Point-Set Topology & Measure Theory

Shuailong Zhu

Contents

1 Basic Topology

1.1 Topology and Topological Basis

Definition 1.1.1 (Topology and open sets). *Let X be any non-empty set. Then a topology on the set X is a collection τ of subsets $U \subseteq X$ satisfying the following properties:*
(a) Both $\varnothing \in \tau$ and $X \in \tau$;
(b) τ is closed under unions and finite intersections.
The sets $U \in \tau$ are called open sets and the pair (X, τ) is called a topological space.

Definition 1.1.2 (Basis of a topology). *Let X be a set. Then a collection τ^B of subsets of X is called a basis for a topology, if it satisfies two conditions:*
(a) $\cup_{V \in \tau^B} V = X$;
(b) $\forall V_1 \in \tau^B$, $V_2 \in \tau^B$ and $x \in V_1 \cap V_2$, there is some $V_3 \in \tau^B$ with $x \in V_3 \subseteq V_1 \cap V_2$.
Consider τ to be the set of all possible unions of $V \in \tau^B$, together with the empty set. Then τ defines a topology on X.

Proposition 1.1.1 (Criteria for finding a basis). *Let (X, τ) be a topological space. Let $\widetilde{\tau}^B \subseteq \tau$ be a collection of subsets such that any set in τ is an union of sets from $\widetilde{\tau}^B \subseteq \tau$. Then $\widetilde{\tau}^B$ is a basis for some topology, and the topology induced by this basis is τ.*

Definition 1.1.3 (Interior and closure). *Let (X, τ) be a topological space. Then for any subset $A \subseteq X$, $int(A)$ is defined as the largest open set contained in A, and $cl(A)$ is defined as the smallest closed set containing A. Specifically, $int(A)$ is equal to the union of all open sets contained in A, and $cl(A)$ is equal to the intersection of all closed sets containing A.*

Definition 1.1.4 (Limit point in topology space). *Let (X, τ) be a topological space. Then $x \in X$ is called a limit point if for any open set containing x contains at least one point different from x, then x is called a limit point.*

1.2 Metric Topology

Definition 1.2.1 (Metric topology). *Let (X, d) be a metric space. We define $U \subseteq X$ to be open if for every $x \in U$, we can find some $\delta > 0$ such that $B(x, \delta) \subseteq U$ (We set $U \in \tau_d$). Then τ_d is a topology and is called the metric topology.*

Proposition 1.2.1 (The structure of open set in metric topology). *Let (X, d) be a metric space, and let τ_d be the metric topology. Then a set U is open if and only if it can be written as a union of open metric balls.*

Proposition 1.2.2 (Basis of a metric topology). *Let (X, d) be a metric space. Then $\tau^B := \{B(x, \delta) : x \in X, \delta > 0\}$ is a basis for the metric topology τ_d.*

Proposition 1.2.3 (The structure of open sets in $\mathbb{R}^n$). *The balls $B(x, \delta)$ where $x \in \mathbb{Q}_n$ and $\delta \in \mathcal{Q} \cap (0, \infty)$ are a basis for the Euclidean topology on $\mathbb{R}^n$. It suggests that any open set in $\mathbb{R}^n$ could be written as a union of countable metric balls.*

1.3 Convergence, Continuity and Homeomorphism

Definition 1.3.1 (Convergence). *Let (X, τ) be a topological space and $(x_n)_{n \geq 1}$ a sequence of points in X. We define $x_n \to x$ if for any open set U containing x there is some $n_U in \mathbb{N}$ such that $\forall n \geq n_U : x_n \in U$.*

Proposition 1.3.1 (Two counter-intuitive exmaples). *(a) $\mathbb{N}$ with its cofinite topology. Let $x_i = i$, $(x_n)_{n \geq 1}$ converges simultaneously to all $n \in N$. (b) $\mathbb{R}$ with the co-countable topology. All open sets are either countable or $\mathbb{R}$. The closure of $(0, 1)$ is the whole space.*

Definition 1.3.2 (Continuity at a point). *A map $f : X \to Y$ from a topological space (X, τ_X) to a topological space (Y, τ_Y) is continuous at a point $x \in X$, if $\forall$ open set U containing $f(x)$, there is some open set V_U containing x, such that $f(V_U) \subseteq U$. Then for any sequence $(x_n)_{n\geq 1} \to x$, we have that $(f(x_n))_{n\geq 1} \to f(x)$.*

Proof. [3] $\forall$ U containing $f(x)$, $\exists$ open set V_U containing x, thus $\exists$ n_{V_U} such that for all $n \geq n_{V_U}$, $x_n \in V_U$. Then for all $n \geq n_{V_U}$, $f(x_n) \in U$. □

Proposition 1.3.2 (Continuous map). *A map $f : X \to Y$ from a topological space (X, τ_X) to a topological space (Y, τ_Y) is continuous at every point $x \in X$ iff the pre-image of any open set is an open set, i.e. iff for any open set U of (Y, τ_Y) we have that $f^{-1}(U)$ is open in (X, τ_X).*

Definition 1.3.3 (Homeomorphism). *Let (X, τ_X), (Y, τ_Y) be two topological spaces. Then $f : X \to Y$ is called a homeomorphism if f is bijective and both f and f^{-1} are continuous.*

1.4 Subspace

Definition 1.4.1 (Subspace topology). *Let (X, τ_X) be a topological space and A a subset of X. Then define to be the collection of sets of the form $A \cap U$, where $U \in \tau_X$. Then $\tau_{X,A}$ defines a topology on A that is called the subspace topology.*

Theorem 1.4.1 (Subspace in metric space). *Let (X, d) be a metric space, then it induces a topological space (X, τ_X) via the metric topology. Now consider $A \subseteq X$. If we restrict d to $A \times A$, we obtain a metric space (A, d) and this induces a topological space (A, τ_A). We have,*

$$\tau_A = \tau_{X,A}$$

Theorem 1.4.2 (The natural way to define subspace topology). *Let (X, τ_X) be a topological space and $(A, \tau_{X,A})$ a subspace with the subspace topology. Then $\tau_{X,A}$ is the smallest topology $\widetilde{\tau}$ for which the inclusion map $i : (A, \widetilde{\tau}) \to (X, \tau_X)$ defined on A by identity is continuous.*

Proposition 1.4.3 (Several properties of subspace topology). *Let (X, τ_X) be a topological space and $(A, \tau_{X,A})$ a subspace with the subspace topology.*
(a) Prove that if (Y, τ_Y) is another topological space and $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then also f restricted to A is a continuous map from $(A, \tau_{X,A}) \to (Y, \tau_Y)$.
(b) In particular, prove that if $f : (X, \tau_X) \to (Y, \tau_Y)$ is a homeomorphism and $f(A) = B$ for some $B \subseteq Y$, then the restriction of f to A induces a homeomorphism between A and B with their respective subspace topologies.

1.5 Product Topology

Definition 1.5.1 (Product topology on $X \times Y$). *Consider two topological spaces (X, τ_X) and (Y, τ_Y). Define $\tau^B_{X\times Y}$ to be the collection of the subsets of $X \times Y$ of the form $U \times V$, where U is open in X and V is open in Y. Then $\tau^B_{X\times Y}$ is a basis for a topology, and the topology $\tau_{X\times Y}$ induced by it is called the product topology on $X \times Y$.*

Theorem 1.5.1 (The natural way to define finite product topology). *Let (X, τ_X) and (Y, τ_Y) be topological spaces and consider $X \times Y$ with the product topology. Then the product topology $\tau_{X\times Y}$ is the smallest topology $\widetilde{\tau}$ on $X \times Y$ such that the projection maps $p_X : (X \times Y, \widetilde{\tau}) \to (X, \tau_X)$ given by $p_X(x, y) := x$ and $p_Y : (X \times Y, \widetilde{\tau}) \to (Y, \tau_Y)$, given by $p_Y(x, y) := y$ are both continuous.*

Proof. Again, let us first check that p_X, p_Y are continuous for the product topology. For any open set U of (X, τ_X) we have that $p_X^{-1}(U) = U \times Y$, and this belongs to $\tau^B_{X\times Y}$. Similarly, for any open set V of (Y, τ_Y) we have that $p_Y^{-1}(V) = X \times V \in \tau^B_{X\times Y}$, and thus the continuity follows.

Now, suppose $p_X : (X \times Y, \widetilde{\tau}) \to X$ and $p_Y : (X \times Y, \widetilde{\tau}) \to Y$ are continuous. Then, by above all sets of the form $U \times Y$ with $U \in \tau_X$ and $X \times V$ with $V \in \tau_Y$ have to belong to $\widetilde{\tau}$. But then also $(U \times Y) \cap (X \times V) \in \widetilde{\tau}$ and thus in particular $\widetilde{\tau}$ contains the basis $\tau^B_{X\times Y}$. But then, as $\widetilde{\tau}$ is a topology, it has to contain the topology induced by this basis, i.e. $\tau_{X\times Y}$, giving the claim. □

Proposition 1.5.2 (Pointwise continuity and continuity)**.** *Let (X, τ_X) and (Y, τ_Y) be topological spaces and consider $X \times Y$ with the product topology. Let further (Z, τ_Z) be another topological space and $f : (Z, \tau_Z)ß(X \times Y, \tau_{X\times Y})$. Prove that f is continuous if and only if both $f_1 := p_X \circ f : (Z, \tau_Z) \to (X, \tau_X)$ and $f_2 : p_Y \circ f : (Z, \tau_Z) \to (Y, \tau_Y)$ are continuous.*

Definition 1.5.2 (The infinite product topology)**.** *Let I be some infinite index set and $((X_i, \tau_{X_i}))_{i\in I}$ a collection of topological spaces. Let $\tau^B_{\prod_{i\in I} X_i}$ be the collection of subsets of $\prod_{i\in I} X_i$ of the form $\prod_{i\in I} U_i$, where each $U_i \subseteq X_i$ is open in X_i and $U_i \neq X_i$ only for finitely many $i \in I$. Then τ^B is a basis for a topology, and this topology is called the product topology on $\prod_{i\in I} X_i$.*

Theorem 1.5.3 (The natural way to define infinite product topology)**.** *Let now I be some infinite index set and $((X_i, \tau_{X_i}))_{i\in I}$ a collection of topological spaces. Then the product topology is the smallest topology on $\prod_{i\in I} X_i$ such that all coordinate maps are continuous.*

Proof. The direction "$\Rightarrow$" is the same as Theorem 1.5.1. For "$\Leftarrow$" direction, we need the finite intersection to complete the proof, which is guaranteed by the definition of infinite product topology basis.

□

Proposition 1.5.4 (The relationship between pointwise convergence and convergence)**.** *Let now I be some index set and $((X_i, \tau_{X_i}))_{i\in I}$ a collection of topological spaces. Then a sequence $(x_n)_{n\geq 1}$ converges to x in $\prod_{i\in I} X_i$ with the product topology if and only if it converges pointwise, i.e. iff for all $i \in I$, $(x_n(i))_{n\geq 1}$ converges to $x(i)$ in (X_i, τ_{X_i}).*

1.6 Hausdroff Space

Definition 1.6.1 (Hausdorff space)**.** *A topological space (X, τ_X) is called Hausdorff if for any two distinct points x, y we can find two disjoint open sets U_x, U_y such that $x \in U_x$ and $y \in U_y$.*

Theorem 1.6.1 (Several properties of Hausdroff space)**.**
(a) If (X, τ_X) is Hausdorff, then any convergent sequence has a unique limit.
(b) Suppose (X, τ_X) is Hausdorff and $f : (X, \tau_X) \to (Y,\tau_Y)$ a homeomorphism. Then (Y, τ_Y) is also Hausdorff.
(c) Let (X, τ_X) be a Hausdorff topological space. Then $(A, \tau_{X,A})$ is also Hausdorff.

2 Compactness

2.1 Properties of Compactness

Definition 2.1.1 (Compactness)**.** *A topological space (X, τ_X) is called compact if any open cover of X admits a finite subcover, i.e. if I is any index set, U_i are open for all $i \in I$ and $\cup_{i\in I} U_i = X$, then there exists a finite subset $I_0 \subseteq I$ such that $i \in I$ and $\cup_{i\in I_0} U_i = X$.*

Definition 2.1.2 (Sequentially compact)**.** *A topological space (X, τ_X) is called sequentially compact, if any sequence $(x_n)_{n\geq 1}$ in X admits a convergent subsequence.*

Theorem 2.1.1 (Boundedness theorem)**.** *Let (X, τ_X) be a compact topological space and $f : X \to R$ a real-valued continuous function. Then f is bounded on (X, τ_X), i.e. there exist $i, s \in \mathbb{R}$ such that $i \leq f(x) \leq s$ for all $x \in X$.*

The similar result holds for sequentially compact spaces, but the proof argues by contradiction and is not half as neat.

Definition 2.1.3 (Compactness of set)**.** *Let (X, τ_X) be a topological space and consider $K \subseteq X$. Then $(K, \tau_{X,K})$ is compact as a topological space if and only if every covering of K with open sets of X admits a finite subcover.*

Proposition 2.1.2. *Let (X, τ_X) be a topological space, τ_X^B a basis and A some subset. Suppose that any covering of A with sets from τ_X^B admits a finite cover. Then A is compact.*

Theorem 2.1.3 (The preserving of compactness under continuous mapping)**.** *Let (X, τ_X) be a compact topological space and $f : (X, \tau_X) \to (Y, \tau_Y)$ be continuous. Then $f(X)$ is compact.*

Theorem 2.1.4 (Extreme value theorem)**.** *Let (X, τ_X) be a compact topological space and $f : X \to \mathbb{R}$ a real-valued continuous function. Then f is bounded on (X, τ_X) and attains its bounds at some points $x_i, x_s \in X$ such that $f(x_i) \le f(x) \le f(x_s)$ for all x.*

2.2 Compactness with Closed Subsets

Definition 2.2.1 (Compactness defined through closed subsets)**.** *A topological space (X, τ_X) is compact if and only if for any collection $(C_j)_{j\in J}$ of closed subsets of X such that the intersection $\cap_{j\in J} C_j$ is empty, there exists some finite subset $J_c \subseteq J$ such that $\cup_{j\in J_c} C_j$ is empty.*

Theorem 2.2.1 (Cantor's intersection theorem, Nested set property)**.** *Let (X, τ_X) be a compact topological space and $(C_n)_{n\ge1}$ a sequence of nested closed non-empty subsets of X, i.e $\forall n \in \mathbb{N}$: we have $C_n \supseteq C_{n+1}$. Then $\cap_{n\in\mathbb{N}} C_n$ is nonempty.*

Theorem 2.2.2. *Let (X, τ_X) be a topological space. If $T \subseteq X$ is a compact set, then every closed subset of T is compact.*

2.3 Compactness for Hausdorff Space

Proposition 2.3.1. *Let (X, τ_X) be a Hausdorff topological space. Then every compact subset of X is closed.*

Proposition 2.3.2. *A continuous bijection between two compact Hausdorff spaces is a homeomorphism.*

Definition 2.3.1 (Normal Space)**.** *A topological space (X, τ_X) is called normal if for any two closed disjoint sets C_1, C_2 we can find open sets U_1, U_2 such that $C_1 \subseteq U_1, C_2 \subseteq U_2$ and $U_1 \cap U_2 = \varnothing$.*

Theorem 2.3.3. *Any compact Hausdorff space is also normal.*

2.4 Compactness for Finite Product Space

Theorem 2.4.1. *Let $(X_1, \tau_{X_1}), ..., (X_n, \tau_{X_n})$ be compact topological spaces. Then also $X_1 \times \cdots \times X_n$ with its product topology is compact.*

Theorem 2.4.2 (Heine-Borel Theorem)**.** *Consider $\mathbb{R}^n$ with its standard topology. Then a subset $K \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded in the sense that K it is contained in some Euclidean ball $B(0, R)$.*

Proof. Several properties of $\mathbb{R}^n$.
(a) The Euclidean topology on $\mathbb{R}^n$ is the same as the product topology on the product of n copies of $\mathbb{R}$.
(b) The Euclidean topology on $\mathbb{R}^n$ is Hausdorff.
(c) The standard Euclidean distance is continuous.

Suppose K is compact. Then because $\mathbb{R}^n$ is Hausdorff, Proposition 2.3.1 implies that K is closed. Moreover, as mentioned just above the function $d_E(x, 0) : (\mathbb{R}^n, \tau_E) \to (\mathbb{R}, \tau_E)$ is continuous. Thus by the Boundedness Theorem 2.1.1 we know that $d_E(x, 0)$ is bounded on K and hence K is bounded. ☐

2.5 Locally Compactness

Definition 2.5.1 (Locally compact)**.** *Let (X, τ_X) be a topological space. If every point x of X has a compact neighborhood, i.e., we can find an open set U and a compact set K such that $x \in U \subseteq K$, then we say that X is locally compact.*

2.6 Tychonoff's Theorem and Axiom of Choice

3 Metric Space

3.1 Basic Properties of Metric Space

Theorem 3.1.1 (Continuity at a point in metric space in terms of sequence)**.** *Consider a metric space (X, d) and any topological space (Y, τ_Y). Then a function $f : (X, \tau_x) \to (Y, \tau_Y)$ is continuous at x if and only if for any sequence $(x_n)_{n\geq 1} \to x$, we have that $(f(x_n))_{n\geq 1} \to f(x)$.*

Definition 3.1.1 (Complete metric space)**.** *A metric space (X, d) is called complete if every Cauchy sequence converges.*

Theorem 3.1.2. *A metric space (X, d) is compact if and only if it is sequentially compact.*

Theorem 3.1.3. *Every sequentially compact metric space is complete.*

3.2 Properties of Subsets

Definition 3.2.1 (Precompact set and sequentially compact set)**.** *For $A \subseteq X$, $\forall \{x_n\} \in A$, $\exists \{x_{n,k}\}$, s.t. $x_{n,k} \to x_0$. If $x_0 \in X$, then A is a precompact subset. If $x_0 \in A$, then A is a sequentially compact subset.*

Definition 3.2.2 (Totally bounded set)**.** *Let (X, d) be a metric space. $A \subseteq X$ is a totally bounded set if for all $\epsilon > 0$, we can find a finite number of balls of radius ϵ in X covering A. In topology, total-boundedness is a generalization of compactness for circumstances in which a set is not necessarily closed.*

Theorem 3.2.1. *If (X, d) is metric space, A is totally bounded $\Rightarrow$ A is precompact. If (X, d) is complete metric space, A is totally bounded $\iff$ A is compact.*

Definition 3.2.3 (Dense set and nowhere dense set)**.** *Let (X, τ_X) be a topology space. We say $A \subseteq X$ is dense in $B \subseteq X$ if $cl(A) \supseteq B$. We say that $A \subseteq X$ is nowhere dense if A is not dense in any nonempty open set $B \subseteq X$.*

Definition 3.2.4 (Meagre sets)**.** *Let (X, τ_X) be a topological space. A subset $A \subseteq X$ is called meagre if it can be written as a countable union of nowhere dense sets.*

3.3 Baire Category Theorem

Theorem 3.3.1 (Baire Category Theorem)**.** *Every complete metric space is not meagre.*

3.4 Continuous Map

Definition 3.4.1 (Homeomorphism)**.** *Let (X, ρ_X), (Y, ρ_Y) be two metric spaces. Then $T : X \to Y$ is called a homeomorphism if T is bijective and both t and T^{-1} are continuous.*

Definition 3.4.2 (Isometry)**.** *Let (X, ρ_X), (Y, ρ_Y) be two metric spaces. Then $T : X \to Y$ is called an isometry if T is bijective and $\rho(Tx_1, Tx_2) = \rho(x_1, x_2)$ for any $x_1, x_2 \in X$.*

Theorem 3.4.1 (Urysohn's Lemma)**.** *Let X be a normal topological space, and let E and F be disjoint closed subsets of X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ holds at each x in E and $f(x) = 1$ holds at each x in F.*

4 Measure

4.1 Set Family

Proposition 4.1.1 (The relationship among different set families)**.**

$$\pi\text{-system} \to \text{semiring} \to \text{ring} \to \text{field} \to \sigma-\text{field} \tag{4.1.1}$$

$$\text{monotone class} \to d\text{-system} \to \sigma\text{-field} \tag{4.1.2}$$

$$\begin{cases} \mathscr{A} \text{ is a monotone class} \\ \mathscr{A} \text{ is a field} \end{cases} \Rightarrow \mathscr{A} \text{ is a } \sigma\text{-field.} \tag{4.1.3}$$

Theorem 4.1.2 (Monotone class theorem)**.** *If $\mathscr{A}$ is a algebra (field) of sets, then $\sigma(\mathscr{A}) = m(\mathscr{A})$ where $m(\mathscr{A})$ denotes the smallest monotone class containing $\mathscr{A}$.*

We begin with the following lemma.

Lemma 4.1.3. *If $\mathscr{A}$ is a ring and $X \in \mathscr{A}$, then $\mathscr{A}$ is a field.*

Proof. It is easy to show with the properties of ring and field.
Property of ring: $\mathscr{R}$ is a π-system; $A, B \in \mathscr{R} \Rightarrow A \cup B, A \backslash B \in \mathscr{R}$.
Property of field: $\mathscr{F}$ is a π-system; $X \in \mathscr{F}; A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}; A, B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$.
(The difference between semiring and semialgebra: semiring + $X \in \mathscr{R} \Rightarrow$ semialgebra) □

Proof of Theorem 4.1.2. As $m(\mathscr{A})$ is the smallest σ-field containing $\mathscr{A}$, $m(\mathscr{A})$ is also a monotone class. Then $m(\mathscr{A}) \subseteq \sigma(\mathscr{A})$ since $m(\mathscr{A})$ is the smallest monotone class containing $\mathscr{A}$. Thus, it is enough to prove $\sigma(\mathscr{A}) \subseteq m(\mathscr{A})$. Furthermore, it suffices to show $m(\mathscr{A})$ is a filed by means of relationship 4.1.3. Then, according to Lemma 4.1.3, we only need to verify $m(\mathscr{A})$ is a ring.

$\forall A \in \mathscr{A}$, let

$$\mathscr{G}_A = \{B : B, A \cup B, A \backslash B \in m(\mathscr{A})\}$$

We have

- $\mathscr{G}_A$ is a monotone class. $\forall B_i \uparrow \Rightarrow A \cup B_i \uparrow$, $A \backslash B_i \downarrow \Rightarrow \cup B_i, \cup(A \cup B_i), \cup(A \backslash B_i) \in m(\mathscr{A})$ (according to the definition of monotone class)$\Rightarrow \cup B_i$, $A \cup (\cup B_i)$, $A \backslash (\cup B_i) \in m(\mathscr{A}) \Rightarrow \cup B_i \in \mathscr{G}_A$.
- $\mathscr{A} \subseteq \mathscr{G}_A$. Fix $A \in \mathscr{A}$, since $\mathscr{A}$ is a field (it is also a ring and is closed under the formation of finite unions), $\forall A' \in \mathscr{A}$, $A', A \cup A', A \backslash A' \in \mathscr{A}$. Thus, $\mathscr{A} \subset \mathscr{G}_A$ which indicates $m(\mathscr{A}) \subset \mathscr{G}_A$. Furthermore,

$$A \in \mathscr{A}, B \in m(\mathscr{A}) \Rightarrow A \cup B, A \backslash B \in m(\mathscr{A}) \tag{4.1.4}$$

$\forall B \in m(\mathscr{A})$, let

$$\mathscr{H}_B = \{A : A, A \cup B, A \backslash B \in m(\mathscr{A})\}$$

- Similarly, $\mathscr{H}_B$ is a monotone class.
- According to formula 4.1.4, $\mathscr{A} \subseteq \mathscr{H}_B$. It follows that $m(\mathscr{A}) \subseteq \mathscr{H}_B$,

$$A, B \in m(\mathscr{A}) \Rightarrow A \cup B, A \backslash B \in m(\mathscr{A}) \tag{4.1.5}$$

□

Theorem 4.1.4 (π-λ theorem)**.** *If $\mathscr{A}$ is a π-system, then $\sigma(\mathscr{A}) = l(\mathscr{A})$ where $l(\mathscr{A})$ denotes the smallest Dynkin system (λ-system) containing $\mathscr{A}$. It is tantamount to: Let $\mathscr{P}$ be a π-system and $\mathscr{L}$ be a Dynkin system with $\mathscr{P} \subseteq \mathscr{L}$, then $\sigma(\mathscr{P}) \subseteq \mathscr{L}$.*

Example 4.1.1. *$\mathscr{P}_R = \{(-\infty, a] : a \in \mathbb{R}\}$ is a π-system; $\mathscr{R}_R = \{(a, b] : a, b \in \mathbb{R}\}$ is a semiring. The definition of Borel system of sets on $\mathbb{R}$ is*

$$\mathscr{B}_R = \sigma(\mathscr{P}_R) = \sigma(\mathscr{R}_R)$$

4.2 Measurable Mappings

Definition 4.2.1 (Topological measurable space)**.** *For topological space X, we denote the collection of open sets $\mathcal{O}$, we call $\mathscr{B} = \sigma(\mathcal{O})$ Borel algebra (system) of sets on X. And $(X, \mathscr{B})$ is called topological measurable space.*

Proposition 4.2.1 (Properties of preimage of sets)**.** *......*
As a simple corollary of preimage properties, for any set system $\mathscr{E}$ on Y,

$$\sigma(f^{-1}\mathscr{E}) = f^{-1}\sigma(\mathscr{E})$$

Definition 4.2.2 (Measurable mappings)**.** *$(X, \mathscr{A}) \xrightarrow{f} (Y, \mathscr{F})$ is a measurable mapping if*

$$f^{-1}\mathscr{F} \subseteq \mathscr{A}$$

Theorem 4.2.2. *Let $\mathscr{E}$ be any set system on Y, then*

$$(X, \mathscr{A}) \xrightarrow{f} (Y, \sigma(\mathscr{E})) \text{ is a measurable mapping } \iff f^{-1}\mathscr{E} \subseteq \mathscr{A}$$

which can be proved by means of the corollary in Proposition 4.2.1.

Definition 4.2.3 (Measurable function)**.** *The measurable mapping $f : (X, \mathscr{A}) \to (\overline{\mathbb{R}}, \mathscr{B}_{\bar{R}}))$ is called measurable function on $(X, \mathscr{A})$. The measurable mapping $f : (X, \mathscr{A}) \to (\mathbb{R}, \mathscr{B}_R))$ is called random variable (or finite measurable function) on $(X, \mathscr{A})$. Note that for image in $\overline{\mathbb{R}}$ of measurable function, we only care about Borel set.*

4.3 Measure

Definition 4.3.1. *A measure (or a countably additive measure) on $\mathscr{A}$ (could be semiring, ring, algebra, σ-algebra, $\cdots$) is a function $\mu : \mathscr{A} \to [0, +\infty]$ that satisfies $\mu(\varnothing) = 0$ and is countably additive.*

Theorem 4.3.1 (Measure on semiring)**.** *The measure on a semiring exhibits monotonicity, subadditivity, semiadditivity, upper semicontinuity, and lower semicontinuity.*

Lemma 4.3.2. *If $\mathscr{R}$ is a semiring,*

$$r(\mathscr{R}) = \bigcup_{n=1}^{\infty} \left\{ \bigcup_{k=1}^{n} A_k : \{A_k \in \mathscr{R}, k = 1, ..., n\} \text{are disjoint} \right\} \tag{4.3.1}$$

See Theorem 1.3.2 in [2].

Proof of Theorem 4.3.1. Semiadditivity. Assume $A_1, A_2, ... \in \mathscr{R}$ and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{R}$, then $A_1, A_2, ... \in r(\mathscr{R}) \Rightarrow \bigcup_{i=1}^{n-1} A_i \in r(\mathscr{R}) \Rightarrow A_n \backslash \bigcup_{i=1}^{n-1} A_i \in r(\mathscr{R})$ (definition of ring). Thus, according to Lemma 4.3.2, $\exists$ disjoint sets $\{C_{n,k} \in \mathscr{R}, k = 1, ..., k_n\}$ such that,

$$A_n \backslash \bigcup_{i=1}^{n-1} A_i = \bigcup_{k=1}^{k_n} C_{n,k}$$

Similarly,

$$A_n \backslash \bigcup_{k=1}^{k_n} C_{n,k} = \bigcup_{l=1}^{l_n} D_{n,l}$$

We have

$$A_n = (\bigcup_{k=1}^{k_n} C_{n,k}) \cup (\bigcup_{l=1}^{l_n} D_{n,l}) \tag{4.3.2}$$

Then we can easily derive $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \bigcup_{n=1}^{\infty} \mu(A_n)$ by means of additivity of μ. For other properties see Proposition 2.1.4 in [2]. $\square$

4.4 Outer Measure

Definition 4.4.1. *Let X be a set, and let $\mathcal{P}(X)$ be the collection of all subsets of X. An outer measure on X is a function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$ such that it is a monotone and countably subadditive function with $\mu^*(\varnothing) = 0$.*
(a) $\mu^*(\varnothing) = 0$,
(b) if $A \subseteq B \subseteq X$, then $\mu^*(A) \leq \mu^*(B)$, and
(c) if $\{A_n\}$ is an infinite sequence of subsets of X, then $\mu^*(\cup A_n) \leq \sum \mu^*(A_n)$.
Remark: Measure is defined on σ-algebra, while outer measure is defined on a set X.

Theorem 4.4.1 (Construction of outer measure)**.** *Let $\mathscr{E}$ be a set system and $\varnothing \in \mathscr{E}$, $\mu : \mathscr{E} \to [0, +\infty]$ be a non-negative set function satisfying $\mu(\varnothing) = 0$. Then the function $\mu^* : \mathcal{P}(X) \to [0, +\infty]$*

$$\mu^*(A) = inf\left\{\sum_{n=1}^{\infty} \mu(B_n) : B_n \in \mathscr{E}, A \subseteq \bigcup_{i=1}^{\infty} B_n\right\} \tag{4.4.1}$$

is an outer measure.

Definition 4.4.2 (μ^*-measurable)**.** *Let μ^* be an outer measure on X, we say A is μ^*-measurable if*

$$\mu^*(D) = \mu^*(D \cap A) + \mu^*(D \cap A^c)$$

for any $D \in \mathscr{A}$ We denote all μ^*-measurable sets $\mathscr{F}_{\mu^*}$.

Definition 4.4.3 (σ-finiteness)**.** *Let μ be a measure on a measurable space $(X, \mathscr{A})$. Then μ is a finite measure if $\mu(X) < +\infty$ and is a σ-finite measure if X is the union of a sequence $A_1, A_2, ...$ of sets that belong to A and satisfy $\mu(A_i) < +\infty$ for each i.*

Let X be any set with at least two points, take the trivial σ -algebra $\mathscr{F} = \{X, \varnothing\}$, and define μ on F by $\mu(X) = \mu(\varnothing) = 0$.

4.5 The Extension of Measure

Definition 4.5.1 (Completeness)**.** *Let $(X, \mathscr{F}, \mu)$ be a measure space. The measure μ (or the measure space $(X, \mathscr{F}, \mu)$ is complete if the relations $A \in \mathscr{F}$, $\mu(A) = 0$, and $B \subseteq A$ together imply that $B \in A$.*

Theorem 4.5.1 (General extension theorem)**.** *If μ^* is an outer measure, them $\mathscr{F}_{\mu^*}$ is a σ-field, and $(X, \mathscr{F}_{\mu^*}, \mu^*)$ is a complete measurable space.*

We want to extend the measure μ on $\mathscr{E}$ to a larger set system $\mathscr{F}$. Can we construct the outer measure μ^* using Formula 4.4.1 and extend it to $\mathscr{F}$ by means of Theorem 4.5.1?
The answer is no. There is no restriction to ensure $\mathscr{E} \subseteq \mathscr{F}_{\mu^*}$

Theorem 4.5.2 (Carathéodory extension theorem I)**.** *Let μ be a σ-finite measure on an semialgebra $\mathscr{S}$. Then μ has a unique extension to $\sigma(\mathscr{S})$. See [2].*

Theorem 4.5.3 (Carathéodory extension theorem II)**.** *Let μ be a σ-finite measure on an algebra $\mathscr{A}$. Then μ has a unique extension to $\sigma(\mathscr{A})$. See [4].*

In the following of this chapter, we focus on extending the measure on semiring $\mathscr{R}_R$ in Example 4.1.1.

Proposition 4.5.4. *Let F be a nondecreasing right-continuous function on $\mathbb{R}$ and for any $a < b \in \mathbb{R}$,*

$$\mu((a, b]) = F(b) - F(a)$$

otherwise $\mu((a, b]) = 0$. Then μ is a measure on $\mathscr{R}_R$.

Theorem 4.5.5. *Let μ^* be the outer measure generated by measure μ defined on semiring $\mathscr{R}$, then $(X, \mathscr{F}_{\mu^*})$ is the completion of $(X, \sigma(\mathscr{R}), \mu^*)$.*

Definition 4.5.2 (Definition I of Lebesgue measure)**.**

(1) According to Theorem 4.5.2, μ has a unique extension on $\sigma(\mathscr{R}_R) = \mathscr{B}_R$.

(2) According to Theorem 4.5.1, the outer measure μ_F^* is also a measure on $\mathscr{F}_{\mu_F^*}$. We call sets in $\mathscr{F}_{\mu_F^*}$ Lebesgue-Stieljes measurable sets and μ_F^* on $\mathscr{F}_{\mu_F^*}$ Lebesgue measure. If $F(x) = x$, then μ_F^* is called Lesbegue measure.

(3) Note we have $\sigma(\mathscr{R}) \subseteq \mathscr{F}_{\mu_F^*}$. Theorem 2.3.4 in [2] contains more details about the difference between $\sigma(\mathscr{R})$ and $\mathscr{F}_{\mu_F^*}$. Appendix in [4] also provides a slightly different version of the description about this relation. Proposition 2.1.11 in [1] presents a Lebesgue measurable set which is not a Borel set.

(4) Theorem 4.5.5 gives a more accurate description.

Definition 4.5.3 (Definition II of Lebesgue measure)**.** *We can also start with Lebesgue outer measure denoted as λ^*.*

$$\lambda^*(A) = inf\left\{\sum_i (a_i - b_i) : (a_i, b_i) \text{ is a open interval.}\right\} \tag{4.5.1}$$

Lebesgue measurable subset is the set that is λ^*-measurable in accord with Definition 4.4.2, the set of which is denoted as M_{λ^*}. Then Lebesgue measure is λ^* restricted on M_{λ^*} and is denoted by λ.

5 Convergence

5.1 Modes of Convergence

Definition 5.1.1 (Almost everywhere finite and almost everywhere bounded)**.** *Almost everywhere finiteness is a pointwise property while almost everywhere boundedness is a global property.*

Definition 5.1.2 (Modes of convergence)**.**

(1) Uniform convergence;

(2) Pointwise convergence;

(3) Almost everywhere convergence;

(4) Convergence in measure;

(5) Almost uniform convergence (not almost everywhere);

(6) Convergence in mean (generally, convergence in L_p);

(7) Convergence in distribution (in context of probability theory; is in correspondence with weak convergence of cdf [2])

Proposition 5.1.1 (Relations of different convergence modes)**.** *Let $(X, \sigma(\mathscr{A}), \mu^*)$ be a measure space, and let f and f_1, $f_2, \cdots$ be almost everywhere finite $\mathscr{A}$-measurable functions on X to $\bar{\mathbb{R}}$.*

(1)

$$\left\{\begin{array}{c} \mu \text{ is finite} + \text{almost everywhere convergence} \Rightarrow \text{ convergence in measure} \\ \text{convergence in measure} \Rightarrow \text{a subsequence of } \{f_n\} \text{ converges to } f \text{ almost everywhere} \end{array}\right\}$$

(2)

$$\left\{\begin{array}{c} \text{almost uniform convergence} \Rightarrow \text{almost everywhere convergence} \\ \text{(Egoroff's Theorem) } \mu \text{ is finite+ almosteverywhere convergence} \Rightarrow \text{almost uniform convergence} \end{array}\right\}$$

(3) Convergence in L_p $\Rightarrow$ convergence in measure.

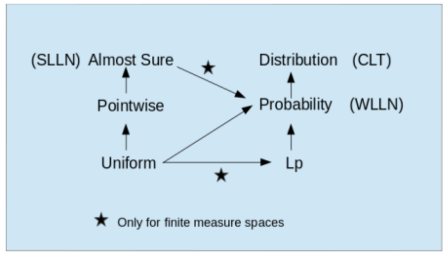

Figure 1: Relations of different convergence modes. [5]

5.2 Measurable Function and Continuous Functions

Lusin's theorem, converse theorem of Lusin's theorem and Frechet's theorem.

5.3 Integral and Limit Theorems

Theorem 5.3.1 (Monotone convergence theorem). *Let $(X, \mathscr{A}, \mu)$ be a measure space, and let f and $f_1, f_2, ...$ be $[0, +\infty]$-valued $\mathscr{A}$-measurable functions on X. Suppose that $f_1(x) \leq f_2(x) \leq ...$ and $f(x) = \lim_{n\to\infty} f_n(x)$ hold at μ-almost every x in X. Then*

$$\int f d\mu = lim_n \int f_n d\mu$$

In this theorem the functions f and $f_1, f_2, ...$ are only assumed to be nonnegative and measurable; there are no assumptions about whether they are integrable. A corollary of monotone convergence theorem is Beppo Levi's theorem.

Theorem 5.3.2 (Dominated convergence theorem.). *Let $(X, \mathscr{A}, \mu)$ be a measure space, let g be $[0, +\infty]$-valued integrable function on X, and let f and $f_1, f_2, ...$ be $[0, +\infty]$-valued $\mathscr{A}$-measurable functions on X and such that $f(x) = \lim_n f_n(x)$ and $|f_n(x)| \leq g(x)$ hold at μ-almost every x in X. Then f and $f_1, f_2, ...$ are integrable, and*

$$\int f d\mu = lim_n \int f_n d\mu$$

Theorem 5.3.3 (Fatou's lemma).

6 Signed Measure

6.1 Hahn Decomposition

Definition 6.1.1 (Signed measure). *A signed measure on $\mathscr{A}$ is a function $\mu : \mathscr{A} \to \bar{\mathbb{R}}$ such that,*
(1) $\mu(\varnothing) = 0$,
(2) μ is countably additive,
(3) Either $\mu(A) \in [-\infty, +\infty)$ or $\mu(A) \in (-\infty, +\infty]$.

Definition 6.1.2 (Positive set). *A subset A of X is a positive set if $A \in \mathscr{A}$ and each $\mathscr{A}$-measurable subset E of A satisfies $\mu(A) \geq 0$.*

Theorem 6.1.1 (Hahn Decomposition Theorem). *Let $(X, \mathscr{A})$ be a measurable space, and let μ be a signed measure on $(X, \mathscr{A})$. Then there are disjoint subsets P and N of X such that P is a positive set for μ, N is a negative set for μ, and $X = P \cup N$. The decomposition in essentially unique.*

Theorem 6.1.2 (Jordan Decomposition Theorem)**.** *Every signed measure is the difference of two positive measures, at least one of which is finite.*
(1) Let μ be a signed measure on $(X, \mathscr{A})$. Choose a Hahn decomposition (P, N) for μ and then define functions μ^+ and μ^- on $\mathscr{A}$ by

$$\mu^+(A) = \mu(A \cap P) \tag{6.1.1}$$
$$\mu^-(A) = -\mu(A \cap N) \tag{6.1.2}$$

(2) Actually,

$$\mu^+(A) = sup\{\mu(B) : B \in \mathscr{A} \text{ and } B \subseteq A)\} \tag{6.1.3}$$
$$\mu^-(A) = sup\{-\mu(B) : B \in \mathscr{A} \text{ and } B \subseteq A)\} \tag{6.1.4}$$

Proposition 6.1.3. *Let $(X, \mathscr{A})$ be a measurable space. Then the spaces $M(X, \mathscr{A}, \mathbb{R})$ (finite signed measure) is complete under the total variation norm $||\mu|| = |\mu|(X)$ where the variation is*

$$|\mu| = \mu^+ + \mu^- \tag{6.1.5}$$

6.2 Radon–Nikodym Theorem

Definition 6.2.1 (Absolutely consinuous)**.** *Let $(X, \mathscr{A})$ be a measurable space, and let μ and υ be positive measures on $(X, \mathscr{A})$. Then υ is absolutely continuous with respect to μ ($\upsilon \ll \mu$) if*

$$\mu(A) = 0 \Rightarrow \upsilon(A) = 0, \forall A \in \mathscr{A}$$

Let υ' be a signed measure on $(X, \mathscr{A})$. We say $\upsilon' \ll \mu$ if $|\upsilon| \ll \mu$.

Theorem 6.2.1 (Radon–Nikodym theorem)**.** *Let $(X, \mathscr{A})$ be a measurable space, and let μ and υ be σ-finite positive measures on $(X, \mathscr{A})$. If $\upsilon \ll \mu$, then there is an $\mathscr{A}$-measurable function $g : X \to [0, +\infty)$ such that $\upsilon(A) = \int_A g d\mu$ holds for each $A \in \mathscr{A}$. The function g is unique up to μ-almost everywhere equality.*

Proof. TBD see [1]. ☐

Theorem 6.2.2 (Radon–Nikodym theorem (signed))**.** *Let $(X, \mathscr{A})$ be a measurable space, let μ be a σ-finite positive measure on $(X, \mathscr{A})$, and let υ be a finite signed measure on $(X, \mathscr{A})$. If $\upsilon \ll \mu$, then there is a function g that belongs to $(X, \mathscr{A}, \mu, \mathbb{R})$ and satisfies $\upsilon(A) = \int_A g d\mu$ for each $A \in \mathscr{A}$. The function g is unique up to μ-almost everywhere equality.*

6.3 Lebesgue Decomposition

Definition 6.3.1 (Sigularity)**.** *Let μ, υ be signed measures. Then μ and υ are mutually singular if $\exists N \in \mathscr{A}$ such that $|\mu|(N) = |\upsilon|(N^c) = 0$.*

Theorem 6.3.1 (Lebesgue decomposition)**.** *Let $(X, \mathscr{A})$ be a σ-finite measurable space, let μ, υ be a σ-finite signed measure on $(X, \mathscr{A})$, then there exists two σ-finite signed measures μ_c, μ_s such that*
(1) $\mu = \mu_c + \mu_s$
(2) $\mu_c \ll \mu$
(3) $\mu_s \perp \upsilon$

Proof. See [2]. ☐

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