Non-linear Eqs

Previously, we looked at non-linear eqs. with one variable.

For example: $x^3 - 2x - 5 = 0$

which we solved using various root finding methods such as:

1. Bisection method.
2. Secant method.
3. Newton's method

All the above methods are iterative methods.

Let's look at one more approach, called as:

Fixed-point iteration:

this method is suitable for equations expressed in form

$$\boxed{x = g(x)}$$

Solve

Example :-

$$f(x) = x^3 - 2x - 5 = 0 \quad \text{in} \quad x \in [2,3]$$

we rewrite $f(x) = 0$ in the form $x = g(x)$

one possible way

$$x = g_1(x) = \frac{x^3 - 5}{2} \quad \text{—} \quad (1)$$

another

$$x = g_2(x) = (2x+5)^{1/3} \quad \text{—} \quad (2)$$

the iterative step is given as:

$$x^{(n+1)} = g_1(x^{(n)})$$

nth $\rightarrow$ iteration

$n = 0, 1, 2, 3.$

similary

$$x^{(n+1)} = g_2(x^{(n)})$$

Table

$n+1$	$g_1(x^{(n+1)})$	$g_2(x^{(n+1)})$
1	1.500	2.080
2	−0.812	2.092
3	−2.768	2.094
4	−13.106	2.094
5	−1128.124	2.094

We see, that series $g_1(x)$ diverges.

whereas, $g_2(x)$ converges to 2.094

which is a good approx. of the root of

$$x^3 - 2x - 5 = 0$$

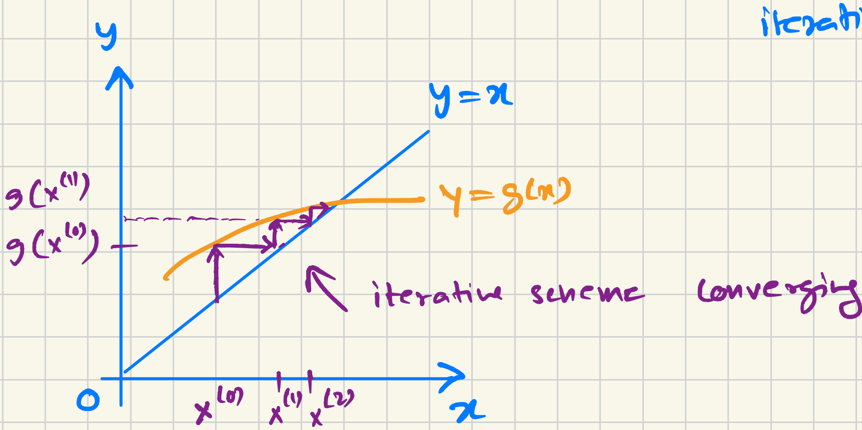
Therefore, iterative scheme

$$x^{(n+1)} = g(x^{(n)})$$

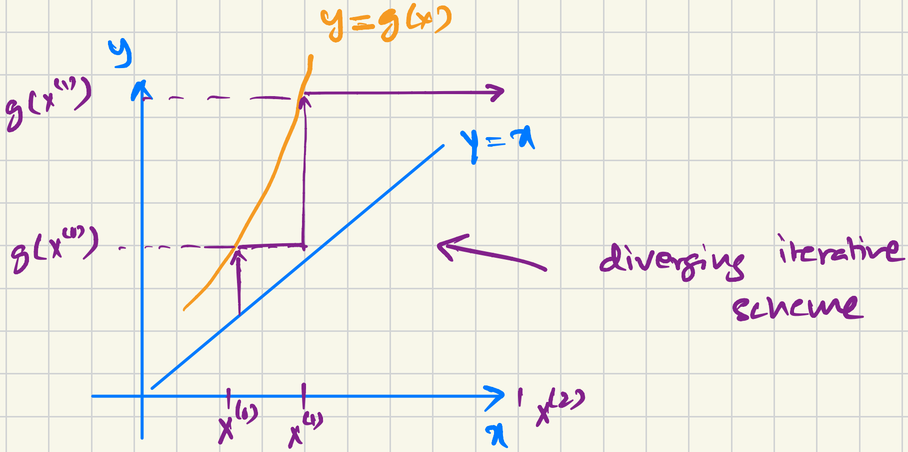
starting, with initial guess $x^{(0)}$

converges to a root 'α'

such, that $\alpha = g(\alpha)$. The scheme is called fixed-point iteration.



" Contraction mapping "



Que How to choose $g(x)$?

Theorem :- Fixed-point iteration converges for a choice of $g(x)$, such that

$$|g'(\alpha)| < 1 \qquad \text{where } \alpha \text{ is the root}$$

or more generally

$$|g'(x)| \le k < 1 \qquad \forall\ x \in [a,b]$$

where $k > 0$

Then $g(x)$ has a unique fixed point $\alpha \in [a,b]$

Note :- $g(x)$ is continuous and differentiable $\forall\ x \in [a,b]$

Note :- Newton Raphson Method (or just Newton's method)

iterative scheme is given as :-

$$x^{(n+1)} = x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})}$$

special case of fixed-point method, with

$$g(x^{(n)}) = x^{(n)} - \frac{f(x^{(n)})}{f'(x^{(n)})}$$

Now consider :- we need to find roots of a non-linear system given by

$$\left.\begin{aligned} x_1^2 + x_2^2 - 1 &= 0 \quad -2.1 \\ x_2 - 3x_1^2 + 2 &= 0 \quad -2.2 \end{aligned}\right\} \text{2 variable problem.}$$

One approach is simple algebraic manipulation.

from 2.2 $\quad x_1^2 = \dfrac{(x_2+2)}{3}$

Substituting in Eq 2.1

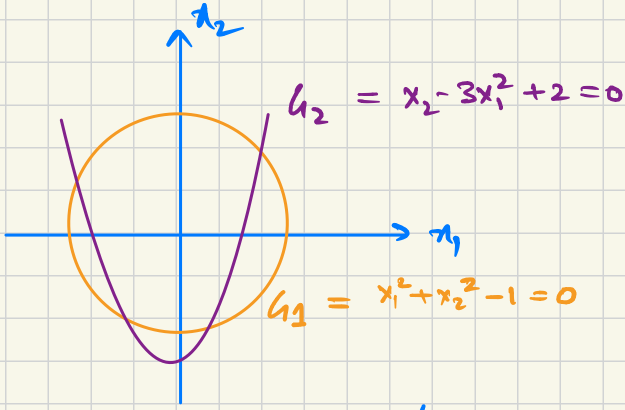
$$\frac{x_2+2}{3} + x_2^2 = 1$$

$$\Rightarrow \quad 3x_2^2 + x_2 - 1 = 0$$

<u>roots</u> $\quad x_2 = -0.767$

$\quad\quad\quad = 0.434$ | then get x_1

Alternatively, can solve using the graph.



Intersection gives = solutions/roots.

However, how to solve them numerically ?

Before we dive into numerical approach.

Some definitions :-

$f \rightarrow$ non-linear function of n independent variables $x_1, x_2, \ldots, x_n$.

set of "n" eqns

$$f_1(x_1, x_2, \ldots, x_n) = 0$$

$$f_2(x_1, x_2, \ldots, x_n) = 0$$

$$\vdots$$

$$f_{n-1}(x_1, x_2, \ldots, x_n) = 0$$

$$f_n(x_1, x_2, \ldots, x_n) = 0$$

We define

$$F(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}$$

$F(x)$ is n-dimensional vector consisting of components f_i

Suppose $F(x)$ is given, and is continuous & differentiable then, we define Jacobian matrix of F as:

$$J_F(X) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}$$

is simply the matrix of first partial derivatives of its component functions $f_i(x)$.

Example:-

$$f_1(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 + 5$$

$$f_2(x_1, x_2, x_3) = x_1 - 3x_2 + x_3^2 - 1$$

$$f_3(x_1, x_2, x_3) = -x_1^3 + 3x_1^2 x_2$$

$$J_F(X) = \begin{bmatrix} 3x_1^2 & 3x_2^2 & 3x_3^2 \\ 1 & -3 & 2x_3 \\ -3x_1^2 + 6x_1x_2 & 3x_1^2 & 0 \end{bmatrix}$$

and

$J_F(x)$ for $x = (x_1, x_2, x_3) = (-1, 2, 0)$

$$J_F(-1,2,0) = \begin{bmatrix} 3 & 12 & 0 \\ 1 & -3 & 0 \\ -15 & 3 & 0 \end{bmatrix}$$

We can solve the system of Non Linear eq^n using iterative methods learned before.

Example 1: **Fixed-point iteration**

$$f_1(x_1, x_2) = 1 + x_1 - x_2^2 = 0$$

$$f_2(x_1, x_2) = x_2 - x_1^3 = 0$$

let's express them in form

$$x_1 = g_1(x_1, x_2) = x_2^2 - 1$$

$$x_2 = g_2(x_1, x_2) = x_1^3$$

<u>Assume</u> $(x_1^{(0)}, x_2^{(0)}) = (1.5, 1.5)$

<u>Table</u>

n	x_1	x_2
0	1.5	1.5
1	1.25	3.37
2	10.39	1.95
3	2.81	1121.82

<u>Iterations do not converge.</u>

<u>Alternatively</u> :-

$(x_1^{(0)}, x_2^{(0)}) = (1.5, 1.5)$

$$x_1 = x_2^{1/2} = g_1(x_1, x_2)$$

$$x_2 = \sqrt{1+x_1} = g_2(x_1, x_2)$$

<u>Table</u>

K	x_1	x_2
0	1.500	1.500
1	1.144	1.581
2	1.164	1.464
3	1.135	1.471
⋮		
9	1.134	1.461

Theorem

$$G(x) = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix}$$

$$J_G(x) = \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} \\ \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} \end{bmatrix}$$

let α be fixed-point of $G(x)$, & $G(x)$ is continuously differentiable in $\alpha \in [a,b]$

$$\| J_G(\alpha) \|_\infty < 1$$

$\infty \to$ norm

then for $x^{(0)}$ {initial guess}, iterative scheme will converge.

iterative scheme :-

$$x_1^{(n+1)} = g_1\left(x_1^{(n)}, x_2^{(n)}\right)$$

$$x_2^{(n+1)} = g_2\left(x_1^{(n)}, x_2^{(n)}\right)$$

$\Rightarrow$ fixed-point iterative method can be accelerated using Gauss-Seidel approach.

$$x_1^{(n+1)} = g_1\left(x_1^{(n)}, x_2^{(n)}\right)$$

$$x_2^{(n+1)} = g_2\left(x_1^{(n+1)}, x_2^{(n)}\right)$$

---------- x ----------

<u>Example 2</u> :- Newton's method.

<u>iterative scheme</u> :-

$$x^{(n+1)} = x^{(n)} - \left[J_f(x^{(n)})\right]^{-1} F(x^{(n)})$$

<u>implementation in code</u> :-

$$x^{(n+1)} = x^{(n)} + dx^{(n)}$$

$$dx^{(n)} = -\left[J_f(x^{(n)})\right]^{-1} F(x^{(n)})$$

$$\Rightarrow \underline{J_f(x^{(n)}) \cdot dx^{(n)} = f(x^{(n)})} \quad \left[\begin{array}{c} Ax=b \\ \text{form} \end{array}\right]$$

Other methods used to solve system of non-linear Eq^s.

- Quasi-newton methods
- Descent techniques.

Quasi-newton approaches :-

Example - approach 1 :- Use newton's approach but evaluate Jacobian matrix once every 'n' ($n = 2, 3, \ldots$) iterations.

- approach 2 :- $x^{(n+1)} = x^{(n)} + \beta \Delta x^{(n)}$

$\beta = 1$ { newton approach }

howeveve; $\beta < 1$ { for certain system help with convergence }

- Broyden's method :- Inspired secant approach that we discussed for finding roots of single variable non-linear eq^s.

Among descent techniques (popular for optimization problem)

a. steepest descent

b. Conjugate gradient.