, ,

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Contents

, Computational Geometry: Algorithms and Applications Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars.

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- **:** 12 ,) ,(
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- **:** , 1-10,
Computational Geometry: Algorithms and Applications.

Geometric Approximation Algorithms.
- **:** , .
- **:**) ,(
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- **?** **?**
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- **?** – , ,
- .

Chapter 1

1.1

1.1.1 ?

, , , ,
. . ,
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, , . ,
:

- **(Preprocessing Time)** - .
- **/ (Storage Space)** - .
- **(Query Time)** - .

1) 1–2).

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, 1.3 .

1.1.2

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,) , -
.(, - $\mathbb{R}^2$,) ,
.(.
“ ” (curse of high dimensionality), ,
. , ,
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, .
. **RAM**
(real RAM). Random Access Machine,
. ,
, , .
: , .
,) , , ,(, .

RAM ,
. ,
. , **CGAL** .

1.2

, .

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- **Ipe-** " Otfried Cheong, .
.
- **Geogebra-** .
- **CGAL -** C++ .

, . :

- **CMSC 754 Computational Geometry**, by David M. Mount.
- **Geometry: Combinatorics and Algorithms**, by Luis Barba Bernd Gärtner, Michael Hoffmann and Emo Welzl.
- **Computational Geometry**, video lectures by Philipp Kindermann.

1.3

1.3.1

. ,
. , .

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1.3.2

1.3.3

1.4

1.4.1

. ,
. , -
, .

.

:

S $p, q \in S$ pq S.

:

S, (Convex Hull) S $CH(S)$.

:

.1 S S. ,) S S.(

2. S - S S.

3. S S.

1.4.2

1.1 , **SlowConvexHull**, . $O(n^3)$, , .

(degeneracies), . , **(robustness)**, (floating point), . .

1.1) 2–5)

1.4.3

ConvexHull 1.1 . , **(Graham Scan)**. , - . .

:

1.1) 6–8).

1.4.4

? , **(Gift Wrapping)** **Jarvis's March**. , . **output sensitive**, .

.

1.5

, **(general position)**. **"** **" (degenerate)**, , x, , . . ,) ,(- x . x? . , , x. , .

(robustness),
. , , . ,
, p_1, p_2, p_3, , $\alpha = \angle p_1 p_2 p_3$, - 180°.
$p_1 p_2$ p_3 ,
. , , $\alpha = 180^\circ$.
, , , . ,
, ,
, .

1.2) 8–10).

Chapter 2

2.1

2.1.1

. (GIS -
Graphic Information Systems), . ,
, :
, , . ,
(map overlay).

, :
(sweep-line).) ,(- DCEL.
, , , .

2) 19–20).

2.2

2.2.1

(sweep-line) . ,
, (line segment intersection).
, , .
.

:

, ” “ ℓ . ” ” (status)
. , -
, “ ” (event points). ,

, , . ,
, .

, $O(n^2)$.
, $\Theta(n^2)$?

( )

, - i , $i-1$,
$i-1$. $\Theta(n^2)$.

2.2.2 -

,
, .

? 2.1 FindIntersections,
, HandleEvenPoint .
$O((n+k)\log n)$, k .

2.1) 20–29)

2.1 , .

n_v , n_e , - n_f , , .
: $n_v - n_e + n_f \geq 2$.

, .

, . ?

( )

, .
$O(n \log n)$.

2.2.3 ?

, . , $S = \overline{pq}$
- $S' = \overline{p'q'}$, ?

:

. ℓ S, ℓ' S'. t ℓ
- ℓ'. t S - S'. , .

:

$$y = mx + c, \qquad m.$$

, $\overline{pq}$ $p = (p_x, p_y)$, $q = (q_x, q_y)$, $m = \frac{q_y - p_y}{q_x - p_x}$.

, c .

$\overline{pq}$, m .

:

, $\ell : y = mx + c$ - $\ell' : y = m'x + c'$, $t = (t_x, t_y)$
. : $t_x = \frac{c - c'}{m' - m}$, $t_y = \frac{cm' - c'm}{m' - m}$.

.

:

$p_x \leq q_x$, $p_y \leq q_y$. t $\overline{pq}$ $t_x \in [p_x, q_y]$, $t_y \in [p_y, q_y]$.

, ?

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, , .

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, , . :

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, , .

:

: S - S' ℓ S " " p' - q' , ℓ'
S' " " p - q.

? : 3 , $a, b, c,$
- a - b - c , , .

: $det \begin{bmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{bmatrix} = a_x b_y + a_y c_x + b_x c_y - a_y b_x - a_x c_y - b_y c_x$

$\triangle abc$.

- b, - b, 0- .

, p' - q' ℓ $\overline{pq}$,
. p - q ℓ' $\overline{p'q'}$.

2.3 DCEL

2.3.1

map overlay , .
, ,) .(
map overlay ,
. , DCEL (Doubly-Connected Edge List).

:

2.2) 29–33)

2.3.2 map overlay

, overlay- . overlay-
.

$\mathcal{S}_1$ - $\mathcal{S}_2$ DCEL, DCEL overlay- $\mathcal{S}_1$ - $\mathcal{S}_1$, $\mathcal{O} = (\mathcal{S}_1, \mathcal{S}_2)$. overlay- , f_1 - $\mathcal{S}_1$ - f_2 - $\mathcal{S}_2$, f $f_1 \cap f_2$.) - f_1 f_2 .(, f f_1 - f_2, :

2.3 MapOverlay, $\mathcal{O} = (\mathcal{S}_1, \mathcal{S}_2)$.

2.3) 33–39)

2.3.3

MapOverlay . 2.4 :
. P_1, P_2, , ,
. ,overlay- - P_1, - P_2,
. , , .

2.4) 39–40)

Chapter 3

3.1

3.1.1

, , .
, , - .
; ,
. , , .
, ? ?
: .

3) .(45

3.1.2

1973- - .
, n P n) n ,(
P. - .
, ( ) g $p \in P$ $\overline{pq}$ P) P.(
, , ” “ .
P? , P.
?) .(
. ?

3.1.3 ?

P , ,
G , $p \in P$ $g \in G$ $\overline{pg}$ P) g

p.(NP- , ,) ,
.(

: ? ,
$g(n)$, : .1 $g(n)$ n , .2 n
$g(n)$.

, .

" " :
" " . 3 3 , .
k ? ?

: $n \geq 3$, n ,
$\lfloor n/3 \rfloor$, - , :

. ,
n $\lfloor n/3 \rfloor$.

, n n $\lfloor n/3 \rfloor$. , $g(n) \geq \lfloor n/3 \rfloor$.
? $\lfloor n/3 \rfloor$ n? .

3.2

3.2.1

, , .
, , .
, ?
, .
, . - .

:
P P .

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, 5 , . .
. , 5 5 .
.

n ?

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: $n-3$) .(
) ,(.2- , $\frac{n(n-3)}{2}$.

3.2.2

, , .

:)
- .( )

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. , .
.

?

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. .
, ? ! ,
4 .

: 4 , .

: P n , v P) , .(
- u, w - v . $\overline{uw}$ P,
P. , P $u, v, w,$ $\overline{uw}$)
$v,$ u, v, w .(v' .
$\overline{vv'}$ P, , - v. , $\overline{vv'}$
P.

: .

, n $n-2$ - $2n-3$.

: P n . n. $n = 3,$ P .
$n > 3,$ $m < n.$, - P . P
$P_1, P_2.$ - $n,$
P_1 - $P_2,$ $P.$
, $P,$ $\overline{uv}$) u, v
P.(P $P_1, P_2,$ m_1, m_2 , . P
u, v - $P_1, P_2,$ u, v . , $m_1 + m_2 = n + 2.$
, P_1 $m_1 - 2,$ P_2 $m_2 - 2.$
$m_1 - 2 + m_2 - 2 = n - 2$.
, P - $\overline{uv}$ $P_1, P_2,$ $\overline{uv}$.
, P $(2m_1-3)+(2m_2-3)-1 = 2(m_1+m_2)-7 =$
$2(n+2) - 7 = 2n - 3.$

.

, .

: , n
$O(n^2).$

, $O(n \log n),$ - $O(n)$.

3.2.3

(!) . ,
.

- T P. G_T T
- T, u, v
.

:
T G_T .

G_T - T, G_T) .(
G_T . ?

: **.**

T, G_T .

: , G_T . , P ,
G_T . , G_T , ,
.

: ! _ , .
, ? ?

3.2.4

n , - $\lfloor \frac{n}{3} \rfloor$. , - $g(n) \geq \lfloor \frac{n}{3} \rfloor$.
? , $\lfloor \frac{n}{3} \rfloor$ n ? ” “
?

- $\lfloor \frac{n}{3} \rfloor$! , $g(n) \leq \lfloor \frac{n}{3} \rfloor$, :

: $g(n) = \lfloor \frac{n}{3} \rfloor$**.**

: (Václav Chvátal),
(Steve Fisk). , “Proofs
from the BOOK” - (Paul Erdős): “a place where God
keeps aesthetically perfect proofs”.

- $\lfloor \frac{n}{3} \rfloor$, .
P n ?

:
P. n P, ?

P, , P. ,
$n-2$, - $n-2$.

. . ,
-) .(
k -) - k (( ) 1 - k
.

: **.**

: n. $n = 3$,. .
$n > 3$, $m < n$. P n , P,
overlineuv) u, v P.(P P_1, P_2.
, P_1 : , , , P_2 : , , .
" " P. ,
P_1 u, v , P_2 .
, , . P .

: $g(n) \leq \lfloor \frac{n}{3} \rfloor$**.**

, P n $\lfloor \frac{n}{3} \rfloor$.

: T P : , , . ,
$\lfloor \frac{n}{3} \rfloor$,) - n.(,
. T- : ,
. , , P.

.

, $\lfloor \frac{n}{4} \rfloor$. : " " ,
. ,)
,(. ,
$\lfloor \frac{n}{4} \rfloor$, ,
.

3.2.5 :

. ,
, .

3.1) 46–49).

3.3

3.3.1 ?

, , .

,
n $O(n^2)$.

: n
$O(n^2)$.) 3.2 (

?

. , ,
, . , $O(n)$.
, : ,
.

, .
$O(n \log n)$, - $O(n)$.

:
P ℓ ℓ' - ℓ, ℓ' . ,
$\ell' \cap P$, , .

- y y. -

y? - x? -

( )

( ) y - x - ( ) x - ( ) y -

3.3.2

MakeMonotone, P n y. -
,2 .

.

: P DCEL,
.2

? MakeMonotone 3.2
. $O(n \log n)$, - $O(n)$.

3.2) 49–55).

3.3.3

n y - $O(n \log n)$,
.

.

3.3 TriangulateMonotonePolygons .

$O(n)$, -
)$O(n^2)$) . , n !

3.3) 55-58).

Chapter 4

4.1

4.1.1

- , ,
. , .
) , , ,(
?

4) 63-64).

4.1.2

P n , P. P
, P .
P, n P, n .
P.

! : ?

) , . - (.
( )

.

castable. P castable,
. ,) P ,(
( ) $\vec{d}$ P $\vec{d}$.

f P , $\hat{f}$. , P
, f P " " " - $\hat{f}$. ,

) (f, $\vec{\mu}(f)$. f f.
$\hat{f}$, $\vec{d}$ ( ) $\vec{d}$ - $\vec{\mu}(f)$ 90°.

: $\vec{d}$ P
90°. P $\vec{d}$. " : $\vec{d}$ - $\vec{\mu}(f)$
- 90° $\vec{d} \cdot \vec{\mu}(f) \leq 0$. $\vec{d}$ $z = 1$, , $\vec{d}$
$(x, y, 1)$ $(x, y, 1)$. $\vec{\mu}(f) = (\mu_x, \mu_y, \mu_z)$
: $\mu_x \cdot x + \mu_y \cdot y + \mu_z \leq 0$.

$\vec{d}$ P , (x, y) , ,
. :

n , , .
$O(n)$, $O(n^2)$.

4.1) 64-66).

4.2

4.2.1

, , .
, . ,( ) .( )

, .
4.2 IntersectHalfPlanes, H ,
. : H ,
,
IntesectConvexRegions.

IntersectConvexRegions .

4.2) 66–68).

4.2.2

2) ,( ) (
O ((n + k) log n), k , MapOverlay.

n ?
.

( )

C_1 - C_2 n , $2n$.
e C_1 C_2: e C_2, C_2
.

C_1 - C_2 C_1 C_2:
n .

MapOverlay , $O(n \log n)$,
, $k \leq 2n$. , (Master Theorem),
IntersectHalfPlanes $O(n \log^2 n)$. ?
!) , $O(n \log n)$...)

MapOverlay , . $(n + k)$ –
. $\log n$ – , ,) $O(n)$,(
. ?

, 4 .
) .(y, -
( ) .

, .
$O(n)$,
- $O(n \log n)$.

4.2) 71–68).

4.3

4.3.1 ?

, . ,
.
, . ,
, .

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4.3) 71–72).

4.3.2

4.3 2DBoundedLP,
. . ,
$m_1, m_2,$ $\vec{c},$
. .

$C_0 = m_1 \cap m_2,$ $h_1, \ldots, h_n$.
- $i,$ v_i $C_i = m_1 \cap m_2 \cap h_1 \cap h_2 \cap \cdots \cap h_i.$
, v_i:

$v_{i-1} \in h_i$) ,(h_i $v_i = v_{i-1}$. , h_i
- C_i v_{i-1}) .(, C_i , v_i
ℓ_i h_i.

, $O(n^2),$ ℓ_i
$m_1, m_2, h_1, \ldots, h_{i-1}$. , , - i $O(i)$.

? : ,
v_i $O(i)$?

) 4.3 (

, .

4.3) 73–76).

4.3.3

, $O(n)$, 2DRandom-
izedBoundedLP . ,
: ,
RandomPermutation.

, , :
. ,
, $O(n)$.

4.3.3.1 :

4.3) 66–68).

4.3.4

, , m_1, m_2.
, , h_1, h_2
m_1, m_2.

, ρ C. p
$\vec{d}$. rho $\vec{d}$
$\vec{c}$ - 90°. , h, $\vec{d}$ $\vec{\mu}(h)$ h) (.
. , $\vec{\mu}(h)$ - $\vec{d}$)h_1, h_5 ,(.

ρ .
, , $O(n)$. 4.5
.

4.4) 79–82).

:

ρ, h_i $\vec{c}$ - $\vec{\mu}(h_i)$.

h_j ρ . $j \neq i$ ρ $\ell_1 \cap h_j$
$\vec{c}$,- $\vec{c}$ C- - $\vec{c}$.

, h_j ρ - $\vec{c}$,- h_i - h_j - C - $\vec{c}$.
$\vec{c}$ - $\vec{\mu}(h_j)$ $\vec{c}$ - $\vec{\mu}(h_i)$, h_j - $\vec{c}$ - 90°.

Chapter 5

5.1

5.1.1

. ,
, .

- d : ,
.) ,(
R, , d - . ,
.

P - d , :
R, P R?

5) .(96 95

5.1.2

,
, . ,
, P , ,
. , R equals left square bracket x comma x'
right square bracket.

P . left square bracket x comma x'
right square bracket, ,x-
, .'x- O(log n plus k)
k . O(n log n) O(n).
: ,) ,(.
, 5.1 , .

. T left square bracket x comma x ' right square bracket, 1DRangeQuery x- x- ' , bold- italic x, bold- italic x bold'. x ' .x-

, .

5.1) 96–99).

5.2

5.2.1 kd-

KD , .kd- , .

:

kd- n O(n log n). O(n), $O(\sqrt{n}+k)$, k , . 5.2 BuildKdTree , SearchKdTree. , , .kd-

5.2) 99–105).

, , .kd-

( )

(TODO)

kd- , :

.1) (2 SearchKdTree?
2. ReportSubtree?

( )

(TODO)

.

. R , :

.1 R .
.2 R .

R c , ? ? R ?

5.2.2 (Range Trees)

, range tree.
, : ,x-
.y-

:

(TODO)

, kd- - $O(log^2 n + k)$,
$O(\sqrt{n} + k)$ .kd- – $O(nlogn)$ $O(n)$. 5.3
Build2DRangeTree , 2DRangeQuery.
, , .

5.3) 105–109).

.

log n, Fractional Cascading .
, 5.6 .

5.3

5.3.1

5.2 kd- $d > 2$. d ,
$O(n \log n)$ $O(n)$. $O(n^{1-\frac{1}{d}} + k)$,
- $O(n)$ - d . 5.4 $d > 2$.
$O(n \log^{d-1} n)$, $O(\log^d n + k)$.

5.4) 109–110).

5.3.2

5.3 5.1 x y.
, , ,
. ,
, . 5.5 .

5.5) 110–111).

Chapter 6

6.1

6.1.1

, , , , .
. : ,
?

6) .(122 121

6.2

6.2.1 :

$\mathcal{S}$, 2 : $\mathcal{S}$, .
: q , q .

q , ( ) $\mathcal{S}$ q " " . ,
6.1 , (slabs):
$\mathcal{S}$. . $\mathcal{S}$ n,
q $O(\log n)$. $\mathcal{S}$,
q " " , $O(\log n)$.

, $O(\log n)$, ?
$\Theta(n^2)$.(?)

6.1) 122–124).

6.2.2

$\mathcal{S}'$)
.(/ $\mathcal{S}'$ (refinement) $\mathcal{S}$, , $\mathcal{S}'$
$\mathcal{S}$, $\mathcal{S}'$, $\mathcal{S}$.
) ,(.
$\mathcal{S}$, .

:

(TBD)

- $leftp$?

( )

(TODO)

.

x. ,
, - ,
. 6.3 .

,) (?

( )

$\Theta(n)$

: 6.1) 124–128).

6.3

6.3.1

$\mathcal{S}$ n .

$\mathcal{T}(\mathcal{S})$,) .(? ,
, . ?

$\mathcal{D}$ $\mathcal{D}$. , -
$\mathcal{T}(\mathcal{S})$ $\mathcal{D}$.

$\mathcal{D}$ (DAG), :

- ( ) x - - $\mathcal{S}$) .(
- ( ) y - - $\mathcal{S}$) .(
- - $\mathcal{T}(\mathcal{S})$) .(

, , $\mathcal{D}$. , C .

kd- :5 ,
.

q, :

- - x p, : “ q .”?p
  ” “ - , .
- - y s, : “ q s.”? ” “ - ,
  . , q s.

, .q

! . .

:

- p_1.
- q_1.
- s_1.

?

:

- q_2.
- s_2.

?

( )

TBD

6.3.2

$\mathcal{D}$ $\mathcal{T}(\mathcal{S})$.
$\mathcal{S}$, .

,
- 4 . .

:

(TBD)

6.2) 128–133).

:

s_i?

( )

, .

( )

6.3.3

TrapezoidalMap $\mathcal{D}$. ,
, $\mathcal{D}$, . ,
- 4 ,
, , $O(\log n)$.

? n ,) (
$O(n^2)$.

) .(

q . $\mathcal{D}$ q)
.(, $\mathcal{D}$.

, , 3 .
. ,) $n!$,(
.

- X_i) $1 \leq i \leq n$(- i ,) s_i.(,
X_i , q $E[\sum_{i=1}^{n} X_i]$
$\sum_{i=1}^{n} E[X_i]$.

3 , $X_i \leq 3$. , - P_i
- i, $E[X_i] \leq 3P_i$.

- i q s_i
q s_i. , ,4 q
- $i-1$ - i (backward analysis). ,
s_i, Δ q s_i. Δ
)$top(\Delta)$, $bottom(\Delta)$, $leftp(\Delta)$, $rightp(\Delta)$) . - s_i
$top(\Delta)$, 1 over i, $bottom(\Delta)$, $leftp(\Delta)$,
$rightp(\Delta)$. , $P_i \leq \frac{4}{i}$, $O(\log n)$.

6.2) 133–137).

Chapter 7

7.1

7.1.1 :) (

) ,( ) .(
? ? ? (Voronoi
diagram) , , . ,
) ,(
.

: , -
, , , , , . TODO

7) .(148 147

7.2

7.2.1

$P = (p_1, ..., p_n)$ n .

:

P - n , P, q
p_i $\|p_i - q\| < \|p_j - q\|$ $p_j \in P$, $j \neq i$.

- Vor(P) P, $p_i \in P$ - $\mathcal{V}(p_i)$ - p_i.

, p_1, p_2. ?

$\overline{p_1 p_2}$. bisector- p_1 - p_2. bisector-
- p_1 - p_2.

bisector- p_1 - p_2 . - $h(p_1, p_2)$ p_1.
- $h(p_1, p_2)$, - p_1 - p_2.

7.2.2

?

:

: $\mathcal{V}(p_i)$ $n-1$,) (
$n-1$ - $n-1$.

?

, , ,
.) (.
:

: P Vor(P) - $n-1$. , Vor(P) ,
.( )

.

$\Theta(n)$, .

? , ,3
Vor(P) $O(n)$) .(

: Vor(P) $O(n)$.

.

.
q, $C_P(q)$ - q ( ) P.
, . .

:

.1 q Vor(P) $C_P(q)$ P .
.2 bisector- p_i - p_j Vor(P) q bisector- - $C_P(q)$
p_i - p_j , P .

, .

( )

.

7.1) .(151 148

7.3

7.3.1

. p_i
$h(p_i, p_j)$ $j \neq i$,4 - p_i.
$O(n \log n)$, $O(n^2 \log n)$. ? ,
.... $O(n \log n)$,
.

7.3.1.1 .

7.3.2 Fortune

7.2 Fortune .
. , ,
:

: n $O(n \log(n)$.

.

) Kevin Schaal) :

: 7.2) .(159 151

7.4

L_2, , .
: .
.

7.4.0.1

L_1

7.4.0.2

7.4.0.3

- k) (