Nonparamtric Estimation of Drift and/or Diffusion Coefficient(s) of SDE

Présented by :

ONANA ASSOUGA Florent

Under the supervision of:

Pr TAKAM SOH Patrice

UNIVERSITY OF YAOUNDÉ I

DÉPARTEMENT OF MATHÉMATICS

26 juillet 2023

We are interested on the identification of continuous time models, with dynamics governed by a local mean (drift) $b(\cdot)$ and local variance (diffusion coefficient, or volatility) $\sigma(\cdot)$ on the state space $\mathcal{S} = \mathbb{R}$ or $\mathcal{S} \subset \mathbb{R}$ with appropriate boundary condition. The dynamics are described by an Itô-type stochastic differential equation in the interior of $\mathcal{S}$ like :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

where the driving process $(W_t)_{t\geq 0}$ is standard Brownian motion. We want to construct the estimators of $b(\cdot)$ and $\sigma(\cdot)$ under the conditions of existence and uniqueness of strong/weak or stationary/nonstationary solutions of (1) .

Objectives

- Present the area of nonparametric estimation of SDE and its historical evolution by naming some forerunners Banon (1978), Prakasa(1983), Florens-Zmirou(1993), Aït Sahalia(1996), Bandi and Philipps (2003, 2010)
- Present three methods of Nonparametric Estmation : Kernel method(Florens Zmirou 1964, Bandi and Philipps), spectral method (Gobet, Hoffmann and M. Reiss) and the method of Sieves (Prakasa)
- Give some orientations of our future tasks

plan de l'exposé

1. Preleminaries and basic Notations
   - History
   - Preleminaries
2. Three methods of Nonparametric Estimation in SDE
   - Kernel Method
   - Spectral Method
   - Method of Sieves
3. Remarks and Open problems

plan de l'exposé

1. Preleminaries and basic Notations
   - History
   - Preleminaries
2. Three methods of Nonparametric Estimation in SDE
   - Kernel Method
   - Spectral Method
   - Method of Sieves
3. Remarks and Open problems

plan de l'exposé

plan de l'exposé

plan de l'exposé

1. Preleminaries and basic Notations
   - History
   - Preleminaries
2. Three methods of Nonparametric Estimation in SDE
   - Kernel Method
   - Spectral Method
   - Method of Sieves
3. Remarks and Open problems

From continuous sampling to discrete sampling

In the late 1970s, a statistician was able to characterize qualitatively the properties of parametric ergodic diffusion model based on the continuous observation of a sample path

$$X^T = \{X_t, 0 \leq t \leq T\}$$

of the trajectory, as $T \to \infty$.
The 1980s explored some various discretization schemes of the the continuous time model : the data X^T could progressively be replaced by the more realistic observation

$$X^{(N,\Delta_N)} = (X_{n\Delta_N}, n = 0, 1, \cdots, N) \tag{2}$$

with asymptotics taken as $N \to \infty$. We have two type discretization :

- **Low frequency** : that is the fixed case, $\Delta_N = \Delta$
- **High frequency** : that is $\Delta \to 0$ and $N\Delta_N \to \infty$ as $N \to \infty$ in order to guarantee the closeness of X^T and $X^{(N,\Delta)}$.

From stationarity to nonstationarity

Florens-Zmirou (1993) was the first to provide the asymptotic theory for the nonparametric estimation of diffusion $\sigma(\cdot)$ with discrete sampling and high frequency (she considered the sde of the form : $dX_t = bdt + \sigma(X_t)dW_t$ by assuming b to be known) ; however she did not consider the problem of estimating the drift term $b(\sigma)$. Jiang and Knight (1997) considered nonparametric estimation of both $b(\cdot)$ and $\sigma(\cdot)$ on stationary case. Bandi and Philipps 2003, 2010 were the first to provide the asymptotic distribution theory for the nonparametric drift estimator and drift estimator under general conditions(eg : allowing for nonstationarity).

plan de l'exposé

1. Preleminaries and basic Notations
   - History
   - Preleminaries
2. Three methods of Nonparametric Estimation in SDE
   - Kernel Method
   - Spectral Method
   - Method of Sieves
3. Remarks and Open problems

Définition 1 (Type of Solutions)

A diffusion process X_t is a

- **Strong solution** if given the BM W_t, the process satisfies (1) such that X_t is adapted to the filtration generated by W_t.
- **Weak solution** if there exists a BM W_t and (1) holds.
- **Weak uniqueness** if whenever (X_t, W_t) and (X_t', W_t') are two weak solutions, then the joint law of the process (X_t, W_t) and $\left(X_t', W_t'\right)$ are equal.

Lemme 2.1

- *If $\sigma(\cdot)$ and $b(\cdot)$ are lipschitz and bounded, then there exists a* **unique** *pathwise solution to the SDE (1).*
- *If $\sigma(\cdot)$ and $b(\cdot)$ are lipschitz there exists a constant C such that*

$$|\sigma(x)| + |b(x)| \leq C(1 + |x|), 0 \tag{3}$$

  then there exists a pathwise solution to (1) and the solution is pathwise unique.
- *If $\sigma(\cdot)$ and $b(\cdot)$ are lipschitz, then the solution to (1) is a strong solution.*
- *If $\sigma(\cdot)$ and $b(\cdot)$ fail to be lipschitz, then the weak uniqueness holds for (1).*

Définition 2 (Infinitesimal generator($\emptyset ksendal$1985))

The infinitesimal generator $\mathcal{L}$ of X_t, solution of (1) is the operator defined by :

$$\begin{aligned} \mathcal{L}f(x) &= \lim_{\rho \to t} \frac{\mathbb{E}\left(f(X_\rho) | X_t = x\right) - f(x)}{\rho - t} \\ &= b(x)f^{'}(x) + \frac{1}{2}\sigma^2(x)f^{''}(x) \end{aligned} \tag{4}$$

with f a twice differentiable function.

Remark

Since $\mathcal{L}$ is an operator, we have for all integer $k > 1$,

$$\mathcal{L}^k f = \mathcal{L}^{k-1}(\mathcal{L}f). \tag{5}$$

Définition 3 (Transition density)

The transition density of the diffusion process X_t solution of (1) is defined for $s, t \in [0, T]$ by

$$p(t, y|s, x) = \mathbb{P}\left(X_t = y|X_s = s\right).$$

Remark

We can also use the notation below :

$$p(s, y|x) = \mathbb{P}\left(X_{t+s} = y|X_s = s\right).$$

Proposition 2.1 (Kolmogorov Backward and Forward Equations)

- *Kolmogorov forward equation*

$$\begin{aligned}\frac{\partial}{\partial t}p(t,y\mid s,x) &= -\frac{\partial}{\partial y}\left(b(y)p(t,y\mid s,x)\right)\\ &+ \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(y)p(t,y\mid s,x)\right). \qquad (6)\end{aligned}$$

- *Kolmogorov backward equation*

$$\begin{aligned}-\frac{\partial}{\partial s}p(t,y\mid s,x) &= b(x)\frac{\partial}{\partial x}p(t,y\mid s,x)\\ &+ \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}p(t,y\mid s,x). \qquad (7)\end{aligned}$$

Proposition 2.2 (Itô-Taylor Expansion)

Let f be a function such thatits first two derivatives f' and f'' have exponential growth. Then, for every $x_0 \in \mathbb{R}$, there exists $\delta \in [0, \Delta]$ such that :

$$\mathbb{E}\left[f(X_{t+\Delta}) \mid X_t = x_0\right] = \sum_{k=0}^{K} \mathcal{L}^k f(x_0) \frac{\Delta^k}{k!} + \underbrace{\mathbb{E}\left[\mathcal{L}^{K+1} f(X_{t+\delta}) \mid X_t = x_0\right] \frac{\Delta^{K+1}}{(K+1)!}}_{Reste} \quad (8)$$

In addition, there exists a constant P_K depending on K but not on f and δ such that :

$$\left|\mathbb{E}\left[\mathcal{L}^{K+1} f(X_{t+\delta}) \mid X_t = x_0\right]\right| \leq P_K.$$

Définition 4 (Transition density Operator)

The transition density operator p_Δ of X_t, solution of (1) is the operator defined by :

$$p_\Delta f(x) = \mathbb{E}\left(f(X_\Delta) \mid X_0 = x\right) \tag{9}$$

where f is a twice differentiable function.

Remark : Relation between the infinitesimal generator and the Transition density Operator

If the transition density p exists, then the infinitesimal generator is unbounded, self-adjoint negative on $L^2(p)$ and in operator sense , we have :

$$p_\Delta = exp(\Delta\mathcal{L}). \tag{10}$$

plan de l'exposé

1. Preleminaries and basic Notations
   - History
   - Preleminaries
2. Three methods of Nonparametric Estimation in SDE
   - Kernel Method
   - Spectral Method
   - Method of Sieves
3. Remarks and Open problems

plan de l'exposé

1. Preleminaries and basic Notations
   - History
   - Preleminaries
2. Three methods of Nonparametric Estimation in SDE
   - Kernel Method
   - Spectral Method
   - Method of Sieves
3. Remarks and Open problems

The equation (8) can be written as :

$$\mathbb{E}\left[f(X_{t+\Delta}) \mid X_t = x_0\right] = \sum_{k=0}^{K} \mathcal{L}^k f(x_0) \frac{\Delta^k}{k!} + \bigcirc(\Delta^{K+1}).$$

And the following form can be obtained :

$$\mathcal{L}f(X_t) = \frac{1}{\Delta}\left\{\mathbb{E}\left[f(X_{t+\Delta}) \mid X_t\right] - f(X_t)\right\} - \sum_{k=2}^{K} \mathcal{L}^k f(X_t) \frac{\Delta^k}{k!} + \bigcirc(\Delta^{K+1}).$$

We deduce that the first order approximation for $\mathcal{L}f(X_t)$, is :

$$\mathcal{L}f(X_t) = \frac{1}{\Delta}\left\{\mathbb{E}\left[f(X_{t+\Delta}) \mid X_t\right] - f(X_t)\right\} + \bigcirc(\Delta). \qquad (11)$$

Given suitable choices of the function f, equation (11) can then be used to construct approximations of $b(\cdot)$ and $\sigma(\cdot)$ of first order.

Remark

Bandi and Nguyen (1999) argued that approximations to the drift and diffusion of any order display the same rate of convergence and have the same asymptotic variance, so that asymptotic arguments in conjunction with computational issues suggest simply using the first order approximations in practice.

Constructing Approximations to $b(\cdot)$ and $\sigma(\cdot)$

To derive approximation to the drift, consider the function

$$f_1(x) = x.$$

From the definition of $\mathcal{L}$, we have $\mathcal{L}f_1(x) = b(x)$. And so :

$$b(X_t) = \frac{1}{\Delta}\mathbb{E}\left\{X_{t+\Delta} - X_t \mid X_t\right\} + \bigcirc(\Delta). \tag{12}$$

To construct approximation to the diffusion, consider the function :

$$f_2(x) = (x - X_t)^2.$$

So we have $\mathcal{L}f_2(X_t) = \sigma^2(X_t)$. And so :

$$\sigma(X_t) = \sqrt{\frac{1}{\Delta}\mathbb{E}\left\{(X_{t+\Delta} - X_t)^2 \mid X_t\right\}} + \bigcirc(\Delta). \tag{13}$$

Proposition : Some Local Constant Estimators

- If we have a continuous sampling X^T :

$$\hat{b}(x) = \frac{\int_0^T K\left(\frac{X_s - x}{h}\right) dX_s}{\int_0^T K\left(\frac{X_s - x}{h}\right) ds} \tag{14}$$

in the sense of Skorokhod, where K is a kernel function and h is the bandwidth. To deduce an estimation of the drift, we can use the Kolmogorov backward equation in wich we turn t tp $-\infty$ to get a relation between the drift and the diffusion.

Local Constant Estimator

Proposition : Some local Constant Estimator

- If we have a discrete sample $X^{N,\Delta}$:

$$\hat{b}(x) = \frac{\frac{1}{Nh}\sum_{i=1}^{N} K\left(\frac{X_{i\Delta}-x}{h}\right)\left(\frac{X_{(i+1)\Delta}-X_{i\Delta}}{\Delta}\right)}{\sum_{i=1}^{N} K\left(\frac{X_{i\Delta}-x}{h}\right)} \tag{15}$$

$$\sigma^2(x) = \frac{\frac{1}{Nh}\sum_{i=1}^{N-1} K\left(\frac{X_{i\Delta}-x}{h}\right)\left(\frac{X_{(i+1)\Delta}-X_{i\Delta}}{\sqrt{\Delta}}\right)^2}{\sum_{i=1}^{N-1} K\left(\frac{X_{i\Delta}-x}{h}\right)}. \tag{16}$$

Local Linear Estimator

Proposition : Cfr Aït Sahalia 2015

contenu...

plan de l'exposé

1. Preleminaries and basic Notations
   - History
   - Preleminaries
2. Three methods of Nonparametric Estimation in SDE
   - Kernel Method
   - Spectral Method
   - Method of Sieves
3. Remarks and Open problems

Equation (10) suggests a consistent statistical program starting from the observed Markov chain $X^{N,\Delta}$, estimate its transition density operator p_Δ and infer about the pair Constructing Approximations to $(b(\cdot);\sigma(\cdot))$ via the equation (4).

$$Data = X^{N,\Delta} \to p_\Delta \to \mathcal{L} \to (b(\cdot);\sigma(\cdot)). \tag{17}$$

We can refer to Kessler and S∅rensen (1999),, Hansen, Scheinkman and Touzi (1998).

Définition 5

An estimator of the pair $(b(\cdot);\sigma(\cdot))$ is an $\mathcal{F}_N$-mesurable function on Ω with values in $L^2([0,T]) \times L^2([0,T])$.

Définition 6

For $s > 1$ and given constants $C \geq c > 0$, we consider the class $\Theta_s = \Theta(s, C, c)$ defined by :

$$\left\{(b(\cdot); \sigma(\cdot)) \in H^{s-1}([0, T]) \times H^s([0, T]) \mid\right.$$

$$\text{——}\sigma||_{H^s} \leq C, ||b||_{H^{s-1}} \leq C, \underset{x}{\sigma(x)} \geq c (18)$$

plan de l'exposé

plan de l'exposé

A word on New method such as : Dynamic Programming, Galerkin Method

A word on Time effects and and Generalization of SDE : Fractional BM, Levy and Hawks Processes

contenu...

Bibliographie

- Y. Aït-Sahalia, Y. Joon Park (2015), Bandwidth Selection and Asymptotic Properties of Local Nonparametric Estimators in Possibly Nonstationary Continuous-Time Model, Elsevier.
- Bandi, F.M., Phillips, P.C.B., (2003), Fully Nonparametric Estimation of Scalar diffusion models, Econometrica.
- D Florens Zmirou (1993), On Estimating the diffusion coefficient from discrete observations, Journal of applied probability.
- B.L.S. Prakasa Rao (1983), Estimation of the Drift for Diffusion Process, statistics 16 (1985) 2,263-275.

NOUS VOUS REMERCIONS

POUR VOTRE AIMABLE

ATTENTION !