Computer Assigment

1. A simple model of a vehicle moving in one dimension is given by

$$\begin{bmatrix} s_1(t+1) \\ s_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0.95 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \qquad t = 0, 1, 2, \ldots$$

$s_1(t)$ is the position at time t, $s_2(t)$ is the velocity at time t, and $u(t)$ is the actuator input. Roughly speaking, the equations state that the actuator input affects the velocity, which in turn affects the position. The coefficient 0.95 means that the velocity decays by 5% in one sample period (for example, because of friction), if no actuator signal is applied. We assume that the vehicle is initially at rest at position $0: \ s_1(0) = s_2(0) = 0$. We will solve the minimum energy optimal control problem: for a given time horizon N, choose inputs $u(0), \ldots, u(N-1)$ so as to minimize the total energy consumed, which we assume is given by

$$E = \sum_{t=0}^{N-1} u(t)^2$$

In addition, the input sequence must satisfy the constraint $s_1(N) = 10, s_2(N) = 0$. Your task therefore is to bring the vehicle to the final position $s_1(N) = 10$ with final velocity $s_2(N) = 0$, as efficiently as possible.

a) Formulate the minimum energy optimal control problem as a least norm problem

$$\text{minimize} \|x\|^2 \qquad \text{subject to} Cx = d.$$

Clearly state what the variables x, and the problem data C and d are.

b) Solve the problem for $N = 30$. Plot the optimal $u(t)$, the resulting position $s_1(t)$, and velocity $s_2(t)$.

c) Solve the problem for $N = 2, 3, ..., 29$. For each N calculate the energy E consumed by the optimal input sequence. Plot E versus N. (The plot looks best if you use a logarithmic scale for E)

d) Suppose we allow the final position to deviate from 10. However, if $s_1(N) \neq 10$, we have to pay a penalty, equal to $(s_1(N) - 10)^2$. The problem is to find the input sequence that minimizes the sum of the energy E consumed by the input and the terminal position penalty,

$$\sum_{t=0}^{N-1} u(t)^2 + (s_1(N) - 10)^2,$$

subject to the constraint $s_2(N) = 0$.

Formulate this problem as a least norm problem, and solve it for $N = 30$. Plot the optimal input signals $u(t)$, the resulting position $s_1(t)$ and the resulting velocity $s_2(t)$.

2. Two vehicles are moving along a straight line. For the first vehicle we use the same model as in the previous question i.e

$$\begin{bmatrix} s_1(t+1) \\ s_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0.95 \end{bmatrix} \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t), \qquad t = 0, 1, 2, \dots$$

assuming that the vehicle is initially at rest at position $0 : s_1(0) = s_2(0) = 0$. The model for the second vehicle is

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} v(t), \qquad t = 0, 1, 2, \dots$$

$p_1(t)$ is the position at time $t, p_2(t)$ is the velocity at time t, and $v(t)$ is the actuator input. We assume that the second vehicle is initially at rest at position $1 : p_1(0) = 1, p_2(0) = 0$. Formulate the following problem as a least norm problem, and solve it . Find the control inputs $u(0), u(1), \dots, u(19)$ and $v(0), v(1), \dots, v(19)$ that minimize the total energy

$$\sum_{t=0}^{19} u(t)^2 + \sum_{t=0}^{19} v(t)^2$$

and satisfy the following three conditions:

$$s_1(20) = p_1(20) \quad s_2(20) = 0, \quad p_2(20) = 0.$$

In other words, at time t = 20 the two vehicles must have velocity zero, and be at the same position. (The final position itself is not specied, i.e., you are free to choose any value as long as s1(20) = p1(20).) Plot the positions $s_1(t)$ and $p_1(t)$ of the two vehicles, for $t = 1, 2, \dots, 20$.