Solutions to Sample Problems - Solow Model

Problem 1

Consider a Solow economy that is on its balanced growth path. Assume for simplicity that there is no technological progress. Now suppose that the rate of population growth falls.

(a). What happens to the balanced-growth-path values of capital per worker, output per worker, and consumption per worker? Sketch the paths of these variables as the economy moves to its new balanced growth path.

The fall in the population growth rate makes the break-even investment line ($(\delta + n)\, k$) flatter (see Figure 1). In the absence of technological progress, the change in capital per worker over time is given by

$$k_{t+1} - k_t = \triangle k_{t+1} = s k_t^{\alpha} - (\delta + n)\, k_t$$

Since $\triangle k_{t+1}$was zero before the fall in the population growth because the economy was in steady state (on balanced growth path), the decrease in n causes $\triangle k_{t+1}$ to become positive. At k^*, the steady-state level of capital per worker before the population growth rate falls, actual investment per worker, $s\,(k^*)^{\alpha}$, exceeds break even investment per worker, $(\delta + n_{new})\, k^*$. Thus $\triangle k_{t+1}$ becomes positive and capital accumulation starts in the economy and k moves to k^*_{new} (see Figure 1).

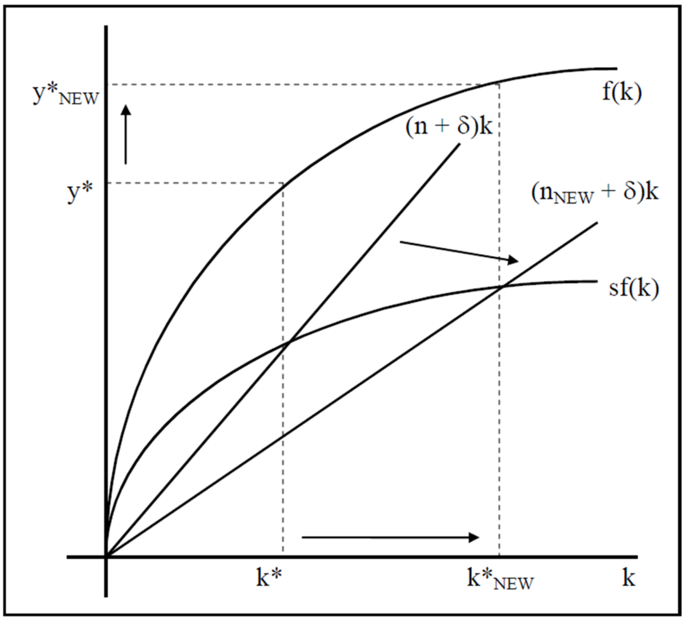

Figure 1:

As capital per worker, k, rises, output per worker, y, rises as well because $y = k^{\alpha}$. Since a constant fraction of output is saved, consumption per worker, c, rises as y rises. This is summarized in Figure 2.

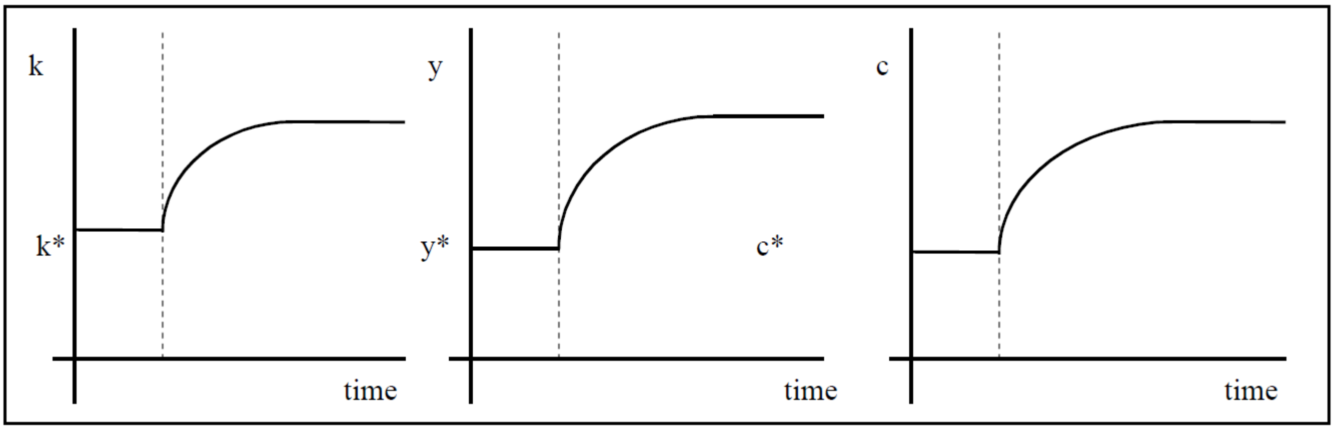

Figure 2:

(b). Describe the effect of the fall in population growth on the path of output (that is, total output, not output per worker).

Output can be written as $Y_t = y_t L_t$. The growth rate of the output is given by

$$\begin{aligned}
\ln Y_t &= \ln y_t - \ln L_t \\
\ln Y_{t+1} &= \ln y_{t+1} - \ln L_{t+1} \\
\ln Y_{t+1} - \ln Y_t &= \ln y_{t+1} - \ln y_t - (\ln L_{t+1} - \ln L_t) \\
g_Y &= g_y - g_L
\end{aligned}$$

On the initial balanced growth path. growth rate of output per worker, g_y, is equal to zero. So, the growth rate of output, g_Y, is equal to growth rate of labor, g_L, that is equal to n. On the final balanced growth path, growth rate of output per worker, g_y, is equal to zero once again. So, the growth rate of output, g_Y, is equal to new growth rate of labor, g_L, that is equal to n_{new}. In the end, output will be growing at permanently lower rate since $n_{new} < n$.

What happens during the transition? Examine the production function $Y_t = K_t^{\alpha} L_t^{1-\alpha}$. Repeating the similar steps above, the growth rate of output can be expressed as:

$$g_Y = \alpha g_K + (1 - \alpha)\, g_L$$

On the initial balanced growth path. growth rate of output, g_Y, capital, g_K,and labor, g_L, are all equal to n. Now, the growth rate of g_L is equal to n_{new}. The growth rate of capital during the transition is equal to

$$\begin{aligned}
K_{t+1} &= sY_t + (1 - \delta)\, K_t \\
K_{t+1} - K_t &= sK_t^{\alpha} L_t^{1-\alpha} + -\delta K_t \\
\frac{K_{t+1} - K_t}{K_t} &= s\frac{K_t^{\alpha} L_t^{1-\alpha}}{K_t} + -\delta\frac{K_t}{K_t} \\
g_K &= sk_t^{\alpha-1} - \delta
\end{aligned}$$

During the transition $k_t > k^*_{new}$ which implies that $g_K > n_{new}$. So, the growth rate of output is larger than n_{new} during transition. However, the growth rate of output falls during the transition and converges to n_{new} along the new balanced growth path. (See Figure 3 and remember $\frac{\mathrm{d} \ln Y_t}{\mathrm{d}t}$gives growth rate of Y)

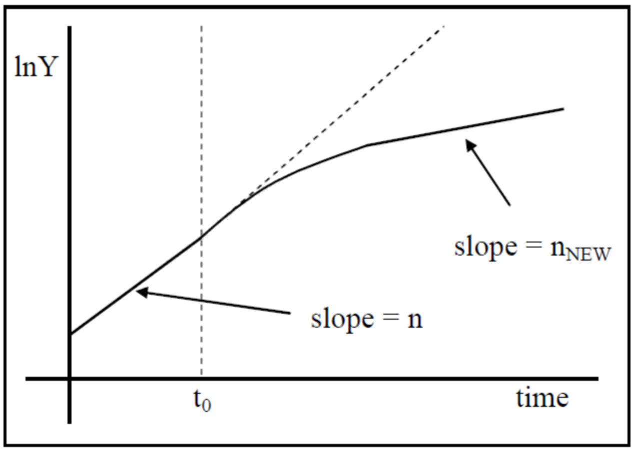

Figure 3:

Problem 2

Let's examine the role of taxes in the Solow-Swan model. Imagine that the behavior of an economy may be summarized by the following three equations:

$$K_{t+1} = I_t - (1-\delta) K_t$$
$$I_t = s(1-\tau) Y_t$$
$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha}$$

Assume that population grows at rate n and technology at rate g, so that $L_{t+1}/L_t = N_{t+1}/N_t = 1+n$ and $A_{t+1}/A_t = 1+g$, respectively. Income in this economy is taxed with rate τ and the tax revenues are used for government consumption which is useless from the point of view of households.

(a). Transform the three equations into per effective labor form, i.e. divide them by $A_t L_t$. Make use of notational convention $\hat{x}_t \equiv X_t / (A_t L_t)$.

$$\frac{K_{t+1}}{A_t L_t} \frac{A_{t+1} L_{t+1}}{A_{t+1} L_{t+1}} = s(1-\tau) \frac{K_t^{\alpha} (A_t L_t)^{1-\alpha}}{A_t L_t} + (1-\delta) \frac{K_t}{A_t L_t}$$

$$(1+g)(1+n) \hat{k}_{t+1} = s(1-\tau) \hat{k}_t^{\alpha} + (1-\delta) \hat{k}_t$$

$$\frac{I_t}{A_t L_t} = s(1-\tau) \frac{K_t^{\alpha} (A_t L_t)^{1-\alpha}}{A_t L_t}$$

$$\hat{i}_t = s(1-\tau) \hat{k}_t^{\alpha}$$

$$\frac{Y_t}{A_t L_t} = \frac{K_t^{\alpha} (A_t L_t)^{1-\alpha}}{A_t L_t}$$

$$\hat{y}_t = \hat{k}_t^{\alpha}$$

(b). Find the balanced growth path level of capital per effective labor $\hat{k}^*$ in this economy.

$$(1+g)(1+n) \hat{k}^* = s(1-\tau) \left(\hat{k}^*\right)^{\alpha} + (1-\delta) \hat{k}^*$$

$$(1+g)(1+n) = s(1-\tau) \left(\hat{k}^*\right)^{\alpha-1} + (1-\delta)$$

$$s(1-\tau) \left(\hat{k}^*\right)^{\alpha-1} = \delta + n + g$$

$$\left(\hat{k}^*\right)^{\alpha-1} = \frac{\delta+n+g}{s(1-\tau)}$$

$$\hat{k}^* = \left(\frac{\delta+n+g}{s(1-\tau)}\right)^{\frac{1}{\alpha-1}}$$

$$\hat{k}^* = \left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}$$

(c). Discuss the effects of changes in parameters δ, n, g, s, τ on the economy's balanced growth path level of capital per effective labor $\hat{k}^*$.

Higher s increases the BGP level of $\hat{k}^*$.

Higher δ, n, g, τ decreases the BGP level of $\hat{k}^*$.

(d). Discuss the effects of changes in parameters δ, n, g, s, τ on the economy's balanced growth path level of consumption per effective labor $\hat{c}^*$.

$$\hat{y}^* = \left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\hat{y}^* = \hat{c}^* + \hat{i}^* + \hat{g}^*$$
$$\hat{y}^* = \hat{c}^* + \hat{i}^* + \tau\hat{y}^*$$
$$(1-\tau)\,\hat{y}^* = \hat{c}^* + \hat{i}^*$$
$$(1-\tau)\,\hat{y}^* = \hat{c}^* + s(1-\tau)\,\hat{y}^*$$
$$\hat{c}^* = (1-s)(1-\tau)\,\hat{y}^*$$
$$\hat{c}^* = (1-s)(1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} \tag{1}$$

Higher δ, n, g decreases the BGP level of $\hat{c}^*$.

Let's look at impact of change in s on $\hat{c}^*$

$$\frac{\partial\hat{c}^*}{\partial s} = -(1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}\frac{(1-\tau)}{\delta+n+g}(1-s)(1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}-1}$$

$$\frac{\partial\hat{c}^*}{\partial s} = -(1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}\frac{(1-\tau)}{\delta+n+g}(1-s)(1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}\frac{\delta+n+g}{s(1-\tau)}$$

$$\frac{\partial\hat{c}^*}{\partial s} = -(1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha}\frac{(1-\tau)(1-s)}{s}\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{\partial\hat{c}^*}{\partial s} = (1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}\left[-1 + \frac{\alpha}{1-\alpha}\frac{(1-s)}{s}\right]$$

$$\frac{\partial\hat{c}^*}{\partial s} = \left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}\left[-1 + \frac{\alpha}{1-\alpha}\frac{1-s}{s}\right]$$

$$-1 + \frac{\alpha}{1-\alpha}\frac{1-s}{s} = 0$$

$$\frac{\alpha}{1-\alpha}\frac{1-s}{s} = 1$$

$$s_{GR} = \alpha$$

If $s < s_{GR}$, higher s increases the BGP level of $\hat{c}^*$and if $s > s_{GR}$, lower s increases the BGP level of $\hat{c}^*$

$$\frac{\partial \hat{c}^*}{\partial \tau} = -(1-s)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}} - \frac{\alpha}{1-\alpha}\frac{s}{\delta+n+g}(1-s)(1-\tau)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}-1}$$

$$\frac{\partial \hat{c}^*}{\partial \tau} = (1-s)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}\left[-1-\frac{\alpha}{1-\alpha}\frac{s}{\delta+n+g}(1-\tau)\frac{\delta+n+g}{s(1-\tau)}\right]$$

$$\frac{\partial \hat{c}^*}{\partial \tau} = (1-s)\left(\frac{s(1-\tau)}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}\left[-1-\frac{\alpha}{1-\alpha}\right] < 0$$

Higher τ decreases the BGP level of $\hat{c}^*$.

(e). Households care about the level of consumption per capita, i.e. c_t. This variable grows at rate g once the economy reaches its balanced growth path. Discuss whether low g or high g is better from the point of view of households.

$$c_t^* = A_t \hat{c}^*$$

Let's assume that there are two economies with the same parameter values except technological growth rate, g. For simplicity assume $\tau = 0$. Denote $g_1$and g_2 for the growth rate of technology for the first economy and second economy, respectively and $g_1 < g_2$. Assume that we are at the steady-state of the both economies. Since $g_1 < g_2$, the first economy will have higher consumption per effective labor, compared to the second economy, $\hat{c}_1^* > \hat{c}_2^*$ (see Eq. 1). However, since c_t^* is growing at rate g, the growth of consumption per capita in the second economy will be higher than the growth of consumption per capita in the second economy since $g_2 > g_1$. This means that at some point in time, denote as t_1, the level of consumption per capita in the second economy will exceed the the level of consumption per capita in the first economy and the difference will increase over time due to the higher technological growth in the second economy. Since there is no discounting of future consumption and we only concentrate on the balanced growth path, the consumers in second economy are better off compared to the consumers in the first economy (assuming the time horizon is large enough so the higher c^*in the second economy after t_1 offsets the lower c^*before t_1).

Problem 3

Consider an economy with technological progress but without population growth that is on its balanced growth path. Now suppose there is a one-time jump in the number of workers without change in the population growth rate (e.g., due to an immigration inflow)

(a). At the time of the jump, does output per unit of effective labor rise, fall, or stay the same? Why?

At some time, call it t_0 , there is a discrete upward jump in the number of workers. This reduces the amount of capital per unit of effective labor from k^* to k_{NEW} . We can see this by simply looking at the definition, $k = K/AL$. An increase in L without a jump in K or A causes k to fall. Since $f'(k) > 0$, this fall in the amount of capital per unit of effective labor reduces the amount of output per unit of effective labor as well. In the figure below, y falls from y^* to y_{NEW}.

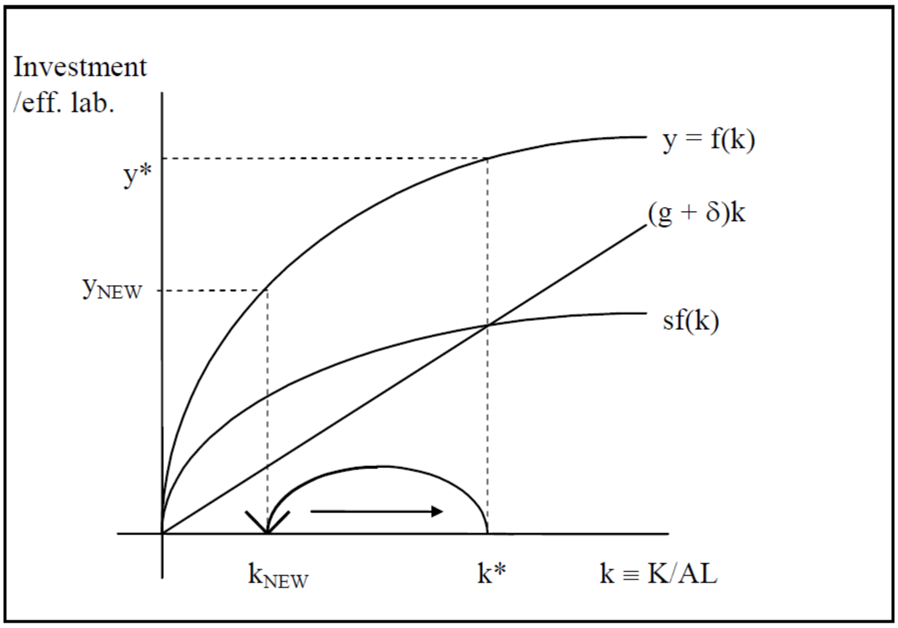

Figure 4:

(b). After the initial change (if any) in output per unit of effective labor when the new workers appear, is there any further change in output per unit of effective labor? If so, does it rise or fall? Why?

Now at this lower k_{NEW}, actual investment per unit of effective labor exceeds break-even investment per unit of effective labor. That is, $sf(k_{NEW}) > (g+)k_{NEW}$. The economy is now saving and investing more than enough to offset depreciation and technological progress at this lower k_{NEW}. Thus k begins rising back toward k^*. As capital per unit of effective labor begins rising, so does output per unit of effective labor. That is, y begins rising from y_{NEW} back toward y^*.

(c). Once the economy has again reached a balanced growth path, is output per unit of effective labor higher, lower, or the same as it was before the new workers appeared? Why?

Capital per unit of effective labor will continue to rise until it eventually returns to the original level of k^*. At k^*, investment per unit of effective labor is again just enough to offset technological progress and depreciation and keep k constant. Since k returns to its original value of k^* once the economy again returns to a balanced growth path, output per unit of effective labor also returns to its original value of $y^* = f(k^*)$.