Project 1 Numerical Methods

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Problem 3

Consider the following function $f(x)$

$$f(x) = \sqrt{x+1}$$

Using Taylors Theorem we know

$$f(x) = P_n(x) + R_n(x)$$

where $P_n(x)$ and $R_n(x)$ are defined as

$$P_n(x) = \sum_{k=0}^{n} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$$

$$R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x)$$

where $x > \xi_x > x_0$

Given $x_0 = 0$ and $n = 1$ these formulas simplify to

$$P_1(x) = \sum_{k=0}^{1} \frac{x^k}{k!} f^{(k)}(0) = \frac{x^0}{0!} f(0) + \frac{x^1}{1!} f'(0) = f(0) + x f'(0)$$

$$R_1(x) = \frac{x^2}{2!} f^{''}(\xi_x) = \frac{x^2}{2} f''(\xi_x)$$

Where $x > \xi_x > 0$

Recall that f(x) = $\sqrt{x+1}$

$$f(x) = \sqrt{x+1}$$

$$f'(x) = \frac{1}{2\sqrt{x+1}}$$

$$f''(x) = -\frac{1}{4\sqrt{(x+1)^3}}$$

This means

$$P_1(x) = \sqrt{0+1} + \frac{x}{2\sqrt{0+1}} = 1 + \frac{x}{2}$$

$$R_1(x) = \frac{x^2}{2}\left(-\frac{1}{4\sqrt{(\xi_x+1)^3}}\right) = -\frac{x^2}{8\sqrt{(\xi_x+1)^3}} = O\left(\frac{x^2}{\sqrt{(\xi_x+1)^3}}\right)$$

Since $\frac{1}{\sqrt{(\xi_x+1)^3}}$ is a decreasing function from $(-1,\infty)$ and $\xi_x \in [0,\infty)$ the maximum of this function is at $\xi_x = 0$ and therefore bounded at that point

$$\frac{1}{\sqrt{(0+1)^3}} = 1$$

$$O(x^2 * 1) = O\left(\frac{x^2}{\sqrt{(\xi_x+1)^3}}\right)$$

$$O(x^2) = O(x^2 * 1)$$

Therefore, by plugging into Taylors Theroem we get

$$\sqrt{x+1} = 1 + \frac{x}{2} + O(x^2)$$

Problem 4

Consider the following function $f(x)$

$$f(x) = \frac{1}{x+1}$$

Using Taylors Theorem with $x_0 = 0$ and $n = 2$

$$P_2(x) = \sum_{k=0}^{2} \frac{x^k}{k!} f^{(k)}(0) = \frac{x^0}{0!}f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) = f(0) + xf'(0) + \frac{x^2}{2}f''(0)$$

$$R_2(x) = \frac{x^3}{6}f'''(\xi_x)$$

Using $f(x) = \frac{1}{x+1}$ we know

$$f(x) = \frac{1}{x+1}$$

$$f'(x) = -\frac{1}{(x+1)^2}$$

$$f''(x) = \frac{2}{(x+1)^3}$$

$$f'''(x) = -\frac{6}{(x+1)^4}$$

Using this we know

$$P_2(x) = 1 - x + \frac{x^2}{2} * 2 = 1 - x + x^2$$

$$R_2(x) = \frac{x^3}{6} * -\frac{6}{(\xi_x + 1)^4} = O(\frac{x^3}{6} * -\frac{6}{(\xi_x + 1)^4}) = O(\frac{x^3}{(\xi_x + 1)^4})$$

As, $O(\frac{1}{(\xi_x+1)^4})$ is an decreasing function from $(-1, \infty)$ and $\xi_x \in [0, \infty)$ the maximum value is at $\xi_x = 0$

$$\frac{1}{(0+1)^4} = 1$$

$$O(x^3) = O(x^3 * 1) = O(\frac{x^3}{(\xi_x + 1)^4})$$

Therefore, using Taylors Theroem we know

$$\frac{1}{x+1} = 1 - x + x^2 + O(x^3)$$

Problem 5

Consider the following function $f(x)$

$$f(x) = \sin(x)$$

Using Taylors Theorem with $x_0 = 0$ and $n = 0$

$$f(x) = P_0(x) + R_0(x)$$

$$P_0(x) = \frac{x^0}{0!} f(0) = f(0) = \sin(0) = 0$$

$$R_0(x) = \frac{(x-0)^1}{(0+1)!} f'(\xi_x) = x f'(\xi_x)$$

Using the original statement we know

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

We then know,

$$R_0(x) = x \cos(\xi_x)$$

$$R_0(x) = O(x\cos(\xi_x))$$

Since the function $\cos(x)$ is bounded between [-1,1] the function $R_0(x)$ can be represented as

$$R_0(x) = O(x * 1) = O(x)$$

Since, we know for $\forall x > 1, x^3 > x$ This can then be used to say $O(x^3) > O(x)$ This means

$$R_0(x) = O(x^3)$$

Therefore,

$$\sin(x) = 1 + O(x^3)$$

Problem 6

Consider the following function $f(x)$

$$f(x) = \sum_{k=0}^{n} r^k$$

Using the given sumation formula we know

$$f(x) = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} = \frac{O(1 - r^{n+1})}{1 - r}$$

Since we know the $\max(1, -r^{n+1}) = -r^{n+1}$ we can use definition of $O()$ to remove 1 meaning

$$\frac{O(1 - r^{n+1})}{1 - r} = \frac{O(-r^{n+1})}{1 - r}$$

Since, $O()$ allows for $\forall$ coefficient C multiplying current C by -1 will allow us to remove coefficeints Therefore,

$$\frac{O(-r^{n+1})}{1 - r} = \frac{O(r^{n+1})}{1 - r}$$

Problem 15