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SOLOW-SWAN MODEL

1 Model Environment

1.1 Firms and Production

A production function describes how capital K_t and labor L_t, using technology A_t, is transformed into output Y_t:

$$Y_t = F(K_t, A_t L_t) \tag{1}$$

A neoclassical production function has the following properties:

1. Continuous and at least twice differentiable
2. Positive marginal products of K_t and L_t :

$$\frac{\partial F(K_t, A_t L_t)}{\partial K_t} = F_K > 0, \frac{\partial F(K_t, A_t L_t)}{\partial L_t} = F_L > 0 \tag{2}$$

3. Diminishing marginal products of K_t and L_t:

$$\frac{\partial F_K(K_t, A_t L_t)}{\partial K_t} = F_{KK} < 0, \frac{\partial F_L(K_t, A_t L_t)}{\partial L_t} = F_{LL} < 0 \tag{3}$$

4. Constant returns to scale in K_t and L_t. Multiplying both capital and labor inputs by a certain proportion z translates to multiplying produced output by that same proportion z:

$$F(zK_t, zA_t L_t) = zF(K_t, A_t L_t) \tag{4}$$

Technology grows at constant rate g:

$$A_{t+1} = (1+g)A_t \tag{5}$$

Cobb-Douglas production function is the most common type of production function:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} \tag{6}$$

Task: Show that Cobb-Douglas function given in Eq. (6) satisfies the properties given in Eqs.(2)-(4).
The representative firm rents capital K_t at rate R_t and labor L_t at w_t from the households and use production function given in Eq. (6) to produce. The objective of the firm is to maximize the profits:

$$\max_{K_t, L_t} \Pi_t = Y_t - w_t L_t - R_t K_t \tag{7}$$

Using Eq. (6) in Eq. (7) gives:

$$\max_{K_t, L_t} \Pi_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} - w_t L_t - R_t K_t \tag{8}$$

First-order conditions:

$$\frac{\partial \Pi_t}{\partial K_t} = 0 : \alpha K_t^{\alpha-1}(A_t L_t)^{1-\alpha} = R_t \Rightarrow R_t = \alpha \frac{Y_t}{K_t} \tag{9}$$

$$\frac{\partial \Pi_t}{\partial L_t} = 0 : (1-\alpha) K_t^{\alpha}(A_t L_t)^{1-\alpha} (L_t)^{-1} = w_t \Rightarrow w_t = (1-\alpha) \frac{Y_t}{L_t} \tag{10}$$

Since there is perfect competition with constant returns to scale production function, the profits are equal to zero, and GDP equals to total factor payments:

$$Y_t = w_t L_t + R_t K_t \tag{11}$$

This can be verified by plugging Eqs. (9) and (10) into (11):

$$Y_t = w_t L_t + R_t K_t = (1-\alpha)\frac{Y_t}{L_t} L_t + \alpha \frac{Y_t}{K_t} K_t = (1-\alpha) Y_t + \alpha Y_t = Y_t.$$

What is the share of GDP that is paid to factors (labor and capital)?

$$\text{Labor}: \frac{w_t L_t}{Y_t} = \frac{(1-\alpha)\frac{Y_t}{L_t} L_t}{Y_t} = 1-\alpha$$

$$\text{Capital}: \frac{R_t K_t}{Y_t} = \frac{\alpha \frac{Y_t}{K_t} K_t}{Y_t} = \alpha$$

Cobb-Douglas function implies constant shares of labor and physical capital in income (GDP).

1.2 Household and Saving

Own factors of production (capital and labor) and earn income from renting them to firms. Each households supplies one unit of labor:

$$L_t = N_t,$$

and population grows at a rate n:

$$N_{t+1} = (1+n) N_t.$$

Capital accumulates from investment I_t and depreciates at rate δ:

$$K_{t+1} = (1-\delta) K_t + I_t. \tag{12}$$

The budget constraint of the household is:

$$C_t + I_t = w_t L_t + R_t K_t. \tag{13}$$

Eqs. (11) and (13) imply that total expenditure ($C_t + I_t$) equals total income ($w_t L_t + R_t K_t$) which equals to GDP (Y_t).
Households don't optimize, save a constant fraction s of income:

$$I_t = s Y_t, \tag{14}$$

This also implies constant consumption share of income:

$$C_t = (1-s) Y_t. \tag{15}$$

1.3 Dynamics

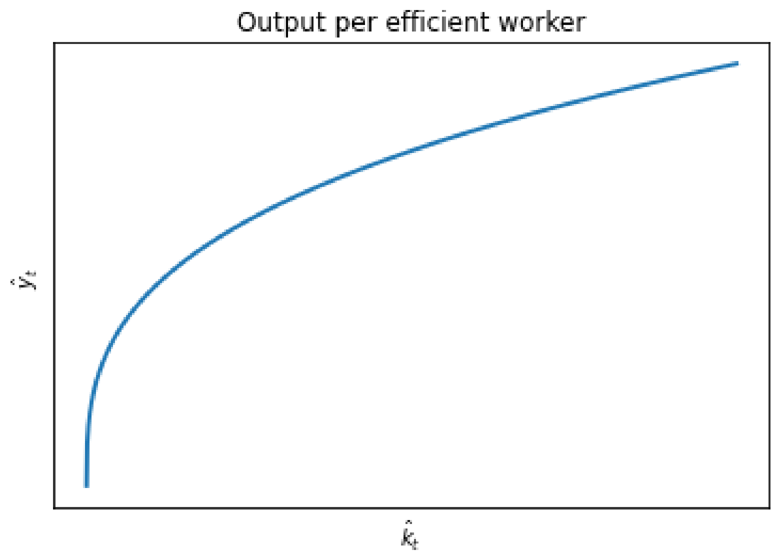
First, let's define capital per unit of effective labor (the variables in small letters with hat denotes the variables in per unit of effective labor and obtained by dividing the variable to technology and labor, $A_t L_t$):

$$\hat{k}_t = \frac{K_t}{A_t L_t}. \tag{16}$$

It is also useful to define output per unit of effective labor:

$$\hat{y}_t = \frac{1}{A_t}\frac{Y_t}{L_t} = \hat{k}_t^{\alpha} = f\left(\hat{k}_t\right). \tag{17}$$

The blule line in the following figure shows the output per unit of effective labor with capital and output per efficient labor are on the x and y-axis, respectively. The increase in capital brings larger increase in output when capital is small as a result of the properties of the neoclassical production function.



Usually we are interested in output (GDP) per capita (or per labor, the variables in small letters denotes the variables in per unit of capita (labor) and obtained by dividing the variable to labor, L_t):

$$y_t = \frac{Y_t}{L_t} = \frac{K_t^{\alpha}(A_tL_t)^{1-\alpha}}{L_t} = \frac{K_t^{\alpha}(A_tL_t)^{1}(A_tL_t)^{-\alpha}}{L_t} = A_t\left(\frac{K_t}{A_tL_t}\right)^{\alpha} = A_t\hat{k}_t^{\alpha} = A_tf\left(\hat{k}_t\right). \tag{18}$$

Output per capita is function of technology and capital per effective labor and it increases due to improvements in technology and due to higher investment.
The first derivative of the production function in Eq. (17) is positive and the second derivative is negative. This implies that output per capita increases with $\hat{k}$, but the size of the increase falls with $\hat{k}$.
We can reexpress capital accumulation equation by plugging Eq. (14) into Eq. (12):

$$K_{t+1} = (1-\delta)K_t + sY_t. \tag{19}$$

Capital per unit of effective labor accumulation is then equal to:

$$\frac{K_{t+1}}{A_tL_t} = \frac{(1-\delta)K_t}{A_tL_t} + s\frac{Y_t}{A_tL_t}$$

$$\frac{K_{t+1}}{A_tL_t}\frac{A_{t+1}L_{t+1}}{A_{t+1}L_{t+1}} = \frac{(1-\delta)K_t}{A_tL_t} + s\frac{Y_t}{A_tL_t}$$

$$\frac{K_{t+1}}{A_{t+1}L_{t+1}}\frac{A_{t+1}L_{t+1}}{A_tL_t} = \frac{(1-\delta)K_t}{A_tL_t} + s\frac{Y_t}{A_tL_t}$$

$$\hat{k}_{t+1}(1+g)(1+n) = (1-\delta)\hat{k}_t + s\hat{y}_t$$

$$\hat{k}_{t+1} = \frac{(1-\delta)\hat{k}_t + s\hat{k}_t^{\alpha}}{(1+g)(1+n)}. \tag{20}$$

Assuming $gn\hat{k}_{t+1} \approx 0$ Eq. (20) can be expressed in following way:

$$(1+g+n)\hat{k}_{t+1} = (1-\delta)\hat{k}_t + s\hat{k}_t^{\alpha},$$

$$(1+g+n)\hat{k}_{t+1} - (g+n)\hat{k}_t = (1-\delta)\hat{k}_t + s\hat{k}_t^{\alpha} - (g+n)\hat{k}_t,$$

$$(1+g+n)\left(\hat{k}_{t+1} - \hat{k}_t\right) = s\hat{k}_t^{\alpha} - (\delta+g+n)\hat{k}_t,$$

$$(1+g+n)\left(\triangle\hat{k}_{t+1}\right) = s\hat{k}_t^{\alpha} - (\delta+g+n)\hat{k}_t. \tag{21}$$

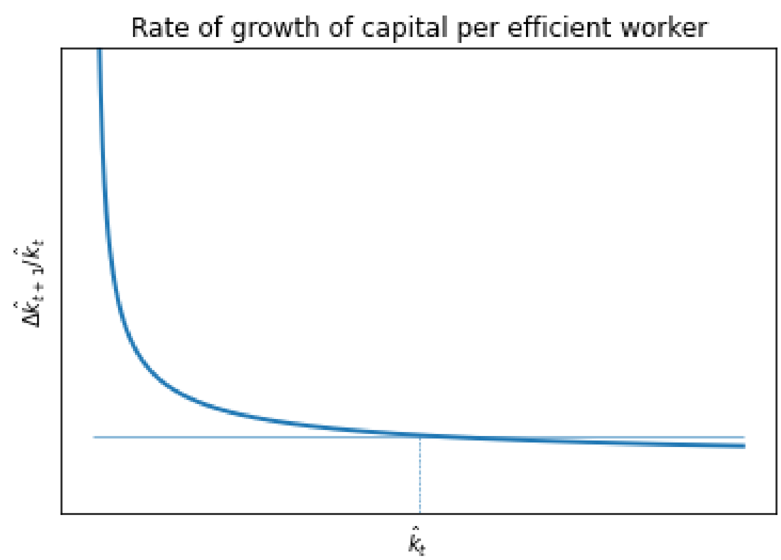
Assuming $g\triangle\hat{k}_{t+1} \approx 0$ and $n\triangle\hat{k}_{t+1} \approx 0$ gives the capital per unit of effective labor accumulation as following:

$$\triangle\hat{k}_{t+1} = s\hat{k}_t^{\alpha} - (\delta+g+n)\hat{k}_t. \tag{22}$$

The growth rate of capital per unit of effective labor, $g_{\hat{k}}$, is then equal to:

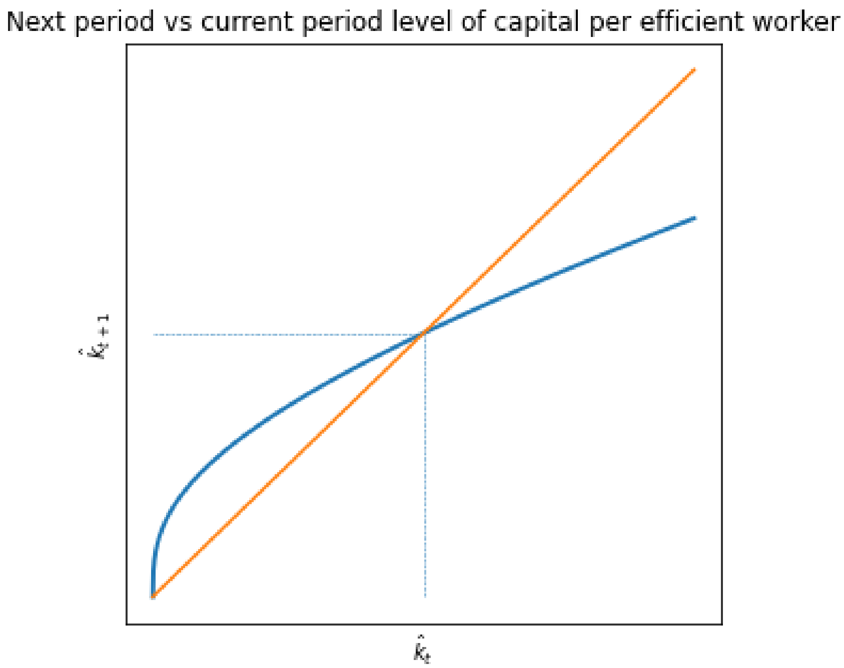
$$g_{\hat{k}} = \frac{\hat{k}_{t+1} - \hat{k}_t}{\hat{k}_t} = \frac{\triangle\hat{k}_{t+1}}{\hat{k}_t} = s\hat{k}_t^{\alpha-1} - (\delta+g+n). \tag{23}$$

The following graph shows the growth rate of capital per unit of effective labor as function of capital per unit of effective labor. The growth is equal to zero on the horizontal blue line; hence, after certain level of capital, the growth rate is negative.



1.4 Balanced Growth Path

A balanced growth path (BGP) is a path $\{Y_t, C_t, K_t\}_{t=0}^{\infty}$ along which the quantities Y_t, C_t and K_t are positive and grow at constant rates, which we denote g_y, g_c and g_k, respectively. The BGP is also the growth rate of the variables at the steady-state.



From the plot we can easily see that there is a level of k for which the next period and current period level of capital are the same. We will call this level a steady state level of capital per worker, and we will denote it with k^*. The properties of the neoclassical production function guarantee that there is only one, positive level of k^*.

Let us find the expression for the steady state level of capital per worker under the Cobb-Douglas production function, by setting $\hat{k}_{t+1} = \hat{k}_t = k^*$ which imply that the left-hand side of the Eq. (22) is zero. Then capital per effective labor at the steady-state is equal to

$$s\left(\hat{k^*}\right)^{\alpha} = (\delta + g + n)\hat{k^*},$$

$$\left(\hat{k^*}\right)^{\alpha-1} = \left(\frac{\delta + g + n}{s}\right),$$

$$\hat{k^*} = \left(\frac{\delta + g + n}{s}\right)^{\frac{1}{\alpha-1}},$$

$$\hat{k^*} = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}, \tag{24}$$

GDP per effective labor at the steady-state is equal to

$$\hat{y^*} = \left(\frac{s}{\delta + g + n}\right)^{\frac{\alpha}{1-\alpha}}. \tag{25}$$

Along the balanced growth path (BGP) variables per capita (worker) grow together with increases in technology:

$$y_t^* = A_t\hat{y^*} \rightarrow g_y^* = \frac{\triangle y_{t+1}^*}{y_t^*} = \frac{\triangle A_{t+1}\hat{y^*}}{A_t\hat{y^*}} = g.$$

Task: Show that consumption and capital per capita grows at the rate g along the BGP.
Task: Show that aggregate variables like aggregate capital and GDP grow at the sum of rates of increase in population and technology.
Solow-Swan model predicts that the BGP (steady-state) level of GDP per worker

$$y_t^* = A_t\left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}} \tag{26}$$

is higher in countries with higher investment share of GDP s and higher technology level A_t, and lower in countries with higher population growth rate n.
The consumption per effective labor at the BGP is equal to:

$$\hat{c^*} = (1-s)\hat{y^*} = (1-s)\left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}}.$$

The level of saving rate that gives the highest level of consumption per effective labor at the steady-state is know as the golden-rule saving, s_G. The value of s_G is given by the solution of the following problem:

$$\max_s (1-s)\left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}}.$$

The FOC is given by:

$$-\left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}\left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}-1}\frac{1}{\delta+g+n} = 0,$$

$$\left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{s}{\delta+g+n}\right)^{\frac{\alpha}{1-\alpha}-1}\left[(1-s)\frac{\alpha}{1-\alpha}\frac{1}{\delta+g+n}\right],$$

$$\frac{s}{\delta+g+n} = (1-s)\frac{\alpha}{1-\alpha}\frac{1}{\delta+g+n},$$

$$1-s = \frac{1-\alpha}{\alpha} \Rightarrow s_G = \alpha.$$

Task: How does the consumption respond to a one-time, permanent increase in s when the s before and after change is below s_G. And how does the consumption respond to a one-time, permanent decrease in s when the s before and after change is above s_G. Which one do you think is beneficial for the consumers? First or second case, or both? Hint: Think both short and long-run.

1.5 Transitional Dynamics

We are also interested in the determinants of growth rates in GDP per worker outside of the BGP. Start with growth rate of GDP per effective labor:

$$g_{\hat{y}} = \ln\hat{y}_{t+1} - \ln\hat{y}_t = \ln\hat{k}_{t+1}^\alpha - \ln\hat{k}_t^\alpha = \alpha\left(\ln\hat{k}_{t+1} - \ln\hat{k}_t\right) = \alpha g_{\hat{k}} \tag{27}$$

Plug Eq. (23) into Eq. (27) to get the growth rate of GDP per effective labor

$$g_{\hat{y}} = \alpha\left[s\hat{k}_t^{\alpha-1} - (\delta+g+n)\right] \tag{28}$$

To obtain growth rate of GDP per worker, add the growth rate of technology g to Eq. (28):

$$g_y = \alpha \left[s\hat{k}_t^{\alpha-1} - (\delta + g + n) \right] + g \tag{29}$$

An increase in s or a decrease in n temporarily increases the growth rate of GDP per worker. Why temporarily? Because GDP per effective labor converges to steady-state and GDP per worker grows at rate of g afterwards.
Using $\hat{k^*}$ as capital per effective labor along the BGP, let us revisit factor prices:

$$R_t^* = \alpha \frac{Y_t^*}{K_t^*} \frac{A_t^* L_t^*}{A_t^* L_t^*} = (1-\alpha) \left(\hat{k^*}\right)^{\alpha-1},$$

$$w_t^* = (1-\alpha) \frac{Y_t^*}{L_t^*} \frac{A_t^*}{A_t^*} = (1-\alpha) A_t \left(\hat{k^*}\right)^{\alpha}.$$

The model predicts that along the BGP the interest rates are constant while hourly wages grow at the same rate as GDP per hour.
Solow-Swan model predicts that if countries have access to the same technology and share the same steady state, then ones that are poorer should grow faster.
Task: Can you explain why poorer countries should grow faster? Hint: Think about the properties given in Eqs. (1), (2), (4), and (5).

2 Convergence

To compute the speed of convergence, first, express $g_{\hat{y}}$ in Eq. (29) only as a function of $\hat{y}$ and parameters:

$$y_t = A_t \hat{k}_t^{\alpha} \Rightarrow \hat{k}_t = \hat{y}_t^{1/\alpha} =$$

$$g_{\hat{y}} = \alpha \left[s \left(\hat{y}_t^{1/\alpha}\right)^{\alpha-1} - (\delta + g + n) \right]$$

$$g_{\hat{y}} = \alpha \left[s \hat{y}_t^{\frac{\alpha-1}{\alpha}} - (\delta + g + n) \right]$$

Now take the first-order Taylor approximation of $g_{\hat{y}}$ in Eq. (23) around the steady state $\hat{y^*}$:

$$g_{\hat{y}} \approx g_{\hat{y}^*} + \frac{\partial g_{\hat{y}}}{\partial \hat{y}^*} \left(\hat{y}_t - \hat{y}^*\right),$$

$$g_{\hat{y}} \approx g_{\hat{y}^*} + \frac{\alpha - 1}{\alpha} \alpha s \left(y^*\right)^{\frac{\alpha-1}{\alpha}} \left(\frac{\hat{y}_t - \hat{y}^*}{\hat{y^*}} \right). \tag{30}$$

Using the fact that $g_{\hat{y}^*} = 0$ and plugging $\hat{y^*}$ from Eq. (25) into Eq. (30) gives:

$$g_{\hat{y}} \approx \frac{\alpha - 1}{\alpha} \alpha s \left(\left(\frac{s}{\delta + g + n} \right)^{\frac{\alpha}{1-\alpha}} \right)^{\frac{\alpha-1}{\alpha}} \left(\frac{\hat{y}_t - \hat{y}^*}{\hat{y}^*} \right)$$

$$g_{\hat{y}} \approx -\underbrace{(1-\alpha)(\delta + g + n)}_{\lambda} \left(\ln \hat{y}_t - \ln \hat{y}^*\right)$$

$$g_{\hat{y}} \approx \underbrace{(1-\alpha)(\delta + g + n)}_{\lambda} \left(\ln \hat{y}^* - \ln \hat{y}_t\right) \tag{31}$$

The term $(1-\alpha)(\delta + g + n)$ in Eq. (31) measures the speed of convergence denoted by λ. It measures how quickly $\hat{y}_t$ increases when $\hat{y}_t < \hat{y}^*$.

Econometric studies both on individual countries and states within USA find that $\lambda \approx 0.02$, meaning that it takes about 35 years to close half of the gap between the current income and the steady state.
Given sensible parameter values: $\alpha = 0.33, \delta = 0.05, n = 0.01, g = 0.02$, the model generates $\lambda = 0.053$, implying that it would take about 13 years to close half of the gap, an unrealistic number.
Adding human capital allows the model to assign lower weight to raw labor and be consistent with slow convergence.

3 Human Capital (MRW, 1992)

The production function is given by:

$$Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}, \tag{32}$$

where H is the stock of human capital. Let s_k be the fraction of income invested in physical capital and s_h the fraction invested in human capital. The physical and human capital accumlation equations are

$$\hat{k}_{t+1} = s_k . \hat{y}_t - (\delta + g + n)\, \hat{k}_t$$
$$\hat{h}_{t+1} = s_h . \hat{y}_t - (\delta + g + n)\, \hat{h}_t$$

where all variables denoted in hat are quantities per effective unit of labor. It is assumed that one unit of consumption can be costlessly converted to one unit of physical capital or one unit of human capital. The BGP level (steady state) of physical and human capital per effective unit of labor are given by

$$\hat{k}_t^* = \left(\frac{s_k^{1-\beta} s_h^{\beta}}{\delta + g + n} \right)^{\frac{1}{1-\alpha-\beta}} \tag{33}$$

$$\hat{h}_t^* = \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{\delta + g + n} \right)^{\frac{1}{1-\alpha-\beta}} \tag{34}$$

The higher saving rates in both types of capital generates higher steady-state levels of physical and human capital. One can show that

$$g_{\hat{y}} \approx \underbrace{(1 - \alpha - \beta)\,(\delta + g + n)}_{\lambda} (\ln \hat{y}^* - \ln \hat{y}_t)\,. \tag{35}$$

The only difference in the the speed of convergence denoted by λ is the term in Eq. (35) compared to the Eq. (31), the one withour human capital. Mankiw, Romer, and Weil (1992) find $\lambda \approx 0.02$ among OECD countries and the capital share $\alpha = 0.33$. Hence, the results from the model augmented with human capital are more consistent with the data.