IML - Ex-3

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1. Let $f(x) = ax + b$ we will show that f is convex. Let $x, y \in \mathbb{R}, t \in [0, 1]$.

$$f\left(tx + (1-t)\,y\right) = atx + a(1-t)y + b = atx + a(1-t)y + tb + (1-t)b = t * (f(x)) + (1-t)f(y)$$

2. We will prove that $f(x) = ax^2 + bx + c$ is convex $\iff a \geq 0$:

   (a) $f(x)$ is convex $\implies a \geq 0$. we will take $x = 1$ $y = -1$ and $t = \frac{1}{2}$then we know that

   $$f(tx+(1-t)y) = f(0) = c \leq \frac{1}{2}f(-1) + \frac{1}{2}f(1) = \frac{a1^2 - bc + c + ac^2 + bc + c}{2} = a + c \implies 0 \leq a$$

   (b) $a \geq 0 \implies f(x)$ is convex Let $x, y \in \mathbb{R}$and $t \in [0, 1]$. we will show that

   $$\Delta = t(f(x) + (1-t)f(y) - [f(tx + (1-t)y)] \geq 0$$
   $$f(tx + (1-t)y) = a(tx + (1-t)y)^2 + b(tx + (1-t)y) + c$$

   like in section 1 the $b(tx + (1-t)y) + c$ will be cancelled from both side in Δ. Thus we get

   $$\Delta = \left[at^2x^2 - 2at(t-1)xy + a(1-t)^2y^2\right] - tax^2 - (1-t)ay^2$$

   we pull out a and group by x^2, y^2, xy

   $$\Delta = a\left[x^2\left(t^2 - t\right) + 2xy\left(t^2 - t\right) + y^2 \underbrace{\left[(1-t)^2 - (1-t)\right]}_{=1-2t+t^2-1+t=t^2-t}\right] = \underbrace{a}_{\geq 0} * \underbrace{t}_{\geq 0} * \underbrace{(1-t)}_{\geq 0}\underbrace{(x-y)^2}_{\geq 0} \geq 0$$

3. $f(x) = e^x$ is convex. The most standard way to prove this is by second derivative since $f''(x) = e^x \geq 0$. Since we only use the algebraic definition, Let $x, y \in \mathbb{R}, t \in [0, 1]$.

$$f(tx + (1-t)y) = e^{tx+(1-t)y} = e^{tx} * e^{(1-t)y}$$

both $e^x = u,\ e^y = v$ are non-negative then by the young inequality:

$$e^{tx} * e^{(1-t)y} = u^t v^{1-t} \leq tu * (1-t)v = tf(x) + (1-t)y$$

4. Let $c \in \mathbb{R} f(x) = \max(x, c) = \begin{cases} c & x \leq c \\ x & else \end{cases}$. Let $x, y \in \mathbb{R}, t \in [0, 1]$.

   (a) case 1: $x \leq c, y \leq c \implies tx + (1-t)y \leq c \implies f(tx + (1-t)y) = c$ thus:
   $$t(f(x)) + (1-t)f(y) = c \geq c$$

   (b) case 2: $x, y \geq c$ this is a linear function which we already proved is convex.

   (c) case 3: WLG $x \leq c$ and $y \geq c$ thus $f(x) = c \; f(y) = y$.
   $$tf(x) + (1-t)f(y) = tc + (1-t)y$$
   but also
   $$tc + (1-t)y \geq tc + (1-t)c = c$$
   Thus
   $$\max(c, tc + (1-t)y) \leq tf(x) + (1-t)f(y)$$

5. We will show a counter example to $f(x) = cos(x)$ with $t = \frac{1}{2}$ and $x = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ thus on the left side we get $cos(0) = 1$ and on the right side we get
$$\frac{cos(-\frac{\pi}{2}) + cos(\frac{\pi}{2})}{2} = 0 \ngeq 1$$