Project 2 Numerical Methods

Dustin O'Brien

September 10, 2024

Problem 24

Given,

$$\ln(1+x), x \in [-\frac{1}{2}, \frac{1}{2}]$$

Use Mean Value Theorem to find a value M s.t.

$$f(x_1) - f(x_2) \leq M|x_1 - x_2|$$

$$f(x) = \ln(1+x)$$

using MVT. $f'(\xi_x) = \frac{f(x_1)-f(x_2)}{x_1-x_2}$ where $\xi_x \in [-\frac{1}{2}, \frac{1}{2}]$

$$f'(\xi_x)(x_1 - x_2) = f(x_1) - f(x_2)$$

$$f(x_1) - f(x_2) = f'(\xi_x)(x_1 - x_2)$$
$$f(x_1) - f(x_2) \leq f'(\xi_x)|x_1 - x_2|$$
$$f(x) = \ln(1+x)$$
$$f'(x) = \frac{1}{1+x}$$

Since $\frac{1}{1+x}$ is a decreasing function on the intervals $[-\frac{1}{2}, \frac{1}{2}]$ therefore the maximum must be the smallest term in our interval $-\frac{1}{2}$ must be the input that give the maximum output

$$f'(-\frac{1}{2}) = \frac{1}{1+(-\frac{1}{2})}$$

$$f'(-\frac{1}{2}) = \frac{1}{\frac{1}{2}}$$

$$f'(-\frac{1}{2}) = 2$$

Therefore the maximum output of $f'(x) = 2$

$$f(x_1) - f(x_2) \leq 2|x_1 - x_2|$$

Which is what you asked for :)

Problem 25

A function is monotone if on an interveral the derivative is strictly positive or negative suppose f is continous and monotone on interval [a,b] and $f(a)f(b) < 0$ prove there is exactly one value $\alpha \in$ [a,b] s.t. f(α)=0

Consider a function $f(x)$ that meets the defintions of the function in the question

Since, $f(a) * f(b) < 0$ one function $f(a)$ or $f(b)$ must be negative and the other positive as if they were both positive the product would be positive and if both were negative it would also be positive.

Hence, either

$$f(a) \leq 0 \leq f(b)$$

$$f(b) \leq 0 \leq f(a)$$

must be true Using intermediate value theorem there must exist a point α s.t. $f(\alpha) = 0$

Problem 3

Use Taylor Series to show

$$\sqrt{1+x} = 1 - x + x^2 + O(x^3)$$

for x sufficently small

Let $f(x) = \sqrt{1+x}$

$$f(x) = \sqrt{1+x}$$

$$f'(x) = \frac{1}{2\sqrt{1+x}}$$

$$f''(x) = -\frac{1}{4\sqrt{1+x}^3}$$

Using Taylor Theroem we can derive at $x_0 = 0$

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{x^2}{(3)!} * -\frac{1}{4\sqrt{1+\xi_x}^3}$$

$$\sqrt{1+x} \leq 1 + \frac{1}{2}x + |\frac{x^2}{3!} * -\frac{1}{4\sqrt{1+\xi_x}^3}|$$

$$\sqrt{1+x} \leq 1 + \frac{1}{2}x + |\frac{x^2}{3!} * \frac{1}{4\sqrt{1+\xi_x}^3}|$$

As the function $\frac{1}{4\sqrt{1+\xi_x}^3}$ is a decreasing function on the intervals $\xi_x \in [-1, \infty)$ and $\xi_x \in [0, x]$ the maximum value of $\frac{1}{4\sqrt{1+\xi_x}^3}$ is at $\xi_x = 0$

$$\frac{1}{4\sqrt{1+0}^3} = \frac{1}{4}$$

To continue

$$\sqrt{1+x} \leq 1 + \frac{1}{2}x + |\frac{x^2}{3!} * \frac{1}{4}|$$

$$\sqrt{1+x} \leq 1 + \frac{1}{2}x + |\frac{x^2}{12}|$$

$$\sqrt{1+x} \leq 1 + \frac{1}{2}x + O(\frac{x^2}{12})$$

Since $\frac{1}{12} > 0$ it can be declared the C and $x^2 = \beta(x)$ and using defintions of O()

$$\sqrt{1+x} \leq 1 + \frac{1}{2}x + O(x^2)$$

Problem 6

Recall that

$$1 + r + r^2 + r^3 ... r^n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

Prove that

$$\sum_{k=0}^{n} r^k = \frac{1}{1-r} + O(r^{n+1})$$

$$1 + r + r^2 + r^3 ... r^n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

$$\frac{1 - r^{n+1}}{1 - r} \leq \frac{1 + | - r^{n+1}|}{1 - r}$$

$$\frac{1-r^{n+1}}{1-r} \leq \frac{1+|r^{n+1}|}{1-r}$$
$$\frac{1-r^{n+1}}{1-r} \leq \frac{1+O(r^{n+1})}{1-r}$$
$$\frac{1-r^{n+1}}{1-r} \leq \frac{1}{1-r} + \frac{O(r^{n+1})}{1-r} \leq \frac{1}{1-r} + O(r^{n+1})$$

Using defintion of O()

$$\frac{1}{1-r} + O(r^{n+1}) = \frac{1-r^{n+1}}{1-r}$$

Therefore,

$$1 + r + r^2 + r^3 ... r^n = \frac{1}{1-r} + O(r^{n+1})$$

Problem 8

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}$$

Use this to show

$$\sum_{k=0}^{n} k = \frac{1}{2}n^2 + O(n)$$

$$\frac{n(n+1)}{2} = \frac{n^2+n}{2}$$
$$\frac{n^2+n}{2} = \frac{n^2}{2} + \frac{n}{2}$$
$$\frac{n^2}{2} + \frac{n}{2} = \frac{n^2}{2} + O(\frac{n}{2})$$

Using the definition of O() at C = $\frac{1}{2}$ we know

$$\frac{n^2}{2} + O(\frac{n}{2}) = \frac{n^2}{2} + O(n)$$

Therefore,

$$\sum_{k=0}^{n} k = \frac{1}{2}n^2 + O(n)$$