Theorem 27

Every permutation can be written as a product of two cycles

$(a_1 a_m)(a_1 a_{m-1}) \dots (a_1 a_4)(a_1 a_3)(a_1 a_2) = (a_1 a_2 a_3 \dots a_m)$

Example 39 Write $(4\ 6\ 1\ 3\ 5)(2\ 8\ 7)$ as a product of two cycles

$(4\ 5)(4\ 3)(4\ 1)(4\ 6)(2\ 7)(2\ 8) = (4\ 6\ 1\ 3\ 5)(2\ 8\ 7)$

alt $(4\ 6)(6\ 1)(1\ 3)(3\ 5)(2\ 8)(8\ 7) = (4\ 6\ 1\ 3\ 5)(2\ 8\ 7)$

alt $(4\ 6)(6\ 1)(1\ 3)(3\ 5)(2\ 8)(8\ 7)(1\ 2)(2\ 1)$ 8 cycles

Theorem 28

i) If $e = c_1 c_2 \cdots c_r$ where c_i are two cycles then r is even

ii) If $\alpha = \beta_1 \beta_2 \dots \beta_s = \gamma_1 \gamma_2 \dots \gamma_s$ where β_i and γ_j are two-cycles then either both s and t are even or both are odd

Definition

i) A permutation is S_n that can be written as a product of an even # of two-cycles is called an even permutation

ii) If α can be written as a product of an odd # of two-cycles is called an odd permutation

iii) The signiture of α is $sgn(\alpha) = \{ 1 \text{ if } \alpha \text{ is even} \mid -1 \text{ if } \alpha \text{ is odd} \}$

Example 38 Is $\alpha = (1\ 2\ 3)(4\ 5\ 6\ 7)$ even or odd?

$\alpha = (1\ 2)(2\ 3)(4\ 5)(5\ 6)(6\ 7)$

$sgn(\alpha) = -1$, so α is odd

Definition

For any integer $n \geq 3$, let $A_n = \{\alpha \in S_n \mid sgn(\alpha) = 1\}$

$= $ the set of even permutations

This is called the Alternating Group of Degree n

Theorem 29

Let $n \geq 3$ for $n \in \mathbb{Z}$

a) A_n is a subgroup of S_n

b) $|A_n| = \frac{n!}{2}$

a i) $A_n \neq \emptyset$ because $(1\ 2\ 3) = (1\ 2)(2\ 3)$, $(1\ 2\ 3) \in A_n$

$\therefore A_n$ is a subset of S_n

ii) Let $\alpha = \alpha_1 \alpha_2 \dots \alpha_s$

$\beta = \beta_1 \beta_2 \dots \beta_t$ $\qquad$ (α_i, β_i two-cycles)

Then $\alpha\beta = \underbrace{\alpha_1 \alpha_2 \dots \alpha_s \beta_1 \beta_2 \dots \beta_t}_{\text{product of } s+t}$ two-cycles

and $s+t$ is even, so $\alpha\beta \in A_n$

iii) If $\alpha \in A_n$, $\alpha = \alpha_1 \alpha_2 \dots \alpha_{\underset{\text{even}}{s}}$ then $\alpha^{-1} = \alpha_s^{-1} \alpha_{s-1}^{-1} \dots \alpha_1^{-1}$

$= \alpha_s \alpha_{s-1} \dots \alpha_1 \in A_n$

s two-cycles so s is even, $\therefore \alpha^{-1} \in A_n$

b) If $\alpha \in A_n$ is an even permutation then $(1\ 2)\alpha$ is an odd permutation

$\alpha = (3\ 4\ 5) = (3\ 4)(4\ 5)$ even # 2-cycles

Then $(1\ 2)\alpha = (1\ 2)(3\ 4)(4\ 5)$ odd # 2-cycles

Also if $\alpha \neq \beta$ then $(1\ 2)\alpha \neq (1\ 2)\beta$

So # odd permutations $\geq$ # even permutations

Similarly if α is odd then $(1\ 2)\alpha$ is even. If $\alpha \neq \beta$ then $(1\ 2)\alpha \neq (1\ 2)\beta$

even perm. $\geq$ # odd perm. so # even perm = # odd perm

$$|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$$

Example How many elements are there in S_n of order 14?

Need $\text{lcm}(\dots) = 14$

$\text{lcm}(7, 2) = 14$ ✓

$\text{lcm}(14) = 14$ ×

$\text{lcm}(7, 2, 2) = 14$ ✓

Case 1 $(1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9) \in S_{12}$

Cycle structure $[7, 2]$

(12choices 4 choices 10choices — — — 6choices)(5choices 6choices)

But $(1\ 2) = (2\ 1)$

$(3\ 7) = (7\ 3)$

Must divide by 2

$(3\ 4\ 7\ 6\ 2\ 1\ 5) = (4\ 7\ 6\ 2\ 1\ 5\ 3) = (7\ 6\ 2\ 1\ 5\ 3\ 4)$

7 ways to write each 7-cycle, so divide by 7

Total for case 1: $\frac{12!}{3! \cdot 2 \cdot 7}$

Case 2 Cycle structure 7, 2, 2

(12 choices 11 10 9 8 7 6)(5 4)(3 2 choices)

$(1\ 2\ 3\ 4\ 5\ 6\ 7)(10\ 11)(8\ 9) = (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9)(10\ 11)$

Need to divide by 2 again, Total for case 2: $\frac{12!}{7\cdot 2\cdot 2\cdot 2}$

Total: $\frac{12!}{3!\cdot 7\cdot 2} + \frac{12!}{7\cdot 2^3} = 14256000$

Example 40 How many elements in A_{12} have order 14?

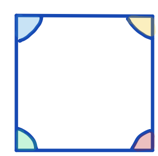
$7,2: (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9) = (1\ 2)(2\ 3)(3\ 4)(5\ 6)(7\ 8)(8\ 9) \notin A_{12}$

$7,2,2: (1\ 2\ 3\ 4\ 5\ 6\ 7)(8\ 9)(10\ 11) = (1\ 2)(2\ 3)(3\ 4)(4\ 5)(5\ 6)(6\ 7)(8\ 9)(10\ 11) \in A_{12}$

Total for A_{12} = total for case 2 only = $\frac{12!}{7\cdot 2^3}$

Chapter 6 Isomorphisms

Example 41 Consider the symmetries of a rectangle. Write down its Cayley Table.



$G = \{R_0, R_{180}, V, H\}$

Example 42 Write down the Cayley Table of $U(8)$

$U(8) = \{1, 3, 5, 7\}$

$*$ mod 8	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

$1 \approx R_0$ $\quad 7 \approx H$ $\quad R_{180} \approx 3$

$5 \approx V$

Definition

Suppose G and $\bar{G}$ are groups

i) A bijection ϕ from G onto $\bar{G}$ such that

$$\phi(ab) = \phi(a)\ \phi(b) \qquad \forall a,b \in G$$

ϕ is operation preserving

then ϕ is an isomorphism from G to $\bar{G}$

ii) If there exists an isomorphism from G to $\bar{G}$ then they are isomorphic, written $G \approx \bar{G}$

Example 43 $U(8) \approx$ symmetry group of rectangle

$\phi(1) = R_0$ $\quad \phi(7) = H$ $\quad \phi(3) = R_{180}$ $\quad \phi(5) = V$

Example 44 Define $\phi : (\mathbb{R}^+, \times) \to (\mathbb{R}, +)$ by $\phi(x) = \ln(x)$, is it an isomorphism?

One-to-one: Suppose $\phi(x) = \phi(y)$, show $x = y$

Suppose $\phi(x) = \phi(y)$

so $\ln(x) = \ln(y)$

$e^{\ln(x)} = e^{\ln(y)}$

$x = y$ $\qquad$ so ϕ is one-to-one

Onto: $\phi : G \to \bar{G}$, suppose $y \leftarrow \bar{G}$, show $\exists x \in G$ st $\phi(x) = y$

Suppose $y \in \mathbb{R}$, let $x = e^y$, then $\phi(x) = \phi(e^y) = \ln e^y = y$

So $\phi(x) = y$, ϕ is onto

OP: Let $x, y \in G$, show $\phi(xy) = \phi(x)\phi(y)$

Let $x, y \in \mathbb{R}^+$, WTS $\phi(x\ y) = \phi(x)\ \phi(y)$

Let $x, y \in \mathbb{R}^+$, WTS $\phi(x \cdot y) = \phi(x) + \phi(y)$

$$\phi(x \cdot y) = \ln(x y) = \ln x + \ln y = \phi(x) + \phi(y), \quad \therefore \text{ OP}$$

So $(\mathbb{R}^+, \times) \approx (\mathbb{R}, +)$

Example 45 $(\mathbb{R}, +) \to (\mathbb{R}, +)$, $\phi(x) = \sqrt[3]{x}$, is this an isomorphism?

OP Let $x, y \in \mathbb{R}$, then $\phi(x + y) = \sqrt[3]{x + y}$

$$\phi(x) + \phi(y) = \sqrt[3]{x} + \sqrt[3]{y}$$

Not equal, so it is not OP, so it is not an isomorphism