Topology & Differential Manifolds

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1 Basic Topology

1.1 Topology and Topological Basis

Definition 1.1.1 (Topology and open sets). *Let X be any non-empty set. Then a topology on the set X is a collection τ of subsets $U \subseteq X$ satisfying the following properties:*
(a) Both $\varnothing \in \tau$ and $X \in \tau$;
(b) τ is closed under unions and finite intersections.
The sets $U \in \tau$ are called open sets and the pair (X, τ) is called a topological space.

Definition 1.1.2 (Basis of a topology). *Let X be a set. Then a collection τ^B of subsets of X is called a basis for a topology, if it satisfies two conditions:*
(a) $\cup_{V \in \tau^B} V = X$;
(b) $\forall V_1 \in \tau^B$, $V_2 \in \tau^B$ and $x \in V_1 \cap V_2$, there is some $V_3 \in \tau^B$ with $x \in V_3 \subseteq V_1 \cap V_2$.
Consider τ to be the set of all possible unions of $V \in \tau^B$, together with the empty set. Then τ defines a topology on X.

Proposition 1.1.1 (Criteria for finding a basis). *Let (X, τ) be a topological space. Let $\widetilde{\tau}^B \subseteq \tau$ be a collection of subsets such that any set in τ is an union of sets from $\widetilde{\tau}^B \subseteq \tau$. Then $\widetilde{\tau}^B$ is a basis for some topology, and the topology induced by this basis is τ.*

Definition 1.1.3 (Interior and closure). *Let (X, τ) be a topological space. Then for any subset $A \subseteq X$, $int(A)$ is defined as the largest open set contained in A, and $cl(A)$ is defined as the smallest closed set containing A. Specifically, $int(A)$ is equal to the union of all open sets contained in A, and $cl(A)$ is equal to the intersection of all closed sets containing A.*

Definition 1.1.4 (Limit point in topological space). *Let (X, τ) be a topological space. Then $x \in X$ is called a limit point if for any open set containing x contains at least one point different from x, then x is called a limit point. We can use limit point to define closed set in general topology: A closed set is a set that contains all its limit points.*

1.2 Metric Topology

Definition 1.2.1 (Metric topology). *Let (X, d) be a metric space. We define $U \subseteq X$ to be open if for every $x \in U$, we can find some $\delta > 0$ such that $B(x, \delta) \subseteq U$ (We set $U \in \tau_d$). Then τ_d is a topology and is called the metric topology.*

Proposition 1.2.1 (The structure of open set in metric topology). *Let (X, d) be a metric space, and let τ_d be the metric topology. Then a set U is open if and only if it can be written as a union of open metric balls, which suggests the basis of topology for metric topology.*

Proposition 1.2.2 (Basis of a metric topology). *Let (X, d) be a metric space. Then $\tau^B := \{B(x, \delta) : x \in X, \delta > 0\}$ is a basis for the metric topology τ_d.*

Proposition 1.2.3 (The structure of open sets in $\mathbb{R}^n$). *The balls $B(x, \delta)$ where $x \in \mathbb{Q}_n$ and $\delta \in \mathcal{Q} \cap (0, \infty)$ are a basis for the Euclidean topology on $\mathbb{R}^n$. It suggests that any open set in $\mathbb{R}^n$ could be written as a union of countable metric balls.*

1.3 Convergence, Continuity and Homeomorphism

Definition 1.3.1 (Convergence). *Let (X, τ) be a topological space and $(x_n)_{n \geq 1}$ a sequence of points in X. We define $x_n \to x$ if for any open set U containing x there is some $n_U in \mathbb{N}$ such that $\forall n \geq n_U : x_n \in U$.*

Proposition 1.3.1 (Two counter-intuitive exmaples). *(a) $\mathbb{N}$ with its cofinite topology. Let $x_i = i$, $(x_n)_{n \geq 1}$ converges simultaneously to all $n \in N$. (b) $\mathbb{R}$ with the co-countable topology. All open sets are either countable or $\mathbb{R}$. The closure of $(0, 1)$ is the whole space.*

Definition 1.3.2 (Continuity at a point). *A map $f : X \to Y$ from a topological space (X, τ_X) to a topological space (Y, τ_Y) is continuous at a point $x \in X$, if $\forall$ open set U containing $f(x)$, there is some open set V_U containing x, such that $f(V_U) \subseteq U$. Then for any sequence $(x_n)_{n\geq 1} \to x$, we have that $(f(x_n))_{n\geq 1} \to f(x)$.*

Proof. [1] $\forall$ U containing $f(x)$, $\exists$ open set V_U containing x, thus $\exists$ n_{V_U} such that for all $n \geq n_{V_U}$, $x_n \in V_U$. Then for all $n \geq n_{V_U}$, $f(x_n) \in U$. ☐

Proposition 1.3.2 (Continuous map). *A map $f : X \to Y$ from a topological space (X, τ_X) to a topological space (Y, τ_Y) is continuous at every point $x \in X$ iff the pre-image of any open set is an open set, i.e. iff for any open set U of (Y, τ_Y) we have that $f^{-1}(U)$ is open in (X, τ_X).*

Remark. The proof of direction "the preimage of open set is open" $\Leftarrow$ "continuous map" is easy.

Proof. Let $U \subseteq Y$ be open. For any $x \in f^{-1}(U)$, we can find a open set $V_x \in \tau_X$ such that $f(V_x) \in U$, and thus $V_x \subseteq f^{-1}(U)$. By definition of open set, $f^{-1}(U)$ is open. ☐

The definition of open map is: the image of open set is open. The property of continuous map is kind of in the opposite direction of open map.

Definition 1.3.3 (Homeomorphism). *Let (X, τ_X), (Y, τ_Y) be two topological spaces. Then $f : X \to Y$ is called a homeomorphism if f is bijective and both f and f^{-1} are continuous.*

1.4 Subspace

Definition 1.4.1 (Subspace topology). *Let (X, τ_X) be a topological space and A a subset of X. Then define to be the collection of sets of the form $A \cap U$, where $U \in \tau_X$. Then $\tau_{X,A}$ defines a topology on A that is called the subspace topology.*

Theorem 1.4.1 (Subspace in metric space). *Let (X, d) be a metric space, then it induces a topological space (X, τ_X) via the metric topology. Now consider $A \subseteq X$. If we restrict d to $A \times A$, we obtain a metric space (A, d) and this induces a topological space (A, τ_A). We have,*

$$\tau_A = \tau_{X,A}$$

Theorem 1.4.2 (The natural way to define subspace topology). *Let (X, τ_X) be a topological space and $(A, \tau_{X,A})$ a subspace with the subspace topology. Then $\tau_{X,A}$ is the smallest topology $\widetilde{\tau}$ for which the inclusion map $i : (A, \widetilde{\tau}) \to (X, \tau_X)$ defined on A by identity is continuous.*

Proposition 1.4.3 (Several properties of subspace topology). *Let (X, τ_X) be a topological space and $(A, \tau_{X,A})$ a subspace with the subspace topology.*
(a) Prove that if (Y, τ_Y) is another topological space and $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then also f restricted to A is a continuous map from $(A, \tau_{X,A}) \to (Y, \tau_Y)$.
(b) In particular, prove that if $f : (X, \tau_X) \to (Y, \tau_Y)$ is a homeomorphism and $f(A) = B$ for some $B \subseteq Y$, then the restriction of f to A induces a homeomorphism between A and B with their respective subspace topologies.

1.5 Product Topology

Definition 1.5.1 (Product topology on $X \times Y$). *Consider two topological spaces (X, τ_X) and (Y, τ_Y). Define $\tau^B_{X\times Y}$ to be the collection of the subsets of $X \times Y$ of the form $U \times V$, where U is open in X and V is open in Y. Then $\tau^B_{X\times Y}$ is a basis for a topology, and the topology $\tau_{X\times Y}$ induced by it is called the product topology on $X \times Y$.*

Interestingly, the basis of subspace topology could be generated through the basis of original subspace topology, while the basis of product topology can not be generated in such a way.

Theorem 1.5.1 (The natural way to define finite product topology). *Let (X, τ_X) and (Y, τ_Y) be topological spaces and consider $X \times Y$ with the product topology. Then the product topology $\tau_{X\times Y}$ is the smallest topology $\widetilde{\tau}$ on $X \times Y$ such that the projection maps $p_X : (X \times Y, \widetilde{\tau}) \to (X, \tau_X)$ given by $p_X(x, y) := x$ and $p_Y : (X \times Y, \widetilde{\tau}) \to (Y, \tau_Y)$, given by $p_Y(x, y) := y$ are both continuous.*

Proof. Again, let us first check that p_X, p_Y are continuous for the product topology. For any open set U of (X, τ_X) we have that $p_X^{-1}(U) = U \times Y$, and this belongs to $\tau^B_{X\times Y}$. Similarly, for any open set V of (Y, τ_Y) we have that $p_Y^{-1}(V) = X \times V \in \tau^B_{X\times Y}$, and thus the continuity follows.

Now, suppose $p_X : (X \times Y, \widetilde{\tau}) \to X$ and $p_Y : (X \times Y, \widetilde{\tau}) \to Y$ are continuous. Then, by above all sets of the form $U \times Y$ with $U \in \tau_X$ and $X \times V$ with $V \in \tau_Y$ have to belong to $\widetilde{\tau}$. But then also $(U \times Y) \cap (X \times V) \in \widetilde{\tau}$ and thus in particular $\widetilde{\tau}$ contains the basis $\tau^B_{X\times Y}$. But then, as $\widetilde{\tau}$ is a topology, it has to contain the topology induced by this basis, i.e. $\tau_{X\times Y}$, giving the claim. □

Proposition 1.5.2 (Pointwise continuity and continuity). *Let (X, τ_X) and (Y, τ_Y) be topological spaces and consider $X \times Y$ with the product topology. Let further (Z, τ_Z) be another topological space and $f : (Z, \tau_Z)$ß$(X \times Y, \tau_{X\times Y})$. Prove that f is continuous if and only if both $f_1 := p_X \circ f : (Z, \tau_Z) \to (X, \tau_X)$ and $f_2 : p_Y \circ f : (Z, \tau_Z) \to (Y, \tau_Y)$ are continuous.*

Definition 1.5.2 (The infinite product topology). *Let I be some infinite index set and $((X_i, \tau_{X_i}))_{i\in I}$ a collection of topological spaces. Let $\tau^B_{\prod_{i\in I} X_i}$ be the collection of subsets of $\prod_{i\in I} X_i$ of the form $\prod_{i\in I} U_i$, where each $U_i \subseteq X_i$ is open in X_i and $U_i \neq X_i$ only for finitely many $i \in I$. Then τ^B is a basis for a topology, and this topology is called the product topology on $\prod_{i\in I} X_i$.*

Theorem 1.5.3 (The natural way to define infinite product topology). *Let now I be some infinite index set and $((X_i, \tau_{X_i}))_{i\in I}$ a collection of topological spaces. Then the product topology is the smallest topology on $\prod_{i\in I} X_i$ such that all coordinate maps are continuous.*

Proof. The direction "$\Rightarrow$" is the same as Theorem 1.5.1. For "$\Leftarrow$" direction, we need the finite intersection to complete the proof, which is guaranteed by the definition of infinite product topology basis.

□

Proposition 1.5.4 (The relationship between pointwise convergence and convergence). *Let now I be some index set and $((X_i, \tau_{X_i}))_{i\in I}$ a collection of topological spaces. Then a sequence $(x_n)_{n\geq 1}$ converges to x in $\prod_{i\in I} X_i$ with the product topology if and only if it converges pointwise, i.e. iff for all $i \in I$, $(x_n(i))_{n\geq 1}$ converges to $x(i)$ in (X_i, τ_{X_i}).*

1.6 Hausdroff Space

Definition 1.6.1 (Hausdorff space). *A topological space (X, τ_X) is called Hausdorff if for any two distinct points x, y we can find two disjoint open sets U_x, U_y such that $x \in U_x$ and $y \in U_y$.*

Theorem 1.6.1 (Several properties of Hausdroff space).
(a) If (X, τ_X) is Hausdorff, then any convergent sequence has a unique limit.
(b) Suppose (X, τ_X) is Hausdorff and $f : (X, \tau_X) \to (Y, \tau_Y)$ a homeomorphism. Then (Y, τ_Y) is also Hausdorff.
(c) Let (X, τ_X) be a Hausdorff topological space. Then $(A, \tau_{X,A})$ is also Hausdorff.
(d) Let (X, τ_X) be a Hausdorff topological space. Then every compact subset of X is closed.

2 Compactness

2.1 Properties of Compactness

Definition 2.1.1 (Compactness). *A topological space (X, τ_X) is called compact if any open cover of X admits a finite subcover, i.e. if I is any index set, U_i are open for all $i \in I$ and $\cup_{i\in I} U_i = X$, then there exists a finite subset $I_0 \subseteq I$ such that $i \in I$ and $\cup_{i\in I_0} U_i = X$.*

Definition 2.1.2 (Sequentially compact)**.** *A topological space (X, τ_X) is called sequentially compact, if any sequence $(x_n)_{n\geq 1}$ in X admits a convergent subsequence.*

Theorem 2.1.1 (Boundedness theorem)**.** *Let (X, τ_X) be a compact topological space and $f : X \to R$ a real-valued continuous function. Then f is bounded on (X, τ_X), i.e. there exist $i, s \in \mathbb{R}$ such that $i \leq f(x) \leq s$ for all $x \in X$.*

The similar result holds for sequentially compact spaces, but the proof argues by contradiction and is not half as neat.

Definition 2.1.3 (Compactness of set)**.** *Let (X, τ_X) be a topological space and consider $K \subseteq X$. Then $(K, \tau_{X,K})$ is compact as a topological space if and only if every covering of K with open sets of X admits a finite subcover.*

Proposition 2.1.2. *Let (X, τ_X) be a topological space, τ_X^B a basis and A some subset. Suppose that any covering of A with sets from τ_X^B admits a finite cover. Then A is compact.*

Theorem 2.1.3 (The preserving of compactness under continuous mapping)**.** *Let (X, τ_X) be a compact topological space and $f : (X, \tau_X) \to (Y, \tau_Y)$ be continuous. Then $f(X)$ is compact.*

Theorem 2.1.4 (Extreme value theorem)**.** *Let (X, τ_X) be a compact topological space and $f : X \to \mathbb{R}$ a real-valued continuous function. Then f is bounded on (X, τ_X) and attains its bounds at some points $x_i, x_s \in X$ such that $f(x_i) \leq f(x) \leq f(x_s)$ for all x.*

2.2 Compactness with Closed Subsets

Definition 2.2.1 (Compactness defined through closed subsets)**.** *A topological space (X, τ_X) is compact if and only if for any collection $(C_j)_{j\in J}$ of closed subsets of X such that the intersection $\cap_{j\in J} C_j$ is empty, there exists some finite subset $J_c \subseteq J$ such that $\cup_{j\in J_c} C_j$ is empty.*

Theorem 2.2.1 (Cantor's intersection theorem, Nested set property)**.** *Let (X, τ_X) be a compact topological space and $(C_n)_{n\geq 1}$ a sequence of nested closed non-empty subsets of X, i.e $\forall n \in \mathbb{N}$: we have $C_n \supseteq C_{n+1}$. Then $\cap_{n\in\mathbb{N}} C_n$ is nonempty.*

Theorem 2.2.2. *Let (X, τ_X) be a topological space. If $T \subseteq X$ is a compact set, then every closed subset of T is compact. Remark. Check Proposition 3.13 in EPFL lecture notes. We also use this theorem in Exercise sheet 8 of Smooth Manifolds course.*

2.3 Compactness for Hausdorff Space

Proposition 2.3.1. *Let (X, τ_X) be a Hausdorff topological space. Then every compact subset of X is closed. Remark. We have used this property in Corollary 5.6 in Lee's textbook [4]. We can check the proof at Proposition 3.14 in the EPFL lecture note.*

Proposition 2.3.2. *A continuous bijection between two compact Hausdorff spaces is a homeomorphism.*

Definition 2.3.1 (Normal Space)**.** *A topological space (X, τ_X) is called normal if for any two closed disjoint sets C_1, C_2 we can find open sets U_1, U_2 such that $C_1 \subseteq U_1, C_2 \subseteq U_2$ and $U_1 \cap U_2 = \varnothing$.*

Theorem 2.3.3. *Any compact Hausdorff space is also normal.*

2.4 Compactness for Finite Product Space

Theorem 2.4.1. *Let $(X_1, \tau_{X_1}), ..., (X_n, \tau_{X_n})$ be compact topological spaces. Then also $X_1 \times \cdots \times X_n$ with its product topology is compact.*

Theorem 2.4.2 (Heine-Borel Theorem). *Consider $\mathbb{R}^n$ with its standard topology. Then a subset $K \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded in the sense that K it is contained in some Euclidean ball $B(0, R)$.*

Proof. Several properties of $\mathbb{R}^n$.
(a) The Euclidean topology on $\mathbb{R}^n$ is the same as the product topology on the product of n copies of $\mathbb{R}$.
(b) The Euclidean topology on $\mathbb{R}^n$ is Hausdorff.
(c) The standard Euclidean distance is continuous.

Suppose K is compact. Then because $\mathbb{R}^n$ is Hausdorff, Proposition **??** implies that K is closed. Moreover, as mentioned just above the function $d_E(x, 0) : (\mathbb{R}^n, \tau_E) \to (\mathbb{R}, \tau_E)$ is continuous. Thus by the Boundedness Theorem 2.1.1 we know that $d_E(x, 0)$ is bounded on K and hence K is bounded. □

2.5 Locally Compactness

Definition 2.5.1 (Locally compact). *Let (X, τ_X) be a topological space. If every point x of X has a compact neighborhood, i.e., we can find an open set U and a compact set K such that $x \in U \subseteq K$, then we say that X is locally compact.*

2.6 Tychonoff's Theorem and Axiom of Choice

One useful case is for the proof of non-separable version of Banach-Alaoglu Theorem.

3 Metric Space

3.1 Basic Properties of Metric Space

Theorem 3.1.1 (Continuity at a point in metric space in terms of sequence). *Consider a metric space (X, d) and any topological space (Y, τ_Y). Then a function $f : (X, \tau_x) \to (Y, \tau_Y)$ is continuous at x if and only if for any sequence $(x_n)_{n\geq 1} \to x$, we have that $(f(x_n))_{n\geq 1} \to f(x)$.*

Definition 3.1.1 (Complete metric space). *A metric space (X, d) is called complete if every Cauchy sequence converges.*

Theorem 3.1.2. *A metric space (X, d) is compact if and only if it is sequentially compact.*

Theorem 3.1.3. *Every sequentially compact metric space is complete.*

3.2 Properties of Subsets

Definition 3.2.1 (Precompact set and sequentially compact set). *For $A \subseteq X$, $\forall \{x_n\} \in A$, $\exists \{x_{n,k}\}$, s.t. $x_{n,k} \to x_0$. If $x_0 \in X$, then A is a precompact subset. If $x_0 \in A$, then A is a sequentially compact subset.*

Definition 3.2.2 (Totally bounded set). *Let (X, d) be a metric space. $A \subseteq X$ is a totally bounded set if for all $\epsilon > 0$, we can find a finite number of balls of radius ϵ in X covering A. In topology, total-boundedness is a generalization of compactness for circumstances in which a set is not necessarily closed.*

Theorem 3.2.1. *If (X, d) is metric space, A is totally bounded $\Rightarrow$ A is precompact. If (X, d) is complete metric space, A is totally bounded $\iff$ A is compact.*

Definition 3.2.3 (Dense set and nowhere dense set). *Let (X, τ_X) be a topology space. We say $A \subseteq X$ is dense in $B \subseteq X$ if $cl(A) \supseteq B$. We say that $A \subseteq X$ is nowhere dense if A is not dense in any nonempty open set $B \subseteq X$.*

Definition 3.2.4 (Meagre sets). *Let (X, τ_X) be a topological space. A subset $A \subseteq X$ is called meagre if it can be written as a countable union of nowhere dense sets.*

3.3 Baire Category Theorem

Theorem 3.3.1 (Baire Category Theorem)**.** *Every complete metric space is not meagre.*

3.4 Continuous Map

Definition 3.4.1 (Homeomorphism)**.** *Let (X, ρ_X), (Y, ρ_Y) be two metric spaces. Then $T : X \to Y$ is called a homeomorphism if T is bijective and both t and T^{-1} are continuous.*

Definition 3.4.2 (Isometry)**.** *Let (X, ρ_X), (Y, ρ_Y) be two metric spaces. Then $T : X \to Y$ is called an isometry if T is bijective and $\rho(Tx_1, Tx_2) = \rho(x_1, x_2)$ for any $x_1, x_2 \in X$.*

Theorem 3.4.1 (Urysohn's Lemma)**.** *Let X be a normal topological space, and let E and F be disjoint closed subsets of X. Then there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ holds at each x in E and $f(x) = 1$ holds at each x in F.*

4 A Glimpse of Smooth Manifolds and Smooth Maps

The most important concept is to deal with manifolds locally.

4.1 Local Properties

Open Submanifold. More generally, let M be a smooth n-manifold and let $U \subseteq M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U.\} \tag{4.1.1}$$

We can show this is a naturally smooth atlas, and the structure induced by this atlas is a smooth structure for the open subset.

Smoothness of Mapping is Local, Proposition 2.6 in [4] Let $F : M \to N$ be a smooth map. (a) If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, then F is smooth. (b) Conversely, if F is smooth, then its restriction to every open subset is smooth.

Restriction of Diffeomorphism. The restriction of a diffeomorphism to an open submanifold a diffeomorphism onto its image.

The Tangent Space to an Open Manifold, Proposition 3.9 in [4]. Let M be a smooth manifold, let $U \subseteq M$ be an open subset, and let $l : U \to M$ be the inclusion map. For every $p \in U$, the differential $dl_p : T_pU \to T_pM$ is an isomorphism.

Smooth immersions and submersions behave locally, Proposition 4.1 in [4]**.** Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is surjective/injective, then p has a neighborhood U such that $F|_U$ is a submersion/immersion.

Diffeomorphism behave locally, Proposition 4.5 in [4]. Inverse Function Theorem for Manifolds. Suppose $F : M \to N$ is a smooth map. If $p \in M$ is a point such that dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

Necessary and Sufficient Conditions for Local Diffeomorphisms, Proposition 4.8 in [4].Suppose M and N are smooth manifolds and $F : M \to N$ is a map.
(a) F is a local diffeomorphism if and only if it is both a smooth immersion and a smooth submersion.
(b) If $dim\ M = dim\ N$ and F is either a smooth immersion or a smooth submersion, then it is a local diffeomorphism.

4.2 Different Ways of Defining Tangent Space

Definition 4.2.1 (Definition through Curves). *Given a point $p \in M$, the tangent space T_pM can be defined as*

$$T_pM := \{\gamma'(0) | \gamma : J \to M, \gamma(0) = p\}$$

where I contains 0.

4.3 Immersion/Submersion/Embedding and Rank Theorem

Theorem 4.3.1 (Rank Theorem). *Suppose $F : M \to N$ is a smooth map with constant rank r. For each $p \in M$ there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form,*

$$\hat{F}(x^1, ..., x^r, x^{r+1}, ...x^m) = (x^1, ..., x^r, 0..., 0)$$

One of the most fundamental theorems is Rank Theorem. The following are two important corollaries. One is for smooth immersion and another is for smooth submersion.

Proposition 4.3.2 (Local Embedding Theorem, Proposition 4.21 in [4]). *Suppose M and N are smooth manifolds without boundary, and $F : M \to N$ is a smooth map. Then F is a smooth immersion if and only if every point in M has a neighborhood $U \subseteq M$ such that $F|_U : U \to N$ is a smooth embedding.*

In the proof of theorem 4.25 in Lee's textbook, why can we say that, there exists a precompact neighborhood U of p such that $\bar{U}$ is in U_1?

> Now let $p \in M$ be arbitrary, and let U_1 be a neighborhood of p on which F is injective. There exists a precompact neighborhood U of p such that $\bar{U} \subseteq U_1$. The restriction of F to $\bar{U}$ is an injective continuous map with compact domain, so it is a topological embedding by the closed map lemma. Because any restriction of a topological embedding is again a topological embedding, $F|_U$ is both a topological embedding and a smooth immersion, hence a smooth embedding. □

My proof is:

(1) For every point p, when can find a (U, ϕ), such that U is homeomorphic to an open subset $\hat{U} \subset \mathbb{R}^n$. We can find a sufficiently small ball $B_{\hat{p},\epsilon} \subset \hat{U}$, where $\hat{p}$ is the image of p.

(2) Since homeomorphism preserves compactness, openness, and closedness, we can argue the preimage of $B_{\hat{p},\epsilon} \subset \hat{U}$ is also precompact in U.

Figure 1: A small clarification for proof of Theorem 4.25 in [4]

Definition 4.3.1 (Section and Local Section). *If $\pi : M \to N$ is any continuous map, a section of π is a continuous right inverse for π, i.e., a continuous map $\sigma : M \to N$ such that $\pi \circ \sigma = Id_N$. A local section of π is a continuous map $\sigma : U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = Id_U$.*

Proposition 4.3.3 (Local Section Theorem, Proposition 4.26 in [4]). *Suppose $F : M \to N$ is a smooth map. Then F is a smooth submersion if and only if every point of M is in the image of a smooth local section of F.*

4.4 Submanifolds

Definition 4.4.1 (Immersed/Embeded Submanifolds). *An immersed (embeded) submanifold of M is a subset $S \subseteq M$ endowed with a topology (subspace topology) with respect to which it is a topological manifold, and a smooth structure with respect to which the inclusion map $S \to M$ is a smooth immersion (embedding).*

Proposition 4.4.1 (Images of Immersions/Embeddings as Submanifolds)**.** *Suppose $F : M \to N$ is a injective smooth immersion (smooth embedding). Let $S = F(M)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of N with the property that $F : M \to S$ is a diffeomorphism onto its image*

Proposition 4.4.2 (Immersed Submanifolds Are Locally Embedded, Proposition 5.22 in [4])**.** *If M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold, then for each $p \in S$ there exists a neighborhood U of p in S that is an embedded submanifold of M.*

Local Slice Criterion, Regular Level Set Theorem are important theorems for embedded submanifolds, which can be used to prove that a manifold is an embedded submanifold.

Some Theorems from Topology. Theorem 2.2.2 and Proposition 2.3.1 are important for Corollary 5.6 and Proposition A.53 in [4]. Also, some arguments are pretty crucial for Proposition 4.22 in [4] such as closed map lemma.

5 Some Structures on Manifolds

5.1 Vector Bundles

Definition 5.1.1 (Vector Bundles)**.** *Let M be a topological space. A (real) vector bundle of rank k over M is a topological space E together with a surjective continuous map $\pi : E \to M$ satisfying several conditions. Tangent Bundle and Cotangent Bundle are specific vector bundles.*

Proposition 5.1.1 (Tangent Bundle as Vector Bundle)**.** *Let M be a smooth n-manifold with or without boundary, and let TM be its tangent bundle. With its standard projection map,*

[Projection Map]

$$\pi(p, v) = p, \text{ for } p \in M,\ v \in T_pM \tag{5.1.1}$$

its natural vector space structure on each fiber, and the topology and smooth structure,

[Smooth Structure of TM] Given any smooth chart $(U, \varphi) = (U, (x^i))$ for M, $(\pi^{-1}(U), \widetilde{\varphi})$ would be smooth charts for TM, where $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$

$$\widetilde{\varphi}(v^i \frac{\partial}{\partial x^i}|_p) = (x^1(p), ..., x^n(p), v^1, ..., v^n)$$

TM (with $\pi : TM \to M$) is a smooth vector bundle of rank n over M.

[Local Trivialization of TM] $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$

$$\Phi(v^i \frac{\partial}{\partial x^i}|_p) = (p, v^1, ..., v^n)$$

(1) $\pi_U \circ \Phi = \pi$
(2) Restriction of Φ to T_pM is an isomorphism from T_pM to $\{p\} \times \mathbb{R}^n$

Assume $(\frac{\partial}{\partial x^i}|_p)$ is the basis of T_pM, then the dual basis is $(\lambda_i|_p)$. [not that accurate, And (tangent) covector $w \in T_p^*M$ could be written as

$$w = w_i \lambda^i|p$$

where $w_i = w(\frac{\partial}{\partial x^i}|_p)$.]

Proposition 5.1.2 (Cotangent Bundle as Vector Bundle)**.** .

[Projection Map] $\pi : T^*M \to M$
$$\pi(\xi_i \lambda^i|_p) = p$$

[Smooth Structure of T^*M] Given any smooth chart $(U, \varphi) = (U, (x^i))$ for M, $(\pi^{-1}(U), \widetilde{\varphi})$ would be smooth charts for T^*M, where $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$
$$\widetilde{\varphi}(\xi_i \lambda^i|_p) = (x^1(p), ..., x^n(p), \xi_1, ..., \xi_n)$$

[Local Trivialization of T^*M] $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$
$$\Phi(\xi_i \lambda^i|p) = (p, \xi_1, ..., \xi_n)$$

Definition 5.1.4 (Local Frame for Vector Bundles)**.** *Assume we have a vector bundle $E \to M$. A local frame for E over $U \subseteq M$ is an ordered k-tuple $(\sigma_1, ..., \sigma_k)$ of linearly indepndent local sections over U that span E.*

5.2 Vector Fields and Covector Fields

Vector Fields and Covector Fields are the local section of corresponding vector bundles.

Definition 5.2.1 (Vector Fields)**.** *A rough/continuous/smooth vector field is a rough/continuous/smooth map $X : M \to TM$, usually written as $p \to X_p$, with the property that*
$$\pi \circ X = Id_M (\iff X_p \in T_pM)$$

Let M be a smooth manifold, $X \in \mathfrak{X}(M)$, $f \in C^{\infty(M)}$,
(1)$X_p = X^i(p) \dfrac{\partial}{\partial x^i}|_p$
(2)$(fX)_p = f(p)X_p, \quad X = X^i \dfrac{\partial}{\partial x^i}$

Definition 5.2.2 (Covector Fields)**.**

Let $w : M \to T^*M$ (local formulation) be a covector field, X be a vector field, $w_p : T_pM \to \mathbb{R}$
(1) $w = w_i \lambda^i$, $w_i(p) = w_p(\dfrac{\partial}{\partial x^i}|_p)$
(2) $W(X)(p) = W_p X_p, W(X) = w_i X^i$

We can easily define (Local) Frame and (Local) Coframe that corresponds to tangent bundle and cotangent bundle.

Definition 5.2.3 (Derivation)**.** *A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a derivation at p if it satisfies*
$$v(fg) = f(p)vg + g(p)vf$$
for all $f, g \in C^\infty(M)$.

A map $X : C^\infty(M) \to C^\infty(M)$ is called a derivation if it is linear over $\mathbb{R}$ and satisfies,
$$X(fg) = fXg + gXf \tag{5.2.1}$$
for all $f, g \in C^\infty(M)$.

Vector Fields as Derivations of $C^\infty(M)$. An essential property of vector fields is that they define operators on the space of smooth real-valued functions. If X is a vector field and f is a smooth real-valued function defined on an open subset $U \subseteq M$; we obtain a new function $Xf : U \to \mathbb{R}$, defined by

$$Xf(p) = X_p f$$

Clearly, it is linear and satisfies Eq 5.2.1.

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields.

Proposition 5.2.1. *Let M be a smooth manifold. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some smooth vector field X.*

5.3 Curves and Flows

Definition 5.3.1 (Integral Curve)**.** *If V is a vector field on M. An integral curve of V is a differentiable curve $\gamma : J \to M$ whose velocity at each point is equal to the value of V at that point:*

$$\gamma'(t) = V_{\gamma(t)}, \textit{for all } t \in J$$

Definition 5.3.2 (Flow)**.** *If M is a manifold, a flow domain for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set $\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t,p) \in \mathcal{D}\}$ is an open interval containing 0.*

A flow on M is a continuous map $\theta : \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, that satisfies the following group laws: for all $p \in M$,

$$\theta(0, p) = p$$

and for all $s \in \mathcal{D}^{(p)}$ and $t \in \mathcal{D}^{(\theta(s,p))}$ such that $s + t \in \mathcal{D}^{(p)}$,

$$\theta(t, \theta(s, p)) = \theta(t + s, p)$$

A maximal integral curve is one that cannot be extended to an integral curve on any larger open interval, and a maximal flow is a flow that admits no extension to a flow on a larger flow domain.

Fundamental Theorem on Flows, Theorem 9.12 in [4] We have seen that every smooth global flow gives rise to a smooth vector field whose integral curves are precisely the curves defined by the flow. Conversely, we would like to be able to say that every smooth vector field is the infinitesimal generator of a smooth global flow. However, it is easy to see that this cannot be the case. With global flow replaced by maximal flow, the Fundamental theorem on Flows indeed shows this relation.

6 Stokes Theorem

We will leave the tasks of organizing the notes of Differential Forms, Orientation, and Stokes Theorem to the future.

6.1 Differential Forms

6.2 Intergration on Manifolds

7 Useful Results for Optimal Transport

Proposition 7.0.1. *Given a probability measure $\mu \in \mathcal{P}(X)$, the following statements hold [2]:*
(1) For any $\epsilon > 0$, there is a compact set $K_\epsilon \subseteq X$ such that $\mu(K_\epsilon) \geq 1 - \epsilon$.
(2) For any $\epsilon > 0$, there is $\eta_\epsilon \in C_c(X)$ with $0 \leq \eta \leq 1$ such that $\int \eta_\epsilon \geq 1 - \epsilon$.

Proof. (1) Since X is separable, there is a countable sequence of points $(x_n)_{n\in N}$ that is dense in X. Hence, for any $r > 0$, we have

$$\bigcup \overline{B(x_n, r)} = X$$

Therefore, given $\epsilon > 0$, for any $k \in \mathbb{N}$ there exists $\eta_{k,\epsilon}$ such that

$$\mu\Big(\bigcup_{1\le n\le n_{k,\epsilon}} \overline{B(x_n, k^{-1})}\Big) \ge 1 - \frac{\epsilon}{2^k} \tag{7.0.1}$$

Let us consider the subset $K_\epsilon \subseteq X$ defined as,

$$K_\epsilon = \bigcap_{k\in\mathbb{N}} \bigcup_{1\le n\le n_{k,\epsilon}} \overline{B(x_n, k^{-1})}) \tag{7.0.2}$$

Being intersection of finite unions of closed sets, K_ϵ is closed. Also, K_ϵ is totally bounded. Since X is complete, K_ϵ is totally bounded, by Theorem 3.2.1, it is compact. And $\mu(K_\epsilon) \ge 1 - \epsilon$.
(2) Let K_ϵ be the compact set provided by the previous step (compact = bounded, closed in metric space). Since X is locally compact, there exists a compact set H_ϵ such that $K_\epsilon \subset int(H_\epsilon)$ and $H_\epsilon \subset B_R$. Thus, Urysohm Lemma 3.4.1 gaurantees the existence of continuous function η_ϵ such that $\eta_\epsilon = 1$ in K_ϵ, $\eta_\epsilon = 0$ in $B_R \backslash H_\epsilon$, and $\epsilon \in [0, 1]$. This function satisfies the requirements. □

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