~Picture~

If $D_4/Z(D_4) \approx Z_4$ then $D_4/Z(D_4)$ would be cyclic so then by the G/Z theorem would mean that D_4 would be abelian!

Contradiction, so $D_4/Z(D_4) \approx Z_2 \oplus Z_2$

Chapter 10 Group Homomorphisms

Suppose G and H are groups. A map from ϕ from G to H is called a homomorphism if $\phi(ab) = \phi(a)\phi(b)\ \forall a,b \in G$

ie ϕ is operation preserving

Note: A homomorphism which is also a bijection would be an isomorphism

Definitions Suppose $\phi\ G \to H$ is a homomorphism and G and H are groups

i) $\ker\phi = \{x \in G \mid \phi(x) = e\}$ is called the kernel

ii) $\operatorname{Im}\phi = \{\phi(x) \mid x \in G\}$

Example 72 Which of these are homomorphisms?

What is $\ker\phi$ and $\operatorname{Im}\phi$? Are any of them isomorphisms?

a) $\phi_1 : (\mathbb{C}^*, \times) \to (\mathbb{R}, x)$

$\phi_1(z) = |z|$

b) $\phi_2 : Z \to Z_{100}$, $\phi_2(x) = x^2 \bmod 100$

c) $\phi_3 : S_5 \to (\mathbb{Q}^*, \times)$, $\phi_3(\alpha) = \operatorname{sgn}(\alpha)$

a) $z = a + bi$ then $|z| = \sqrt{a^2+b^2}$

Is it OP? $\phi(z_1 z_2) = \phi(z_1)\phi(z_2)$

$z_1 = a + bi \quad z_2 = c + di$

$\phi(z_1 z_2) = \phi(ac + adi + bci - bd)$

$= \sqrt{(ac-bd)^2 + (ad+bc)^2}$

$= \sqrt{b^2d^2 + a^2d^2 + b^2c^2}$

$\phi(a+bi)\,\phi(c+di) = \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}$

$= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$

So ϕ is a homom

$\ker \phi = \{ z \in \mathbb{C}^* \mid |z| = 1 \}$

$\operatorname{Im} \phi = \mathbb{R}^+$

$\phi_2(x) = x^2 \bmod 100$

Let $x, y \in \mathbb{Z}$

$\phi_2(x+y) = (x+y)^2$

$\phi_2(x) + \phi_2(y) \bmod 100 = x^2 + y^2 \bmod 100$

Not equal so ϕ_2 is not a homomorphism

c) $\phi_3 : S_5 \to (\mathbb{Q}^*, \times)$

$\phi_3(\alpha) = \operatorname{sgn}(\alpha)$

$\alpha, \beta = S_5$

$\phi(\alpha\beta) = \operatorname{sgn}(\alpha\beta)$

$\phi_3(\alpha)\,\phi_3(\beta) = \operatorname{sgn}(\alpha)\,\operatorname{sgn}(\beta)$

So ϕ_3 is a homomorphism

$\ker \phi_3 = A_5 \qquad \operatorname{Im} \phi_3 = \{1, -1\}$

Not an isomorphism

Proposition 52 Suppose ϕ is a homomorphism from G to H, suppose $g \in G$,

i) $\phi(e_G) = e_H$

ii) $\phi(g^n) = (\phi(g))^n \quad \forall n \in \mathbb{Z}$

iii) If $|g|$ is finite then $|\phi(g)|$ divides $|g|$

i) ii) Same as for isomorphisms (Prop 31)

iii) Suppose $|g| = n$, then $g^n = e$

Then $e_H = \phi(e_G) = \phi(g^n) = (\phi(g))^n \Rightarrow e_H = (\phi(g))^n$

$\Rightarrow |\phi(a)|$ divides n

Example 73 Can there be a homomorphism from $\mathbb{Z}_9 \oplus \mathbb{Z}_2$ onto $\mathbb{Z}_4$?

Suppose $\phi : \mathbb{Z}_9 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4$ is an onto homomorphism

$\mathbb{Z}_4 = \langle 1 \rangle$ (cyclic) order of 1 is 4. $|1| = 4$ in $\mathbb{Z}_4$

So if $|\phi(g)| = 4$ then $|g|$ could be $4, 8, 12, 16, \ldots$

$4 \mid (g)$

In $\mathbb{Z}_9 \oplus \mathbb{Z}_2$ which elements have order that is divisible by 4?

$\underset{\text{orders } 1,3,9}{\mathbb{Z}_9} \oplus \underset{\text{orders } 1,2}{\mathbb{Z}_2}$ $\qquad$ $\text{lcm}(\underset{1/3/9}{\_}, \underset{1/2}{\_}) = 4/8/12$

Impossible $\Rightarrow$ contradiction $\quad \therefore$ no such homomorphism exists

Proposition 53 Suppose ϕ is a homomorphism from G to H. Then $\ker \phi$ is a subgroup of G and $\text{Im}(G) = \phi(G)$ is a subgroup of H

Prove $\ker \phi \leq G$

i) $\ker \phi = \{ g \in G, \mid \phi(g) = e \}$

$e_G \in \ker \phi$ and $\ker \phi \subseteq G$

ii) Let $a, b \in \ker \phi$

$\phi(a) = e \qquad \phi(b) = e$

So $\phi(ab) = \phi(a)\phi(b) = ee = e$ so $ab \in \ker \phi$

iii) Let $a \in \ker \phi$ so $\phi(a) = e$ WTS $a^{-1} \in \ker \phi$ WTS $\phi(a^{-1}) = e$

$\phi(a^{-1}) = (\phi(a))^{-1} = e^{-1} = e$

Proposition 54 Suppose ϕ is a homomorphism from G to H. Then $\ker \phi$ is a normal subgroup of G

Already know $\ker \phi$ is a subgroup of G WTS $x \ker \phi x^{-1} \subseteq \ker \phi \quad \forall x \in G$

Let $x \in G$, let $g \in \ker \phi$ WTS $xgx^{-1} \in \ker \phi$

$\phi(xgx^{-1}) = \phi(x)\,\phi(g)\,\phi(x^{-1})$ $\quad g \in \ker \phi$ so $\phi(g) = e$

$= \phi(x)\, e\, \phi(x)$

$= \phi(x)\phi(x^{-1})$

$= \phi(xx^{-1}) = \phi(e) = e$

So $\phi(xgx^{-1}) = e$

$xgx^{-1} \in \ker \phi$

So $x \ker \phi x^{-1} \subseteq \ker \phi \quad \forall x \in G$

$\therefore$ $\ker \phi$ is normal

Unfortunately $\operatorname{Im} \phi$ is not always normal in H

Proposition 55 Let $\phi: G \to G$ be a homomorphism. Let H be a subgroup of G then:

i) If H is abelian then $\phi(H)$ is Abelian $\qquad \phi(H) = \{\phi(x) \mid x \in H\}$

ii) If H is cyclic then $\phi(H)$ is cyclic

iii) If H is normal subgroup of G then $\phi(H)$ is a normal subgroup of $\phi(G) = \text{Im}\,\phi$

iv) $\phi(a) = \phi(b) \iff a\ker\phi = b\ker\phi$

v) If $\phi(g) = h$ then $\phi^{-1}(h) = \{x \in G \mid \phi(x) = h\} = g\ker\phi$

iv) $a, b \in G$

$$\begin{aligned}\phi(a) = \phi(b) \iff e &= (\phi(a))^{-1}\phi(b)\\ &= \phi(a^{-1})\,\phi(b)\\ &= \phi(a^{-1}b) \qquad \iff a^{-1}b \in \ker\phi\\ &\qquad\qquad\qquad \iff a\ker\phi = b\ker\phi\end{aligned}$$

Example 74 Suppose $\phi: \mathbb{Z}_{12} \to \mathbb{Z}_6 \oplus \mathbb{Z}_2$ is onto. Prove that ϕ can not be a homomorphism

Proof by contradiction, suppose ϕ is a homomorphism

$\mathbb{Z}_{12}$ is cyclic so then by prop 55(ii), $\phi(\mathbb{Z}_{12})$ is cyclic. Also ϕ is onto.

So $\phi(\mathbb{Z}_{12}) = \mathbb{Z}_6 \oplus \mathbb{Z}_2$

But $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ is not cyclic $\qquad \mathbb{Z}_6 \oplus \mathbb{Z}_2 \not\approx \mathbb{Z}_{12} \qquad \gcd(6, 2) \neq 1$

So there is a contradiction

Theorem 56: First Isomorphism Suppose ϕ is a homomorphism from G to H.
Define $\psi : G/\ker\phi \longrightarrow \mathrm{Im}(G)$ by $\psi(g\ker\phi) = \phi(g)$
Then ψ is an isomorphism. So $G/\ker\phi \approx \phi(G)$
Proof later

Example 75 Define $\phi: \mathbb{Z} \to \mathbb{Z}_6$ by $\phi: x \to x \bmod 6$
This is a homomorphism, what does 1^{st} Iso Thrm tell us in this case?
$\mathbb{Z}/\ker\phi \approx \phi(\mathbb{Z})$ Here $\ker\phi = \{$ multiples of $6\}$
$\phi(\mathbb{Z}) = \mathbb{Z}_6$
$\ker\phi = \{6n \mid n \in \mathbb{Z}\} = \langle 6 \rangle$
So 1^{st} Iso says $\mathbb{Z}/\langle 6 \rangle \approx \mathbb{Z}_6$

Notation textbook uses $\mathbb{Z}/n\mathbb{Z}$ for $\mathbb{Z}_n$

Example 76 Prove that A_n is a normal subgroup of S_n and that $S_n/A_n \approx \mathbb{Z}_2$
Use 1^{st} Iso, we want a homomorphism $\phi: S_n \to \mathbb{Z}_2$
With $\ker\phi = A_n$
Image $= \phi(S_n) = \mathbb{Z}_2$
Let $\alpha = S_n$

$$\mathrm{Sgn}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is even} \\ -1 & \text{if } \alpha \text{ is odd} \end{cases}$$

So define $\phi: S_n \to \{1, -1\}$ by $\phi(\alpha) = \mathrm{sgn}(\alpha)$
ϕ is a homomorphism (Example 72)
So 1^{st} Iso says $S_n/\ker\phi \approx \{1,-1\} \approx \mathbb{Z}_2$
Here $\ker\phi = \{$ even permutations $\} = A_n$

So $S_n / A_n \approx \{1, -1\} \approx \mathbb{Z}_2$

Also $A_n = \ker \phi$ which we know is normal so A_n is normal in S_n