Recall $aH = \{ah \mid h \in H\}$ left coset

Lemma 33, Example 52 $H = \{1, 5, 21, 25\}$ is a subgroup of $U(26)$ (mult mod 26). Are the left cosets $9H$ and $17H$ equal or diffirent?

Option 1 Lemma 33 (v): $aH = bH \iff a^{-1}b \in H$

Check $9^{-1}7 \in H$? $9 \times 3 \equiv 1 \pmod{26} \implies 9^{-1} = 3$

$27 \equiv 1 \pmod{26}$

$9^{-1} \cdot 17 = 3 \cdot 17 = 51 \equiv 25 \pmod{26} \in H$, so $9H = 17H$

Option 2 Lemma 33 (iv) $aH = bH \iff a \in bH$

$9H = \{9, 9\cdot 5, 9\cdot 21, 9\cdot 25\} = \{9, 19, 7, 17\}$

$17 \in 9H$ so $17H = 9H$

Example 53 Let $G = GL(2, \mathbb{R})$, Let $H = \{a \in G \mid \det a = \pm 1\}$, a subgroup of G

If $a, b \in G$ and $aH = bH$ what can be said about $\det a$ and $\det b$?

$aH = bH \iff a^{-1}b \in H \iff \det(a^{-1}b) = \pm 1$

$\iff \det(a^{-1}) \cdot \det b = \pm 1$

$\iff \frac{1}{\det a} \cdot \det b = \pm 1$

$\iff \det b = \pm \det a$

$\iff |\det b| = |\det a|$

Definition Suppose G is a group, H is a subgroup of G. Then the number of distinct left cosets of H in G is called the index of H in G denoted $|G : H|$

Lagranges Theorem 34 If G is a finite group and H is a subgroup then $|H|$ divides $|G|$ and $|G:H| = \frac{|G|}{|H|}$

Let $a_1H, a_2H, \ldots, a_nH$ be the distinct left cosets of H in G

Claim $G = a_1H \cup a_2H \cup \ldots \cup a_nH$

Let $a \in G$ then $a \in aH$ and $aH = a_iH$ for some i

So $a \in a_1H \cup a_2H \cup \ldots \cup a_nH$

So $G = a_1H \cup a_2H \cup \ldots \cup a_nH$

Now $|G| = |a_1H \cup a_2H \ldots \cup a_nH|$

(vi) $= |a_1H| + |a_2H| + \ldots + |a_nH|$ In general $|A \cup B| \neq |A| \cup |B|$

(vii) $= |H| + |H| + \ldots + |H|$

$= r \cdot |H|$

So $|a| = r|H|$ so $|H|$ divides $|a|$

Also $|G:H|$ = # distinct cosets $= r = \frac{|G|}{|H|}$

Example 54 Suppose H is a subgroup of a group G and $|G|$, what could $|H|$ be?

$|H|$ could be $12, 6, 4, 3, 1, 2$

Remark ① converse of lagrange's theorem is False

eg $|A_4| = 12$ but A_4 has no subgroup of order 6

② A group can have many subgroups of the same order

Corollary 35 Suppose G is a finite group and $a \in G$, then the order of a divides the order of G

$\langle a \rangle$ a subgroup of G

$|\langle a \rangle| = |a|$ so $|a|$ divides $|G|$

Corollary 36 If $|G| = p$ a prime, then G is cyclic

Suppose $|G| = p$, let $a \in G$ and $a \neq e$

$|a|$ must divide $|G| = p$ (by cor 35)

So $|a| = 1$ or p but $|a| \neq 1$ since $a \neq e$

So $|a| = p$ and $|\langle a \rangle| = p = |a|$ so $\langle a \rangle = G$ so G is cyclic.

Corollary 37 Suppose G is a finite group $a \in G$, then $a^{|G|} = e$

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Fermat's little theorem If $a \in \mathbb{Z}^*$ and p is prime then $a^p \equiv a \pmod{p}$

Example 55 Suppose K is a proper subgroup of H and H is a proper subgroup of G and $|K| = 42$, $|G| = 420$. What could $|H|$ be?

$K \leq H \leq G \Rightarrow 42 \leq H \leq 420$

$42 \mid |H|$ and $|H| \mid 420$

$|H|$ could be ~~42~~, 84, 210, ~~420~~ proper

Suppose also $a \in H$ and $|a| = 5$, what could $|H|$ be?

So $|a| \mid |H|$ so $5 \mid |H|$ so $|H| = 210$

Definition If H and K are finite subgroups of G then define

$HK = \{hk \mid h \in H, k \in K\}$

Theorem 39 Then $|HK| = \frac{|H| \cdot |K|}{|H \cap K|}$

Stabilizers and Orbits

Suppose G is a permutation group ie a group consisting of permutations of some set S

i) For $i \in S$ define the stabilizer of i in G by

$$stab_G(i) = \{ \alpha \in G \mid \alpha(i) = i \}$$ subset of G

ii) For $i \in S$ define the orbit of i in G by

$$orb_G(i) = \{ \alpha(i) \mid \alpha \in G \}$$ subset of S

Proposition 40 $Stab_G(i)$ is a subgroup of G Exercise

Example 57 Let $G = \{ e, (1\,2\,3), (1\,3\,2), (4\,5), (1\,2\,3)(4\,5), (1\,3\,2)(4\,5) \}$

G is a permutation group on $S = \{1, 2, 3, 4, 5\}$

$stab_G(1) = \{e, (4\,5)\}$ $\quad orb_G(1) = \{1, 2, 3\}$

$stab_G(2) = \{e, (4\,5)\}$ $\quad orb_G(2) = \{1, 2, 3\}$

$stab_G(3) = \{e, (4\,5)\}$ $\quad orb_G(3) = \{1, 2, 3\}$

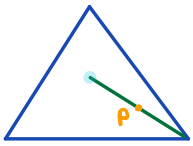
$stab_G(4) = \{e, (1\,2\,3), (1\,3\,2)\}$ $\quad orb_G(4) = \{4, 5\}$

$stab_G(5) = \{e, (1\,2\,3), (1\,3\,2)\}$ $\quad orb_G(5) = \{4, 5\}$

Note $|stab_G(i)| \cdot |orb_G(i)| = |G|$

Example 58 The elements of D_3 permute the points on an equilateral triangle

$stab_{D_3}(P) = \{R_0, F_R\}$



$orb_{D_3}(P) =$  3 points

Orbit stabilizer Theorem 41 Suppose G is a finite group of permutations of some set S. If $i \in S$ then $|stab_G(i)| \cdot |orb_G(i)| = |G|$

We want to show $|orb_G(i)| = \frac{|G|}{|stab_G(i)|}$

$\frac{|G|}{|stab_G(i)|} = |G : stab_G(i)| =$ # of distinct left cosets of $stab_G(i)$ in G

We will show $|orb_G(i)| =$ # cosets of $stab_G(i)$ in G

We will do this by finding a bijection between them

Define T: $\{$distinct left cosets of $stab_G(i)$ in $G\} \to orb_G(i)$

$T(\alpha\, stab_G(i)) = \alpha(i) \in orb_G(i)$

Well-defined Need to check whenever there is more than one way to write the same thing

T is well defined Suppose $\alpha, \beta \in G$, $\alpha\, stab_G(i) = \beta\, stab_G(i)$

WTS $T(\alpha\, stab_G(i)) = T(\beta\, stab_G(i))$

$\alpha\, stab_G(i) = \beta\, stab_G(i) \Rightarrow \alpha^{-1}\beta \in stab_G(i)$ $\quad aH = bH \Rightarrow a^{-1}b \in H$ (L33)

$\Rightarrow \alpha^{-1}\beta(i) = i$

$\Rightarrow \beta(i) = \alpha(i) \Rightarrow T(\beta\, stab_G(i)) = T(\alpha\, stab_G(i))$

One-to-one Reverse steps in "well defined", and we see that everything does go back

Onto If $j \in orb_G(i)$ then $j = \alpha(i)$ for some $\alpha \in G$ so $T(\alpha\, stab_G(i)) = j$

So T is a bijection, so $|\{\text{left coset}\}| = |orb_G(i)|$

Definition The rotational symmetry group of a 3D object S is the group of rotations of $\mathbb{R}^3$ which maps S to S

Proposition 42 consider the rotational symmetry group of a Dodecahedron S_D

i) $|S_D| = 60$

ii) $S_D \approx A_5$

Let one side be x, $|orb_{S_D}(x)| = 12$ — sides

$|stab_{S_D}(x)| = 5$ — facing front

So $|S_D| = 12 \cdot 5 = 60$

$|A_5| = \frac{5!}{2} = 60$

It is possible to inscribe a cube in the Docahedron. There are 5 ways of doing this ie 5 positions of the cube 

Lable the cubes 1, 2, 3, 4, 5

- Rotation through the vertex: a 3-cycle eg $(1\,2\,3) \to$ even $\in A_5$
- Rotation through a face: a 5-cycle eg $(1\,2\,3\,4\,5) \to$ even ✓ $\in A_5$
- Rotation through an edge: 2 2-cycles $\to (\_\_)(\_\_) \in A_5$

So $S_D \leq A_5$ Also $|S_D| = 60 = |A_5|$

So $S_D \approx A_5$

Example 59 Find the order of the symmetry group of a truncated icosahedron in 2 different ways. soccer ball

OST $|orb_G(\pentagon)| \cdot |stab_G(\pentagon)| =$

$12 \cdot 5 = 60$

or $|orb_G(\pentagon)| \cdot |stab_G(\pentagon)| = 20 \cdot 3 = 60$