Triangle-degree and triangle-distinct graphs

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Abstract

In the presented work authors introduce an algorithm for construction of triangle-distinct graphs and several theorems about their structural properties. I will provide an example implementation of their method and its complexity analysis. After that I will go through their theoretical findings and illustrate them with appropriate examples.

Notation

$G(V, E)$ - graph (V vertices set, E - edges set)
$V(G)$ - vertices of G, $E(G)$ - edges of G
$n(G) = |V(G)|$, $e(G) = |E(G)|$
$N_G(v)$ - vertices adjacent to v, $N_G[v] = N_G(v) \cup \{v\}$
$d_G(v) = |N_G(v)|$, $t_G(v)$ - triangle degree of vertex v
$G[S]$ - subgraph induced on $S \subseteq V(G)$, $\overline{G}$ - complement of graph
$\overline{S} = V(G) \setminus S$, $G - S = G[V(G) \setminus S]$
$E_G(V_1, V_2)$ - set of edges with ends in disjoint sets V_1, V_2
$e_G(V_1, V_2) = |E_G(V_1, V_2)|$

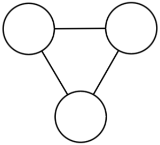
Intro

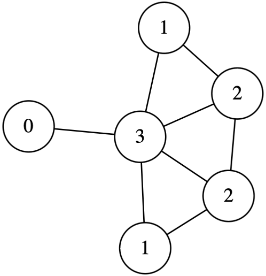
Complement Graph - graph over the same vertices to given where two distinct vertices are adjacent if and only if they are not adjacent in given graph

Triangle - complete K_3 graph

Triangle-Degree - number of triangles in G that contain given vertex

Triangle-Distinct Graph - graph where each vertex a has different triangle-degree





Triangle-Degree Formula

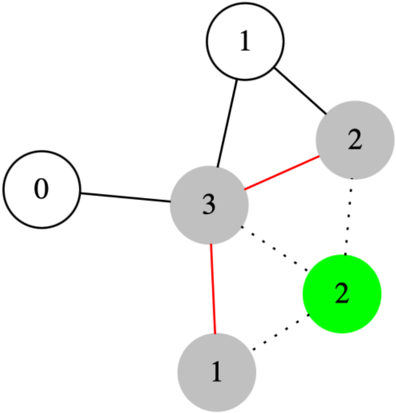
Theorem

$v \in G$
$t_G(v) = e(G[N_G(v)])$

Proof

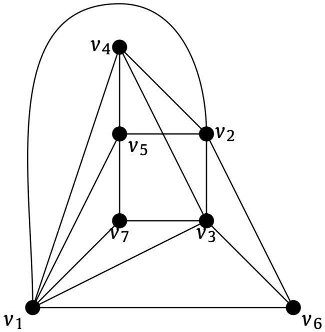
Let's consider all vertices used to build triangles with v - T.
First we notice that $\forall_{u \in T} (v, u) \in E(G)$. So $T = N_G(v)$.
Next we can notice that in order to build a triangle with v other vertices $u_1, u_2 \in T$ have to be adjacent.
Now it's easy to notice that the triangle-degree of v is the number of edges ending in it's neighbours.

Triangle-Degree Formula



Smallest Triangle-Distinct Graph - G_7

Through computer-search we can see that we need at least 7 vertices to build a triangle-distinct graph.



It has 7 vertices, 15 edges. Sequences of vertices degrees and triangle-degrees ordered by labels are: (6, 5, 5, 4, 4, 3, 3) and (9, 7, 6, 5, 4, 3, 2).

Construction of G_n

Note: Family constructed below does not cover all triangle-distinct graphs.

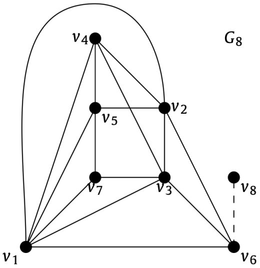
Starting with G_7 we define G_n graphs for all n.

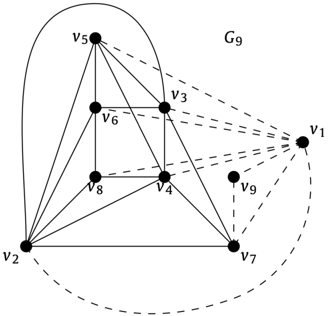
Let's suppose that G_n has been defined for some odd $n \geq 7$ with vertices labeled so that $t(v_1) > \ldots > t(v_n)$.

Let $1 \leq k \leq n$ be the smallest integer such that $d_n(v_k)$ is the minimum degree in G_n.

- To construct G_{n+1} add a vertex labeled v_{n+1} and edge (v_k, v_{n+1}) to G_n.
- To construct G_{n+2} relabel each vertex in G_{n+1}, increasing each subscript by 1. Then add a single vertex v_1 and edges (v_1, v_i) for $2 \leq i \leq n+1$.

Construction of G_n





Implementation & Complexity Analysis

Implementation presented below uses adjacency matrix for graph representation.

- $O(n^3)$ to check whether a graph is triangle-distinct
- $O(n^2)$ to calculate triangle-degree for given vertex
- $O(n^2)$ to construct G_n
- $O(n)$ neighbours lookup
- $O(n)$ to add v_1 or $v_n + 1$ vertex
- $O(1)$ to add new edge

Add New Vertex

```
def prepend_vertex(self) -> None:
    for l in self.matrix:
        l.insert(0, [])
    self.N += 1
    self.matrix.insert(0, [False]*self.N)

def append_vertex(self) -> int:
    for l in self.matrix:
        l.append([])
    self.N += 1
    self.matrix.append([False]*self.N)
    return self.N-1
```

Get Neighbours & Traingle-Degree

```
def get_neighbours(self, i: int) -> List[int]:
    return  [ j for j, x in
enumerate(self.matrix[i]) if x == True ]

def get_traingle_degree(self, i: int) -> int:
    neighbours = self.get_neighbours(i)
    sub_matrix = [ self.matrix[x] for x in neighbours ]
    for i, l in enumerate(sub_matrix):
        sub_matrix[i] = [ l[x] for x in neighbours ]
    sub_graph = Graph(sub_matrix)
    return sum(map(sub_graph.get_degree,
list(range(len(sub_graph)))))//2
```

Triangle-Distinct Check

```python
def check_traingle_distinct(graph: Graph) -> bool:
    R = True
    tdegs = set()
    for i in range(len(graph)):
        tdeg = graph.get_traingle_degree(i)
        if tdeg in tdegs:
            R = False
            break
        tdegs.add(tdeg)
    return R
```

Construction

```
def construct(N: int) -> Optional[Graph]:
    if N <= 7:
        return Graph.G7() if N == 7 else None
    else:
        graph = Graph.G7()
        k = 5
        for i in range(8, N+1):
            if i % 2 == 0:  # G_n+1
                j = graph.append_vertex()
                graph.add_edge(k, j)
                k = j
            else:   # G_n+2
                graph.prepend_vertex()
                for j in range(1, len(graph)):
                    graph.add_edge(0, j)
        return graph
```

G_n are Triangle-Distinct

Theorem

Each G_n where $n \geq 7$ is triangle-distinct, $t_n(v_1) > \ldots > t_n(v_n)$ and $d_n(v_1) \geq \ldots d_n(v_n)$. If n is odd, $t_n(v_n) > 0$, $d_n(v_n) > 0$, $e(G_n) = t_n(v_1) + dn(v_1)$, and G_n is not regular.

Proof

First, we can quickly verify above properties for G_7.

For induction, suppose that for some odd $n \geq 7$, G_n has all of the properties stated in the theorem. We will prove that G_{n+1} and $G_n + 2$ both satisfy the conditions in the theorem.

-> Based on the construction of G_{n+1}, we have $t_{n+1}(v_{n+1}) = 0$, $d_{n+1}(v_{n+1}) = 1$, $t_{n+1}(v_k) = t_n(v_k)$, and $d_{n+1}(v_k) = d_n(v_k) + 1$. Also for all $1 \leq i \leq n, i \neq k$ $t_{n+1}(v_i) = t_n(v_i)$ and $d_{n+1}(v_i) = d_n(v_i)$.

G_n are Triangle-Distinct

Proof

Recall that $t_n(v_n) > 0$ which gives us the following:
$t_{n+1}(v_1) > t_{n+1}(v_2) > \ldots > t_{n+1}(v_{n+1}) = 0$.
Based on choice of k and $d_n(v_n) > 0$ we have:
$d_{n+1}(v_1) \geq d_{n+1}(v_2) \geq \ldots \geq d_{n+1}(v_{n+1}) = 1$.
-> Now let's consider G_{n+2}. Since G_n is not regular $k \neq 1$ so $t_{n+1}(v_1) = t_n(v_1)$ and $d_{n+1}(v_1) = d_n(v_1)$. Observe that for each $2 \leq i \leq n+2$ we have $t_{n+2}(v_i) = t_{n+1}(v_{i-1}) + d_{n+1}(v_{i-1})$. After adding new v_1 each vertex forms new triangles with it's neighbours. Substituting above into our result from G_{n+1} about triangle-degrees sequence we get: $t_{n+2}(v_2) > t_{n+2}(v_3) > \ldots > t_{n+2}(v_{n+2}) = 1$.
Because $t_{n+1}(v_1) = e(G_{n+1}) = e(G_n) + 1$ and
$t_{n+2}(v_2) = t_{n+1}(v_1) + d_{n+1}(v_1) = t_n(v_1) + d_n(v_1) = e(G_n)$ we get
$t_{n+2}(v_1) > t_{n+2}(v_2)$ which completes our sequence.

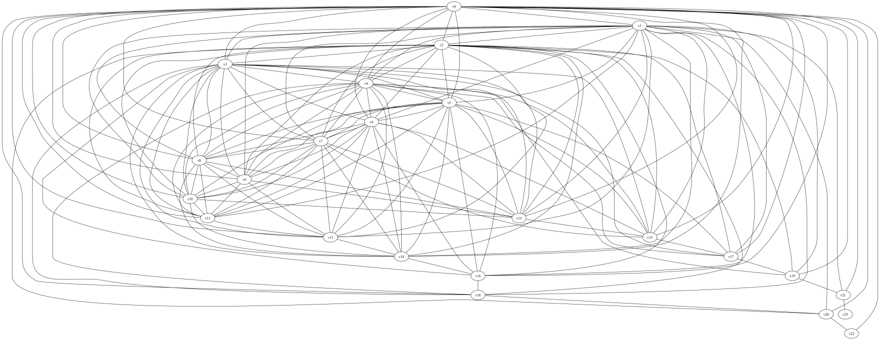
G_n are Triangle-Distinct

Proof

For sequence of degrees we can notice that $d_{n+2}(v_1) = n + 1$ and for the rest $d_{n+2}(v_i) = d_{n+1}(v_i) + 1$. So:
$d_{n+2}(v_1) \geq d_{n+2}(v_2) \geq \ldots \geq d_{n+2}(v_{n+2}) = 2$.
Also notice how $t_{n+2}(v_1) + d_{n+2}(v_1) = e(G_{n+1}) + n + 1 = e(G_{n+2})$ which was another property we desired. To show that G_{n+2} is not regular we can use $t_{n+2}(v_1) > t_{n+2}(v_{n+2}) = 1$ and the fact that vertex with triangle degree of at least 2 has degree of at least 3. Which concludes that v_1 and v_{n+2} have different degrees.

Based on proof above it's easy to see that we can get k while constructing G_n by updating it to $n + 1$ in each even step.

G_{24}



v0
v1
v2
v3
v4
v5
v6
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v14
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v16
v17
v18
v19
v20
v21
v22
v23

Structural Properties of Triangle-Distinct Graphs

Theorem (Lemma 1)

Let G be a graph on n vertices and $u \in V(G)$. If G has two distinct vertices u and v with the same degree and

$$\mathrm{t}_{\bar{G}}(u) + e_{\bar{G}}\left(N_G(u), \overline{N_G[u]}\right) = \mathrm{t}_{\bar{G}}(v) + e_{\bar{G}}\left(N_G(v), \overline{N_G[v]}\right)$$

then G is not triangle-distinct.

Proof

Note that $\mathrm{t}_{\bar{G}}(u)$ counts the number of edges xy in $\bar{G}$ with $x, y \notin N_G(u)$. $\binom{n-1-d_G(u)}{2} = |\{xy \notin E(G) : x, y \notin N_G[u]\}| + |\{xy \in E(G) : x, y \notin N_G[u]\}| = \mathrm{t}_{\bar{G}}(u) + e(G - N_G[u])$

Structural Properties of Triangle-Distinct Graphs

Proof

Let's write $e(G)$ as sum of: edges between u and it's neighbourhood, edges between neighbours of u, edges between neighbourhood of u and rest of graph (excluding u) and edges between verticies not adjacent to u.

$$e_G\left(u, N_G(u)\right)+e\left(G\left[N_G(u)\right]\right)+e_G\left(N_G(u), \overline{N_G[u]}\right)+e\left(G-N_G[u]\right)$$

$$=\mathrm{d}_G(u)+\mathrm{t}_G(u)+e_G\left(N_G(u), \overline{N_G[u]}\right)+\left(\binom{n-1-d_G(u)}{2}-\mathrm{t}_{\bar{G}}(u)\right)$$

Solving the above equation for $\mathrm{t}_G(u)$, we find:

$$\mathrm{t}_G(u)=e(G)-d_G(u)-e_G\left(N_G(u), \overline{N_G[u]}\right)-\binom{n-1-d_G(u)}{2}+\mathrm{t}_{\bar{G}}(u)$$

Based on the construction of a graph complement, $n-1-d_G(u)=d_{\bar{G}}(u)$. Now let's consider a bipartite graph with one set of size $d_G(u)$ and the other of size $d_{\bar{G}}(u)$:

Structural Properties of Triangle-Distinct Graphs

Proof

$d_{\bar{G}}(u)d_G(u) = e_G\left(N_G(u), \overline{N_G[u]}\right) + e_{\bar{G}}\left(N_G(u), \overline{N_G[u]}\right)$

The intuition is as follows: number of edges in complete bipartite graph - size of one set times size of another, since they correlate with degree of u in G and $\overline{G}$, so number of adjacent and not adjacent nodes in G we can sum edges between adjacent and not adjacent vertices. Substituting, we get: $(*)\mathrm{t}_G(u) = e(G) - d_G(u) - \left(d_{\bar{G}}(u)d_G(u) - e_{\bar{G}}\left(N_G(u), \overline{N_G[u]}\right)\right) - \binom{d_{\bar{G}}(u)}{2} + \mathrm{t}_{\bar{G}}(u) = e(G) - (1 + d_{\bar{G}}(u))\, d_G(u) - \binom{d_{\bar{G}}(u)}{2} + \mathrm{t}_{\bar{G}}(u) + e_{\bar{G}}\left(N_G(u), \overline{N_G[u]}\right)$

Now let's consider two distinct vertices u, v with the same degree and fulfilling assumptions of this lemma:

$\mathrm{t}_{\bar{G}}(u) + e_{\bar{G}}\left(N_G(u), \overline{N_G[u]}\right) = \mathrm{t}_{\bar{G}}(v) + e_{\bar{G}}\left(N_G(v), \overline{N_G[v]}\right)$

Structural Properties of Triangle-Distinct Graphs

Proof

Since $d_G(u) = d_G(v)$ and consequently $d_{\bar{G}}(u) = d_{\bar{G}}(v)$, then:

$e(G) - (1 + d_{\bar{G}}(u))\, d_G(u) - \binom{d_{\bar{G}}(u)}{2} =$

$e(G) - (1 + d_{\bar{G}}(v))\, d_G(v) - \binom{d_{\bar{G}}(v)}{2}$

And finally based on (*) we get that they have the same triangle-degree.

Structural Properties of Triangle-Distinct Graphs

Theorem

Let G be a graph of order $n \geq 3$. Then we have the following:

- *(a)If $\Delta(G) \leq \sqrt{2n}$, then G has distinct vertices $u, v \in V(G)$ with $\mathrm{t}_G(u) = \mathrm{t}_G(v)$.*
- *(b) If $\delta(G) > n - 1 - (2n/3)^{1/3}$, then G has two distinct vertices $u, v \in V(G)$ with $d(u) = d(v)$ and $\mathrm{t}_G(u) = \mathrm{t}_G(v)$.*
- *(c) If G is d-regular and triangle-distinct, then $\sqrt{2n} < d \leq n - \sqrt{2n/3}$*

Proof

For (a), note that for each vertex $v \in V(G)$, there are at most $\binom{\sqrt{2n}}{2}$ edges in $G[N_G(v)]$. Since $\binom{\sqrt{2n}}{2} < n - 1$, the triangle-degree for each vertex is between 0 and $n - 2$.

Structural Properties of Triangle-Distinct Graphs

Proof

However, the graph G has n vertices, the Pigeonhole Principle guarantees us two vertices with the same triangle-degree.
For (b), since $\delta(G) > n - 1 - (2n/3)^{1/3}$, we have $\Delta(\bar{G}) < (2n/3)^{1/3}$. Because the degree in $\bar{G}$ of each vertex is in $\left\{0, 1, \ldots, (2n/3)^{1/3} - 1\right\}$, there are at least $\left(\frac{3}{2}\right)^{1/3} n^{2/3}$ vertices with the same degree in $\bar{G}$. Let v be an arbitrary one of these vertices. Observe:
$$\mathrm{t}_{\bar{G}}(v) + e_{\bar{G}}\left(N_G(v), \overline{N_G(v)}\right) \leq \binom{(2n/3)^{1/3}}{2} + (2n/3)^{2/3} < \left(\frac{3}{2}\right)^{1/3} n^{2/3}$$
By the Pigeonhole Principle, we can find two distinct vertices which satisfy the hypotheses in Lemma 1, so G is not triangle-distinct.

Structural Properties of Triangle-Distinct Graphs

Proof

For (c), the lower bound follows directly from (a). We now show the upper bound. Suppose for contradiction that $d > n - \sqrt{2n/3}$. Every vertex in $\bar{G}$ has degree $n - 1 - d < \sqrt{2n/3}$. Thus, for each vertex v, we have:

$$\mathrm{t}_{\bar{G}}(v) + e_{\bar{G}}\left(N_G(v), \overline{N_G(v)}\right) < \binom{\sqrt{2n/3}}{2} + (\sqrt{2n/3})^2 < n$$

By the Pigeonhole Principle, there are two distinct vertices which satisfy the conditions in Lemma 1, so G is not triangle-distinct, a contradiction.

Structural Properties of Triangle-Distinct Graphs

Now we turn our attention from triangle-degrees to the number of edges in a triangle-distinct graph. Following theorem is introduced.

Theorem

Let G be a triangle-distinct graph of order n. Then

$$\frac{\sqrt{2}}{3} n^{3/2}(1 - o(1)) < e(G) \leq \binom{n}{2} - \omega(n)$$

Since a planar graph can have at most $3n - 6$ edges as a consequence of above theorem no triangle-distinct graph is planar.

Open problems

- Does there exist a regular graph that is triangle-distinct?
- Are $\frac{\sqrt{2}}{3} n^{3/2}(1 - o(1)) < e(G) \leq \binom{n}{2} - \omega(n)$ tight? If not how can we sharpen them?

Bibliography

Zhanar Berikkyzy, Beth Bjorkman, Heather Smith Blake, Sogol Jahanbekam, Lauren Keough, Kevin Moss, Danny Rorabaugh, Songling Shan: Triangle-degree and triangle-distinct graphs