Markov Property

When a system has markov property associated with a state then we can say that the state transition based on action on the state depends only on the current state of the system an it doesn't depend entire state, action, reward ~~[illegible]~~ trajectory in the past

Ex- Chess Game

In Reinforcement Learning, the goal is the agent has to take action in the environment where the states are markovian that means it will just look into the current state of the system and it will take an action and the goal of the agent is to increase the total reward of the system.

The Bellman Equation

The Bellman equation can be written as :

$$V(s) = \max [R(s, a) + \gamma V'(s')]$$

where,

$V(s)$ = value calculated at a particular point

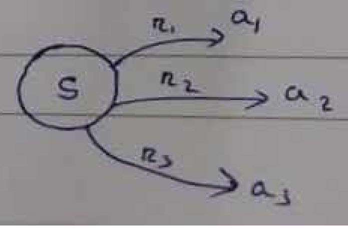
$R(s, a)$ = Reward at particular state s by performing an action

γ = Discount Factor

$V'(s)$ = The value of the previous state

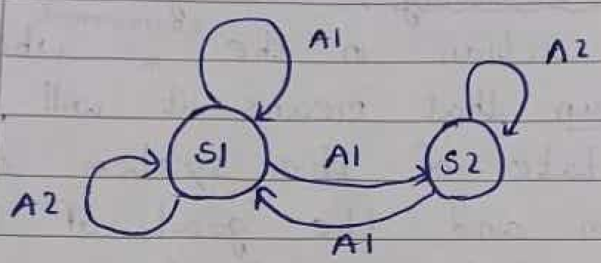
→ Value Iteration :

$$V(s) = \max_a \sum P(s', r | s, a) [r + \gamma V_{old}(s')]$$



Q) Given the following MDP with 2 states ($S1$ & $S2$) and 2 actions ($A1$ & $A2$).

State	Action	Next State	Reward	Prob ~~Rob~~
S1	A1	S1	4	0.5
S1	A1	S2	2	0.5
S2	A2	S1	3	1
S2	A1	S1	0	1
S2	A2	S2	1	1



Assume a discount vector $\gamma = 0.9$

Initialize $V(S1) = V(S2) = 0$

a) Perform one iteration of the value iteration algorithm.

b) What is the best action in S1 after one iteration.

Ans- For state S1 :

Action AI : $Q(S1, A1) = 0.5[4 + 0.9 \times 0]$
$+ 0.5[2 + 0.7 \times 0] = 3$

~~For state S2~~ :

Action A2 : $Q(S1, A2) = 1[3 + 0.9 \times 0] = 3$

$V(S1) = \max\{3, 3\} = 3$

___/___/______

Policy Iteration

$$v(s) = \sum_{s', r} P(s', r \mid s, \pi(s))\left[r + \gamma v_{old}(s')\right]$$

; $s' \to$ next state ; $\pi \to$ policy

Q) You are given an MDP with states : $\{s_1, s_2\}$ Action = $\{a\}$. Transition probability :

$P(s_1 \mid s_1, a) = 0.6$ reward = 2
$P(s_2 \mid s_1, a) = 0.4$ reward = 3
$P(s_1 \mid s_2, a) = 1$ reward = 1

The policy π always select action 'a', discount factor $\gamma = 0.8$. Initial values $v(s_1) = v(s_2) = 0$
Perform two iterations of policy evaluation and find the value of $v(s_1)$ and $v(s_2)$ after two iteration

Ans - 1st iteration
$$v_1(s_1) = 0.6\left[2 + 0.8 \times 0\right] + 0.4\left[3 + 0.8 \times 0\right] = 2.4$$
$$v_1(s_2) = 1\left[1 + 0.8 \times 0\right] = 1$$

$$v_2(s_1) = 0.6\left[2 + 0.8 \times v_1(s_1)\right] + 0.4\left[3 + 0.8 \times v_1(s_2)\right]$$
$$= 0.6\left[2 + 0.8 \times 2.4\right] + 0.4\left[3 + 0.8 \times 1\right]$$
$$= 3.872$$

$$v_2(s_2) = 1\left[1 + 0.8 \times v_1(s_1)\right] = 1\left[1 + 0.8 \times 2.4\right] = 2.92$$

Chapter -13
AI and Partial Differential Equation

09/05/2025

Differential Equation

A differential equation is an equation that relates one and more unknown functions and their derivation / derivatives

$$\frac{dy}{dx} = f(x)$$; $x \Rightarrow$ Independent variable
$y \Rightarrow$ Dependent variable

★ ⇒ Ordinary Differential Equation

An ODE involves only one independent variable and it's derivatives.

General Formula : $F(x, y, y', y'' \dots y^n) = 0$

★ ⇒ Partial Differential Equation

PDE involves function of several variables along with any of it's partial derivatives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$x, y \Rightarrow$ Independent variable
$u \Rightarrow$ Dependent variable

Analytical Technique / Solution :

1) Separation of variables
2) Method of variables
3) Fundamental solution
4) Changes of variable and many more

Numerical Method :

1) Finite element method
2) Finite volume method
3) Finite difference method

Discretization Technique

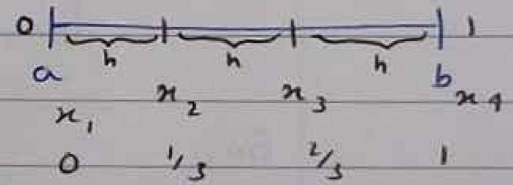
Machines like to compute functions on discrete case, so we need to discrete our continum equations. We can discreteze the differential equation using one of these method :

① Finite Method
② Finite Element
③ Variation on Energy method
④ Monte Carlo Method

> **Finite Method** : We use finite difference to numerically approximate the derivation / derivatives of the function that appears in PDE's.

In finite difference approximations, we replace the derivation/derivates with linear combination of function values at discrete points in the domain. Now, let's discretize one-dimenstional interal $[a, b]$ then write down finite difference approximation for $f(x)$ defined over the interval $[a, b]$.

Here, mesh size $h = \frac{b-a}{n}$

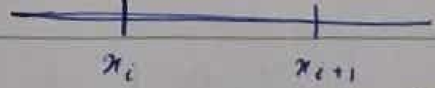


We can now, evaluate "f" at any of these $n+1$ discrete points

x_1, x_2, x_3, x_4 = points
a, b = interval

So, for some x_i,

$$f_{i+1} = f(x_i + h)$$
$$f_{i+2} = f(x_i + 2h)$$
$$f_{i-1} = f(x_i - h)$$



$$\frac{dy}{dx} = \frac{f(x+h) - f(x)}{h}$$

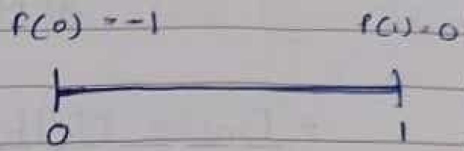
1) Forward difference approximation : $f'(x_i) \approx \frac{f_{i+1} - f_i}{h}$

2) Backward difference approximation : $f'(x_i) \approx \frac{f_i - f_{i-1}}{h}$

3) Central difference approximation:

$$f'(x) \approx \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2}$$

Q) Solve $y''(x) = 1$ on $[0, 1]$ with boundary condition $y(0) = -1$ and $y(1) = 0$



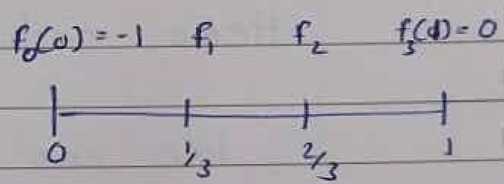
Integration

$$y''(x) = 1$$
$$y'(x) = x + c$$
$$y(x) = \frac{x^2}{2} + c_1 x + c_2$$

$x = 0,\ y = -1$: $\boxed{-1 = c_2}$

$x = 1,\ y = 0$: $0 = \frac{1}{2} + c_1 - 1 \Rightarrow \boxed{c_1 = \frac{1}{2}}$

Hence; $\boxed{y(x) = \frac{x^2}{2} + \frac{x}{2} - 1}$



So,

$$y''(x_i) \approx \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$$

Let $h = 1/3$

$x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$

$$\Rightarrow 1 = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$$

$\Rightarrow f_{i+1} - 2f_i + f_{i-1} = h^2$

$\Rightarrow f_2 - 2f_1 + f_0 = h^2$

$\Rightarrow f_2 - 2f_1 - 1 = \left(\frac{1}{3}\right)^2 \Rightarrow f_2 - 2f_1 = \frac{10}{9} \quad \ldots (1)$

__/__/__

Similarly; $f_3 - 2f_2 + f_1 = h^2$

$\Rightarrow 0 - 2f_2 + f_1 = \left(\frac{1}{3}\right)^2 \Rightarrow -2f_2 + f_1 = \frac{1}{9}$ ..... (2)

{Solve for f_1 & f_2}

Date : 15/05/2025

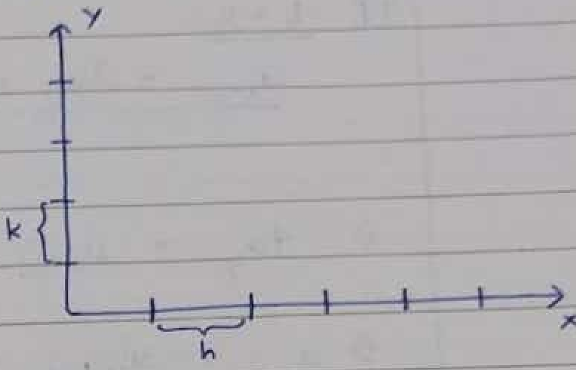
PDE

discretize

$u(x, y)$ $\quad$ $x, y \to$ Independent

$u \to$ Dependent

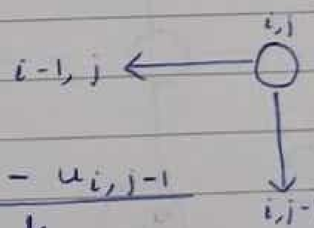
$$x_i \nearrow \; u_{ij} \; \nwarrow y_j$$



These are for Forward

So; $$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j}}{k}$$



Backward Difference

Also; $$\frac{\partial u}{\partial x} = \frac{u_{i,j} - u_{i-1,j}}{h}$$

$$\frac{\partial u}{\partial y} = \frac{u_{i,j} - u_{i,j-1}}{k}$$

Similarly; $$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2}$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j-1}}{2h}$$

Date :

Q) Discretize the laplace equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

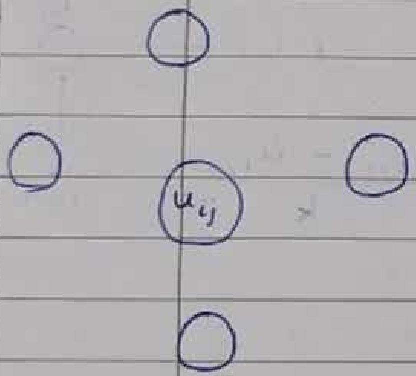
Ans - $$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{k^2} = 0$$

If $h = k$:

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j} + u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{h^2} = 0$$

$$\Rightarrow 4u_{ij} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}$$

$$\Rightarrow u_{ij} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j+1}}{4} \quad \text{(Ans)}$$



⇒ 'u_{ij}' depends on 4-points , if their values are given we can find the actual values.

Q) Discretize the one dimensional heat equation $U_t = \alpha U_{xx}$ in the interval $x \in (0,1)$

Here; $U_{xx} = \frac{\partial^2 u}{\partial x^2}$ $\quad U_t = \frac{\partial u}{\partial t}$ → 1st order forward diff.

$x \Rightarrow i$ $\quad t \Rightarrow j$

↳ h $\quad k$

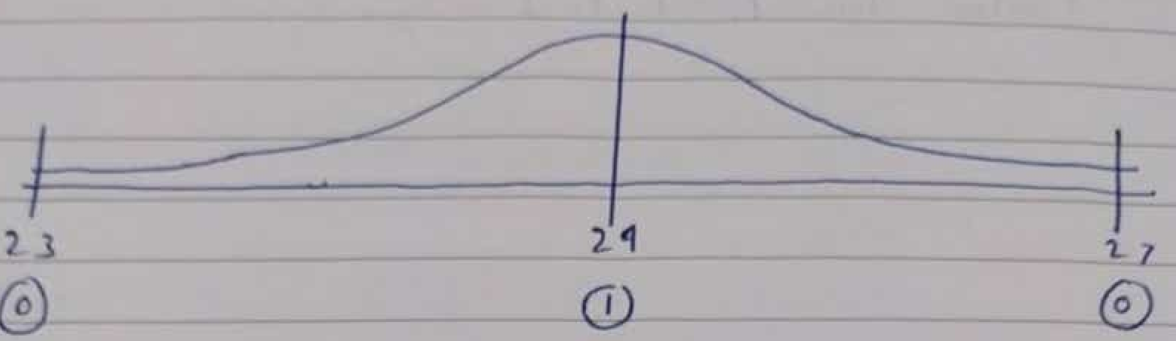
So,

$$\frac{u_{i,j+1} - u_{ij}}{k} = \alpha \frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2}$$

16/05/2025

Fuzzy Logic

Here a single value is not appropriate.
So, we use a "range"



AC (automated) temperature

"The chance of raining tomorrow." → Probability
"The quality of rain" → Fuzzy Logic

Fuzzy Logic

Fuzzy logic is a mathematical approach to reasoning that mimics human decision-making by handling uncertainty and partial truth.

⇒ Work with degrees of truth rather than binary partial truths (true/false) logic.

⇒ Uses linguistic variables like "warm", "hot" and "cold"

⇒ Fuzzy logic is fundamental to AI systems requiring approximate reasoning such as control system.

Some industry oriented use-cases of Fuzzy logic

1) Air Conditions
2) Face Pattern Recognition
3) Vacuum Cleaners
4) Transmission System
5) Control of Subway System

Difference between Probability and Fuzzy Logic

Fuzzy Logic	Probability
1) Trying to understand the concepts of vagueness	1) Main association with events and to check wethear the event will occur or not
2) This captures the meaning of partial truth	2) This captures partial knowledge.
3) Here, everything is a matter of degree.	3) It is specific with the range between 0 and 1
4) Used by quatitative analysts for improvisation of the algorithm	4) Not capable to capture any type of uncertainties