第四次作业

6.1. 证明如下: $\because \underline{\underline{A}}\,\underline{\theta} + \underline{\epsilon} = \underline{x}$

$\therefore \underline{\epsilon} = \underline{x} - \underline{\underline{A}}\,\underline{\theta}$

代入加权误差函数 $Q(\theta)$

得 $Q(\theta) = (\underline{x} - \underline{\underline{A}}\,\underline{\theta})^T \underline{\underline{W}} (\underline{x} - \underline{\underline{A}}\,\underline{\theta})$

$= (\underline{x}^T \underline{\underline{W}} - \underline{\theta}^T \underline{\underline{A}}^T \underline{\underline{W}})(\underline{x} - \underline{\underline{A}}\,\underline{\theta})$

$= \underline{x}^T \underline{\underline{W}}\,\underline{x} - \underline{\theta}^T \underline{\underline{A}}^T \underline{\underline{W}}\,\underline{x} - \underline{x}^T \underline{\underline{W}}\,\underline{\underline{A}}\,\underline{\theta} + \underline{\theta}^T \underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\underline{A}}\,\underline{\theta}$

$\therefore$ 用 θ 对 $Q(\theta)$ 求导可得

$$\frac{\partial Q(\theta)}{\partial \theta} = -\underline{\underline{A}}^T \underline{\underline{W}}\,\underline{x} - (\underline{x}^T \underline{\underline{W}}\,\underline{\underline{A}})^T + \underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\underline{A}}\,\underline{\theta} + (\underline{\theta}^T \underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\underline{A}})^T$$

$$= 0$$

$\because W$ 对称

$\therefore 2\underline{\underline{A}}^T \underline{\underline{W}}\,\underline{x} = 2\underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\underline{A}}\,\underline{\theta}$

$\therefore \theta_{WLS} = (\underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\underline{A}})^{-1} \underline{\underline{A}}^T \underline{\underline{W}}\,\underline{x}$ 得证

$\therefore$ 代入 $\underline{x} = \underline{\underline{A}}\,\underline{\theta} + \underline{\epsilon}$ 到 θ_{WLS} 中可得

$$\theta_{WLS} = (\underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\underline{A}})^{-1} \underline{\underline{A}}^T \underline{\underline{W}} (\underline{\underline{A}}\,\underline{\theta} + \underline{\epsilon})$$

$$= \underline{\theta} + (\underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\underline{A}})^{-1} \underline{\underline{A}}^T \underline{\underline{W}}\,\underline{\epsilon}$$

$\therefore$ 求协方差矩阵

$C = E\{(\theta - \theta_{WLS})(\theta - \theta_{WLS})^T\}$

$= (A^T W A)^{-1} A^T W R W A (A^T W A)^{-1}$

$\therefore$ 用 W 对 $tr(C)$ 求导

满足 $\frac{\partial tr(C)}{\partial W} = 0$ 时, $A(A^T W A)^{-2} A^T W R W A (A^T W A)^{-1} A^T = R W A (A^T W A)^{-1}$

$\therefore$ 此时可得 $W = R^{-1}$

6.4. $\because$ 求 $\min \frac{1}{2}\|\underline{\underline{A}}\underline{x}-\underline{b}\|_2^2-\frac{1}{2}\lambda\|x\|_2^2$

$\therefore \frac{1}{2}\|\underline{\underline{A}}\underline{x}-\underline{b}\|_2^2-\frac{1}{2}\lambda\|x\|_2^2$

$$=\frac{1}{2}\left[(\underline{\underline{A}}\underline{x}-\underline{b})^H(\underline{\underline{A}}\underline{x}-\underline{b})-\lambda x^H x\right]$$

$\therefore$ 对上式用 x^H 求导可得

$$\frac{\partial\, \frac{1}{2}\|\underline{\underline{A}}\underline{x}-\underline{b}\|_2^2-\frac{1}{2}\lambda\|x\|_2^2}{\partial x^H}=\underline{\underline{A}}^H\underline{\underline{A}}\underline{x}-\underline{\underline{A}}^H\underline{b}-\lambda\underline{x}=0$$

$\therefore$ $\underline{x}=(\underline{\underline{A}}^H\underline{\underline{A}}-\lambda\underline{\underline{I}})^{-1}\underline{\underline{A}}^H\underline{b}$ 得证

6.7. 数据点为 $(1,3),(3,1),(5,7),(4,6),(7,4)$

① 总体最小二乘拟合:

$\because \bar{x}=4,\ \bar{y}=4.2.$ $\therefore M=\begin{bmatrix} x_1-\bar{x} & y_1-\bar{y} \\ x_2-\bar{x} & y_2-\bar{y} \\ x_3-\bar{x} & y_3-\bar{y} \\ x_4-\bar{x} & y_4-\bar{y} \\ x_5-\bar{x} & y_5-\bar{y}\end{bmatrix}=\begin{bmatrix} -3 & -1.2 \\ -1 & -3.2 \\ 1 & 2.8 \\ 0 & 1.8 \\ 3 & -0.2\end{bmatrix}$

$\therefore M^T M=\begin{bmatrix} 20 & 9 \\ 9 & 22.8\end{bmatrix}$, 再对其进行特征值分解得

特征多项式 $\lambda^2-42.8\lambda+375=0$

解 $\lambda_1=30.508,\ \lambda_2=12.292.$

$$\therefore M^T M=\begin{bmatrix} 0.6505 & -0.7595 \\ 0.7595 & 0.6505\end{bmatrix}\begin{bmatrix} 30.508 & 0 \\ 0 & 12.292\end{bmatrix}\begin{bmatrix} 0.6505 & -0.7595 \\ 0.7595 & 0.6505\end{bmatrix}$$

$\therefore \begin{bmatrix} a \\ b\end{bmatrix}=\begin{bmatrix} -0.7595 \\ 0.6505\end{bmatrix}$ $\therefore$ 直线为 $-0.7595(x-4)+0.6505(y-4.2)=0$

$\therefore D_{TLS}=12.2918.$

② 一般最小二乘法

$$D_{LS}^{(1)}=\sum_{i=1}^{5}[m(x_i-\bar{x})+(y_i-\bar{y})]^2$$

$$=(-3m-1.2)^2+(-m-3.2)^2+(m+2.8)^2+1.8^2+(3m-0.2)^2$$

$\therefore \frac{\partial D_{LS}^{(1)}}{\partial m}=40m+18=0$, $m=-0.45$

$\therefore d_1^2 = \dfrac{D^{(1)}_{LS}}{1+m^2} = 15.5965$

$\because D^{(2)}_{LS} = \sum_{i=1}^{5}[(x-\bar{x})+m(y-\bar{y})]^2$

$= (-3-1.2m)^2+(-1-3.2m)^2+(1+2.8m)^2+(1.8m)^2+(3-0.2m)^2$

$\therefore \dfrac{\partial D^{(2)}_{LS}}{\partial m} = 45.6m+18 = 0 \Rightarrow m = -0.3947$

$\therefore$ 直线方程 $(x-4)-0.3947(y-4.2)=0$

$\therefore d_2^2 = \dfrac{D^{(2)}_{LS}}{1+m^2} = 14.2305.$