Applied Mathematics 364

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// Problem 1

// (a)

Using Mathemetica to evaluate the integrals, I got

$$\begin{aligned} A(\alpha) &= \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx \\ &= \int_{-1}^{1}(1-x)\cos(\alpha x)dx = \frac{2\sin(\alpha)}{\alpha} \end{aligned}$$

And

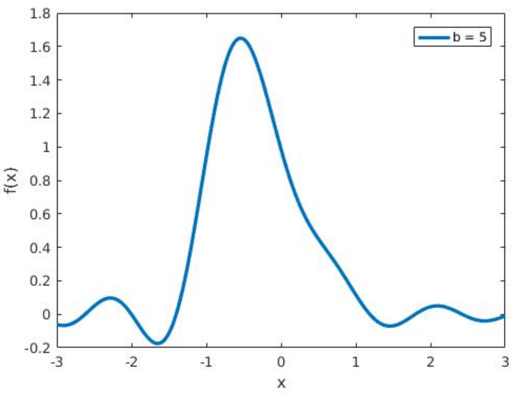
$$\begin{aligned} B(\alpha) &= \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx \\ &= \int_{-1}^{1}(1-x)\sin(\alpha x)dx = \frac{2\alpha\cos(\alpha)-2\sin(\alpha)}{\alpha^2} \end{aligned}$$

Therefore The Fourier integral representation of f(x) is

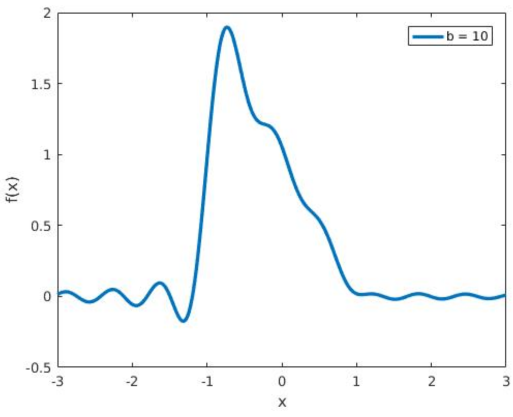
$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{2\sin(\alpha)}{\alpha}\cos(\alpha x) + \left(\frac{2\alpha\cos(\alpha)-2\sin(\alpha)}{\alpha^2}\right)\sin(\alpha x)\right]d\alpha$$

// (b)

Using matlab, the plots of the Fourier integral of f(x) with different upper bounds are given below

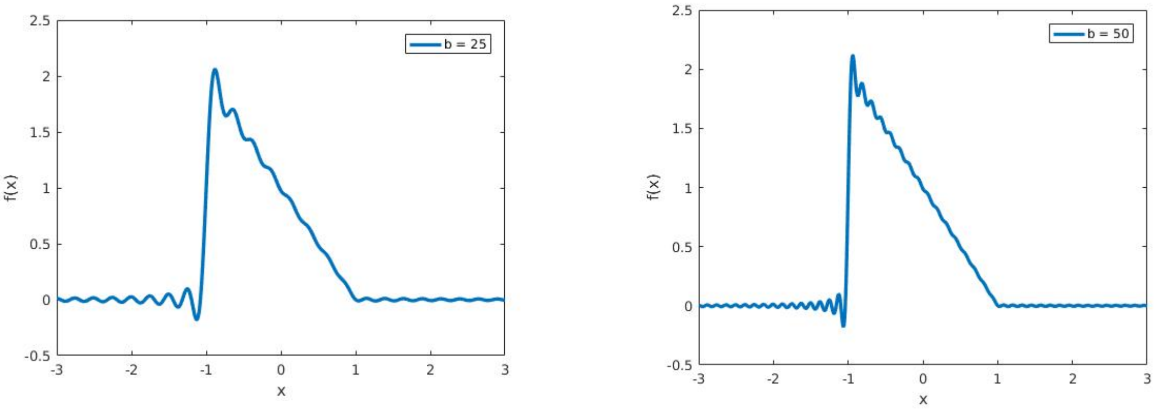


(a) b = 5



(b) b = 10

Figure 1: Fourier integral of f(x)



(a) b = 25

(b) b = 50

Figure 2: Fourier integral of f(x)

Since f is continous and f' is piecewise continous on $(-\infty, \infty)$ and f is Lebesgue integrable(i.e absolutely integrable)

$$\int_{-\infty}^{\infty} |f(x)| = \int_{-\infty}^{\infty} f(x)$$

Then Fourier integral converges to f(x) at all point continous hence

$$\frac{1}{2}\bigg(f(0) + f(2)\bigg) = \frac{2}{2} = 1$$

The Gibbs phenomenon decreases as the upper bound of the Fourier integral get larger (closer to ∞)

// Problem 2

Its suffice to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x,1) = e^{-x^2}, u_y(x,0) = 0, -\infty < x < \infty$$

$$\mathscr{F}\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\} = \mathscr{F}\{0\} \longrightarrow \frac{d^2U}{dy^2} - \alpha^2 U = 0$$

Therefore we end up with the 2^{nd} ODE

$$\frac{d^2U}{dy^2} - \alpha^2 U = 0$$

We know from the lecture slide the solution of the above ODE is given by

$$U(y,\alpha) = c_1 \cosh(\alpha y) + c_2 \sinh(\alpha y) \Longrightarrow \frac{\partial}{\partial y} U = \frac{c_1}{\alpha} \sinh(\alpha y) + \frac{c_2}{\alpha} \cosh(\alpha y)$$

Next we tranform the boundary conditions

$$\mathscr{F}\{u(x,1)\} = U(\alpha,1) = \int_{-\infty}^{\infty} e^{-x^2} e^{i\alpha x} dx$$

Using Mathematica to evaluate the above integral I got

$$U(\alpha,1) = \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y)$$

And the other boundary condition becomes

$$\begin{aligned}\mathscr{F}\{u_y(x,0)\} = \mathscr{F}\{0\} &= \int_{-\infty}^{\infty} \frac{\partial u}{\partial y} e^{i\alpha x} dx \\ &= \frac{\partial}{\partial y} \int_{-\infty}^{\infty} u e^{i\alpha x} dx \\ &= \frac{\partial}{\partial y} U(\alpha, 0) = 0\end{aligned}$$

Now solving for coefficients

$$\frac{\partial}{\partial y} U(\alpha, 0) = 0 \Longrightarrow 0 = c_1/\alpha + c_2/\alpha \Longrightarrow c_2 = 0$$

$$U(\alpha, 1) = c_1 \quad \Longrightarrow \sqrt{\pi} e^{-\alpha^2/4} = c_1 \cosh(\alpha) \quad \Longrightarrow c_1 = \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)}$$

Now our ODE becomes

$$U(y, \alpha) = \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y)$$

Our ODE becomes

$$\begin{aligned}u(x,y) = \mathscr{F}^{-}\{U(y,\alpha)\} = &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y) e^{-i\alpha x} d\alpha \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y) \cos(\alpha x) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y) \sin(\alpha x)\end{aligned}$$

since $\sin(\alpha x) = 0$ then

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y) \sin(\alpha x) = 0$$

Therefore our our solution becomes

$$\begin{aligned}u(x,y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{\pi} e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y) \cos(\alpha x) \\ &= \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha y) \cos(\alpha x)\end{aligned}$$

Due to Matlab not being able to correctly integrate until infinity, I chose the upper bound of the above integral to be 1000 and got the steady-state temperature at the center of the plate (at x=0 and y=1\2) to be

$$u(0, 1/2) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-\alpha^2/4}}{\cosh(\alpha)} \cosh(\alpha/2) = 0.7035$$