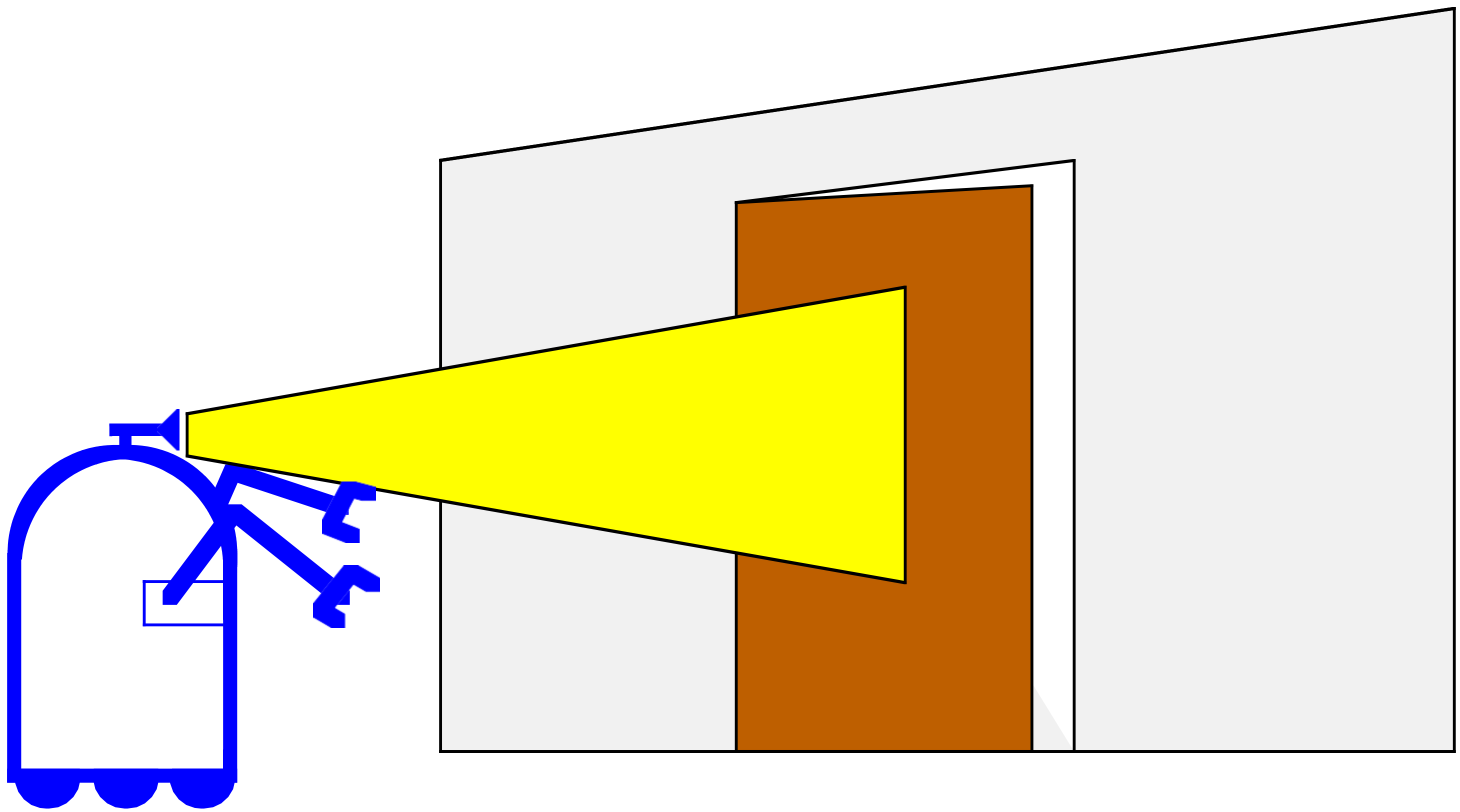
$$P(x|y) = \int P(x|y,z)P(z|y)dz$$

- Bayes rule and background knowledge:

$$P(x|y,z) = \frac{P(y|x,z)P(x|z)}{P(y|z)}$$

Example of Sensing

- Suppose a robot obtains sensor measurement about <u>z = "*door is perceived as open*"</u>
- What is $P(open|z)$?



Causal vs. Diagnostic Reasoning

- $P(open|z)$ is diagnostic.
- $P(z|open)$ is causal, and we call it sensor model.
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:

Count frequencies!

$$P(open|z) = \frac{P(z|open)P(open)}{P(z)}$$

Example

- $P(z \mid open) = 0.6 \qquad P(z \mid \neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open|z) = \frac{P(z|open)P(open)}{P(z|open)P(open) + P(z|\neg open)P(\neg open)}$$

$$= \frac{0.6 \times 0.5}{0.6 \times 0.5 + 0.3 \times 0.5} = \frac{2}{3} = 0.67$$

- *z* raises the probability that the door is open.

Combining Evidence

- Suppose our robot obtains another observation z_2.
- How can we integrate this new information?
- More generally, how can we estimate $P(x \mid z_1,...,z_n)$?

Example: Second Measurement

- $P(z_2 \mid open) = 0.5 \qquad P(z_2 \mid \neg open) = 0.6$
- $P(open \mid z_1) = 2/3$

$$P(open|z_2, z_1) = \frac{P(z_2|open)P(open|z_1)}{P(z_2|open)P(open|z_1) + P(z_2|\neg open)P(\neg open|z_1)}$$

$$= \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{1}{2} \times \frac{2}{3} + \frac{3}{5} \times \frac{1}{3}} = \frac{5}{8} = 0.625$$

- z_2 lowers the probability that the door is open.

Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by

  change the world.

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants **grow** over **time**…

- Actions are **never carried out with absolute certainty**.
- In contrast to measurements, **actions generally increase the uncertainty**.

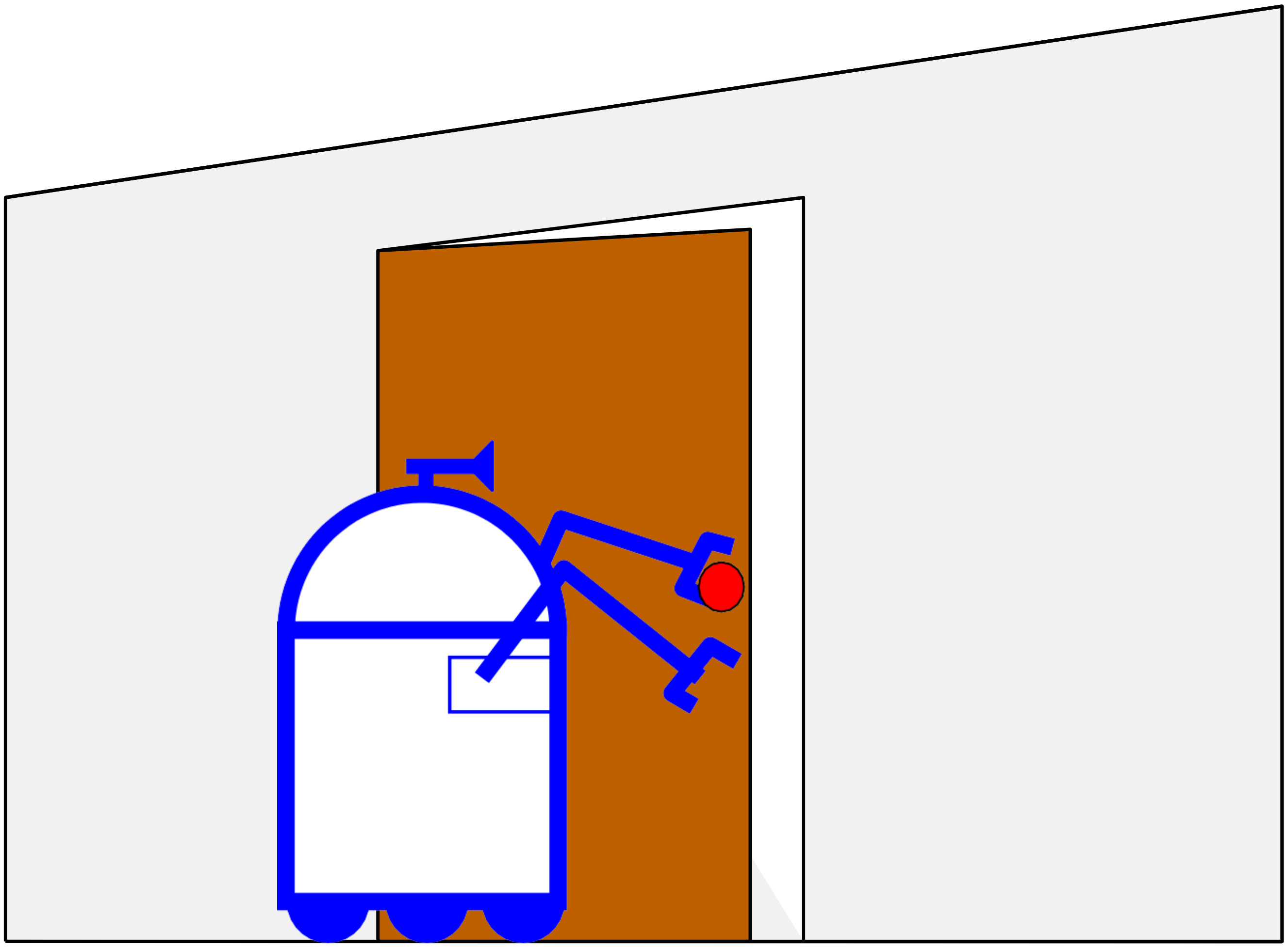
Modeling Actions

- To incorporate the outcome of an action *u* into the current “belief”, we use the conditional pdf

$$P(x|u,x')$$

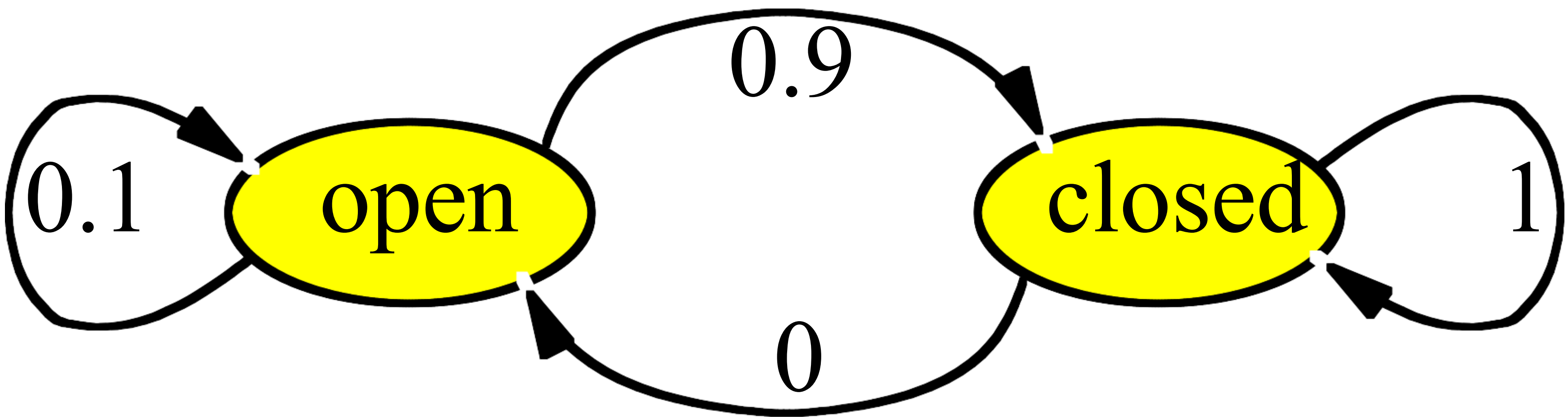
- This term specifies the pdf that executing *u* changes the state from *x’* to *x*.

Example: Closing the door



State Transitions (Action model)

$P(x|u,x')$ for u = “close door”:



If the door is open, the action “close door” succeeds in 90% of all cases.

Integrating the Outcome of Actions

Continuous case:

$$P(x|u) = \int P(x|u,x')P(x')dx'$$

Discrete case:

$$P(x|u) = \sum_{x'} P(x|u,x')P(x')$$

Example: Resulting belief after action *u* = "*close door*"

P(*open*) = 5/8
P(*closed*) = 3/8
From the calculation of the last example

$$P(closed \mid u) = \sum P(closed \mid u, x')P(x')$$

$$= P(closed \mid u, open)P(open)$$

$$+ P(closed \mid u, closed)P(closed)$$

$$= \frac{9}{10} \times \frac{5}{8} + \frac{1}{1} \times \frac{3}{8} = \frac{15}{16}$$

$$P(open \mid u) = \sum P(open \mid u, x')P(x')$$

$$= P(open \mid u, open)P(open)$$

$$+ P(open \mid u, closed)P(closed)$$

$$= \frac{1}{10} \times \frac{5}{8} + \frac{0}{1} \times \frac{3}{8} = \frac{1}{16}$$

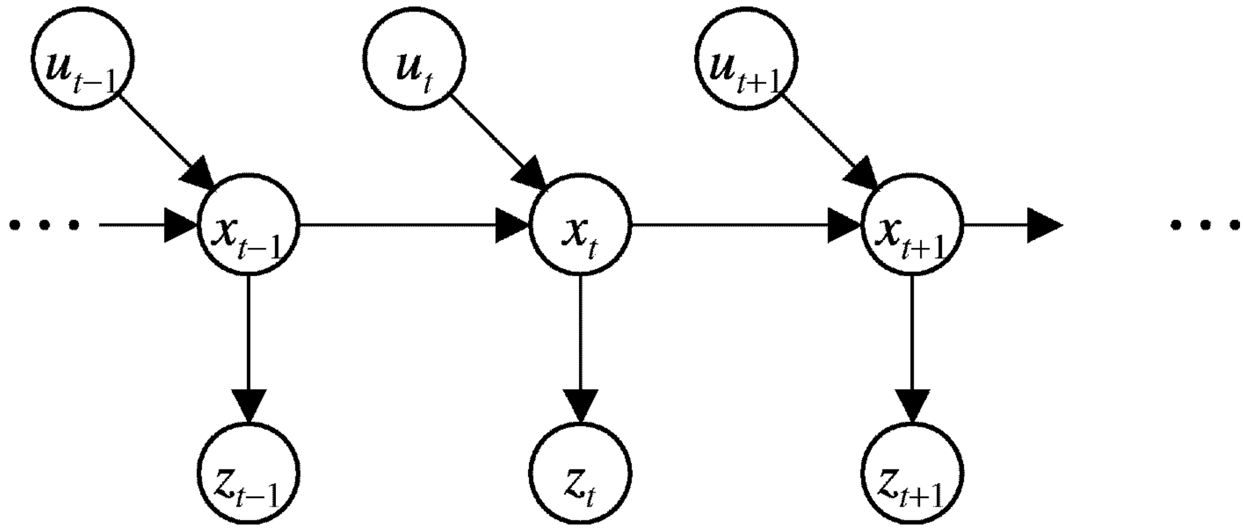
$$= 1 - P(closed \mid u)$$

Bayes Filters: Framework

- **Given:**
  - Stream of sensor observations z and actions u:
    $$d_t = \{ u_t, z_t \}$$
  - Sensor model $P(z|x)$
  - Action model $P(x|u,x')$
  - Prior probability of the state $P(x)$
- **Wanted:**
  - Estimate of the state x of a dynamic system
  - The posterior of the state is also called **Belief**:

$$\begin{aligned} Bel(x_t) &= P(x_t \mid d_1, \ldots, d_t) \\ &= P(x_t \mid u_1, z_1, \ldots, u_t, z_t) \end{aligned}$$

Markov Assumption



Correction-> $P(z_t \mid x_{1:t}, z_{1:t-1}, u_{1:t}) = P(z_t \mid x_t)$ Sensor Model

Prediction-> $P(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = P(x_t \mid x_{t-1}, u_t)$ Action Model

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Recursive Bayesian Updating

- Perceptual (Observation) Update for z_t

Markov assumption: z_t is independent of $d_1,\ldots,d_{t-1}, u_t$ **if we know x_t**

$$d_t = \{ u_t, z_t \}$$

$$\begin{aligned}
Bel(x_t) &= P(x_t | d_1, \ldots, d_{t-1}, d_t) \\
&= P(x_t | d_1, \ldots, d_{t-1}, u_t, z_t) \\
&= \frac{P(z_t | x_t, d_1, \ldots, d_{t-1}, u_t) P(x_t | d_1, \ldots, d_{t-1}, u_t)}{P(z_t | d_1, \ldots, d_{t-1}, u_t)} \\
&= \frac{P(z_t | x_t) P(x_t | d_1, \ldots, d_{t-1}, u_t)}{P(z_t | d_1, \ldots, d_{t-1}, u_t)} \\
&= \eta P(z_t | x_t) P(x_t | d_1, \ldots, d_{t-1}, u_t) \\
&= \eta P(z_t | x_t) \boxed{\overline{Bel}(x_t)}
\end{aligned}$$

Intermediate Belief before receiving z_t

Recursive Bayesian Updating

- Action (Dynamic) Update for u_t

$$d_t = \{ u_t , z_t \}$$

$$\boxed{\overline{Bel}(x_t)} = P(x_t|d_1, ..., d_{t-1}, u_t)$$

Continue from last page

$$= \int P(x_t|d_1, ..., d_{t-1}, u_t, x_{t-1})P(x_{t-1}|d_1, ..., d_{t-1}, u_t)dx_{t-1}$$

$$= \int P(x_t|u_t, x_{t-1})P(x_{t-1}|d_1, ..., d_{t-1})dx_{t-1}$$

$$= \int P(x_t|u_t, x_{t-1})\boxed{Bel(x_{t-1})}dx_{t-1}$$

Belief of last time stamp

Bayes Filter Algorithm

Algorithm **Bayes_Filter**($Bel(x),d$):

$\eta = 0$

If d is a perceptual data z then

    For all x do

$$Bel'(x) = P(z|x)Bel(x)$$

$$\eta = \eta + Bel'(x)$$

    For all x do

$$Bel'(x) = \eta^{-1} Bel'(x)$$

Else if d is an action data u then

    For all x do

$$Bel'(x) = \int P(x|u,x')Bel(x')dx'$$

Return $Bel'(x)$

Discrete Bayes Filter Algorithm

1. Algorithm **Discrete_Bayes_Filter**($Bel(x), d$):
2. $\eta = 0$
3. If d is a perceptual data z then
4. For all x do
5. $Bel'(x) = P(z|x)Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all x do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. Else if d is an action data u then
10. For all x do
11. $Bel'(x) = \sum_{x'} P(x|u,x')Bel(x')$
12. Return $Bel'(x)$

Bayes Filter

z = observation

u = action

x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes $$= \eta \; P(z_t \mid x_t, u_1, z_1, \ldots, u_t) \, P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov $$= \eta \; P(z_t \mid x_t) \, P(x_t \mid u_1, z_1, \ldots u_t)$$

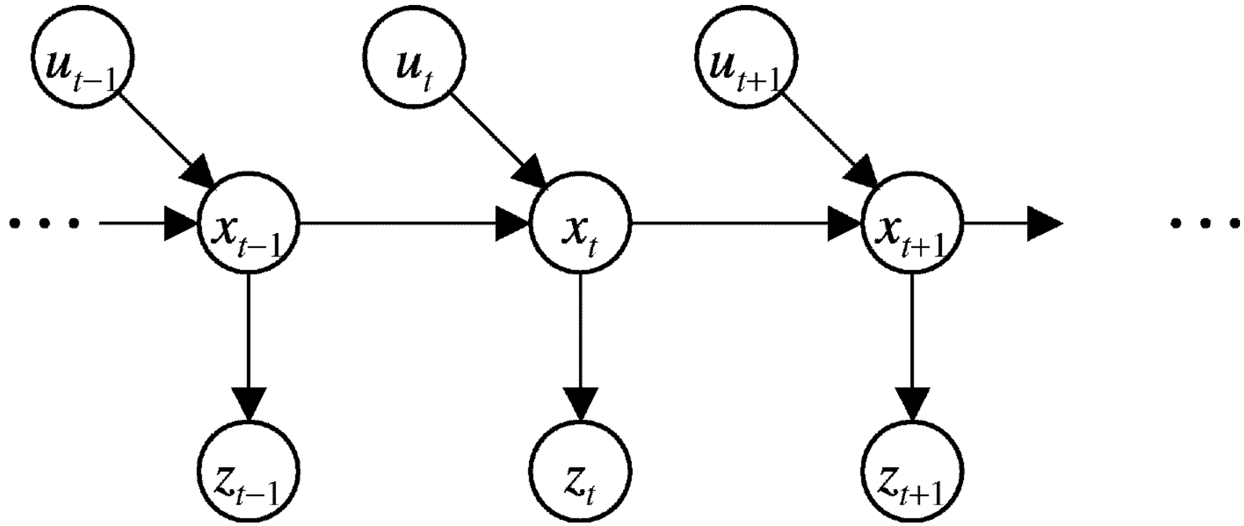
Total prob. $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots u_t) \, dx_{t-1}$$

Markov $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \ldots u_t) \, dx_{t-1}$$

Markov $$= \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \ldots z_{t-1}) \, dx_{t-1}$$

$$\boxed{= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}}$$

Recap – Bayes Filter

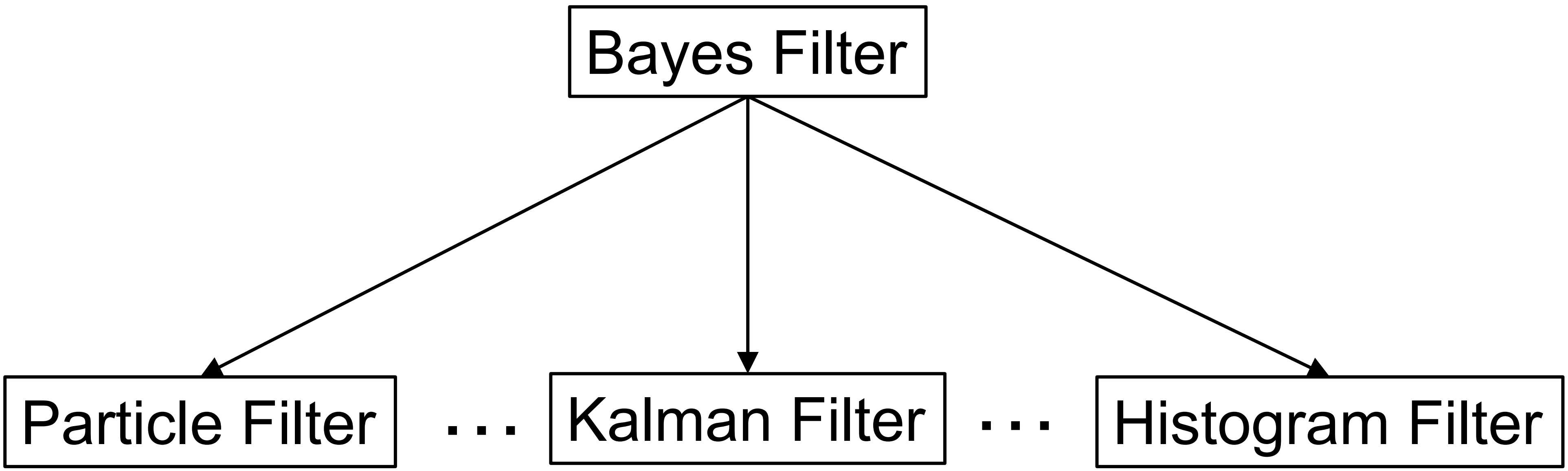


$P(z_t \mid x_{1:t}, z_{1:t\text{-}1}, u_{1:t}) = P(z_t \mid x_t)$ **Sensor Model**

$P(x_t \mid x_{1:t\text{-}1}, z_{1:t\text{-}1}, u_{1:t}) = P(x_t \mid x_{t\text{-}1}, u_t)$ **Action Model**

How do we get *sensor model* and *action model* in a <u>more general</u> way?

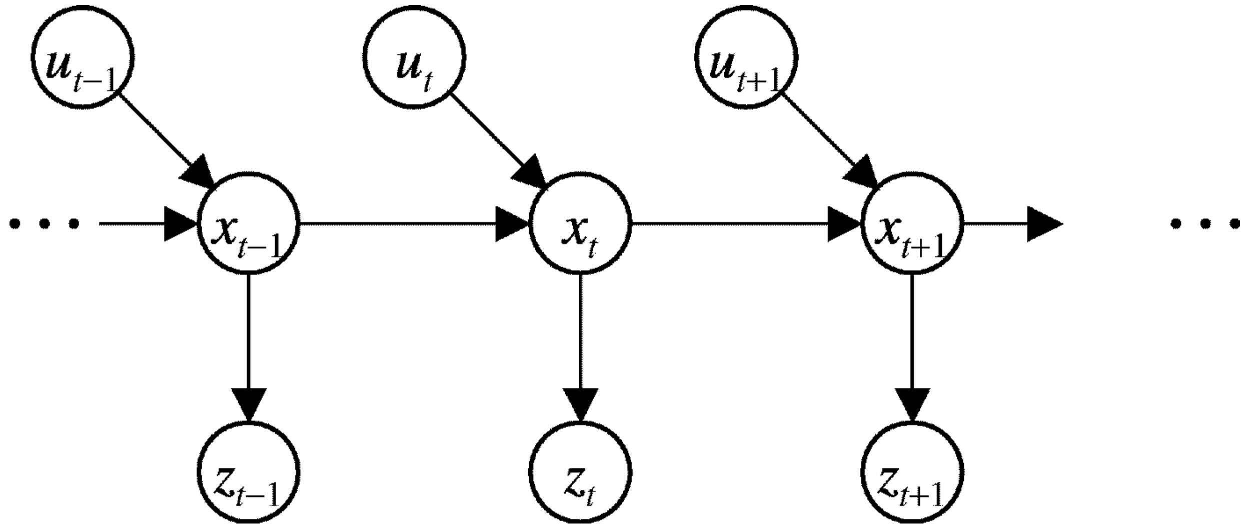
Recap – Bayes Filter



How do we get *sensor model* and *action model* in a more general way?

Kalman Filter

A special case of Bayes Filter with <u>linear Gaussian dynamics and sensory models</u>



$$P(z_t \mid x_{1:t}, z_{1:t-1}, u_{1:t}) = P(z_t \mid x_t)$$ Sensor Model

$$P(x_t \mid x_{1:t-1}, z_{1:t-1}, u_{1:t}) = P(x_t \mid x_{t-1}, u_t)$$ Action Model

Kalman Filter

Prediction-> $$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$ Action Model (Dynamics)

Correction-> $$z_t = C_t x_t + \delta_t$$ Sensor Model (Observation)

$$Bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$ State Initialization

Components of a Kalman Filter

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

$$z_t = C_t x_t + \delta_t$$

$$Bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

$\boxed{A_t}$ Matrix (n x n) that describes how the state evolves from t-1 to t without actions or noise.

$\boxed{B_t}$ Matrix (n x l) that describes how the action u_t changes the state from t-1 to t.

$\boxed{C_t}$ Matrix (k x n) that describes how to map the state x_t to an observation z_t.

$\boxed{\varepsilon_t}$ $\boxed{\delta_t}$ Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t respectively.

Multivariate Gaussians Properties

$$\left.\begin{array}{l} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array}\right\} \quad \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}} \right)$$

- We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations.

Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$Bel(x_0) = N\left(x_0; \mu_0, \Sigma_0\right)$$

Linear Gaussian Systems: Dynamics

- Dynamics are linear function of state and action plus additive noise:

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t}$$

$$\boxed{p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right)}$$

$$\boxed{\begin{array}{ccc} \overline{Bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) & & Bel(x_{t-1})\, dx_{t-1} \\ \Downarrow & & \Downarrow \\ \sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) & & \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right) \end{array}}$$

Linear Gaussian Systems: Dynamics

$$\overline{Bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad Bel(x_{t-1})\ dx_{t-1}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

$$\Downarrow$$

$$\overline{Bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{Bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Linear Gaussian Systems: Observations

- Observations are linear function of state plus additive noise:

$$\boxed{z_t = C_t x_t + \delta_t}$$

$$\boxed{p(z_t \mid x_t) = N\left(z_t; C_t x_t, Q_t\right)}$$

$$\boxed{\begin{array}{llll} Bel(x_t) = & \eta & p(z_t \mid x_t) & \overline{Bel}(x_t) \\ & & \Downarrow & \Downarrow \\ & & \sim N\left(z_t; C_t x_t, Q_t\right) & \sim N\left(x_t; \overline{\mu}_t, \overline{\Sigma}_t\right) \end{array}}$$

Linear Gaussian Systems: Observations

$$Bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{Bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N(z_t; C_t x_t, Q_t) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$Bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$Bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Kalman Filter Algorithm

1. Algorithm **Kalman_Filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$:

2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

Recap - Kalman Filter

A special case of Bayes Filter with linear Gaussian dynamics and sensory models

- Highly computationally efficient
- Optimal for linear Gaussian systems!
- But does this linearity/Gaussian distribution always hold in robotic systems?

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

Action Model

$$z_t = C_t x_t + \delta_t$$

Sensor Model

$$Bel(x_0) = N(x_0; \mu_0, \Sigma_0)$$

Extensions

- Extended Kalman Filter
- Unscented Kalman Filter
- Learning-based Kalman Filter

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