

Daniel Bernoulli and the St. Petersburg Paradox ¹

Herold G. Dehling
*Mathematisch Instituut,
Rijksuniversiteit Groningen,
Blauwborgje 3, 9747 AC Groningen*

1. DANIEL BERNOULLI

If somebody in Groningen has to choose a famous local mathematician from the past as subject of a talk, the choice is not hard. Until the beginning of this century there had been only one mathematician of international reputation teaching at this university, namely Johann Bernoulli. However, talking year after year about the same person becomes a bit boring, so we slightly stretch our definition of what a Groningen mathematician is to include Johann's son Daniel. Daniel Bernoulli was born in this town on the 8th of February in the year 1700, actually in a house in the Oude Boteringestraat, not far from this lecture hall. Daniel's affiliation with Groningen was short, as he left town already in 1705 when his father became professor in Basel. Nevertheless, his 5 years in Groningen brought him a spot on the stained-glass windows in this hall where he is shown standing next to his father.

Daniel Bernoulli went to study mathematics himself and in 1725 became professor of mathematics at the St. Petersburg Academy of Sciences, where he staid for 8 years until also he could return as professor to Basel in 1733. He remained in Basel ever since then and died there on March 17, 1782. Daniel Bernoulli is probably best known for his contributions to hydrodynamics where he found the so-called Bernoulli equation

$$p + \frac{1}{2}\rho v^2 = \text{const},$$

relating pressure and velocity in a horizontally flowing incompressible fluid.

2. THE ST. PETERSBURG PARADOX

Today we want to direct our attention to Daniel Bernoulli's clarification of a famous paradox in probability theory, concerning the St. Petersburg game. The rule of this game is very simple: the player may toss a fair coin until Head shows up for the first time. The payoff is 2 guilders if this occurs at the first toss, 4 if it occurs at the second, 8 if it occurs at the third, 16 if it occurs at

¹ Talk presented on May 12, 1997 as part of the 1996/97 Johann Bernoulli lecture in Groningen

the fourth and so on. A little thought reveals that the probability of getting the first Head at the first toss is $\frac{1}{2}$, at the second toss $\frac{1}{4}$, at the third toss $\frac{1}{8}$ and so on.

The St. Petersburg paradox concerns the value of this game to a potential player. How much should a reasonable man be willing to pay for being allowed to play this game and receive the resulting payoff? The generally accepted formula for the fair price of a game, or actually any claim contingent on the outcome of some random experiment, is the expected value – multiply possible payoffs with the respective probabilities and compute the sum of these products. For the St. Petersburg game, there is a probability of $\frac{1}{2}$ to win 2 guilders, of $\frac{1}{4}$ to win 4 guilders, of $\frac{1}{8}$ to win 8 guilders, and so on, resulting in the expected value

$$2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} + 8 \cdot \frac{1}{8} + \dots = \infty$$

On the other hand, would anybody in the audience be willing to pay say 100 guilders for being allowed to play the game? A quick computation shows that in order to receive at least your payment back, the first 6 tosses must all yield Tails, an event having probability $\frac{1}{64}$. Hence your odds of losing some amount in this game is 63 to 1, too high for most people to risk. This apparent paradox of a game that according to standard textbook formulas has an infinite value but for which no reasonable man would want to pay more than say 100 guilders is known as the St. Petersburg paradox.

3. HISTORICAL BACKGROUND

The early commotion about the St. Petersburg paradox has to be seen in the light of the birth of probability theory then only about half a century ago. Around 1650 two famous French mathematicians, Fermat and Pascal were engaged in computing the value of all kinds of games of chance. Indirectly their ideas, which they exchanged in letters, inspired a fellow countryman of ours, Christiaan Huygens, who visited Paris in 1656. Huygens shortly afterwards devoted a short paper with the modest title *Some computations regarding games of chance* to solving some of their problems, and discussing the foundations of probability theory at the same time. Huygens pointed out very clearly that though the outcome of a single game is unpredictable, one can quantify the probabilities of the possible outcomes.

Huygens introduced the notion of ‘value’ of a game, beginning with the proposal ‘if I have equal chances to obtain a or b, this has the same value to me as $(a + b)/2$ ’. Although he did not yet use probabilities in the present day sense, his ‘value’, i.e. the weighted average of the possible payoffs, can be readily transformed by the modern reader into expected values. Quickly the idea of value of a game was extended to the computation of the value of any monetary claim conditional upon the occurrence of some contingent events. This found widespread application during the 2nd half of the 17th century to fields as annuities and life insurances, among others by Johan de Witt and Edmund Halley. It is against this historical background of the emergence of

probability as a scientific discipline that the St. Petersburg paradox has to be seen as an example questioning the foundations of the subject.

4. HISTORY OF THE ST. PETERSBURG PARADOX

The St. Petersburg problem was raised for the first time in 1713 by Nikolaus Bernoulli, a cousin of Daniel's, in a letter to the French mathematician De Montmort. Nikolaus also gave his own solution to the problem involving the idea that events with extremely small probabilities should be regarded as morally impossible because the frequency of their occurrence is so low that no human mind is bothered by them. During the strongest storms we feel comfortable behind our seadikes, though knowing that there is a 1 in 10000 years chance of them breaking. Regarding events of probability $\leq 1/10,000$ as morally impossible, we may neglect the possibility that the first 14 tosses will all result in Tails. In this way the infinite series defining the value of the St. Petersburg game becomes a finite sum involving only the first 14 terms and thus resulting in an expected value of 14:

$$\sum_{k=1}^{14} 2^k \frac{1}{2^k} = 14$$

The next person to become involved with the St. Petersburg game was Gabriel Cramer (1704 – 1752), the Swiss mathematician who was to become immortal by inventing Cramer's determinant rule for solving systems of linear equations. Cramer's solution is contained in a letter that he wrote in 1728 to Nikolaus Bernoulli '*The mathematical expectation is rendered infinite by the enormous amount which I can win if the cross does not fall upward until rather late, perhaps at the hundredth or thousands throw. Now, as a matter of fact if I reason as a sensible man, this sum is worth no more to me, causes me no more pleasure and influences me no more to accept the game than does a sum amounting only to ten or twenty million guilders*'

Cramer then pursues with the assumption that any sum above 2^{24} guilders (which is about 16 million) is worth the same to him. In this way the value of the game to him becomes 25:

$$\sum_{k=1}^{24} 2^k \frac{1}{2^k} + 2^{24} \frac{1}{2^{24}} = 25$$

5. DANIEL BERNOULLI'S IDEA OF MORAL EXPECTATION

Only 2 years after Cramer, Daniel Bernoulli, aware of the paradox through letters from his cousin Nikolaus, conducted a thorough investigation of the matter. The resulting paper '*Exposition of a new theory on the measurement of risk*' was read to the St. Petersburg Academy of Sciences, thus giving the game and the paradox their present name. Daniel's paper can be seen as an elaboration of Cramer's ideas which he apparently became aware of only later.

Central to Daniel's solution is the proposal to measure the utility of wealth by a utility function $u(x)$ and then to compute not the expected payoff, but

its expected utility. The further assumption that the utility of an additional guilder is inversely proportional to the wealth previously possessed, leads Bernoulli to propose the logarithm of wealth as the utility function.

For a person with present wealth a , Bernoulli defines the moral expectation of a game as the weighted average of possible utilities after the game where the weights are the respective probabilities:

$$\sum_i p_i \log(a + x_i).$$

The value of the game for this person is then that fixed amount v whose utility equals the moral expectation of the game i.e. the solution to $\log(a + v) = \sum_i p_i \log(a + x_i)$. An explicit solution is given by

$$v = \prod_i (a + x_i)^{p_i} - a$$

a weighted geometric average of possible posterior wealths.

An interesting feature of Daniel's solution is that the value of a game thus computed depends on the initial wealth of the player. This explains why some games of chance can be profitable to both sides, like an insurance contract to insurer and insured.

For the St. Petersburg Daniel Bernoulli's proposal leads to the table of values depending on initial wealth shown below:

Value of St. Petersburg game					
Initial wealth	0	10	100	1000	10,000
Value of game	4	5.5	7.9	11	14.2

6. FELLER'S LAW OF LARGE NUMBERS

More than 200 years after Daniel Bernoulli's paper the case of the St. Petersburg problem was reopened by William Feller, one of the greatest probabilists of our century. Refusing to leave the frequentist paradise created by Kolmogorov's foundations of probability theory, Feller insisted that the question for the fair price of a game of chance only made sense in connection with the law of large numbers. According to this law, the average payoff in a long series of independent repetitions of the same game converges towards the expected value. Hence if the price charged per game is only slightly smaller than the expected value, the game in the long run becomes profitable for the player.

The traditional law of large numbers still holds for the St. Petersburg game and states here that the average payoff in a series of repeated games will converge to infinity:

$$\frac{1}{n}(X_1 + \dots + X_n) \rightarrow \infty$$

However, this convergence to infinity can be extremely slow. If you bet a fixed amount on every instance of the game, you are guaranteed to win in the long run. However you might have to wait until the end of your days before a large

payoff washes away all your previous losses. This gave Feller the idea that one should consider a variable price, dependent on the number of repetitions you are allowed to plan. Feller supported this idea by a Law of Large Numbers essentially stating that the average payoff per game in the long run is about $\log_2(n)$

$$\frac{1}{n \log_2 n} (X_1 + \dots + X_n) \rightarrow 1,$$

leading to the conclusion that the value of the St. Petersburg game ought to be $\log_2 n$ for someone who may play the game n consecutive times.

Price per game according to Feller				
Number of games	10	100	1000	10,000
Price per game	3.3	6.6	10	13.3

7. CONCLUSION

You have now seen several solutions to the St. Petersburg problem. Which one to resort to, will depend also on your circumstances. If you have only a once in a lifetime chance to play the game, Feller's law of large numbers is not very helpful. On the other hand I made sure to study it when earlier in the talk I offered the St. Petersburg game for 100 guilders to everyone in the audience.

Research inspired by the St. Petersburg paradox is continuing until today, attempting to unravel a bit more of the mysteries of games of chance with infinite expectation.

Here in Groningen we have to regret that Johann Bernoulli's chair was not inherited by Daniel – otherwise chances would have been that every student of probability theory today would have to learn about the Groningen paradox.

ACKNOWLEDGEMENT. I would like to thank Jan van Maanen and Willem Schaafsma for critical reading of an earlier version of this manuscript and for their suggestions for improvement.

REFERENCES

1. DANIEL BERNOULLI, 1738, Specimen Theoriae Novae de Mensura Sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* **V**, 175–192. (English translation 'Exposition of a new theory on the measurement of risk' in *Econometrica* **22**, 23–36 (1954).)
2. JACQUES DUTKA, 1988, On the St. Petersburg paradox. *Archive for History of Exact Sciences* **39**, 13–39.
3. WILLIAM FELLER, 1945, Note on the law of large numbers and fair games. *Annals of Mathematical Statistics* **16**, 301–304.