1^{st} Isomorphism Theorem Suppose ϕ is a homomorphism from G to H

Define $\psi : G/\ker\phi \longrightarrow \phi(G)$ by $\psi(g \ker\phi) = \phi(g)$ $\forall g \in G$

Then ψ is an isomorphism so $G/\ker\phi \approx \phi(G)$

Well Defined Suppose $x \ker\phi = y \ker\phi$ $\quad x, y \in G$

WTS $\psi(x \ker\phi) = \psi(y \ker\phi)$ $\underset{\text{prop 55 (iv)}}{\Longleftrightarrow}$ $\phi(x) = \phi(y)$

$\Longrightarrow \psi(x \ker\phi) = \psi(y \ker\phi)$

$\therefore$ It is well defined

One To One Suppose $\psi(x \ker\phi) = \psi(y \ker\phi)$

WTS $x \ker\phi = y \ker\phi$ $\rightarrow$ reverse the steps of well defined

$\psi(x \ker\phi) = \psi(y \ker\phi)$ $\quad x, y \in G$

$\Rightarrow \phi(x) = \phi(y)$

$\Leftrightarrow x \ker\phi = y \ker\phi$ $\qquad \therefore \psi$ is one-to-one

Onto Let $y \in \phi(G) = \{\phi(g) \mid g \in G\}$

So $y = \phi(g)$ for some $g \in G$

Then $\psi(g \ker\phi) = \phi(g) = y$ $\qquad \therefore \psi$ is onto

OP Let $x\ker\phi, y\ker\phi \in G/\ker\phi$ so $x, y \in G$

Then $\psi(x \ker\phi \;\; y \ker\phi) = \psi(xy \ker\phi)$ definition of operation in factor group

definition of ψ $= \phi(xy)$

ϕ is OP $= \phi(x)\,\phi(y)$

$= \psi(x \ker\phi)\,\psi(\ker\phi)$

$\therefore \psi$ is OP $\Rightarrow \psi$ is a homomorphism

Proposition 57 Suppose G is a group with normal subgroup H. Then define

$\gamma(a) = aH \quad a \in G$

γ is called the natural homomorphism from G to G/H. It is a homomorphism and $\ker \gamma = H$

Note So any normal subgroup is the kernel of some homomorphism

$$\begin{tikzcd} G \arrow[rr, bend left, "\varnothing"] & & \varnothing(G) \arrow[dl, "\psi"] \\ & G/\ker \psi \arrow[ul, "\gamma"] & \end{tikzcd}$$

Chapter 11

Theorem 58: Fundamental theorem of finite abelian groups

Suppose G is a finite abelian group. Then:

i) G is the direct product of cyclic groups of prime power order

ii) The number of terms in this direct product and the orders of the cyclic groups is uniquely determined by G

This means $G \approx \mathbb{Z}_{p_1 n_1} \oplus \mathbb{Z}_{p_2 n_2} \oplus \dots \oplus \mathbb{Z}_{p_k n_k}$

p_i are primes, $n_i \in \mathbb{Z}^+$

Example 77 Find all abelian groups of order 25 (up to isomorphism)

FTFAG: $G \approx \mathbb{Z}_{p_1 n_1} \oplus \mathbb{Z}_{p_2 n_2} \oplus \dots \oplus \mathbb{Z}_{p_k n_k}$

order is 25 is $p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_k^{n_k} = 25 = 5^2$

So all $p_i = 5$

$5^1 \cdot 5^1 = 25$

$5^2 = 25$

$G \approx \mathbb{Z}_5 \oplus \mathbb{Z}_5$

$G = \mathbb{Z}_{25}$

G must be one of these 2

Example 78 List all abelian groups of order 32 (up to isomorphism)

G Abelian $|G| = 32 = 2^5$

$G \approx \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_1^{n_1}} \oplus \dots \oplus \mathbb{Z}_{p_k^{n_k}}$

$2^5 = |\mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_1^{n_1}} \oplus \dots \oplus \mathbb{Z}_{p_k^{n_k}}| = |\mathbb{Z}_{p_1^{n_1}}| \cdot |\mathbb{Z}_{p_2^{n_2}}| \cdot \ldots \cdot |\mathbb{Z}_{p_k^{n_k}}|$

$= p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_k^{n_k} = 2^5$

$\Longrightarrow p_1 = p_2 = \dots = p_k = 2$

$2^{n_1} \cdot 2^{n_2} \cdot \ldots \cdot 2^{n_k} = 2^5$

$5 \longrightarrow G \approx \mathbb{Z}_{2^5} = 32$

$4 + 1 \longrightarrow G \approx \mathbb{Z}_{16} \oplus \mathbb{Z}_2$

$3 + 2 \longrightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_4$

$3 + 1 + 1 \longrightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$2 + 2 + 1 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$

$2 + 1 + 1 + 1 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

$1 + 1 + 1 + 1 + 1 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Example 79 How many nonisomorphic Abelian groups of order 2400 are there?

$2400 = 2^5 \cdot 3 \cdot 5^2$

$G \approx \mathbb{Z}_{p_1^{n_1}} \oplus \dots \oplus \mathbb{Z}_{p_k^{n_k}}$

$p_i = 2$ or 3 or 5

$G \approx \underbrace{\mathbb{Z}_{2^{n_1}} \oplus \dots \mathbb{Z}_{2^{n_k}}}_{\text{one of the 7 from 78}} \oplus \mathbb{Z}_3 \oplus \underbrace{\quad\quad}_{\text{one of the 2 from 77}}$

So $7 \cdot 2 = 14$ possibilities

Example 80 Suppose G is an abelian group and $|G| = 2400$ and G has an element of order 16, no element of order 32, and exactly 24 elements of order 5. Which group in 79 is G isomorphic to?

$G \approx \underbrace{\mathbb{Z}_{16} \oplus \mathbb{Z}_2}_{78} \oplus \mathbb{Z}_3 \oplus \underbrace{\mathbb{Z}_5 \oplus \mathbb{Z}_5}_{77}$

78 Only one with an element of order 6, from remaining 6 after removing $\mathbb{Z}_{32}$

since $|(x_1, x_2, x_3)| = \text{lcm}(|x_1|, |x_2|, |x_3|)$ no element of order 32 so not $\mathbb{Z}_{32}$

77: $|\mathbb{Z}_{25}| = 25 \quad |\mathbb{Z}_5 \oplus \mathbb{Z}_5| = 25 \to$ all elements except $(0,0)$ has order 5

Example 81 $G = \left\{ A \in GL(2, \mathbb{R}) \mid A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \ a_i = \pm 1 \right\}$

Subgroup of $GL(2, \mathbb{R})$ What is the isomorphism class of G?

$|G| = 4$

Claim G is abelian, Let $A, B \in G$

$AB = \dots = \begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_2 b_2 \end{pmatrix}$

$BA = \dots = \begin{pmatrix} a_1 b_1 & 0 \\ 0 & a_1 b_1 \end{pmatrix} \quad \therefore$ abelian

So $G \approx \mathbb{Z}$

FTFAG $\quad \mathbb{Z} \oplus \mathbb{Z}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ squaring any gives identity

$\implies$ they all have order 2

So $G \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Example 82 Suppose we have a box with dimensions $10\text{cm} \times 20\text{cm} \times 5\text{cm}$

What is the rotational symmetry group of this box?

$$|G| = |\operatorname{orb}(\underset{20}{\square}^{10})| \cdot |\operatorname{stab}(\square)|$$

$$= 2 \cdot 2 = 4$$

So $G \approx \mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$