Operations Research

Books:

Operations Research : An introduction

by Hamdy A Taha Ninth ed Pearson

Today, optimization methods are used everywhere in business, industry, government, engineering, and computer science.

You will find optimization taught in various departments of engineering, computer science, business, mathematics, and economics;

<u>The desire for optimality (perfection) is inherent for humans</u>

Optimization techniques used today have trace their origins during World War II to improve the effectiveness of the war efforts

For example:

What is the optimum allocation of gasoline supplies among competing campaigns? What is the best search and bombing pattern for anti-submarine patrols? How to manage in the best way the limited food.

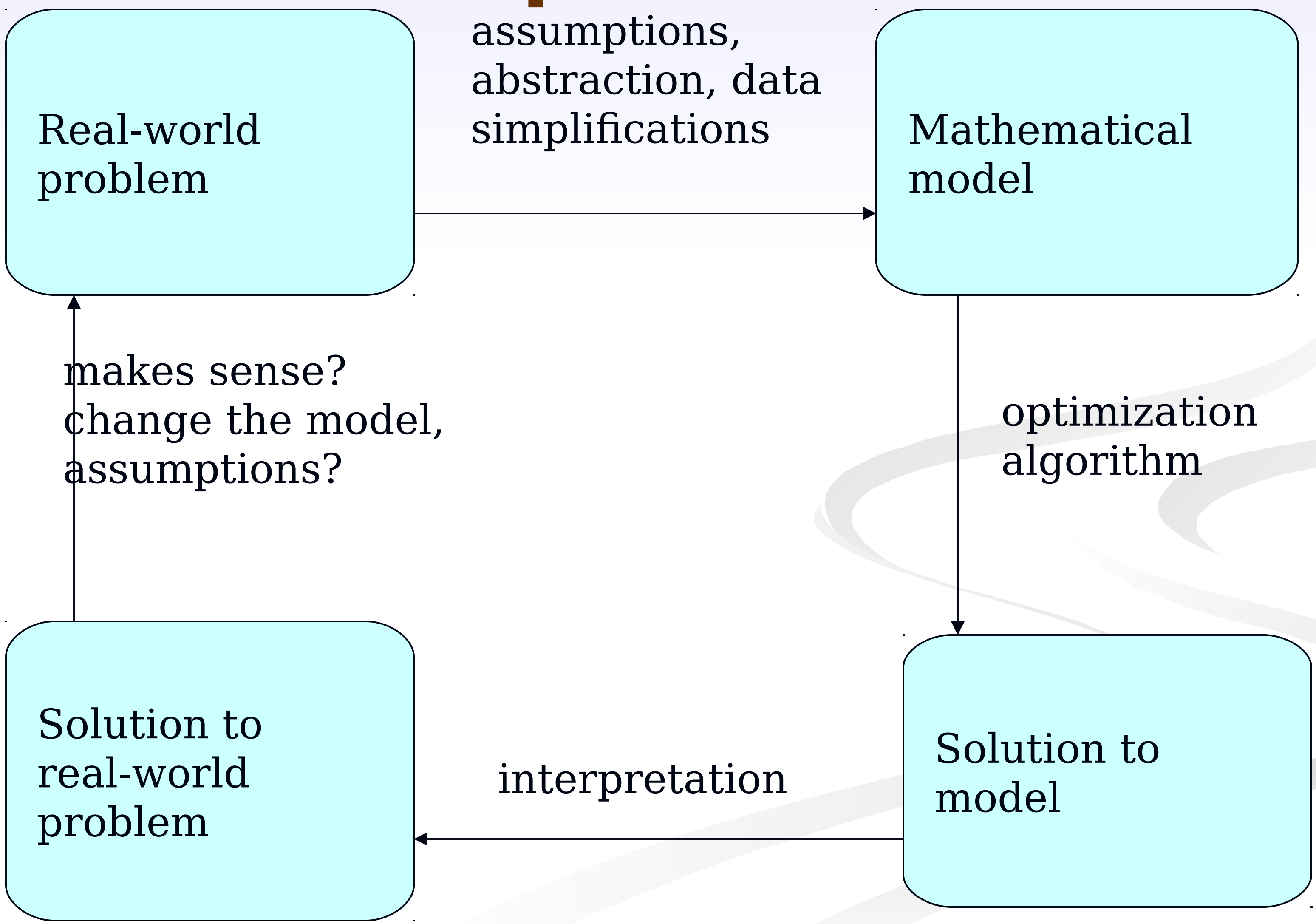
Here are a few more examples of optimization problems:

- How should the transistors and other devices be laid out in a new computer chip so that the layout takes up the least area?
- What is the smallest number of warehouses, and where should they be located so that the maximum travel time from any retail sales outlet to the closest warehouse is less than 6 hours?
- How should telephone calls be routed between two cities to permit the maximum number of simultaneous calls?

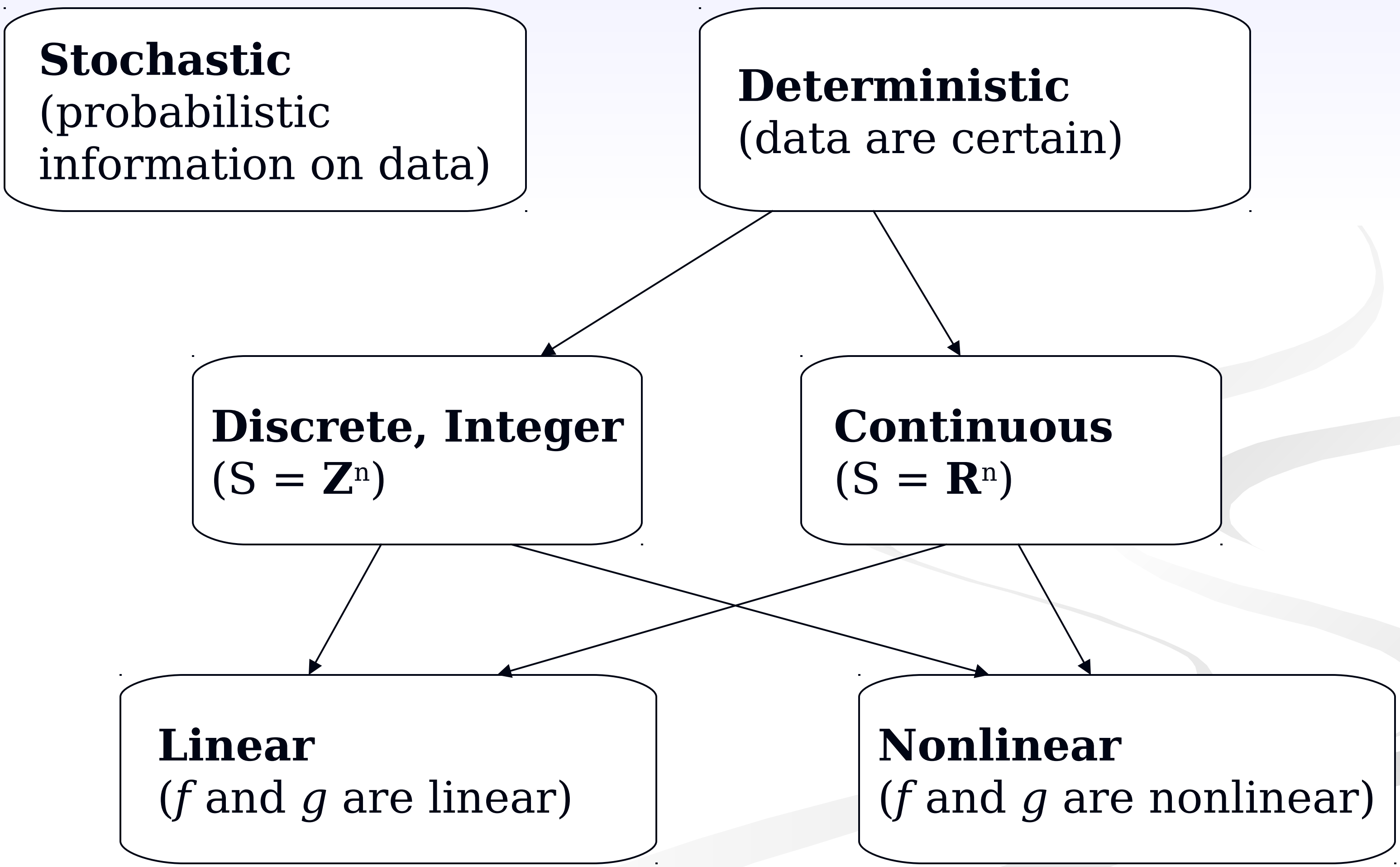
TABLE 1.1 Methods of Operations Research

Mathematical Programming Techniques	Stochastic Process Techniques	Statistical Methods
Calculus methods	Statistical decision theory	Regression analysis
Calculus of variations	Markov processes	Cluster analysis, pattern recognition
Nonlinear programming	Queueing theory	Design of experiments
Geometric programming	Renewal theory	Discriminate analysis (factor analysis)
Quadratic programming	Simulation methods	
Linear programming	Reliability theory	
Dynamic programming		
Integer programming		
Stochastic programming		
Separable programming		
Multiobjective programming		
Network methods: CPM and PERT		
Game theory		
Simulated annealing		
Genetic algorithms		
Neural networks		

A schematic view of modeling/optimization process



Types of Optimization Models



What is Optimization process?

- ***Optimization*** is an iterative process by which a desired solution (max/min) of the problem can be found while satisfying all its constraints or bounded conditions.

  - Optimization problem could be linear or non-linear.

- The search procedure is termed as ***algorithm***.

Mathematical models in Optimization

- The general form of an *optimization model*:

  min or *max* $f(x_1,\ldots,x_n)$ (objective function)

  subject to $g_i(x_1,\ldots,x_n) \geq 0$ (functional constraints)

  $x_1,\ldots,x_n \in S$ (set constraints)

- $x_1,\ldots,x_n$ are called decision variables
- *In words*, the goal is to find
- --- $x_1,\ldots,x_n$ that satisfy the constraints;

  --- achieve min (max) objective function value.

Optimization Problems

Unconstrained optimization

$\min\{f(x) : x \in R^n\}$

Constrained optimization

$\min\{f(x) : c_i(x) \leq 0, i \in I, c_j(x) = 0, j \in E\}$

Quadratic programming

$\min\{1/2x^TQx + c^Tx : a_i^Tx \leq b_i, i \in I, a_i^Tx = b_j, j \in E\}$

Zero-One programming

$\min\{c^Tx : Ax = b, x \in \{0,1\}^n, c \in R^n, b \in R^m\}$

Mixed Integer Programming

$\min\{c^Tx : Ax \leq b, x \geq 0, x_i \in Z^n, i \in I, x_r \in R^n, r \in R\}$

Linear programming is the most widely used of the major techniques for constrained optimization.

“Programming” here does not imply computer programming –it is an old word for “planning”.

In linear programming, all of the underlying models of the real-world processes are linear, hence you can think of “linear programming” as “planning

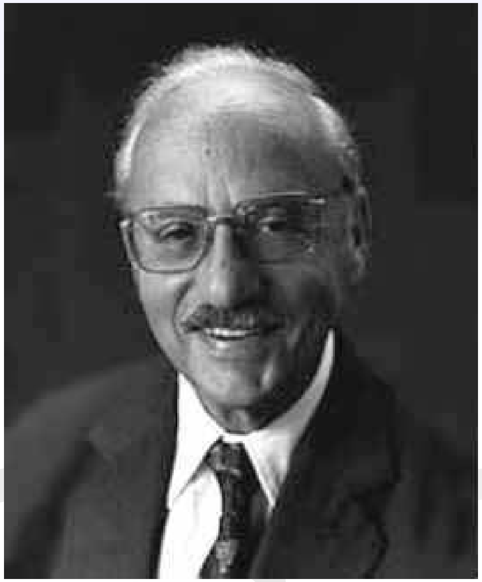
Introduction

- Problems of this kind are called “linear programming problems” or “LP problems” for short; linear programming is the branch of applied mathematics concerned with these problems.
- A *linear programming problem* is the problem of maximizing (or minimizing) a linear function subject to a finite number of linear constraints.
- Standard form:

$$\text{maximize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\le b_i \quad (i = 1, 2, \ldots, m) \\ x_j &\ge 0 \quad (j = 1, 2, \ldots, n) \end{aligned}$$

History of Linear Programming



- It started in 1947 when G.B.Dantzig design the “simplex method” for solving linear programming formulations of U.S. Air Force planning problems.
- It soon became clear that a surprisingly wide range of apparently unrelated problems in production management could be stated in linear programming terms and solved by the simplex method.
- Later, it was used to solve problems of management. It’s algorithm can also used to network flow problems.

History of Linear Programming

- On Oct.14th,1975, the Royal Sweden Academy of Science awarded the Nobel Prize in economic science to L.V.Kantorovich and T.C.Koopmans ”for their contributions to the theory of optimum allocation of resources”

- The breakthrough in looking for a theoretically satisfactory algorithm to solve LP problems came in 1979 when L.G.Khachian published a description of such an algorithm.

Applicatio

- Efficient allocation of scarce resources
  - diet problem
- Scheduling production and inventory
  - multistage scheduling problems
- The cutting-stock problem
  - find a way to cut paper or textiles rolls by complicated summary of orders
- Approximating data by linear functions
  - find approximate solutions to possibly unsolvable systems of linear equations

Linear Programming: An Overview

- Objectives of business decisions frequently involve ***maximizing profit*** or ***minimizing costs***.
- Linear programming uses ***linear algebraic relationships*** to represent a firm's decisions, given a business ***objective***, and resource ***constraints***.
- Steps in application:
  1. Identify problem as solvable by linear programming.
  2. Formulate a mathematical model of the unstructured problem.

Introduction

- A Diet Problem

eg: Polly wonders how much money she must spend on food in order to get all the energy (1,000 kcal), protein (55 g), and calcium (800 mg) that she needs every day. She choose six foods that seem to be cheap sources of the nutrients:

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (c)
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole Milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

Introduction

- Servings-per-day limits on all six foods:

Oatmeal	at most 4 servings per day
Chicken	at most 3 servings per day
Eggs	at most 2 servings per day
Milk	at most 8 servings per day
Cherry pie	at most 2 servings per day
Pork with beans	at most 2 servings per day

- Now there are so many combinations seem promising that one could go on and on, looking for the best one. Trial and error is not particularly helpful here.

Introduction

- A new way to express this—using inequalities:

minimize $3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6$

subject to

$$
\begin{aligned}
0 &\le x_1 \le 4 \\
0 &\le x_2 \le 3 \\
0 &\le x_3 \le 2 \\
0 &\le x_4 \le 8 \\
0 &\le x_5 \le 2 \\
0 &\le x_6 \le 2
\end{aligned}
$$

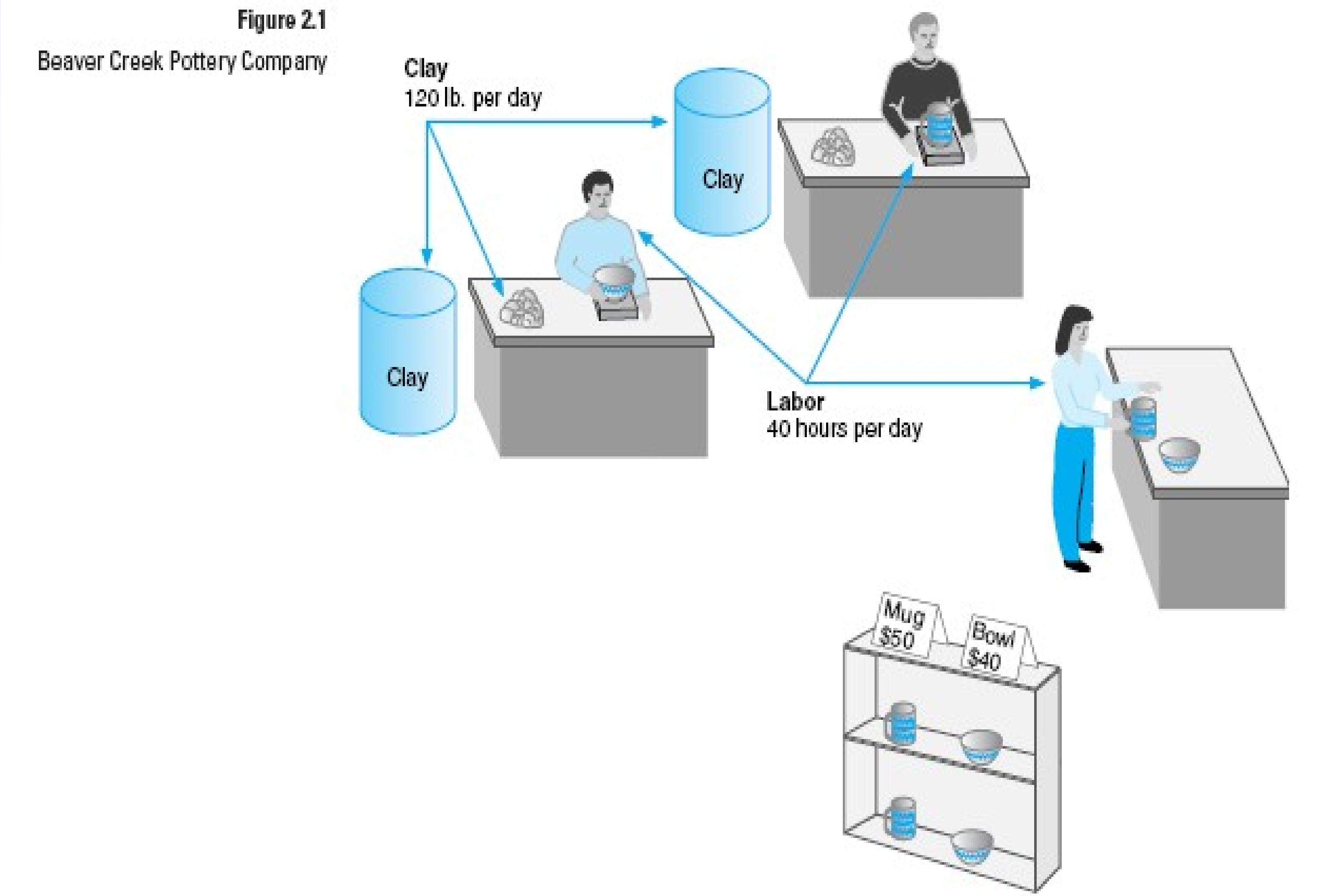
$$
\begin{aligned}
110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\ge 1{,}000 \\
4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\ge 55 \\
2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\ge 800
\end{aligned}
$$

LP Model Formulation
A Maximization Example (1 of 4)

- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

	Resource Requirements		
Product	Labor (Hr./Unit)	Clay (Lb./Unit)	Profit ($/Unit)
Bowl	1	4	40
Mug	2	3	50

LP Model Formulation
A Maximization Example (2 of 4)



LP Model Formulation
A Maximization Example (3 of 4)

Resource Availability:
40 hrs of labor per day
120 lbs of clay

Decision Variables:
x_1 = number of bowls to produce per day
x_2 = number of mugs to produce per day

Objective Function:
Maximize $Z = \$40x_1 + \$50x_2$
Where Z = profit per day

Resource Constraints:
$1x_1 + 2x_2 \leq 40$ hours of labor
$4x_1 + 3x_2 \leq 120$ pounds of clay

Non-Negativity
$x_1 \geq 0; x_2 \geq 0$

LP Model Formulation
A Maximization Example (4 of 4)

Complete Linear Programming Model:

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Feasible Solutions

A ***feasible solution*** does not violate ***any*** of the constraints:

Example: $x_1 = 5$ bowls

$x_2 = 10$ mugs

$Z = \$40x_1 + \$50x_2 = \$700$

Labor constraint check: $1(5) + 2(10) = 25 \leq 40$ hours

Clay constraint check: $4(5) + 3(10) = 50 \leq 120$ pounds

Infeasible Solutions

An ***infeasible solution*** violates ***at least one*** of the constraints:

Example: $x_1 = 10$ bowls

$x_2 = 20$ mugs

$Z = \$40x_1 + \$50x_2 = \$1400$

Labor constraint check: $1(10) + 2(20) = 50 > 40$ hours

Graphical Solution of LP Models

- Graphical solution is limited to linear programming models containing ***only two decision variables*** (can be used with three variables but only with great difficulty).

- Graphical methods provide ***visualization of how*** a solution for a linear programming problem is obtained.

Coordinate Axes Graphical Solution of Maximization Model (1 of 12)

X2 is mugs

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

X1 is bowls

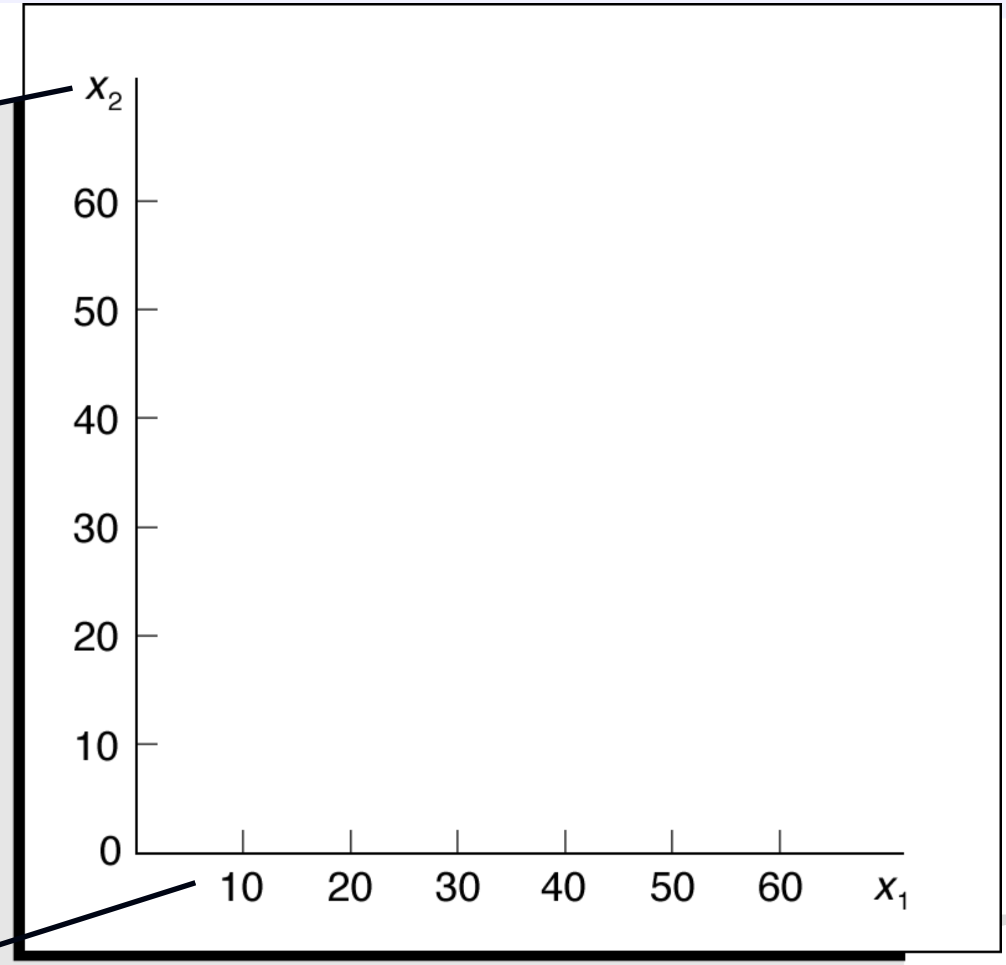


Figure Coordinates for Graphical Analysis

Labor Constraint Graphical Solution of Maximization Model (2 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

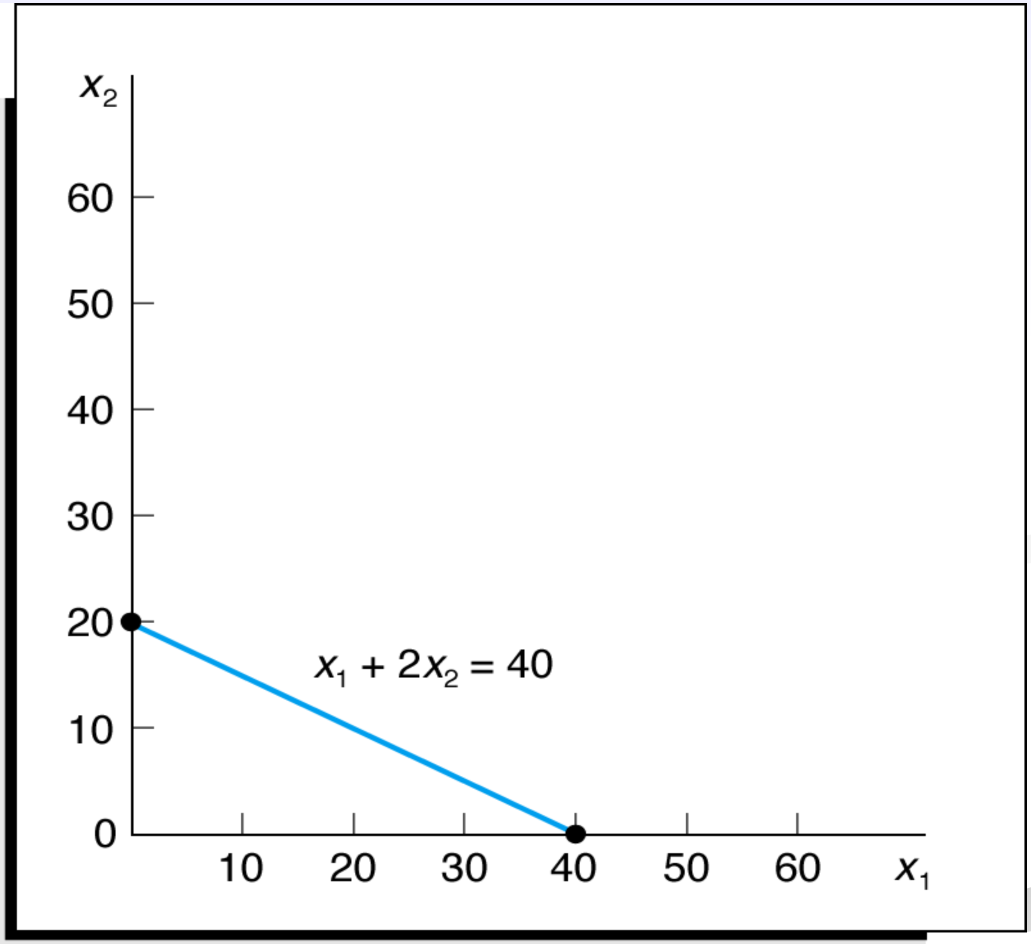


Figure Graph of Labor Constraint

Labor Constraint Area Graphical Solution of Maximization Model (3 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

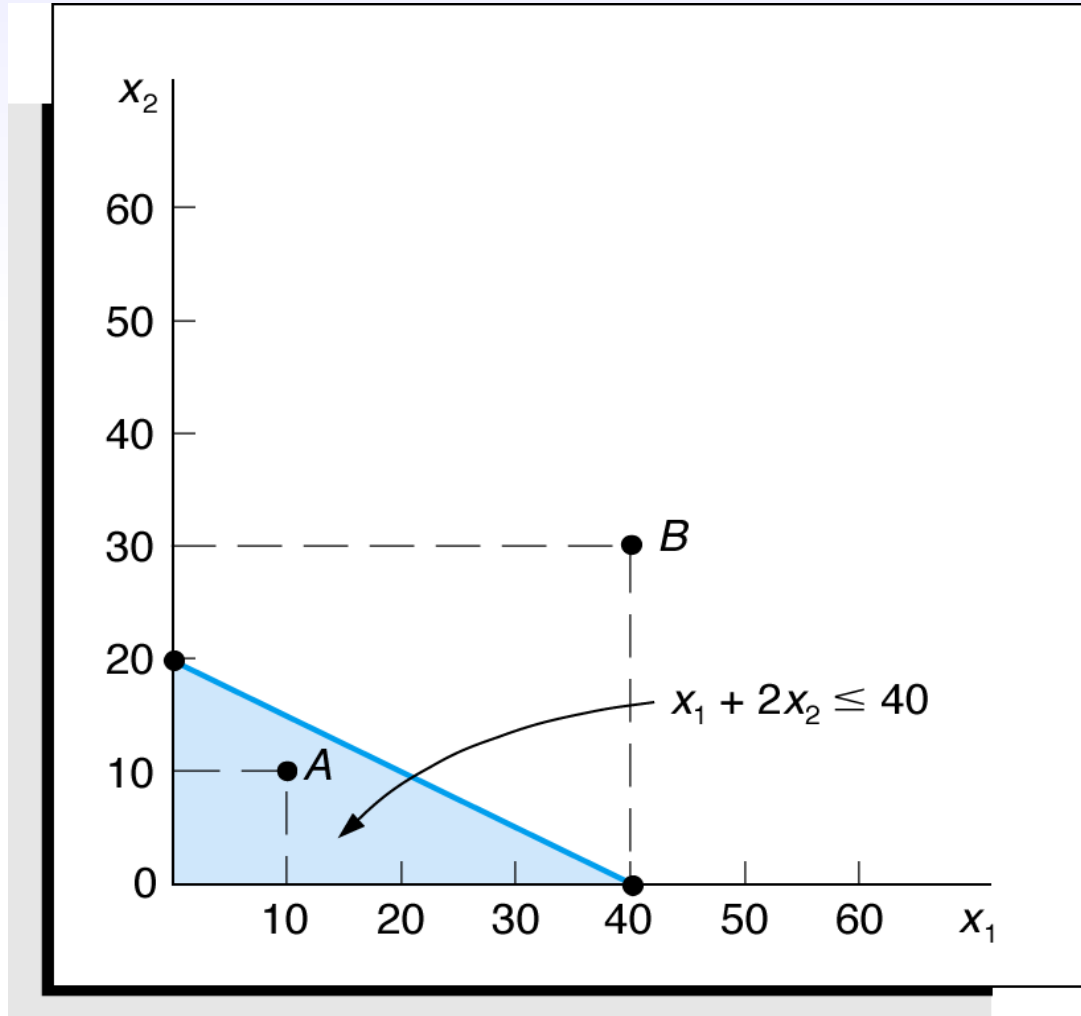


Figure Labor Constraint Area

Clay Constraint Area Graphical Solution of Maximization Model (4 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

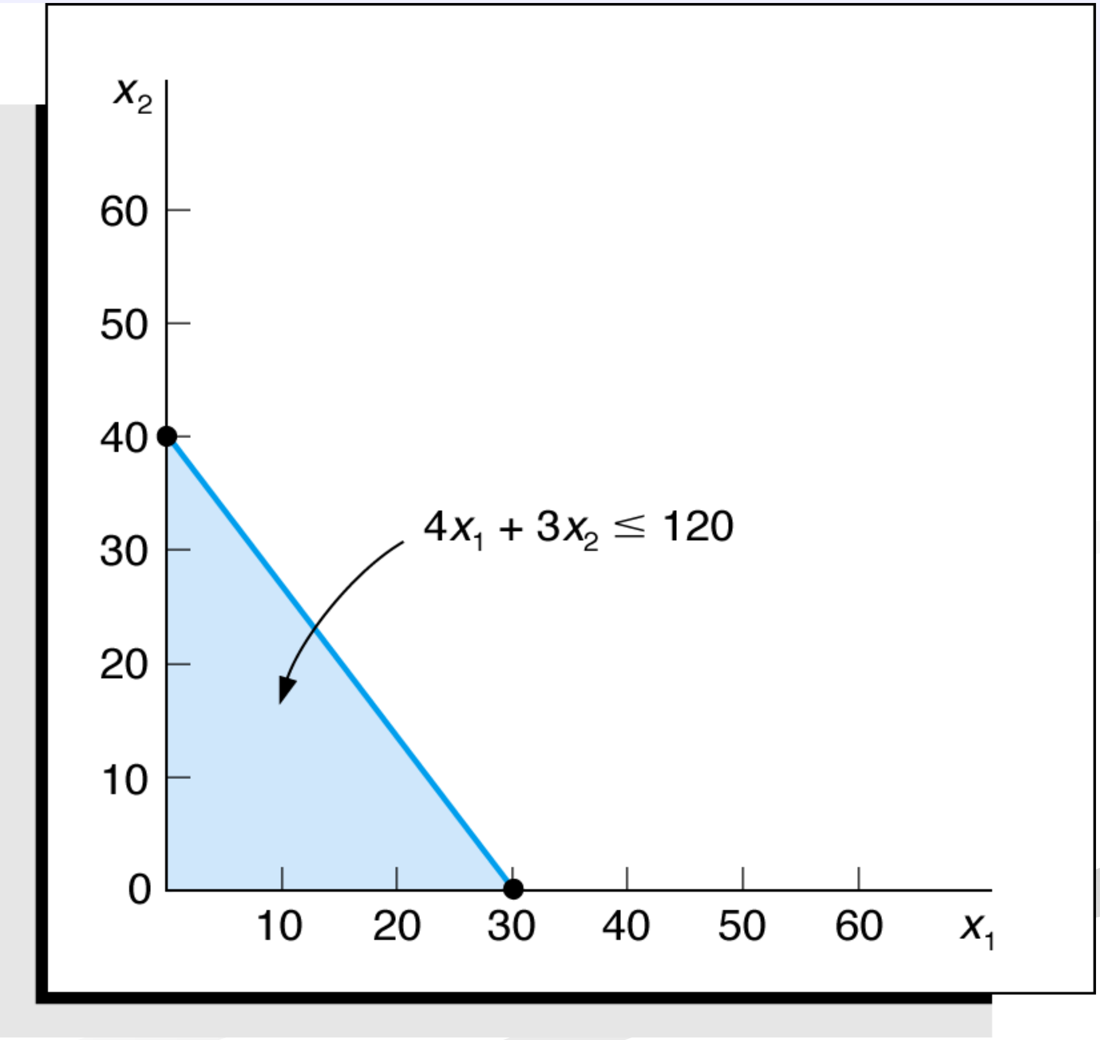


Figure Clay Constraint Area

Graphical Solution of Maximization Model (5 of 12)

Both Constraints

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

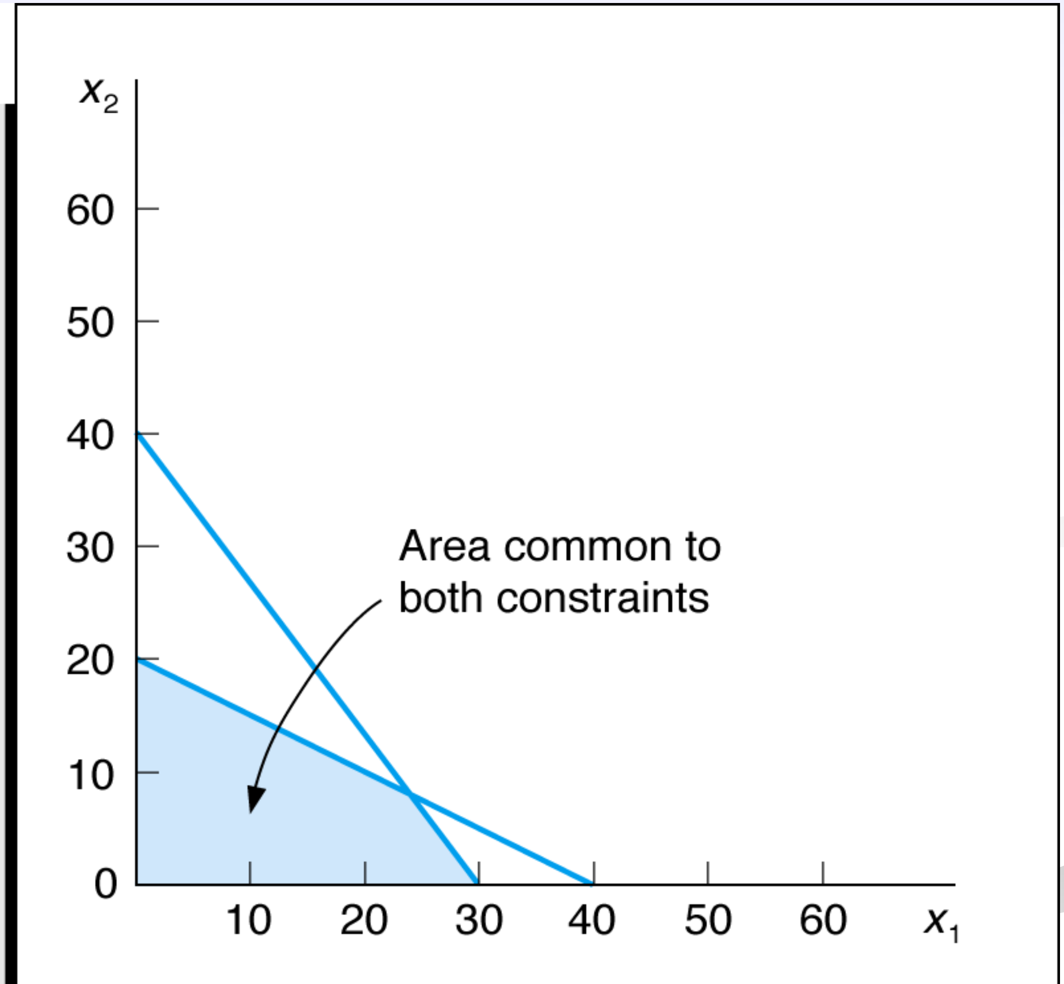


Figure Graph of Both Model Constraints

Feasible Solution Area Graphical Solution of Maximization Model (6 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

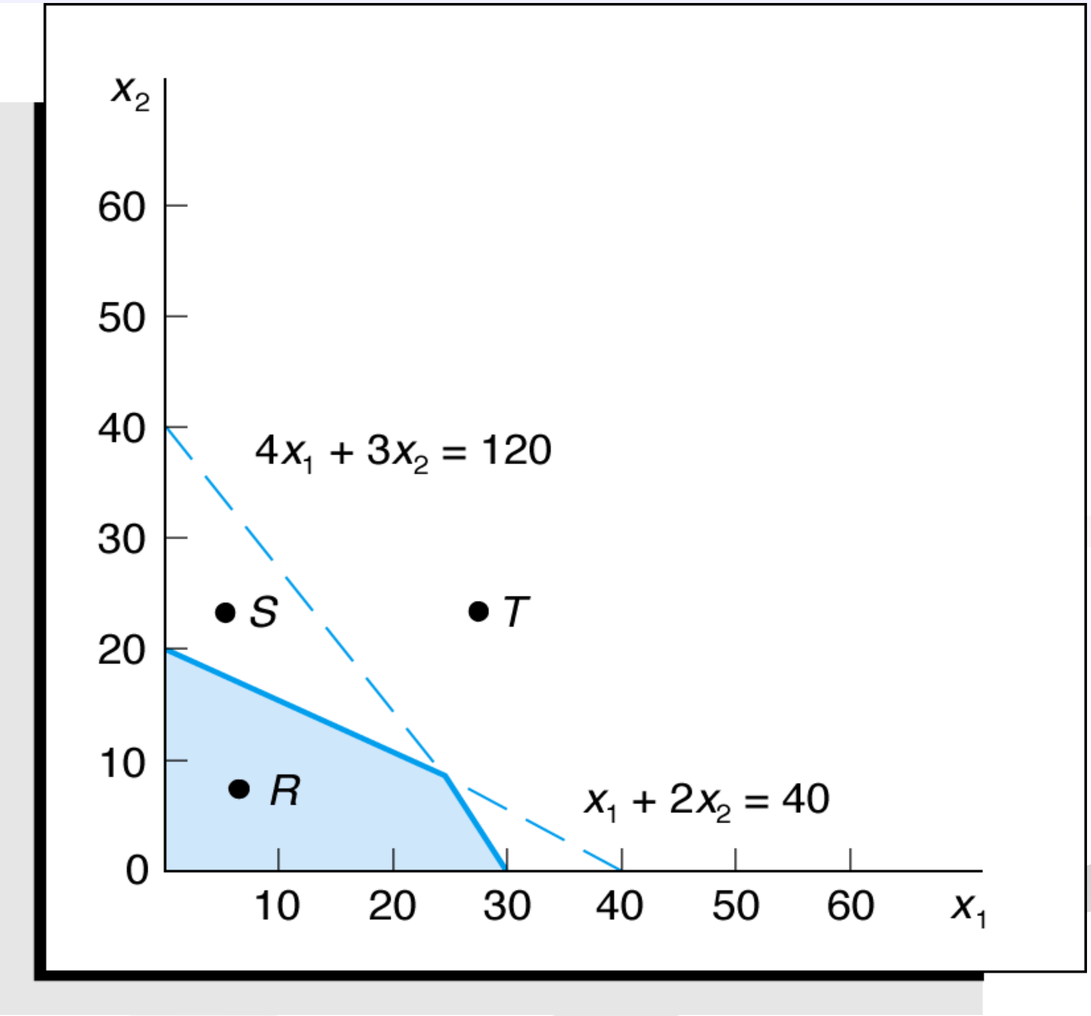


Figure Feasible Solution Area

Objective Function Solution = $800 Graphical Solution of Maximization Model (7 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

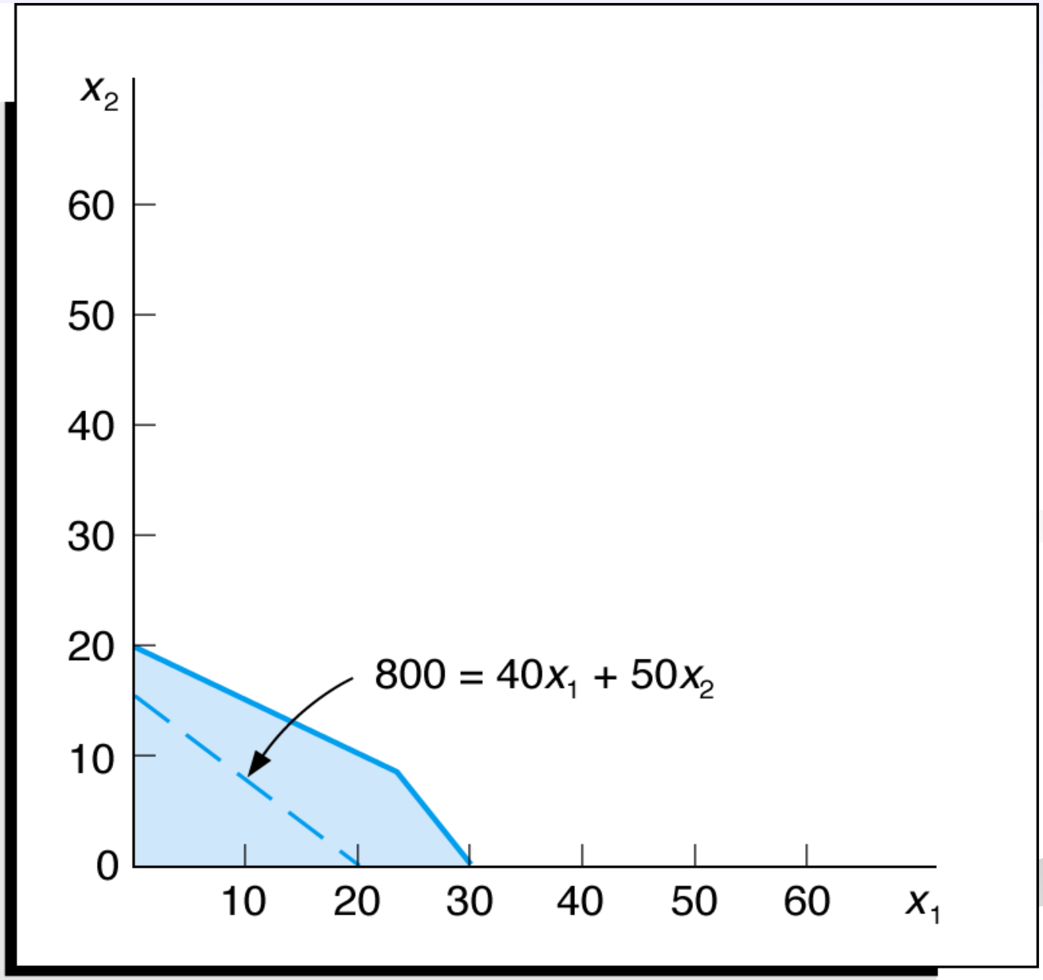


Figure Objection Function Line for Z = $800

Alternative Objective Function Solution Lines
Graphical Solution of Maximization Model (8 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

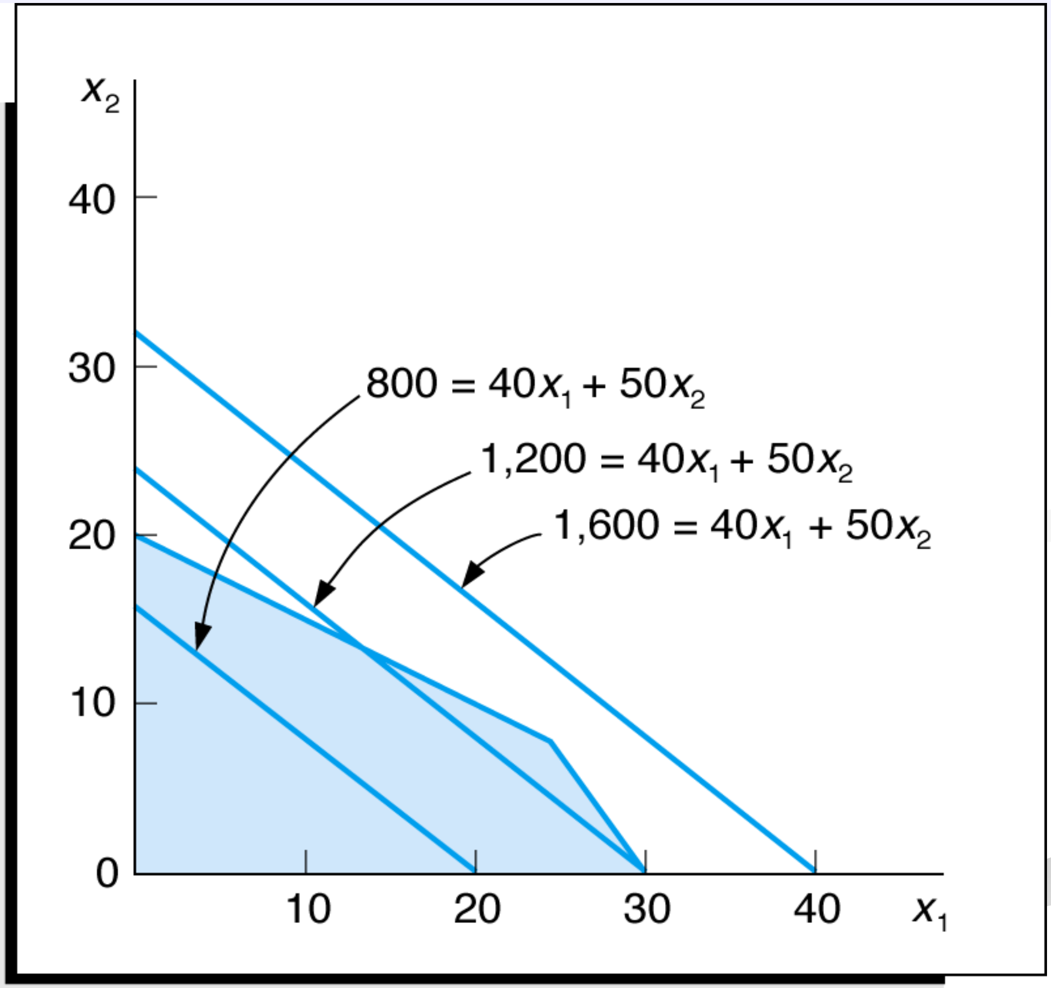


Figure Alternative Objective Function Lines

Optimal Solution Graphical Solution of Maximization Model (9 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

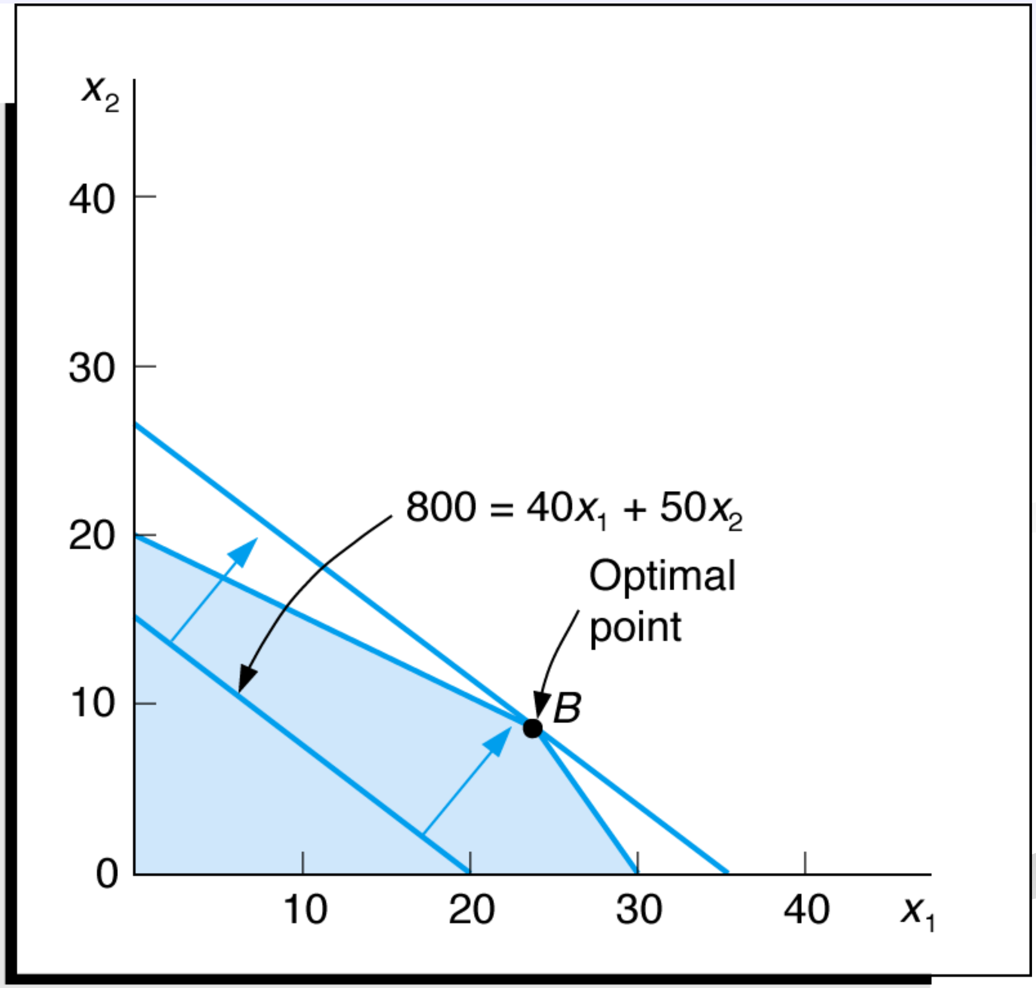


Figure Identification of Optimal Solution Point

Optimal Solution Coordinates Graphical Solution of Maximization Model (10 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \le 40$

$4x_1 + 3x_2 \le 120$

$x_1, x_2 \ge 0$

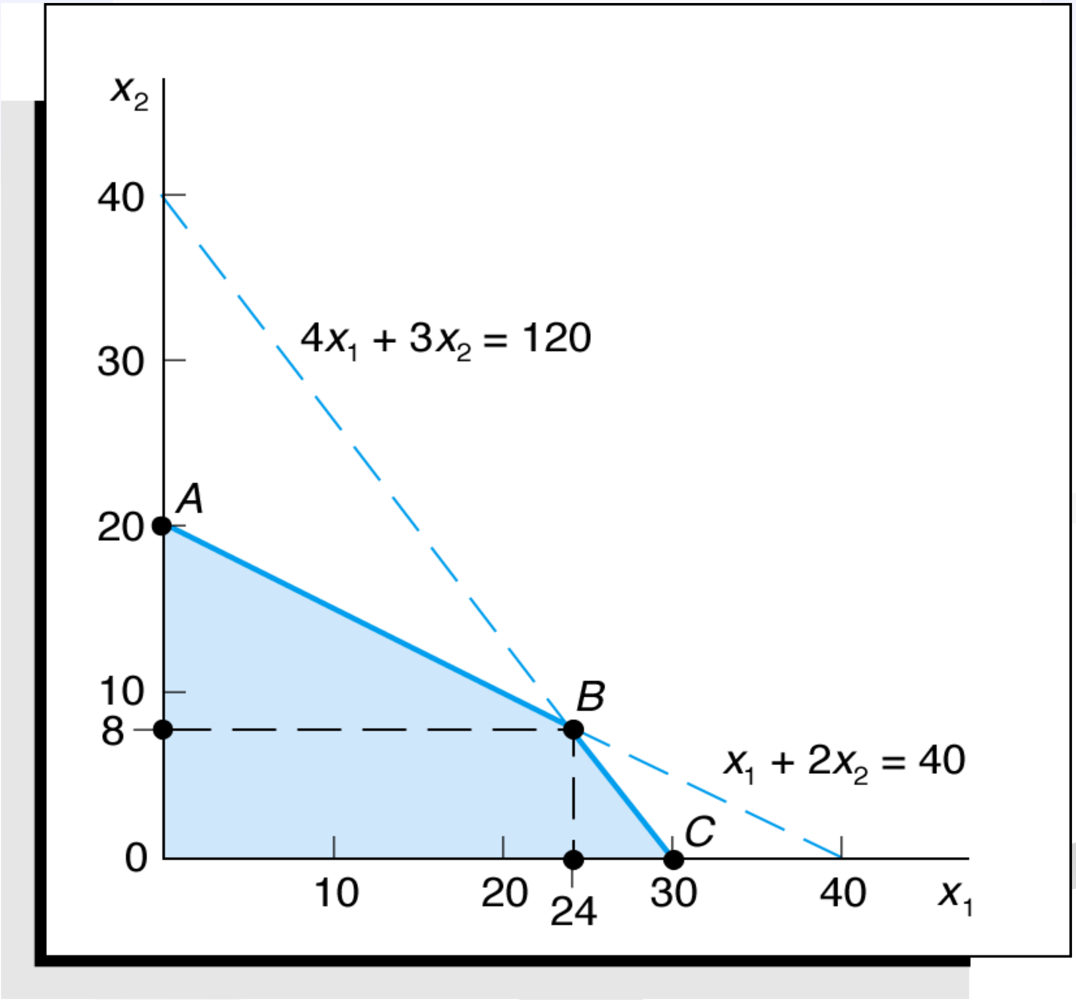


Figure Optimal Solution Coordinates

Extreme (Corner) Point Solutions Graphical Solution of Maximization Model (11 of 12)

Maximize $Z = \$40x_1 + \$50x_2$

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

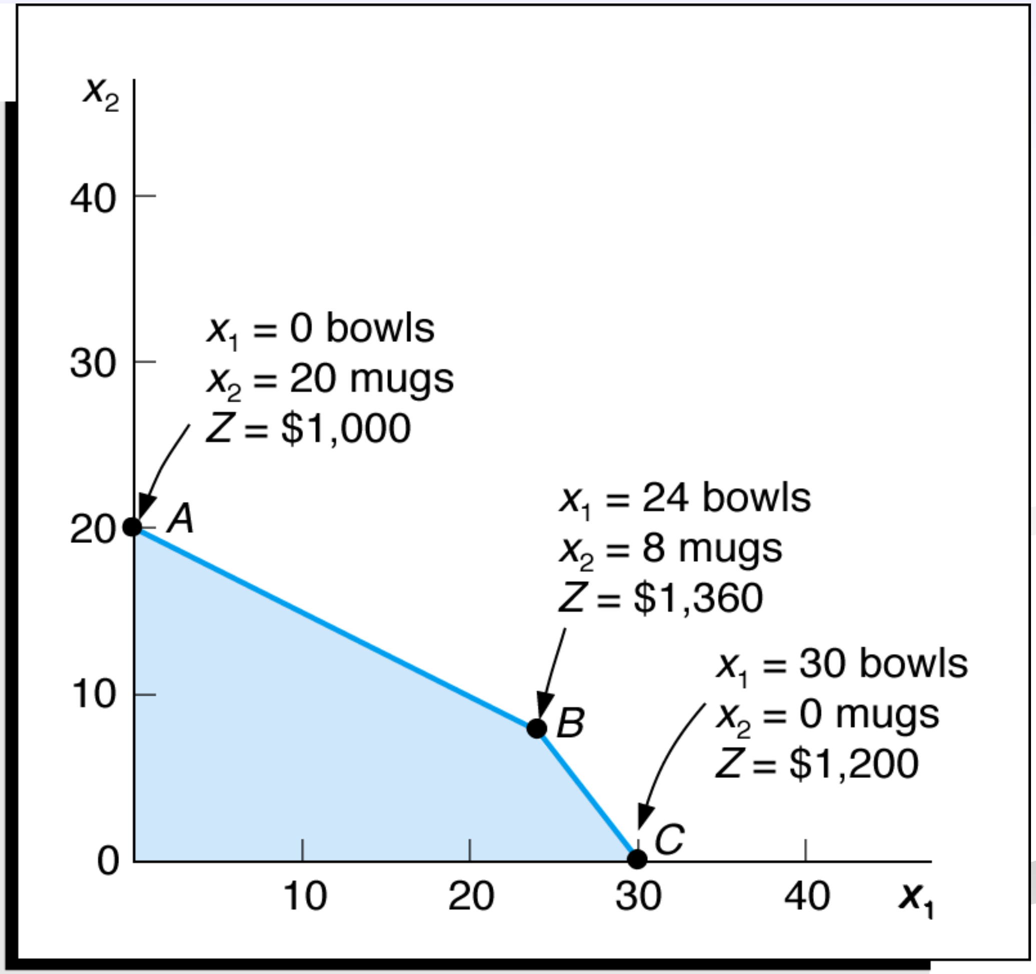


Figure Solutions at All Corner Points

Optimal Solution for New Objective Function

Maximize Z **= $70x_1$ + $20x_2$**

subject to: $1x_1 + 2x_2 \leq 40$

$4x_1 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

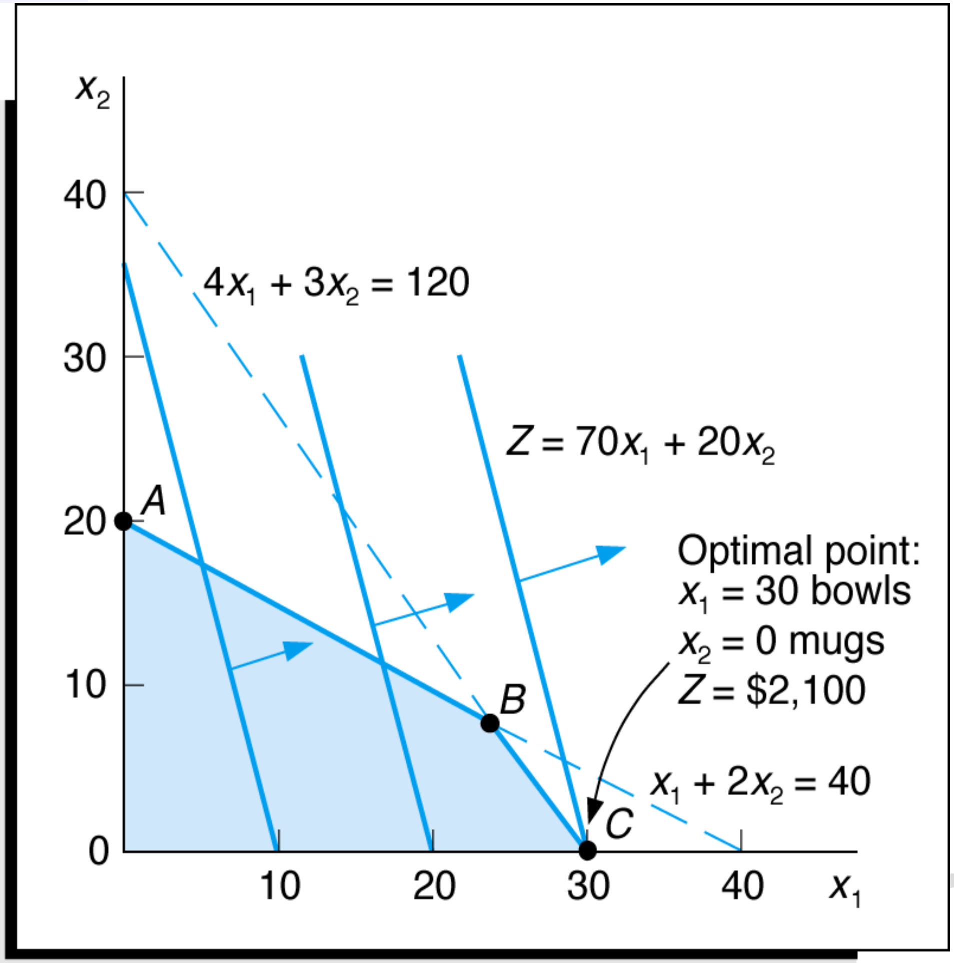


Figure Optimal Solution with $Z = 70x_1 + 20x_2$

Slack Variables

- Standard form of a linear programming problem requires that all constraints be in the form of equations (equalities).
- A slack variable is ***added to a ≤ constraint*** (weak inequality) to convert it to an equation (=).
- A slack variable typically represents an ***unused resource***.
- A slack variable ***contributes nothing*** to the objective function value.i.e**. such variables are added in the objective**

Linear Programming Model: Standard Form

Max Z = $40x_1 + 50x_2 + 0s_1 + 0s_2$

subject to: $1x_1 + 2x_2 + s_1 = 40$

$4x_1 + 3x_2 + s_2 = 120$

$x_1, x_2, s_1, s_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

s_1, s_2 are slack variables

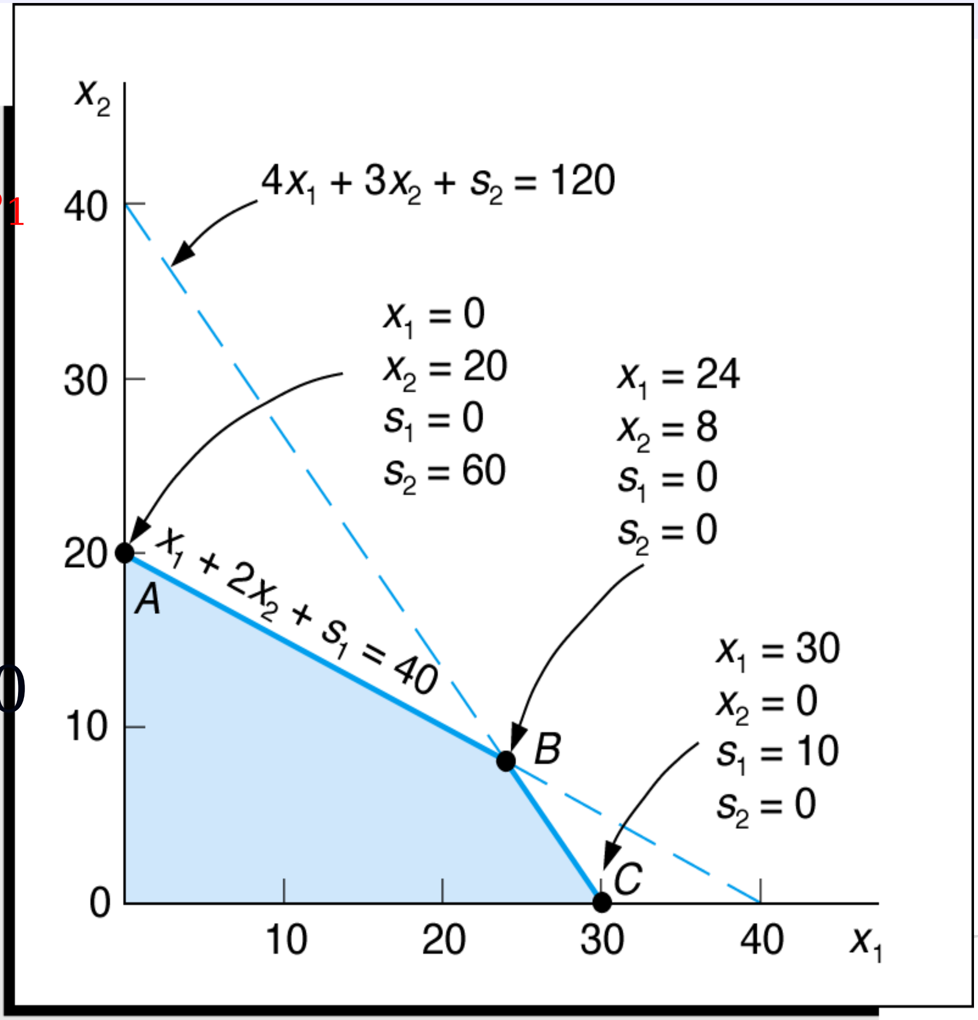


Figure Solution Points A, B, and C with Slack

LP Model Formulation – Minimization (1 of 8)

- Two brands of fertilizer available - Super-gro, Crop-quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-gro costs $6 per bag, Crop-quick $3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data ?

	Chemical Contribution	
Brand	Nitrogen (lb/ bag)	Phosphate (lb/ bag)
Super-gro	2	4
Crop-quick	4	3

LP Model Formulation – Minimization (2 of 8)

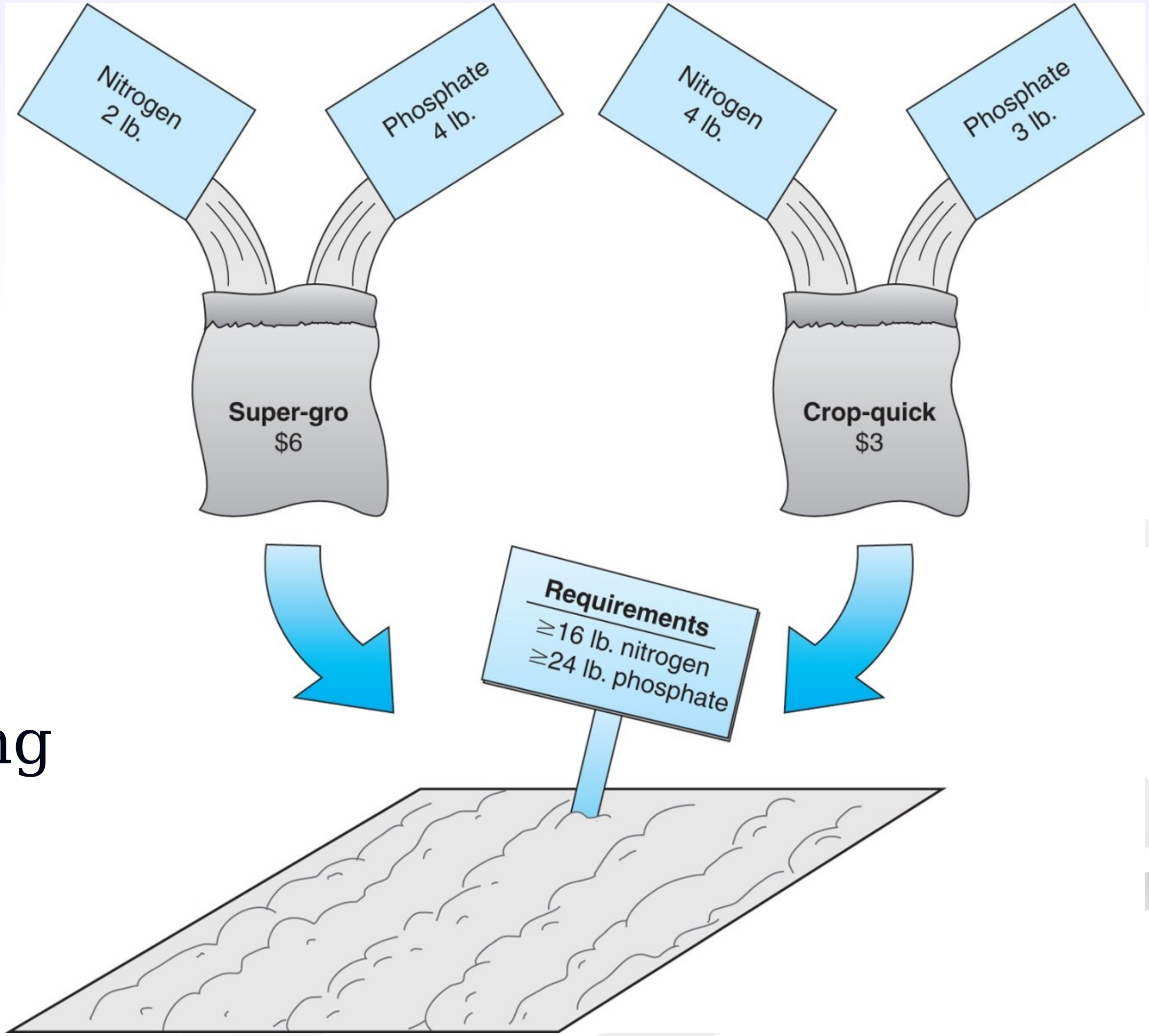


Figure Fertilizing farmer’s field

LP Model Formulation – Minimization (3 of 8)

Decision Variables:

x_1 = bags of Super-gro

x_2 = bags of Crop-quick

The Objective Function:

Minimize $Z = \$6x_1 + 3x_2$

Where: $\$6x_1$ = cost of bags of Super-Gro

$\$3x_2$ = cost of bags of Crop-Quick

Model Constraints:

$2x_1 + 4x_2 \geq 16$ lb (nitrogen constraint)

$4x_1 + 3x_2 \geq 24$ lb (phosphate constraint)

$x_1, x_2 \geq 0$ (non-negativity constraint)

Constraint Graph – Minimization (4 of 8)

Minimize $Z = \$6x_1 + \$3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

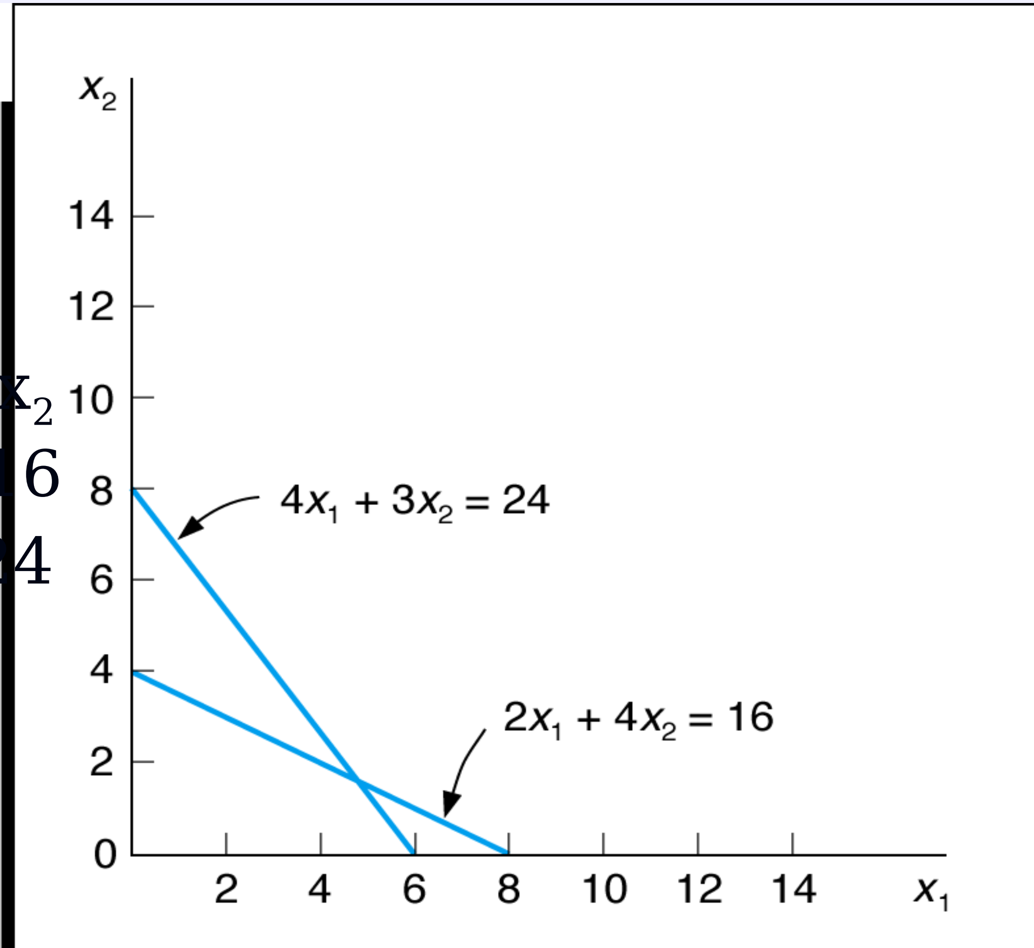


Figure Graph of Both Model Constraints

Feasible Region– Minimization (5 of 8)

Minimize $Z = \$6x_1 + \$3x_2$

subject to: $2x_1 + 4x_2 \geq 16$

$4x_1 + 3x_2 \geq 24$

$x_1, x_2 \geq 0$

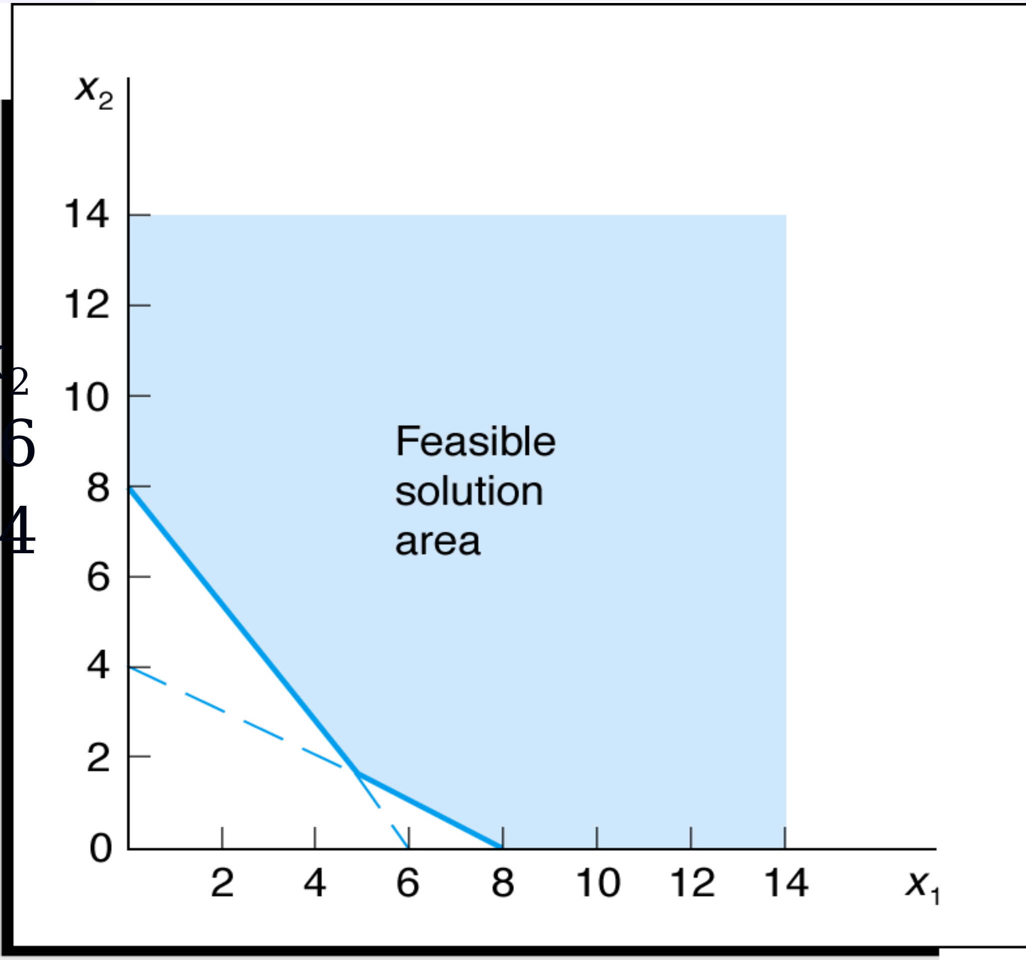


Figure Feasible Solution Area

Optimal Solution Point – Minimization (6 of 8)

$$\text{Minimize } Z = \$6x_1 + \$3x_2$$

$$\text{subject to: } 2x_1 + 4x_2 \geq 16$$

$$4x_1 + 3x_2 \geq 24$$

$$x_1, x_2 \geq 0$$

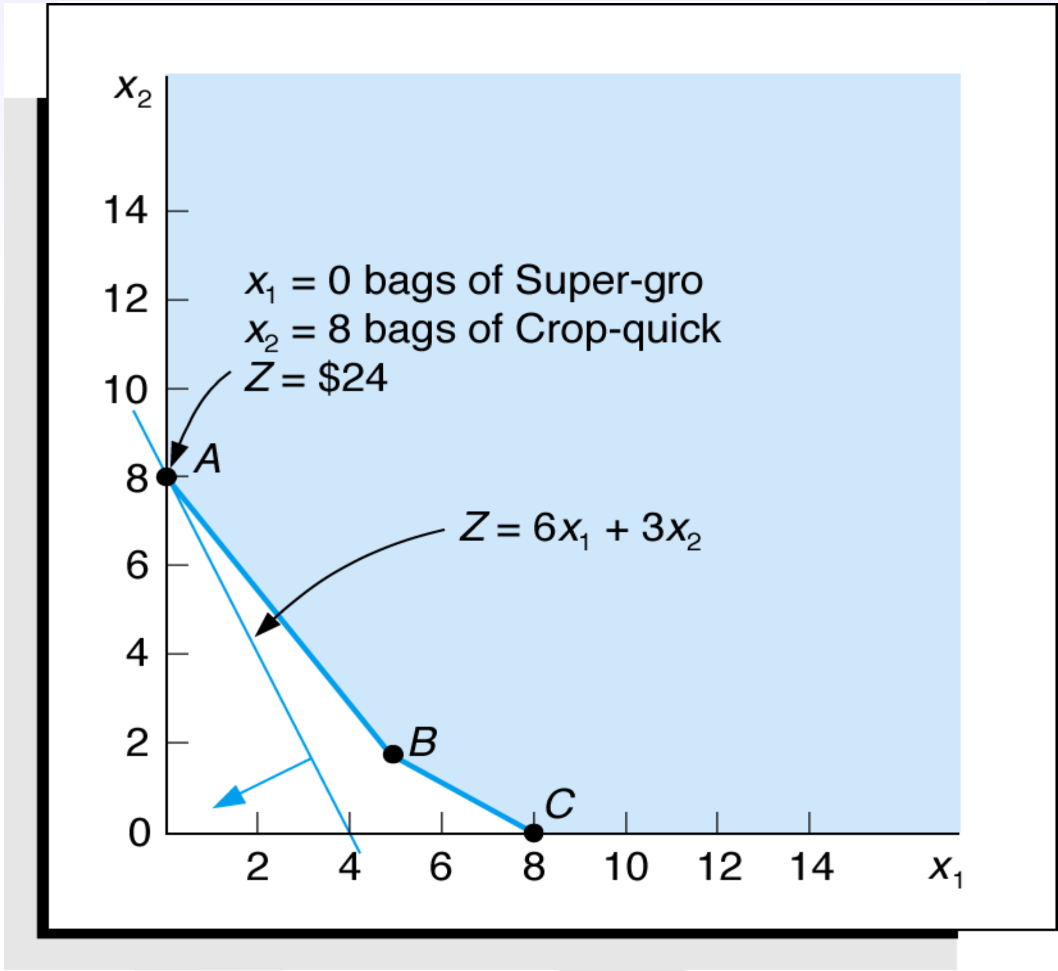


Figure Optimum Solution Point

Surplus Variables – Minimization (7 of 8)

To convert lin prog problem into standard form

- A surplus variable is ***subtracted from a ≥ constraint*** to convert it to an equation (=).
- A surplus variable ***represents an excess*** above a constraint requirement level.
- A surplus variable ***contributes nothing*** to the calculated value of the objective function. **such variables are added in the objective function with zero cost/profit**
- Subtracting surplus variables in the farmer problem constraints:

$$2x_1 + 4x_2 - s_1 = 16$$

(nitrogen)

Graphical Solutions – Minimization (8 of 8)

Minimize $Z = \$6x_1 + \$3x_2 + 0s_1 + 0s_2$

subject to: $2x_1 + 4x_2 - s_1 = 16$

$4x_1 + 3x_2 - s_2 = 24$

$x_1, x_2, s_1, s_2 \geq 0$

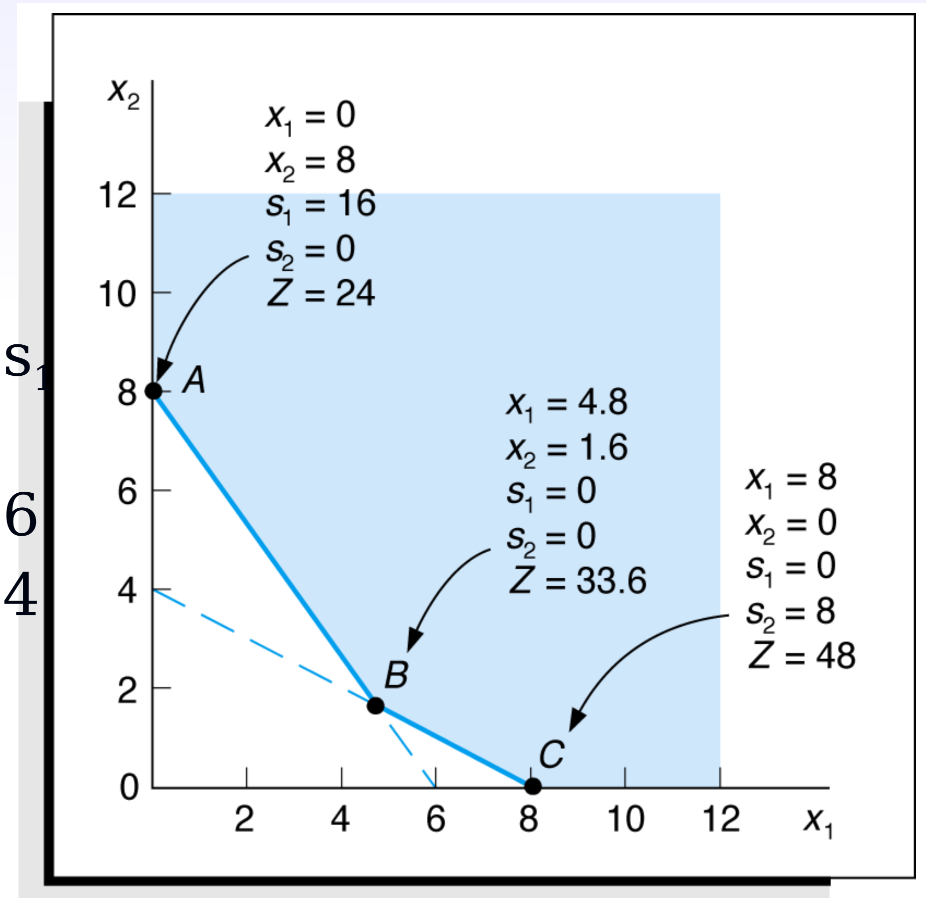


Figure Graph of Fertilizer Example

Irregular Types of Linear Programming Problems

- **Special types of problems include those with:**
  - **Multiple optimal solutions**
  - **Infeasible solutions**
  - **Unbounded solutions**

Multiple Optimal Solutions

The objective function is parallel to a constraint line.

Maximize $Z=\$40x_1 + 30x_2$
subject to: $1x_1 + 2x_2 \leq 40$
$4x_1 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

Where:
x_1 = number of bowls
x_2 = number of mugs

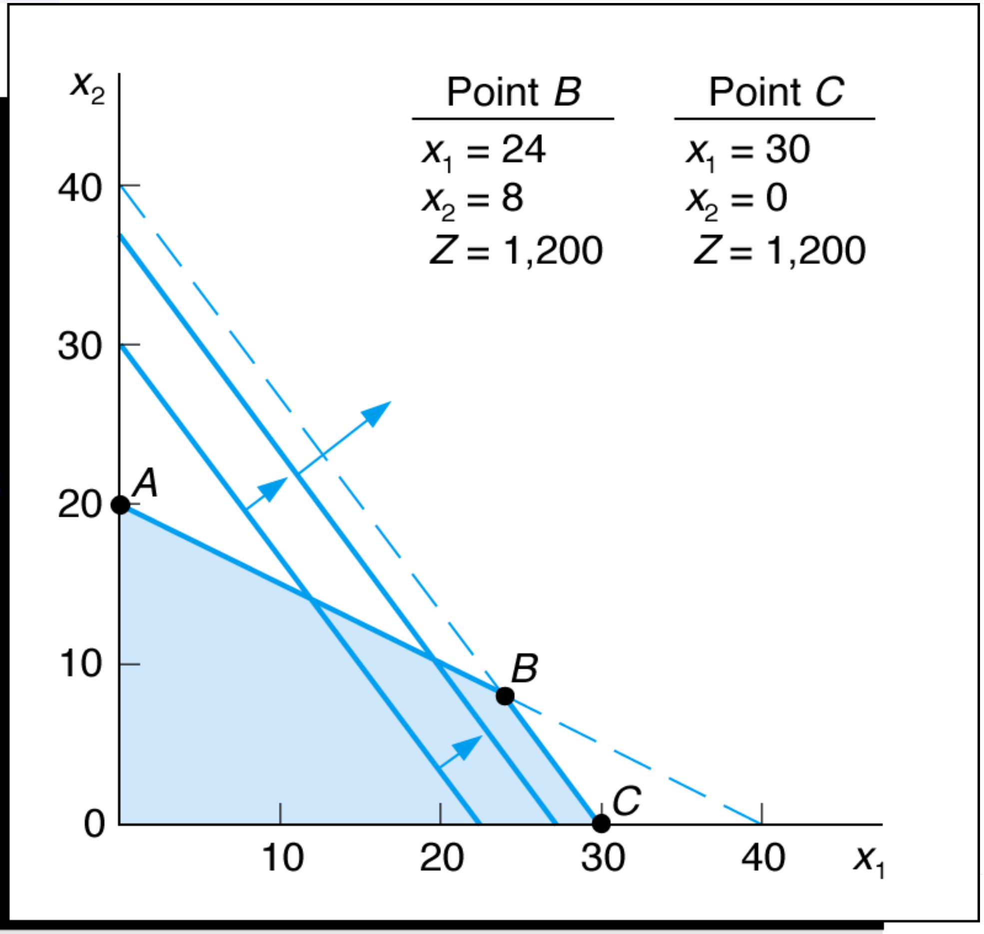


Figure Example with Multiple Optimal Solutions

An Infeasible Problem

Every possible solution **violates** at least one constraint:

Maximize $Z = 5x_1 + 3x_2$

subject to: $4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

$x_2 \geq 6$

$x_1, x_2 \geq 0$

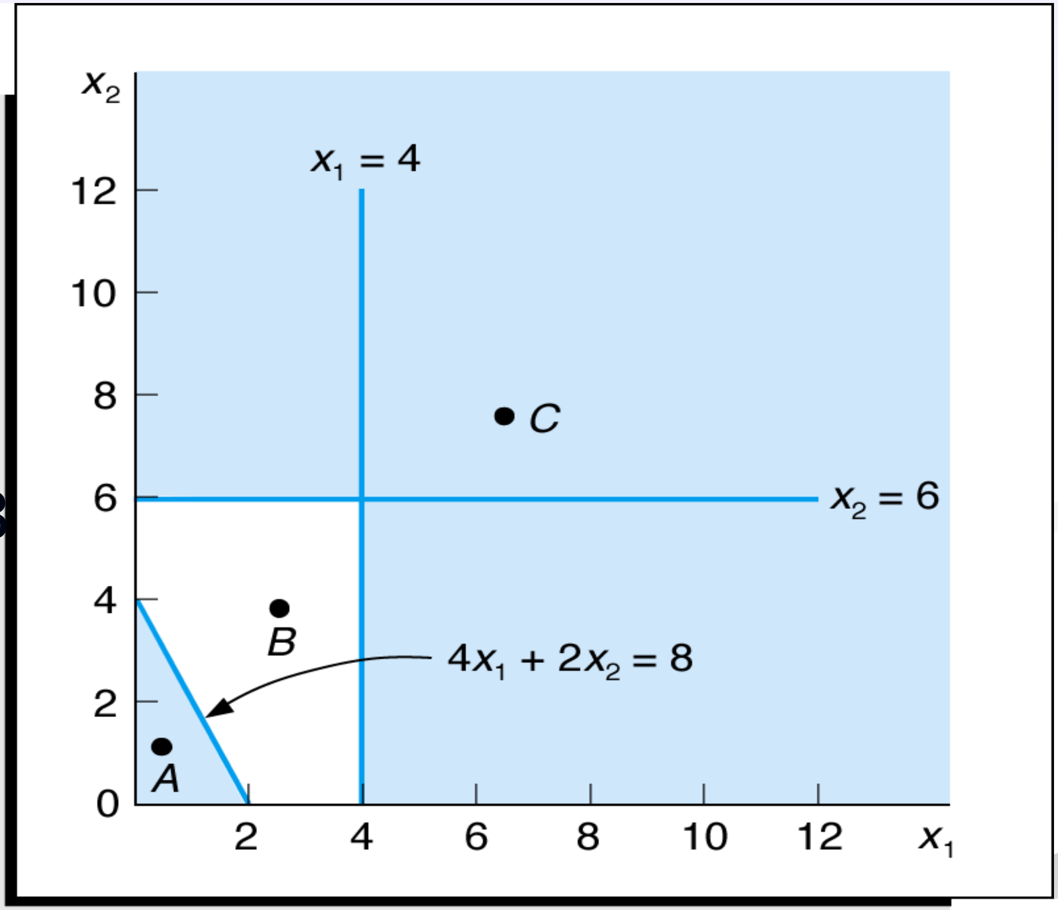


Figure Graph of an Infeasible Problem

An Unbounded Problem

Value of the objective function increases indefinitely:

Maximize $Z = 4x_1 + 2x_2$

subject to: $x_1 \geq 4$

$x_2 \leq 2$

$x_1, x_2 \geq 0$

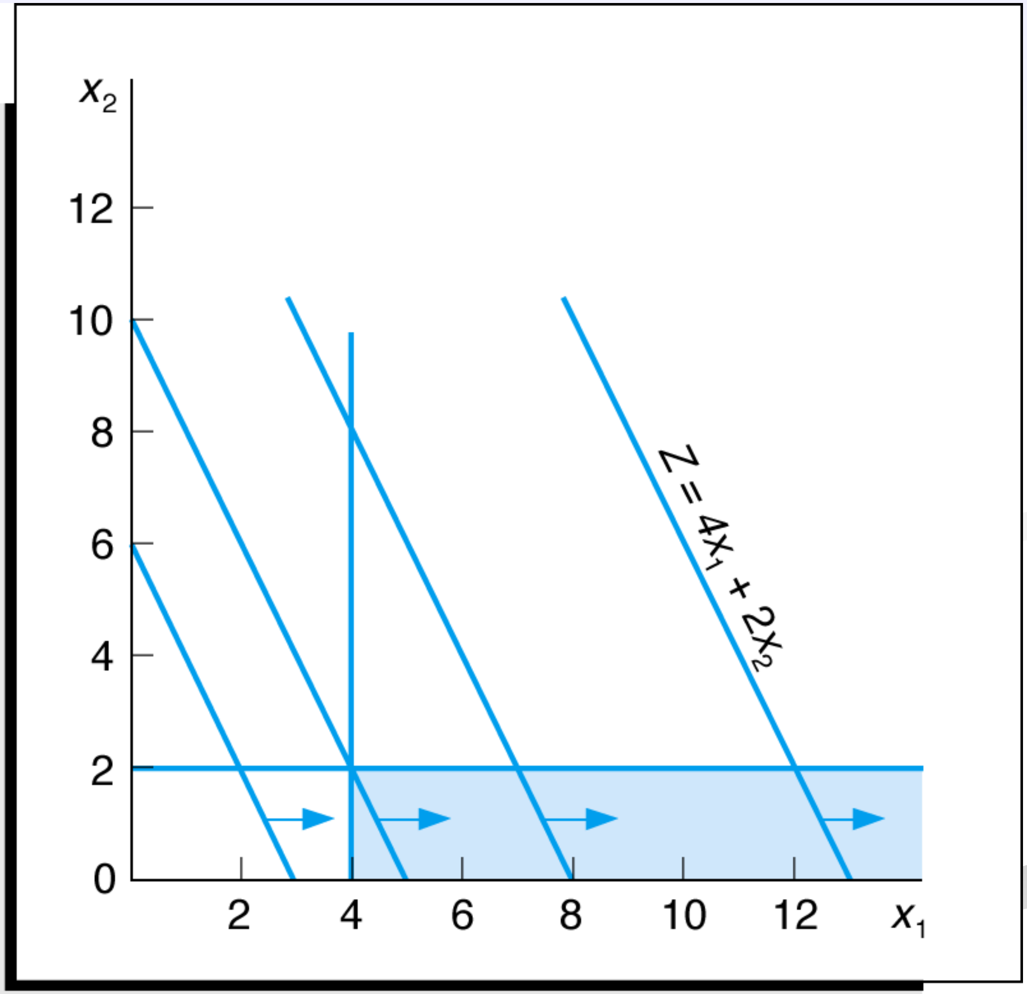


Figure Graph of an Unbounded Problem

Properties(assumptions) of Linear Programming Models

- **Proportionality** - The rate of change (slope) of the objective function and constraint equations is constant.
- **Additivity** - Terms in the objective function and constraint equations must be additive.
- **Divisibility** -Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.
- **Certainty** - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).

Problem Statement Example Problem No. 1 (1 of 3)

- Hot dog mixture in 1000-pound batches.
- Two ingredients, chicken ($3/lb) and beef ($5/lb).
- Recipe requirements:

  at least 500 pounds of "chicken"

  at least 200 pounds of "beef"
- Ratio of chicken to beef must be at least 2 to 1.

Solution
Example Problem No. 1 (2 of 3)

Step 1:

Identify decision variables.

x_1 = lb of chicken in mixture

x_2 = lb of beef in mixture

Step 2:

Formulate the objective function.

Minimize $Z = \$3x_1 + \$5x_2$

where Z = cost per 1,000-lb batch

$\$3x_1$ = cost of chicken

$\$5x_2$ = cost of beef

Solution
Example Problem No. 1 (3 of 3)

Step 3:

Establish Model Constraints

$$x_1 + x_2 = 1{,}000 \text{ lb}$$
$$x_1 \geq 500 \text{ lb of chicken}$$
$$x_2 \geq 200 \text{ lb of beef}$$
$$x_1/x_2 \geq 2/1 \text{ or } x_1 - 2x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

The Model: Minimize $Z = \$3x_1 + 5x_2$

subject to: $x_1 + x_2 = 1{,}000$ lb

$$x_1 \geq 500$$
$$x_2 \geq 200$$
$$x_1 - 2x_2 \geq 0$$

SIMPLEX METHOD

- Linear programming models could be solved algebraically.
- The most widely used algebraic procedure for solving linear programming problem is called the_Simplex Method.
- The simplex method is a general-purpose linear-programming algorithm widely used to solve large scale problems. Although it lacks the intuitive appeal of the graphical approach, its ability to handle problems with more than two decision variables makes it extremely valuable for solving problems often encountered in production/operations management.
- Thus simplex method offers an efficient means of solving more complex linear programming problems.

Characteristics of Simplex Method

- In the simplex method, the computational routine is an iterative process. To *iterate* means to repeat; hence, in working toward the optimum solution, the computational routine is repeated over and over, following a standard pattern.
- Successive solutions are developed in a systematic pattern until the best solution is reached.

- Each new solution will yield a value of the objective function as large as or larger than the previous solution. This important feature assures us that we are always moving closer to the optimum answer. Finally, the method indicates when the optimum solution has been reached.

Characteristics of Simplex Method

The simplex method requires simple mathematical operations (addition, subtraction, multiplication, and division), but the computations are lengthy and tedious,
The student will discover that it is better not to use his/her calculator in working through these problems because rounding can easily distort the results.

Instead, it is better to work with numbers in fractional form.

Skip slides upto 107

Why we should study the Simplex Method?

- It is important to understand the ideas used to produce solution. The simplex approach yields not only the optimal solution to the xi variables, and the maximum profit (or minimum cost) but valuable economic information as well.

- We begin by solving a maximization problem using the simplex method. We then tackle a minimization problem.

SUMMARY OF THE SIMPLEX METHOD

Step 1. **Formulate a LP model of the problem.**

Step 2. **Add slack variables to each constraint to obtain standard form**.

Step 3. **Set up the initial simplex tableau**.

Step 4. Choose the nonbasic variable with the largest entry in the net evaluation row . This identifies the pivot (key) column; the column associated with the incoming variable.

Step 5. Choose as the pivot row that row with the smallest ratio of “bi/ aij”, for aij >0 where j is the pivot column. This identifies the pivot row, the row of the variable leaving the basis when variable j enters.

Step 6. a). Divide each element of the pivot row by the pivot element.

b). According to the entering variable, find the new values for remaining variables.

Step 7. Test for optimality. If in the net evaluation row all are negative for all columns, we have the optimal solution. If not, return to step 4.

Example 1

A Furniture Ltd., wants to determine the most profitable combination of products to manufacture given that its resources are limited. The Furniture Ltd., makes two products, *tables* and *chairs*, which must be processed through *assembly* and *finishing departments*. Assembly has 60 hours available; Finishing can handle up to 48 hours of work.
Manufacturing one table requires 4 hours in assembly and 2 hours in finishing.
Each chair requires 2 hours in assembly and 4 hours in finishing. Profit is $8 per table and $6 per chair.

	Hours required for 1 unit of product		Total hours available
	Tables	*Chairs*	
Assembly	4	2	60
Finishing	2	4	48
Profit per unit	$8	$6	

Tabular solution for Example 1/1

Stated algebraically, the Ltd., problem is

Maximise! Profit $Z = 8X_1 + 6X_2$

Subject to :

Assembly	$4X_1 + 2X_2 \leq 60$
Finishing	$2X_1 + 4X_2 \leq 48$
	All variables ≥ 0

- The first step is to convert the inequalities into equations.

The best combination of tables and chairs may not necessarily use all the time available in each department. We must therefore add to each inequality a variable, which will take up the slack, i.e. the time not used in each department. This variable is called a slack variable.

By adding the slack variables we convert the constraint inequalities in the problem into equations. The slack variable in each department takes on whatever value is required to make the equation relationship hold.

∴ The final form is

Maximize Profit $Z = 8X_1 + 6X_2 + 0S_1 + 0S_2$

Subject to

$$4X_1 + 2X_2 + S_1 = 60$$
$$2X_1 + 4X_2 + S_2 = 60$$

All variables ≥ 0

Tabular solution for Example 1/2

- The 2nd step is to put the equations into tabular form, called ***tableaus***.

Product mix column

Profit per unit column

constant column (quantities of product in the mix)

Variable columns

Cj			\$8	6	0	0	Cj row
	Product mix	Quantity	X_1	X_2	S_1	S_2	Variable row
\$0	S_1	60	4	2	1	0	
0	S_2	48	2	4	0	1	

Real products

slack time

The simplest starting solution is to make no tables or chairs, have all unused time and earn no profit. This solution is technically feasible but not financially attractive. (Because the variables X_1 and X_2 do not appear in the mix, they are equal to zero.)

To find the profit for each solution and to determine whether the solution can be improved upon, we need to add two more rows to the initial simplex tableau: a Z_j row and a $C_j - Z_j$ row.

Column Z_j = Total profit from this particular solution

Tabular solution for Example 1/3

The four values for Z_j under the variable columns (all 0 $) are the amounts by which profit would be reduced if 1 unit of any of the variables were added to the mix.

C_j			$8	6	0	0	
	Product mix	Quantity	X_1	X_2	S_1	S_2	
$0	S_1	60	4	2	1	0	
0	S_2	48	2	4	0	1	
	Z_j	$0	0	0	0	0	
	$C_j - Z_j$		8	6	0	0	

Max

Z_j represents the gross profit given up by adding 1 unit of this variable into the current solution (profit loss per unit). $C_j - Z_j$ is net profit from the introduction 1 unit of each variable into the solution.

By examining the numbers in the $C_j - Z_j$ row we can see that total profit can be increased by 48 for each unit of X_1 (tables). Positive number indicates that profits can be improved for each unit added. We select the largest positive value. Max $C_j - Z_j$ value showing the variable that should be added, replacing one of the variables present in the mix.

Tabular solution for Example 1/4

- The next step is to determine which variable will be replaced.

  This is done in the following manner:

  Divide quantity column values by their corresponding numbers in the maximum (optimum) column and select the row with the smallest nonnegative ratio as the row to be replaced.

  S_1 row 60/4 = 15 units of Table (X_1) → minimum replaced row.

  S_2 row 48/4 = 24 units of Table (X_1)

1st Simplex Tableau

Cj			$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	b_i/a_{ij}
0	S_1	60	4	2	1	0	60/4 = 15
0	S_2	48	2	4	0	1	48/2 = 24
	Zj	$0	$0	0	0	0	Intersectional elements
	Cj – Zj		8	6	0	0	(key #)

↑ Max. (optimum entering variable)

Tabular solution for Example 1/5

a_{ij} = coefficient associated with variable j in the constraint i

For 2nd Simplex Tableau

X_1 = 60/4 = 15, 4/4 = 1, 2/4 = 1/2, 1/4 = 1/4 , 0/4 = 0

Thus new X_1 row should be (15, 1 , 1/2 , 1/4 , 0),

The new values for remaining rows:

[elements in old row] - [key #] x [corresponding elements in replacing row] = new row

Elements in old row	–	key #	x	replacing row	=	new row
48	–	2	x	15	=	18
2	–	2	x	1	=	0
4	–	2	x	1/2	=	3
0	–	2	x	1/4	=	-1/2
1	–	2	x	0	=	1

The computation of Z_j row for 2nd tableau is as follows.

$$\left.\begin{array}{ll} Z_j \text{ for } X_1 & 8 \times 1 + 0\ (0) = 8 \\ Z_j \text{ for } X_2 & 8\ (1/2) + 0\ (3) = 4 \\ Z_j \text{ for } S_1 & 8\ (1/4) + 0\ (-1/2) = 2 \\ Z_j \text{ for } S_2 & 8\ (0) + 0\ (1) = 0 \end{array}\right\}$$

Profit given up by introducing 1 unit of these variables

Z_j (total profit) = 8 (15) + 0 (18) = \$120

Tabular solution for Example 1/6

2^{nd} Simplex Tableau

C_j			\$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	b_i/a_{ij}
\$8	X_1	15	1	1/2	1/4	0	15/ ½ = 30
0	S_2	18	0	3	-1/2	1	18/3 = 6 → min! leaving
	Z_j	\$120	\$8	4	2	0	
	$C_j - Z_j$		\$0	2	-2	0	

Max! entering

X_2 will enter in the product mix and S_2 is leaving.

New X_2 values: 18/3 = 6, 0/3 = 0, 3/3 = 1, -½ /3 = -1/6, 1/3 = 1/3

Thus new X_2 (replacing row) values = 6, 0, 1, -1/6, 1/3 (Assumes same row position as the replaced row)

Tabular solution for Example 1/7

New Values for X_1 :

Elements in old X_1 row	–	key #	x	replacing row	=	new X_1 row
15	–	1/2	x	6	=	12
1	–	1/2	x	0	=	1
1/2	–	1/2	x	1	=	0
1/4	–	1/2	x	-1/6	=	1/3
0	–	1/2	x	1/3	=	-1/6

New Z_j values :

Z_j (total profit) = 8 (12) + 6 (6) = $132

Z_j for X_1 8 (1) + 6 (0) = 8

Z_j for X_2 8 (0) + 6 (1) = 6

Z_j for S_1 8 (1/3) + 6 (-1/6) = 5/3

Z_j for S_2 8 (- 1/6) + 6 (1/3) = 2/3

Tabular solution for Example 1/8

3rd Simplex Tableau

C_j			$8	6	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	
$8	X_1	12	1	0	1/3	-1/6	
6	X_2	6	0	1	-1/6	1/3	
	Z_j	$132	$8	6	5/3	2/3	
	$C_j - Z_j$		$0	0	-5/3	-2/3	

There is no positive "$C_j - Z_j$" value, no further profit improvement is possible. Thus the optimum solution is obtained. Profit will be maximized by making 12 tables and 6 chairs and having no unused time in either department (because slack variables do no appear in the product-mix column and are equal to zero). Optimum profit is $132 .

Verification:

Objective function

$$Z_j = 8\ X_1 + 6\ X_2 + 0\ (S_1 + S_2)$$
$$Z_j = 8\ (12) + 6\ (6) + 0 = \$132$$

Constraints :

Assembly $4\ X_1 + 2\ X_2 \leq 60 \rightarrow 4\ (12) + 2\ (6) \leq 60 \rightarrow 60 \leq 60$

Finishing $2\ X_1 + 4\ X_2 \leq 48 \rightarrow 2\ (12) + 4\ (6) \leq 48 \rightarrow 48 \leq 48$

Example 2

PAR Inc. produces golf equipment and decided to move into the market for standard and deluxe golf bags. Each golf bag requires the following operations:

Cutting and dyeing the material,

Sewing,

Finishing (inserting umbrella holder, club separators etc.),

Inspection and packaging.

Each standard golf-bag will require 7/10 hr. in the cutting and dyeing department, 1/2 hr. in the sewing department, 1 hr. in the finishing department and 1/10 hr. in the inspection & packaging department.

Deluxe model will require 1 hr. in the cutting and dyeing department, 5/6 hr. for sewing, 2/3 hr. for finishing and 1/4 hr. for inspection and packaging

The profit contribution for every standard bag is 10 MU and for every deluxe bag is 9 MU.

In addition the total hours available during the next 3 months are as follows:

Cutting & dyeing dept	630 hrs
Sewing dept	600 hrs
Finishing	708 hrs
Inspection & packaging	135 hrs

The company's problem is to determine how many standard and deluxe bags should be produced in the next 3 months?

Example 2

Let X_1 = number of standard bags
X_2 = number of deluxe bags
Z = the total profit contribution

Objective function:

$$\text{Max!} \quad Z = 10X_1 + 9X_2$$

Subject to constraints

$7/10\ X_1 + 1X_2 \leq 630$	cutting and dying
$1/2\ X_1 + 5/6\ X_2 \leq 600$	sewing
$1X_1 + 2/3\ X_2 \leq 708$	finishing
$1/10\ X_1 + 1/4\ X_2 \leq 135$	inspection & packaging
$X_1 \geq 0$, $X_1 \geq 0$	Nonnegative constraints

In linear programming terminology, any unused or idle capacity for a ≤ constraint is referred to as the slack associated with the constraint. Often variables, called slack variables, are added to the formulation of a linear programming problem to represent the slack or idle capacity. Unused capacity makes no contribution to profit; thus slack variables have coefficients of zero in the objective function. Whenever a linear program is written in a form with all constraints expressed as equalities, it is said to be written in standard form.

After the addition of slack variables to the mathematical statement, the mathematical model becomes

$$\text{Max!} \quad 10X_1 + 9X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$$

Subject to

$$\begin{aligned} 7/10\ X_1 + 1X_2 + 1S_1 &= 630 \\ 1/2\ X_1 + 5/6\ X_2 + 1S_2 &= 600 \\ 1X_1 + 2/3\ X_2 + 1S_3 &= 708 \\ 1/10\ X_1 + 1/4\ X_2 + 1S_4 &= 135 \end{aligned}$$

$$X_1, X_2, S_1, S_2, S_3, S_4 \geq 0$$

Tabular solution for Example 2/1

Initial Tableau

C_j			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	b_i / a_{ij}
0 MU	S_1	630	7/10	1	1	0	0	0	630/ 7/10 =900
0 MU	S_2	600	½	5/6	0	1	0	0	600/ ½ = 1200
0 MU	S_3	708	1	2/3	0	0	1	0	708/1 = 708 (min leaving)
0 MU	S_4	135	1/10	¼	0	0	0	1	135/ 1/10 = 1350
	Z_j	0 MU	0 MU	0 MU	0 MU	0 MU	0 MU	0 MU	
	$C_j - Z_j$		10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	

Max. (Entering)

For the 2nd Simplex Tableau

708/1 = 708, 1/1 = 1, 2/3/1 = 2/3, 0, 0, 1, 0

∴ new X_1 value are : 708, 1, 2/3, 0, 0, 1, 0

Tabular solution for Example 2/2

Elements in old S_1 row	−	key #	x	new X_1 row	=	new S_1 row
630	−	7/10	x	708	=	134.4
7/10	−	7/10	x	1	=	0
1	−	7/10	x	2/3	=	8/15
1	−	7/10	x	0	=	1
0	−	7/10	x	0	=	0
0	−	7/10	x	1	=	-7/10
0	−	7/10	x	0	=	0

Elements in old S_2 row	−	key #	x	new X_1 row	=	new S_2 row
600	−	1/2	x	708	=	246
1/2	−	1/2	x	1	=	0
5/6	−	1/2	x	2/3	=	1/2
0	−	1/2	x	0	=	0
1	−	1/2	x	0	=	1
0	−	1/2	x	1	=	-1/2
0	−	1/2	x	0	=	0

Tabular solution for Example 2/3

Elements in old S_4 row	–	key #	x	new X_1 row	=	new S_4 row
135	–	1/10	x	708	=	64.2
1/10	–	1/10	x	1	=	0
1/4	–	1/10	x	2/3	=	11/60
0	–	1/10	x	0	=	0
0	–	1/10	x	0	=	0
0	–	1/10	x	1	=	-1/10
1	–	1/10	x	0	=	1

2nd Tableau

C_j			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	b_i / a_{ij}
0 MU	S_1	134.4	0	8/15	1	0	-7/10	0	252 (min leaving)
0 MU	S_2	246	0	½	0	1	-1/2	0	492
10 MU	X_1	708	1	2/3	0	0	1	0	1062
0 MU	S_4	64.2	0	11/60	0	0	-1/10	1	3852/11
	Z_j	7080 MU	10	20/3	0	0	10	0	
	$C_j - Z_j$		0	7/3	0	0	-10	0	

Max. (Entering)

New X_2 values are: $\frac{134.4}{77} = 252,\ 0,\ 1,\ 15/8,\ 0,\ -21/16,\ 0$

Tabular solution for Example 2/4

Elements in old S_2 row	–	key #	x	new X_2 row	=	new S_2 row
246	–	1/2	x	252	=	120
0	–	1/2	x	0	=	0
1/2	–	1/2	x	1	=	0
0	–	1/2	x	15/8	=	-15/16
1	–	1/2	x	0	=	1
-1/2	–	1/2	x	-21/16	=	5/32
0	–	1/2	x	0	=	0

Elements in old X_1 row	–	key #	x	new X_2 row	=	new X_1 row
708	–	2/3	x	252	=	540
1	–	2/3	x	0	=	1
2/3	–	2/3	x	1	=	0
0	–	2/3	x	15/8	=	-5/4
0	–	2/3	x	0	=	0
1	–	2/3	x	-21/16	=	15/8
0	–	2/3	x	0	=	0

Elements in old S_4 row	–	key #	x	new X_2 row	=	new S_4 row
64.2	–	11/60	x	252	=	18
0	–	11/60	x	0	=	0
11/60	–	11/60	x	1	=	0
0	–	11/60	x	15/8	=	-11/32
0	–	11/60	x	0	=	0
-1/10	–	11/60	x	-21/10	=	45/320
1	–	11/60	x	0	=	1

Tabular solution for Example 2/5

3^{rd} Tableau

Cj			10 MU	9 MU	0 MU	0 MU	0 MU	0 MU
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4
9 MU	X_2	252	0	1	15/8	0	-21/16	0
0 MU	S_2	120	0	0	-15/16	1	5/32	0
10 MU	X_1	540	1	0	-5/4	0	15/8	0
0 MU	S_4	18	0	0	-11/32	0	45/320	1
	Zj	7668 MU	10	9	135/8+(5/4) = 35/8	0	-189/16+ 300/16=111/16	0
	Cj – Zj		0	0	-35/8	0	-111/16	0

There is no positive Cj – Zj value in the simplex tableau. Therefore no further profit improvement is possible. Thus the optimum solution is obtained.

- Thus: Standard bag production (X_1) = 540 bags .
  Deluxe bag production (X_2) = 252 bags .
- Maximum profit = Z = \$10 (540) + \$9 (252) = \$7668
- Unused hours in Sewing department = 120 hours
  Inspection and packaging department = 18 hours

Example 3

High Tech industries import components for production of two different models of personal computers, called deskpro and portable. High Tech's management is currently interested in developing a weekly production schedule for both products.

The deskpro generates a profit contribution of $50/unit, and portable generates a profit contribution of $40/unit. For next week's production, a max of 150 hours of assembly time is available. Each unit of deskpro requires 3 hours of assembly time. And each unit of portable requires 5 hours of assembly time.

High Tech currently has only 20 portable display components in inventory; thus no more than 20 units of portable may be assembled. Only 300 sq. feet of warehouse space can be made available for new production. Assembly of each Deskpro requires 8 sq. ft. of warehouse space, and each Portable requires 5 sq. ft. of warehouse space.

	X_1 - Deskpro	X_2 – Portable	Capacity
Assembly line	3	5	150
Portable Ass	-	1	20
Space	8	5	300
Profit Cont.	$50/unit	$40/unit	

Tabular solution for Example 3/1

	X_1 - Deskpro	X_2 – Portable	Capacity
Assembly line	3	5	150
Portable Ass	-	1	20
Space	8	5	300
Profit Cont.	$50/unit	$40/unit	

X_1 = *number of units of the Deskpro* X_2 = *number of units of the Portable*

Objective Function : Max! $Z = 50 X_1 + 40 X_2$

Subject to :

$$3 X_1 + 5 X_2 \leq 150 \quad \text{Assembly time}$$
$$1 X_2 \leq 20 \quad \text{Portable display}$$
$$8 X_1 + 5 X_2 \leq 300 \quad \text{Warehouse capacity}$$
$$X_1, X_2 \geq 0$$

Adding a slack variable to each of the constraints permits us to write the problem in standard form:

Objective Function: Max! $Z = 50 X_1 + 40 X_2 + 0 S_1 + 0S_2 + 0S_3$

Subject to:

$$3 X_1 + 5 X_2 + 1S_1 = 150$$
$$1 X_2 + 1S_2 = 20$$
$$8 X_1 + 5 X_2 + 1S_3 = 300$$
$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Tabular solution for Example 3/2

– **Initial Tableau**

C_j			\$50	\$40	\$0	\$0	\$0	
	Product mix	Quantity bi	X_1	X_2	S_1	S_2	S_3	b_i / a_{ij}
\$0	S_1	150	3	5	1	0	0	150/3 = 50
\$0	S_2	20	0	1	0	1	0	--
\$0	S_3	300	8	5	0	0	1	300/8 = 37.5 (min. leaving)
	Z_j	\$0	\$0	\$0	\$0	\$0	\$0	
	$C_j - Z_j$		\$50	\$40	\$0	\$0	\$0	

Max. (entering)

New X_1 value : 300/8 = 37.5, 8/8 = 1, 5/8, 0, 0, 1/8

Old S_1 row	–	key #	x	new X_1 row	=	new S_1 row
150	–	3	x	75/2	=	37.5
3	–	3	x	1	=	0
5	–	3	x	5/8	=	25/8
1	–	3	x	0	=	1
0	–	3	x	0	=	0
0	–	3	x	1/8	=	-3/8

Old S_2 row	–	key #	x	new X_1 row	=	new S_2 row
20	–	0	x	75/2	=	20
0	–	0	x	1	=	0
1	–	0	x	5/8	=	1
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	1/8	=	0

Tabular solution for Example 3/3

2nd Tableau

Cj			$50	$40	$0	$0	$0	
	Product mix	Quantity bi	X_1	X_2	S_1	S_2	S_3	b_i / a_{ij}
$0	S_1	75/2	0	25/8	1	0	-3/8	75/2 / 25/8 = 12 (min leaving)
$0	S_2	20	0	1	0	1	0	20/1=20
$50	X_1	75/2	1	5/8	0	0	1/8	75/2 / 5/8 = 60
	Zj	$1875	$50	$250/8	$0	$0	$50/8	
	Cj – Zj		$0	$70/8	$0	$0	$-50/8	

Max. (Entering)

New X_2 values : 12, 0, 1, 8/25, 0, -3/25

old S_2 row	–	key #	x	new X_2 row	=	new S_2 row
20	–	1	x	12	=	8
0	–	1	x	0	=	0
1	–	1	x	1	=	0
0	–	1	x	8/25	=	-8/25
1	–	1	x	0	=	1
0	–	1	x	-3/25	=	3/25

old X_1 row	–	key #	x	new X_2 row	=	new X_1 row
75/2	–	5/8	x	12	=	30
1	–	5/8	x	0	=	1
5/8	–	5/8	x	1	=	0
0	–	5/8	x	8/25	=	-1/5
0	–	5/8	x	0	=	0
1/8	–	5/8	x	-3/25	=	1/5

Tabular solution for Example 3/3

3rd Tableau

Cj			\$50	\$40	\$0	\$0	\$0
	Product mix	Quantity Bi	X_1	X_2	S_1	S_2	S_3
\$40	X_2	12	0	1	8/25	0	-3/25
\$0	S_2	8	0	0	-8/25	1	3/25
\$50	X_1	30	1	0	-1/5	0	1/5
	Z_j	\$1980	\$50	\$40	\$14/5	\$0	\$26/5
	$C_j - Z_j$		\$0	\$0	\$-14/5	\$0	\$-26/5

The optimal solution to a linear programming problem has been reached when all of the entries in the net evaluation row $C_j - Z_j$ are zero or negative. In such cases, the optimal solution is the current basic feasible solution.

Thus:

Units of Deskpro production (X_1) = 30 units
Units of Portable production (X_2) = 12 units
S_2 = 8 units

Management should note that there would be eight unused Portable display units.
Maximum profit is \$1980.

Tabular solution for Example 4/1

- Suppose that in the high-tech industries problem, management wanted to ensure that the combined total production for both models would be at least 25 units.
- Thus,
- Objective Function $\text{Max } Z = 50X_1 + 40X_2$

Subjective to :

	$3X_1 + 5X_2 \leq 150$	Assembly time
	$1X_1 \leq 20$	Portable display
	$8X_1 + 5X_2 \leq 300$	Warehouse space
	$1X_1 + 1X_2 \geq 25$	Min. total production
	$X_1, X_2 \geq 0$	

Tabular solution for Example 4/2

- First, we use three slack variables and one surplus variable to write the problem in std. Form.
- Max $Z = 50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4$
- Subject to $3X_1 + 5X_2 + 1S_1 = 150$
- $1X_2 + 1S_2 = 20$
- $8X_1 + 5X_2 + 1S_3 = 300$
- $1X_1 + 1X_2 - 1S_4 = 25$
- All variables ≥ 0
- For the initial tableau $X_1 = 0$ $X_2 = 0$
- $S_1 = 150$ $S_2 = 20$
- $S_3 = 300$ $S_4 = -25$

Tabular solution for Example 4/3

- Clearly this is not a basic feasible solution since S4 = -25 violates the nonnegativity requirement.

∴ We introduce new variable called ARTIFICIAL VARIABLE.

- Artificial variables will be eliminated before the optimal solution is reached. We assign a very large cost to the variable in the objective function.

∴ Objective function

$$50X_1 + 40X_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 - MA_4$$

Tabular solution for Example 4/4

Initial Tableau

C_j			50	40	0	0	0	0	-M	
	Product mix	Quantit bi	X_1	X_2	S_1	S_2	S_3	S_4	A_4	b_i / a_{ij}
0	S_1	150	3	5	1	0	0	0	0	150/3=50
0	S_2	20	0	1	0	1	0	0	0	--
0	S_3	300	8	5	0	0	1	0	0	300/8=37.5
-M	A_4	25	1	1	0	0	0	-1	1	25 Min. leaving
	Z_j	-25M	-M	-M	0	0	0	M	-M	
	$C_j – Z_j$		50+M	40+M	0	0	0	-M	0	

Max. (Entering)

New X_1 values = 25, 1, 1, 0, 0, 0, -1, 1

Tabular solution for Example 4/5

Old S_1 row	–	key #	x	new X_1 values	=	new S_1 row
150	–	3	x	25	=	75
3	–	3	x	1	=	0
5	–	3	x	1	=	2
1	–	3	x	0	=	1
0	–	3	x	0	=	0
0	–	3	x	0	=	0
0	–	3	x	-1	=	3
0	–	3	x	1	=	-3
Old S_2 row	**–**	**key #**	**x**	**new X_1 values**	**=**	**new S_2 row**
20	–	0	x	25	=	20
0	–	0	x	1	=	0
1	–	0	x	1	=	1
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	-1	=	0
0	–	0	x	1	=	0
Old S_3 row	**–**	**key #**	**x**	**new X_1 values**	**=**	**new S_3 row**
300	–	8	x	25	=	10
8	–	8	x	1	=	0
5	–	8	x	1	=	-3
0	–	8	x	0	=	0
0	–	8	x	0	=	0
1	–	8	x	0	=	1
0	–	8	x	-1	=	8
0	–	8	x	1	=	-8

Tabular solution for Example 4/6

2nd Tableau

Cj			$50	40	0	0	0	0	~~-M~~	
	Prodt mix	Quant b_i	X_1	X_2	S_1	S_2	S_3	S_4	~~A_4~~	b_i / a_{ij}
$0	S_1	75	0	2	1	0	0	3	~~-3~~	75/3=25
0	S_2	20	0	1	0	1	0	0	~~0~~	--
0	S_3	100	0	-3	0	0	1	8	~~-8~~	100/8=12.5 Min,leaving
50	X_1	25	1	1	0	0	0	-1	~~1~~	--
	Z_j	$1250	50	50	0	0	0	-50	~~50~~	
	$C_j - Z_j$		0	-10	0	0	0	50	~~-M-50~~	

ew S4 values : 100/8 = 25/2, 0, -3/8, 0, 0, 1/8, 1

Max. (Entering)

Tabular solution for Example 4/7

- IMPORTANT!!
- Since A4 is an artificial variable that was added simply to obtain an initial basic feasible solution, we can drop its associated column from the simplex tableau.
- Indeed whenever artificial variables are used, they can be dropped from the simplex tableau as soon as they have been eliminated from the basic feasible solution.

Tabular solution for Example 4/8

Old S_1 row	–	key #	x	new S_4 values	=	new S_1 row
75	–	3	x	25/2	=	75/2
0	–	3	x	0	=	0
2	–	3	x	-3/8	=	25/8
1	–	3	x	0	=	1
0	–	3	x	0	=	0
0	–	3	x	1/8	=	-3/8
3	–	3	x	1	=	0

Old S_2 row	–	key #	x	new S_4 values	=	new S_2 row
20	–	0	x	25/2	=	20
0	–	0	x	0	=	0
1	–	0	x	-3/8	=	1
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	1/8	=	0
0	–	0	x	1	=	0

Old X_1 row	–	key #	x	new S_4 values	=	new X_1 row
25	–	-1	x	25/2	=	75/2
1	–	-1	x	0	=	1
1	–	-1	x	-3/8	=	5/8
0	–	-1	x	0	=	0
0	–	-1	x	0	=	0
0	–	-1	x	1/8	=	1/8
-1	–	-1	x	1	=	0

Tabular solution for Example 4/9

3rd Tableau

C_j			50	40	0	0	0	0	
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4	b_i / a_{ij}
0	S_1	75/2	0	25/8	1	0	-3/8	0	12 Min. leaving
0	S_2	20	0	1	0	1	0	0	20
0	S_4	25/2	0	-3/8	0	0	1/8	1	---
50	X_1	75/2	1	5/8	0	0	1/8	0	60
	Z_j	1875	50	250/8	0	0	50/8	0	
	$C_j - Z_j$		0	70/8	0	0	-50/8	0	

Max. (Entering)

One more iteration is required. This time X2 comes into the solution and S1 is eliminated. After performing this iteration, the following simplex tableau shows that the optimal solution has been reached.

Tabular solution for Example 4/10

C_j			50	40	0	0	0	0
	Product mix	Quantity b_i	X_1	X_2	S_1	S_2	S_3	S_4
40	X_2	12	0	1	8/25	0	-3/25	0
0	S_2	8	0	0	-8/25	1	3/25	0
0	S_4	17	0	0	3/25	0	2/25	1
50	X_1	30	1	0	-5/25	0	5/25	0
	Z_j	1980	50	40	14/5	0	26/5	0
	$C_j - Z_j$		0	0	-14/5	0	-26/5	0

It turns out that the optimal solution has been reached. (All $C_j - Z_j \leq 0$ and all artificial variables have been eliminated.)

EQUALITY CONTRAINTS
NEGATIVE RIGHT-HAND SIDE VALUES

- Simply add an artificial variable A1 to create a basic feasible solution in the initial simplex tableau.

  $6X1 + 4X2 - 5X3 = 30 \Rightarrow 6X1 + 4X2 - 5X3 + 1A1 = 30$
- One of the properties of the tableau form of a linear program is that the values on the right-hand sides of the constraints have to be nonnegative.
- e.g. # of units of the portable model (X2) has to be less than or equal to the # of units of the deskpro model (X1) after setting aside 5 units of the deskpro for internal company use.

  $X2 \leq X1 - 5$
- $- X1 + X2 \leq -5$
- (Min)Multiply by –1 ⇒ (Max) $X1 - X2 \geq 5$
- We now have an acceptable nonnegative right-hand-side value. Tableau form for this constraint can now be obtained by subtracting a surplus variable and adding an artificial variable.

Tabular solution for Example 5/1

- Livestock Nutrition Co. produces specially blended feed supplements. LNC currently has an order for 200 kgs of its mixture.
- This consists of two ingredients

  X_1 (a protein source)

  X_2 (a carbohydrate source)
- The first ingredient, X_1 costs LNC 3MU a kg. The second ingredient, X_2 costs LNC 8MU a kg. The mixture can't be more than 40% X_1 and it must be at least 30% X_2.
- LNC's problem is to determine how much of each ingredient to use to minimize cost.

Tabular solution for Example 5/2

- The cost function can be written as

  Cost = $3X_1 + 8X_2$ Min!
- LNC must produce 200 kgs of the mixture – no more, no less.

  $X_1 + X_2 = 200$ kgs
- The mixture can't be more than 40% X_1, so we may use less than 80 kgs. (40% X 200 = 80). However, we must not exceed 80 kgs.

  $X_1 \leq 80$ kgs
- The mixture must be at least 30% X_2. Thus we may use more than 60 kgs, not less than 60 kgs. (30% X 200 = 60)
- $X_2 \geq 60$ kgs

Tabular solution for Example 5/3

$$\text{Minimize}: \quad \text{Cost} = 3MU\ X1 + 8MU\ X2$$

$$\text{Subject to} \quad X1 + X2 = 200 \text{ kgs}$$

$$X1 \leq 80 \text{ kgs}$$

$$X2 \geq 60 \text{ kgs}$$

$$X1, X2 \geq 0$$

- An initial solution: $X1 + X2 = 200$ kgs

$$\Rightarrow \quad X1 + X2 + A1 = 200$$

$$\Downarrow$$

- *Artificial variable* : A very expensive substance must not be represented in optimal solution.

Tabular solution for Example 5/4

- ***An artificial Variable*** is only of value as a computational device; it allows 2 types of restrictions to be treated.
  - The equality type
  - $\geq$ type
  - $X_1 \leq$ 80 kgs constraint on protein

$$\Rightarrow \quad X_1 + S_1 = 80 \text{ kgs}$$

$X_2 \geq$ 60 kgs constraint on carbohydrates

$$\Rightarrow \quad X2 - S2 + A2 = 60$$

- $X1, X2, S1, S2, A1, A2 \geq 0$

$\Downarrow \quad \Downarrow$ □ □

0MU 0MU M M

Tabular solution for Example 5/5

Minimize : Cost = 3X1 + 8X2 + 0S1 + 0S2 + MA1 + MA2

Subject to : X1 + X2 + A1 =
200

X1 + S1 = 80

X2 - S2 + A2 = 60

All variables $\geq$ 0

Tabular solution for Example 5/6

Initial Tableau

C_j			3 MU	8 MU	M MU	0 MU	0 MU	M MU	
	Product mix	Quantity b_i	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	200	1	1	1	0	0	0	200/1=200
0	S_1	80	1	0	0	1	0	0	--
M	A_2	60	0	1	0	0	-1	1	60/1=60 Min. replaced row
	Z_j	260M	M	2M	M	0	-M	M	
	$C_j - Z_j$		3-M	8-2M	0	0	M	0	

Optimal column

Tabular solution for Example 5/7

Computation for 2nd tableau:

Replacing row = new X_2 values : 60/1=60, 0/1=0, 1/1=1, 0/1=0, -1/1=-1, 1/1=1

Old A_1 row	–	key #	x	new X_2 values	=	new A_1 row
200	–	1	x	60	=	140
1	–	1	x	0	=	1
1	–	1	x	1	=	0
1	–	1	x	0	=	1
0	–	1	x	0	=	0
0	–	1	x	-1	=	1
0	–	1	x	1	=	-1

Old S_1 row	–	key #	x	new X_2values	=	new S_1 row
80	–	0	x	60	=	80
1	–	0	x	0	=	1
0	–	0	x	1	=	0
0	–	0	x	0	=	0
1	–	0	x	0	=	1
0	–	0	x	-1	=	0
0	–	0	x	1	=	0

Tabular solution for Example 5/8

2nd Tableau

C_j			3	8	M	0	0	M	
	Product mix	Quantity b_i	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	140	1	0	1	0	1	-1	140/1=140
0	S_1	80	1	0	0	1	0	0	80/1=80 replaced row, min.
8	X_2	60	0	1	0	0	-1	1	60/0= --
	Z_j	140M + 480	M	8	M	0	M-8	8-M	
	$C_j - Z_j$		3-M	0	0	0	8-M	2M-8	

Optimal column

Tabular solution for Example 5/9

Computations for 3rd Tableau

Replacing row = new X_1 values: 80/1=80, 1/1=1, 0/1=0, 0/1=0, 1/1=1, 0/1=0, 0/1=0

Old A_1 row	–	key #	x	new X_1 values	=	new A_1 row
140	–	1	x	80	=	60
1	–	1	x	1	=	0
0	–	1	x	0	=	0
1	–	1	x	0	=	1
0	–	1	x	1	=	-1
1	–	1	x	0	=	1
-1	–	1	x	0	=	-1

Old X_2 row	–	key #	x	new X_1 values	=	new X_2 row
60	–	0	x	80	=	60
0	–	0	x	1	=	0
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	1	=	0
-1	–	0	x	0	=	-1
1	–	0	x	0	=	1

Tabular solution for Example 5/10

3rd Tableau

C_j			3	8	M	0	0	M	
	Product mix	Quantity bi	X_1	X_2	A_1	S_1	S_2	A_2	b_i / a_{ij}
M	A_1	60	0	0	1	-1	1	-1	60/1=60 replaced row
3	X_1	80	1	0	0	1	0	0	80/0= --
8	X_2	60	0	1	0	0	-1	1	60/-1= - 60 not considered
	Z_j	60M – 720	3	8	M	3-M	M-8	8-M	
	$C_j – Z_j$		0	0	0	M-3	8-M	2M-8	

Optimal column

Tabular solution for Example 5/11

Computations for the 4th Tableau

Replacing row = new S_2 values: 60/1=60, 0/1=0, 0/1=0, 1/1=1, -1/1=-1, 1/1=1, -1/1=-1

Old X_1 row	–	key #	x	new S_2 values	=	new X_1 row
80	–	0	x	60	=	80
1	–	0	x	0	=	1
0	–	0	x	0	=	0
0	–	0	x	1	=	0
1	–	0	x	-1	=	1
0	–	0	x	1	=	0
0	–	0	x	-1	=	0

Old X_2 row	–	key #	x	new S_2 values	=	new X_2 row
60	–	-1	x	60	=	120
0	–	-1	x	0	=	0
1	–	-1	x	0	=	1
0	–	-1	x	1	=	1
0	–	-1	x	-1	=	-1
-1	–	-1	x	1	=	0
1	–	-1	x	-1	=	0

Tabular solution for Example 5/12

4th Tableau

C_j			3	8	M	0	0	M
	Product mix	Quantity bi	X_1	X_2	A_1	S_1	S_2	A_2
0	S_2	60	0	0	1	-1	1	-1
3	X_1	80	1	0	0	1	0	0
8	X_2	120	0	1	1	-1	0	0
	Z_j	1200	3	8	8	-5	0	0
	$C_j - Z_j$		0	0	M-8	5	0	M

$\because$ No negative values remain in the $C_j - Z_j$ row, we have reached the OPTIMAL solution.

It is to use 80 kgs of X_1 and 120 kgs of X_2. This results in a cost of 1200MU.
S_2 represents the amount of X_2 used over the minimum quantity required (60kg)

$$X_2 - S_2 + A_2 = 60 \longrightarrow 120 - 60 + 0 = 60 \longrightarrow 60 = 60 \because A_2 = 0.$$

Basic Solution

Definition: A ***basic solution*** to the system of equations A x = b is obtained as:

If there are m constraints and n no of variables n>=m

setting n-m variables equal to 0,

solving for the *m* remaining variables, and

confirming that the solution for these *m* remaining variables is unique.

- **Comment: condition 3 is equivalent to requiring that the *m* columns of matrix A that correspond to these *m* variables be *linearly independent*. This means that none of these *m* columns from A can be expressed as a linear combination of the remaining columns.**
-

- The variables that are fixed equal to 0 are called **nonbasic variables (NBV).** The remaining *m* variables are called **basic variables (BV).** Note that, in general, different choices of nonbasic and basic variables will yield different (basic) solutions to the system of equations .A basic solution corresponds to a corner point(may be feasible or non feasible)
- Thus maximum no of corner points
- $C_m^{\,n} = n!/\, m!(n-m)!$
-

Example: Find all basic solutions to the system

$$\begin{aligned} x_1 \quad +x_2 \qquad\quad &= 3 \\ -x_2 \quad +x_3 &= -1 \end{aligned}$$

- NBV = $\{x_3\}$, BV= $\{x_1, x_2\}$ $\quad x_1 = 2, x_2 = 1, x_3 = 0$
- NBV = $\{x_2\}$, BV= $\{x_1, x_3\}$ $\quad x_1 = 3, x_2 = 0, x_3 = -1$
- NBV = $\{x_1\}$, BV= $\{x_2, x_3\}$ $\quad x_1 = 0, x_2 = 3, x_3 = 2$

Basic Feasible solution

- **Definition:** A nonnegative basic solution is a called a ***basic feasible solution* (bfs)** if it is feasible and basic .
-

- **Theorem:** the basic feasible solutions of are the extreme points of the polyhedral set defined by .
-

$$\begin{aligned} x_1 \quad &+x_2 \quad \leq 40 \\ 2x_1 \quad &+x_2 \quad \leq 60 \\ &x_1, x_2 \quad \geq 0 \end{aligned}$$

By drawing the polyhedron in (x_1, x_2) space, we see that its extreme points are:

(0,0), (30,0), (20,20), (0,40).

Consider the following LP with two variables:

$$\text{Maximize } z = 2x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 4$$

$$x_1 + 2x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

Ax + NS_B = b

Figure 2 provides the graphical solution space for the problem.

Algebraically, the solution space of the LP is represented by the following $m = 2$ equations and $n = 4$ variables:

$$2x_1 + x_2 + s_1 = 4$$

$$x_1 + 2x_2 + s_2 = 5$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The basic solutions are determined by setting $n - m = 4 - 2 = 2$ variables equal to zero and solving for the remaining $m = 2$ variables. For example, if we set $x_1 = 0$ and $x_2 = 0$, the equations provide the unique basic solution

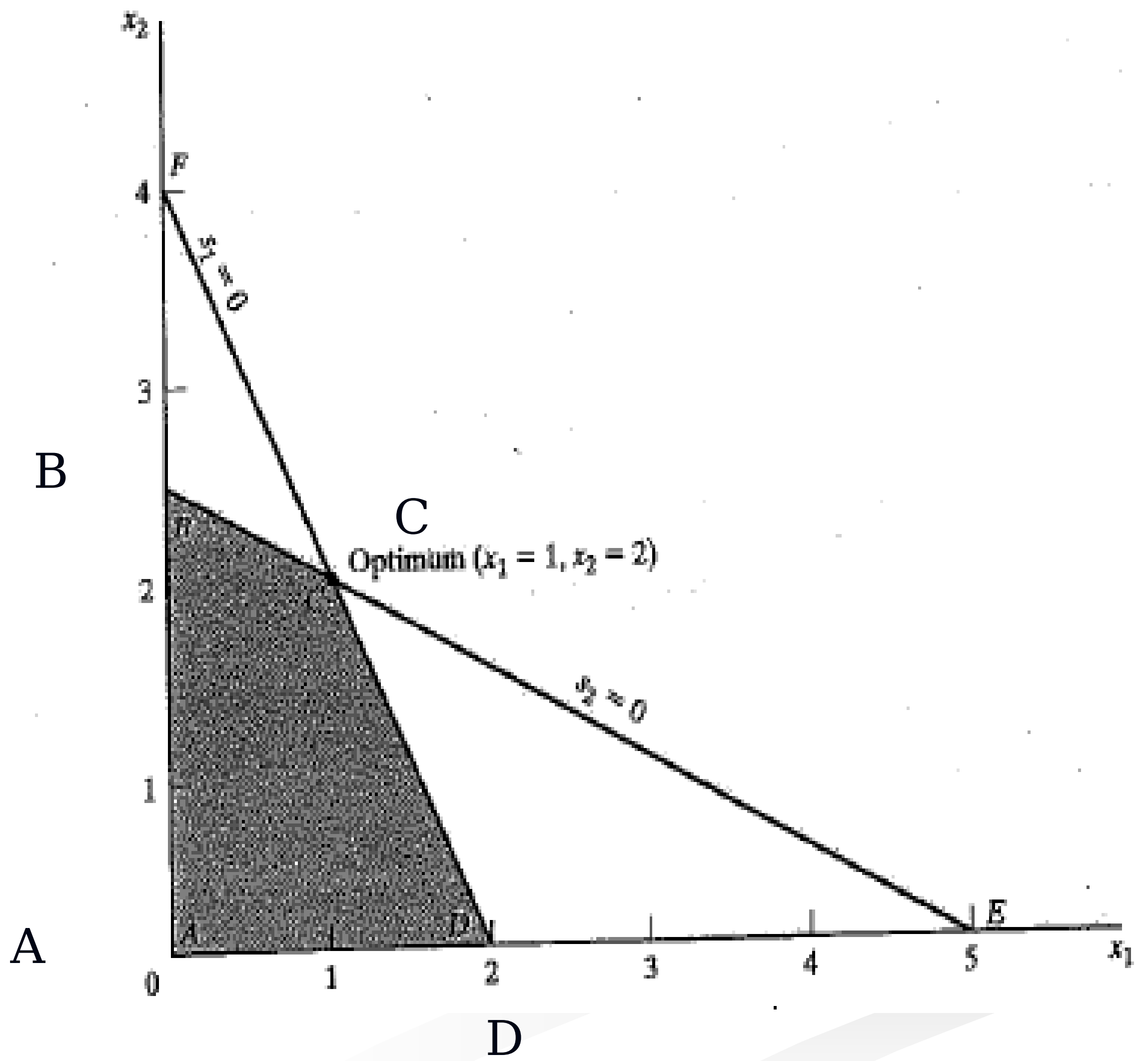
$$s_1 = 4, s_2 = 5$$

This solution corresponds to point A in Figure 2 (convince yourself that $s_1 = 4$ and $s_2 = 5$ at point A). Another point can be determined by setting $s_1 = 0$ and $s_2 = 0$ and then solving the resulting two equations

$$2x_1 + x_2 = 4$$

$$x_1 + 2x_2 = 5$$

The associated basic solution is $(x_1 = 1, x_2 = 2)$, or point C in Figure 2.

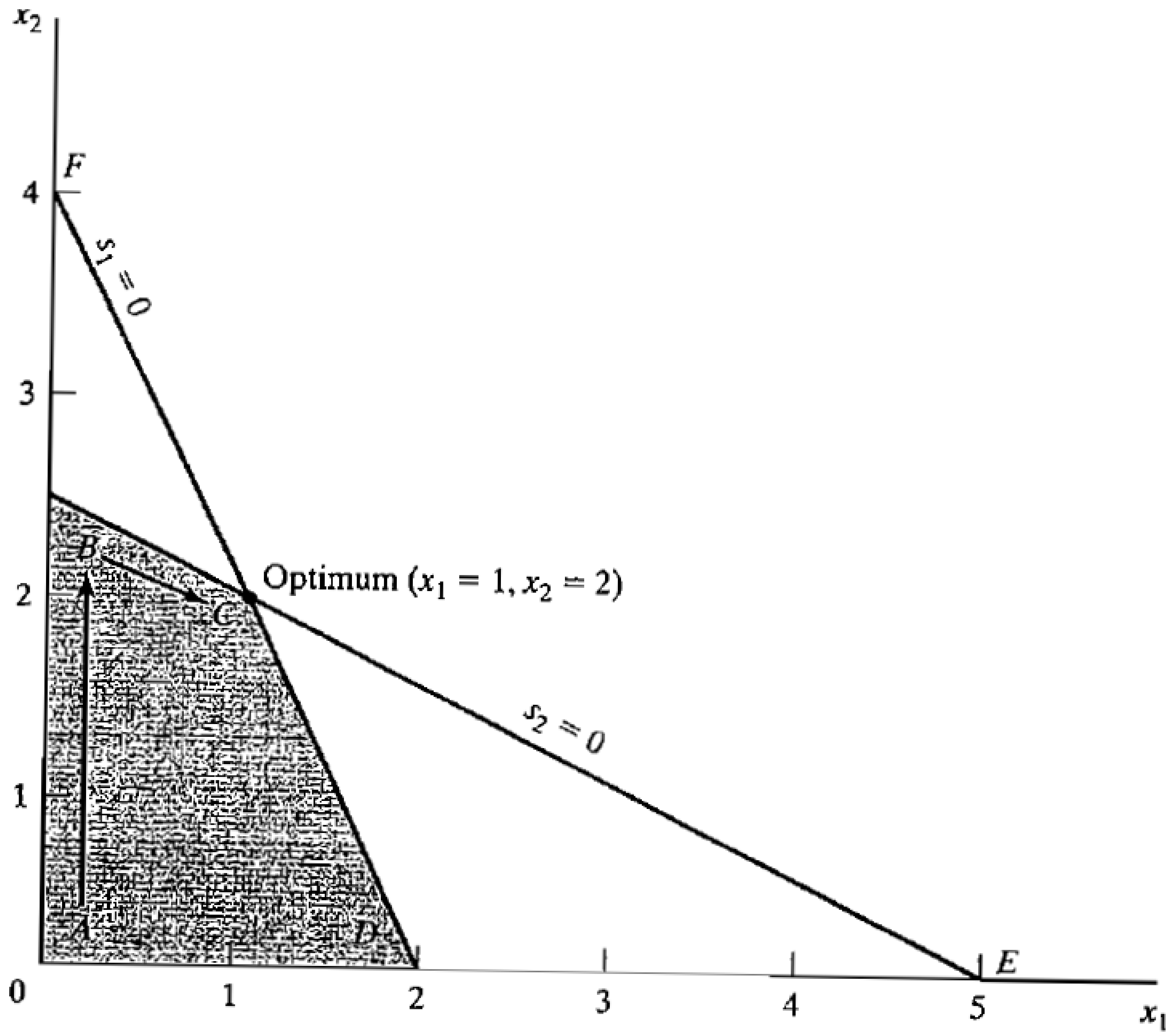


x_2
F
4
$s_1 = 0$
3
B
C
Optimum ($x_1 = 1, x_2 = 2$)
2
$s_2 = 0$
1
A
D
E
0
1
2
3
4
5
x_1
D

You probably are wondering which $n - m$ variables should be set equal to zero to target a specific corner point. Without the benefit of the graphical solution space (which is available only for at most three variables), we cannot specify the $(n - m)$ zero variables associated with a given corner point. But that does not prevent enumerating *all* the corner points of the solution space. Simply consider *all* combinations in which $n - m$ variables equal zero and solve the resulting equations. Once done, the optimum solution is the *feasible* basic solution (corner point) with the best objective value.

[illegible] Looking at Figure 2, we can

Nonbasic (zero) variables	Basic variables	Basic solution	Associated corner point	Feasible?	Objective value, z
(x_1, x_2)	(s_1, s_2)	(4, 5)	A	Yes	0
(x_1, s_1)	(x_2, s_2)	(4, − 3)	F	No	–
(x_1, s_2)	(x_2, s_1)	(2.5, 1.5)	B	Yes	7.5
(x_2, s_1)	(x_1, s_2)	(2, 3)	D	Yes	4
(x_2, s_2)	(x_1, s_1)	(5, − 6)	E	No	–
(s_1, s_2)	(x_1, x_2)	(1, 2)	C	Yes	8 (optimum)


x_2
F
4
$s_1 = 0$
3
B
Optimum ($x_1 = 1, x_2 = 2$)
2
C
$s_2 = 0$
1
A
D
E
0
1
2
3
4
5
x_1

Simplex method

Two requirements on the constraints to stream line the simplex method

1- All the constraints are in equation form with non negative RHS

2- All variables are non negative

first restriction is satisfied by adding slack and surplus variables. After making THS non negative

Second restriction is made satisfied by replacing any unrestricted variable as the difference of two new non negative variables

Let x be unrestricted then

X is replaced by x1 – x2 where x1 andx2 both are non negative

Problem solved by simplex method

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

subject to

$$\begin{aligned}
6x_1 + 4x_2 + s_1 &= 24 \quad \text{(Raw material } M1\text{)} \\
x_1 + 2x_2 + s_2 &= 6 \quad \text{(Raw material } M2\text{)} \\
-x_1 + x_2 + s_3 &= 1 \quad \text{(Market limit)} \\
x_2 + s_4 &= 2 \quad \text{(Demand limit)} \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

Z – 5x1 -4x2-0s1-0s2-0s3-0s4=0

Z - 5x1 -4x2-0s1-0s2-0s3-0s4=0

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

subject to

$$\begin{aligned}
6x_1 + 4x_2 + s_1 &= 24 && \text{(Raw material } M1) \\
x_1 + 2x_2 + s_2 &= 6 && \text{(Raw material } M2) \\
-x_1 + x_2 + s_3 &= 1 && \text{(Market limit)} \\
x_2 + s_4 &= 2 && \text{(Demand limit)} \\
x_1, x_2, s_1, s_2, s_3, s_4 &\geq 0
\end{aligned}$$

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
z	1	−5	−4	0	0	0	0	0	z-row
s_1	0	6	4	1	0	0	0	24	s_1-row
s_2	0	1	2	0	1	0	0	6	s_2-row
s_3	0	-1	1	0	0	1	0	1	s_3-row
s_4	0	0	1	0	0	0	1	2	s_4-row

SUMMARY OF THE SIMPLEX METHOD

Step 1. Formulate a LP model of the problem.

Step 2. Add slack variables to each constraint to obtain standard form.

Step 3. Set up the initial simplex tableau.

Step 4. Choose the nonbasic variable with the neagtive largest entry in the net evaluation row

. This identifies the **pivot (key)** column; the column associated with the incoming variable.

Step 5. Choose as the **pivot row** that row with the smallest ratio of “bi/ aij”, for aij >0 where j is the pivot column. This identifies the pivot row, the row of the variable leaving the basis when variable j enters.

Step 6. a). Divide each element of the pivot row by the pivot element.

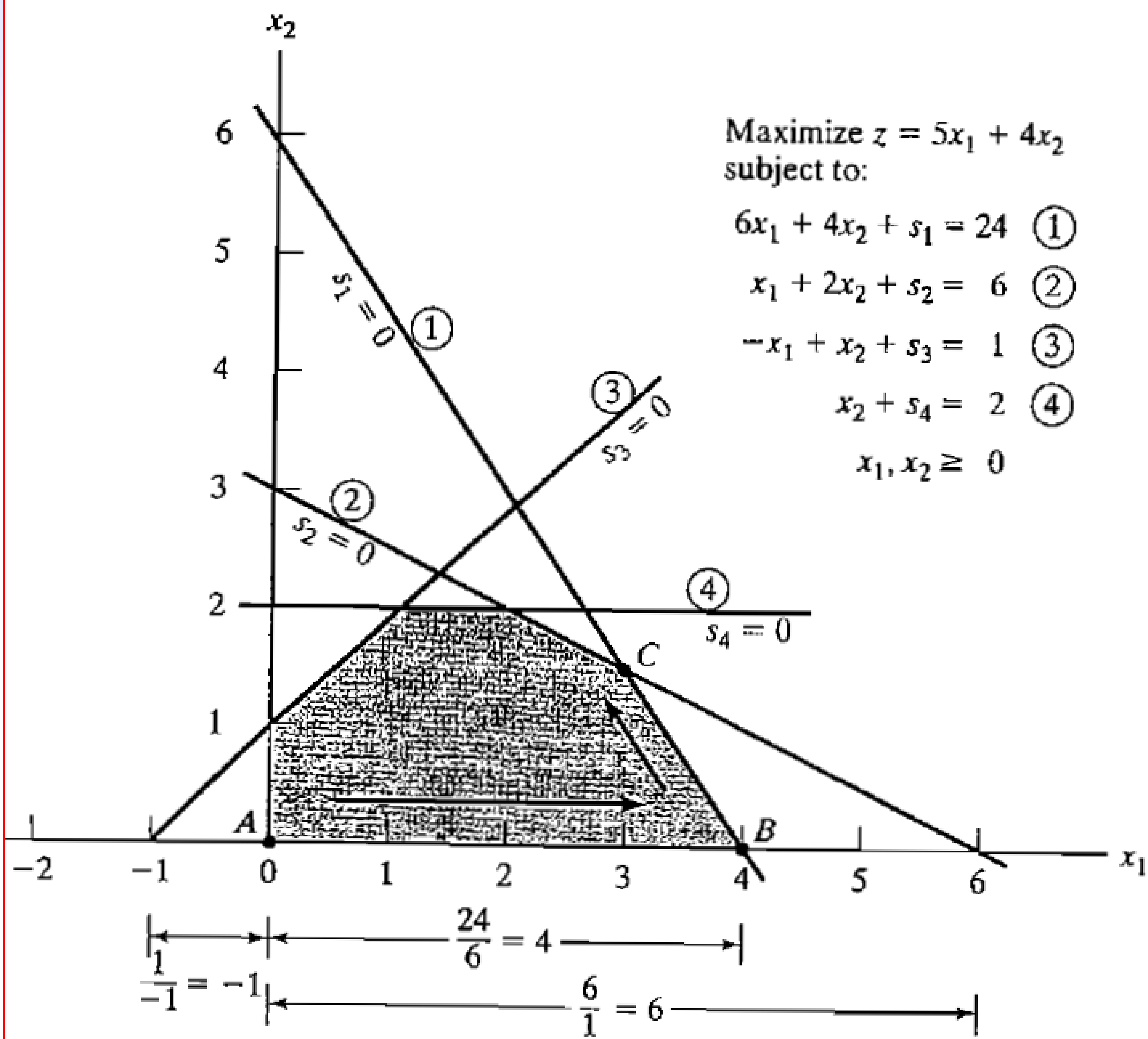
b). According to the entering variable, find the new values for remaining variables.

Step 7. Test for optimality. If in the net evaluation row all are negative for all columns, we have the optimal solution. If not, return to step 4.

Entering variable

Basic	Entering x_1	Solution	Ratio (or Intercept)
s_1	6	24	$x_1 = \frac{24}{6} = 4 \leftarrow$ minimum
s_2	1	6	$x_1 = \frac{6}{1} = 6$
s_3	−1	1	$x_1 = \frac{1}{-1} = -1$ (ignore)
s_4	0	2	$x_1 = \frac{2}{0} = \infty$ (ignore)
Conclusion: x_1 enters and s_1 leaves			

Leaving variable



Maximize $z = 5x_1 + 4x_2$
subject to:

$$6x_1 + 4x_2 + s_1 = 24 \quad (1)$$
$$x_1 + 2x_2 + s_2 = 6 \quad (2)$$
$$-x_1 + x_2 + s_3 = 1 \quad (3)$$
$$x_2 + s_4 = 2 \quad (4)$$
$$x_1, x_2 \geq 0$$

	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	
			Enter ↓							
	z	1	−5	−4	0	0	0	0	0	
Leave ←	s_1	0	6	4	1	0	0	0	24	Pivot row
	s_2	0	−1	2	0	1	0	0	6	
	s_3	0	−1	1	0	0	1	0	1	
	s_4	0	0	1	0	0	0	1	2	
			Pivot column							

SECOND SOL

			↓						
	Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
	z	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
	x_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
←	s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
	s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
	s_4	0	0	1	0	0	0	1	2

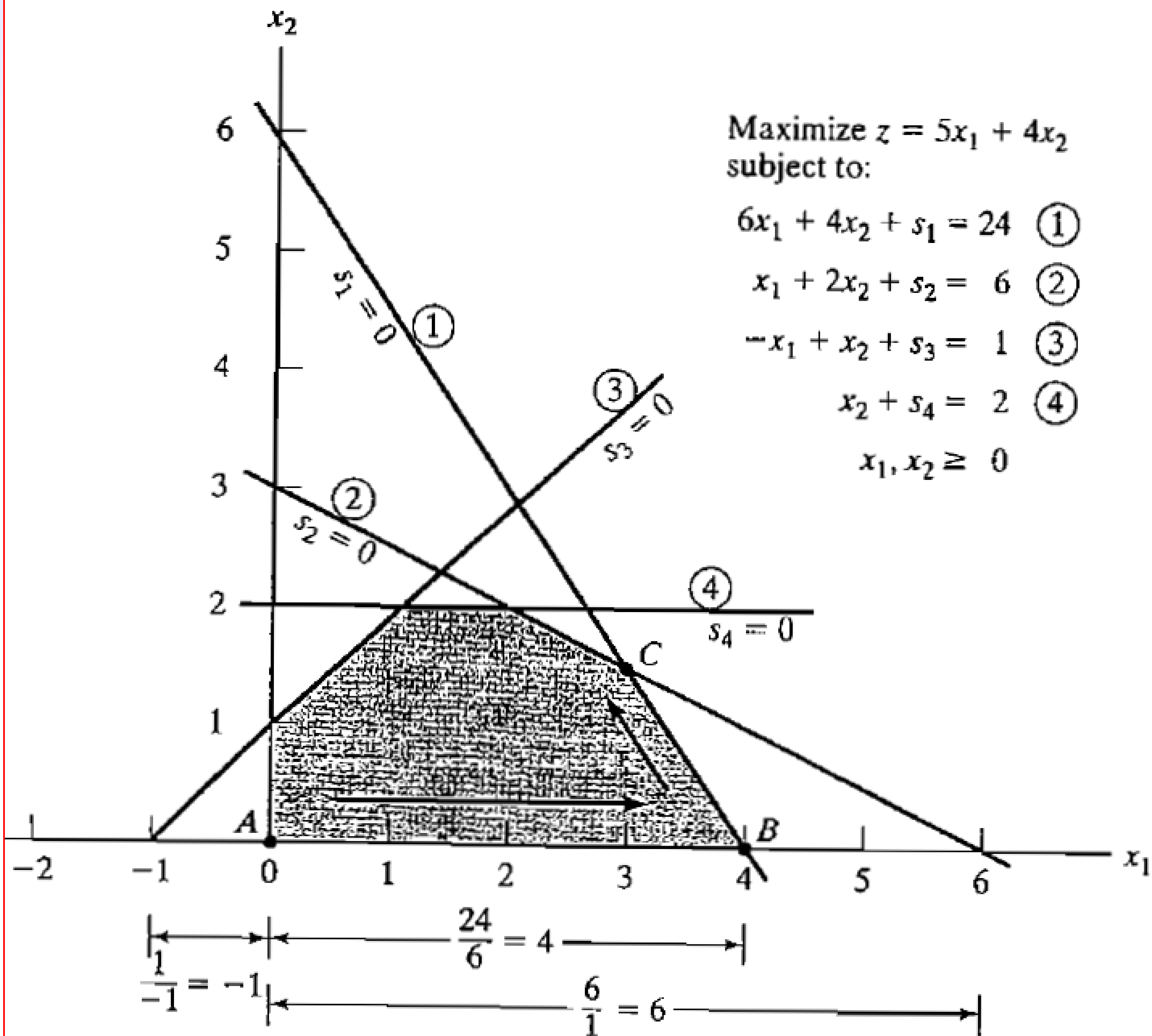
THIRD SOLUTION

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x_1	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
x_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Test for optimality. If in the net evaluation row all are non negative for all columns,

we have the optimal solution

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	$\frac{3}{2}$	Produce 1.5 tons of interior paint daily
z	21	Daily profit is $21,000


Maximize z = 5x1 + 4x2
subject to:
6x1 + 4x2 + s1 = 24 ①
x1 + 2x2 + s2 = 6 ②
−x1 + x2 + s3 = 1 ③
x2 + s4 = 2 ④
x1, x2 ≥ 0
x2
x1
s1 = 0
s2 = 0
s3 = 0
s4 = 0
A
B
C
24/6 = 4
1/−1 = −1
6/1 = 6

problem

$$\max\ z = 2x_1 + x_2 - 3x_3 + 5x_4$$

$X1 + 2x2 + 2x3 + 4x4 \leq 40$

$2x1 - x2 + x3 + 2x4 \leq 8$

$4x1 - 2x2 + x3 - x4 \leq 10$

$X1, x2, x3, x4 \geq 0$

Basic	x1	x2	x3	x4	sx5	sx6	sx7	Solution
z	-2.00	-1.00	3.00	-5.00	0.00	0.00	0.00	0.00
1)sx5	1.00	2.00	2.00	4.00	1.00	0.00	0.00	40.00
2)sx6	2.00	-1.00	1.00	2.00	0.00	1.00	0.00	8.00
3)sx7	4.00	-2.00	1.00	-1.00	0.00	0.00	1.00	10.00
z	3.00	-3.50	5.50	0.00	0.00	2.50	0.00	20.00
1)sx5	-3.00	4.00	0.00	0.00	1.00	-2.00	0.00	24.00
2)x4	1.00	-0.50	0.50	1.00	0.00	0.50	0.00	4.00
3)sx7	5.00	-2.50	1.50	0.00	0.00	0.50	1.00	14.00
z	0.38	0.00	5.50	0.00	0.88	0.75	0.00	41.00
1)x2	-0.75	1.00	0.00	0.00	0.25	-0.50	0.00	6.00
2)x4	0.62	0.00	0.50	1.00	0.12	0.25	0.00	7.00
3)sx7	3.12	0.00	1.50	0.00	0.62	-0.75	1.00	29.00

Summary

- **Optimality condition**:
- The entering variable in maximization(minimization)

problem is the nonbasic variable with most negative(positive) coeff in the z-row.

Ties are broken arbitrarily.

Optimal sol is reached when all coeff in z row are (nonnegative(positive)

- **Feasibility condition**:for both maximization and minimization problems the leaving var is the basic variable associated with smallest nonnegative ratio with strictly positive denominator. **Ties are broken arbitrarily**.

Tableau Form : The Special Case

- **Obtaining tableau form is somewhat more complex** if the LP contains ≥ constraints, = constraints,
- Here we will explain how to develop tableau form for each of these situations.

For each constraint having ≥ sign a new surplus variable is subtracted to convert it into equation.

Artificial Starting solution

- Lin prog problem with = or ≥ constraints , method of slack variables will not work.
- **We use Artificial variables in these cases. These variables are disposed of later on**. There are two methods for it **big M method, 2 -phase method**

$$\text{Minimize } z = 4x_1 + x_2$$

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

M-Method

If I th constraint does not have slack variable an artificial variable is added to get a starting solution.

To put them at zero level at the end we penalize these variables by associating a sufficiently large cost as:

-M in maximization problem

+M in minimization problem

$$\text{Minimize } z = 4x_1 + x_2$$

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Using x_3 as a surplus in the second constraint and x_4 as a slack in the third constraint, th equation form of the problem is given as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 \phantom{{}- x_3 + x_4} &= 3 \\ 4x_1 + 3x_2 - x_3 \phantom{{}+ x_4} &= 6 \\ x_1 + 2x_2 \phantom{{}- x_3} + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The third equation has its slack variable, x_4, but the first and second equations do not. Thu we add the artificial variables R_1 and R_2 in the first two equations and penalize them in the ol jective function with $MR_1 + MR_2$ (because we are minimizing). The resulting LP is given as subject to **Minimize Z = 4x1+ x2 +100 R1 +100R2**

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - x_3 + R_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

The associated starting basic solution is now given by $(R_1, R_2, x_4) = (3, 6, 4)$.

Using $M = 100$, the starting simplex tableau is given as follows (for convenience, the z-column is eliminated because it does not change in all the iterations):

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	−4	−1	0	−100	−100	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Co
as

Before proceeding with the simplex method computations, we need to make the z-row consistent with the rest of the tableau. Specifically, in the tableau, $x_1 = x_2 = x_3 = 0$, which yields the starting basic solution $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$. This solution yields $z = 100 \times 3 + 100 \times 6 = 900$ (instead of 0, as the right-hand side of the z-row currently shows). This inconsistency stems from the fact that R_1 and R_2 have nonzero coefficients $(-100, -100)$ in the z-row (compare with the all-slack starting solution in Example 3.3-1, where the z-row coefficients of the slacks are zero).

We can eliminate this inconsistency by substituting out R_1 and R_2 in the z-row using the appropriate constraint equations. In particular, notice the highlighted elements $(= 1)$ in the R_1-row and the R_2-row. Multiplying *each* of R_1-row and R_2-row by 100 and adding the *sum* to the z-row will substitute out R_1 and R_2 in the objective row—that is,

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (100 \times R_1\text{-row} + 100 \times R_2\text{-row})$$

The modified tableau thus becomes (verify!)

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	696	399	-100	0	0	0	900
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
x_4	1	2	0	0	0	1	4

Notice that $z = 900$, which is consistent now with the values of the starting basic feasible solution: $R_1 = 3$, $R_2 = 6$, and $x_4 = 4$.

Now this table is ready to contiue with simplex method

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
z	0	167	−100	−232	0	0	204
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	−1	$-\frac{4}{3}$	1	0	2
x_4	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

This shows that x2 and R2 are entering and leaving variables resp. After two more iterations answer is obtained as (try on your own and verify the solution given below) X1 = 2/5 x2 = 9/5 and z = 17/5

M- method may result round off error. So two phase method is a better option

- **Two Phase Method**
- **Phase I**: add artificial var to get a starting basic feasible sol.
- Find a basic feasible sol that minimizes sum of artificial variables(irrespective whether original problem is minimization/maximization)
- **If the min value is positive it implies original problem has no feasible sol at all, otherwise move to phase II**
- **Phase II**: use the feasible sol obtained from phase I as a starting sol for the original problem and iterate to get final optimal sol

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

P

Phase I

$$\text{Minimize } r = R_1 + R_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 \quad + R_1 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 \quad + R_2 \quad &= 6 \\ x_1 + 2x_2 \quad + x_4 &= 4 \end{aligned}$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

The associated tableau is given as

The associated tableau is given as

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	−1	−1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
x_4	1	2	0	0	0	1	4

Modify r-row as required in simplex method.
coeff of basic var in r row should be all zero

Moving as in simplex method final sol is as

Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution
r	0	0	0	-1	-1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	-1	1	1

Minimum r = 0 phase I produces basic feasible sol as x1 = 3/5, x2 = 6/5 and x4 = 1

Note the art var are zero
thus their work is over we can eliminate their column forever to go to phase II

After deleting the artificial columns, we write the *original* problem as

$$\text{Minimize } z = 4x_1 + x_2$$

subject to

$$x_1 + \tfrac{1}{5}x_3 = \tfrac{3}{5}$$
$$x_2 - \tfrac{3}{5}x_3 = \tfrac{6}{5}$$
$$x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The starting table of phase II is thus

Basic	x_1	x_2	x_3	x_4	Solution
z	-4	-1	0	0	0
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

Here it is imp to see that Z row has non zero coeff for basic variables so

$$\text{New } z\text{-row} = \text{Old } z\text{-row} + (4 \times x_1\text{-row} + 1 \times x_2\text{-row})$$

The initial tableau of Phase II is thus given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
x_4	0	0	1	1	1

X3 should enter into the basic sol. After one iteration final sol is obtained as

X1= 2/5 x2 = 9/5 and z= 17/5

The removal of the artificial variables and their columns at the end of Phase I can take place only when they are all *nonbasic* (as Example 3.4-2 illustrates). If one or more artificial variables are *basic* (at *zero* level) at the end of Phase I, then the following additional steps must be undertaken to remove them prior to the start of Phase II.

Step 1. Select a zero artificial variable to leave the basic solution and designate its row as the *pivot row*. The entering variable can be *any* nonbasic (nonartificial) variable with a *nonzero* (positive or negative) coefficient in the pivot row. Perform the associated simplex iteration.

Step 2. Remove the column of the (just-leaving) artificial variable from the tableau. If all the zero artificial variables have been removed, go to Phase II. Otherwise, go back to Step 1.

Special cases in simplex method

- Degeneracy
- Alternate optima
- Unbounded solution
- Nonexisting sol(infeasible)

Degeneracy

- **At least one basic variable is zero**
- **It occurs when there is a tie for minimum ratio**
- **At least one basic variable will be zero in the next iteration**

- **Degeneracy can cause to cycle indefinitely(only theoretically)**
- **This reveals at least one redundant constraint**

Example

- Max $z = 3x + 9y$
- S.t. $X + 4y \leq 8$
- $x + 2y \leq 4$
- $x, y \geq 0$

- Solve by simplex and check

$$\text{Maximize } z = 3x_1 + 9x_2$$

subject to

$$x_1 + 4x_2 \leq 8$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Given the slack variables x_3 and x_4, the following tableaus provide the simplex iterations of the problem:

Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-3	-9	0	0	0
x_2 enters	x_3	1	4	1	0	8
x_3 leaves	x_4	1	2	0	1	4
1	z	$-\frac{3}{4}$	0	$\frac{9}{4}$	0	18
x_1 enters	x_2	$\frac{1}{4}$	1	$\frac{1}{4}$	0	2
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0
2	z	0	0	$\frac{3}{2}$	$\frac{3}{2}$	18
(optimum)	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	2
	x_1	1	0	-1	2	0

X3 and x4 there is a tie for leaving the basis.

Next sol is degenerate one basic var is zero obj fn does not improve in next iteration

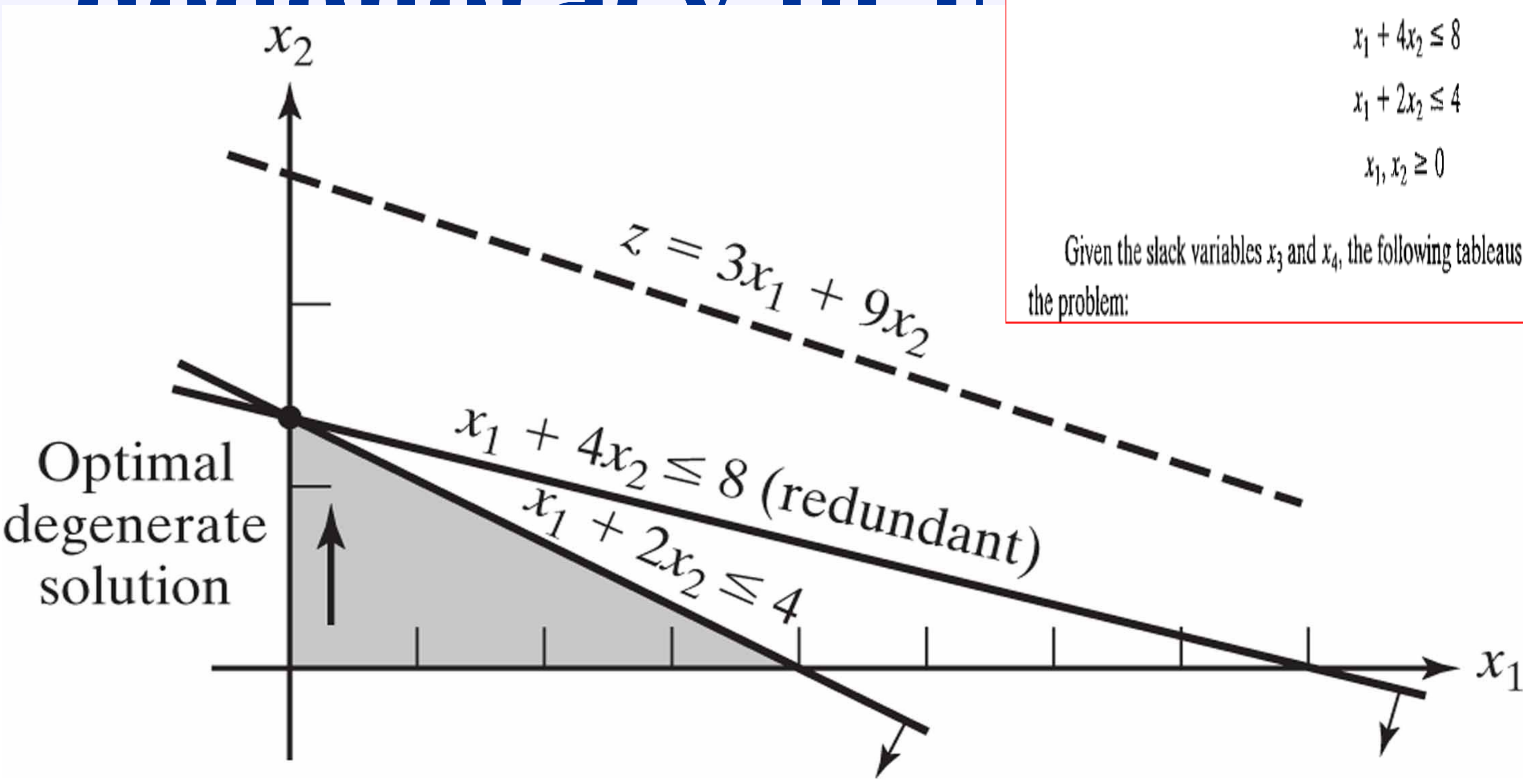
Figure 3.7 degeneracy in F

Maximize $z = 3x_1 + 9x_2$

subject to

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Given the slack variables x_3 and x_4, the following tableaus provide the problem:



Iteration one and iteration two correspond to the same corner point

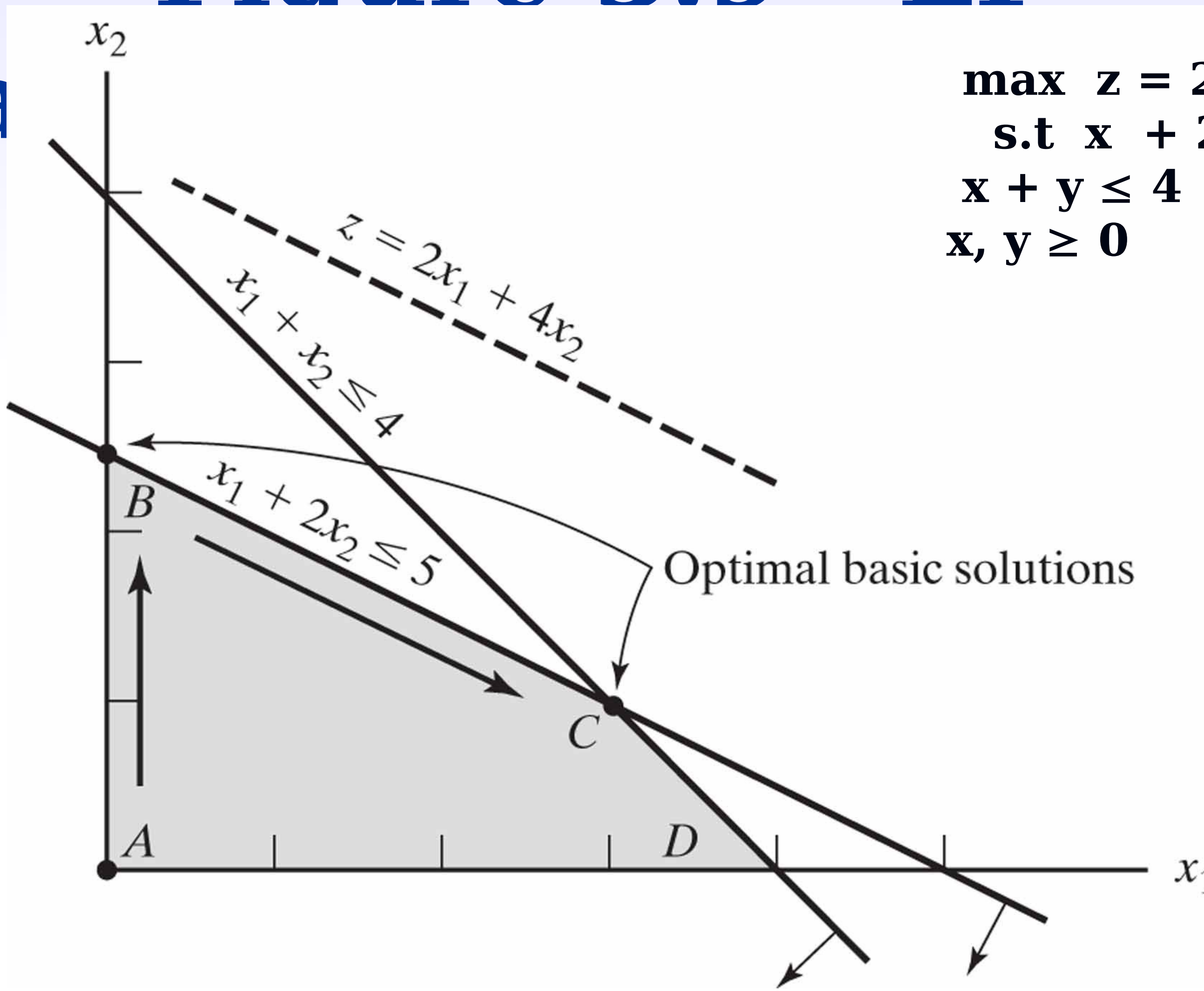
Alternate optima

- **Zero coef of non basic x in z row indicates that x can be made basic, altering the values of the basic variables without changing the value of z.**
- **max $z = 2x + 4y$**

  s.t $x + 2y \leq 5$
- **$x + y \leq 4$**
- **$x, y \geq 0$**

Figure 3.9 LP

$$\max\ z = 2x + 4y$$
$$\text{s.t } x + 2y \le 5$$
$$x + y \le 4$$
$$x, y \ge 0$$



Iteration	Basic	x_1	x_2	x_3	x_4	Solution
0	z	-2	-4	0	0	0
x_2 enters	x_3	1	2	1	0	5
x_3 leaves	x_4	1	1	0	1	4
1 (optimum)	z	0	0	2	0	10
x_1 enters	x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{5}{2}$
x_4 leaves	x_4	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{3}{2}$
2	z	0	0	2	0	10
(alternative optimum)	x_2	0	1	1	-1	1
	x_1	1	0	-1	2	3

This is corner point B

This is corner point C

Iteration 1 gives the optimum sol as x1 = 0,x2 = 5/2 and z = 10
In Z row coeff of non basic var is zero.when this non basic var is made basic a new sol is obtained without effecting the val of objective function.
Iteration 2 does this. Using x1 as entering and x4 as leavingvar a new sol is obtained as x1 = 3 x2 = 1 z=10 thus no change in the optimal val of obj fn

Iteration 1 gives the optimum sol as $x1 = 0, x2 = 5/2$ and $z = 10$
In Z row coeff of non basic var is zero.
when this non basic var is made basic a new sol is obtained without effecting the val of objective function.

Iteration 2 does this.
Using x1 as entering and x4 as leaving var a new sol is obtained as $x1 = 3$ $x2 = 1$ $z=10$

thus no change in the optimal val of obj fn

Thus two sols both optimal

Unbounded solution

- If in the pivot(key) col all the coeff s are non positive
- New var can be increased indefinitely without violating any of the constraints
- Hence z can be increases indefinitely

example

- Max z = 2x + y
- S.t x – y ≤10

  2x ≤ 40

  x, y ≥ 0

Starting Iteration

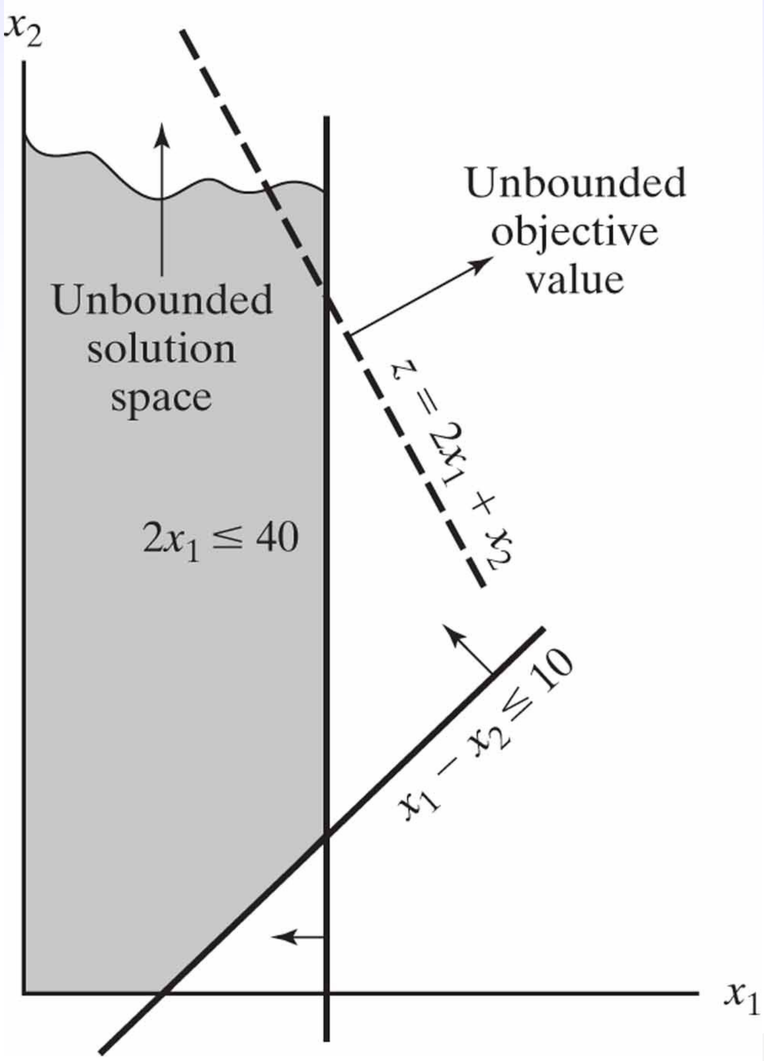
Basic	x_1	x_2	x_3	x_4	Solution
z	−2	−1	0	0	0
x_3	1	−1	1	0	10
x_4	2	0	0	1	40

Starting Iteration

Basic	x_1	x_2	x_3	x_4	Solution
z	−2	−1	0	0	0
x_3	1	−1	1	0	10
x_4	2	0	0	1	40

In the starting tableau, both x_1 and x_2 have negative z-equation coefficients. Hence either one can improve the solution. Because x_1 has the most negative coefficient, it is normally selected as the entering variable. However, *all* the *constraint* coefficients under x_2 (i.e., the denominators of the ratios of the feasibility condition) are *negative* or *zero*. This means that there is no leaving variable and that x_2 can be increased indefinitely without violating any of the constraints (compare with the graphical interpretation of the minimum ratio in Figure 3.5). Because each unit increase in x_2 will increase z by 1, an infinite increase in x_2 leads to an infinite increase in z. Thus, the problem has no bounded solution. This result can be seen in Figure 3.10. The solution space is unbounded in the direction of x_2, and the value of z can be increased indefinitely.

LP unbounded solution in Example



- Max z = 2x1 + x2
- S.t x1 – x2 ≤10

  2x1 ≤ 40

  x1, x2 ≥ 0

Infeasible sol

Case does not occur if all constraints are

less than equal to

If at least one artificial variable is positive in the optimum iteration the LP has no feasible sol

LP models with inconsistent constraints have no feasible solution. This situation can never occur if *all* the constraints are of the type $\leq$ with nonnegative right-hand sides because the slacks provide a feasible solution. For other types of constraints, we use artificial variables. Although the artificial variables are penalized in the objective function to force them to zero at the optimum, this can occur only if the model has a feasible space. Otherwise, at least one artificial variable will be *positive* in the optimum iteration. From the practical standpoint, an infeasible space points to the possibility that the model is not formulated correctly.

example

- Max $z = 3x1 + 2\ x2$

- $2x1 + x2 \le 2$
- $3x1 + 4\ x2 \ge 12$
- $x1, x2 \ge 0$

- Check by solving

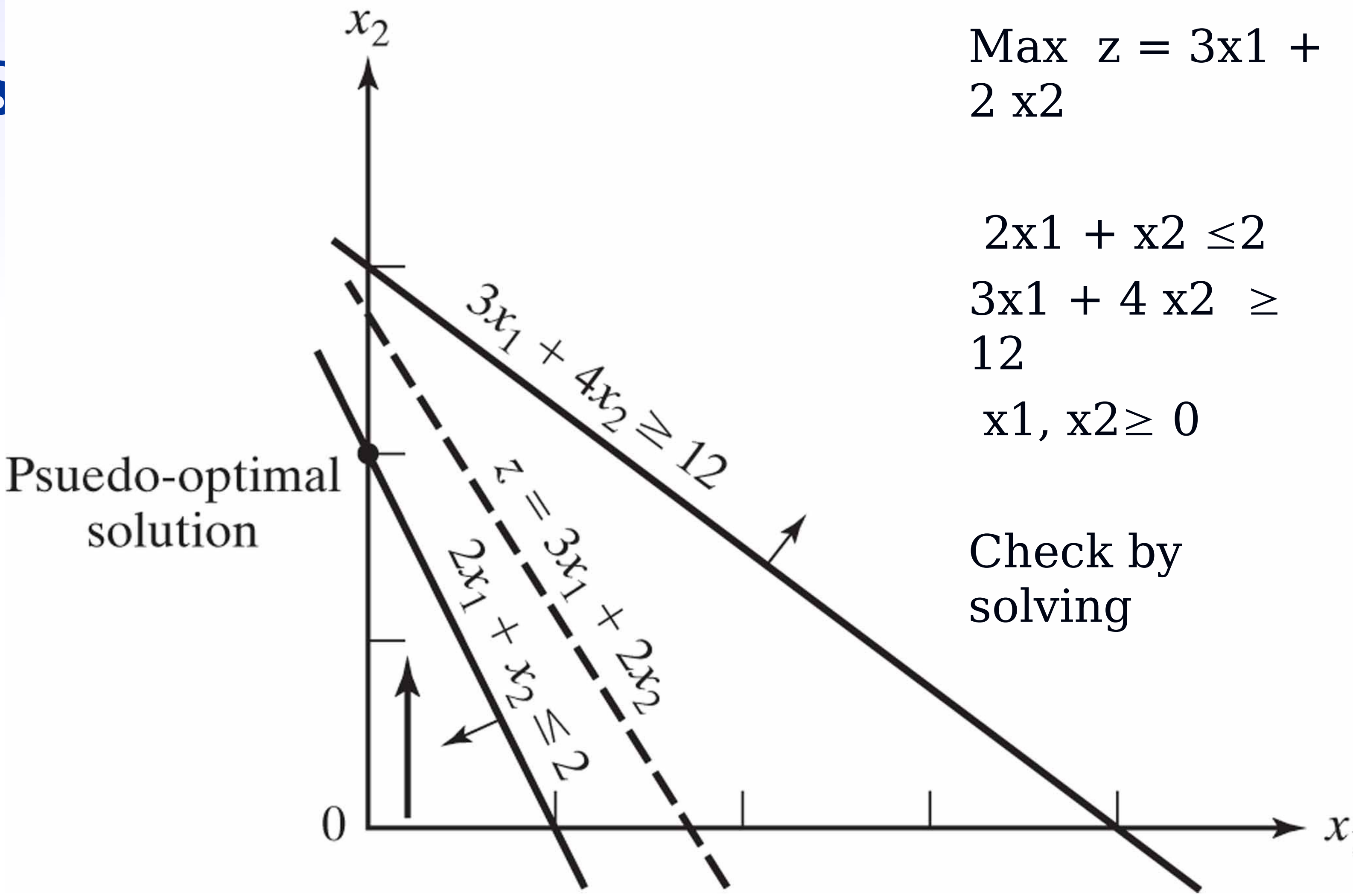
Using the penalty $M = 100$ for the artificial variable R, the following tableaux provide the simplex iterations of the model.

Iteration	Basic	x_1	x_2	x_4	x_3	R	Solution
0	z	−303	−402	100	0	0	−1200
x_2 enters	x_3	2	1	0	1	0	2
x_3 leaves	R	3	4	−1	0	1	12
1	z	501	0	100	402	0	−396
(pseudo-optimum)	x_2	2	1	0	1	0	2
	R	−5	0	−1	−4	1	4

Optimum iteration 1 shows that the artificial variable R is *positive* (= 4), which indicates

That given problem is infeasible.

Figure 3.11 Infeasible s... 4



Max $z = 3x1 + 2\ x2$

$2x1 + x2 \leq 2$

$3x1 + 4\ x2 \geq 12$

$x1, x2 \geq 0$

Check by solving

Duality Theory

The theory of duality is a very elegant and important concept within the field of operations research.

In general no of iterations depends upon no of constraints so if in a problem no of constraints are much more than no of variables it is better to use dual for solving

Every linear program has associated with it a related linear program called its **dual**. **The original problem is termed the primal.**

The two problems are closely related, in the sense that optimal sol of one provides optimal sol to the other.

Forming of dual problem

We assume here that

primal problem is in equation form(all the constraints are equations with non negative RHS and all the variables are non negative.)

$$\text{Maximize or minimize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \; i = 1, 2, \ldots, m$$

$$x_j \geq 0, \; j = 1, 2, \ldots, n$$

The variables x_j, $j = 1, 2, \ldots, n$, include the surplus, slack, and artificial variables, if any.

1. A dual variable is defined for each primal (constraint) equation.
2. A dual constraint is defined for each primal variable.
3. The constraint (column) coefficients of a primal variable define the left-hand-side coefficients of the dual constraint and its objective coefficient define the right-hand side.
4. The objective coefficients of the dual equal the right-hand side of the primal constraint equations.

TABLE 4.1 Construction of the Dual from the Primal

		Primal variables					
		x_1	x_2	$\cdots$	x_j	$\cdots$	x_n	
Dual variables		c_1	c_2	$\cdots$	c_j	$\cdots$	c_n	Right-hand side
Dual variables	y_1	a_{11}	a_{12}	$\cdots$	a_{1j}	$\cdots$	a_{1n}	b_1
	y_2	a_{21}	a_{22}	$\cdots$	a_{2j}	$\cdots$	a_{2n}	b_2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	y_m	a_{m1}	a_{m2}	$\cdots$	a_{mj}	$\cdots$	a_{mn}	b_m
					↑ jth dual constraint			↑ Dual objective coefficients

TABLE 4.2 Rules for Constructing the Dual Problem

Primal problem objective[a]	Dual problem: Objective	Dual problem: Constraints type	Dual problem: Variables sign
Maximization	Minimization	$\geq$	Unrestricted
Minimization	Maximization	$\leq$	Unrestricted

Example 4.1-1

Primal	Primal in equation form	Dual variables
Maximize $z = 5x_1 + 12x_2 + 4x_3$	Maximize $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 + x_3 \leq 10$	$x_1 + 2x_2 + x_3 + x_4 = 10$	y_1
$2x_1 - x_2 + 3x_3 = 8$	$2x_1 - x_2 + 3x_3 + 0x_4 = 8$	y_2
$x_1, x_2, x_3 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Minimize } w = 10y_1 + 8y_2$$

subject to

$$\begin{aligned} y_1 + 2y_2 &\geq 5 \\ 2y_1 - y_2 &\geq 12 \\ y_1 + 3y_2 &\geq 4 \\ \left.\begin{aligned} y_1 + 0y_2 &\geq 0 \\ y_1, y_2 &\text{ unrestricted} \end{aligned}\right\} &\Rightarrow (y_1 \geq 0,\ y_2 \text{ unrestricted}) \end{aligned}$$

Example 4.1-2

Primal	Primal in equation form	Dual variables
Minimize $z = 15x_1 + 12x_2$	Minimize $z = 15x_1 + 12x_2 + 0x_3 + 0x_4$	
subject to	subject to	
$x_1 + 2x_2 \geq 3$	$x_1 + 2x_2 - x_3 + 0x_4 = 3$	y_1
$2x_1 - 4x_2 \leq 5$	$2x_1 - 4x_2 + 0x_3 + x_4 = 5$	y_2
$x_1, x_2 \geq 0$	$x_1, x_2, x_3, x_4 \geq 0$	

Dual Problem

$$\text{Maximize } w = 3y_1 + 5y_2$$

subject to

$$y_1 + 2y_2 \leq 15$$

$$2y_1 - 4y_2 \leq 12$$

$$\left.\begin{array}{rr} -y_1 & \leq 0 \\ y_2 & \leq 0 \\ y_1, y_2 \text{ unrestricted} & \end{array}\right\} \Rightarrow (y_1 \geq 0,\ y_2 \leq 0)$$

Primal	Primal in equation form	Dual variables

Maximize $z = 5x_1 + 6x_2$

subject to

$$x_1 + 2x_2 = 5$$
$$-x_1 + 5x_2 \geq 3$$
$$4x_1 + 7x_2 \leq 8$$
$$x_1 \text{ unrestricted}, x_2 \geq 0$$

substitute $x_1 = x_1^- - x_1^+$

Max z = $5x_1^- - 5x_1^+ + 6x_2$

s.T

$x_1^- - x_1^+ + 2x_2 = 5$

$-x_1^- + x_1^+ + 5x_2 - x_3 = 3$

$4x_1^- - 4x_1^+ + 7x_2 + x_4 = 8$

$x_1^-, x_1^+, x_2, x_3, x_4 \geq 0$

Dual Problem

Minimize $z = 5y_1 + 3y_2 + 8y_3$

subject to

$$\left.\begin{array}{r} y_1 - y_2 + 4y_3 \geq 5 \\ -y_1 + y_2 - 4y_3 \geq -5 \end{array}\right\} \Rightarrow (y_1 - y_2 + 4y_3 = 5)$$

$$2y_1 + 5y_2 + 7y_3 \geq 6$$

$$\left.\begin{array}{r} -y_2 \geq 0 \\ y_3 \geq 0 \\ y_1, y_2, y_3 \text{ unrestricted} \end{array}\right\} \Rightarrow (y_1 \text{ unrestricted}, y_2 \leq 0, y_3 \geq 0)$$

Thus corresponding to unrestricted variables dual constraint is in equation form

- **corresponding to unrestricted variables dual constraint is in equation form**

Conversion

Multiply through the greater-than-or-equal-to inequality constraint by -1

Use the approach described above to convert the equality constraint to **a pair of inequality constraints.**

Replace the variable unrestricted in sign, , by the **difference** of two nonnegative variables.

Streamlining the conversion …

An equality constraint in the primal generates a dual variable that is **unrestricted in sign.**

An unrestricted in sign variable in the primal generates an **equality constraint** in the dual.

TABLE 4.3 Rules for Constructing the Dual Problem

Maximization problem		Minimization problem
Constraints		*Variables*
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		*Constraints*
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

Primal-Dual relationship

Primal Problem	Dual Problem
opt=max	opt=min
Constraint i :	Variable i :
<= form	y_i >= 0
= form	y_i urs
Variable j:	Constraint j:
x_j >= 0	>= form
x_j urs	= form

Example

$$\max_{x} Z = 5x_1 + 4x_2$$

$$3x_1 - 8x_2 \geq -6$$

$$x_1 + 6x_2 = -5$$

$$8x_1 = 10$$

$$x_2 \geq 0; \quad x_1 \text{ urs}$$

equivalent non-standard form

$$\max_{x} Z = 5x_1 + 4x_2$$

$$\begin{aligned} -3x_1 + 8x_2 &\le 6 \\ x_1 + 6x_2 &= -5 \\ 8x_1 \quad &= 10 \\ x_2 \ge 0;\ \ x_1 &\text{ urs} \end{aligned}$$

Dual

$$\min_{y} w = 6y_1 - 5y_2 + 10y_3$$

$$
\begin{aligned}
-3y_1 + y_2 + 8y_3 &= 5 \\
8y_1 + 6y_2 &\geq 4 \\
y_1 \geq 0;\ y_2, y_3 &\text{ urs}
\end{aligned}
$$

Important Observation

FOR ANY PRIMAL LINEAR PROGRAM, THE DUAL OF THE DUAL IS THE PRIMAL

Exercise

Maximize $z = 3x + 9y$

such that $x + 4y \leq 8$

$$x + 2y \leq 4$$

$$x, y \geq 0$$

Determine its dual ,Solve both the problems and compare the results

Max $3x + 9y$

$x + 4y \le 8$

$x + 2y \le 4$

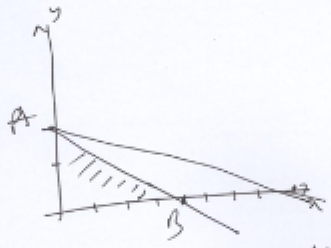
$x, y \ge 0$

Min $8y_1 + 4y_2$

$y_1 + y_2 \ge 3$

$4y_1 + 2y_2 \ge 9$

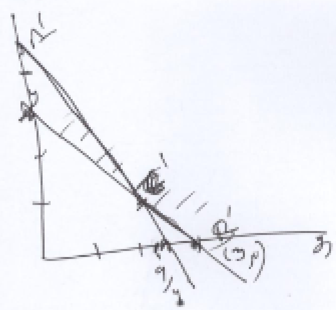
$y_1, y_2 \ge 0$



Max $z = 18$

A $(0,2)$ $z = 18$ } — A is optimal

B $(4,0)$ $z = 12$



$A' = (0, \frac{9}{2})$ $z' = \frac{9}{2} \times 4 = 18$

$B' = (3, 0)$ $z' = 24$

$C' = (\frac{3}{2}, \frac{3}{2})$ $z' = \frac{8 \times 3}{2} + \frac{4 \times 3}{2}$

$= 12 + 6$

$= 18$

Min $z' = 18$

$2y_1 + 2y_2 = 6$

$4y_1 + 2y_2 = 9$

$-2y_1 = -3$

$y_1 = 3/2, \; y_2 = \frac{3}{2}$

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Dual Simplex Method

Simplex method starts with feasible solution and continues to be feasible until optimal sol is obtained.

Dual simplex method starts with infeasible solution but better than optimal) and remains infeasible until feasibility is restored.

The main advantage of dual simplex over the usual simplex method is that we do not require any artificial variables in the dual simplex method. Hence a lot of labor is saved whenever

Dual feasibility condition. The leaving variable, x_r, is the basic variable having the most negative value (ties are broken arbitrarily). If all the basic variables are nonnegative, the algorithm ends.

Dual optimality condition. Given that x_r is the leaving variable, let $\bar{c}_j$ be the reduced cost of nonbasic variable x_j and α_{rj} the constraint coefficient in the x_r-row and x_j-column of the tableau. The entering variable is the nonbasic variable with $\alpha_{rj} < 0$ that corresponds to

$$\min_{\text{Nonbasic } x_j} \left\{ \left| \frac{\bar{c}_j}{\alpha_{rj}} \right|, \alpha_{rj} < 0 \right\}$$

(Ties are broken arbitrarily.) If $\alpha_{rj} \geq 0$ for all nonbasic x_j, the problem has no feasible solution.

To start the LP optimal and infeasible, two requirements must be met:

1. The objective function must satisfy the optimality condition of the regular simplex method (Chapter 3).

2. All the constraints must be of the type ($\leq$).

The second condition requires converting any ($\geq$) to ($\leq$) simply by multiplying both sides of the inequality ($\geq$) by -1. If the LP includes ($=$) constraints, the equation can be replaced by two inequalities. For example,

$$x_1 + x_2 = 1$$

is equivalent to

$$x_1 + x_2 \leq 1, x_1 + x_2 \geq 1$$

or

$$x_1 + x_2 \leq 1, -x_1 - x_2 \leq -1$$

After converting all the constraints to ($\leq$), the starting solution is infeasible if at least one of the right-hand sides of the inequalities is strictly negative.

Starting solution (optimal but infeasible)

$$\text{Minimize } z = 3x_1 + 2x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 + x_2 + x_3 &\geq 3 \\ -3x_1 + 3x_2 + x_3 &\geq 6 \\ x_1 + x_2 + x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In the present example, the first two inequalities are multiplied by -1 to convert them to $(\leq)$ constraints. The starting tableau is thus given as:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	-2	-1	0	0	0	0
x_4	-3	-1	-1	1	0	0	-3
x_5	3	-3	-1	0	1	0	-6
x_6	1	1	1	0	0	1	3

The tableau is optimal because all the reduced costs in the z-row are ≤ 0 ($\bar{c}_1 = -3, \bar{c}_2 = -2, \bar{c}_3 = -1, \bar{c}_4 = 0, \bar{c}_5 = 0, \bar{c}_6 = 0$). It is also infeasible because at least one of the basic variables is negative ($x_4 = -3, x_5 = -6, x_6 = 3$).

According to the dual feasibility condition, x_5 ($= -6$) is the leaving variable. The next table shows how the dual optimality condition is used to determine the entering variable.

	$j = 1$	$j = 2$	$j = 3$
Nonbasic variable	x_1	x_2	x_3
z-row ($\bar{c}_j$)	-3	-2	-1
x_5-row, α_{4j}	3	-3	-1
Ratio, $\left\lvert\frac{\bar{c}_j}{\alpha_{5j}}\right\rvert, \alpha_{5j} < 0$	—	$\frac{2}{3}$	1

The ratios show that x_2 is the entering variable. Notice that a nonbasic variable x_j is a candidate for entering the basic solution only if its α_{rj} is strictly negative. This is the reason x_1 is excluded in the table above.

The next tableau is obtained by using the familiar row operations, which give

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-5	0	$-\frac{1}{3}$	0	$-\frac{2}{3}$	0	4
x_4	-4	0	$-\frac{2}{3}$	1	$-\frac{1}{3}$	0	-1
x_2	-1	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	0	2
x_6	2	0	$\frac{2}{3}$	0	$\frac{1}{3}$	1	1
Ratio	$\frac{5}{4}$	—	$\frac{1}{2}$	—	2	—	

The preceding tableau shows that x_4 leaves and x_3 enters, thus yielding the following tableau, which is both optimal and feasible:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	-3	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{9}{2}$
x_3	6	0	1	$-\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$
x_2	-3	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_6	-2	0	0	1	0	1	0

Notice how the dual simplex works. In all the iterations, optimality is maintained (all reduced costs are ≤ 0). At the same time, each new iteration moves the solution toward feasibility. At iteration 3, feasibility is restored for the first time and the process ends with the optimal feasible solution given as $x_1 = 0$, $x_2 = \frac{3}{2}$, $x_2 = \frac{3}{2}$, and $z = \frac{9}{2}$.

Examples

There is a small company in Melbourne which has recently become engaged in the production of office furniture.

The company manufactures tables, desks and chairs. The production of a table requires 8 kgs of wood and 5 kgs of metal and is sold for $80; a desk uses 6 kgs of wood and 4 kgs of metal and is sold for $60; and a chair requires 4 kgs of both metal and wood and is sold for $50.

We would like to determine the revenue maximizing strategy for this company, given that their resources are limited to 100 kgs of wood and 60 kgs of metal.

Problem P1

$$\max_{x} Z = 80x_1 + 60x_2 + 50x_3$$

$$8x_1 + 6x_2 + 4x_3 \leq 100$$

$$5x_1 + 4x_2 + 4x_3 \leq 60$$

$$x_1, x_2, x_3 \geq 0$$

Now consider that there is a much bigger company in Melbourne which has been the lone producer of this type of furniture for many years. They don't appreciate the competition from this new company; so they have decided to tender an offer to buy all of their competitor's resources and therefore put them out of business.

The challenge for this large company then is to develop a linear program which will determine the appropriate amount of money that should be offered for each unit of each type of resource, such that the offer will be acceptable to the smaller company while minimizing the expenditures of the larger company.

Problem P1

$$\max_{x} Z = 80x_1 + 60x_2 + 50x_3$$

$$8x_1 + 6x_2 + 4x_3 \le 100$$

$$5x_1 + 4x_2 + 4x_3 \le 60$$

$$x_1, x_2, x_3 \ge 0$$

$$\min_{y} w = 100y_1 + 60y_2$$

$$8y_1 + 5y_2 \ge 80$$

$$6y_1 + 4y_2 \ge 60$$

$$4y_1 + 4y_2 \ge 50$$

$$y_1, y_2 \ge 0$$

A Diet Problem

An individual has a choice of two types of food to eat, meat and potatoes, each offering varying degrees of nutritional benefit. He has been warned by his doctor that he must receive at least 400 units of protein, 200 units of carbohydrates and 100 units of fat from his daily diet. Given that a kg of steak costs $10 and provides 80 units of protein, 20 units of carbohydrates and 30 units of fat, and that a kg of potatoes costs $2 and provides 40 units of protein, 50 units of carbohydrates and 20 units of fat, he would like to find the minimum cost diet which satisfies his nutritional requirements

Problem P2

$$\min_{x} Z = 10x_1 + 2x_2$$

$$80x_1 + 40x_2 \geq 400$$

$$20x_1 + 50x_2 \geq 200$$

$$30x_1 + 20x_2 \geq 100$$

$$x_1, x_2 \geq 0$$

Now consider a chemical company which hopes to attract this individual away from his present diet by offering him synthetic nutrients in the form of pills. This company would like determine prices per unit for their synthetic nutrients which will bring them the highest possible revenue while still providing an acceptable dietary alternative to the individual.

- Sensitivity analysis deals with determining the conditions that will keep the current optimal solution unchanged.
- Postoptimality analysis **deals with finding a new optimal solution when the data of the model are changed**

Optimal dual solution

The primal and dual solutions are closely related,

Optimal sol of one yields optimal sol of other.

In general no of iterations depends upon no of constraints so if in a problem no of constraints are much more than no of variables it is better to use dual for solving

Method 1.

$$\begin{pmatrix}\text{Optimal value of} \\ \textit{dual} \text{ variable } y_i\end{pmatrix} = \begin{pmatrix}\text{Optimal primal } z\text{-coefficient of } \textit{starting} \text{ variable } x_i \\ + \\ \textit{Original} \text{ objective coefficient of } x_i\end{pmatrix}$$

Consider the following LP:

$$\text{Maximize } z = 5x_1 + 12x_2 + 4x_3$$

subject to

$$x_1 + 2x_2 + x_3 \leq 10$$

$$2x_1 - x_2 + 3x_3 = 8$$

$$x_1, x_2, x_3 \geq 0$$

Primal	Dual
Maximize $z = 5x_1 + 12x_2 + 4x_3 - MR$	Minimize $w = 10y_1 + 8y_2$
subject to	subject to
$x_1 + 2x_2 + x_3 + x_4 = 10$	$y_1 + 2y_2 \geq 5$
$2x_1 - x_2 + 3x_3 + R = 8$	$2y_1 - y_2 \geq 12$
$x_1, x_2, x_3, x_4, R \geq 0$	$y_1 + 3y_2 \geq 4$
	$y_1 \geq 0$
	$y_2 \geq -M$ ($\Rightarrow y_2$ unrestricted)

TABLE 4.4 Optimal Tableau of the Primal of Example 4.2-1

Basic	x_1	x_2	x_3	x_4	R	Solution
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5}+M$	$54\frac{4}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

Method 1. In Table 4.4, the starting primal variables x_4 and R uniquely correspond to the dual variables y_1 and y_2, respectively. Thus, we determine the optimum dual solution as follows:

Starting primal basic variables	x_4	R
z-equation coefficients	$\frac{29}{5}$	$-\frac{2}{5}+M$
Original objective coefficient	0	$-M$
Dual variables	y_1	y_2
Optimal dual values	$\frac{29}{5}+0=\frac{29}{5}$	$-\frac{2}{5}+M+(-M)=-\frac{2}{5}$

Method 2.

$$\begin{pmatrix}\text{Optimal values}\\ \text{of } \textit{dual} \text{ variables}\end{pmatrix} = \begin{pmatrix}\text{Row vector of}\\ \text{original objective coefficients}\\ \text{of optimal } \textit{primal} \text{ basic variables}\end{pmatrix} \times \begin{pmatrix}\text{Optimal } \textit{primal}\\ \text{inverse}\end{pmatrix}$$

Obtained directly from final table

The elements of the row vector must appear in the same order in which the basic variables are listed in the *Basic* column of the simplex tableau.

TABLE 4.4 Optimal Tableau of the Primal of Example 4.2-1

Basic	x_1	x_2	x_3	x_4	R	Solution
z	0	0	$\frac{3}{5}$	$\frac{29}{5}$	$-\frac{2}{5} + M$	$54\frac{4}{5}$
x_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{12}{5}$
x_1	1	0	$\frac{7}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{26}{5}$

$$\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}$$

- This is is inverse of the matrix corresponding to basic sol (x2 ,x1) which is

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

Original objective coeff = coeff of x2,coeff of x2 =(12 , 5)

$$\begin{bmatrix} 2/5 & -1/5 \\ 1/5 & 2/5 \end{bmatrix}$$

Thus, the optimal dual values are computed as

$$(y_1, y_2) = \begin{pmatrix} \text{Original objective} \\ \text{coefficients of } x_2, x_1 \end{pmatrix} \times (\text{Optimal inverse})$$

$$= (12, 5)\begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

$$= \left(\frac{29}{5}, -\frac{2}{5}\right)$$

Primal-dual objective values. Having shown how the optimal dual values are determined, next we present the relationship between the primal and dual objective values. For any pair of *feasible* primal and dual solutions,

$$\begin{pmatrix}\text{Objective value in the}\\ \textit{maximization}\text{ problem}\end{pmatrix} \leq \begin{pmatrix}\text{Objective value in the}\\ \textit{minimization}\text{ problem}\end{pmatrix}$$

At the optimum, the relationship holds as a strict equation. The relationship does not specify which problem is primal and which is dual. Only the sense of optimization (maximization or minimization) is important in this case.

	Primal	Dual
1	*Primal feasible* Maximize $z = 2x_1 + x_2$, subject to: $x_1 + x_2 \le 4$, $x_1 - x_2 \le 2$, $x_1 \ge 0, \quad x_2 \ge 0$.	*Dual feasible* Minimize $v = 4y_1 + 2y_2$, subject to: $y_1 + y_2 \ge 2$, $y_1 - y_2 \ge 1$, $y_1 \ge 0, \quad y_2 \ge 0$.
2	*Primal feasible and unbounded* Maximize $z = 2x_1 + x_2$, subject to: $x_1 - x_2 \le 4$, $x_1 - x_2 \le 2$, $x_1 \ge 0, \quad x_2 \ge 0$.	*Dual infeasible* Minimize $v = 4y_1 + 2y_2$, subject to: $y_1 + y_2 \ge 2$, $-y_1 - y_2 \ge 1$, $y_1 \ge 0, \quad y_2 \ge 0$.
3	*Primal infeasible* Maximize $z = 2x_1 + x_2$, subject to: $-x_1 - x_2 \le -4$, $x_1 + x_2 \le 2$, $x_1 \ge 0, \quad x_2 \ge 0$.	*Dual feasible and unbounded* Minimize $v = -4y_1 + 2y_2$, subject to: $-y_1 + y_2 \ge 2$, $-y_1 + y_2 \ge 1$, $y_1 \ge 0, \quad y_2 \ge 0$.
4	*Primal infeasible* Maximize $z = 2x_1 + x_2$, subject to: $-x_1 + x_2 \le -4$, $x_1 - x_2 \le 2$, $x_1 \ge 0, \quad x_2 \ge 0$.	*Dual infeasible* Minimize $v = -4y_1 + 2y_2$, subject to: $-y_1 + y_2 \ge 2$, $y_1 - y_2 \ge 1$, $y_1 \ge 0, \quad y_2 \ge 0$.

Example 1

Minimize $Z = 2x1 + x2$

Subject to

$3x1 + x2 \geq 3$

$4x1 + 3x2 \geq 6$

$x1 + 2x2 \geq 3$

and $x1 \geq 0$, $x2 \geq 0$

Solution

Step 1 – Rewrite the given problem in the form

Maximize $Z' = -- 2x1 - x2$

Subject to

$-3x1 - x2 \leq -3$

$-4x1 - 3x2 \leq -6$

$-x1 - 2x2$

$x1, x2 \geq 0$

Step 2 – Adding slack variables to each constraint

Maximize $Z' = -2\,x1 - x2$

Subject to

$-3x1 - x2 + s1 = -3$

$-4x1 - 3x2 + s2 = -6$

$-x1 - 2x2 + s3 = -3$

$x1, x2, s1, s2, s3 \geq 0$

Step

Step 3 – Construct the simplex table

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-3	-3	-1	1	0	0	
s_2	0	-6	-4	$\boxed{-3}$	0	1	0	$\rightarrow$ outgoing
s_3	0	-3	-1	-2	0	0	1	
	$Z' = 0$		2	$\uparrow$ 1	0	0	0	$\leftarrow \Delta_j$

Step 4 – To find the leaving vector
Min (-3, -6, -3) = -6. Hence s2 is outgoing vector
Step 5 – To find the incoming vector
Max (Δ1 / x21, Δ2 / x22) = (2/-4, 1/-3) = -1/3. So x2 is incoming vector
Step 6 –The key element is -3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
s_1	0	-1	$\boxed{-5/3}$	0	1	-1/3	0	$\rightarrow$ outgoing
x_2	-1	2	4/3	1	0	-1/3	0	
s_3	0	1	5/3	0	0	-2/3	1	
	$Z' = -2$		$\uparrow$ 2/3	0	0	1/3	0	$\leftarrow \Delta_j$

Step 7 – To find the leaving vector

Min (-1, 2, 1) = -1. Hence s1 is outgoing vector

Step 8 – To find the incoming vector

Max (Δ1 / x11, Δ4 / x14) = (-2/5, -1) = -2/5. So x1 is incoming vector

Step 9 –The key element is -5/3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	S_3	
x_1	-2	3/5	1	0	-3/5	1/5	0	
x_2	-1	6/5	0	1	4/5	-3/5	0	
s_3	0	0	0	0	1	-1	1	
	$Z' = -12/5$		0	0	2/5	1/5	0	$\leftarrow \Delta_j$

Step 10 – $\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max $Z' = -12/5$, $Z = 12/5$, and $x_1 = 3/5$, $x_2 = 6/5$

Example 2

Minimize $Z = 3x_1 + x_2$

Subject to

$x_1 + x_2 \geq 1$

$2x_1 + 3x_2 \geq 2$

and $x_1 \geq 0$, $x_2 \geq 0$

Solution

Maximize $Z' = -3x_1 - x_2$

Subject to

$-x_1 - x_2 \le -1$

$-2x_1 - 3x_2 \le -2$

$x_1, x_2 \ge 0$

SLPP

Maximize Z'

$= -3x_1 - x_2$

Subject to

$-x_1 - x_2 + s_1 = -1$

$-2x_1 - 3x_2 + s_2 = -2$

$x_1, x_2, s_1, s_2 \ge 0$

	$C_j \rightarrow$		-3	-1	0	0	
Basic variables	C_B	X_B	X_1	X_2	S_1	S_2	
s_1	0	-1	-1	-1	1	0	
s_2	0	-2	-2	$\boxed{-3}$	0	1	$\rightarrow$
	$Z' = 0$		3	$\uparrow$ 1	0	0	$\leftarrow \Delta_j$
s_1	0	-1/3	-1/3	0	1	$\boxed{-1/3}$	$\rightarrow$
x_2	-1	2/3	2/3	1	0	-1/3	
	$Z' = -2/3$		7/3	0	0	$\uparrow$ 1/3	$\leftarrow \Delta_j$
s_2	0	1	1	0	-3	1	
x_2	-1	1	1	1	-1	0	
	$Z' = -1$		2	0	1	0	$\leftarrow \Delta_j$

$\Delta_j \geq 0$ and $X_B \geq 0$, therefore the optimal solution is Max $Z' = -1$, $Z = 1$, and $x_1 = 0$, $x_2 = 1$

Problem 1

Maximize $2X1 + x2 - 3x3 + 5x4$

s.t.

$X1 + 2x2 + 2x3 + 4x4 \leq 40$

$2x1 - x2 + x3 + 2x4 \leq 8$

$4x1 - 2x2 + x3 - x4 \leq 10$

$X1, x2, x3, x4 \geq 0$