Fréchet Distance Library

The Fréchet distance is a useful and well known similarity measure for polygonal curves. It is generally described as follows; Consider a person and a dog connected by a leash, each walking along a different curve from its starting point to its end point. Both are allowed to control their speed but they cannot backtrack. The Fréchet distance between the two curves is the minimum length of a leash that is sufficient for traversing both curves in this manner.

See this web-site for more details and a Fréchet distance applet.

Here is a summery of the different variants and applications of the Fréchet distance metric which have been studied in the literature.

Contents

1 Continuous Fréchet distance

- The Fréchet distance was first defined by Maurice Fréchet (1878-1973) [Fre06].
- Alt and Godau [AG95] showed that the Fréchet distance of two polygonal curves with a total of n edges can be computed, using dynamic programming, in $O(n^2 \log n)$ time.
- Buchin et al. [BBMM14] recently improved the bound of Alt and Godau and showed how to compute the Fréchet distance in $O(n^2(\log n)^{1/2}(\log\log n)^{3/2})$ time on a pointer machine, and in $O(n^2(\log\log n)^2)$ time on a word RAM.

- Bringmann [Bri14] showed that the (discrete and continuous) Fréchet distance has no strongly subquadratic algorithms unless SETH fails. He also showed that there is no strongly subquadratic 1.001-approximation algorithm unless SETH' fails.

2 Discrete Fréchet distance

The *discrete Fréchet distance* is a simpler variant that arises when we replace each of the input curves by a sequence of densely sampled points. Intuitively, the discrete Fréchet distance replaces the curves by two sequences of points, and replaces the person and the dog by two frogs. At each move, the frogs can jump from their current point to the next. The frogs are not allowed to backtrack. We are interested in the minimum length of a leash that connects the frogs and allows them to traverse both curves in this manner. The resulting discrete distance is considered as a good approximation of the actual continuous distance.

- Eiter and Mannila [EM94] showed that the discrete Fréchet distance can be computed, also using dynamic programming, in $O(mn)$ time.
- Agarwal et al. [AAKS14] showed how to compute the discrete Fréchet distance in $O\left(\frac{mn \log\log n}{\log n}\right)$ time.
- Jiang et al [JXZ08] successfully applied the discrete Fréchet distance for aligning the backbones of proteins, which are represented as chains of atoms in 2D or 3D. In this application, the discrete variant makes more sense than the continuous because matching points that does not represent atoms is false biologically.

3 Variants of Fréchet distance

3.1 Fréchet distance with shortcuts

Anne Driemel and Sariel Har-Peled. Jaywalking your dog: Computing the Fréchet distance with shortcuts, 2013. Rinat Ben Avraham, Omrit Filtser, Haim Kaplan, Matthew J. Katz, and Micha Sharir. The discrete Fréchet distance with shortcuts via approximate distance counting and selection, 2014. Maike Buchin, Anne Driemel, and Bettina Speckmann. Computing the Fréchet distance with shortcuts is NP-hard, 2014.

3.2 Weak Fréchet distance

Helmut Alt and Michael Godau. Computing the Fréchet distance between two polygonal curves, 1995. Sariel Har-Peled and Benjamin Raichel. The Fréchet distance revisited and extended, 2011.

3.3 Partial Fréchet distance

Helmut Alt and Michael Godau. Computing the Fréchet distance between two polygonal curves, 1995. Kevin Buchin, Maike Buchin, and Yusu Wang. Exact algorithms for partial curve matching via the Fréchet distance, 2009. Anne Driemel and Sariel Har-Peled. Jaywalking your dog: Computing the Fréchet distance with shortcuts, 2013. Jean-Lou De Carufel, Amin Gheibi, Anil Maheshwari, Jörg-RÃŒdiger Sack, Christian Scheffer. Similarity of Polygonal Curves in the Presence of Outliers, 2012.

3.4 Fréchet distance of a set of curves

Adrian Dumitrescu and Günter Rote. On the Fréchet distance of a set of curves, 2004. Sariel Har-Peled and Benjamin Raichel. The Fréchet distance revisited and extended, 2011.

3.5 Average and summed Fréchet distance

Sotiris Brakatsoulas, Dieter Pfoser, Randall Salas, and Carola Wenk. On map-matching vehicle tracking data, 2005. Alon Efrat, Quanfu Fan and Suresh Venkatasubramanian. Curve matching, time warping, and light fields: New algorithms for computing similarity between curves, 2007.

3.6 Fréchet distance with speed limits

3.7 Strong Fréchet distance

3.8 Locally Correct Fréchet Matchings

4 Fréchet distance in different metric spaces

4.1 Homotopic Fréchet distance

Erin W. Chambers, Éric Colin de Verdière, Jeff Erickson, Sylvain Lazard, Francis Lazarus, and Shripad Thite. Homotopic Fréchet distance between curves or, walking your dog in the woods in polynomial time, 2010.

4.2 Geodesic Fréchet distance

Atlas F. Cook and Carola Wenk. Geodesic Fréchet distance inside a simple polygon, 2008.

5 Fréchet distance for realistic curves

H. Alt, C. Knauer, and Carola Wenk. Comparison of distance measures for planar curves, 2003. Anne Driemel, Sariel Har-Peled, and Carola Wenk. Approximating the Fréchet distance for realistic curves in near linear time. G. Rote. Computing the Fréchet distance between piecewise smooth curves, 2007. B. Aronov, S. Har-Peled, C. Knauer, Y. Wang, and C. Wenk. Fréchet distance for curves, revisited.

6 Applications of Fréchet distance

6.1 Chain simplification

Pankaj K. Agarwal, Sariel Har-Peled, Nabil H. Mustafa, and Yusu Wang. Near-linear time approximation algorithms for curve simplification, 2005. Sergey Bereg, Minghui Jiang, Wencheng Wang, Boting Yang, and Binhai Zhu. Simplifying 3d polygonal chains under the discrete Fréchet distance, 2008. Reza Dorrigiv, Stephane Durocher, Arash Farzan, Robert Fraser, Alejandro López-Ortiz, J. Ian Munro, Alejandro Salinger, and Matthew Skala. Finding a hausdorff core of a polygon: On convex polygon containment with bounded hausdorff distance, 2009. Tim Wylie and Binhai Zhu. Protein chain pair simplification under the discrete Fréchet distance, 2013. Omrit Filtser, Matthew J. Katz, Tim Wylie, and Binhai Zhu. On the chain pair simplification problem, 2014.

6.2 Mean and median curve

Kevin Buchin, Maike Buchin, Marc J. van Kreveld, Maarten Löffler, Rodrigo I. Silveira, Carola Wenk, and Lionov Wiratma. Median trajectories, 2013.

6.3 Matching

H. Alt, A. Efrat, G. Rote, and C. Wenk. Matching planar maps, 2003. Sotiris Brakatsoulas, Dieter Pfoser, Randall Salas, and Carola Wenk. On map-matching vehicle tracking data, 2005. Daniel Chen, Anne Driemel, Leonidas J. Guibas, Andy Nguyen, and Carola Wenk. Approximate map matching with respect to the Fréchet distance, 2011.

6.4 Voronoi diagram

Sergey Bereg, Kevin Buchin, Maike Buchin, Marina Gavrilova and Binhai Zhu. Voronoi Diagram of Polygonal Chains Under the Discrete Fréchet Distance, 2007.

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