第四章作业

4.3 (1) $\because \underline{y} = \underline{\underline{A}}\,\underline{\theta} + \underline{e}$，加权误差平方和定义 $E_w = \underline{e}^H \underline{\underline{W}}\,\underline{e}$.

$$\therefore E_w = (\underline{y} - \underline{\underline{A}}\,\underline{\theta})^H \underline{\underline{W}} (\underline{y} - \underline{\underline{A}}\,\underline{\theta})$$

$$= (\underline{y}^H - \underline{\theta}^H \underline{\underline{A}}^H) \underline{\underline{W}} (\underline{y} - \underline{\underline{A}}\,\underline{\theta})$$

$$= (\underline{y}^H \underline{\underline{W}} - \underline{\theta}^H \underline{\underline{A}}^H \underline{\underline{W}})(\underline{y} - \underline{\underline{A}}\,\underline{\theta})$$

$$= \underline{y}^H \underline{\underline{W}}\,\underline{y} - \underline{\theta}^H \underline{\underline{A}}^H \underline{\underline{W}}\,\underline{y} - \underline{y}^H \underline{\underline{W}}\,\underline{\underline{A}}\,\underline{\theta} + \underline{\theta}^H \underline{\underline{A}}^H \underline{\underline{W}}\,\underline{\underline{A}}\,\underline{\theta}$$

$$\therefore \frac{\partial E_w}{\partial \theta^*} = -\underline{\underline{A}}^H \underline{\underline{W}}\,\underline{y} + \underline{\underline{A}}^H \underline{\underline{W}}\,\underline{\underline{A}}\,\underline{\theta}$$

$\therefore$ 当 E_w 取 min 时，$\hat{\theta}_{LS} = (\underline{\underline{A}}^H \underline{\underline{W}}\,\underline{\underline{A}})^{-1} \underline{\underline{A}}^H \underline{\underline{W}}\,\underline{y}$

(2) 假设误差向量 $\underline{e}$ 的协方差矩阵为 $C_e = Var(e) = \sigma^2 \underline{\underline{V}}$，其中 V 为实对称正定矩阵

$$\therefore \underline{\underline{V}} = \underline{\underline{P}} \cdot \underline{\underline{P}}^T$$

$\therefore$ 设 $\underline{z} = \underline{\underline{P}}^{-1}\underline{y}$，$\underline{\epsilon} = \underline{\underline{P}}^{-1}\underline{e}$

$$\therefore \underline{z} = \underline{\underline{P}}^{-1}\underline{y} = \underline{\underline{P}}^{-1}(\underline{\underline{A}}\,\underline{\theta} + \underline{e}) = \underline{\underline{B}}\,\underline{\theta} + \underline{\epsilon}$$

$$\because Var(\epsilon) = var(\underline{\underline{P}}^{-1}\underline{e}) = \underline{\underline{P}}^{-1} Var(\underline{e}) \underline{\underline{P}}^{-T}$$

$$= \underline{\underline{P}}^{-1} \sigma^2 \underline{\underline{P}}\,\underline{\underline{P}}^T \underline{\underline{P}}^{-T} = \sigma^2 \underline{\underline{I}}$$

$\therefore \epsilon$ 为白噪声

$$\therefore \hat{\theta}_{LS} = (\underline{\underline{B}}^H \underline{\underline{B}})^{-1} \underline{\underline{B}}^H \underline{z}$$

$$= (\underline{\underline{A}}^H \underline{\underline{V}}^{-1} \underline{\underline{A}})^{-1} \underline{\underline{A}}^H \underline{\underline{V}}^{-1} \underline{z}$$

$\therefore$ 加权最小二乘 $W = V^{-1}$，$\therefore$ 其等价于把 e 白化.

4.4. $f(w) = w^H R_e w$ s.t. $Re(w^H x) = b$

$\therefore$ 约束条件可化为 $w^H x + x^H w = 2b$

$\therefore$ 用 Lagrange 乘子法可得

$$J(w,\lambda)= w^H R_e w + \lambda(2b - w^H x - x^H w)$$

令 $\begin{cases} \dfrac{\partial J}{\partial w^*} = R_e w - \lambda x = 0 \\ \dfrac{\partial J}{\partial \lambda} = 2b - w^H x - x^H w = 0 \end{cases} \Rightarrow \begin{cases} R_e w = \lambda x \\ w^H x + x^H w = 2b \end{cases}$

① 当 R_e 满秩时，$w = \lambda R_e^{-1} x$

代入上式约束条件中，可解得

$$\lambda \cdot x^H (R_e^{-1})^H x + \lambda x^H R_e^{-1} x = 2b$$

$$2\lambda x^H R_e^{-1} x = 2b$$

$$\lambda = \frac{b}{x^H R_e^{-1} x}$$

代入可得 $w = \dfrac{b R_e^{-1} x}{x^H R_e^{-1} x}$

② R_e 奇异时，必有一个零特征值对应的特征向量，将该向量放大一定比例，必能使约束条件满足，最优解为 $w=$ 零特征值对应的特征向量

4.13 $\min[\mathrm{tr}(A^T A) - 2\mathrm{tr}(A)]$ s.t. $AX = 0$ $\longrightarrow \hat{A} = I - XX^\dagger$

用Lagrange乘子法，$J(A) = \mathrm{tr}(A^T A) - 2\mathrm{tr}(A) + 2\mathrm{tr}(LAX)$

$\therefore d(J(A)) = 2\mathrm{tr}((A^T - 2I + 2XL)dA)$

$\therefore D_A J(A) = 2A^T - 2I + 2XL \Rightarrow \hat{A} = I - L^T X^T$

代入 $AX = 0$

$\therefore \Rightarrow (I - L^T X^T)X = 0$

$\Rightarrow L = (X^T X)^\dagger X^T$

$\Rightarrow \hat{A} = I - (XX^\dagger)^T$

4.18 $\min \frac{1}{2} x^T x$ s.t. $Cx = b$ $\longrightarrow$ 唯一解 $x^* = C^\dagger b$

将约束优化问题化成无约束优化问题.

$$J(x,\lambda) = \frac{1}{2} x^T x + \lambda^T (b - Cx)$$

分别对 $J(x,\lambda)$ 用 x 和 λ 求导

$$\begin{cases} \dfrac{\partial J(x,\lambda)}{\partial x} = x - C^T\lambda = 0 \quad \Rightarrow \quad x = C^T\lambda \\ \dfrac{\partial J(x,\lambda)}{\partial \lambda} = Cx = b \end{cases}$$

$\therefore$ 两式联立得 $CC^T\lambda = b$

$\because$ CC^T 不是列满秩

$\therefore$ $\lambda = (CC^T)^+ b + [I - (CC^T)^+ CC^T]y$，$y$ 是任意向量

$\therefore$ 代入 λ 到 $x = C^T\lambda$ 中可得

$$\begin{aligned} x &= C^T(CC^T)^+ b + [C^T - C^T(CC^T)^+ CC^T]y \\ &= C^+ b \qquad + 0 \\ &= C^+ b \end{aligned}$$