WNE UW

University of Warsaw
Faculty of Economic Sciences

Topic 2.2

The Constrained Optimization - the foundation of modern microeconomics

Overview of today lecture

One optimization problem in two forms — primal and dual — is sufficient to move from modeling the “atom” (consumer, producer) to modeling the whole economy.

A brief overview of solving constrained optimization problems using the Lagrangian method; Primary and dual problems

Foundations of General Equilibrium Models and Computable General Equilibrium (CGE) Models

Overview of today lecture

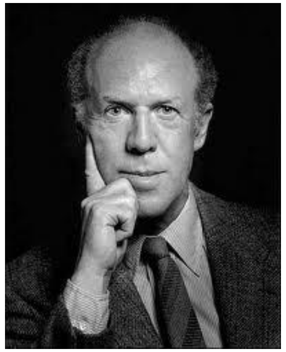
Most economic theories assume that economic agents optimize their behavior to achieve the best possible outcome. This behavior is translated into a model, framed as an optimization problem: consumers aim to maximize utility, while firms focus on minimizing costs and maximizing profit.

In the last lectures, I discussed unconstrained optimization. Today, we will explore constrained optimization, using examples from consumer and producer choices. Ninety percent of economic calculations involve either unconstrained or constrained optimization. I'll demonstrate that a solid understanding of these concepts is enough to construct a simple General Equilibrium Model.

We'll go into more detail as we dive deeper into producer and consumer theory..

Welcome to the Arrow–Debreu world (general equilibrium)

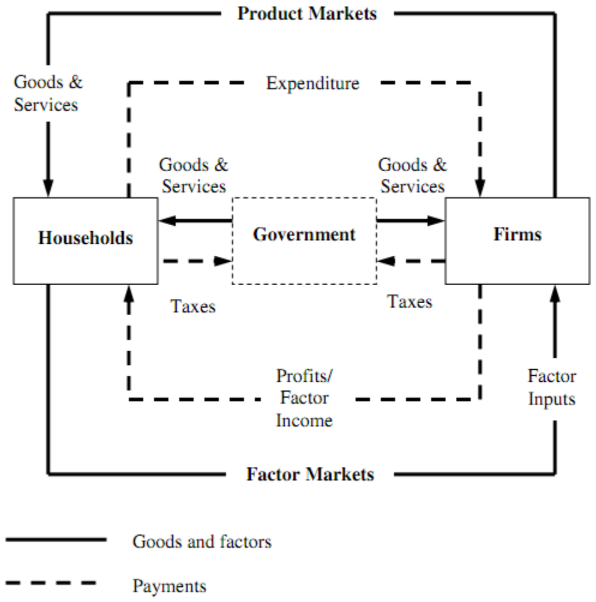
Kenneth Arrow



Gérard Debreu

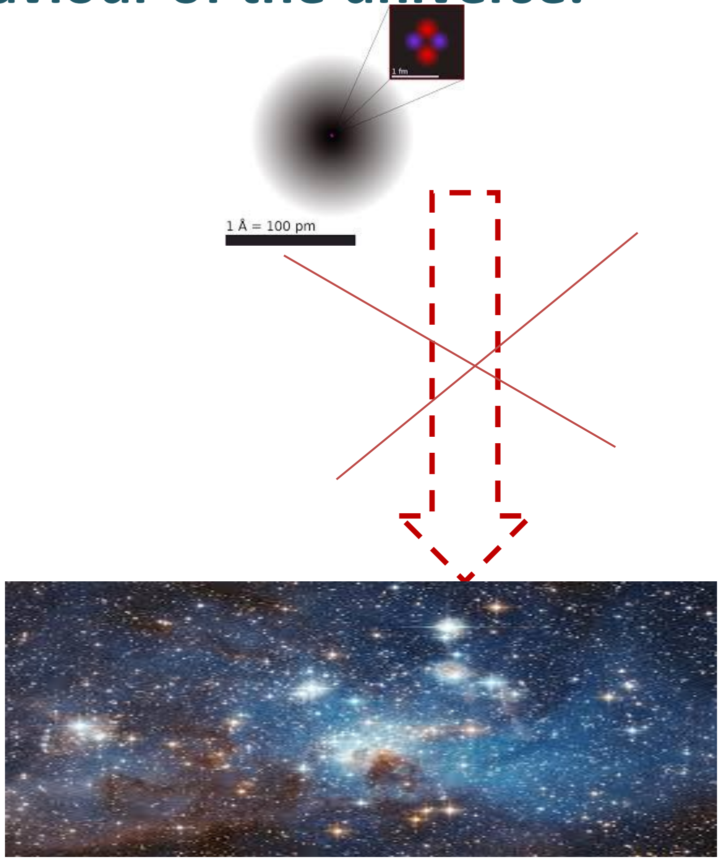
Today, general-equilibrium models dominate macroeconomics. We start from individual agents (homo oeconomicus) and end with the whole economy. Kenneth Arrow and Gérard Debreu did what physics could not: a model of atomic behaviour does not explain the behaviour of the universe.

Figure 1. The Circular Flow of the Economy



Economics consists of:
- N1 consumers
- N2 goods and services
- N3 factors of production
- N2 markets for goods and services and their prices
- N3 markets for factors of production and their prices

and a mechanism that ties it all together.



Witamy w świecie Arrow–Debreu (równowagi ogólnej)

Axioms of rational choice

Utility function representing preferences

Maximization of U(x1, x2) subject to the budget constrain

Demand for goods and supply of factors of production

Consumers (the role they play)
They own production factors K and L, sell them on the market, and consume them, maximizing utility.

Markets connecting both sides:
Markets for factors of production and for goods and services

Producers (the role they take) buy K and L to produce to maximise profit.

Supply of goods and services and demand for factors of production

Axioms defining technology

Production function

Cost minimization for a given output Y(K, L)

Overview of today lecture

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Lagrangian Multiplier Method

Objective: Maximize/Minimize $y = f(x_1, x_2, \dots, x_n)$

Subject to: $g(x_1, x_2, \dots, x_n) = 0$ [Sometimes, we can add more conditions or add inequalities [very rarely]]

This structure represents a typical constrained optimization problem, where the goal is to maximize/minimize the (objective, goal) function y subject to the constraint $g(x_1, x_2, \dots, x_n) = 0$, which limits the values of the decision variables $x_1, x_2, \dots, x_n$

Lagrangian Multiplier Method

The **Lagrangian multiplier method** begins by setting up the following expression (we ad zero to our goal function):

$$\boldsymbol{L = f(x_1, x_2, \ldots, x_n) + \lambda \cdot g(x_1, x_2, \ldots, x_n)}$$

Where λ is an additional variable called the **Lagrangian multiplier**. When the constraint holds, $L = f$, because $g(x_1, x_2, \ldots, x_n) = 0$ (we ad zero to our goal function).

Lagrangian Multiplier Method; First-Order Conditions

For a function $f(x_1, x_2, \ldots, x_n)$ subject to the constraint $g(x_1, x_2, \ldots, x_n) = 0$, the **first-order conditions** are derived as follows:

$$\frac{\partial L}{\partial x_1} = f_1 + \lambda g_1 = 0$$
$$\frac{\partial L}{\partial x_2} = f_2 + \lambda g_2 = 0$$
$$\vdots$$
$$\frac{\partial L}{\partial x_n} = f_n + \lambda g_n = 0$$

$$\frac{\partial L}{\partial \lambda} = g(x_1, x_2, \ldots, x_n) = 0$$

Lagrangian Multiplier Method

First-Order Conditions and Solution Properties

First-Order Conditions can be solved for $x_1, x_2, \dots, x_n$ and λ.

The solution will have properties:

The x's will **satisfy the constraint**.

These x's will **maximize the value of L** and therefore the goal function f.

The **Lagrangian multiplier** λ provides a measure of how the relaxation in the constraint will affect the value of *y*.

Lagrangian Multiplier Method

Second-order conditions that describe the curvature of the function must be checked. For minimization, the Hessian of the Lagrangian must be positive semidefinite, while for maximization, it must be negative semidefinite.

However, we typically select a goal function that satisfy these conditions (our discussions about the production function).

For those interested in more mathematical rigor, refer to the following link for a detailed explanation of the Kuhn–Tucker conditions: https://www.cs.cmu.edu/~ggordon/10725-F12/slides/16-kkt.pdf

[As a final paper, you can prepare a cheat sheet based on Polish materials [previous final papers] in Maxima, Python, (symbolic/numerical calculation) and in R (only numerical)]

Lagrangian Multiplier Method

A **constrained optymalization** problem can have an associated **dual** problem that focuses on the constraints in the original problem.

Primal problem: Consumers maximize utility subject to a budget constraint

Dual problem: consumers minimize the expenditure needed to achieve a given level of utility

Primal problem: Firms minimize the cost of inputs to produce a given level of output

Dual problem: firms maximize output for a given cost of inputs purchased

Typically, for consumers, we use the primal problem (maximizing utility), while for firms, we use the dual problem (minimizing costs).

Utility maximization - preferences

Rational choice theory makes some basic assumptions (axioms) about the preferences of homo economicus:

- **Completeness** - all alternatives can be ranked. If A and B are any two alternatives, an individual can always specify exactly one of these possibilities: i) A is preferred to B ii) B is preferred to A iii) A and B are indifferent
- **Transitivity** - if alternative A is preferred to B, and alternative B is preferred to C, then A is preferred to C.

We can add more axioms, but they mainly have technical implications and do not constitute a core of rationality assumption.

Utility maximization - preferences

- **Non-Satiation** (Monotonicity) -> More of a good is always better (or at least not worse) than less, assuming the additional amount is desirable.
- **Continuity** -> Small changes in outcomes result in small changes in preferences. This means that if A is preferred over B and B over C, there exists some mix of A and C that is equally as preferred as B.
- **Independence of Irrelevant Alternatives** -> Preferences between options should remain consistent, even if additional options are introduced. If A is preferred over B in a choice set, then introducing a third option C should not change the preference for A over B.
- **Consistency** -> Choices are consistent over time and context; if a decision-maker prefers A over B today, they should prefer A over B in similar situations in the future.

Utility maximization - preferences

- **Convexity** (for preferences) -> means that individuals prefer averages or mixes of extreme choices. For example, if an agent is indifferent between A and B, they would also be indifferent to a mix of A and B.
- **Independence from Social or Group Preferences** – Individual preferences are unaffected by others’ choices.

…. More ……

Together these assumptions ensure that preferences are consistent --> we can predict the behaviour of homo oeconomicus and need some of them to have the possibility of obtaining a "nice” mathematical representation of preferences as a utility function.

Utility maximization – utility function

We assume that people are able to rank in order all possible situations from least desirable to most. We can find a function, which maps the ranking -> **utility function**. If A is preferred to B, then the utility assigned to A exceeds the utility assigned to B

$$U(A) > U(B)$$

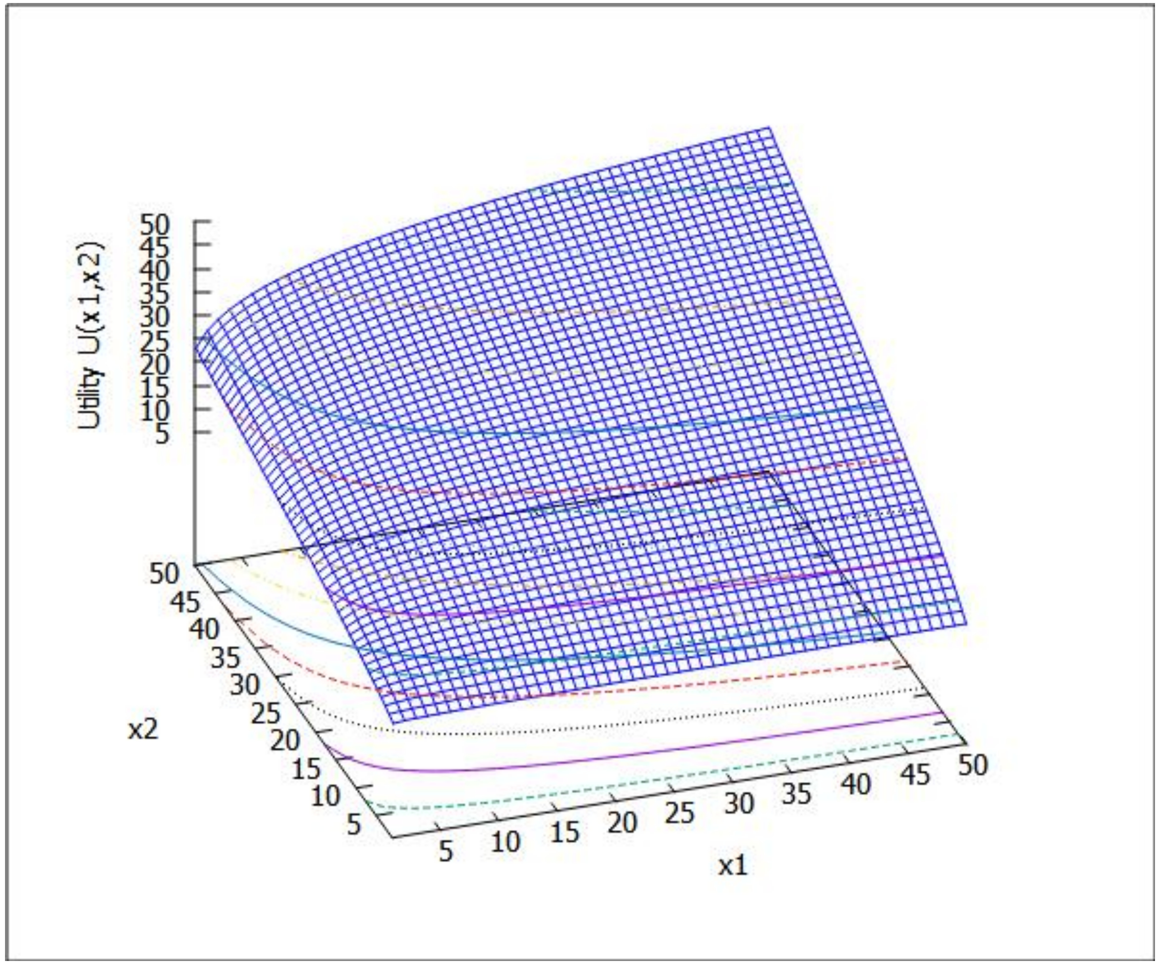
Well-behaving preferences give us well-behaving utility function, which is continuous and monotonically increasing, the second derivative of the function is negative => the law of diminishing marginal utility [mixture of technical and bahavioral assumption]

Lagrangian Multiplier Method

The properties of the utility function are similar to those discussed for the production function (we will use the same Cobb-Douglas function), but its interpretation is different.

The utility function represents **ordinal preferences** (e.g., A is better than B, but we do not measure by how much), so there is no need to analyze scale effects or homogeneity. The utility function can also be transformed monotonically, and such transformations do not change the ordinal ranking of preferences. A common trick to make life easier is to take the logarithm of a utility function, this makes the function easier to manipulate and can help find extrema.

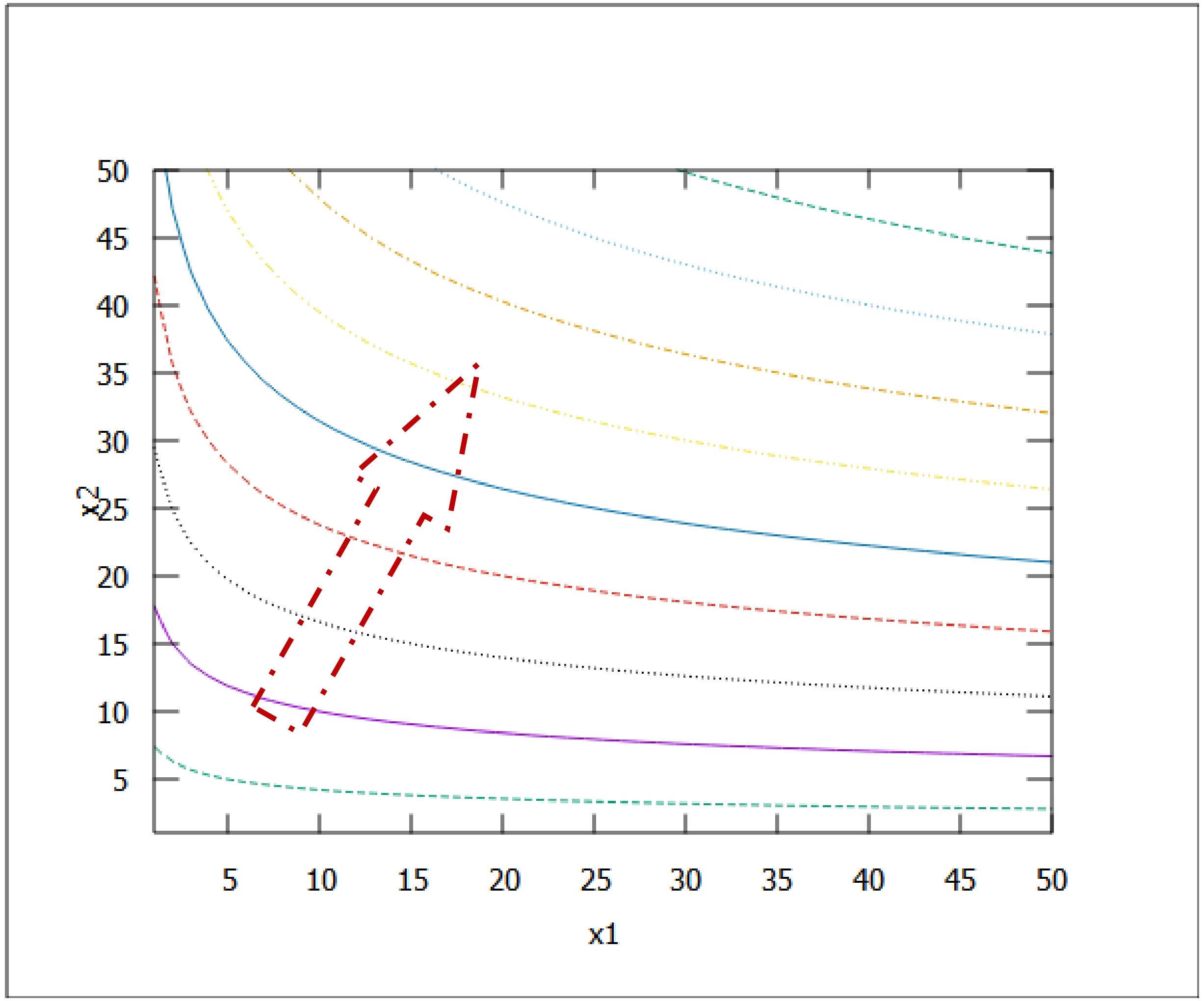
Utility maximization – utility function



The choice space is reduced to **two dimensions**. The consumer chooses between good $x_1 \, and \, x_2$. The consumer's preferences are represented by the utility function $U \; = f(x_1, x_2)$

Mapping the utility function on choice space gives us the **indifference curve**: each points (a basket of two goods $x_1 \, and \, x_2$) provides the same utility.

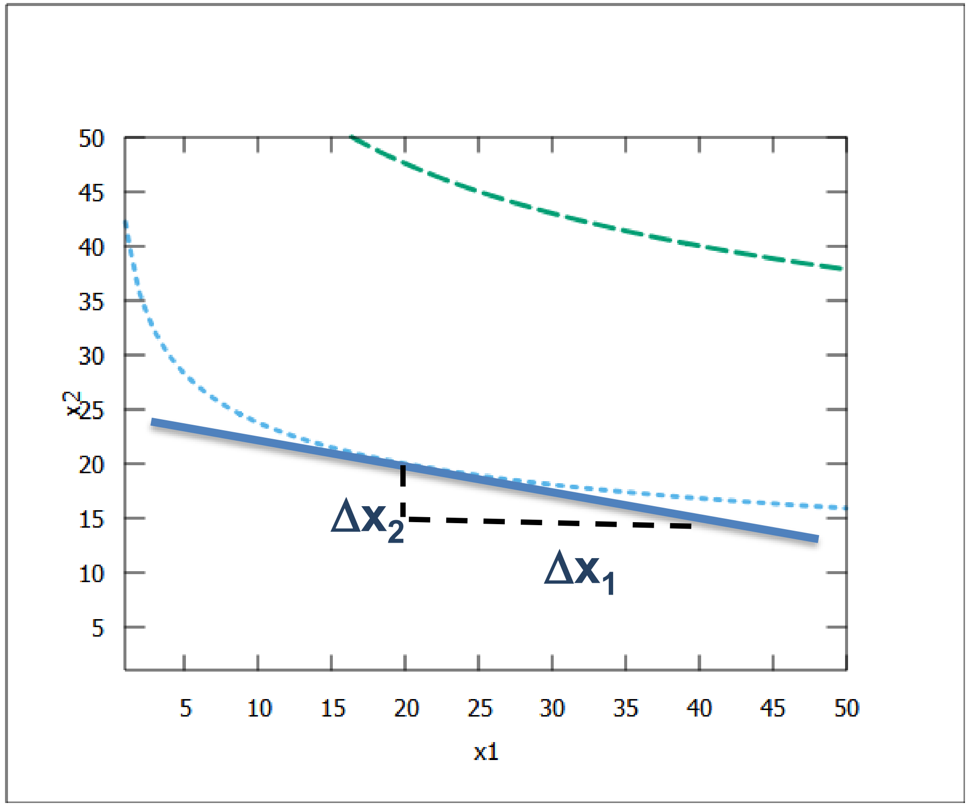
Utility maximization – utility function



Points (a basket of two goods x_1 and x_2) lying on the **indifference curve** are **equally good** for the consumer.

Baskets lying on higher indifference curves are **preferred** to those on lower curves. However, we cannot determine **by how much** one basket is better than another. For example, we cannot say that one basket is "120 times better" than another on a lower indifference curve.

Utility maximization – utility function

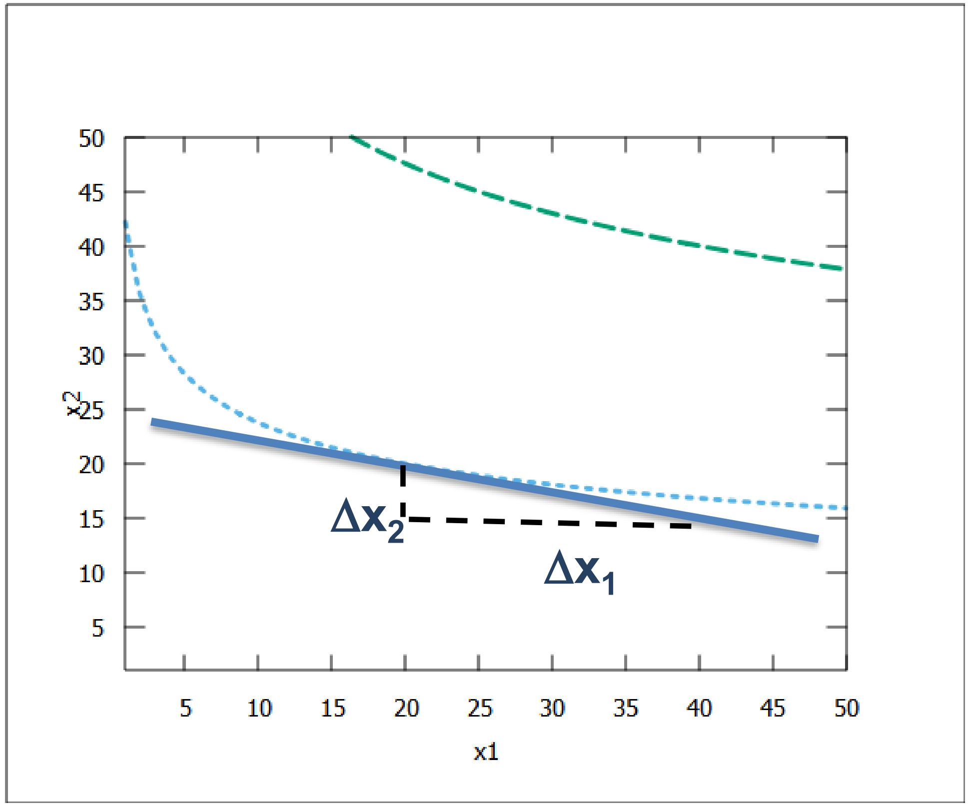


Moving along the indifference curve does not change the utility. We can calculate the change in utility, ΔU, using the total differential.

Since utility remains constant along the indifference curve, ΔU = 0.

The Marginal Rate of Substitution (MRS) is essentially a measure of how much of one good a person is willing to give up to gain an additional unit of another good, keeping the utility level constant.

Utility maximization – utility function



Marginal Rate of Substitution (MRS) derive from total differential:

$$\Delta U = 0 = \Delta x_1 \frac{\delta U}{\delta X_1} + \Delta x_2 \frac{\delta U}{\delta X_2}$$

Ratio of Marginal Utilities:

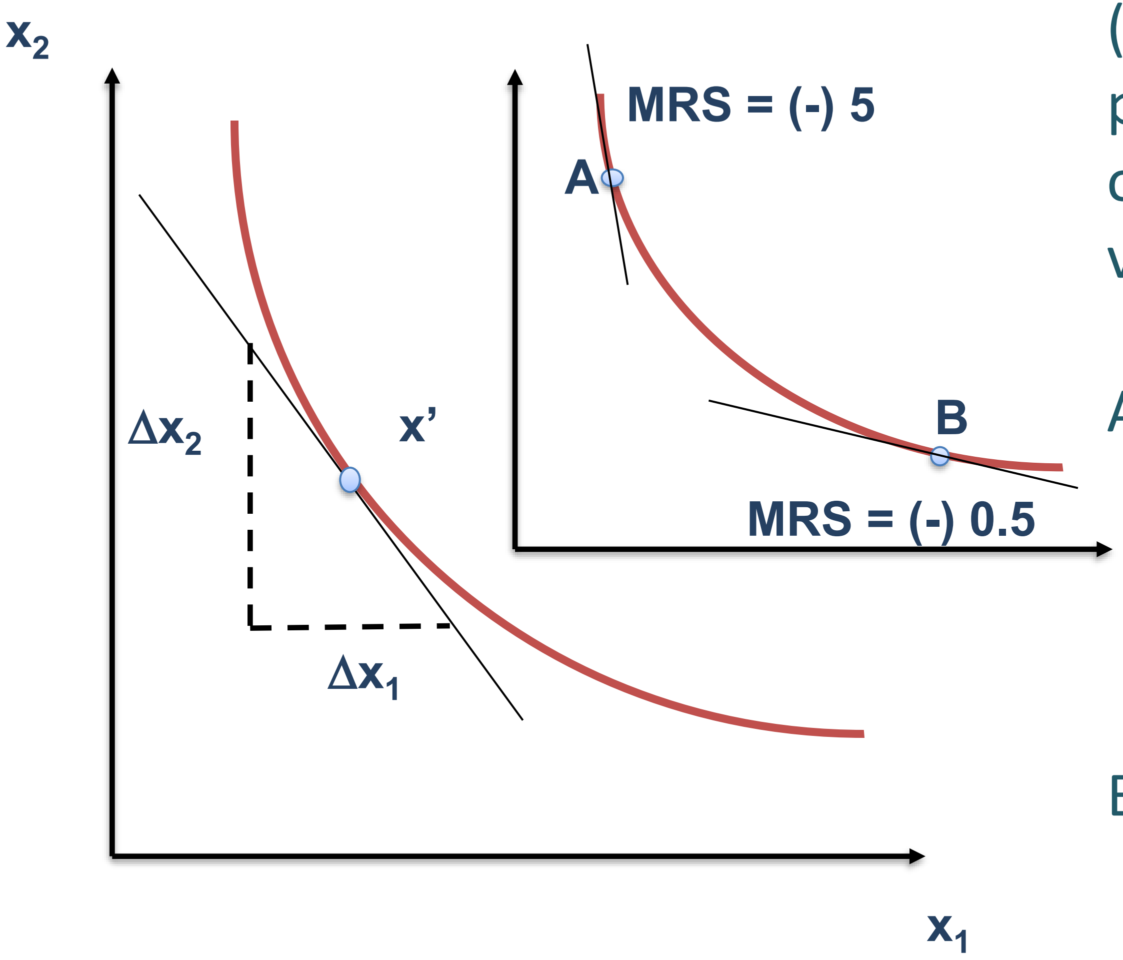
$$\frac{\Delta x_2}{\Delta x_1} = \frac{\frac{\delta U}{\delta x_1}}{\frac{\delta U}{\delta x_2}} = \frac{MU_1}{MU_2}$$

ΔU the change in utility,

Δx_1 and Δx_2 are changes in goods x1and x2,

$\frac{\delta U}{\delta x_1} = MU_1$ and $\frac{\delta U}{\delta x_2} = MU_2$ the marginal utilities

Utility maximization Psychological Perspective on MRS



MRS defines our psychological exchange relations (we can substuitute one good by another in given proportion and it not change our sytuation) . The law of diminishing MRS fits with human behavior – we value what we lack.

A: We have a lot of x2 and a little x1

- Psychological Insight: We value what we lack – x1 is valued more.
- Behavior: Willing to trade 5 units of x2 for 1 unit of x1

B: We have a lot of x1 and a little x2

- Psychological Insight: Now x2 is more valuable.
- Behavior: Willing to trade only 1/2 unit of x2 for 1 unit of x1.

Utility maximization - budget constraint

The consumers have finite resources. We assume that consumers spend all their income (there are no savings in this model; if we want to allow for savings, we treat them as an additional variable).

Therefore, consumer spending can be represented by budget constraint of the form: $\mathbf{p_1x_1 + p_2x_2 +\ldots+ p_nx_n = M}$.

The budget constraint is defined by the market: M and prices p1, p2 are given - the external world defines HO possibility of consumption.

Utility - the maximization problem

The consumer choice is simplified to maximization problem with constraints.

$$\text{Max } U(x_1, x_2, \ldots, x_n)$$

subject to

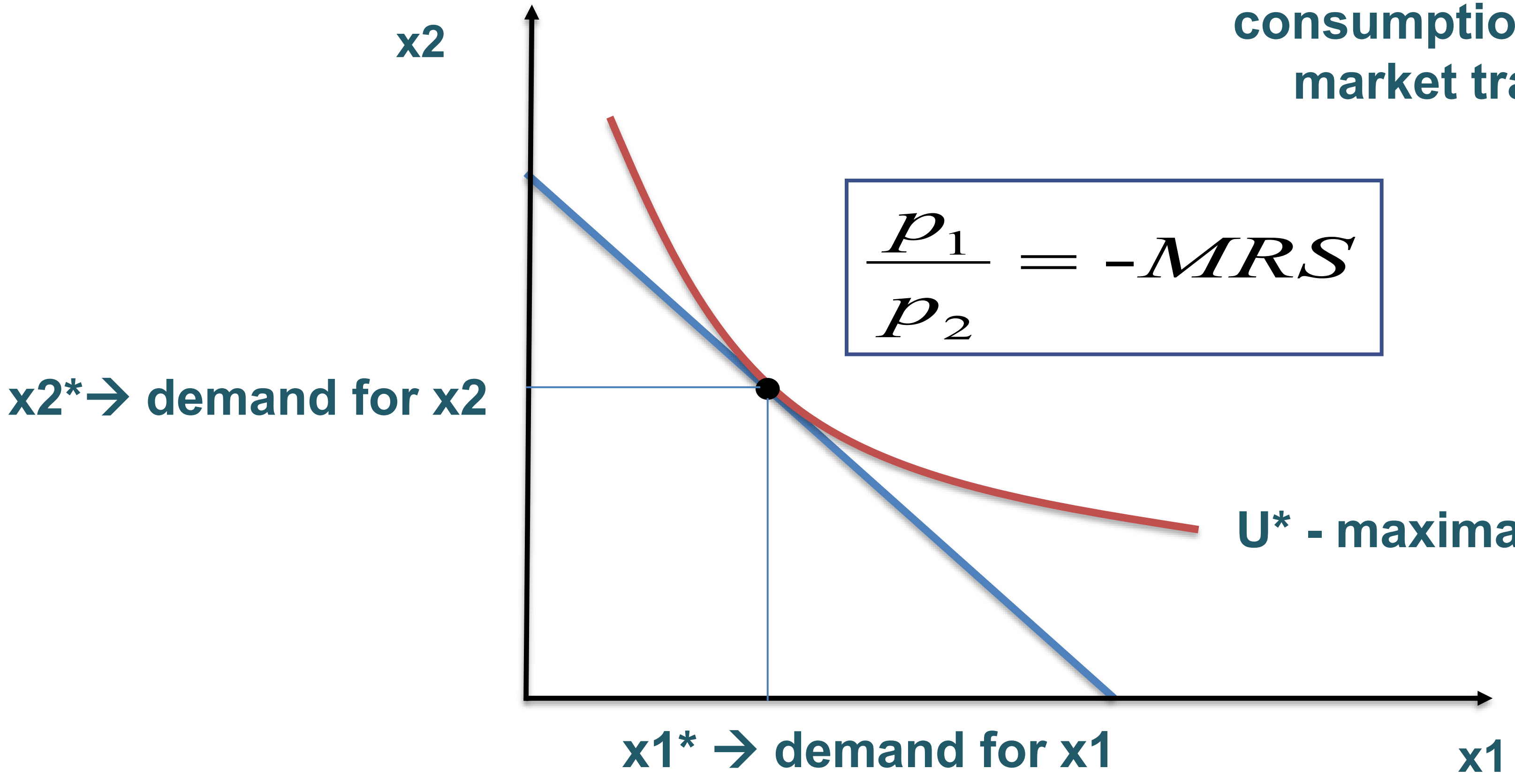
$$p_1x_1 + p_2x_2 + \ldots + p_nx_n = M$$

Utility - the equilibrium

We know the consumer choice. It is defined by an equilibrium condition.

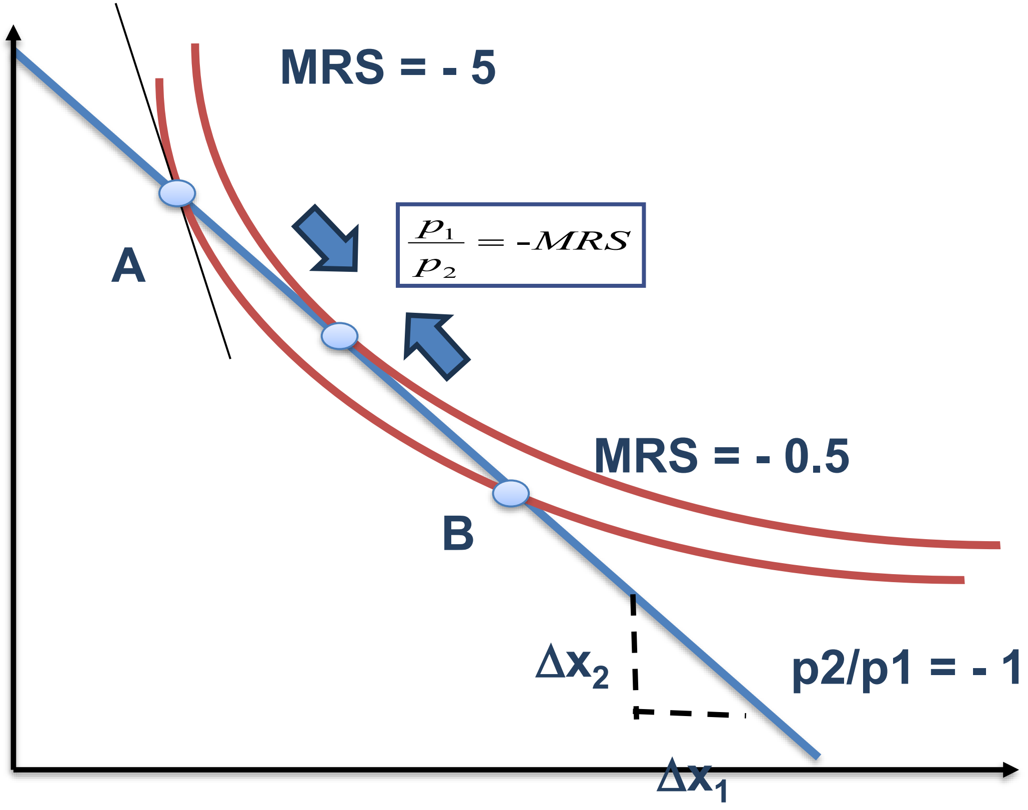
The psychological trade-off between consumption x1 and x2 (MRS) is equal the market trade-off which is set by price relations (p1/p2)

$$\frac{p_1}{p_2} = -MRS$$



U* - maximal level of utility for given p1,p2 and M

Utility maximization subject to a budget constraint



Can we find **the candy experiment** in this case?

Let's assume the market exchange ratio (the slope of the budget constraint) equals (-) 1 --> we can exchange a unit of x1 for a unit of x2 on the "market"

We are at point(basket) A --> MRS = (-)5 --> we are willing to give up 5 units of x2 for 1 unit of x1 and the market tradeoff is 1 to 1 --> it pays to increase x1, because psychologically we gain from the difference in exchange ratios and Marginal Benefit = 5 ≠ Marginal Cost = 1 from exchange.

Point B -> by analogy

The maximization problem

Cobb-Douglas: $U(x1,x2) = x1^{a}x2^{b}$

Objective function (utility)

Lagrangian:

$$L = x1^{a}x2^{b} + \lambda(p1x1 + p2x2 - M)$$

(F.O.C)

$$\partial L/\partial x1 = a x1^{a-1}x2^{b} + \lambda p1 \quad = 0$$

$$\partial L/\partial x2 = b x1^{a}x2^{b-1} + \lambda p2 \quad = 0$$

$$\partial L/\partial \lambda = p1x1 + p2x2 - M \quad = 0$$

We add zero --> the equation gives zero when our choices satisfy the budget constraint, and there are no savings.

S.O.C

Hessian of the Lagrangian must be negative semidefinite

```
kill(all)$  /* Clear all previous variables and functions */
assume(x1>0, x2>0, p1>0, p2>0, u>0, a>0, a<1, b>0,b<1, M>0,lambda>0)$  /* Assume all variables are positive to simplify the solution */
L: (x1^a * x2^b) + lambda * (p1 * x1 + p2 * x2 - M);
/* Define the Lagrange function with a utility function and budget constraint */

/* First-order conditions */
eq1: diff(L, x1, 1) = 0;  /* Take the first derivative of L with respect to x1 and set it to zero */
eq2: diff(L, x2, 1) = 0;  /* Take the first derivative of L with respect to x2 and set it to zero */
eq3: diff(L, lambda, 1) = 0;  /* Take the first derivative of L with respect to lambda and set it to zero (budget constraint) */

/* Solve the system of equations for x1, x2, and lambda */
solution: solve([eq1, eq2, eq3], [x1, x2, lambda]);

/* Second-order conditions - Compute the Hessian matrix */
Hessian: matrix(
   [diff(L, x1, 2), diff(diff(L, x1), x2)],
   [diff(diff(L, x2), x1), diff(L, x2, 2)]
);

/* Evaluate the Hessian at the solution values for x1, x2, and lambda */
Hessian_at_solution: subst(solution, Hessian);

/* Check if the Hessian is negative semi-definite for maximization */
 compare(determinant(''Hessian_at_solution),0);
```

Utility - the equilibrium

$$\left[\left[x1 = \frac{Ma}{(b+a)p1}, x2 = \frac{Mb}{(b+a)p2}, \lambda = -\left(\frac{M^{b+a-1}a^a b^b (b+a)^{-b-a+1}}{p1^a p2^b}\right)\right]\right]$$

Lambda Interpretation: Increase in utility due to budget increase by 1

Consumption x1, 2 for prices p1 p2 and income M maximizing utility. For the general form --> demand functions x1 = f(p1,p,M), x1 = f(p1,p,M). It is worth noting:

- Income Effect: Demand for both goods positively depends on income M (linear relationship).
- Price Elasticity: Demand functions are nonlinear to the price of the good itself.
- Cross-Price Independence: Demand for x1 does not depend on p2, and demand for x2 does not depend on p1 in this setup.

Question:

How the expenses for the two goods are related? (Consider the total spending on each good and the impact of price and income changes.)

The next lecture will discuss the issues and whether they are realistic.

Utility - the equilibrium

ACS programs sometime have trouble solving a system of equations like the one above. In “difficult” cases, you can solve it by:

1) By using the MRS = p1/p2 equilibrium and solve the system of equations: MRS=p1/p2 and p1*X1+p2*X2 = M

and / or

2) By substitution -> from the budget constraint, we can obtained one variable as a function of the other and inserted into the objective function to solve a simple one-variable optimization problem.

Utility - the equilibrium

$$\partial L/\partial x1 = ax1^{a-1}x2^{b} + \lambda p1 \quad = 0$$
$$\partial L/\partial x2 = bx1^{a}x2^{b-1} + \lambda p2 \quad = 0$$
$$\partial L/\partial \lambda \quad = p1x1 + p2x2 - M \quad = 0$$

$$ax1^{a-1}x2^{b} \;= -\lambda p1\; 0$$
$$bx1^{a}x2^{b-1} \;= -\lambda p2\; 0$$
$$p1x1 + p2x2 - M \quad = 0$$

$$\frac{ax1^{a-1}x2^{b}}{bx1^{a}x2^{b-1}} = \frac{-\lambda p1}{-\lambda p2}$$

$$\frac{ax1^{a-1}x2^{b}}{bx1^{a}x2^{b-1}} = \frac{p1}{p2}$$

$$\frac{MU1}{MU2} = \frac{p1}{p2} = MRS$$

$$p1x1 + p2x2 - M \quad = 0$$

```
ill(all); /* Clear all previous definitions and variables */

assume(x1>0, x2>0, p1>0, p2>0, U>0, a>0, b>0, a<1, b<1, lambda>0);
/* Set assumptions for the variables: x1, x2, p1, p2, U, a, b, and lambda are all positive,
  and exponents a and b are between 0 and 1 */

$declare(a, noninteger, b, noninteger);
/* Declare 'a' and 'b' as non-integer values for further symbolic manipulation */

U: x1^a * x2^b;
/* Define the utility function U as a Cobb-Douglas function with exponents 'a' and 'b' */

eq1: diff(U, x1, 1) / diff(U, x2, 1) = p1 / p2;
/* First equation: Set the ratio of marginal utilities equal to the price ratio (p1/p2) */

eq2: p1 * x1 + p2 * x2 - M;
/* Second equation: Budget constraint where total spending (p1*x1 + p2*x2) equals
income (M) */

solve([eq1, eq2], [x1, x2]);
/* Solve the system of equations (eq1 and eq2) for the variables x1 and x2 */
```

By using the MRS = p1/p2 equilibrium and solve the system of equations:

MRS=p1/p2

and

p1*X1+p2*X2 = M

Utility - the equilibrium

```
kill(all)$
/* Step 1: Define variables and parameters */
assume(x1 > 0, x2 > 0, p1 > 0, p2 > 0, M > 0, a > 0, a < 1, A > 0, M > 0)$
declare(a, constant)$
declare(ro, constant)$
declare(M, constant)$
declare(A, constant)$

/* Step 2: Define the CES utility function U */
U: A*( a * x1^(-ro) + (1 - a) * x2^(-ro) )^(-1 / ro )$

/* Step 3: Compute the marginal utilities MU1 and MU2 */
MU1: diff(U, x1)$
MU2: diff(U, x2)$

/* Step 4: Simplify marginal utilities */
MU1_simplified: ratsimp(MU1)$
MU2_simplified: ratsimp(MU2)$

/* Step 5: Compute the Marginal Rate of Substitution */
MRS: MU1_simplified / MU2_simplified$
MRS_simplified: (a / (1 - a)) * (x2 / x1)^(ro + 1)$

/* Step 6: Set MRS equal to the price ratio */
eq_mrs: MRS_simplified = p1 / p2$

/* Step 7: Solve for x1/x2 */
x1_over_x2_expr: (a / (1 - a)) * (p2 / p1)$
x1_over_x2: x1_over_x2_expr^(1 / (ro + 1))$

/* Step 8: Display the ratio x1/x2 */
x1_over_x2_simplified: ratsimp(x1_over_x2)$
print("The ratio x1/x2 is:", x1_over_x2_simplified)$

/* Step 9: Substitute x1 in the budget constraint */
x1_expr: x1_over_x2_simplified * x2$
budget_constraint: p1 * x1_expr + p2 * x2 = M$

/* Step 10: Solve for x2 */
denominator: p1 * x1_over_x2_simplified + p2$
x2_sol: x2 = M / denominator$

/* Step 11: Compute x1 using x1 = (x1/x2) * x2 */
x1_sol: x1 = x1_over_x2_simplified * x2_sol$

/* Step 12: Simplify the demand functions */
x1_demand: ratsimp(x1_sol)$
x2_demand: ratsimp(x2_sol)$

/* Step 13: Display the demand functions */
print("Demand function for x1:", x1_demand)$
print("Demand function for x2:", x2_demand)$
```

By using the MRS = p1/p2 equilibrium and solve the system of equations: MRS=p1/p2 and p1*X1+p2*X2 = M

and

By substitution ->from the budget constraint, one variable is obtained as a function of the other and inserted into the objective function to solve a simple one-variable optimization problem.

The CES utility finction.
This is not my code; it's just based on my codes created by Chat GPT. Chat (version o1 4o) works quite well with Maxima. This code didn't work the first time, but I just pasted the results and information about the error and obtained the corrected code.

The cost (expenditure) minimization problem - by analogy

We assume that firms (owners??) are cost-effective. For given prices of factors of production and a given technology Y(), they will employ such a combination production factors to minimize the cost of producing a given volume of output **Y**. How much money do I need to reach a certain level of production?

We can treat entrepreneurs as consumers; they "consume" factors of production to obtain the desired production volume. So, we have a dual problem to the previously analyzed one, but ...

The cost (expenditure) minimization problem - by analogy

Problem: Creating a Universal Production Model

To approximate technology, we use a production function with several simplifying assumptions.

Single Product(Key Requirement: Use a Function, Not a Functional) In macro models, GDP represents the economy's total output. In microeconometrics, however, we have to aggregate multiple products into a single variable (Samsung production = smrtphones + tanks + ……+ insurace).

Multiple Production Factors: Traditional factors are land, capital, and labor, with additional "new" factors like intellectual and social capital.

The cost (expenditure) minimization problem - by analogy

What is and is not a Production Factor? Ideologies influence this concept, and it is not clearly defined in textbooks.

- Marxist View: Labor is the primary production factor, while other factors are labour outcomes. Capital is viewed as a surplus taken over by capitalists from workers.
- Modern View: New Factors, even culture capital, are frequently added, which raises measurement challenges (Capital measured by parent's knowledge of a foreign language)
- In research, factors are treated technically based on production type and available data (e.g., education production function: output = exam scores, capital = school equipment, labour = a number of teachers).

The cost (expenditure) minimization problem - by analogy

Measurement and data

While we can roughly assign a measure to capital or labor, how do we measure intellectual capital or, even more so, social or culture capital? In research, certain approximations of these quantities are used.

The textbook approach does not consider these nuances. In research, the factors of production must be redefined depending on the purpose, subject matter, and availability of data.

The cost (expenditure) minimization problem - by analogy

Dynamics of Change in Factors of Production

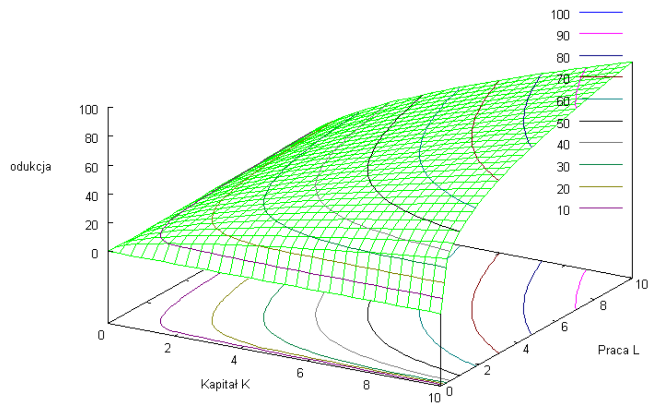
Consumption vs. Accumulation: Some production factors are consumed, while others accumulate in the production proces.

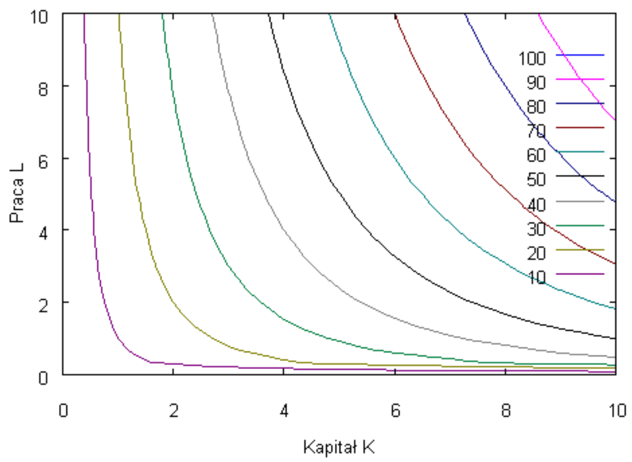
Accumulating Factors: Financial, intellectual, and social capital grow through use – the more you „use”, the more they increase.

Non-accumulating Factors: Land, raw materials, and labor wear out in the production process without accumulation.

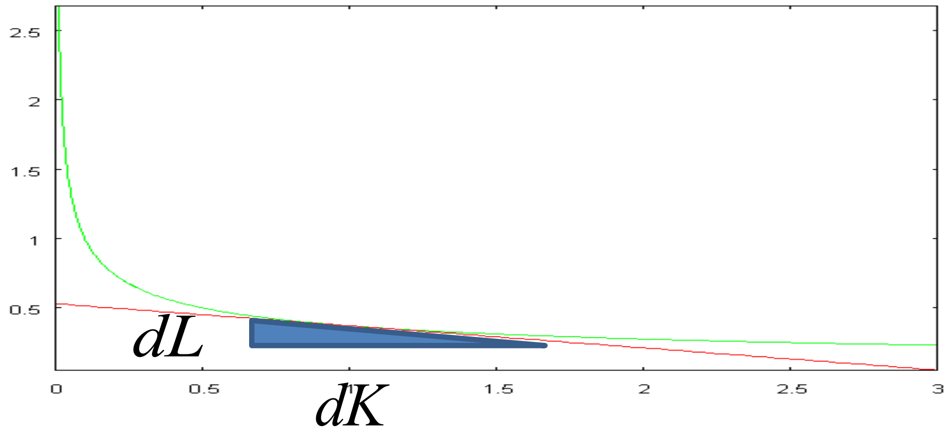
The cost (expenditure) minimization problem - by analogy

The choice of firm space is reduced to **two dimensions**. The firm chooses between factors K(capital) and L (Labour). The technology are represented by the production function $Y = f(K, L)$ Mapping the production function onto choice space creates an isoquant map: each point (a combination of capital, K, and labour, L) gives the same production level. **Now we care about how much these isoquants differ in value --> production is measurable. The economy of scale matters.**





The cost (expenditure) minimization problem - by analogy



$$dY = dK\frac{\partial Y}{\partial K} + dL\frac{\partial Y}{\partial L}$$

$$dY = 0$$

$$0 = dK\frac{\partial Y}{\partial K} + dL\frac{\partial Y}{\partial L}$$

$$\frac{dL}{dK} = \frac{\frac{\partial Y}{\partial K}}{\frac{\partial Y}{\partial L}} = \frac{MP_K}{MP_L} = MRTS$$

Marginal Rate of Technical Substitution (MRTS)

MRTS measures how one factor can be substituted for another while keeping production constant.

The isoquant shows combinations of K and L that yield the same production level.

The slope of the isoquant indicates the amount of L that can be reduced for each additional unit of K to maintain the same production level.

The cost (expenditure) minimization problem - by analogy

Cobb-Douglas vs. CES Production Function

- Cobb-Douglas: Widely used due to simplicity and fixed factor shares.
- The CES function is more universal but not easy to use in models [but you can find all results of calculations in textbooks].CES (Constant Elasticity of Substitution) offers:

adjustable factor shares
flexibility in modelling substitution of production factors.

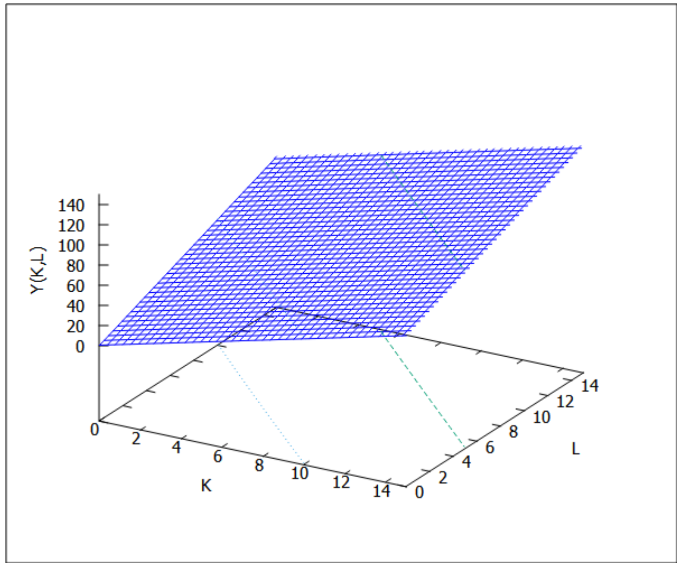
$$Y = A\left[\delta K^{-\rho} + (1-\delta)L^{-\rho}\right]^{-\frac{1}{\rho}}$$

A - efficiency parameter; represents the state of technology A>0
δ - parameter of relative share of individual factors in production 0<δ<1 (similarly as in the C-D alpha (1-alpha) function)
ρ - is the parameter of substitution of one factor for another -1 < ρ

The cost (expenditure) minimization problem - by analogy

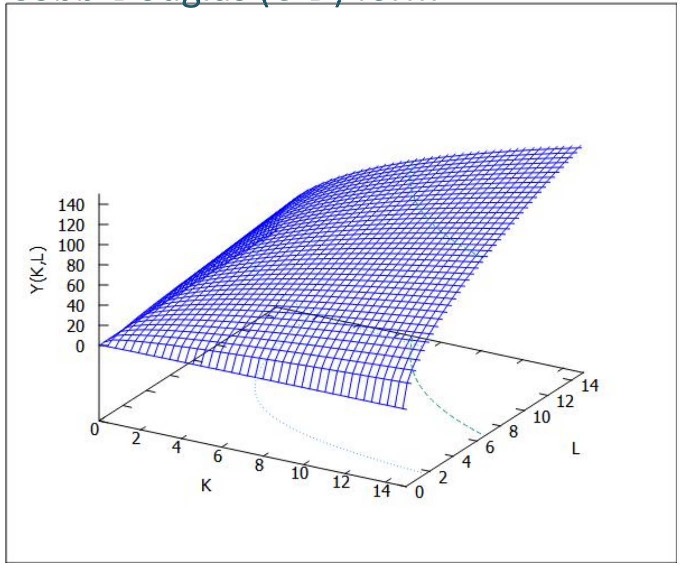
```
kill(all); /* Clear all previous definitions and variables
*/

load(draw); /* Load the drawing package for 3D
plotting */

Y: 10 * (a * (K^(-ro)) + (1 - a) * (L^(-ro)))^(-1/ro);
/* Define the CES production function Y with elasticity
parameter 'ro'
  and distribution parameter 'a' (controls the share of K
and L in production) */

a: 0.5; /* Set the distribution parameter 'a' (share of
capital K in production) */
ro: 23; /* Set the elasticity of substitution parameter
'ro' range (-1, 0) and (0,inf) */

draw3d(
   yrange = [0, 15], /* Define range for the y-axis
(output Y) */
   xrange = [0, 15], /* Define range for the x-axis
(capital K) */
   zrange = [0, 150], /* Define range for the z-axis
(output Y) */
   xlabel = "K", /* Label for x-axis (Capital K) */
   ylabel = "L", /* Label for y-axis (Labor L) */
   zlabel = "Y(K,L)", /* Label for z-axis (Output Y as a
function of K and L) */
   contour_levels = 5, /* Set the number of contour
levels for the plot */
   contour = both, /* Enable contour lines on both the
base and surface */
   explicit(''Y, K, 0.001, 19, L, 0.001, 19), /* Plot the CES
function over specified ranges of K and L */
   color = red, /* Set the plot color to red */
   label(["ro =", 5, 5,20]),label([string(ro),5,5,22.5]) /*
Add a label to indicate the value of 'ro' on the plot */
) $
```

Case of perfect substitution (ro = -1): The CES function becomes linear



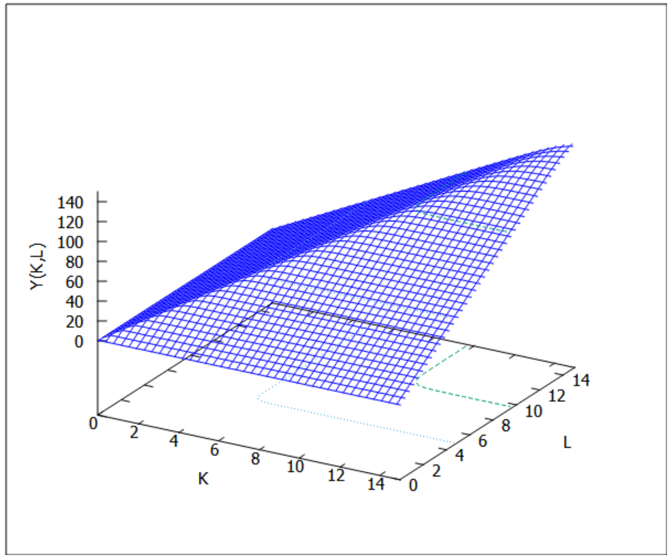

```
/* Case of perfect substitution (ro = -1): The CES function becomes linear */
kill(all); /* Clear all previous definitions and variables */
Y: A * (delta * (K^(-ro)) + (1 - delta) * (L^(-ro)))^(-1/ro);
/* Define the CES production function Y with elasticity parameter 'ro'
  and distribution parameter 'delta' for perfect substitution */
delta: 0.5; /* Set the share parameter 'delta' for factor K */
A: 10$ /* Set the scaling factor 'A' for the production function */
radcan(limit(''Y, ro, -1, plus));
/* Calculate the limit of Y as ro approaches -1 (perfect substitution),
  simplifying the CES function to a linear form */
(%o4) /* Expected output: 5 * L + 5 * K */;
```

Case of partial substitution (ro -> 0): The CES function converges to the Cobb-Douglas (C-D) form




```
/* Case of partial substitution (ro -> 0): The CES function converges to the Cobb-Douglas (C-D)
form */
kill(all); /* Clear all previous definitions and variables */
Y: A * (delta * (K^(-ro)) + (1 - delta) * (L^(-ro)))^(-1/ro);
/* Define the CES production function Y with elasticity parameter 'ro' and
  distribution parameter 'delta' for factors K and L */
delta: 0.5; /* Set the share parameter 'delta' for factor K */
A: 10$ /* Set the scaling factor 'A' for the production function */
radcan(limit(radcan(''Y), ro, 0, plus));
/* Calculate the limit of Y as ro approaches 0 (partial substitution),
  simplifying the CES function to the Cobb-Douglas form */
(%o4) /* Expected output: 10 * sqrt(K) * sqrt(L) */;
```

As ρρ approaches infinity, the CES function converges to the Leontief form: Y = max(aK,(1-a)L)

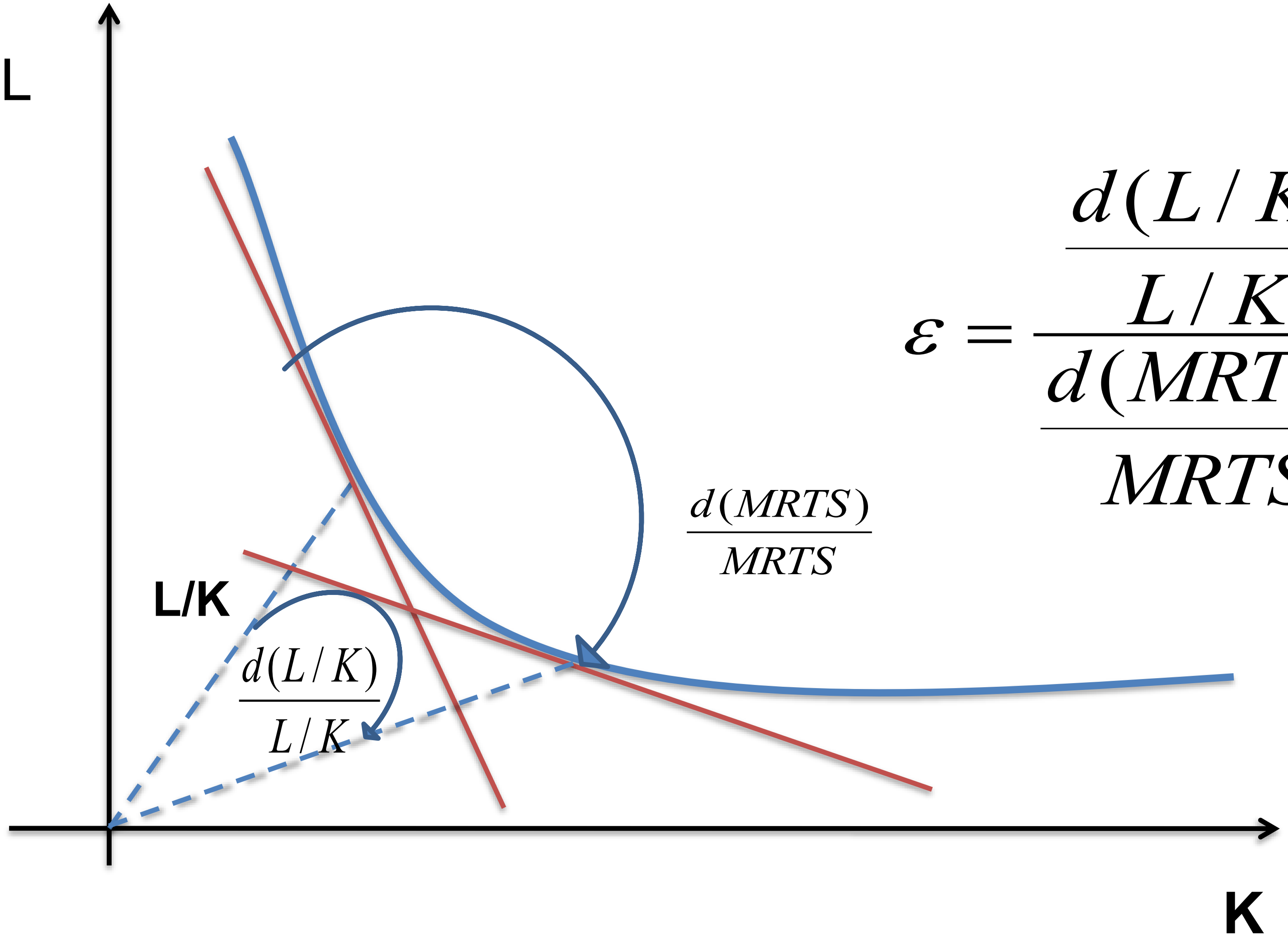


The cost (expenditure) minimization problem - by analogy

The name of this function is also its characteristic: It is a function with constant elasticity of substitution. Elasticity of substitution is a measure of the percentage change in the proportion of production factors in relation to the percentage change in MRTS. It is a measure of the curvature of the isoquant - it is constant and defined by the parameter ρ

$$\varepsilon = \frac{\dfrac{d(L/K)}{L/K}}{\dfrac{d(MRTS)}{MRTS}} = \frac{1}{1-\rho}$$

The cost (expenditure) minimization problem - by analogy



$$\varepsilon = \frac{\dfrac{d(L/K)}{L/K}}{\dfrac{d(MRTS)}{MRTS}}$$

The cost minimization problem

Firms are cost-effective, i.e.: given:

- prices of factors of production r, in
- a technology Y(K,L)

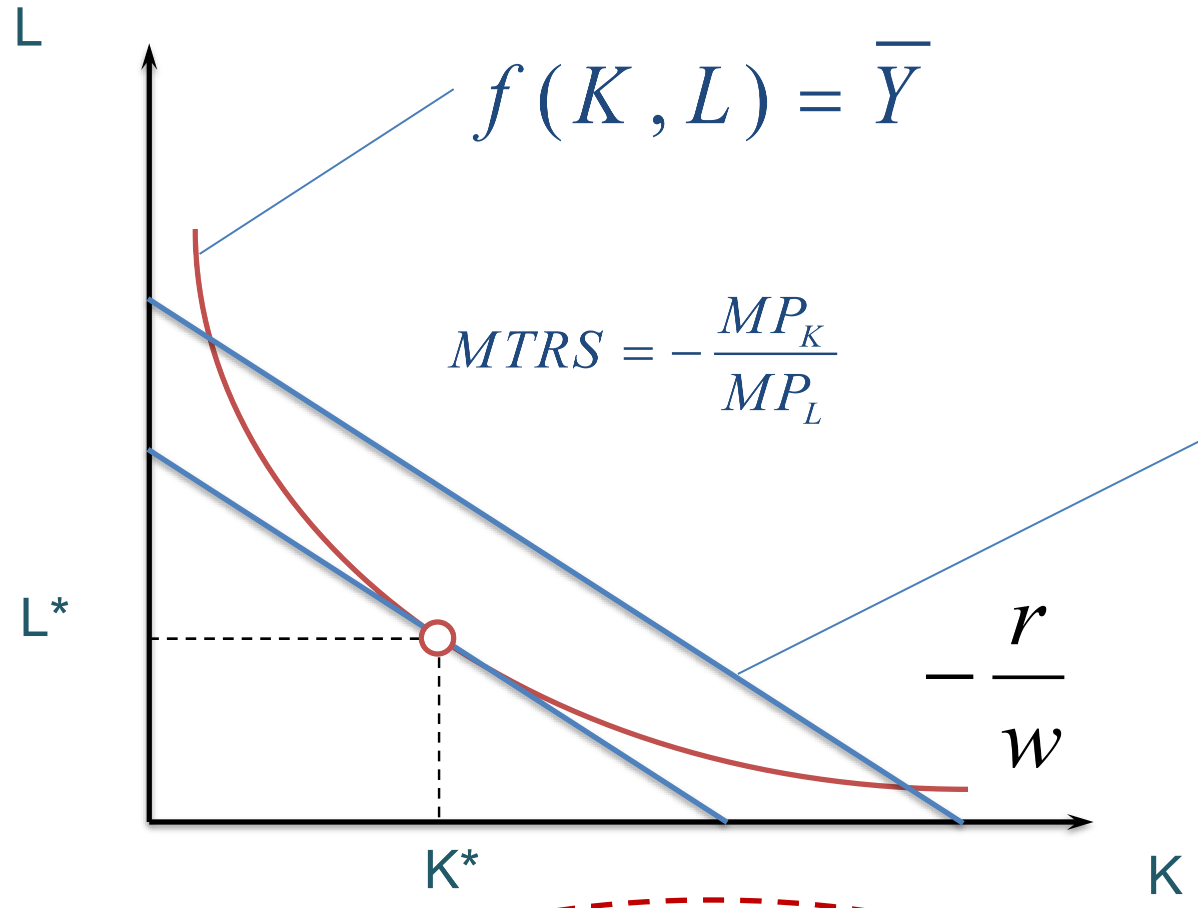
they will employ such a combination of factors of production as to **minimize the cost** of producing a given **quantity of output Y** → the primary problem takes the form:

$$\min_{K,L\geq 0} C = rK + wL$$

subject to constraint

$$Y(K,L) = \bar{Y}$$

The cost minimization problem



$$C = rK + wL$$

$$L = \frac{C}{w} + \frac{r}{w}K$$

$$-\frac{MP_K}{MP_L} = MRTS = -\frac{r}{w}$$

The cost minimization problem

$$\min_{K,L} C = rK + wL \qquad \text{subject to} \qquad Y(K,L) = \overline{Y}$$

$$\mathrm{L}(K,L,\lambda) = (rK + wL) - \lambda\left(Y(K,L) - \overline{Y}\right)$$

Objective function

We use K and L to achieve the desired level of production with a given technology.

F.O.C
$$\begin{cases} \dfrac{\mathrm{L}(K,L,\lambda)}{\partial L} = 0 \\ \dfrac{\mathrm{L}(K,L,\lambda)}{\partial K} = 0 \\ \dfrac{\mathrm{L}(K,L,\lambda)}{\partial \lambda} = 0 \end{cases}$$

The cost minimization problem

```
kill(all)$ /* Clear all previous definitions and variables */

assume(K>0, L>0, r>0, w>0, Y>0, lambda>0, a>0, b>0, a<1, b<1, _Y>0, A>0)$
/* Set assumptions for variables: K, L, r, w, Y, lambda, a, b, _Y as positive.
   Also, a and b are between 0 and 1 */

declare(a, constant, b, constant, A, constant)$
declare(a, noninteger, b, noninteger);
/* Declare 'a' and 'b' as non-integer values for further symbolic manipulation */
/* Declare 'a', 'b', and 'A' as constants for symbolic manipulation */

Y: A * (K^(a)) * (L^(b))$
/* Define the production function Y as a function of K and L with scaling factor A and exponents a and b
*/

Lagr: (r * K + w * L) + lambda * (_Y - Y)$
/* Define the Lagrangian function with the objective function (r * K + w * L)
   and the constraint (_Y - Y) multiplied by lambda */

eq1: diff(Lagr, K, 1) = 0$ /* First-order condition with respect to K */
eq2: diff(Lagr, L, 1) = 0$ /* First-order condition with respect to L */
eq3: diff(Lagr, lambda, 1) = 0$ /* First-order condition with respect to lambda (constraint) */

/* Solve the equations sequentially since direct substitution does not work */

/* From the third equation, solve for K */
solve(eq3, K)$ K: rhs(%[1])$ /* Extract the value of K from the solution */

/* From the second equation, solve for lambda */
solve(''eq2, lambda)$ lambda: ''rhs(%[1])$ /* Extract the value of lambda from the solution */

/* From the first equation, solve for L */
solve(''eq1, L)$ L: rhs(%[1])$ /* Extract the value of L from the solution */

/* Display the final values of L, K, and lambda */
print("Optimal value of L:", ''L)$
print("Optimal value of K:", ''K)$
print("Optimal value of lambda:", ''lambda)$
```

$$\mathsf{L}(K, L, \lambda) = (rK + wL) - \lambda(Y(K, L) - \overline{Y})$$

(%o8) $(120 - K^{0.6} L^{0.1})\, lambda + w\, L + r\, K$

(%o9) $r - \frac{0.6\, L^{0.1}\, lambda}{K^{0.4}} = 0$

(%o10) $w - \frac{0.1\, K^{0.6}\, lambda}{L^{0.9}} = 0$

$$\begin{cases} \frac{\mathsf{L}(K, L, \lambda)}{\partial L} = 0 \\ \frac{\mathsf{L}(K, L, \lambda)}{\partial K} = 0 \\ \frac{\mathsf{L}(K, L, \lambda)}{\partial \lambda} = 0 \end{cases}$$

We solve the system of 3 equations by substitution. In textbooks it is done in such a way that we get interpretable conditions along the way, i.e.:
MRTS = price ratio

The cost minimization problem

Optimal value of L: $$\frac{b^{\frac{a}{b+a}}\,\_Y^{\frac{1}{b+a}}\,r^{\frac{a}{b+a}}}{A^{\frac{1}{b+a}}\,a^{\frac{a}{b+a}}\,w^{\frac{a}{b+a}}}$$

Optimal value of K: $$\frac{a^{\frac{b}{b+a}}\,A^{\frac{b}{a\,(b+a)}-1/a}\,\_Y^{1/a-\frac{b}{a\,(b+a)}}\,w^{\frac{b}{b+a}}}{b^{\frac{b}{b+a}}\,r^{\frac{b}{b+a}}}$$

Optimal value of lambda: $$\frac{b^{\frac{a}{b+a}-1}\,\_Y^{\frac{1}{b+a}-1}\,r^{\frac{a}{b+a}}\,w^{1-\frac{a}{b+a}}}{A^{\frac{1}{b+a}}\,a^{\frac{a}{b+a}}}$$

Employing K* and L* gives the minimum cost of producing Y.

There are also functions of demand for production factors (analogous to consumer choice)

Conditional Demands:
K = f(r,w,Y) and L = f(r,w,Y)

The interpretation of lambda is Marginal Cost: how much will the cost increase when we „relax" the constraint and increase the production of Y by a unit

The cost minimization problem

We can insert the obtained solutions K*(r,w,Y) and L*(r,w,Y) into the cost function and we will get an intermediate cost function that depends on the market price variables: r,w, on the parameters defining the technology and the production volume Y.

$$C(r,w,Y) = rK^{*}(r,w,Y) + rL^{*}(r,w,Y)$$

K*(r,w,Y)

L*(r,w,Y)

$$C = \frac{b^{\frac{a}{b+a}} Y^{\frac{1}{b+a}} r^{\frac{a}{b+a}} w^{1-\frac{a}{b+a}}}{A^{\frac{1}{b+a}} a^{\frac{a}{b+a}}} + \frac{Y^{1/a} r}{A^{1/a} L^{b/a}}$$

The cost minimization problem

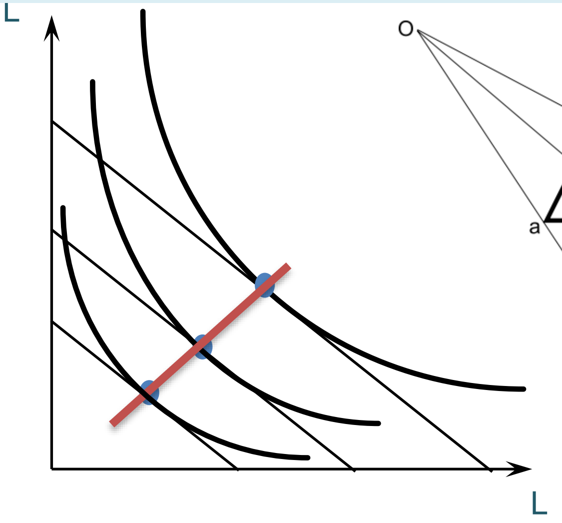
```
kill(all)$ /* Clear all previous definitions and
variables */

assume(X1>0, X2>0, p>0, w1>0, w2>0, MP1>0,
MP2>0, a>0, b<0)$
/* Set assumptions for variables: X1, X2, p,
w1, w2, MP1, MP2 as positive,
  a as positive, and b as negative. */

Y: A * (K^a) * (L^b);
/* Define the production function Y as a
function of capital (K) and labor (L)
  with parameters A, a, and b, where a and b
represent the output elasticities of K and L. */

MP_K: diff(Y, K, 1)$
/* Calculate the marginal product of capital
(MP_K) by taking the first derivative of Y with
respect to K */

MP_L: diff(Y, L, 1)$
/* Calculate the marginal product of labor
(MP_L) by taking the first derivative of Y with
respect to L */

eq1: MRTS: MP_K / MP_L = r / w;
/* Define the Marginal Rate of Technical
Substitution (MRTS) as the ratio of the
marginal products
  (MP_K / MP_L) and set it equal to the ratio
of input prices (r/w).
  This indicates that the spending on capital
(K) and labor (L) is in the same proportion to
achieve optimal production. */

print("Production function Y:", Y);
print("Marginal Product of Capital (MP_K):",
MP_K);
print("Marginal Product of Labor (MP_L):",
MP_L);
print("MRTS Condition (eq1):", eq1);
```

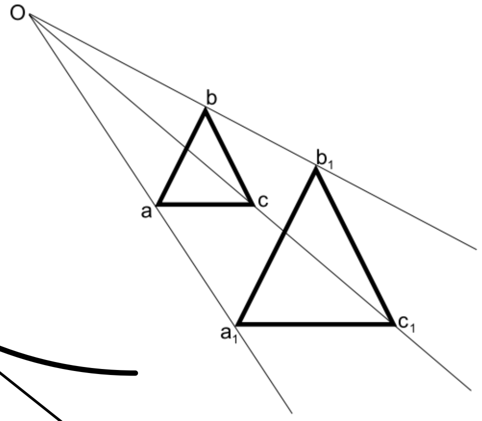
$$Y(K,L) = AK^{a}L^{\beta}$$

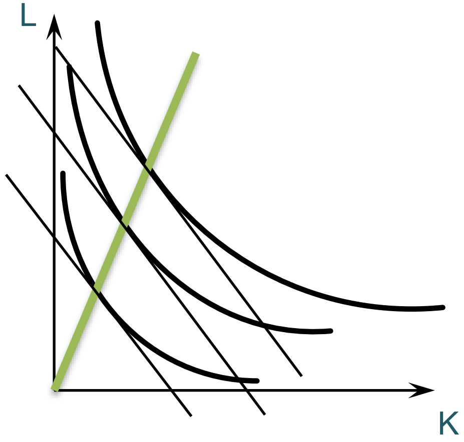
$$-\frac{MP_K}{MP_L} = MRTS = -\frac{r}{w}$$

$$\frac{\alpha L}{\beta K} = \frac{r}{w} \qquad S = \frac{rK}{wL} = \frac{\alpha}{\beta}$$

Constant proportion of expenditure on K to expenditure on L depending on the parameters of the production function
--> S= a/b







„In production theory, homothetic functions are a class of functions where the ratio of input factors remains constant along expansion paths. This means that if inputs are scaled by a certain factor, output also scales by a function of that factor, preserving the production structure. Homothetic functions are particularly useful for analyzing production processes where returns to scale are consistent. These functions allow for a straightforward analysis of proportional changes in input levels, which makes them valuable for studying cost functions and input substitution in economic models."

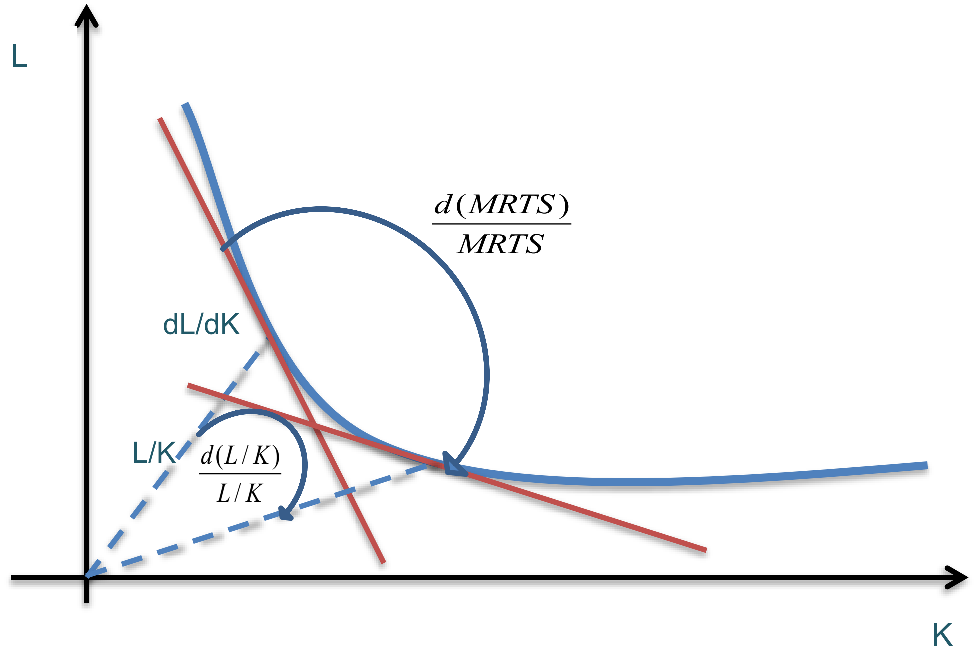
The cost minimization problem

For the C-D function, elasticity of substitution = 1

$$\varepsilon = \frac{\dfrac{d(L/K)}{L/K}}{\dfrac{d(MRTS)}{MRTS}} = \frac{\dfrac{d(L/K)}{L/K}}{\dfrac{d(\frac{r}{w})}{\frac{r}{w}}} = 1$$

The % change in factor prices is fully compensated by the % change in the factor ratio (technology.

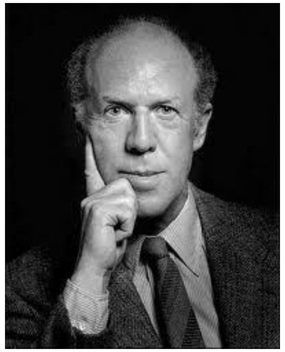
The % change in the market price ratio



Restrictive assumption for C-D: The division of total income between factors of production depends on technology parameters.

Contemporary Microeconomic Models of Macroeconomics - Welcome to Arrow–Debreu World

Kenneth Arrow



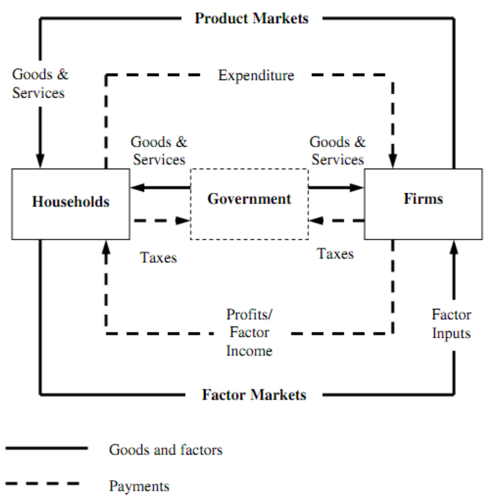
Gérard Debreu

General equilibrium models - we start with individuals (homo oeconomicus [assumptions - axioms]) and end with the entire economy.

Kenneth Arrow, Gérard Debreu did what physics failed to do [the model of the behavior of atoms does not explain the behavior of the universe].

We can start with, two optimization problems and a few tautological equalities from the scheme of circulation in the economy are enough, to do it.

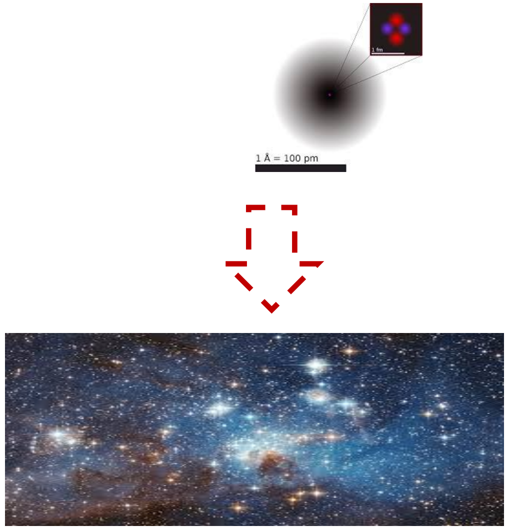
Figure 1. The Circular Flow of the Economy



The economy is:
N1 consumers
N2 goods and services
N3 factors of production
N2 markets for goods and services
and their prices
N3 markets for factors of production and their prices

and …

the mechanism
that makes it all clear



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Computable General Equilibrium – Instead of Tedious Calculations

Assumptions

2X2X2 Model:

- **Two Goods:**
  X_1 and X_2 – one productive and one non-productive good.
- **Two Factors of Production:**
  K (Capital) and L (Labor), with respective rewards r (interest) and w (wages).
- **Two Types of Households (Consumers):**
  - **Capitalists (Wealthy):** Own capital K_{rich} and labor L_{rich}.
  - **Workers (Poor):** Own capital K_{poor} and labor L_{poor}.

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Consumer Households Maximize Approximate Utility: Cobb-Douglas Function

$$U_{\text{poor}}(x_1, x_2) = x_1^{\alpha_{\text{poor}}} x_2^{\beta_{\text{poor}}}$$

$$U_{\text{rich}}(x_1, x_2) = x_1^{\alpha_{\text{rich}}} x_2^{\beta_{\text{rich}}}$$

They Face Budget Constraints:

$$p_1 x_{1\text{poor}} + p_2 x_{2\text{poor}} = r\overline{K_{\text{poor}}} + w\overline{L_{\text{poor}}} = M_{\text{poor}}$$

$$p_1 x_{1\text{rich}} + p_2 x_{2\text{rich}} = r\overline{K_{\text{rich}}} + w\overline{L_{\text{rich}}} = M_{\text{rich}}$$

Demand Function:

$$x_{1\text{poor}} = \frac{\alpha_{\text{poor}}}{\alpha_{\text{poor}} + \beta_{\text{poor}}} \cdot \frac{M_{\text{poor}}}{p_1}$$

$$x_{1\text{rich}} = \frac{\alpha_{\text{rich}}}{\alpha_{\text{rich}} + \beta_{\text{rich}}} \cdot \frac{M_{\text{rich}}}{p_1}$$

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Firms Hire Production Factors to Minimize Production Costs with a Technology Approximated by a Cobb-Douglas Function:

$$x_1(K, L) = A_1 K^{\delta_1} L^{1-\delta_1}$$

$$x_2(K, L) = A_2 K^{\delta_2} L^{1-\delta_2}$$

Conditional Demand for Production Factors:

$$K_1 = \left[A_1^{-1} \left(\frac{\delta_1}{1-\delta_1}\right)^{1-\delta_1} \left(\frac{w}{r}\right)^{1-\delta_1}\right] X_1$$

$$L_1 = \left[A_1^{-1} \left(\frac{1-\delta_1}{\delta_1}\right)^{\delta_1} \left(\frac{r}{w}\right)^{\delta_1}\right] X_1$$

$$K_2 = \left[A_2^{-1} \left(\frac{\delta_2}{1-\delta_2}\right)^{1-\delta_2} \left(\frac{w}{r}\right)^{1-\delta_2}\right] X_2$$

$$L_2 = \left[A_2^{-1} \left(\frac{1-\delta_2}{\delta_2}\right)^{\delta_2} \left(\frac{r}{w}\right)^{\delta_2}\right] X_2$$

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Goal: Find prices p_1, p_2, r, w and quantities X_1, X_2, K_1, L_1, K_2, L_2 for which the system clears, i.e.:

Factor Market Equilibrium:

$$K_1 + K_2 = K = K_{\text{poor}} + K_{\text{rich}}$$

$$L_1 + L_2 = L = L_{\text{poor}} + L_{\text{rich}}$$

Goods and Services Market Equilibrium (Demand = Supply):

$$x_{1\text{poor}} + x_{1\text{rich}} = X_1$$

$$x_{2\text{poor}} + x_{2\text{rich}} = X_2$$

Perfect Competition Conditions: Profit is zero, implying Cost = Revenue. All factors are compensated, leaving no surplus:

$$w \cdot L + r \cdot K = p_1 \cdot X_1 + p_2 \cdot X_2$$

Model Parameters

$$\alpha_{\text{poor}}, \alpha_{\text{rich}}, \beta_{\text{poor}}, \beta_{\text{rich}}, \delta_1, \delta_2, A1, A2$$

Resources

$$K_{\text{poor}}, K_{\text{rich}}, L_{\text{poor}}, L_{\text{rich}}$$

Numeraire: $w = 1$

Comparison to the price of labor; all values expressed in terms of labor cost, reducing computational complexity.

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Calibration Process

Households	α	β	K	L
Capitalists (rich)	0.5	0.5	4	0.25
Workers (poor)	0.5	0.5	0.1	200

Goods	δ	A
Productive ($x1$)	0.8	1
Non-productive ($x2$)	0.1	1.5

Interesting facts: 20 years ago, in large NBP models, these parameters for Poland were taken from the US economy :)

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Numerical Solution: Minimizing the sum of squared deviations from equilibrium

Equilibrium conditions:

$$S1 = (K_1 + K_2) - K = 0$$

$$S2 = (L_1 + L_2) - L = 0$$

$$S3 = (x_{1\text{poor}} + x_{1\text{rich}}) - X1 = 0$$

$$S4 = (x_{2\text{poor}} + x_{2\text{rich}}) - X2 = 0$$

$$S5 = (w \cdot L + r \cdot K) - p1 \cdot X1 = 0$$

Objective: Minimize $\sum (S_i)^2$

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Comparative Statics:

We find the equilibrium in the base model (Model 0) and compare it with changes introduced by economic policy tools—taxes and subsidies (Model 1).

Tax	Modification in Model	Value
On sales of X_1	$(1+t_{x1}) \cdot p_1$	0.2
On sales of X_2	$(1+t_{x2}) \cdot p_2$	0.1
On capital income	$(1-t_k) \cdot r \cdot K_{\text{poor}}; (1-t_k) \cdot r \cdot K_{\text{rich}}$	0.2
On labor income	$(1-t_l) \cdot w \cdot L_{\text{poor}}; (1-t_l) \cdot w \cdot L_{\text{rich}}$	0.2
On revenue	$(p_1 \cdot X_1 + p_2 \cdot X_2) \cdot (1-t_p)$	0.2

Equations for Tax Calculations:

1. **Taxes on Goods Sales:**

$$T_{x12} = t_{X1} \cdot p_1 \cdot x_{1\text{poor}} + t_{X1} \cdot p_1 \cdot x_{1\text{rich}} + t_{X2} \cdot p_2 \cdot x_{2\text{poor}} + t_{X2} \cdot p_2 \cdot x_{2\text{rich}}$$

2. **Taxes on Income from Production Factors:**

$$T_{kl} = t_k \cdot r \cdot K_{\text{poor}} + t_l \cdot w \cdot L_{\text{poor}} + t_k \cdot r \cdot K_{\text{rich}} + t_l \cdot w \cdot L_{\text{rich}}$$

3. **Revenue Tax:**

$$T_p = (p_1 \cdot X_1 + p_2 \cdot X_2) \cdot t_p$$

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Additional Condition S6

Tax Revenues = Total Transfers T

$$S6: T_{x12} + T_{kl} + T_p - T = 0$$

Transfers are distributed to households in specified proportions ($s_{\text{poor}} + s_{\text{rich}} = 1$), increasing their income.

$$M_{\text{poor}} = (1 - t_k) \cdot r \cdot K_{\text{poor}} + (1 - t_l) \cdot w \cdot L_{\text{poor}} + s_{\text{poor}} \cdot T$$

$$M_{\text{rich}} = (1 - t_k) \cdot r \cdot K_{\text{rich}} + (1 - t_l) \cdot w \cdot L_{\text{rich}} + s_{\text{rich}} \cdot T$$

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```
kill(all)$
load(draw)$

/**********************************************************************/
/* Economy Parameters ********/
/*The model is very sensitive to parameter changes try :) */
/**********************************************************************/

/* Households */

alpha_rich: 0.6$
beta_rich: 0.4$
K_rich: 4$
L_rich: 0.25$

alpha_poor: 0.5$
beta_poor: 0.5$
K_poor: 0.5$
L_poor: 7$

/* Production */

delta_1: 0.4$
A1: 1.5$
delta_2: 0.8$
A2: 1$

/* Numeraire w = 1 */
w: 1$
```

Open Maxima's **CGE2.wxm** file and run it cell by cell.

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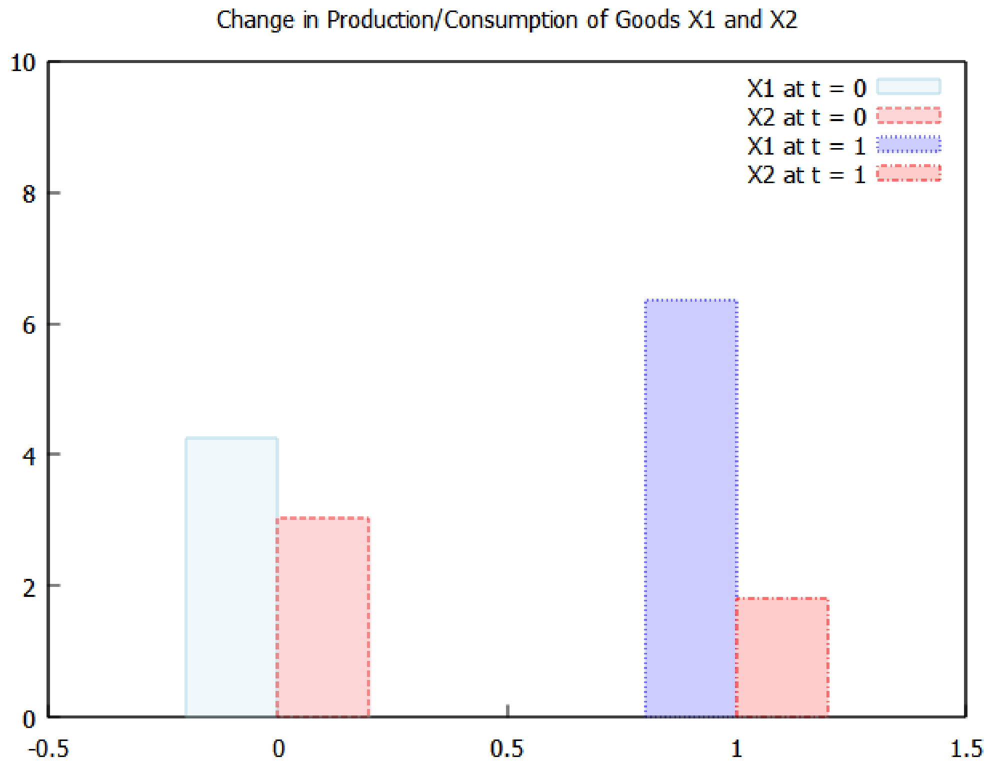

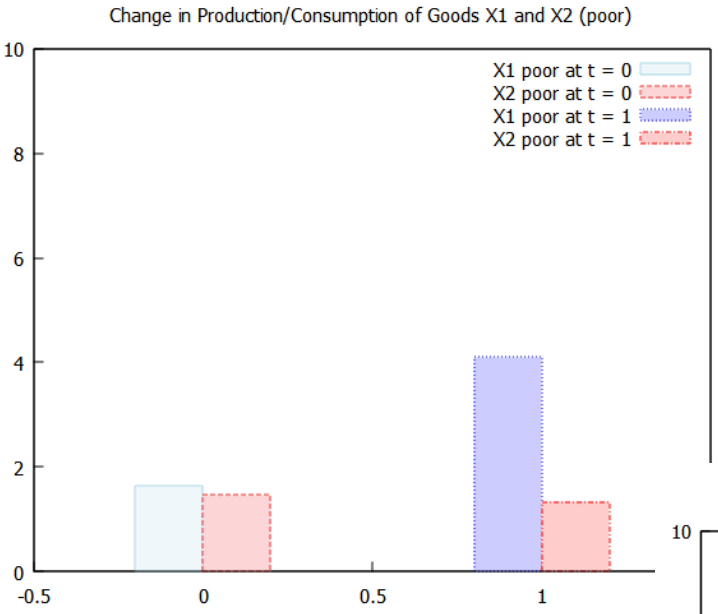

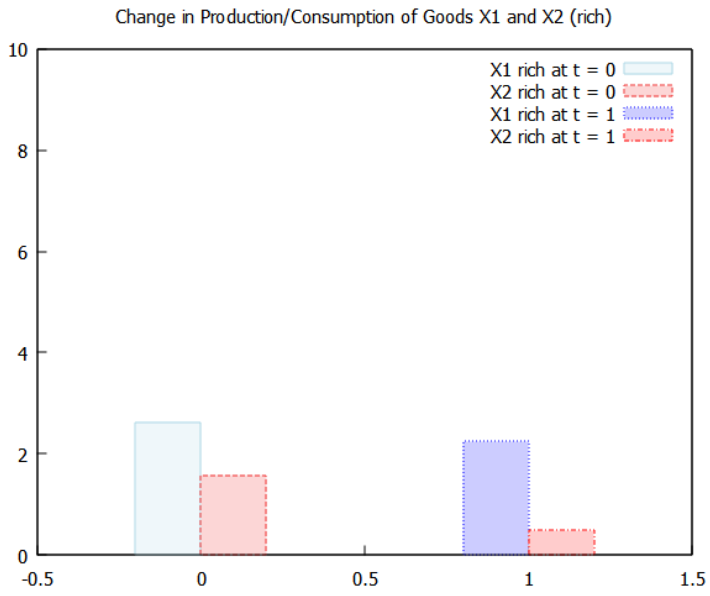

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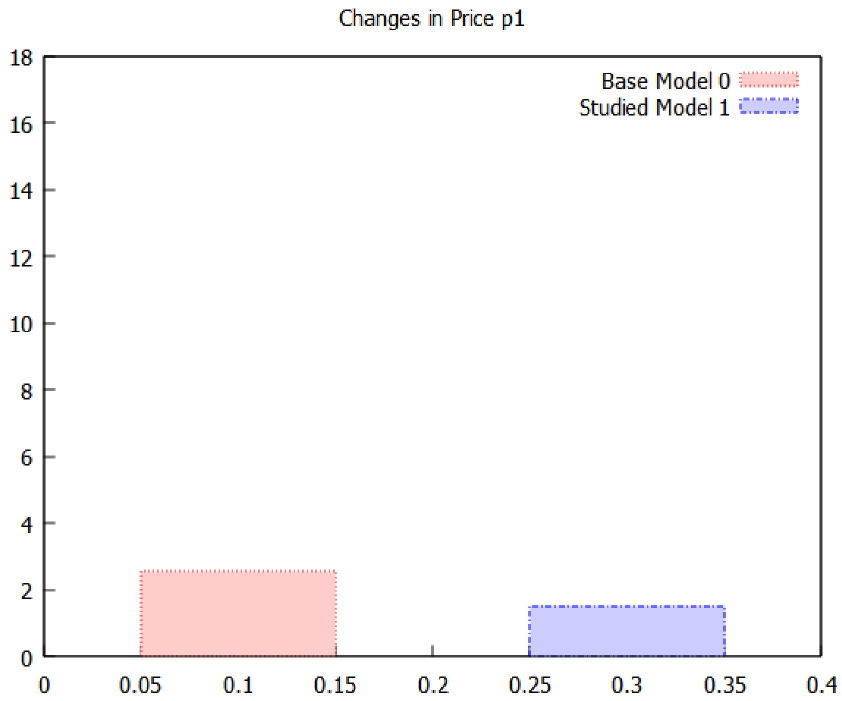

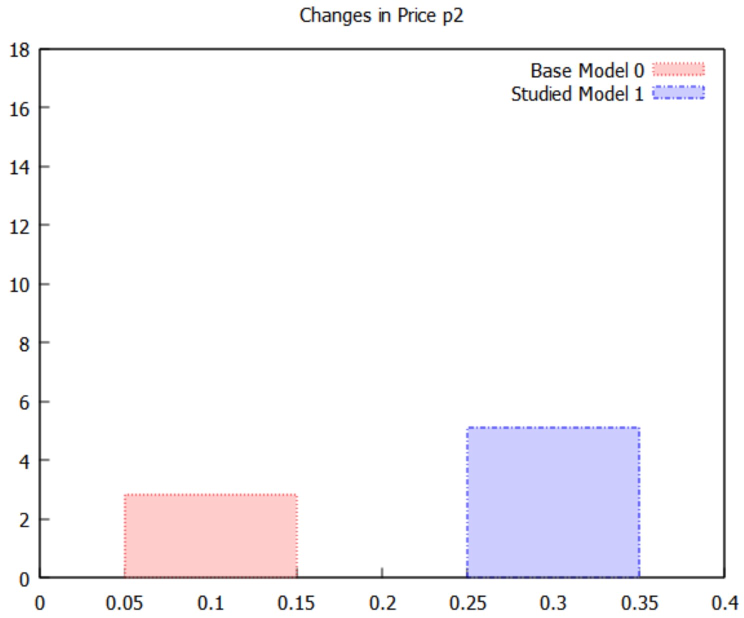

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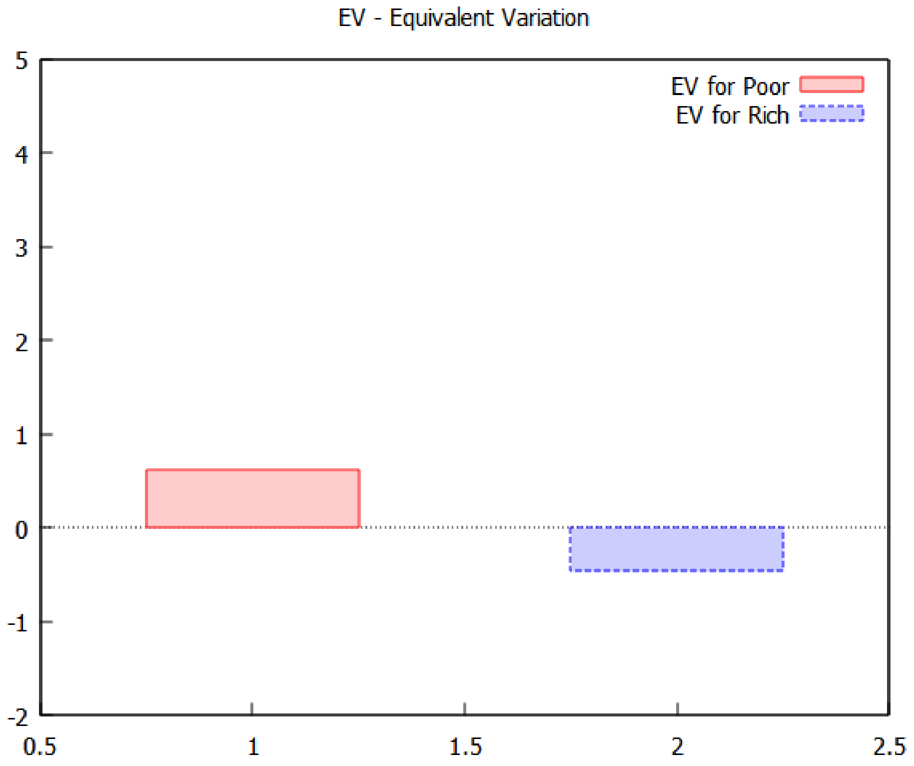
Change in Utility Measured by Equivalent Variation for Two Types of Households

(Income—how much money would need to be taken from the consumer before the implementation of economic policy (taxes and subsidies) to leave them as well off as they were before the policy introduction.)

In the chart:

- **EV for the Poor (red)** represents the equivalent variation for lower-income households.
- **EV for the Rich (blue)** represents the equivalent variation for higher-income households.



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