The Gate-Cover Problem in Thin Polyominoes

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Abstract

We introduce a new variant of the art gallery problem, namely, the *Gate-Cover Problem in Thin Polyominoes*. We show that the VC-dimension of the problem is 3, describe an efficient greedy algorithm and present selected experimental results. We also discuss some of the open questions that we are considering in this ongoing research project.

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1 Introduction

The Art Gallery Problem (AGP) is a classic problem in computational geometry [4, 5, 6]. In this problem, a polygon P is given, and the goal is to find the smallest set of points to guard P, where a point $p \in P$ is guarded by a point $g \in P$ if and only if the line segment $\overline{gp}$ is contained in P. The motivation for AGP, as implied by its name, is placing a small set of cameras that allow detecting an intruder in an art gallery. However, sometimes guarding the entire gallery is unnecessary, and it is enough to guard a certain set of "gates" in P, so that one can output an approximate location of an intruder, by following it from the moment it enters the gallery. For example, suppose we have a city where a thief roams the streets, and we would like to delimit his location to a certain street, even without knowing his exact location. To represent the city, we use a thin polyomino P. A polyomino P is a polygon formed by joining together $|P| = n$ unit squares on the square lattice, and a polyomino is thin if it does not contain a 2×2 block of unit squares. In this analogy, a street would be a sequence of unit squares in P, such that every pair of consecutive unit squares share an edge. Thus, to be able to tell in which street the thief is located at every moment in time, we simply need to guard all the "junctions" of P, i.e., edges of squares that separates two streets, and "entrances" to the city, which are edges on the boundary of P. We therefore introduce a new variant of AGP in thin polyominoes, which we call *The Gate-Cover Problem in Thin Polyominoes*, and where the goal is to guard a set of gates in a thin polyomino P. We assume that cameras are placed on the center points of unit squares, and that a camera c sees a gate g if and only if g belongs to a unit square visible to c in the same row/column as c. The range of the camera can be unbounded or within some hop-distance.

▶ **Problem 1** (The Gate-Cover Problem in Thin Polyominoes). *Given a thin polyomino $\mathcal{P}$ and a set G of gates, find the minimum set of cameras that cover all gates.*

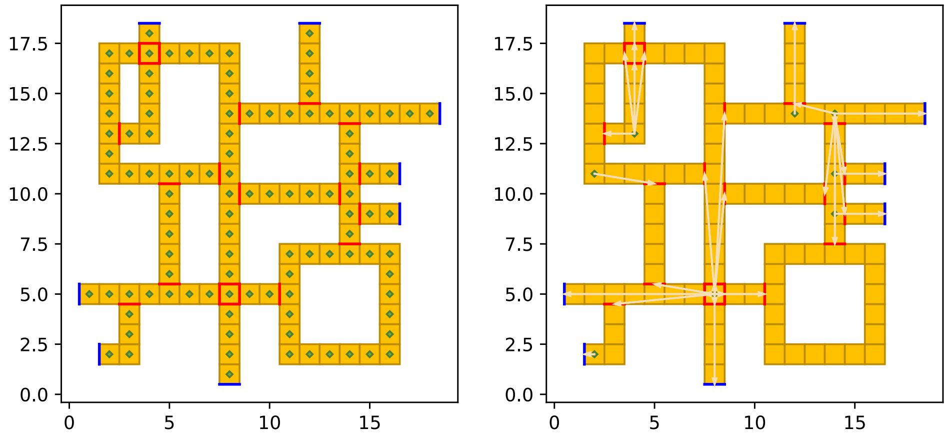

Figure 1 Right: Center camera in each square. Left: Greedy solution with infinite camera.

2 VC-dimension

We show that the VC-dimension of the gate cover problem in thin polyominoes is 3. Let P be a polyomino and G a set of gates. For a camera (unit square) $c \in P$, denote by $G(c)$ the set of gates that c covers. Consider the set system $(G, \mathcal{R})$ where $\mathcal{R} = \{G(c) \mid c \in P\}$. The set G is shattered by $\mathcal{R}$ if for every subset of G, there exists a camera c that covers exactly this subset. We show that there exists a set of three gates that can be shattered (Figure 2), and that no set of four gates can be shattered. Therefore the VC-dimension is 3, which implies a $O(\log \mathrm{OPT})$-approximation algorithm in near linear time by Brönnimann-Goodrich [1].

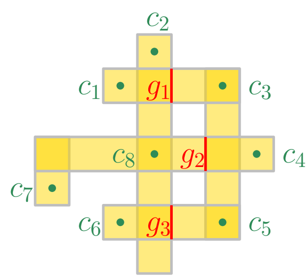

Figure 2 A polymonio with 3 gates that can be shattered.

3 Exact algorithms

To obtain an exact algorithm, we use orthogonal line separators. Recently, separators have been used to obtain exponential time exact algorithms for various problems (see, e.g., [3]). Carmi et al. [2] show that there exists either a horizontal or vertical line on the grid that intersect $O(\sqrt{n})$ unit squares of P, and such that on each side of it there are at most $4/5n$ unit squares of P. Given a vertical line separator ℓ, we can divide the problem into $3^{\sqrt{n}}$ subproblems of size at most $4/5n$, as follows. For each unit square edge that lies on the separator, we have three options: (1) there is a camera above it (in the same column), (2) there is a camera below it, and (3) there is no guarantee that a camera is placed in this column. For the third type of subproblems, we do not have to update the polyomino. For the first (resp. second) type of subproblems, we add a single unit square with an entrance gate below (rep. above) the line, and remove all the gates in the column below it (resp. above it).

Note that for simply connected thin polyominoes (without holes), this approach gives a polynomial time algorithm, because the dual graph of the polyomino in this case is a tree, and thus there is a single vertex that separates the polyomino.

4 A greedy approximation algorithm

Let P be a thin polyomino with n unit squares, and G a set of gates. Consider the following greedy algorithm: Initialize an empty set S. While G is not empty, find a camera c that maximizes $|G(c)|$ (If there is more than one such camera, choose one arbitrarily). Add c to the set S, and remove $G(c)$ from G. The algorithm stops when all gates are covered, and thus S is a gate cover for G.

Running time. To bound the running time, we first observe that not all centers of unit squares have to be considered for placing cameras. More precisely, we can consider only "dominating" cameras (a camera c is dominating if there is no camera c' for which $G(c) \subseteq G(c')$). Dominating cameras are those placed in the unit squares that contain gates, or in "corner" unit squares that cover at least two gates (see Figure 1). Thus there are $O(|G|)$ such cameras. The running time of the algorithm is therefore $O(n + |G| \log |G|)$.

Approximation factor. In Figure 3 we show a polyomino in which the above algorithm output a set of 3 cameras, while the optimal solution has 2 cameras. In an ongoing work we formalized several claims that we hope would lead to an upper bound of 3/2 on the approximation factor. Next, we present experimental results that support this direction.

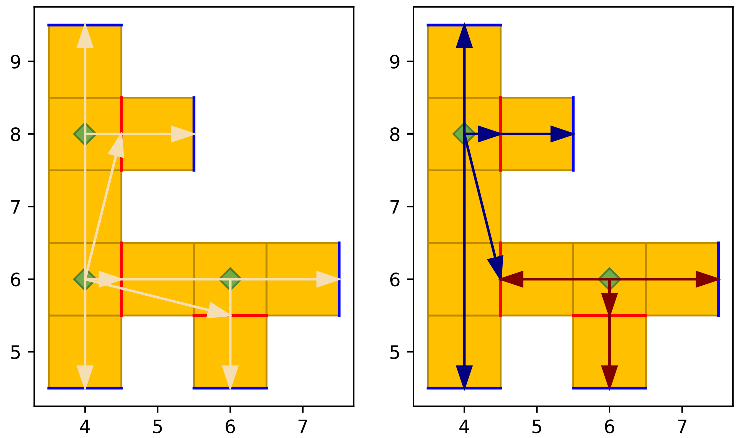

Figure 3 Right: Greedy results. Left: Brute force result - OPT solution.

5 Experimental results

We have constructed random "cities" with increasing size and used the following algorithms to solve the gate cover problem: (i) The greedy algorithm described above, (ii) A brute force algorithm (up to a reasonable size for running), and (iii) 0-1 Integer-Linear-Programming.

The results for unbounded-range cameras and 5-hop distance cameras are presented in Appendix A. Note that comparing to brute force (in small cities) and the $0-1ILP$ algorithm (in larger cities), the greedy algorithm achieves close to optimal results, and always within a 3/2-factor from the optimum. In our ongoing work, we are trying to prove that $0-1ILP$ is optimal by computing the dual problem.

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A Experiments

In general, one can place a *gate* on any edge of a unit square in P. For our purposes, we place one "entrance" gate on every unit square that has exactly one neighbour (see blue edges in Figure 1), a single gate for every unit square with 3 neighbors (a T-junction), and 4 gates for every unit square with 4 neighbours (see red edges in Figure 1).

Table 1 Appendix A, Infinite cameras experimental results summary

#	City size [Squares] (Cameras to select from, non-reduced)	Cameras to select from, reduced	Gates and entrances in the problem	Reduced + Greedy (1000 trials)	Brute force	$0-1LP$
0	18	7	6	1	1	1
1	27	10	12	2	2	2
2	50	17	22	3	3	3
3	56	17	22	2	2	2
4	105	27	48	5	4	4
5	100	31	58	5	5	5
6	147	33	46	4	4	4
7	134	32	60	5	4	4
8	234	60	124	8	7	7
9	312	71	164	10		8
10	304	72	158	7	7	7
11	370	80	188	9		9
12	422	100	258	8	8	8
13	452	107	238	14		11
14	570	118	298	11		9
15	692	148	376	14		12
16	659	149	386	12		11
17	762	148	366	15		13
18	824	174	484	14		12
19	840	169	452	15		12
20	935	177	522	12		12
21	984	171	426	16		14
22	978	192	518	15		14
23	1122	233	650	19		15
24	1210	231	642	20		17
25	1328	242	676	18		16
26	1377	271	778	19		18
27	1637	316	906	22		19
28	1403	248	686	19		16
29	1651	320	942	22		19
30	1678	307	894	22		20
31	2082	371	1138	22		20
32	1880	337	1012	23		19
33	2170	350	1000	28		22
34	2077	374	1078	29		24

Table 2 Appendix B, 5-hop distance cameras experimental results summary

#	City size [Squares] (Cameras to select from, non-reduced)	Cameras to select from, reduced	Gates and entrances in the problem	Reduced + Greedy (1000 trials)	Brute force	$0-1LP$
0	18	10	6	1	1	1
1	27	14	12	2	2	2
2	50	28	22	3	3	3
3	56	27	22	2	2	2
4	105	61	48	7	7	7
5	100	56	58	8	7	7
6	147	65	46	8	7	7
7	134	53	60	10	9	9
8	234	108	124	16		12
9	312	171	164	21		18
10	304	165	158	16		15
11	370	198	188	24		20
12	422	211	258	20		20
13	452	258	238	34		29
14	570	301	298	31		30
15	692	328	376	35		33
16	659	358	386	37		32
17	762	331	366	38		34
18	824	467	484	47		41
19	840	418	452	44		39
20	935	461	522	52		42
21	984	455	426	45		42
22	978	459	518	49		44
23	1122	584	650	65		56
24	1210	654	642	68		58
25	1328	697	676	73		63
26	1377	698	778	75		63
27	1637	844	906	86		72
28	1403	697	686	73		62
29	1651	815	942	89		78
30	1678	908	894	94		78
31	2082	966	1138	96		81
32	1880	920	1012	99		83
33	2170	996	1000	104		88
34	2077	994	1078	107		93