Uniform Convergence of Bernstein Polynomial Sequence and its Derivatives

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January 26, 2024

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Introduction

- We are interested on sequences of polynomials named after their creator Sergei Natanovich Bernstein.

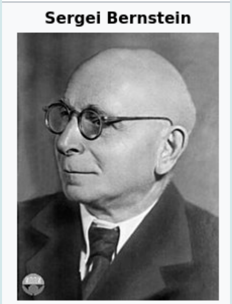

Figure: March 1880-October 1968

He proved :

- The Weierstrass approximation theorem
- And the Bernstein's theorem (approximation theory) .

Motivation

- One might wonder why Bernstein created “new” polynomials for a constructive proof of Weierstrass’s Theorem , instead of using polynomials such as Taylor polynomials or interpolating polynomials that were already known to mathematics.

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- For even setting aside questions of convergence, Taylor polynomials are applicable only to functions that are infinitely differentiable, and not to all continuous functions.

Objectives

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- Study the uniform convergence of Bernstein polynomial and its derivatives (based on additional assumptions)
- See that although the convergence of the Bernstein polynomials is slow, they have compensating “shape-preserving” properties.

Definition (Bernstein polynomial)

Given a function f on $[0, 1]$, we define the Bernstein polynomial as

$$B_n(f, x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \tag{1}$$

for each positive integer n.

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Example

- $B_n(1, x) = 1,\ B_n(x, x) = x,\ B_n(x^2, x) = x^2 + \frac{x(1-x)}{n}$
- $B_n(e^{\alpha x}, x) = \left(xe^{\frac{\alpha}{n}} + (1-x)\right)^n$

Theorem (Weierstrass's first Theorem)

If $f \in \mathcal{C}([0,1])$*, then for every* $\epsilon > 0$*, there exists a polynomial* p *such that* $||f - p||_\infty < \epsilon$*.*

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Proof.

It uses the Bernstein polynomial sequence and the Lemma of Korovkin. □

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Lemma(Korovkin Lemma)

Suppose $H_n, n = 1, 2, 3, \cdots$ is a sequence of positive linear operators on $\mathcal{C}[a, b]$. If for $k = 1, 2, 3$ one has $\lim_{n\to\infty} ||H_n(x^k, x) - x^k||_\infty = 0$, then it follows that for all $f \in \mathcal{C}[a, b]$

$$\lim_{n\to\infty} ||H_n(f, x) - f(x)||_\infty = 0. \tag{2}$$

Corollary

- If $f \in \mathcal{C}[0,1]$, then
$$B_n(f,\cdot) \xrightarrow[{[0,1]}]{CU} f. \tag{3}$$
- If $f \in \mathcal{C}[a,b]$, we define the bijection $\begin{cases} h : [a,b] & \to [0,1] \\ & t \mapsto h(t) = \frac{t-a}{b-a} \end{cases}$, thus $f \circ h^{-1} \in \mathcal{C}[0,1]$ and
$$B_n(f \circ h^{-1}, h(\cdot)) \xrightarrow[{[a,b]}]{CU} f. \tag{4}$$

Another representation of Bernstein polynomial

Definition (Forward difference operator)

Let $f \in \mathcal{C}[0,1]$ $x \in [0,1]$ and $h > 0$ such that $x + h \in [0,1]$.

- $\Delta_h f(x) = f(x+h) - f(x)$
- $\Delta_h^0 f(x) = f(x)$
- $\Delta_h^r f(x) = \Delta_h^{r-1}(\Delta_h f(x)) = \Delta_h^{r-1} f(x+h) - \Delta_h^{r-1} f(x) \quad \text{if} \quad r \geq 1$

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Proposition

The Bernstein polynomial may be expressed in the form

$$B_n(f,x) = \sum_{r=0}^{n} \binom{n}{r} \Delta_{\frac{1}{n}}^{r} f(0) x^r. \tag{5}$$

Derivatives of Bernstein Polynomials

- The derivative of the Bernstein polynomial $B_{n+1}(f, x)$ may be expressed in the form

$$B'_{n+1}(f, x) = (n+1) \sum_{r=0}^{n} \Delta_h f \left(\frac{r}{n+1} \right) \binom{n}{r} x^r (1-x)^{n-r} \qquad (6)$$

for $n \geq 0$ and $h = \frac{1}{n+1}$.

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for $n \geq 0$ and $h = \frac{1}{n+1}$.

- For any integer $k \geq 0$, the kth derivative of $B_{n+k}(f,x)$ may be expressed in terms of the kth differences of f as

$$B^{(k)}_{n+k}(f,x) = \frac{(n+k)!}{n!}\sum_{r=0}^{n} \Delta_h^k f\left(\frac{r}{n+k}\right)\binom{n}{r} x^r (1-x)^{n-r} \tag{7}$$

for $n \geq 0$ and $h = \frac{1}{n+k}$.

Theorem

If $f \in \mathcal{C}^k[0,1]$, for some integer $k \geq 0$ then $B_n^{(k)}(f,x)$ converges uniformly to $f^{(k)}$ on $[0,1]$.

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Proof.

See [2] $\square$

- We now state results concerning the Bernstein polynomials for a convex function f .
- First we suppose that the definition convexity and its connection with second-order divided differences are known.

Shape-preservation

Theorem

1. *If f is convex on* $[0,1]$,

$$B_n(f,x) \geq f(x), \quad 0 \leq x \leq 1 \quad \forall n \geq 1 \tag{8}$$

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$$B_{n-1}(f, x) \geq B_n(f, x), \quad 0 \leq x \leq 1 \quad \forall n \geq 2 \tag{9}$$

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remark

If $f \in \mathcal{C}[0, 1]$, the inequality in (9) is strict for $0 < x < 1$.

Results

We take the example of: $B_n(e^{\alpha x}, x) = \left(xe^{\frac{\alpha}{n}} + (1-x)\right)^n$ for $\alpha = 2$.

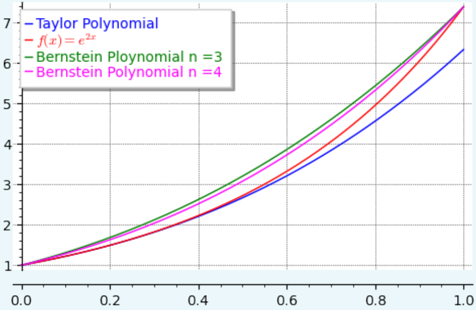

Figure: Comparison of f and $B_n(f, \cdot)$

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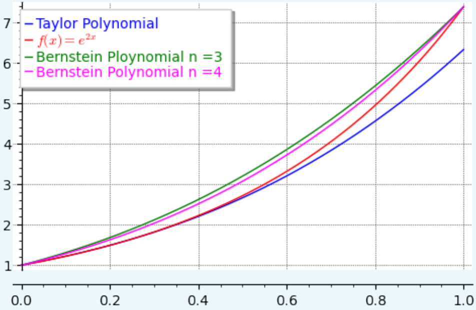

Figure: Comparison of f and $B_n(f, \cdot)$

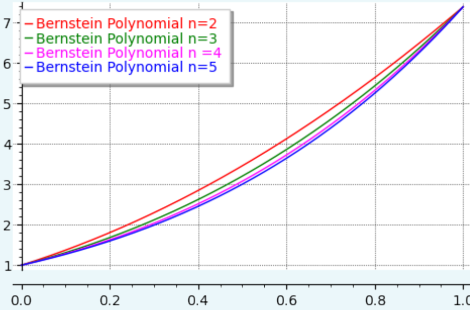

Figure: $B_{n-1}(f, \cdot)$ Versus $B_n(f, \cdot)$

Conclusion

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- Several generalizations have been proposed for Bernstein polynomials: a generalization based on the q-integers

References

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- Michael S. Floater, *On the convergence of derivatives of Bernstein approximation* , Journal of Approximation Theory 134 (2005) 130 – 135

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