Project 3 Numerical Methods

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Problem 3c

Python Code

```
#Allows usage of math functions
import math
#Allows User to type in as Fraction
from fractions import Fraction

#Function f(x)
def f(x):
return math.sin(math.pi * x)

#Takes in user input uses Fraction to
    convert fraction into something usable
    than converts that into a decimal
x = float(Fraction(input("Please Input
    Your x\n")))
h = float(Fraction(input("Please Input
    Your h\n")))

#Formulas for calculating these things
    Based on Forward Difference
approx1 = (f(x + h) - f(x)) / h
error1 = (math.pi ** 2 * h) / 2

#Formulas for calculating Based on Central
     Difference
approx2 = (f(x + h) - f(x - h)) / (2 * h)
error2 = (math.pi ** 3 * h ** 2) / 6

#Outputs
print("Forward Difference Approximation: "
    , approx1, " Error: ", error1)
```

```
        print("Central Difference Approximation :
          ", approx2, " Error: ", error2)
```

Problem 6a

Consider the following equation where A, B, C are all constants

$$Af(x) + Bf(x+h) + Cf(x+2h) = f'(x)$$

Using Taylors Theroem we know

$$f(x+h) = f(x) + hf'(x) + \frac{h^2 f''(\xi_1)}{2}$$

$$f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(\xi_2)$$

Thus,

$$Af(x)+Bf(x+h)+Cf(x+2h) = Af(x)+Bf(x)+Bhf'(x)+\frac{Bh^2 f''(\xi_1)}{2}+Cf(x)+2Chf'(x)+2Ch^2 f''(\xi_2)$$

$$Af(x)+Bf(x+h)+Cf(x+2h) = Af(x)+Bf(x)+Cf(x)+Bhf'(x)+2Chf'(x)+2Ch^2 f''(\xi_2)+\frac{Bh^2 f''(\xi_1)}{2}$$

$$Af(x)+Bf(x+h)+Cf(x+2h) = Af(x)+Bf(x)+Cf(x)+Bhf'(x)+2Chf'(x)+O(h^2)$$

Since, there is no f(x) in final product and 1 f'(x) we know

$$A + B + C = 0$$

$$Bh + 2Ch = 1$$

$$B = \frac{1 - 2Ch}{h}$$

$$A + \frac{1 - 2Ch}{h} + C = 0$$

$$A + \frac{1 - 2Ch + Ch}{h} = 0$$

$$A + \frac{1 - Ch}{h} = 0$$

$$A = -\frac{1 - Ch}{h}$$

$$A = \frac{Ch - 1}{h}$$

Let $C = 1$

$$A = \frac{h - 1}{h}$$

$$B = \frac{1 - 2h}{h}$$

$$C = 1$$

Problem 8

Consider the function

$$8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)$$

Using Taylors Theroem we know,

$$f(x+h) = f(x)+hf'(x)+h^2f''(x)\frac{1}{2}+h^3f'''(x)\frac{1}{6}+h^4f^{(4)}(x)\frac{1}{24}+h^5f^{(5)}(\xi_1)\frac{1}{120}$$

$$f(x-h) = f(x)-hf'(x)+h^2f''(x)\frac{1}{2}-h^3f'''(x)\frac{1}{6}+h^4f^{(4)}(x)\frac{1}{24}-h^5f^{(5)}(\xi_2)\frac{1}{120}$$

$$f(x+2h) = f(x)+2hf'(x)+4h^2f''(x)\frac{1}{2}+8h^3f'''(x)\frac{1}{6}+16h^4f^{(4)}(x)\frac{1}{24}+h^5f^{(5)}(\xi_3)\frac{1}{120}$$

$$f(x-2h) = f(x)-2hf'(x)+4h^2f''(x)\frac{1}{2}-8h^3f'''(x)\frac{1}{6}+16h^4f^{(4)}(x)\frac{1}{24}-h^5f^{(5)}(\xi_4)\frac{1}{120}$$

Inserting these into the original function we get and simplifying all h^5 terms which are only multiplied by constants to $O(h^5)$ we get

$$8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h) = 12hf'(x) + O(h^5)$$

$$\frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h) + O(h^5)}{12h} = f'(x)$$

$$\frac{8f(x+h) - 8f(x-h) - f(x+2h) + f(x-2h)}{12h} + O(h^4) = f'(x)$$

Problem 11

Looking at graph at bottom of page we notice the staticy error begins to occur around 600 however its worth mentioning that Python doesn't use floating point artihmetic and instead uses arbitrary-precision arthemtic therefor this is a different but likely similar sort of error being caused

Problem 12

Consider the function

$$-2f(x) + f(x+h) + f(x-h)$$

Using Taylor Series we know

$$f(x+h) = f(x) + hf'(x)\frac{1}{2} + h^2 f''(x) + h^3 f'''(x)\frac{1}{6} + h^4 f^{(4)}(\xi_1)\frac{1}{24}$$

$$f(x-h) = f(x) - hf'(x)\frac{1}{2} + h^2 f''(x) - h^3 f'''(x)\frac{1}{6} + h^4 f^{(4)}(\xi_1)\frac{1}{24}$$

Thus,

$$h^2 f''(x)\frac{1}{2} + h^4 f^{(4)}(\xi_1)\frac{1}{24} + h^4 f^{(4)}(\xi_2)\frac{1}{24}$$

$$h^2 f''(x)\frac{1}{2} + h^4(f^{(4)}(\xi_1)\frac{1}{24} + f^{(4)}(\xi_2)\frac{1}{24})$$

$$h^2 f''(x)\frac{1}{2} + O(h^4) = -2f(x) + f(x+h) + f(x-h)$$

$$f''(x) = \frac{-4f(x) + 2f(x+h) + 2f(x-h) + O(h^4)}{h^2}$$

Therefore,

$$f''(x) = \frac{-4f(x) + 2f(x+h) + 2f(x-h)}{h^2} + O(h^2)$$

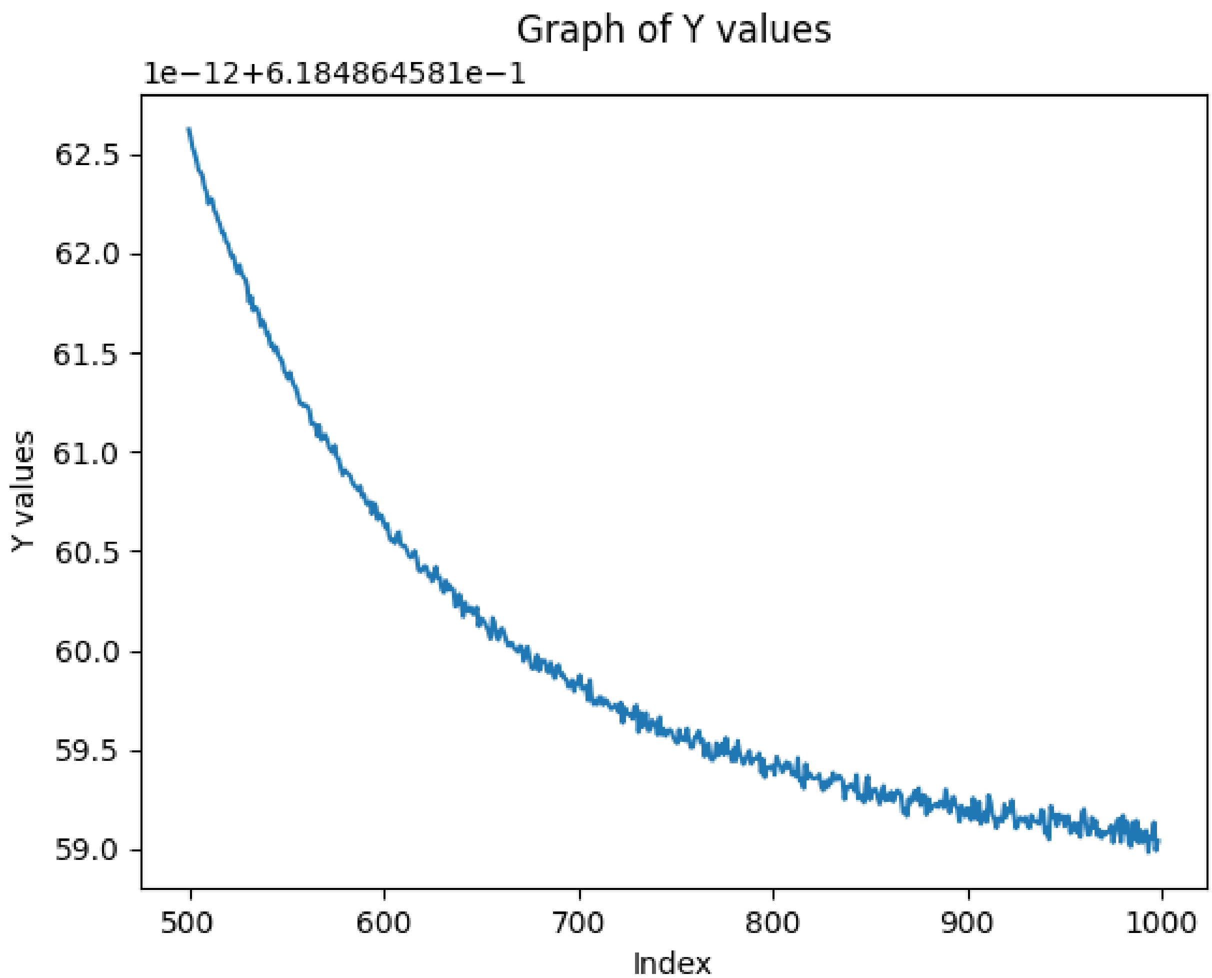

Figure 1: Approximation of derivative arctan.