Project 1 Numerical Methods

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Problem 3

Consider the function $f_x(x)$ where

$$f(x) = \sqrt{1+x^2}$$

Using Taylors Theorem we know

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0)$$

Let n = 6 then

$$P_6(x) = \sum_{k=0}^{6} \frac{x^k}{k!} f^{(k)}(0)$$

Using the defined function we know the following

$$f(x) = \sqrt{1+x^2} = (1+x^2)^{\frac{1}{2}}$$

$$f'(x) = x(1+x^2)^{-\frac{1}{2}}$$

$$f''(x) = (1+x^2)^{-\frac{1}{2}} - x^2(1+x^2)^{-\frac{3}{2}}$$

$$f'''(x) = -3x(1+x^2)^{-\frac{3}{2}} + 3x^3(1+x^2)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = -3(1+x^2)^{-\frac{3}{2}} + 18x^2(1+x^2)^{-\frac{5}{2}} - 15x^4(1+x^2)^{-\frac{7}{2}}$$

$$f^{(5)}(x) = 45x(1+x^2)^{-\frac{5}{2}} - 150x^3(1+x^2)^{-\frac{5}{2}} - 1056x^4(1+x^2)^{-\frac{7}{2}}$$

$$f^{(6)}(x) = 45(1+x^2)^{-\frac{5}{2}} - 675x^2(1+x^2)^{-\frac{7}{2}} + 1575x^4(1+x^2)^{-\frac{9}{2}} - 945x^6(1+x^2)^{-\frac{11}{2}}$$

And $P_6(x)$ expands too

$$P_6(x) = \frac{x^0}{0!}f(0) + \frac{x^1}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(0) + \frac{x^6}{6!}f^{(6)}(0)$$

$$P_6(x) = f(0) + f'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0) + \frac{x^4}{24}f^{(4)}(0) + \frac{x^5}{120}f^{(5)}(0) + \frac{x^6}{720}f^{(6)}(0)$$

Using the previously derived equations this can then be turned into

$$P_6(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24}(-3) + \frac{x^6}{720}45$$

$$P_6(x) = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16}$$

There fore the Taylor Series Polynomial is

$$P_6(x) = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16}$$

Problem 4

Given $R(x) = \frac{|x|^6}{6!}e^{\xi}$ where $0 < \xi < x$ and $x \in [-1, 1]$ find upperbound of $|R(x)|$

$$R(x) = \frac{|x|^6}{6!}e^{\xi}$$

$$|R(x)| = |\frac{|x|^6}{6!}e^{\xi}|$$

since the max value of $|x|$ can only be 1 and $\frac{x}{6!}$ is an increasing function the $|R(x)| \leq |\frac{|1|^6}{6!}e^{\xi}|$

$$|R(x)| \leq |\frac{|1|^6}{6!}e^{\xi}|$$

$$|R(x)| \leq |\frac{1}{720}e^{\xi}|$$

And since the max value of ξ is also 1 and e^{ξ} is also an increasing function $|R(x)| \leq |\frac{1}{720}e^1|$

$$|R(x)| \leq |\frac{1}{720}e^1|$$

$$|R(x)| \leq |\frac{e}{720}|$$

Problem 5

Given $R(x) = \frac{|x|^6}{6!}e^{\xi}$ where $0 < \xi < x$ and $x \in [-\frac{1}{2}, \frac{1}{2}]$ find upperbound of $|R(x)|$

$$R(x) = \frac{|x|^6}{6!}e^{\xi}$$

$$|R(x)| = |\frac{|x|^6}{6!}e^{\xi}|$$

since the max value of $|x|$ can only be $\frac{1}{2}$ and $\frac{x}{6!}$ is an increasing function the $|R(x)| \leq |\frac{|\frac{1}{2}|^6}{6!}e^{\xi}|$

$$|R(x)| \leq |\frac{|\frac{1}{2}|^6}{6!}e^{\xi}|$$

$$|R(x)| \leq |\frac{\frac{1}{64}}{720}e^{\xi}|$$

$$|R(x)| \leq |\frac{1}{16384}e^{\xi}|$$

And since the max value of ξ is also $\frac{1}{2}$ and e^{ξ} is also an increasing function $|R(x)| \leq |\frac{1}{16384}e^{\frac{1}{2}}|$

$$|R(x)| \leq |\frac{1}{16384}e^{\frac{1}{2}}|$$

$$|R(x)| \leq |\frac{e^{\frac{1}{2}}}{16384}|$$

Therefore $|R(x)|$ is bounded by $\frac{e^{\frac{1}{2}}}{16384}$

Problem 6

Given $R(x) = \frac{|x|^4}{4!}e^{\xi}$ where $0 < \xi < x$ and $x \in [-\frac{1}{2}, \frac{1}{2}]$ find upperbound of $|R(x)|$

$$R(x) = \frac{|x|^4}{4!}e^{\xi}$$

$$|R(x)| = |\frac{|x|^4}{4!}e^{\xi}|$$

since the max value of $|x|$ can only be $\frac{1}{2}$ and $\frac{x}{4!}$ is an increasing function the $|R(x)| \leq |\frac{|\frac{1}{2}|^4}{4!}e^{\xi}|$

$$|R(x)| \leq |\frac{|\frac{1}{2}|^4}{4!}e^{\xi}|$$

$$|R(x)| \leq |\frac{\frac{1}{16}}{24}e^{\xi}|$$

$$|R(x)| \leq |\frac{1}{384}e^{\xi}|$$

And since the max value of ξ is also $\frac{1}{2}$ and e^{ξ} is also an increasing function $|R(x)| \leq |\frac{1}{384}e^{\frac{1}{2}}|$

$$|R(x)| \leq |\frac{1}{384}e^{\frac{1}{2}}|$$

$$|R(x)| \leq |\frac{e^{\frac{1}{2}}}{384}|$$

Therefore $|R(x)|$ is bounded by $\frac{e^{\frac{1}{2}}}{384}$

Problem 15

Given The statement

$$\arctan(\frac{1}{239}) = 4\arctan(\frac{1}{5}) - \arctan(1)$$

derive formula for π and Find The order of polynomial s.t. The errors is less than 10^{-100} and 10^{-1000}

$$\arctan(\frac{1}{239}) = 4\arctan(\frac{1}{5}) - \arctan(1)$$

$$\arctan(1) = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239})$$

$$\pi = 16\arctan(\frac{1}{5}) - 4\arctan(\frac{1}{239})$$

Using the Gregory Series we know

$$\arctan(x) = \sum_{k=0}^{n}((-1)^k\frac{x^{2k+1}}{(2k+1)}) + \int_0^x \frac{t^{2n+2}}{1+t^2}dt$$

For simplicity Let their be a function $P_n(x)$ s.t.

$$P_n(x) = \sum_{k=0}^{n}((-1)^k\frac{x^{2k+1}}{(2k+1)})$$

Then,

$$\arctan(x) = P_n(x) + \int_0^x \frac{t^{2n+2}}{1+t^2}dt$$

Since $1+t^2$ is always positive and always greater than 1 as long as $t \geq 0$ than it will as long as $x \geq 0$

$$\arctan(x) \leq P_n(x) + \int_0^x t^{2n+2}dt$$

$$\arctan(x) \leq P_n(x) + \frac{x^{2n+3}}{2n+3}$$

Then,

$$\pi \leq 16(P_n(\frac{1}{5}) + \frac{1}{5^{2n+3}(2n+3)}) - 4(P_n(\frac{1}{239}) + \frac{1}{239^{2n+3}(2n+3)}))$$

$$\pi \leq 16P_n(\frac{1}{5}) + \frac{16}{5^{2n+3}(2n+3)} - 4P_n(\frac{1}{239}) + \frac{4}{239^{2n+3}(2n+3)})$$

$$\pi - 16P_n(\frac{1}{5}) + 4P_n(\frac{1}{239}) \leq \frac{16}{5^{2n+3}(2n+3)} + \frac{4}{239^{2n+3}(2n+3)})$$

$$\pi - 16P_n(\frac{1}{5}) + 4P_n(\frac{1}{239}) \leq |\frac{16}{5^{2n+3}(2n+3)} + \frac{4}{239^{2n+3}(2n+3)})|$$

Let there exist another function $R_n(x)$ s.t.

$$R_n(x) = |\frac{16}{5^{2n+3}(2n+3)} + \frac{4}{239^{2n+3}(2n+3)})|$$

$$\pi - 16P_n(\frac{1}{5}) + 4P_n(\frac{1}{239}) \leq R_n(x)$$

$$\pi - 16P_{71}(\frac{1}{5}) + 4P_{71}(\frac{1}{239}) \leq R_{71}(x) \leq 10^{-101}$$

$$\pi - 16P_{714}(\frac{1}{5}) + 4P_{714}(\frac{1}{239}) \leq R_{714}(x) \leq 10^{-1001}$$

Therefore 71 terms are required to have 100 digits of accuracy and roughlt 714 are needed for 1000 digits of accuracy