10

Defuzzification

Learning Objectives

- Need for defuzzification process.
- How lambda-cuts for fuzzy sets and fuzzy relations can be carried out.
- Various types of defuzzification methods.
- To know how λ-cut relation of a fuzzy tolerance and fuzzy equivalence relation results in crisp tolerance and crisp equivalence relation respectively.
- An example provided to depict how the various defuzzification methods are used to crisp outputs.

10.1 Introduction

In fuzzification process, we have made the conversion from crisp quantities to fuzzy quantities; however, in several applications and engineering area, it is necessary to "defuzzify" the fuzzy results we have generated through the fuzzy set analysis, i.e., it is necessary to convert fuzzy results into crisp results. Defuzzification is a mapping process from a space of fuzzy control actions defined over an output universe of discourse into a space of crisp (nonfuzzy) control actions. This is required because in many practical applications crisp control actions are needed to actuate the control. A defuzzification process produces a nonfuzzy control action that best represents the possibility distribution of an inferred fuzzy control action. The defuzzification process has the capability to reduce a fuzzy set into a crisp single-valued quantity or into a crisp set; to convert a fuzzy matrix into a crisp matrix; or to convert a fuzzy number into a crisp number. Mathematically, the defuzzification process may also be termed as "rounding it off." Fuzzy set with a collection of membership values or a vector of values on the unit interval may be reduced to a single scalar quantity using defuzzification process. Enormous defuzzification methods have been suggested in the literature; although no method has proved to be always more advantageous than the others. The selection of the method to be used depends on the experience of the designer. It may be done on the basis of the computational complexity involved, applicability to the situations considered and plausibility of the outputs obtained based on engineering point of view. In this chapter we will discuss the various defuzzification methods employed for converting fuzzy variables into crisp variables.

10.2 Lambda-Cuts for Fuzzy Sets (Alpha-Cuts)

Consider a fuzzy set $\underset{\sim}{A}$. The set $A_\lambda (0 < \lambda < 1)$ called the lambda (λ)-cut (or alpha [α]-cut) set is a crisp set of the fuzzy set and is defined as follows:

$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) \geq \lambda\}; \quad \lambda \in [0, 1]$$

The set A_λ is called a weak lambda-cut set if it consists of all the elements of a fuzzy set whose membership functions have values greater than or equal to a specified value. On the other hand, the set A_λ is called a strong lambda-cut set if it consists of all the elements of a fuzzy set whose membership functions have values strictly greater than a specified value. A strong λ-cut set is given by

$$A_\lambda = \{x \mid \mu_{\underset{\sim}{A}}(x) > \lambda\}; \quad \lambda \in [0, 1]$$

All the λ-cut sets form a family of crisp sets. It is important to note the λ-cut set A_λ (or A_α, if α-cut set) does not have a tilde score, because it is a crisp set derived from parent fuzzy set $\underset{\sim}{A}$. Any particular fuzzy set $\underset{\sim}{A}$ can be transformed into an infinite number of λ-cut sets, because there are infinite number of values λ can take in the interval [0, 1].

The properties of λ-cut sets are as follows:

1. $(\underset{\sim}{A} \cup \underset{\sim}{B})_\lambda = A_\lambda \cup B_\lambda$
2. $(\underset{\sim}{A} \cap \underset{\sim}{B})_\lambda = A_\lambda \cap B_\lambda$
3. $(\bar{\underset{\sim}{A}})_\lambda \neq (\bar{A}_\lambda)$ except when $\lambda = 0.5$
4. For any $\lambda \leq \beta$, where $0 \leq \beta \leq 1$, it is true that $A_\beta \subseteq A_\lambda$, where $A_0 = X$.

The fourth property is essentially used in graphics. Figure 10-1 shows a continuous-valued fuzzy set with two λ-cut values.

In Figure 10-1, notice that for $\lambda = 0.2$ and $\beta = 0.5$, $A_{0.2}$ has a greater domain than $A_{0.5}$, i.e., for $\lambda \leq \beta (0.2 \leq 0.5)$, $A_{0.5} \subseteq A_{0.2}$.

Figure 10-2 shows the features of the membership functions.

The core of $\underset{\sim}{A}$ is the $\lambda = 1$-cut set A_1. The support of $\underset{\sim}{A}$ is the λ-cut set A_{0^+}, where $\lambda = 0^+$, and it can be defined as

$$A_{0^+} = \{x \mid \mu_{\underset{\sim}{A}}(x) > 0\}$$

The interval $[A_{0^+}, A_1]$ forms the boundaries of the fuzzy set $\underset{\sim}{A}$, i.e., the regions with the membership values between 0 and 1, i.e., for $\lambda = 0$ to 1.

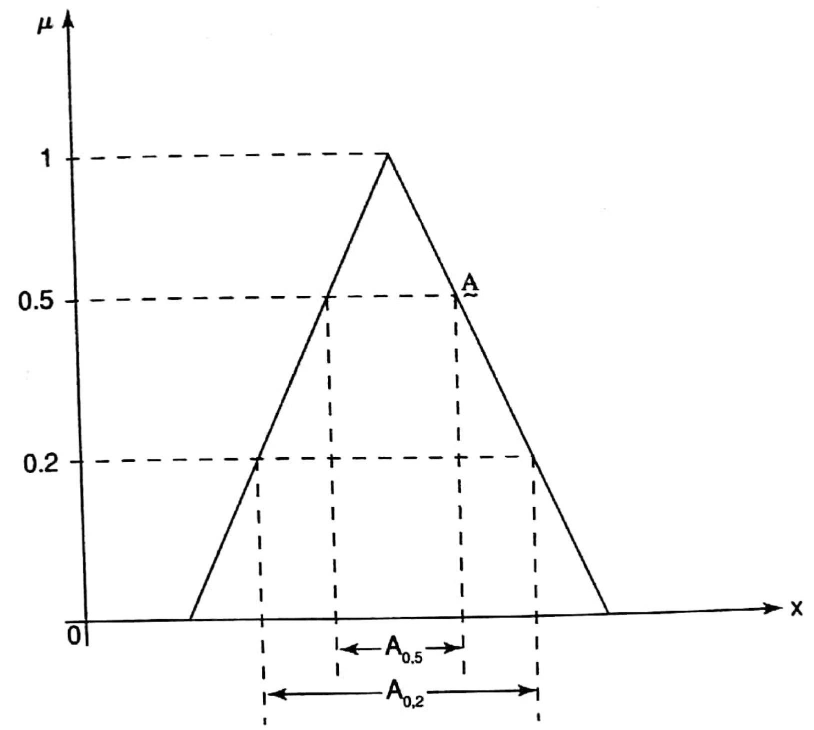

Figure 10-1 Two different λ-cut sets for a continuous-valued fuzzy set.

10.3 Lambda-Cuts for Fuzzy Relations

The λ-cut for fuzzy relations is similar to that for fuzzy sets.

Let $\underset{\sim}{R}$ be a fuzzy relation where each row of the relational matrix is considered a fuzzy set. The jth row in a fuzzy relation matrix $\underset{\sim}{R}$ denotes a discrete membership function for a fuzzy set $\underset{\sim}{R}_j$. A fuzzy relation can be converted into a crisp relation in the following

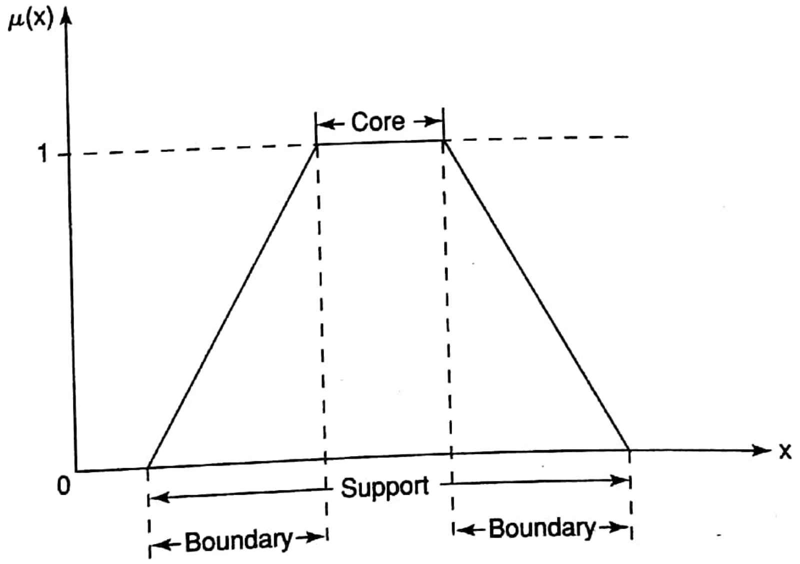

Figure 10-2 Features of the membership functions.

manner:

$$R_\lambda = \{(x, y) | \mu_{\underset{\sim}{R}}(x, y) \geq \lambda\}$$

where R_λ is a λ-cut relation of the fuzzy relation $\underset{\sim}{R}$. Since here $\underset{\sim}{R}$ is defined as a two-dimensional array, defined on the universes X and Y, therefore any pair $(x, y) \in R_\lambda$ belongs to $\underset{\sim}{R}$ with a relation greater than or equal to λ.

Similar to the properties of λ-cut fuzzy set, the λ-cuts on fuzzy relations also obey certain properties. They are listed as follows.

For two fuzzy relations $\underset{\sim}{R}$ and $\underset{\sim}{S}$ the following properties should hold:

1. $(\underset{\sim}{R} \cup \underset{\sim}{S})_\lambda = R_\lambda \cup S_\lambda$
2. $(\underset{\sim}{R} \cap \underset{\sim}{S})_\lambda = R_\lambda \cap S_\lambda$
3. $(\bar{\underset{\sim}{R}})_\lambda \neq (\bar{R}_\lambda)$ except when $\lambda = 0.5$
4. For any $\lambda \leq \beta$, where $0 \leq \beta \leq 1$, it is true that $R_\beta \subseteq R_\lambda$.

10.4 Defuzzification Methods

Defuzzification is the process of conversion of a fuzzy quantity into a precise quantity. The output of a fuzzy process may be union of two or more fuzzy membership functions defined on the universe of discourse of the output variable.

Consider a fuzzy output comprising two parts: the first part, $\underset{\sim}{C}_1$, a triangular membership shape (as shown in Figure 10-3(A)), the second part, $\underset{\sim}{C}_2$, a trapezoidal shape (as shown in Figure 10-3(B)). The union of these two membership functions, i.e., $\underset{\sim}{C} = \underset{\sim}{C}_1 \cup \underset{\sim}{C}_2$ involves the max-operator, which is going to be the outer envelope of the two shapes shown in Figures 10-3(A) and (B); the final shape of $\underset{\sim}{C}$ is shown in Figure 10-3(C).

A fuzzy output process may involve many output parts, and the membership function representing each part of the output can have any shape. The membership function of the fuzzy output need not always be normal. In general, we have

$$\underset{\sim}{C}_n = \bigcup_{i=1}^{n} \underset{\sim}{C}_i = \underset{\sim}{C}$$

Defuzzification methods include the following:

1. Max-membership principle
2. Centroid method
3. Weighted average method
4. Mean-max membership
5. Center of sums
6. Center of largest area
7. First of maxima, last of maxima.

Now we discuss the methods listed above.

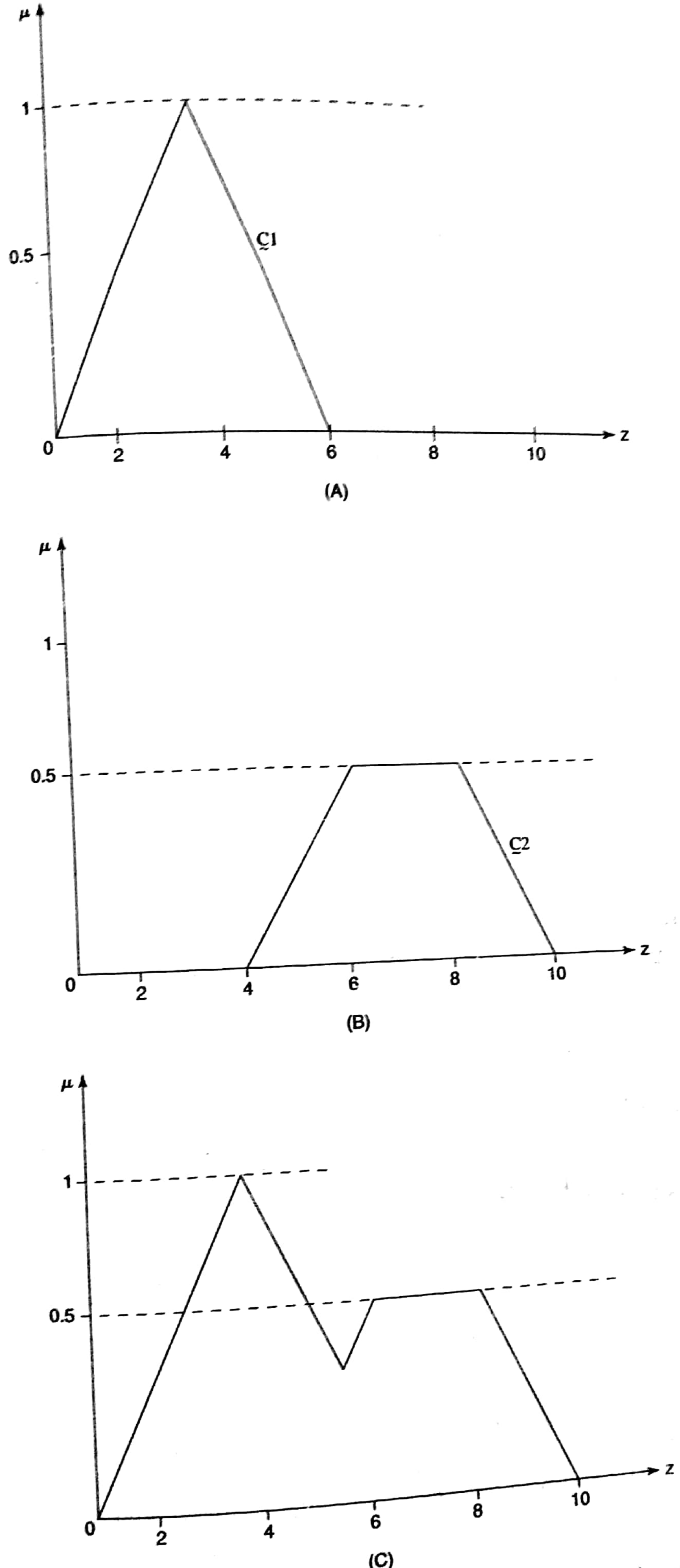

Figure 10-3 (A) First part of fuzzy output, (B) second part of fuzzy output, (C) union of parts (A) and (B).

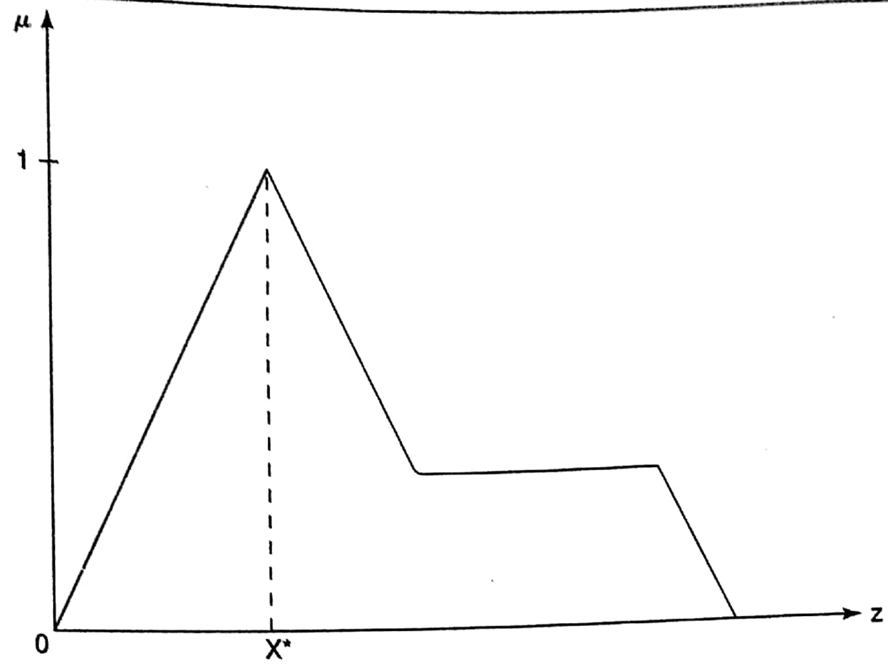

Figure 10-4 Max-membership defuzzification method.

10.4.1 *Max-membership Principle*

This method is also known as height method and is limited to peak output functions. This method is given by the algebraic expression

$$\mu_{\underset{\sim}{C}}(x^*) \geq \mu_{\underset{\sim}{C}}(x) \text{ for all } x \in X$$

The method is illustrated in Figure 10-4.

10.4.2 *Centroid Method*

This method is also known as center of mass, center of area or center of gravity method. It is the most commonly used defuzzification method. The defuzzified output x^* is defined as

$$x^* = \frac{\int \mu_{\underset{\sim}{C}}(x) \cdot x dx}{\int \mu_{\underset{\sim}{C}}(x) dx}$$

where the symbol $\int$ denotes an algebraic integration. This method is illustrated in Figure 10-5.

10.4.3 *Weighted Average Method*

This method is valid for symmetrical output membership functions only. Each membership function is weighted by its maximum membership value. The output in this case is given by

$$x^* = \frac{\sum \mu_{\underset{\sim}{C}}(\bar{x}_i) \cdot \bar{x}_i}{\sum \mu_{\underset{\sim}{C}}(\bar{x}_i)}$$

where $\sum$ denotes algebraic sum and $\bar{x}_i$ is the maximum of the ith membership function. The method is illustrated in Figure 10-6, where two fuzzy sets are considered.

From Figure 10-6, we notice that the defuzzified output is given by

$$x^* = \frac{0.5a + 0.8b}{0.5 + 0.8}$$

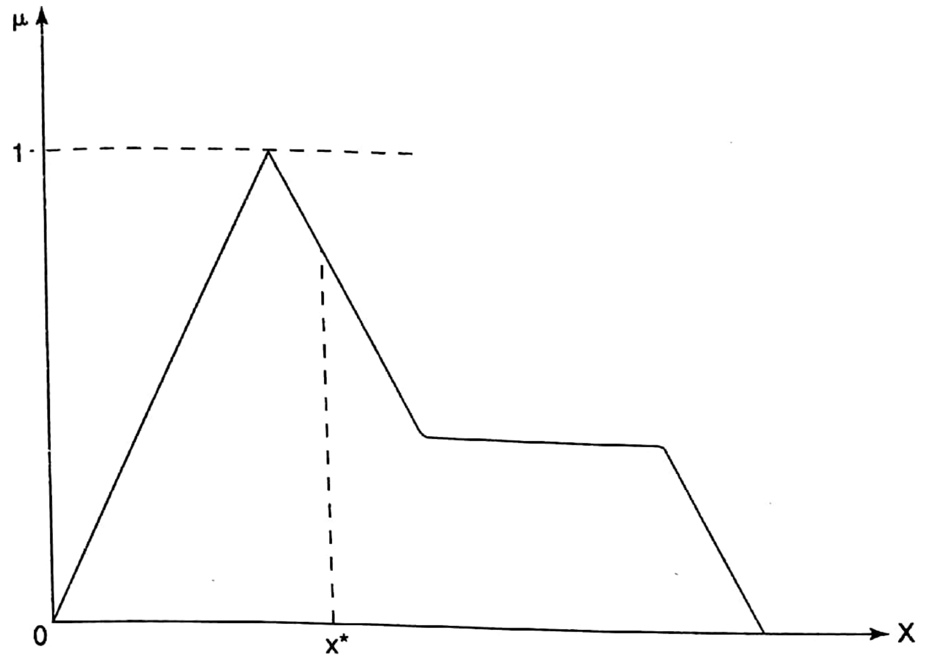

Figure 10-5 Centroid defuzzification method.

As this method is limited to symmetrical membership functions, the values of a and b are the means of their respective shapes.

10.4.4 Mean-max Membership

This method is also known as the middle of the maxima. This is closely related to max-membership method, except that the locations of the maximum membership can be

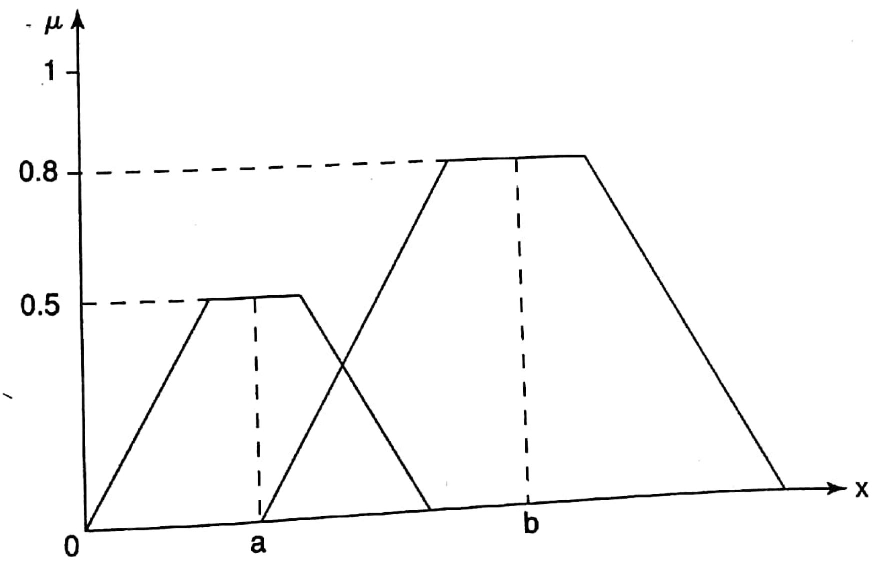

Figure 10-6 Weighted average defuzzification method (two symmetrical membership functions).

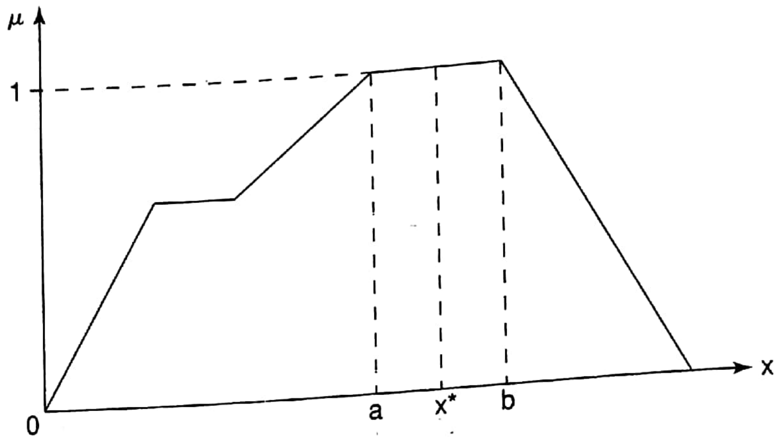

Figure 10-7 Mean-max membership defuzzification method.

nonunique. The output here is given by

$$x^* = \frac{\sum_{i=1}^{n} \bar{x}_i}{n}$$

This is illustrated in Figure 10-7. From Figure 10-7, we notice that the defuzzified output is given by

$$x^* = \frac{a+b}{2}$$

where a and b are as shown in the figure.

10.4.5 Center of Sums

This method employs the algebraic sum of the individual fuzzy subsets instead of their union. The calculations here are very fast, but the main drawback is that intersecting areas are added twice. The defuzzified value x^* is given by

$$x^* = \frac{\int_x x \sum_{i=1}^{n} \mu_{\underset{\sim}{C}_i}(x)dx}{\int_x \sum_{i=1}^{n} \mu_{\underset{\sim}{C}_i}(x)dx}$$

Figure 10-8 illustrates the center of sums method.

In center of sums method, the weights are the areas of the respective membership functions, whereas in the weighted average method the weights are individual membership values.

10.4.6 Center of Largest Area

This method can be adopted when the output consists of at least two convex fuzzy subsets which are not overlapping. The output in this case is biased towards a side of one membership function. When output fuzzy set has at least two convex regions, then the center of gravity of the convex fuzzy subregion having the largest area is used to obtain the defuzzified value

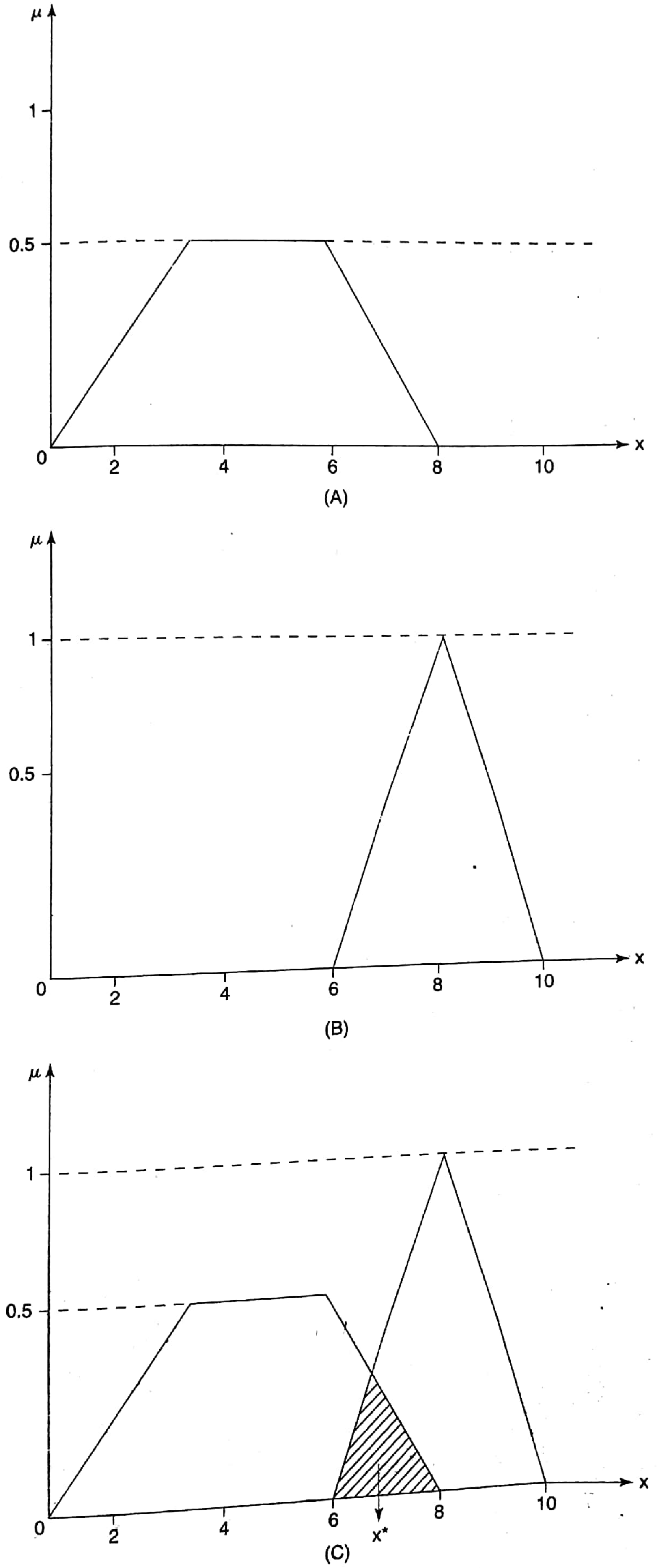

Figure 10-8 (A) First and (B) second membership functions, (C) defuzzification.

x^*. This value is given by

$$x^* = \frac{\int \mu_{c_i}(x) \cdot x dx}{\int \mu_{c_i}(x) dx}$$

where c_j is the convex subregion that has the largest area making up c_j. Figure 10-9 illustrates the center of largest area.

0.4.7 First of Maxima (Last of Maxima)

This method uses the overall output or union of all individual output fuzzy sets c_j for determining the smallest value of the domain with maximized membership in c_j. The steps used for obtaining x^* are as follows:

1. Initially, the maximum height in the union is found:

$$\text{hgt}(c_j) = \sup_{x \in X} \mu_{c_j}(x)$$

   where sup is supremum, i.e., the least upper bound.

2. Then the first of maxima is found:

$$x^* = \inf_{x \in X} \{x \in X \mid \mu_{c_j}(x) = \text{hgt}(c_j)\}$$

   where inf is the infimum, i.e., the greatest lower bound.

3. After this the last maxima is found:

$$x^* = \sup_{x \in X} \{x \in X \mid \mu_{c_j}(x) = \text{hgt}(c_j)\}$$

   where
   sup = supremum, i.e., the least upper bound
   inf = infimum, i.e., the greatest lower bound

This is illustrated in Figure 10-10.

From Figure 10-10, the first maxima is also the last maxima, and since it is a distinct max, it is also the mean-max.

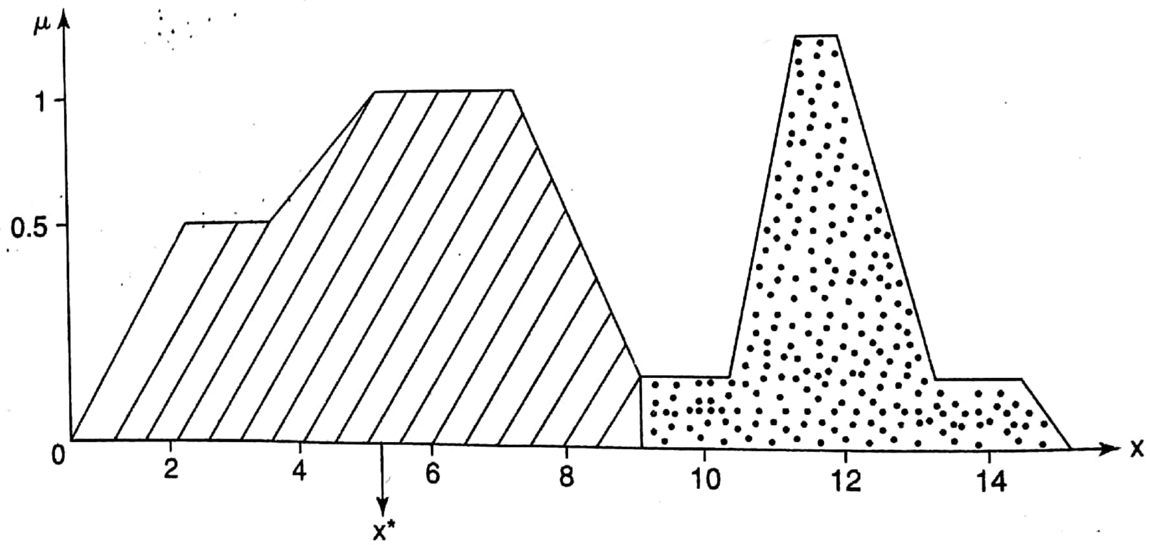

Figure 10-9 Center of largest area method.

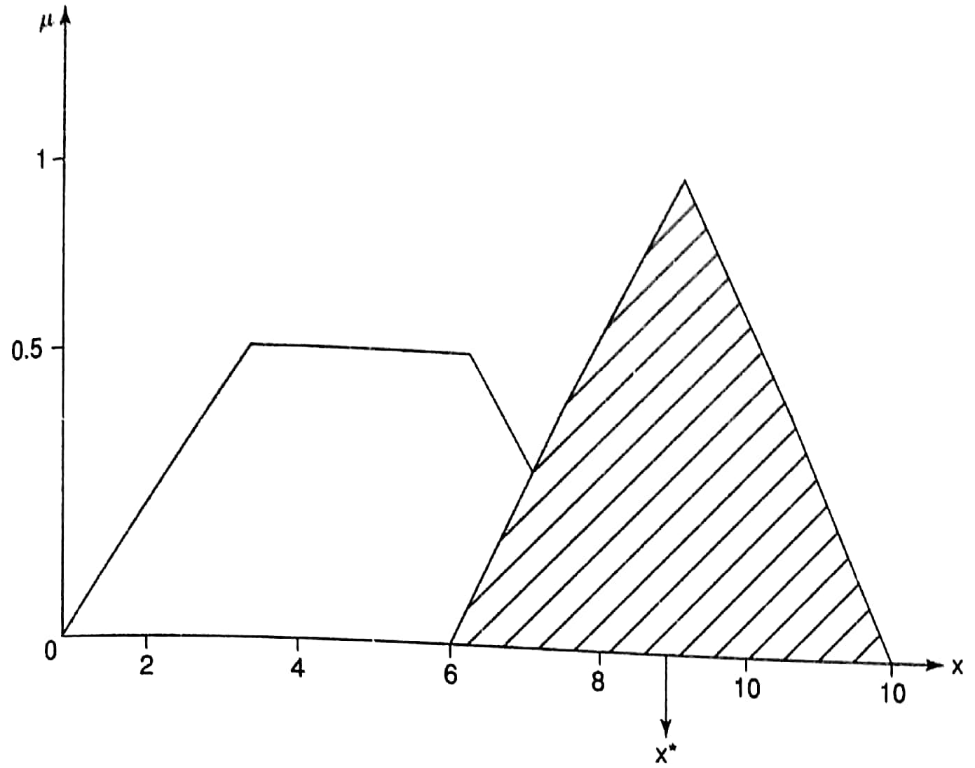

Figure 10-10 First of maxima (last of maxima) method.

10.5 Summary

In this chapter we have discussed the methods of converting fuzzy variables into crisp variables by a process called as defuzzification. Defuzzification process is essential because some engineering applications need exact values for performing the operation. For example, if speed of a motor has to be varied, we cannot instruct to raise it "slightly," "high," etc., using linguistic variables; rather, it should be specified as raise it by 200 rpm or so, a specific amount of raise should be mentioned. Defuzzification is a natural and essential technique. Lambda-cut for fuzzy sets and fuzzy relations were discussed. Apart from the lambda-cut method, seven defuzzification methods were presented. There are analyses going on to justify which of the defuzzification method is the best? The method of defuzzification should be assessed on the basis of the output in the context of data available.

10.6 Solved Problems

1. Consider two fuzzy sets $\underset{\sim}{A}$ and $\underset{\sim}{B}$, both defined on X, given as follows:

$\mu(x_i X)$	x_1	x_2	x_3	x_4	x_5
$\underset{\sim}{A}$	0.2	0.3	0.4	0.7	0.1
$\underset{\sim}{B}$	0.4	0.5	0.6	0.8	0.9