Laplace Transform

CHAPTER OUTLINE

15.1 INTRODUCTION

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

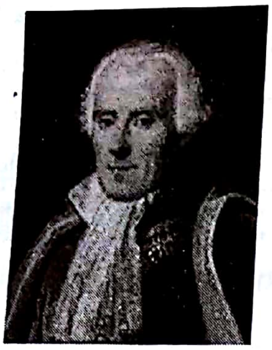
15.2 LAPLACE TRANSFORM

If $f(t)$ is a function of t defined for all $t \geq 0$ then $\int_0^\infty e^{-st} f(t)\, dt$ is defined as Laplace transform of $f(t)$, provided the integral exists and is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

The integral is a function of the parameter s and is denoted by $F(s)$ or $\bar{f}(s)$ or $\phi(s)$.

HISTORICAL DATA



Pierre-Simon, marquis de Laplace (1749–1827) was an influential French scholar whose work was important to the development of mathematics, statistics, physics and astronomy. He summarised and extended the work of his predecessors in his five-volume Mécanique Céleste (Celestial Mechanics). This work translated the geometric study of classical mechanics to one based on calculus, opening up a broader range of problems. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace. He died in Paris in 1827. His brain was removed by his physician, François Magendie, and kept for many years, eventually being displayed in a roving anatomical museum in Britain.

Sufficient Conditions for Existence of Laplace Transform

The Laplace transform of a function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous, i.e. $f(t)$ is continuous in every sub-interval and $f(t)$ has finite limits at the end points of each sub-interval.

(ii) $f(t)$ is of exponential order of α, i.e. there exists M, α such that $|f(t)| \le Me^{\alpha t}$, for all $t \ge 0$. In other words,

$$\lim_{t\to\infty}\left\{e^{-\alpha t} f(t)\right\} = \text{finite quantity}$$

e.g.
- $L\{\tan t\}$ does not exist since $\tan t$ is not piece wise continuous.
- $L\{e^{t^2}\}$ does not exist since e^{t^2} is not of any exponential order.

15.3 LAPLACE TRANSFORM OF ELEMENTARY FUNCTIONS

1. $f(t) = k$, where k is a constant

Proof $L\{k\} = \int_0^\infty e^{-st} k\,dt = k\left|\dfrac{e^{-st}}{-s}\right|_0^\infty = \dfrac{k}{s}$

2. $f(t) = t^n$

Proof $L\{t^n\} = \int_0^\infty e^{-st}\, t^n dt$

Putting $st = x$, $dt = \dfrac{dx}{s}$

$$L\{t^n\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \boxed{\frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\, dx}$$

$$= \frac{\overline{|n+1}}{s^{n+1}} \qquad s>0,\ n+1>0$$

Solve simply and analyse will will get LT

If n is a positive integer, $\overline{|n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. $f(t) = e^{at}$

Proof $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left|\dfrac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \dfrac{1}{s-a}$

4. $f(t) = \sin at$

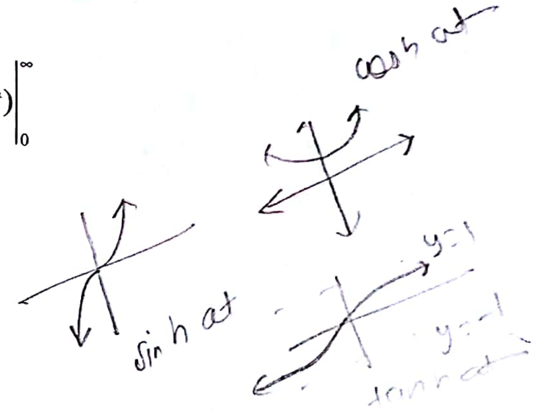
Proof $L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt = \left| \frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at) \right|_0^\infty$

$$= \left[0 - \frac{1}{s^2+a^2}(-a)\right] = \frac{a}{s^2+a^2}$$

5. $f(t) = \cos at$

Proof $L\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt = \left| \frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at) \right|_0^\infty$

$$= \left[0 - \frac{1}{s^2+a^2}(-s)\right] = \frac{s}{s^2+a^2}$$



6. $f(t) = \sinh at$

Proof $L\{\sinh at\} = \int_0^\infty e^{-st} \sinh at \, dt = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right) dt$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt\right] = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

7. $f(t) = \cosh at$

Proof $L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at \, dt = \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t} dt + \int_0^\infty e^{-(s+a)t} dt\right] = \frac{1}{2}\left|\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

Table of Laplace Transform

Sr. No.	$f(t)$	$F(s)$
1	k	$\frac{k}{s}$
2	t	$\frac{1}{s^2}$
3	t^n	$\frac{\overline{\lvert n+1}}{s^{n+1}}$
4	e^{at}	$\frac{1}{s-a}$

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Sr. No.	$f(t)$	$F(s)$
5	$\sin at$	$\frac{a}{s^2+a^2}$
6	$\cos at$	$\frac{s}{s^2+a^2}$
7	$\sinh at$	$\frac{a}{s^2-a^2}$
8	$\cosh at$	$\frac{s}{s^2-a^2}$
9	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$
10	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
11	$e^{-bt}\sinh at$	$\frac{a}{(s+b)^2-a^2}$
12	$e^{-bt}\cosh at$	$\frac{s+b}{(s+b)^2-a^2}$

EXAMPLE 15.1

Find the Laplace transform of $f(t) = 1, \quad 0<t<1$
$\quad = e^t, \quad 1<t<4$
$\quad = 0, \quad t>4$

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}\cdot 1\,dt + \int_1^4 e^{-st}e^t\,dt + \int_4^\infty e^{-st}\cdot 0\,dt$$

$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left|\frac{e^{t(1-s)}}{1-s}\right|_1^4 + 0 = \frac{e^{-s}-1}{-s} + \frac{e^{4(1-s)}-e^{(1-s)}}{1-s} = \frac{1-e^{-s}}{s} + \frac{e^{(1-s)}-e^{4(1-s)}}{s-1}$$

EXAMPLE 15.2

Find the Laplace transform of $f(t) = \cos\left(t-\frac{2\pi}{3}\right), \quad t>\frac{2\pi}{3}$
$\quad = 0, \quad t<\frac{2\pi}{3}$

Solution: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{\frac{2\pi}{3}} e^{-st}\cdot 0\,dt + \int_{\frac{2\pi}{3}}^\infty e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt = \int_{\frac{2\pi}{3}}^\infty e^{-st}\cos\left(t-\frac{2\pi}{3}\right)dt$

Putting $t-\frac{2\pi}{3} = x, \; dt = dx$

When $\quad t = \frac{2\pi}{3}, \qquad x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(t)\} = \int_0^\infty e^{-s\left(x+\frac{2\pi}{3}\right)} \cos x \, dx = e^{-\frac{2\pi}{3}s} \int_0^\infty e^{-xs} \cos x \, dx$$

$$= e^{-\frac{2\pi}{3}s} \left| \frac{e^{-xs}}{s^2+1}(-s\cos x + \sin x) \right|_0^\infty = \frac{e^{-\frac{2\pi}{3}s}}{s^2+1}(0+s) = \frac{se^{-\frac{2\pi}{3}s}}{s^2+1}$$

EXAMPLE 15.3 *Find the Laplace transform of* $f(t) = t, \quad 0 < t < \frac{1}{2}$

$= t - 1; \quad \frac{1}{2} < t < 1$

$= 0, \quad t > 1$

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt = \int_0^{\frac{1}{2}} e^{-st} t \, dt + \int_{\frac{1}{2}}^1 e^{-st}(t-1) \, dt + \int_1^\infty e^{-st} \cdot 0 \, dt$$

$$= \left| \frac{e^{-st}}{-s} t - \frac{e^{-st}}{s^2} \cdot 1 \right|_0^{\frac{1}{2}} + \left| \frac{e^{-st}}{-s}(t-1) - \frac{e^{-st}}{s^2} \cdot 1 \right|_{\frac{1}{2}}^1 + 0$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{2s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{e^{-s}}{s^2} - e^{-\frac{s}{2}}\left(\frac{1}{2s} - \frac{1}{s^2}\right)$$

$$= e^{-\frac{s}{2}}\left(-\frac{1}{s}\right) + \frac{1}{s^2} - \frac{e^{-s}}{s^2} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-\frac{s}{2}}}{s}$$

EXAMPLE 15.4 *Find the Laplace transform of* $\frac{1}{\sqrt{t}}$.

Solution:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^\infty e^{-st} \frac{1}{\sqrt{t}} dt = \int_0^\infty e^{-st} t^{-\frac{1}{2}} dt$$

Putting $st = x$, $dt = \frac{dx}{s}$

When $t = 0$, $x = 0$

When $t \to \infty$, $x \to \infty$

$$L\{f(t)\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^{-\frac{1}{2}} \frac{dx}{s} = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-\frac{1}{2}} dx = \frac{1}{\sqrt{s}} \left\lceil -\frac{1}{2}+1 \right. = \frac{1}{\sqrt{s}} \left\lceil \frac{1}{2} \right.$$

$$= \sqrt{\frac{\pi}{s}} \qquad \left[\because \left\lceil \frac{1}{2} \right. = \sqrt{\pi}\right]$$

EXERCISE 15.1

Find the Laplace transforms of the following functions:

1. $f(t) = t, \quad 0 < t < 3$
   $= 6, \quad t > 3$
   $\left[\textbf{Ans.}: \frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-3s}\right]$

2. $f(t) = t^2, \quad 0 < t < 1$
   $= 1, \quad t > 1$
   $\left[\textbf{Ans.}: \frac{1}{s}(1-e^{-s}) - \frac{2e^{-s}}{s^2} + \frac{2}{s^3}(1-e^{-s})\right]$

3. $f(t) = (t-a)^3, \quad t > a$
   $= 0, \quad t < a$
   $\left[\textbf{Ans.}: \frac{6}{s^4}e^{-as}\right]$

4. $f(t) = 0, \quad 0 \le t \le 1$
   $= t, \quad 1 < t < 2$
   $= 0, \quad t > 2$
   $\left[\textbf{Ans.}: \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} - \left(\frac{1}{s^2} + \frac{2}{s}\right)e^{-2s}\right]$

5. $f(t) = t^2, \quad 0 < t < 2$
   $= t - 1, \quad 2 < t < 3$
   $= 7, \quad t > 3$
   $\left[\textbf{Ans.}: \frac{2}{s^3} - \frac{e^{-2s}}{s^3}(2 + 3s + 3s^2) + \frac{e^{-3s}}{s^2}(5s - 1)\right]$

6. $f(t) = e^t, \quad 0 < t < 1$
   $= 0, \quad t > 1$
   $\left[\textbf{Ans.}: \frac{1}{1-s}(e^{1-s} - 1)\right]$

7. $f(t) = \sin 2t, \quad 0 < t < \pi$
   $= 0, \quad t > \pi$
   $\left[\textbf{Ans.}: \frac{2(1-e^{-\pi s})}{s^2 + 4}\right]$

15.4 BASIC PROPERTIES OF LAPLACE TRANSFORM

15.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{a f_1(t) + b f_2(t)\} = a\,F_1(s) + b\,F_2(s)$, where a and b are constants.

Proof

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^\infty e^{-st}\{af_1(t) + bf_2(t)\}\,dt$$

$$= a\int_0^\infty e^{-st} f_1(t)\,dt + b\int_0^\infty e^{-st} f_2(t)\,dt$$

$$= a\,F_1(s) + b\,F_2(s)$$

EXAMPLE 15.5 *Find the Laplace transform of* $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$.

Solution:

$$L\left\{\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)^3\right\} = L\left\{t^{\frac{3}{2}} - 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}} - t^{-\frac{3}{2}}\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} - 3L\left\{t^{\frac{1}{2}}\right\} + 3L\left\{t^{-\frac{1}{2}}\right\} - L\left\{t^{-\frac{3}{2}}\right\}$$

$$= \frac{\left\lceil\frac{5}{2}\right.}{s^{\frac{5}{2}}} - \frac{3\left\lceil\frac{3}{2}\right.}{s^{\frac{3}{2}}} + \frac{3\left\lceil\frac{1}{2}\right.}{s^{\frac{1}{2}}} - \frac{\left\lceil-\frac{1}{2}\right.}{s^{-\frac{1}{2}}}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.}{s^{\frac{5}{2}}} - \frac{3\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.}{s^{\frac{3}{2}}} + \frac{3\left\lceil\frac{1}{2}\right.}{s^{\frac{1}{2}}} - \frac{\left\lceil\frac{1}{2}\right.}{-\frac{1}{2}s^{-\frac{1}{2}}} \qquad \left[\because \lceil n+1 = n\lceil n,\quad \lceil n = \frac{\lceil n+1}{n}\right]$$

$$= \sqrt{\frac{\pi}{s}}\left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s\right)$$

EXAMPLE 15.6 *Find the Laplace transform of* $\cos(\omega t + b)$.

Solution:

$$L\{\cos(\omega t + b)\} = L\{\cos\omega t\cos b - \sin\omega t\sin b\} = \cos b\,L\{\cos\omega t\} - \sin b\,L\{\sin\omega t\}$$

$$= \cos b\cdot\frac{s}{s^2+\omega^2} - \sin b\cdot\frac{\omega}{s^2+\omega^2}$$

EXAMPLE 15.7 *Find the Laplace transform of* $\sin^5 t$.

Solution:

$$L\{\sin^5 t\} = L\left\{\left(\frac{e^{it}-e^{-it}}{2i}\right)^5\right\}$$

$$= L\left\{\frac{1}{(2i)^5}\left(e^{i5t} - 5e^{i4t}e^{-it} + 10e^{i3t}e^{-i2t} - 10e^{i2t}e^{-i3t} + 5e^{it}e^{-i4t} - e^{-i5t}\right)\right\}$$

$$= \frac{1}{32i}L\left\{(e^{i5t}-e^{-i5t}) - 5(e^{i3t}-e^{-i3t}) + 10(e^{it}-e^{-it})\right\}$$

$$= \frac{1}{16}L\{\sin 5t - 5\sin 3t + 10\sin t\} = \frac{1}{16}\left[L\{\sin 5t\} - 5L\{\sin 3t\} + 10L\{\sin t\}\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+25} - \frac{15}{s^2+9} + \frac{10}{s^2+1}\right] = \frac{5}{16(s^2+25)} - \frac{15}{16(s^2+9)} + \frac{5}{8(s^2+1)}$$

EXAMPLE 15.8 *Find the Laplace transform of* $\frac{\cos\sqrt{t}}{\sqrt{t}}$.

Solution:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-\frac{1}{2}} - \frac{t^{\frac{1}{2}}}{2!} + \frac{t^{\frac{3}{2}}}{4!} - \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\left\{t^{-\frac{1}{2}}\right\} - \frac{1}{2!}L\left\{t^{\frac{1}{2}}\right\} + \frac{1}{4!}L\left\{t^{\frac{3}{2}}\right\} - \cdots$$

$$= \frac{\left|\frac{1}{2}\right.}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\left|\frac{3}{2}\right.}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\left|\frac{5}{2}\right.}{s^{\frac{5}{2}}} - \cdots = \frac{\left|\frac{1}{2}\right.}{s^{\frac{1}{2}}} - \frac{1}{2!}\frac{\frac{1}{2}\left|\frac{1}{2}\right.}{s^{\frac{3}{2}}} + \frac{1}{4!}\frac{\frac{3}{2}\cdot\frac{1}{2}\left|\frac{1}{2}\right.}{s^{\frac{5}{2}}} - \cdots$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \cdots\right] = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{(4s)}}$$

EXERCISE 15.2

Find the Laplace transforms of the following functions:

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$

$$\left[\textbf{Ans.}: \frac{1}{s-2} + \frac{24}{s^4} + \frac{3(s-2)}{s^4+9}\right]$$

2. $e^{2t} + 4t^3 - \sin 2t\cos 3t$

$$\left[\textbf{Ans.}: \frac{1}{s-2} + \frac{24}{s^4} - \frac{5}{2}\cdot\frac{1}{s^2+25} + \frac{1}{2(s^2+1)}\right]$$

3. $3t^2 + e^{-t} + \sin^3 2t$

$$\left[\textbf{Ans.}: \frac{6}{s^3} + \frac{1}{s+1} + \frac{3}{2}\cdot\frac{1}{s^2+4} - \frac{3}{2}\cdot\frac{1}{s^2+36}\right]$$

4. $(t^2+a)^2$

$$\left[\textbf{Ans.}: \frac{a^2s^4 + 4as^2 + 24}{s^5}\right]$$

5. $\sin(\omega t + \alpha)$

$$\left[\textbf{Ans.}: \cos\alpha\cdot\frac{\omega}{s^2+\omega^2} + \sin\alpha\cdot\frac{s}{s^2+\omega^2}\right]$$

6. $\sinh^3 3t$

$$\left[\textbf{Ans.}: \frac{162}{(s^2-81)(s^2-9)}\right]$$

7. $\frac{1+2t}{\sqrt{t}}$

$$\left[\textbf{Ans.}: \sqrt{\frac{\pi}{s}}\left(1+\frac{1}{s}\right)\right]$$

8. $\sin(t+\alpha)\cos(t-\alpha)$

$$\left[\textbf{Ans.}: \frac{1}{s^2+4} + \frac{\sin 2\alpha}{s}\right]$$

15.4.2 Change of Scale

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$.

Proof

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt$$

$$L\{f(at)\}=\int_0^\infty e^{-st}f(at)dt$$

Putting $at=x$, $dt=\dfrac{dx}{a}$

When $t=0$, $x=0$

When $t\to\infty$, $x\to\infty$

$$L\{f(at)\}=\int_0^\infty e^{-s\left(\frac{x}{a}\right)}f(x)\frac{dx}{a}=\frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x}f(x)dx=\frac{1}{a}F\left(\frac{s}{a}\right)$$

EXAMPLE 15.9 *If* $L\{f(t)\}=\log\left(\dfrac{s+3}{s+1}\right)$, *find* $L\{f(2t)\}$.

Solution:

$$L\{f(t)\}=\log\left(\frac{s+3}{s+1}\right)$$

By change-of-scale property,

$$L\{f(2t)\}=\frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right)=\frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$$

EXAMPLE 15.10 $L\{\sin\sqrt{t}\}=\dfrac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$, *find* $L\{\sin 2\sqrt{t}\}$.

Solution:

$$L\{\sin\sqrt{t}\}=\frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{(4s)}}$$

By change-of-scale property,

$$L\{\sin 2\sqrt{t}\}=L\{\sin\sqrt{4t}\}=\frac{1}{4}\frac{\sqrt{\pi}}{2\cdot\frac{s}{4}\sqrt{\frac{s}{4}}}e^{-\frac{1}{4\left(\frac{s}{4}\right)}}=\frac{\sqrt{\pi}}{2s\sqrt{s}}e^{-\frac{1}{s}}$$

EXERCISE 15.3

1. If $L\{f(t)\}=\dfrac{8(s-3)}{(s^2-6s+25)^2}$, find $L\{f(2t)\}$.

   $\left[\textbf{Ans.: } \dfrac{1}{4}\dfrac{(s-6)}{(s^2-12s+100)^2}\right]$

2. If $L\{f(t)\}=\dfrac{2}{s^3}e^{-s}$, find $L\{f(3t)\}$.

   $\left[\textbf{Ans.: } \dfrac{18}{s^3}e^{-\frac{s}{3}}\right]$

15.4.3 First Shifting Theorem

If $L\{f(t)\}=F(s)$ then $L\{e^{-at}f(t)\}=F(s+a)$.

Proof

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)\mathrm{d}t$$

$$L\{e^{-at}f(t)\}=\int_0^\infty e^{-st}e^{-at}f(t)\mathrm{d}t=\int_0^\infty e^{-(s+a)t}f(t)\mathrm{d}t=F(s+a)$$

EXAMPLE 15.11 *Find the Laplace transform of* $e^t\left(1+\sqrt{t}\right)^4$.

Solution:

$$L\left\{\left(1+\sqrt{t}\right)^4\right\}=L\left\{1+4\sqrt{t}+6\left(\sqrt{t}\right)^2+4\left(\sqrt{t}\right)^3+\left(\sqrt{t}\right)^4\right\}$$

$$=L\left\{1+4t^{\frac{1}{2}}+6t+4t^{\frac{3}{2}}+t^2\right\}=\frac{1}{s}+\frac{4\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}}+\frac{6\Gamma 2}{s^2}+\frac{4\Gamma\left(\frac{5}{2}\right)}{s^{\frac{5}{2}}}+\frac{\Gamma 3}{s^3}$$

$$=\frac{1}{s}+\frac{4\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}+\frac{6}{s^2}+\frac{4\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}+\frac{2}{s^3}=\frac{1}{s}+\frac{2\sqrt{\pi}}{s^{\frac{3}{2}}}+\frac{6}{s^2}+\frac{3\sqrt{\pi}}{s^{\frac{5}{2}}}+\frac{2}{s^3}$$

By the first shifting theorem,

$$L\left\{e^t\left(1+\sqrt{t}\right)^4\right\}=\frac{1}{s-1}+\frac{2\sqrt{\pi}}{(s-1)^{\frac{3}{2}}}+\frac{6}{(s-1)^2}+\frac{3\sqrt{\pi}}{(s-1)^{\frac{5}{2}}}+\frac{2}{(s-1)^3}$$

EXAMPLE 15.12 *Find the Laplace transform of* $\dfrac{\cos 2t\sin t}{e^t}$.

Solution:

$$\frac{\cos 2t\sin t}{e^t}=e^{-t}\left(\frac{\sin 3t-\sin t}{2}\right)=\frac{1}{2}(e^{-t}\sin 3t-e^{-t}\sin t)$$

$$L\{\sin t\}=\frac{1}{s^2+1}$$

$$L\{\sin 3t\}=\frac{3}{s^2+9}$$

$$L\left\{\frac{\cos 2t\sin t}{e^t}\right\}=\frac{1}{2}L\left\{e^{-t}\sin 3t-e^{-t}\sin t\right\}$$

By the first shifting theorem,

$$L\left\{\frac{\cos 2t\sin t}{e^t}\right\}=\frac{1}{2}\left[\frac{3}{(s+1)^2+9}-\frac{1}{(s+1)^2+1}\right]=\frac{1}{2}\frac{2s^2+4s-4}{(s^2+2s+10)(s^2+2s+2)}$$

$$=\frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

EXAMPLE 15.13 Find the Laplace transform of $e^{-4t}\sinh t\sin t$.

Solution:

$$e^{-4t}\sinh t\sin t = e^{-4t}\left(\frac{e^{t}-e^{-t}}{2}\right)\sin t = \frac{1}{2}(e^{-3t}\sin t - e^{-5t}\sin t)$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\{e^{-4t}\sinh t\sin t\} = \frac{1}{2}L\{e^{-3t}\sin t - e^{-5t}\sin t\}$$

By the first shifting theorem,

$$L\{e^{-4t}\sinh t\sin t\} = \frac{1}{2}\left[\frac{1}{(s+3)^2+1} - \frac{1}{(s+5)^2+1}\right] = \frac{1}{2}\frac{4s+16}{(s^2+6s+10)(s^2+10s+26)}$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}$$

EXERCISE 15.4

Find the Laplace transforms of the following functions:

1. $t^3 e^{-3t}$

$\left[\textbf{Ans.:} \dfrac{6}{(s+3)^4}\right]$

2. $e^{-t}\cos 2t$

$\left[\textbf{Ans.:} \dfrac{s+1}{s^2+2s+5}\right]$

3. $e^{2t}(3\sin 4t - 4\cos 4t)$

$\left[\textbf{Ans.:} \dfrac{20-4s}{s^2-4s+20}\right]$

4. $(1+te^{-t})^3$

$\left[\textbf{Ans.:} \dfrac{1}{s} + \dfrac{3}{(s+1)^2} + \dfrac{6}{(s+2)^3} + \dfrac{6}{(s+3)^4}\right]$

5. $e^{-t}(3\sinh 2t - 5\cosh 2t)$

$\left[\textbf{Ans.:} \dfrac{1-5s}{s^2+2s-3}\right]$

6. $e^{t}\sin 2t\sin 3t$

$\left[\textbf{Ans.:} \dfrac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}\right]$

7. $e^{-3t}\cosh 5t\sin 4t$

$\left[\textbf{Ans.:} \dfrac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+20)}\right]$

8. $e^{2t}\sin^4 t$

$\left[\textbf{Ans.:} \dfrac{3}{8(s-2)} - \dfrac{s-2}{2(s^2-4s+8)} + \dfrac{s-4}{8(s^2-8s+32)}\right]$

15.4.4 Second Shifting Theorem

If $L\{f(t)\} = F(s)$ and $g(t) = f(t-a), \quad t > a$
$= 0, \quad t < a$

then $L\{g(t)\} = e^{-as} F(s)$

Proof

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\mathrm{d}t$$

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\mathrm{d}t$$

$$= \int_0^a e^{-st} \cdot 0 \, \mathrm{d}t + \int_a^\infty e^{-st} f(t-a)\mathrm{d}t$$

Putting $t - a = x, \mathrm{d}t = \mathrm{d}x$
When $t = a, \quad x = 0$
When $t \to \infty, \quad x \to \infty$

$$L\{g(t)\} = \int_0^\infty e^{-s(a+x)} f(x)\mathrm{d}x = e^{-as}\int_0^\infty e^{-sx} f(x)\mathrm{d}x = e^{-as}\int_0^\infty e^{-st} f(t)\mathrm{d}t = e^{-as} F(s)$$

EXAMPLE 15.14 *Find the Laplace transform of* $g(t) = \cos(t-a), \quad t > a$
$= 0, \quad t < a$

Solution: Let $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-as} \frac{s}{s^2+1}$$

EXAMPLE 15.15 *Find the Laplace transform of* $g(t) = (t-1)^3, \quad t > 1$
$= 0, \quad t < 1$

Solution: Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

By the second shifting theorem,

$$L\{g(t)\} = e^{-s} \frac{3!}{s^4}$$

EXERCISE 15.5

Find the Laplace transforms of the following functions:

1. $f(t) = \cos\left(t - \frac{2\pi}{3}\right), \quad t > \frac{2\pi}{3}$

   $= 0, \quad t < \frac{2\pi}{3}$

   $\left[\textbf{Ans.}: e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}\right]$

2. $f(t) = (t-2)^2, \quad t > 2$

   $= 0, \quad t < 2$

   $\left[\textbf{Ans.}: e^{-2s} \frac{2}{s^3}\right]$

3. $f(t) = 5\sin 3\left(t - \frac{\pi}{4}\right), \quad t > \frac{\pi}{4}$

   $= 0, \quad t < \frac{\pi}{4}$

   $\left[\textbf{Ans.}: e^{-\frac{\pi s}{4}} \frac{15}{s^2+9}\right]$

15.4.5 Derivatives of Laplace Transform (Multiplication by t)

If $L\{f(t)\} = F(s)$ then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$.

Proof $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$

Differentiating both the sides w.r.t. s using Differentiation under integral sign,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt$$

$$= \int_0^\infty (-t e^{-st}) f(t)dt = \int_0^\infty e^{-st} \{-t f(t)\} dt = -L\{t f(t)\}$$

$$L\{t f(t)\} = (-1)\frac{d}{ds}F(s)$$

Similarly,

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}F(s)$$

In general,

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

EXAMPLE 15.16 *Find the Laplace transform of* $t\cos^2 t$.

Solution:

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2}L\{1+\cos 2t\} = \frac{1}{2}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$L\{t\cos^2 t\} = -\frac{d}{ds}L\{\cos^2 t\} = -\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+4}\right)$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2}+\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right] = \frac{1}{2s^2}+\frac{s^2-4}{2(s^2+4)^2}$$

EXAMPLE 15.17 *Find the Laplace transform of* $t\sqrt{1+\sin t}$.

Solution: $$L\{\sqrt{1+\sin t}\} = L\left\{\sin\frac{t}{2}+\cos\frac{t}{2}\right\} = \frac{\frac{1}{2}}{s^2+\frac{1}{4}}+\frac{s}{s^2+\frac{1}{4}} = \frac{1}{2}\cdot\frac{4}{4s^2+1}+\frac{4s}{4s^2+1} = \frac{4s+2}{4s^2+1}$$

$$L\{t\sqrt{1+\sin t}\} = -\frac{d}{ds}L\{\sqrt{1+\sin t}\} = -\frac{d}{ds}\left(\frac{4s+2}{4s^2+1}\right) = -\left[\frac{(4s^2+1)4-(4s+2)8s}{(4s^2+1)^2}\right]$$

$$= \frac{-16s^2-4+32s^2+16s}{(4s^2+1)^2} = \frac{16s^2+16s-4}{(4s^2+1)^2} = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

EXAMPLE 15.18 *Find the Laplace transform of* $t\left(\frac{\sin t}{e^t}\right)^2$.

Solution: $$t\left(\frac{\sin t}{e^t}\right)^2 = te^{-2t}\sin^2 t = te^{-2t}\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}te^{-2t}(1-\cos 2t)$$

$$L\{1-\cos 2t\} = \frac{1}{s}-\frac{s}{s^2+4}$$

$$L\{t(1-\cos 2t)\} = -\frac{d}{ds}L(1-\cos 2t) = -\frac{d}{ds}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

$$= -\left[-\frac{1}{s^2}-\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right] = \frac{1}{s^2}+\frac{4-s^2}{(s^2+4)^2}$$

By the first shifting theorem,

$$L\left\{\frac{1}{2}te^{-2t}(1-\cos 2t)\right\}=\frac{1}{2}\left[\frac{1}{(s+2)^2}+\frac{4-(s+2)^2}{\{(s+2)^2+4\}^2}\right]$$

EXAMPLE 15.19 *Find the Laplace transform of $t^2 e^t \sin 4t$.*

Solution:

$$L\{\sin 4t\}=\frac{4}{s^2+16}$$

$$L\{t^2\sin 4t\}=(-1)^2\frac{d^2}{ds^2}L\{\sin 4t\}=\frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right)$$

$$=-\frac{d}{ds}\left[\frac{4(2s)}{(s^2+16)^2}\right]=-\frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$=-\left[\frac{(s^2+16)^2(8)-8s\cdot 2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$=\frac{-8s^2-128+32s^2}{(s^2+16)^3}=\frac{24s^2-128}{(s^2+16)^3}=\frac{8(3s^2-16)}{(s^2+16)^3}$$

By the first shifting theorem,

$$L\{t^2e^t\sin 4t\}=\frac{8\left[3(s-1)^2-16\right]}{\left[(s-1)^2+16\right]^3}=\frac{8(3s^2-6s-13)}{(s^2-2s+17)^3}$$

EXERCISE 15.6

Find the Laplace transforms of the following functions:

1. $t\cos^3 t$

$$\left[\textbf{Ans.: }\frac{1}{4}\left[\frac{-s^2+9}{(s^2+9)^2}+\frac{s^2+3}{(s^2+1)^2}\right]\right]$$

2. $t\cos(\omega t-\alpha)$

$$\left[\textbf{Ans.: }\frac{(s^2-\omega^2)\cos\alpha+2\omega s\sin\alpha}{(s^2+\omega^2)^2}\right]$$

3. $t\sqrt{1-\sin t}$

$$\left[\textbf{Ans.: }\frac{4(4s^2-4s-1)}{(4s^2+1)^2}\right]$$

4. $t\cosh 3t$

$$\left[\textbf{Ans.: }\frac{s^2+9}{(s^2-9)^2}\right]$$

5. $t\sinh 2t\sin 3t$

$$\left[\textbf{Ans.: }3\left[\frac{s-2}{(s^2-4s+13)^2}-\frac{s-2}{(s^2+4s+13)^2}\right]\right]$$

6. $t(3\sin 2t - 2\cos 2t)$

$$\left[\textbf{Ans.:}\ \frac{8+12s-2s^2}{(s^2+4)^2}\right]$$

7. $t\,e^{3t}\sin 2t$

$$\left[\textbf{Ans.:}\ \frac{4(s-3)}{(s^2-6s+13)^2}\right]$$

8. $t\sqrt{1+\sin 2t}$

$$\left[\textbf{Ans.:}\ \frac{s^2+2s-1}{(s^2+1)^2}\right]$$

9. $t\,e^{2t}(\cos t - \sin t)$

$$\left[\textbf{Ans.:}\ \frac{s^2-6s+7}{(s^2-4s+5)^2}\right]$$

10. $(t^2-3t+2)\sin 3t$

$$\left[\textbf{Ans.:}\ \frac{6s^4-18s^3+126s^2-162s+432}{(s^2+9)^3}\right]$$

11. $(t+\sin 2t)^2$

$$\left[\textbf{Ans.:}\ \frac{2}{s^3}+\frac{s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}\right]$$

12. $(t\sinh 2t)^2$

$$\left[\textbf{Ans.:}\ \frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]\right]$$

13. $t^2e^{-3t}\cosh 2t$

$$\left[\textbf{Ans.:}\ \frac{1}{(s+1)^3}+\frac{1}{(s+5)^3}\right]$$

14. $t^2e^{-2t}\sin 3t$

$$\left[\textbf{Ans.:}\ \frac{18(s^2+4s+1)}{(s^2+4s+13)^2}\right]$$

15. $(t\cos 2t)^2$

$$\left[\textbf{Ans.:}\ \frac{1}{s^3}-\frac{s(48-s^2)}{(s^2+16)^3}\right]$$

16. $t^2\sin t\cos 2t$

$$\left[\textbf{Ans.:}\ \frac{9(s^2-3)}{(s^2+9)^3}+\frac{1-3s^2}{(s^2+1)^3}\right]$$

17. $t^3\cos t$

$$\left[\textbf{Ans.:}\ \frac{6s^4-36s^2+6}{(s^2+9)^3}\right]$$

15.4.6 Integrals of Laplace Transform (Division by t)

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\mathrm{d}s.$

Proof $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^\infty F(s)\mathrm{d}s = \int_s^\infty \int_0^\infty e^{-st} f(t)\mathrm{d}t\,\mathrm{d}s$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s)\mathrm{d}s = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\,\mathrm{d}s\right]\mathrm{d}t = \int_0^\infty \left|\frac{e^{-st}}{-t} f(t)\right|_s^\infty \mathrm{d}t$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = L\left\{\frac{f(t)}{t}\right\}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

EXAMPLE 15.20 *Find the Laplace transform of* $\dfrac{e^{-at}-e^{-bt}}{t}$.

Solution: $L\{e^{-at}-e^{-bt}\} = \dfrac{1}{s+a} - \dfrac{1}{s+b}$

$$L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty L\{e^{-at}-e^{-bt}\}\,ds = \int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$= \left|\log(s+a)-\log(s+b)\right|_s^\infty = \left|\log\frac{s+a}{s+b}\right|_s^\infty = \left|\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right|_s^\infty$$

$$= \log 1 - \log\frac{1+\frac{a}{s}}{1+\frac{b}{s}} = -\log\frac{s+a}{s+b} = \log\frac{s+b}{s+a}$$

EXAMPLE 15.21 *Find the Laplace transform of* $\dfrac{1-\cos t}{t^2}$.

Solution: $L\{1-\cos t\} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$$L\left\{\frac{1-\cos t}{t^2}\right\} = \int_s^\infty\int_s^\infty L\{1-\cos t\}\,ds\,ds = \int_s^\infty\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+1}\right]ds\,ds$$

$$= \int_s^\infty\left|\log s - \frac{1}{2}\log(s^2+1)\right|_s^\infty ds = \int_s^\infty\left|\log\frac{s}{\sqrt{s^2+1}}\right|_s^\infty ds$$

$$= \int_s^\infty\left|0-\log\frac{s}{\sqrt{s^2+1}}\right|ds = -\int_s^\infty \log\frac{s}{\sqrt{s^2+1}}\,ds$$

$$= \int_s^\infty \log\frac{\sqrt{s^2+1}}{s}\,ds = \frac{1}{2}\int_s^\infty \log\left(\frac{s^2+1}{s^2}\right)ds$$

$$= \frac{1}{2}\int_s^\infty \log\left(1+\frac{1}{s^2}\right) ds = \frac{1}{2}\left[\left|s\log\left(1+\frac{1}{s^2}\right)\right|_s^\infty - \int_s^\infty s\frac{1}{\left(1+\frac{1}{s^2}\right)}\left(-\frac{2}{s^3}\right) ds\right]$$

$$= \frac{1}{2}\left[0 - s\log\left(1+\frac{1}{s^2}\right) + 2\int_s^\infty \frac{1}{s^2+1} ds\right] = -\frac{1}{2}s\log\left(1+\frac{1}{s^2}\right) + \left|\tan^{-1} s\right|_s^\infty$$

$$= -\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right) + \frac{\pi}{2} - \tan^{-1} s = -\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right) + \cos^{-1} s$$

EXERCISE 15.7

Find the Laplace transforms of the following functions:

1. $\dfrac{\sin^2 t}{t}$ $\left[\textbf{Ans.: } \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)\right]$

2. $\left(\dfrac{\sin 2t}{\sqrt{t}}\right)^2$ $\left[\textbf{Ans.: } \frac{1}{4}\log\left(\frac{s^2+16}{s^2}\right)\right]$

3. $\dfrac{\sin^3 t}{t}$ $\left[\textbf{Ans.: } \frac{1}{4}\left(3\cot^{-1} s - \cot^{-1}\frac{s}{3}\right)\right]$

4. $\dfrac{1-\cos at}{t}$ $\left[\textbf{Ans.: } \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2}\right)\right]$

5. $\dfrac{\sin t \sin 5t}{t}$ $\left[\textbf{Ans.: } \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)\right]$

6. $\dfrac{2\sin t \sin 2t}{t}$ $\left[\textbf{Ans.: } \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)\right]$

7. $\dfrac{e^{2t}\sin t}{t}$ $\left[\textbf{Ans.: } \cot^{-1}(s-2)\right]$

8. $\dfrac{e^{2t}\sin^3 t}{t}$ $\left[\textbf{Ans.: } \frac{3}{4}\cot^{-1}(s-2) - \frac{1}{4}\cot^{-1}\left(\frac{s-2}{3}\right)\right]$

15.4.7 LAPLACE TRANSFORMS OF DERIVATIVES

If $L\{f(t)\} = F(s)$ then $L\{f'(t)\} = sF(s) - f(0)$

$$L\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

In general,

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{(n-1)}(0)$$

Proof $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$

Integrating by parts,

$$L\{f'(t)\} = \left|e^{-st} f(t)\right|_0^\infty - \int_0^\infty (-se^{-st}) f(t)dt = -f(0) + s\int_0^\infty e^{-st} f(t)dt$$
$$= -f(0) + s\,L\{f(t)\}$$

Similarly, $$L\{f''(t)\} = -f'(0) + s\,L\{f'(t)\} = -f'(0) + s[-f(0) + sL\{f(t)\}]$$
$$= -f'(0) - s\,f(0) + s^2 L\{f(t)\}$$

In general, $$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - f^{n-1}(0)$$

EXAMPLE 15.22 *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of $f(t) = \dfrac{\sin t}{t}$.*

Solution: $$L\{f(t)\} = F(s) = L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}ds = \int_s^\infty \frac{1}{s^2+1}ds = \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$L\{f'(t)\} = sF(s) - f(0) = s\cot^{-1} s - \lim_{t\to 0}\frac{\sin t}{t} = s\cot^{-1} s - 1$$

EXAMPLE 15.23 *Find $L\{f(t)\}$ and $L\{f'(t)\}$ of* $f(t) = t, \quad 0 \le t < 3$

$= 6, \quad t > 3$

Solution: $$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^3 e^{-st} t\, dt + \int_3^\infty e^{-st}\cdot 6\,dt = \left|\frac{e^{-st}}{-s}\cdot t\right|_0^3 - \left|\frac{e^{-st}}{s^2}\right|_0^3 + 6\left|\frac{e^{-st}}{-s}\right|_3^\infty$$

$$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s}e^{-3s} = \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$$

$$L\{f'(t)\} = s\,F(s) - f(0) = s\left[\frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)\right] - 0 = \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$$

EXERCISE 15.8

Find $L\{f'(t)\}$ of the following functions:

1. $f(t)=\left(\frac{1-\cos 2t}{t}\right)$

$$\left[\textbf{Ans.: } s\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$

2. $f(t)=t+1 \quad 0\le t\le 2$
$\quad\; =3 \quad t>2$

$$\left[\textbf{Ans.: } \frac{1}{s}(1-e^{-2s})\right]$$

15.4.8 Laplace Transforms of Integrals

If $L\{f(t)\}=F(s)$ then $L\left\{\int_0^t f(t)\mathrm{d}t\right\}=\frac{F(s)}{s}$.

Proof $L\left\{\int_0^t f(t)\mathrm{d}t\right\}=\int_0^\infty e^{-st}\left\{\int_0^t f(t)\mathrm{d}t\right\}\mathrm{d}t$

Integrating by parts,

$$L\left\{\int_0^t f(t)\mathrm{d}t\right\}=\left|\left\{\int_0^t f(t)\mathrm{d}t\right\}\left(\frac{e^{-st}}{-s}\right)\right|_0^\infty-\int_0^\infty\left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f(t)\mathrm{d}t\right)\right]\mathrm{d}t$$

$$=\int_0^\infty\frac{1}{s}e^{-st}f(t)\mathrm{d}t=\frac{1}{s}L\{f(t)\}=\frac{F(s)}{s}$$

EXAMPLE 15.24 *Find the Laplace transform of* $\int_0^t t\cosh t\,\mathrm{d}t$.

Solution: $L\{t\cosh t\}=L\left\{t\left(\frac{e^t+e^{-t}}{2}\right)\right\}=\frac{1}{2}L\{te^t+te^{-t}\}$

$$=\frac{1}{2}\left[\frac{1}{(s-1)^2}+\frac{1}{(s+1)^2}\right]=\frac{1}{2}\cdot\frac{2(s^2+1)}{(s^2-1)^2}=\frac{s^2+1}{(s^2-1)^2}$$

$$L\left\{\int_0^t t\cosh t\,\mathrm{d}t\right\}=\frac{1}{s}L\{t\cosh t\}=\frac{s^2+1}{s(s^2-1)^2}$$

EXAMPLE 15.25 *Find the Laplace transform of* $e^{-4t}\int_0^t t\sin 3t\,\mathrm{d}t$.

Solution: $L\{t\sin 3t\}=-\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin 3t\}=-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^2+9}\right)=\frac{6s}{(s^2+9)^2}$

$$L\left\{\int_0^t t\sin 3t\right\} = \frac{1}{s}L\{t\sin 3t\} = \frac{6}{(s^2+9)^2}$$

$$L\left\{e^{-4t}\int_0^t t\sin 3t\right\} = \frac{6}{\left[(s+4)^2+9\right]^2} = \frac{6}{(s^2+8s+25)^2}$$

EXAMPLE 15.26 *Find the Laplace transform of* $t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t$.

Solution:

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-4t}\sin 3t\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$$

$$L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = \frac{1}{s}L\{e^{-4t}\sin 3t\} = \frac{3}{s^3+8s^2+25s}$$

$$L\left\{t\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\left\{\int_0^t e^{-4t}\sin 3t\,\mathrm{d}t\right\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{3}{s^3+8s^2+25s}\right) = \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

EXAMPLE 15.27 *Find the Laplace transform of* $\int_0^t\int_0^t\int_0^t t\sin t\,\mathrm{d}t\,\mathrm{d}t\,\mathrm{d}t$.

Solution:

$$L\{t\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}L\{\sin t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$L\left\{\int_0^t t\sin t\,\mathrm{d}t\right\} = \frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t t\sin t\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{\int_0^t t\sin t\,\mathrm{d}t\right\} = \frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\}$$

$$L\left\{\int_0^t\int_0^t\int_0^t t\sin t\,\mathrm{d}t\right\} = \frac{1}{s}L\left\{\int_0^t\int_0^t t\sin t\,\mathrm{d}t\right\} = \frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s}L\{t\sin t\} = \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}$$

EXERCISE 15.9

Find the Laplace transforms of the following functions:

1. $\int_0^t e^{-t}t^4\,\mathrm{d}t$ $\left[\textbf{Ans.: } \dfrac{4!}{s(s+1)^5}\right]$

2. $\int_0^t \frac{1+e^{-t}}{t}\,\mathrm{d}t$ $\left[\textbf{Ans.: } \dfrac{1}{s}\log[s(s+1)]\right]$

3. $\int_0^t \frac{e^t \sin t}{t} dt$

$$\left[\textbf{Ans.:} \frac{1}{s}\cot^{-1}(s-1)\right]$$

4. $\int_0^t t\, e^{-2t} \sin 3t\, dt$

$$\left[\textbf{Ans.:} \frac{1}{s} \cdot \frac{3(2s+4)}{(s^2+4s+13)^2}\right]$$

5. $e^{-3t} \int_0^t t \sin 3t\, dt$

$$\left[\textbf{Ans.:} -\frac{6}{(s^2+6s+18)^2}\right]$$

6. $\int_0^t t^2 \sin t\, dt$

$$\left[\textbf{Ans.:} -\frac{2(1-3s^2)}{s(s^2+1)^3}\right]$$

7. $\int_0^t t \cos^2 t\, dt$

$$\left[\textbf{Ans.:} \frac{1}{2s^3} + \frac{1}{2} \cdot \frac{s^2-4}{s(s^2+4)^2}\right]$$

8. $\int_0^t t\, e^{-3t} \cos^2 2t\, dt$

$$\left[\textbf{Ans.:} \frac{1}{2s(s+3)^2} + \frac{1}{2} \cdot \frac{s^2+6s-7}{s(s^2+6s+25)^2}\right]$$

15.4.9 Initial and Final Value Theorems

1. *Initial Value Theorem*

If $L\{f(t)\} = F(s)$ then $\lim\limits_{t \to 0} f(t) = \lim\limits_{s \to \infty} s\,F(s)$.

Proof

$$L\{f'(t)\} = s\,F(s) - f(0)$$

$$s\,F(s) = L\{f'(t)\} + f(0) = \int_0^\infty e^{-st} f'(t) dt + f(0)$$

$$\lim_{s \to \infty} s\,F(s) = \lim_{s \to \infty}\left[\int_0^\infty e^{-st} f'(t) dt + f(0)\right] = \int_0^\infty \lim_{s \to \infty}\left[e^{-st} f'(t)\right] dt + f(0)$$

$$= 0 + f(0) = f(0) = \lim_{t \to 0} f(t)$$

2. *Final Value Theorem*

If $L\{f(t)\} = F(s)$ then $\lim\limits_{t \to \infty} f(t) = \lim\limits_{s \to 0} s\,F(s)$.

Proof $L\{f'(t)\} = s\,F(s) - f(0)$

$$s\,F(s) = L\{f'(t)\} + f(0) = \int_0^\infty e^{-st} f'(t) dt + f(0)$$

$$\lim_{s \to 0} s\,F(s) = \lim_{s \to 0}\left[\int_0^\infty e^{-st} f'(t) dt + f(0)\right] = \int_0^\infty \lim_{s \to 0}\left[e^{-st} f'(t)\right] dt + f(0)$$

$$= \int_0^\infty f'(t)\, dt + f(0) = \left|f(t)\right|_0^\infty + f(0) = \lim_{t \to \infty} f(t) - f(0) + f(0) = \lim_{t \to \infty} f(t)$$

EXAMPLE 15.28 *Verify the initial value theorem for* $e^{-t}\sin t$.

Solution: Let

$$f(t) = e^{-t}\sin t$$

$$L\{f(t)\} = F(s) = \frac{1}{(s+1)^2+1}$$

$$sF(s) = \frac{s}{(s+1)^2+1}$$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} e^{-t}\sin t = e^0 \cdot 0 = 0$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{s}{(s+1)^2+1} = \lim_{s\to\infty}\frac{1}{2(s+1)} \quad \text{[By L'Hospital's rule]}$$

$$= 0$$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Hence, the initial value theorem is verified.

EXAMPLE 15.29 *Verify the initial and final value theorems for* $e^{-t}(t^2 + \cos 3t)$.

Solution: Let

$$f(t) = e^{-t}(t^2 + \cos 3t)$$

$$L\{f(t)\} = F(s) = \frac{2}{(s+1)^3} + \frac{s+1}{(s+1)^2+9}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}$$

$$\lim_{t\to 0} f(t) = 1$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{\frac{2}{s^2}}{\left(1+\frac{1}{s}\right)^3} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{1}{s}\right)^2 + \frac{9}{s^2}}\right] = 1$$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Hence, the initial value theorem is verified.

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(t^2+\cos 3t)e^{-t} = 0$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{s(s+1)}{(s+1)^2+9}\right] = 0$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Hence, the final value theorem is verified.

EXAMPLE 15.30 *If* $F(s)=\dfrac{3s^2+5s+2}{s^3+4s^2+2s}$, *find* $f(0)$ *and* $f(\infty)$

Solution:

$$F(s)=\frac{3s^2+5s+2}{s^3+4s^2+2s}$$

By the initial value theorem,

$$f(0)=\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\frac{3s^3+5s^2+2s}{s^3+4s^2+2s}=\lim_{s\to\infty}\frac{3+\frac{5}{s}+\frac{2}{s^2}}{1+\frac{4}{s}+\frac{2}{s^2}}=3$$

By the final value theorem,

$$f(\infty)=\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)=\lim_{s\to 0}\frac{3s^3+5s^2+2s}{s^3+4s^2+2s}=\lim_{s\to 0}\frac{3s^2+5s+2}{s^2+4s+2}=\frac{2}{2}=1$$

EXERCISE 15.10

1. Verify the initial value theorem for the functions:

   (i) $3-2\cos t$ (ii) $(2t+3)^2$

   (iii) $t+\sin 3t$

2. Verify the final value theorem for the functions:

   (i) $1+e^{-t}(\sin t+\cos t)$

   (ii) $t^3 e^{-2t}$

15.5 EVALUATION OF INTEGRALS USING LAPLACE TRANSFORM

EXAMPLE 15.31 *Evaluate* $\int_0^\infty e^{-4t}\cosh^3 t\,dt$.

Solution:

$$\int_0^\infty e^{-st}\cosh^3 t\,dt=L\{\cosh^3 t\}=L\left\{\frac{\cosh 3t+3\cosh t}{4}\right\}$$

$$=\frac{1}{4}\frac{s}{s^2-9}+\frac{3}{4}\frac{s}{s^2-1}=\frac{1}{4}\left[\frac{s^3-s+3s^3-27s}{(s^2-9)(s^2-1)}\right]$$

$$=\frac{1}{4}\left[\frac{4s^3-28s}{(s^2-9)(s^2-1)}\right]=\frac{s(s^2-7)}{(s^2-9)(s^2-1)} \quad \text{... (1)}$$

Putting $s=4$ in Eq. (1),

$$\int_0^\infty e^{-4t}\cosh^3 t\,dt=\frac{4(16-7)}{(16-9)(16-1)}=\frac{12}{35}$$

EXAMPLE 15.32 *Show that* $\int_0^\infty e^{-t} t^3 \sin t \, dt = 0.$

Solution:
$$\int_0^\infty e^{-st} t^3 \sin t \, dt = L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{ds^3} L\{\sin t\} = -\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

Differentiating $\frac{1}{s^2+1}$ three times is quite tedious. Hence, rewriting the integral,

$$\int_0^\infty e^{-st} t^3 \sin t \, dt = \int_0^\infty e^{-st} \left[\text{Imaginary part of } e^{it}\right] t^3 \, dt$$
$$= \text{Img}\cdot\text{part} \int_0^\infty e^{-st} \cdot e^{it} t^3 \, dt = \text{Img}\cdot\text{part}\, L\{e^{it} t^3\}$$
$$= \text{Img}\cdot\text{part} \frac{3!}{(s-i)^4} \quad \text{...(1)}$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t} t^3 \sin t \, dt = \text{Img}\cdot\text{part} \frac{6}{(1-i)^4} = \text{Img}\cdot\text{part} \frac{6}{\left(\sqrt{2}\, e^{\frac{-i\pi}{4}}\right)^4}$$
$$= \text{Img}\cdot\text{part} \frac{6}{4} e^{i\pi} = \text{Img}\cdot\text{part}\left[\frac{3}{2}(\cos \pi + i \sin \pi)\right]$$
$$= \text{Img}\cdot\text{part}\left[\frac{3}{2}(-1 + i\cdot 0)\right] = 0$$

EXAMPLE 15.33 *Show that* $\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt = \frac{\pi}{8}.$

Solution:
$$\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} dt = L\left\{\frac{\sin t \sinh t}{t}\right\} = L\left\{\left(\frac{e^t - e^{-t}}{2}\right)\frac{\sin t}{t}\right\}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\} \, ds = \int_s^\infty \frac{1}{s^2+1} ds = \left|\tan^{-1} s\right|_s^\infty = \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t \sinh t}{t} dt = \frac{1}{2}\left[L\left\{e^t \frac{\sin t}{t}\right\} - L\left\{e^{-t} \frac{\sin t}{t}\right\}\right] \quad \left[\begin{array}{l}\text{Using the first shifting} \\ \text{theorem}\end{array}\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1)\right]$$
$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right] \quad \text{...(1)}$$

Putting $s = \sqrt{2}$ in Eq. (1),

$$\int_0^\infty e^{-\sqrt{2}t}\frac{\sin t \sinh t}{t}\,dt = \frac{1}{2}\left[\tan^{-1}(\sqrt{2}+1) - \tan^{-1}(\sqrt{2}-1)\right]$$

$$= \frac{1}{2}\tan^{-1}\left[\frac{\sqrt{2}+1-\sqrt{2}+1}{1+(\sqrt{2}+1)(\sqrt{2}-1)}\right] = \frac{1}{2}\tan^{-1}\left(\frac{2}{1+2-1}\right)$$

$$= \frac{1}{2}\tan^{-1}1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

EXAMPLE 15.34 *Evaluate* $\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u\, du\right)dt.$

Solution: $L\{\sinh u \cosh u\} = L\left\{\frac{1}{2}\sinh 2u\right\}$

$$= \frac{1}{2}\cdot\frac{2}{s^2-4} = \frac{1}{s^2-4}$$

$$L\{u^2 \sinh u \cosh u\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2-4}\right) = \frac{d}{ds}\left[\frac{-2s}{(s^2-4)^2}\right]$$

$$= -2\left[\frac{(s^2-4)^2 - s\cdot 2(s^2-4)\cdot 2s}{(s^2-4)^4}\right] = -2\left[\frac{s^2-4-4s^2}{(s^2-4)^3}\right] = \frac{2(3s^2+4)}{(s^2-4)^3}$$

$$L\left\{\int_0^t u^2 \sinh u \cosh u\, du\right\} = \frac{1}{s}L\{u^2\sinh u\cosh u\} = \frac{2(3s^2+4)}{s(s^2-4)^3}$$

Now, $$\int_0^\infty e^{-st}\left\{\int_0^t u^2 \sinh u \cosh u\, du\right\}dt = \frac{2(3s^2+4)}{s(s^2-4)^3} \qquad \dots (1)$$

Putting $s = 1$ in Eq. (1),

$$\int_0^\infty e^{-t}\left(\int_0^t u^2 \sinh u \cosh u\, du\right)dt = -\frac{14}{27}$$

EXERCISE 15.11

Evaluate the following integrals using the Laplace transform:

1. $\int_0^{\infty} e^{-3t} \cos^2 t \, dt$ $\left[\textbf{Ans.:}\ \frac{11}{39}\right]$

2. $\int_0^{\infty} e^{-5t} \sinh^3 t \, dt$ $\left[\textbf{Ans.:}\ \frac{1}{64}\right]$

3. $\int_0^{\infty} e^{-3t} \cos^3 t \, dt$ $\left[\textbf{Ans.:}\ \frac{4}{15}\right]$

4. $\int_0^{\infty} e^{-2t} t^3 \sin t \, dt$ $\left[\textbf{Ans.:}\ -\frac{576}{25}\right]$

5. $\int_0^{\infty} e^{-3t} t^2 \sinh 2t \, dt$ $\left[\textbf{Ans.:}\ \frac{124}{125}\right]$

6. $\int^{\infty} e^{\ 2}\, t \sin^2 t \, dt$ $\left[\textbf{Ans.:}\ \frac{1}{8}\right]$

7. $\int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} \, dt$ [**Ans.:** log 3]

8. $\int_0^{\infty} e^{-t} \frac{(1-\cos 2t)}{2t} dt$ $\left[\textbf{Ans.:}\ \frac{1}{4}\log 5\right]$

9. $\int_0^{\infty} e^{-t} \frac{(\cos 3t - \cos 2t)}{t} dt$ $\left[\textbf{Ans.:}\ \frac{1}{2}\log\frac{1}{2}\right]$

10. $\int_0^{\infty} e^{-t} \frac{\sin\sqrt{3}t}{t} dt$ $\left[\textbf{Ans.:}\ \frac{\pi}{3}\right]$

11. $\int_0^{\infty} e^{-2t} \frac{\sinh t}{t} dt$ $\left[\textbf{Ans.:}\ \frac{1}{2}\log 3\right]$

12. $\int_0^{\infty} e^{-t} \int_0^{t} t \cos^2 t \, dt \, dt$ $\left[\textbf{Ans.:}\ \frac{12}{50}\right]$

13. $\int_0^{\infty} e^{-t} \left(t \int_0^{t} e^{-4u} \cos u \, du \right) dt$ $\left[\textbf{Ans.:}\ \frac{9}{64}\right]$

14. $\int_0^{\infty} e^{-t} \left(\frac{1}{t} \int_0^{t} e^{-u} \sin u \, du \right) dt$ $\left[\textbf{Ans.:}\ \frac{1}{4}\log 5 - \frac{1}{2}\cot^{-1} 2\right]$

15.6 UNIT STEP FUNCTION

Unit step function (Fig. 15.1) is defined as

$$u(t) = 0, \quad t < 0$$
$$= 1, \quad t > 0$$

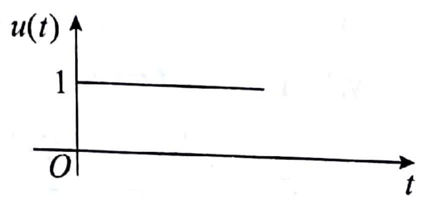

Fig. 15.1 Unit Step Function $u(t)$

The displaced (delayed) unit step function $u(t-a)$ represents the function $u(t)$ which is displaced by a distance a to the right (Fig. 15.2).

$$u(t-a) = 0, \quad t < a$$
$$= 1, \quad t > a$$

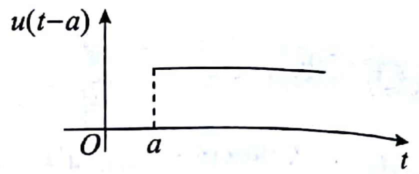

Fig. 15.2 Displaced Unit Step Function $u(t-a)$

Laplace Transform of Unit Step Functions

1. *Laplace Transform of Unit Step Function u(t)*

$$u(t) = 0, \quad t < 0$$
$$= 1, \quad t > 0$$

$$L\{u(t)\} = \int_0^\infty e^{-st} u(t) dt = \int_0^\infty e^{-st} dt = \left| \frac{e^{-st}}{-s} \right|_0^\infty = \frac{1}{s}$$

2. *Laplace Transform of the Displaced Unit Step Function u(t − a)*

$$u(t-a) = 0, \quad t < a$$
$$= 1, \quad t > a$$

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) dt = \int_a^\infty e^{-st} dt = \left| \frac{e^{-st}}{-s} \right|_a^\infty = \frac{1}{s} e^{-as}$$

3. *Laplace Transform of the Function f(t) u(t − a)*

$$f(t)u(t-a) = 0, \quad t < a$$
$$= f(t), \quad t > a$$

$$L\{f(t)u(t-a)\} = \int_0^\infty e^{-st} f(t) u(t-a) dt = \int_a^\infty e^{-st} f(t) dt$$

Putting $t - a = x, \quad dt = dx$

When $t = a, \quad x = 0$

When $t \to \infty, \quad x \to \infty$

$$L\{f(t)u(t-a)\} = \int_0^\infty e^{-s(x+a)} f(x+a) dx = e^{-as} \int_0^\infty e^{-sx} f(x+a) dx$$
$$= e^{-as} \int_0^\infty e^{-st} f(t+a) dt = e^{-as} L\{f(t+a)\}$$

4. *Laplace Transform of the Function f(t − a) u(t − a)*

$$f(t-a)\, u(t-a) = 0, \quad t < a$$
$$= f(t-a), \quad t > a$$

$$L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-st} f(t-a) u(t-a) dt = \int_a^\infty e^{-st} f(t-a) dt$$

Putting $t - a = x, \quad dt = dx$

When $t = a, \quad x = 0$

When $t \to \infty, \quad x \to \infty$

$$L\{f(t-a)u(t-a)\} = \int_0^\infty e^{-s(a+x)} f(x) dx = e^{-as} \int_0^\infty e^{-sx} f(x) dx$$
$$= e^{-as}\, L\{f(x)\} = e^{-as}\, F(s)$$

EXAMPLE 15.35 *Find the Laplace transform of* $(1 + 2t - 3t^2 + 4t^3)\, u(t-2)$ *and hence, evaluate* $\int_0^{\infty} e^{-t}(1+2t-3t^2+4t^3)u(t-2)dt$.

Solution:

$$L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$$

$$L\{(1+2t-3t^2+4t^3)\, u(t-2)\} = e^{-2s} L[1+2(t+2)-3(t+2)^2+4(t+2)^3]$$

$$= e^{-2s} L\{1+2(t+2)-3(t^2+4t+4)+4(t^3+6t^2+12t+8)\}$$

$$= e^{-2s} L\{25+38t+21t^2+4t^3\}$$

$$= e^{-2s}\left(\frac{25}{s}+38\cdot\frac{1}{s^2}+21\cdot\frac{2!}{s^3}+4\cdot\frac{3!}{s^4}\right)$$

$$= e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right)$$

Now,
$$\int_0^{\infty} e^{-st}(1+2t-3t^2+4t^3)\, u(t-2)\, dt = e^{-2s}\left(\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right) \quad \ldots (1)$$

Putting $s = 1$ in Eq. (1),

$$\int_0^{\infty} e^{-t}(1+2t-3t^2+4t^3)\, u(t-2)dt = e^{-2}\left(\frac{25}{1}+\frac{38}{1^2}+\frac{42}{1^3}+\frac{24}{1^4}\right) = \frac{129}{e^2}$$

EXAMPLE 15.36 *Find the Laplace transform of* $\sin t\; u\left(t-\frac{\pi}{2}\right)-u\left(t-\frac{3\pi}{2}\right)$.

Solution:

$$L\{f(t)\, u(t-a)\} = e^{-as} L\{f(t+a)\}$$

$$L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)-u\left(t-\frac{3\pi}{2}\right)\right\} = L\left\{\sin t\, u\left(t-\frac{\pi}{2}\right)\right\}-L\left\{u\left(t-\frac{3\pi}{2}\right)\right\}$$

$$= e^{-\frac{\pi s}{2}} L\left\{\sin\left(t+\frac{\pi}{2}\right)\right\}-\frac{e^{-\frac{3\pi s}{2}}}{s} = e^{\frac{-\pi s}{2}} L\{\cos t\}-\frac{e^{-\frac{3\pi s}{2}}}{s}$$

$$= e^{\frac{-\pi s}{2}}\frac{s}{s^2+1}-e^{-\frac{3\pi s}{2}}\frac{1}{s}$$

EXAMPLE 15.37 *Find the Laplace transform of* $f(t) = \sin 2t, \quad 2\pi < t < 4\pi$
$= 0, \quad$ *otherwise*

Solution: Expressing $f(t)$ in terms of the unit step function,

$$f(t) = \sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)$$

$$L\{f(t)\} = L\{\sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)\}$$

$$= L\{\sin 2t\, u(t-2\pi)\} - L\{\sin 2t\, u(t-4\pi)\}$$

$$= e^{-2\pi s}L\{\sin 2(t+2\pi)\} - e^{-4\pi s} L\{\sin 2(t+4\pi)\}$$

$$= e^{-2\pi s} L\{\sin 2t\} - e^{-4\pi s} L\{\sin 2t\}$$

$$= e^{-2\pi s}\frac{2}{s^2+4} - e^{-4\pi s}\frac{2}{s^2+4} = \frac{2}{s^2+4}(e^{-2\pi s}-e^{-4\pi s})$$

EXAMPLE 15.38 Find the Laplace transform of $f(t) = \cos t, \quad 0 < t < \pi$
$= \cos 2t, \quad \pi < t < 2\pi$
$= \cos 3t, \quad t > 2\pi$

Solution: Expressing $f(t)$ in terms of the unit step function,

$$f(t) = [\cos t\, u(t) - \cos t\, u(t-\pi)] + [\cos 2t\, u(t-\pi) - \cos 2t\, u(t-2\pi)] + \cos 3t\, u(t-2\pi)$$

$$= \cos t\, u(t) + (\cos 2t - \cos t)\, u(t-\pi) + (\cos 3t - \cos 2t)\, u(t-2\pi)$$

$$L\{f(t)\} = L\{\cos t\, u(t)\} + L\{(\cos 2t - \cos t)\, u(t-\pi)\} + L\{(\cos 3t - \cos 2t)\, u(t-2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

EXERCISE 15.12

1. Find the Laplace transforms of the following functions:

(i) $t^4 u(t-2)$

$$\left[\textbf{Ans.: } e^{-4s}\left(\frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{24}{s^5}\right)\right]$$

(ii) $(1 + 3t - 4t^2 + 2t^3)\, u(t-3)$

$$\left[\textbf{Ans.: } e^{-3s}\left(\frac{28}{s} + \frac{33}{s^2} + \frac{28}{s^3} + \frac{12}{s^4}\right)\right]$$

(iii) $t\, e^{-2t}\, u(t-1)$

$$\left[\textbf{Ans.: } e^{-(s+2)} \frac{s+3}{(s+2)^2}\right]$$

(iv) $\cos t\, u(t-1)$

$$\left[\textbf{Ans.: } e^{-s}\left(\frac{s\cos 1 - \sin 1}{s^2+1}\right)\right]$$

(ii) $f(t) = e^t \cos t, \quad 0 < t < \pi$
$= e^t \sin t, \quad t > \pi$

$$\left[\textbf{Ans.: } \frac{s-1}{s^2-2s+2} + e^{-\pi(s-1)} \cdot \frac{s-2}{s^2-2s+2}\right]$$

(iii) $f(t) = \sin t, \quad 0 < t < \pi$
$= \sin 2t, \quad \pi < t < 2\pi$
$= \sin 3t, \quad t > 2\pi$

$$\left[\textbf{Ans.: } \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} + \frac{1}{s^2+1}\right) - e^{-2\pi s}\left(\frac{3}{s^2+9} + \frac{2}{s^2+4}\right)\right]$$

2. Express the following functions in terms of a unit step function and, hence, find the Laplace transforms.

(i) $f(t) = t, \quad 0 < t < 2$
$= t^2, \quad t > 2$

$$\left[\textbf{Ans.: } \frac{1}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)\right]$$

(iv) $f(t) = t-1, \quad 1 < t < 2$
$= 3-t, \quad 2 < t < 3$
$= 0, \quad t > 3$

$$\left[\textbf{Ans.: } \frac{(1-e^{-s})^2}{s^2}\right]$$

(v) $f(t) = \sin t, \quad 0 < t < \pi$
$= t, \quad t > \pi$

$$\left[\textbf{Ans.: } \frac{1+e^{-\pi s}}{s^2+1} + e^{-\pi s}\left(\frac{\pi s+1}{s^2}\right)\right]$$

15.7 DIRAC DELTA OR UNIT IMPULSE FUNCTION

Consider the function $f(t)$ (Fig. 15.3).

$$f(t) = \frac{1}{T}, \qquad -\frac{T}{2} < t < \frac{T}{2}$$
$$= 0, \qquad \text{otherwise}$$

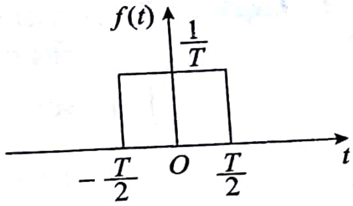

Fig. 15.3 A Function $f(t)$

The width of this function is T and its amplitude is $\frac{1}{T}$.

The area enclosed by the function $f(t)$ and the t-axis is given by

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{-\frac{T}{2}} f(t)\,dt + \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt + \int_{\frac{T}{2}}^{\infty} f(t)\,dt$$

$$= 0 + \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{T}\,dt + 0$$

$$= \frac{1}{T}\left| t \right|_{-\frac{T}{2}}^{\frac{T}{2}} = \frac{1}{T}(T) = 1$$

Hence, the area of this function is one unit. As $T \to 0$, the function becomes a delta function or unit impulse function.

$$\lim_{T \to 0} f(t) = \delta(t)$$

Dirac delta, or *Unit impulse function*, (Fig. 15.4), has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude of the function is infinitely large such that the area under its curve is equal to one unit. Hence, it is defined as

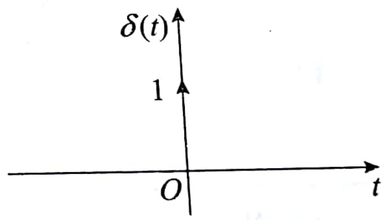

Fig. 15.4 Unit Impulse Function

$$\delta(t) = 0, \qquad t \neq 0$$

and
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1, \qquad t = 0$$

The displaced (delayed) delta or unit impulse function $\delta(t-a)$ represents the function $\delta(t)$ which is displaced by a distance 'a' to the right.(Fig. 15.5)

$$\delta(t-a) = 0, \qquad t \neq a$$

and
$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1, \qquad t = a$$

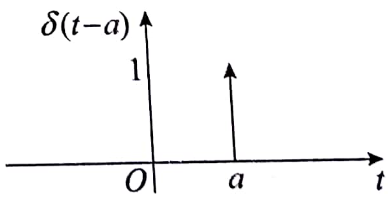

Fig. 15.5 Displaced Unit Impulse Function

Some Properties of Unit Impulse Functions

(i) $\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$

(ii) $\int_{0}^{\infty} f(t)\delta(t)\,dt = f(0)$

(iii) $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a)$

(iv) $\int_{0}^{\infty} f(t)\delta(t-a)\,dt = f(a)$

Laplace Transform of Unit Impulse Functions

1. *Laplace Transform of* $\delta(t)$

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1, \quad t = 0$$

$$L\{\delta(t)\} = \int_0^{\infty} e^{-st}\,\delta(t)\,dt = [e^{-st}]_{t=0} = 1$$

2. *Laplace Transform of* $\delta(t-a)$

$$\delta(t-a) = 0, \quad t \neq a$$

and
$$\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1, \quad t = a$$

$$L\{\delta(t-a)\} = \int_0^{\infty} e^{-st}\delta(t-a)\,dt$$

$$= [e^{-st}]_{t=a}$$ [From Property (iv)]

$$= e^{-as}$$

3. *Laplace Transform of* $f(t)\ \delta(t-a)$

$$f(t)\ \delta(t-a) = 0 \quad t \neq a$$

and
$$\int_0^{\infty} f(t)\delta(t-a)\,dt = f(a), \quad t = a$$

$$L\{f(t)\ \delta(t-a)\} = \int_0^{\infty} e^{-st} f(t)\delta(t-a)\,dt$$

$$= [e^{-st} f(t)]_{t=a}$$ [From Property (iv)]

$$= e^{-as} f(a)$$

EXAMPLE 15.39 *Evaluate* $\int_0^{\infty} \cos 2t\,\delta\left(t - \frac{\pi}{4}\right) dt.$

Solution:

$$\int_0^{\infty} f(t)\delta(t-a)\,dt = f(a)$$

$$\int_0^{\infty} \cos 2t\,\delta\left(t - \frac{\pi}{4}\right) dt = \cos\frac{2\pi}{4} = 0$$

EXAMPLE 15.40 *Evaluate* $\int_0^{\infty} t^m (\log t)^n\ \delta(t-3)\,dt.$

Solution:

$$\int_0^{\infty} f(t)\delta(t-a)\,dt = f(a)$$

$$\int_0^{\infty} t^m (\log t)^n\ \delta(t-3)\,dt = 3^m (\log 3)^n$$

EXERCISE 15.13

1. Find the Laplace transforms of the following functions:

(i) $tu(t-4)-t^2\,\delta(t-2)$

$$\left[\textbf{Ans.: } e^{-4s}\frac{1}{s^2}(1+4s)-4e^{-2s}\right]$$

(ii) $t^2\,u(t-2)-\cosh t\,\delta(t-4)$

$$\left[\textbf{Ans.: } \frac{2e^{-2s}}{s^3}(2s^2+2s+1)-e^{-4s}\cosh 4\right]$$

(iii) $\dfrac{e^{-t}\sin t}{t}\delta(t-3)$

$$\left[\textbf{Ans.: } \frac{1}{3}e^{-(s+3)}\sin 3\right]$$

(iv) $(e^{-4t}+\log t)\,\delta(t-2)$

[**Ans. :** $(e^{-8}+\log 2)\,e^{-2s}$]

2. Evaluate the following integrals:

(i) $\int_0^\infty \sin 4t\,\delta\left(t-\dfrac{\pi}{8}\right)dt$

$$\left[\textbf{Ans.: } e^{\frac{-\pi s}{8}}\right]$$

(ii) $\int_0^\infty e^{-t}\sin\delta(t-a)\,dt$

[**Ans. :** $e^{-a}(\sin a-\cos a)$]

15.8 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

A function $f(t)$ is said to be periodic if there exists a constant $T(T>0)$ such that $f(t+T)=f(t)$, for all values of t.

$$f(t+2T)=f(t+T+T)=f(t+T)=f(t)$$

In general, $f(t+nT)=f(t)$ for all t, where n is an integer (positive or negative) and T is the period of the function.

If $f(t)$ is a piecewise continuous periodic function with period T then

$$L\{f(t)\}=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$$

Proof $L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^T e^{-st}f(t)dt+\int_T^\infty e^{-st}f(t)dt$

In the second integral, putting $t=x+T$, $dt=dx$

When $t=T$, $x=0$

When $t\to\infty$, $x\to\infty$

$$L\{f(t)\}=\int_0^T e^{-st}f(t)dt+\int_0^\infty e^{-s(x+T)}f(x+T)dx=\int_0^T e^{-st}f(t)dt+e^{-Ts}\int_0^\infty e^{-sx}f(x)dx$$

$$=\int_0^T e^{-st}f(t)dt+e^{-Ts}\int_0^\infty e^{-st}f(t)dt=\int_0^T e^{-st}f(t)dt+e^{-Ts}L\{f(t)\}$$

$$(1-e^{-Ts})L\{f(t)\}=\int_0^T e^{-st}f(t)dt$$

$$L\{f(t)\}=\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$$

EXAMPLE 15.41

Find the Laplace transform of

$$f(t) = 1, \quad 0 \le t < a$$
$$= -1, \quad a < t < 2a$$

and $f(t)$ is periodic with period $2a$ (Fig. 15.6).

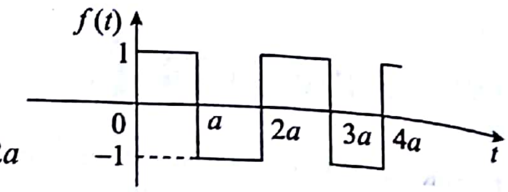

Fig. 15.6

Solution:

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt = \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left|\frac{e^{-st}}{-s}\right|_0^a + \left|\frac{e^{-st}}{s}\right|_a^{2a}\right] = \frac{1}{1-e^{-2as}}\left(-\frac{e^{-as}}{s}+\frac{1}{s}+\frac{e^{-2as}}{s}-\frac{e^{-as}}{s}\right)$$

$$= \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})} = \frac{1}{s}\cdot\frac{e^{\frac{as}{2}}-e^{-\frac{as}{2}}}{\left(e^{\frac{as}{2}}+e^{-\frac{as}{2}}\right)} = \frac{1}{s}\tanh\left(\frac{as}{2}\right)$$

EXAMPLE 15.42

Find the Laplace transform of

$$f(t) = t, \quad 0 < t < a$$
$$= 2a - t, \quad a < t < 2a$$

if $f(t) = f(t+2a)$

Solution: The function $f(t)$ is a periodic function with period $2a$ (Fig. 15.7).

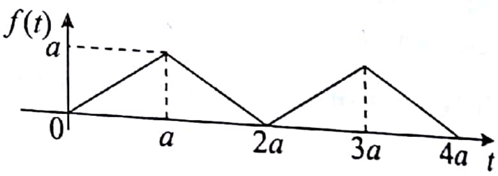

Fig.15.7

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}t\,dt + \int_a^{2a} e^{-st}(2a-t)\,dt\right]$$

$$= \frac{1}{(1-e^{-2as})}\left[\left|\frac{e^{-st}}{-s}t - \frac{e^{-st}}{s^2}\right|_0^a + \left|\frac{e^{-st}}{-s}(2a-t)+\frac{e^{-st}}{s^2}\right|_a^{2a}\right]$$

$$= \frac{1}{(1-e^{-2as})}\left(-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}-\frac{e^{-as}}{s^2}\right)$$

$$= \frac{-2e^{-as}+1+e^{-2as}}{s^2(1-e^{-2as})} = \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})} = \frac{e^{\frac{as}{2}}-e^{\frac{-as}{2}}}{s^2\left(e^{\frac{as}{2}}+e^{\frac{-as}{2}}\right)} = \frac{\tanh\left(\frac{as}{2}\right)}{s^2}$$

EXAMPLE 15.43 *Find the Laplace transform of* $f(t) = |\sin \omega t|, \quad t \ge 0$

Solution: $f\left(t+\frac{\pi}{\omega}\right) = \left|\sin \omega\left(t+\frac{\pi}{\omega}\right)\right| = |\sin(\omega t+\pi)| = |-\sin \omega t| = |\sin \omega t|$

Hence, the function $f(t)$ is periodic with period $\frac{\pi}{\omega}$ (Fig. 15.8).

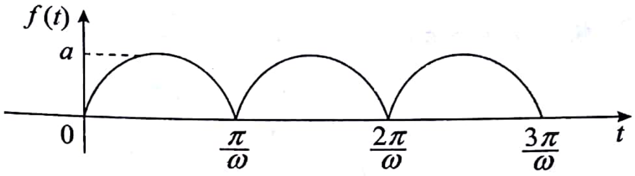

Fig. 15.8

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} |\sin \omega t|\,dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\,dt \qquad \left[\begin{array}{r} \because |\sin \omega t| = \sin \omega t \\ 0 < t < \frac{\pi}{\omega} \end{array}\right]$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left|\frac{e^{-st}}{s^2+\omega^2}(-s\sin \omega t - \omega\cos \omega t)\right|_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\frac{1}{s^2+\omega^2}\left[e^{-\frac{\pi s}{\omega}}(\omega)-(-\omega)\right] = \frac{1}{s^2+\omega^2}\cdot\frac{1}{1-e^{-\frac{\pi s}{\omega}}}\omega\left(1+e^{-\frac{\pi s}{\omega}}\right)$$

$$= \frac{\omega}{s^2+\omega^2}\left(\frac{e^{\frac{\pi s}{2\omega}}+e^{-\frac{\pi s}{2\omega}}}{e^{\frac{\pi s}{2\omega}}-e^{-\frac{\pi s}{2\omega}}}\right) = \frac{\omega}{s^2+\omega^2}\cdot\coth\left(\frac{\pi s}{2\omega}\right)$$

EXERCISE 15.14

Find the Laplace transforms of the following periodic functions:

1. $f(t) = 1, \quad 0 < t < 1$
$\quad = 0, \quad 1 < t < 2$
$\quad = -1, \quad 2 < t < 3$

$f(t) = f(t+3)$

$$\left[\textbf{Ans.: } \frac{1}{s}\left(\frac{3}{1-e^{-3s}} - \frac{1}{1-e^{-s}} - 1\right)\right]$$

2. $f(t) = t, \quad 0 < t < a$
$\quad = \frac{2a-t}{a}, \quad a < t < 2a$

$f(t) = f(t+2a)$

$$\left[\textbf{Ans.: } \frac{1}{as^2}\tanh\frac{as}{2}\right]$$

3. $f(t) = t, \quad 0 < t < a$
$\quad = \pi - t, \quad \pi < t < 2\pi$

$f(t) = f(t+2\pi)$

$$\left[\textbf{Ans.: } \frac{1-(1+\pi s)e^{-\pi s}}{(1+e^{-\pi s})s^2}\right]$$

4. $f(t) = |\cos \omega t|, \quad t > 0$

$$\left[\textbf{Ans.: } \frac{1}{s^2+\omega^2}\left(s+\omega \operatorname{cosech}\frac{\pi s}{2\omega}\right)\right]$$

5. $f(t) = \cos \omega t, \quad 0 < t < \frac{\pi}{\omega}$

$= 0, \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$

$$\left[\textbf{Ans.: } \frac{s}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}\right]$$

6. $f(t) = E, \quad 0 < t < \frac{\pi}{2}$

$= -E, \quad \frac{\pi}{2} < t < \pi$

$f(t) = f(t+\pi)$

$$\left[\textbf{Ans.: } \frac{E}{s}\tanh\left(\frac{\pi s}{4}\right)\right]$$

7. $f(t) = \left(\frac{\pi - t}{2}\right)^2, \quad 0 < t < 2\pi$

$f(t) = f(t+2\pi),$

$$\left[\textbf{Ans.: } \frac{1}{s^3}(2\pi s \coth \pi s - \pi^2 s^2 - 2)\right]$$

15.9 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transforms of simple functions can be found from the properties of Laplace transform.

Table of Inverse Laplace Transform

Sr. No.	$F(s)$	$f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^n}$	$\frac{t^{n-1}}{\lceil n}$
3	$\frac{1}{s-a}$	e^{at}
4	$\frac{1}{(s-a)^n}$	$e^{at}\frac{t^{n-1}}{\lceil n}$
5	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$

(Continued)

(Continued)

Sr. No.	$F(s)$	$f(t)$
6	$\frac{s}{s^2+a^2}$	$\cos at$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sin at$
8	$\frac{s}{s^2-a^2}$	$\cosh at$
9	$\frac{1}{(s+b)^2+a^2}$	$\frac{1}{a}e^{-bt}\sin at$
10	$\frac{s+b}{(s+b)^2+a^2}$	$e^{-bt}\cos at$
11	$\frac{1}{(s+b)^2-a^2}$	$\frac{1}{a}e^{-bt}\sinh at$
12	$\frac{s+b}{(s+b)^2-a^2}$	$e^{-bt}\cosh at$

15.9.1 Linearity

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then $L^{-1}\{aF_1(s)+bF_2(s)\} = af_1(t)+bf_2(t)$, where a and b are constants.

EXAMPLE 15.44 *Find the inverse Laplace transform of* $\frac{2s+1}{s(s+1)}$.

Solution: Let

$$F(s) = \frac{2s+1}{s(s+1)} = \frac{s+(s+1)}{s(s+1)} = \frac{1}{s+1}+\frac{1}{s}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s}\right\} = e^{-t}+1$$

EXAMPLE 15.45 *Find the inverse Laplace transform of* $\frac{4s+15}{16s^2-25}$.

Solution: Let

$$F(s) = \frac{4s+15}{16s^2-25} = \frac{4s+15}{16\left(s^2-\frac{25}{16}\right)} = \frac{1}{4}\frac{s}{s^2-\frac{25}{16}}+\frac{15}{16}\frac{1}{s^2-\frac{25}{16}}$$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{s}{s^2-\frac{25}{16}}\right\}+\frac{15}{16}L^{-1}\left\{\frac{1}{s^2-\frac{25}{16}}\right\} = \frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

EXAMPLE 15.46 *Find the inverse Laplace transform of* $\frac{3(s^2-2)^2}{2s^5}$.

Solution: Let

$$F(s) = \frac{3(s^2-2)^2}{2s^5} = \frac{3}{2}\frac{(s^2-2)^2}{s^5}$$

$$= \frac{3}{2}\frac{s^4-4s^2+4}{s^5} = \frac{3}{2}\left(\frac{1}{s}-\frac{4}{s^3}+\frac{4}{s^5}\right)$$

$$L^{-1}\{F(s)\} = \frac{3}{2}\left[L^{-1}\left\{\frac{1}{s}\right\}-4L^{-1}\left\{\frac{1}{s^3}\right\}+4L^{-1}\left\{\frac{1}{s^5}\right\}\right]$$

$$= \frac{3}{2}\left[1-4\left(\frac{t^2}{2!}\right)+4\left(\frac{t^4}{4!}\right)\right] = \frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]$$

$$= \frac{3}{2}-3t^2+\frac{t^4}{4} = \frac{1}{4}(t^4-12t^2+6)$$

EXERCISE 15.15

Find the inverse Laplace transforms of the following functions:

1. $\frac{2s-5}{s^2-4}$

   $\left[\textbf{Ans.:} 2\cosh 2t-\frac{5}{2}\sinh 2t\right]$

2. $\frac{3s-8}{4s^2+25}$

   $[\textbf{Ans.:} e^{-t}+1]$

3. $\frac{3s-12}{s^2+8}$

   $[\textbf{Ans.:} 3\cos 2\sqrt{2}t-3\sqrt{2}\sin 2\sqrt{2}t]$

4. $\frac{s+1}{s^{\frac{4}{3}}}$

   $\left[\textbf{Ans.:} \frac{t^{-\frac{2}{3}}+3t^{\frac{1}{3}}}{\left|\frac{1}{3}\right.}\right]$

5. $\left(\frac{\sqrt{s}-1}{s}\right)^2$

   $\left[\textbf{Ans.:} 1+t-\frac{4\sqrt{t}}{\sqrt{\pi}}\right]$

6. $\frac{s^2-1}{s^5}$

   $\left[\textbf{Ans.:} 1-t^2-\frac{t^4}{24}\right]$

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15.9.2 Change of Scale

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0.$

EXAMPLE 15.47 *If* $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$, *find* $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$.

Solution: Let

$$F(s) = \frac{s}{(s^2+1)^2}, f(t) = L^{-1}\{F(s)\} = \frac{1}{2}t\sin t$$

$$F(as) = \frac{as}{(a^2s^2+1)^2}$$

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$$

Putting $a = 2$,

$$L^{-1}\{F(2s)\} = \frac{1}{2} f\left(\frac{t}{2}\right)$$

$$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{1}{2}\left(\frac{1}{2}\frac{t}{2}\sin\frac{t}{2}\right)$$

$$L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{t}{2}\sin\frac{t}{2}$$

EXAMPLE 15.48 *Find the inverse Laplace transform of* $\frac{s}{2s^2-8}$.

Solution:

$$\frac{s}{2s^2-8} = \frac{2s}{4s^2-16} = \frac{2s}{(2s)^2-16} = F(2s), \text{ say}$$

Replacing $2s$ by s,

$$F(s) = \frac{s}{s^2-16}, f(t) = L^{-1}\{F(s)\} = \cosh 4t$$

$$L^{-1}\{F(2s)\} = \frac{1}{2} f\left(\frac{t}{2}\right)$$

$$L^{-1}\left\{\frac{2s}{4s^2-16}\right\} = \frac{1}{2}\cosh\frac{4t}{2}$$

$$L^{-1}\left\{\frac{s}{2s^2-8}\right\} = \frac{1}{2}\cosh 2t$$

EXERCISE 15.16

1. Find the inverse Laplace transform of $\frac{3s}{9s^2+16}$.

$\left[\text{Ans.: } \frac{1}{3}\cos\frac{4}{3}t\right]$

2. If $L^{-1}\left\{\frac{2as}{\left(s^2+a^2\right)^2}\right\} = t\sin at$ then find $L^{-1}\left\{\frac{6as}{\left(9s^2+a^2\right)^2}\right\}$.

$\left[\text{Ans.: } \frac{1}{9}t\sin\frac{at}{3}\right]$

15.9.3 First Shifting Theorem

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s+a)\} = e^{-at}f(t)$.

EXAMPLE 15.49 *Find the inverse Laplace transform of* $\frac{s+2}{s^2+4s+8}$.

Solution: Let

$$F(s) = \frac{s+2}{s^2+4s+8}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s+2}{s^2+4s+8}\right\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+4}\right\} = e^{-2t}L^{-1}\left\{\frac{s}{s^2+4}\right\} = e^{-2t}\cos 2t$$

EXAMPLE 15.50 *Find the inverse Laplace transform of* $\frac{s}{(2s+1)^2}$.

Solution: Let

$$F(s) = \frac{s}{(2s+1)^2} = \frac{1}{4}\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2} = \frac{1}{4}\left[\frac{1}{s+\frac{1}{2}} - \frac{1}{2}\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\} - \frac{1}{8}e^{-\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{1}{4}e^{-\frac{t}{2}} - \frac{1}{8}e^{-\frac{t}{2}}t$$

EXAMPLE 15.51 *Find the inverse Laplace transform of* $\frac{1}{\sqrt{2s+3}}$.

Solution: Let $F(s) = \frac{1}{\sqrt{2s+3}} = \frac{1}{\sqrt{2}} \frac{1}{\left(s+\frac{3}{2}\right)^{\frac{1}{2}}}$

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{2}} e^{-\frac{3t}{2}} L^{-1}\left\{\frac{1}{s^{\frac{1}{2}}}\right\} = \frac{1}{\sqrt{2}} e^{-\frac{3t}{2}} \frac{t^{-\frac{1}{2}}}{\left|\frac{1}{2}\right.} = \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}} e^{-\frac{3t}{2}} \qquad \left[\because \left|\frac{1}{2}\right. = \sqrt{\pi}\right]$$

EXERCISE 15.17

Find the inverse Laplace transforms of the following functions:

1. $\frac{5}{(s+2)^5}$ $\left[\textbf{Ans.: } \frac{5}{24} t^4 e^{-2t}\right]$

2. $\frac{4s+12}{s^2+8s+16}$ $\left[\textbf{Ans.: } 4e^{-4t}(1-t)\right]$

3. $\frac{1}{(s^2+2s+5)^2}$ $\left[\textbf{Ans.: } \frac{e^{-t}}{16}(\sin 2t - 2t\cos 2t)\right]$

4. $\frac{s}{(s-2)^6}$ $\left[\textbf{Ans.: } e^{2t}\left(\frac{t^4}{24}+\frac{t^5}{60}\right)\right]$

5. $\frac{s}{s^2+2s+2}$ $\left[\textbf{Ans.: } e^{-t}(\cos t - \sin t)\right]$

6. $\frac{1}{(s+2)^4}$ $\left[\textbf{Ans.: } \frac{1}{6}(e^{-2t}t^3)\right]$

15.9.4 Second Shifting Theorem

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as}F(s)\} = g(t)$, where $g(t) = f(t-a), \quad t > a$
$\qquad = 0, \quad t < a$

The above result can also be expressed as

$$L^{-1}\{e^{-as}F(s)\} = f(t-a), \quad t > a$$
$$= 0, \quad t < a$$

or

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

EXAMPLE 15.52 *Find the inverse Laplace transform of* $e^{-s}\left(\frac{1+\sqrt{s}}{s^3}\right)$.

Solution: Let $F(s)=\left(\frac{1+\sqrt{s}}{s^3}\right)$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s^3}+\frac{1}{s^{\frac{5}{2}}}\right\}=\frac{t^2}{2!}+\frac{t^{\frac{3}{2}}}{\left|\frac{5}{2}\right.}=\frac{t^2}{2}+\frac{t^{\frac{3}{2}}}{\frac{3}{2}\,\frac{1}{2}\left|\frac{1}{2}\right.}=\frac{t^2}{2}+\frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{-s}F(s)\}=\left[\frac{(t-1)^2}{2}+\frac{4(t-1)^{\frac{3}{2}}}{3\sqrt{\pi}}\right]u(t-1)$$

EXAMPLE 15.53 *Find the inverse Laplace transform of* $\frac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$.

Solution: Let $F(s)=\frac{1}{(s+4)^{\frac{5}{2}}}$

$$L^{-1}\{F(s)\}=e^{-4t}\,L^{-1}\left\{\frac{1}{s^{\frac{5}{2}}}\right\}=e^{-4t}\frac{t^{\frac{3}{2}}}{\left|\frac{5}{2}\right.}=\frac{e^{-4t}\,t^{\frac{3}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\left|\frac{1}{2}\right.}=\frac{4e^{-4t}\,t^{\frac{3}{2}}}{3\sqrt{\pi}}$$

$$L^{-1}\{e^{4-3s}F(s)\}=\frac{e^4\cdot 4}{3\sqrt{\pi}}e^{-4(t-3)}\,(t-3)^{\frac{3}{2}}\,u(t-3)=\frac{4}{3\sqrt{\pi}}e^{-4(t-4)}\,(t-3)^{\frac{3}{2}}u(t-3)$$

EXAMPLE 15.54 *Find the inverse Laplace transform of* $\frac{se^{-2s}}{s^2+2s+2}$.

olution: Let $F(s)=\frac{s}{s^2+2s+2}$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{s+1-1}{(s+1)^2+1}\right\}=L^{-1}\left\{\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$=e^{-t}\,L^{-1}\left\{\frac{s}{s^2+1}-\frac{1}{s^2+1}\right\}=e^{-t}(\cos t-\sin t)$$

$$L^{-1}\{e^{-2s}F(s)\}=e^{-(t-2)}\left[\cos(t-2)-\sin(t-2)\right]u(t-2)$$

EXERCISE 15.18

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{e^{-as}}{(s+b)^{\frac{5}{2}}}$

$\left[\textbf{Ans.:} \dfrac{4}{3\sqrt{\pi}} e^{-b(t-a)}(t-a)^{\frac{3}{2}}u(t-a)\right]$

2. $\dfrac{e^{-\pi s}}{s^2+9}$

$\left[\textbf{Ans.:} \dfrac{1}{3}\sin 3(t-\pi)u(t-\pi)\right]$

3. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$

$\left[\textbf{Ans.:} [(t-\pi)+\sin(t-\pi)]u(t-\pi)\right]$

4. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$

$\left[\textbf{Ans.:} \dfrac{e^{\frac{-7(t-4)}{2}}}{\sqrt{2\pi(t-4)}}u(t-4)\right]$

5. $\dfrac{(s+1)e^{-s}}{s^2+s+1}$

$\left[\textbf{Ans.:} e^{\frac{-(t-1)}{2}}\left[\cos\left(\sqrt{3}\dfrac{(t-1)}{2}\right) + \dfrac{1}{\sqrt{3}}\sin\left(\dfrac{\sqrt{3}(t-1)}{2}\right)\right]u(t-1)\right]$

6. $\dfrac{se^{-3s}}{s^2-1}$

$[\textbf{Ans.:} \cosh(t-3)u(t-3)]$

7. $\dfrac{se^{-as}}{s^2+b^2}$

$[\textbf{Ans.:} \cos b(t-a)u(t-a)]$

8. $e^{-s}\left\{\dfrac{1-\sqrt{s}}{s^2}\right\}^2$

$\left[\textbf{Ans.:} \left[\dfrac{(t-1)^3}{6} - \dfrac{16}{15\sqrt{\pi}}(t-1)^{\frac{5}{2}} + \dfrac{(t-1)^2}{2}\right]u(t-1)\right]$

15.9.5 Multiplication by s

If $L^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$ then $L^{-1}\{sF(s)\} = f'(t) = \dfrac{d}{dt}\left[L^{-1}\{F(s)\}\right]$

In general, $L^{-1}\{s^n F(s)\} = f^n(t)$, if $f(0) = f'(0) ... = f^{n-1}(0) = 0$

EXAMPLE 15.55 *Find the inverse Laplace transform of* $\dfrac{s}{s^2-a^2}$.

Solution: Let $F(s) = \dfrac{1}{s^2-a^2}$

$$L^{-1}\{F(s)\} = \frac{1}{a}\sinh at$$

$$L^{-1}\{sF(s)\} = \frac{d}{dt}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \frac{d}{dt}\left[L^{-1}\left\{\frac{1}{s^2-a^2}\right\}\right] = \frac{d}{dt}\left[\frac{1}{a}\sinh at\right] = \frac{1}{a}\cosh at(a) = \cosh at$$

EXAMPLE 15.56 *Find the inverse Laplace transform of* $\frac{s}{(s-4)^5}$.

Solution: Let $F(s) = \frac{1}{(s-4)^5}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s-4)^5}\right\} = e^{4t}L^{-1}\left\{\frac{1}{s^5}\right\} = e^{4t}\frac{t^4}{4!} = \frac{t^4e^{4t}}{24}$$

$$L^{-1}\{sF(s)\} = \frac{d}{dt}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{\frac{s}{(s-4)^5}\right\} = \frac{d}{dt}\left[\frac{1}{24}t^4e^{4t}\right] = \frac{1}{24}\frac{d}{dt}\left(t^4e^{4t}\right) = \frac{1}{24}\left[t^4 4e^{4t} + e^{4t}4t^3\right] = \frac{4t^3e^{4t}}{24}(t+1) = \frac{t^3e^{4t}}{6}(t+1)$$

EXERCISE 15.19

Find the inverse Laplace transforms of the following functions:

1. $\frac{s}{(s+2)^2}$ **[Ans.:** $e^{-2t}(1-2t)$**]**

2. $\frac{s^2}{(s^2+a^2)^2}$ $\left[\textbf{Ans.:} \frac{1}{2a}(\sinh at + at\cosh at)\right]$

3. $\frac{s^2}{(s-1)^3}$ $\left[\textbf{Ans.:} \frac{e^t}{2}(t^2+4t+2)\right]$

4. $\frac{s^2}{(s+4)^3}$ **[Ans.:** $e^{-4t}(8t^2-8t+1)$**]**

5. $\frac{s-3}{(s^2+4s+13)}$ $\left[\textbf{Ans.:} e^{-2t}\left(\cos 3t - \frac{5}{3}\sin 3t\right)\right]$

15.9.6 Division by *s*

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt = \int_0^t L^{-1}\{F(s)\}\,dt$

Similarly, $L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t\int_0^t f(t)dt\,dt$

EXAMPLE 15.57 *Find the inverse Laplace transform of* $\frac{1}{s(s^2+2s+2)}$.

Solution: Let $F(s) = \frac{1}{s^2+2s+2}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\,L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t}\sin t$$

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t L^{-1}\{F(s)\}\,dt$$

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\} = \int_0^t e^{-t}\sin t\,dt = \left|\frac{e^{-t}}{2}(-\sin t-\cos t)\right|_0^t = -\frac{1}{2}\left[\left|e^{-t}(\sin t+\cos t)\right|_0^t\right]$$

$$= -\frac{1}{2}\left[e^{-t}(\sin t+\cos t)-(0+1)\right] = \frac{1}{2}\left[1-e^{-t}(\sin t+\cos t)\right]$$

EXAMPLE 15.58 *Find the inverse Laplace transform of* $\frac{1}{s^2(1+s^2)}$.

Solution: Let $F(s) = \frac{1}{1+s^2}$

$$L^{-1}\{F(s)\} = \sin t$$

$$L^{-1}\left\{\frac{F(s)}{s^2}\right\} = \int_0^t\int_0^t L^{-1}\{F(s)\}dt\,dt$$

$$L^{-1}\left\{\frac{1}{s^2(1+s^2)}\right\} = \int_0^t\int_0^t \sin t\,dt\,dt = \int_0^t \left|-\cos t\right|_0^t\,dt$$

$$= \int_0^t(-\cos t+1)dt = \int_0^t(1-\cos t)\,dt$$

$$= \left|t-\sin t\right|_0^t = (t-\sin t)-(0-0) = t-\sin t$$

EXERCISE 15.20

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s^2+a^2)^2}$

$\left[\textbf{Ans.:} \dfrac{1}{2a^3}(\sin at - at\cos at)\right]$

2. $\dfrac{s^2+2}{s(s^2+4)}$

$\left[\textbf{Ans.:} \dfrac{1}{2}(1+\cos 2t)\right]$

3. $\dfrac{s}{(s^2+4)^2}$

$\left[\textbf{Ans.:} \dfrac{1}{4}t\sin 2t\right]$

4. $\dfrac{1}{s^2(s^2+a^2)}$

$\left[\textbf{Ans.:} \dfrac{1}{a^2}\left(t-\dfrac{1}{a}\sin at\right)\right]$

5. $\dfrac{s+1}{s^2(s^2+1)}$

[**Ans.:** $1+t-\cos t-\sin t$]

6. $\dfrac{1}{s(s^2+4s+5)}$

$\left[\textbf{Ans.:} \dfrac{1}{5}\left[1-e^{-2t}(2\sin t+\cos t)\right]\right]$

15.9.7 Derivatives of Laplace Transform

If $L^{-1}\{F(s)\}=f(t)$ then $L^{-1}\{F'(s)\}=-t\,f(t)$, i.e., $L^{-1}\{F(s)\}=-\dfrac{1}{t}L^{-1}\{F'(s)\}$.

EXAMPLE 15.59 *Find the inverse Laplace transform of* $\log\left(\dfrac{s^2+1}{s^2}\right)$.

Solution: Let

$$F(s)=\log\left(\frac{s^2+1}{s^2}\right)=\log(s^2+1)-\log s^2$$

$$F'(s)=\frac{2s}{s^2+1}-\frac{2s}{s^2}=\frac{2s}{s^2+1}-\frac{2}{s}$$

$$L^{-1}\{F(s)\}=-\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\frac{s^2+1}{s^2}\right\}=-\frac{1}{t}L^{-1}\left\{\frac{2s}{s^2+1}-\frac{2}{s}\right\}=-\frac{2}{t}L^{-1}\left\{\frac{s}{s^2+1}-\frac{1}{s}\right\}=-\frac{2}{t}(\cos t-1)=\frac{2}{t}(1-\cos t)$$

EXAMPLE 15.60 *Find the inverse Laplace transform of* $s\log\left(\frac{s+1}{s}\right)$.

Solution: Let

$$F(s)=\log\left(\frac{s+1}{s}\right)=\log(s+1)-\log s$$

$$F'(s)=\frac{1}{s+1}-\frac{1}{s}$$

$$L^{-1}\{F(s)\}=-\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\log\left(\frac{s+1}{s}\right)\right\}=-\frac{1}{t}L^{-1}\left\{\frac{1}{s+1}-\frac{1}{s}\right\}=-\frac{1}{t}(e^{-t}-1)=-\frac{e^{-t}}{t}+\frac{1}{t}$$

$$L^{-1}\{s\,F(s)\}=\frac{d}{dt}\left[L^{-1}\{F(s)\}\right]$$

$$L^{-1}\left\{s\log\left(\frac{s+1}{s}\right)\right\}=\frac{d}{dt}\left[-\frac{e^{-t}}{t}+\frac{1}{t}\right]=\frac{e^{-t}}{t}+\frac{e^{-t}}{t^2}-\frac{1}{t^2}$$

EXAMPLE 15.61 *Find the inverse Laplace transform of* $\tan^{-1}\left(\frac{s+a}{b}\right)$.

Solution: Let $F(s)=\tan^{-1}\left(\frac{s+a}{b}\right)$

$$F'(s)=\frac{1}{1+\left(\frac{s+a}{b}\right)^2}\cdot\frac{1}{b}=\frac{b}{(s+a)^2+b^2}$$

$$L^{-1}\{F(s)\}=-\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\left\{\tan^{-1}\left(\frac{s+a}{b}\right)\right\}=-\frac{1}{t}L^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\}=-\frac{1}{t}e^{-at}\sin bt$$

EXAMPLE 15.62 *Find the inverse Laplace transform of* $2\tanh^{-1}s$.

Solution: Let

$$F(s)=2\tanh^{-1}s=2\cdot\frac{1}{2}\log\frac{1+s}{1-s}=\log(1+s)-\log(1-s)$$

$$F'(s)=\frac{1}{1+s}+\frac{1}{1-s}$$

$$L^{-1}\{F(s)\}=-\frac{1}{t}L^{-1}\{F'(s)\}$$

$$L^{-1}\{2\tanh^{-1}s\}=-\frac{1}{t}L^{-1}\left\{\frac{1}{1+s}+\frac{1}{1-s}\right\}=-\frac{1}{t}(e^{-t}-e^{t})=\frac{2}{t}\sinh t$$

EXERCISE 15.21

Find the inverse Laplace transforms of the following functions:

1. $\log\left(1+\dfrac{a^2}{s^2}\right)$ $\left[\textbf{Ans.:} \dfrac{2}{t}(1-\cos at)\right]$

2. $\log\dfrac{s^2-4}{(s-3)^2}$ $\left[\textbf{Ans.:} \dfrac{2}{t}(e^{3t}-\cosh 2t)\right]$

3. $\log\left(\dfrac{s^2-4}{s^2}\right)^{\frac{1}{3}}$ $\left[\textbf{Ans.:} \dfrac{2}{3t}(1-\cosh 2t)\right]$

4. $\log\dfrac{1}{s}\left(1+\dfrac{1}{s^2}\right)$ $\left[\textbf{Ans.:} \displaystyle\int_0^t \frac{2(1-\cos t)}{t}\,dt\right]$

5. $\dfrac{1}{s}\log\dfrac{s+1}{s+2}$ $\left[\textbf{Ans.:} \displaystyle\int_0^t \frac{e^{-2t}-e^{-t}}{t}\,dt\right]$

6. $\tan^{-1}(s+1)$ $\left[\textbf{Ans.:} -\dfrac{1}{t}e^{-t}\sin t\right]$

7. $\cot^{-1} as$ $\left[\textbf{Ans.:} \dfrac{1}{t}\sin\dfrac{t}{a}\right]$

8. $\cot^{-1}\left(\dfrac{2}{s^2}\right)$ $\left[\textbf{Ans.:} -\dfrac{2}{7}\sin t\sinh t\right]$

15.9.8 Integrals of Transform

If $L^{-1}\{F(s)\}=f(t)$ then $L^{-1}\left[\int_s^\infty F(s)\,ds\right]=\frac{1}{t}f(t)$, i.e., $L^{-1}\{F(s)\}=t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right]$.

EXAMPLE 15.63 *Find the inverse Laplace transform of* $\dfrac{1}{(s+1)^2}$.

Solution: Let

$$F(s)=\frac{1}{(s+1)^2}$$

$$\int_s^\infty F(s)\,ds=\int_s^\infty \frac{1}{(s+1)^2}\,ds=\left|-\left(\frac{1}{s+1}\right)\right|_s^\infty$$

$$= -\left(0 - \frac{1}{s+1}\right) = \frac{1}{s+1}$$

$$L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)\,ds\right] = t\,L^{-1}\left\{\frac{1}{s+1}\right\} = t\,e^{-t}$$

EXAMPLE 15.64 *Find the inverse Laplace transform of* $\frac{s}{(s^2-a^2)^2}$.

Solution: Let $F(s) = \frac{s}{(s^2-a^2)^2}$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \frac{s}{(s^2-a^2)^2}\,ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2-a^2)^2}\,ds = \frac{1}{2}\left[\left|-\left(\frac{1}{s^2-a^2}\right)\right|_s^\infty\right] = \frac{1}{2}\frac{1}{s^2-a^2}$$

$$L^{-1}\{F(s)\} = t\,L^{-1}\left[\int_s^\infty F(s)ds\right] = t\,L^{-1}\left\{\frac{1}{2}\frac{1}{s^2-a^2}\right\} = \frac{t}{2}\frac{1}{a}\sinh at = \frac{t}{2a}\sinh at$$

EXERCISE 15.22

Find the inverse Laplace transforms of the following functions:

1. $\frac{2s}{(s^2-4)^2}$ $\left[\textbf{Ans.: } \frac{t}{2}\sinh 2t\right]$

2. $\frac{s+2}{(s^2+4s+5)^2}$ $\left[\textbf{Ans.: } \frac{t}{2}e^{-2t}\sin t\right]$

3. $\frac{s}{s^2-a^2}$ $\left[\textbf{Ans.: } \frac{t}{2a}\sinh at\right]$

15.9.9 Partial Fraction Expansion

Any function $F(s) = \frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials in s, can be simplified using partial fraction expansion. To perform partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that of $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I Factors are Linear and Distinct

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II Factors are Linear and Repeated

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \dots + \frac{B_n}{(s+b)^n}$$

Case III Factors are Quadratic and Distinct

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV Factors are Quadratic and Repeated

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1 s+D_1}{s^2+cs+d} + \frac{C_2 s+D_2}{(s^2+cs+d)^2} + \dots + \frac{C_n s+D_n}{(s^2+cs+d)^n}$$

EXAMPLE 15.65 *Find the inverse Laplace transform of* $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution: Let

$$F(s) = \frac{s+2}{s(s+1)(s+3)}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$s+2 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1) \qquad \dots (1)$$

Putting $s = 0$ in Eq. (1), $A = \frac{2}{3}$

Putting $s = -1$ in Eq. (1), $B = -\frac{1}{2}$

Putting $s = -3$ in Eq. (1), $C = -\frac{1}{6}$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

EXAMPLE 15.66 *Find the inverse Laplace transform of* $\frac{s^2+1}{(s+1)(s-2)^2}$.

Solution: Let
$$F(s) = \frac{s^2+1}{(s+1)(s-2)^2}$$

By partial fraction expansion,

$$F(s) = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$s^2 + 1 = A(s-2)^2 + B(s+1)(s-2) + C(s+1) \qquad \ldots(1)$$

Putting $s = -1$ in Eq. (1), $A = \frac{2}{9}$

Putting $s = 2$ in Eq. (1), $C = \frac{5}{3}$

Putting $s = 0$ in Eq. (1),

$$1 = 4A - 2B + C$$

$$1 = 4\left(\frac{2}{9}\right) - 2B + \frac{5}{3}$$

$$B = \frac{7}{9}$$

$$F(s) = \frac{2}{9}\cdot\frac{1}{s+1} + \frac{7}{9}\cdot\frac{1}{s-2} + \frac{5}{3}\cdot\frac{1}{(s-2)^2}$$

$$L^{-1}\{F(s)\} = \frac{2}{9}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{7}{9}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{5}{3}L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= \frac{2}{9}e^{-t} + \frac{7}{9}e^{2t} + \frac{5}{3}e^{2t}\, L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{2}{9}e^{-t} + \frac{7}{9}e^{2t} + \frac{5}{3}te^{2t}$$

EXAMPLE 15.67 *Find the inverse Laplace transform of* $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$.

Solution: Let
$$F(s)=\frac{s^2+2s+3}{(s^2+2s+5)\,(s^2+2s+2)}$$

Let $s^2+2s=x$

$$G(x)=\frac{x+3}{(x+5)\,(x+2)}$$

By partial fraction expansion,

$$G(x)=\frac{A}{x+5}+\frac{B}{x+2}$$

$$x+3=A(x+2)+B(x+5) \qquad \dots (1)$$

Putting $x=-5$ in Eq. (1), $A=\dfrac{2}{3}$

Putting $x=-2$ in Eq. (1), $B=\dfrac{1}{3}$

$$G(x)=\frac{2}{3}\cdot\frac{1}{x+5}+\frac{1}{3}\cdot\frac{1}{x+2}$$

$$F(s)=\frac{2}{3}\cdot\frac{1}{(s^2+2s+5)}+\frac{1}{3}\cdot\frac{1}{(s^2+2s+2)}=\frac{2}{3}\cdot\frac{1}{(s+1)^2+4}+\frac{1}{3}\cdot\frac{1}{(s+1)^2+1}$$

$$L^{-1}\{F(s)\}=\frac{2}{3}L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}=\frac{2}{3}e^{-t}\cdot\frac{1}{2}\sin 2t+\frac{1}{3}e^{-t}\sin t=\frac{1}{3}e^{-t}(\sin 2t+\sin t)$$

EXAMPLE 15.68 *Find the inverse Laplace transform of* $\dfrac{2s}{s^4+4}$.

Solution: Let $F(s)=\dfrac{2s}{s^4+4}=\dfrac{2s}{(s^4+4s^2+4)-4s^2}=\dfrac{2s}{(s^2+2)^2-(2s)^2}$

$$=\frac{2s}{(s^2+2+2s)\,(s^2+2-2s)}=\frac{1}{2}\left[\frac{s^2+2+2s-s^2-2+2s}{(s^2+2+2s)\,(s^2+2-2s)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{s^2+2-2s}-\frac{1}{s^2+2+2s}\right]=\frac{1}{2}\left[\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right]$$

$$L^{-1}\{F(s)\}=\frac{1}{2}\left[L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}-L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\right]$$

$$=\frac{1}{2}\left[e^{t}\,L^{-1}\left\{\frac{1}{s^2+1}\right\}-e^{-t}\,L^{-1}\left\{\frac{1}{s^2+1}\right\}\right]=\frac{1}{2}\left[e^{t}\sin t-e^{-t}\sin t\right]=\sin t\sinh t$$

EXAMPLE 15.69 *Find the inverse Laplace transform of* $\frac{1}{s^3+1}$.

Solution: Let

$$F(s)=\frac{1}{s^3+1}=\frac{1}{(s+1)(s^2-s+1)}$$

By partial fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2-s+1}$$

$$1=A(s^2-s+1)+(Bs+C)(s+1) \quad \dots (1)$$

Putting $s=-1$ in Eq. (1),

$$1=3A \quad \dots (2)$$

Putting $s=0$ in Eq. (1),

$$1=A+C \quad \dots (3)$$

Putting $s=1$ in Eq. (1),

$$1=A+2B+2C \quad \dots (4)$$

Solving Eqs (2), (3), and (4),

$$A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3},$$

$$F(s)=\frac{1}{3}\cdot\frac{1}{s+1}-\frac{1}{3}\cdot\frac{s}{s^2-s+1}+\frac{2}{3}\cdot\frac{1}{s^2-s+1}$$

$$=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\left(\frac{s-2}{s^2-s+1}\right)=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\left[\frac{s-\frac{1}{2}-\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

$$=\frac{1}{3}\frac{1}{s+1}-\frac{1}{3}\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{3}\cdot\frac{\frac{3}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\{F(s)\}=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}L^{-1}\left\{\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

$$=\frac{1}{3}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{3}e^{\frac{t}{2}}L^{-1}\left\{\frac{s}{s^2+\frac{3}{4}}\right\}+\frac{1}{2}e^{\frac{t}{2}}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}} \cos \frac{\sqrt{3}}{2}t + \frac{1}{2}e^{\frac{t}{2}} \frac{2}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t$$

$$= \frac{1}{3}e^{-t} - \frac{1}{3}e^{\frac{t}{2}} \cos \frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}e^{\frac{t}{2}} \sin \frac{\sqrt{3}}{2}t$$

EXAMPLE 15.70 *Find the inverse Laplace transform of* $\dfrac{s^3 - 3s^2 + 6s - 4}{(s^2 - 2s + 2)^2}$.

Solution: Let

$$F(s) = \frac{s^3 - 3s^2 + 6s - 4}{(s^2 - 2s + 2)^2}$$

By partial fraction expansion,

$$F(s) = \frac{As + B}{(s^2 - 2s + 2)} + \frac{Cs + D}{(s^2 - 2s + 2)^2}$$

$$s^3 - 3s^2 + 6s - 4 = (As + B)(s^2 - 2s + 2) + Cs + D$$

$$= As^3 + (B - 2A)s^2 + (2A - 2B + C)s + 2B + D$$

Equating the coefficients of s^3, $A = 1$

Equating the coefficients of s^2,

$$-3 = B - 2A$$
$$B = -3 + 2 = -1$$

Equating the coefficients of s,

$$6 = 2A - 2B + C$$
$$C = 6 - 2 - 2 = 2$$

Equating the coefficients of s^0,

$$-4 = 2B + D$$
$$D = -4 + 2 = -2$$

$$F(s) = \frac{s-1}{(s^2 - 2s + 2)} + \frac{2s - 2}{(s^2 - 2s + 2)^2} = \frac{s-1}{(s-1)^2 + 1} + \frac{2(s-1)}{\left[(s-1)^2 + 1\right]^2}$$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s-1}{(s-1)^2 + 1}\right\} + 2\,L^{-1}\left\{\frac{s-1}{\left[(s-1)^2 + 1\right]^2}\right\}$$

$$= e^t\, L^{-1}\left\{\frac{s}{s^2 + 1}\right\} + 2e^t\, L^{-1}\left\{\frac{s}{(s^2 + 1)^2}\right\}$$

$$= e^t \cos t + 2e^t \frac{t}{2} \sin t = e^t (\cos t + t \sin t)$$

EXERCISE 15.23

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$

$\left[\textbf{Ans.: } -\dfrac{1}{6}e^{-t}-\dfrac{4}{3}e^{2t}+\dfrac{7}{2}e^{3t}\right]$

2. $\dfrac{s+2}{s^2(s+3)}$

$\left[\textbf{Ans.: } \dfrac{1}{9}(1+6t-e^{-3t})\right]$

3. $\dfrac{1}{s(s+1)^2}$

$\left[\textbf{Ans.: } 1-e^{-t}-te^{-t}\right]$

4. $\dfrac{1}{s^2(s+3)^2}$

$\left[\textbf{Ans.: } \dfrac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})\right]$

5. $\dfrac{s^2}{(s+4)^3}$

$\left[\textbf{Ans.: } e^{-4t}(1-8t+8t^2)\right]$

6. $\dfrac{1}{(s-2)^4(s+3)}$

$\left[\textbf{Ans.: } \dfrac{e^{2t}}{6}\left[\dfrac{t^3}{5}-\dfrac{3}{25}t^2+\dfrac{6}{125}t-\dfrac{6}{625}\right]+\dfrac{1}{625}e^{-3t}\right]$

7. $\dfrac{5s^2-7s+17}{(s-1)(s^2+4)}$

$\left[\textbf{Ans.: } 3e^{t}+2\cos 2t-\dfrac{5}{2}\sin 2t\right]$

8. $\dfrac{2s^3-s^2-1}{(s+1)^2(s^2+1)^2}$

$\left[\textbf{Ans.: } \dfrac{1}{2}\sin t+\dfrac{1}{2}t\cos t-te^{-t}\right]$

9. $\dfrac{1}{s^3(s-1)}$

$\left[\textbf{Ans.: } 1-t+\dfrac{t^2}{2}-e^{-t}\right]$

10. $\dfrac{s}{(s+1)^2(s^2+1)}$

$\left[\textbf{Ans.: } \dfrac{1}{2}(\sin t-te^{-t})\right]$

11. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

$\left[\textbf{Ans.: } e^{t}-e^{-t}\cos 2t+\dfrac{3}{2}e^{-t}\sin 2t\right]$

12. $\dfrac{s}{(s^2-2s+2)(s^2+2s+2)}$

$\left[\textbf{Ans.: } \dfrac{1}{2}\sin t\sinh t\right]$

13. $\dfrac{10}{s(s^2-2s+5)}$

$\left[\textbf{Ans.: } 2-e^{t}(2\cos 2t-\sin 2t)\right]$

14. $\dfrac{s^2+8s+27}{(s+1)(s^2+4s+13)}$

$\left[\textbf{Ans.: } 2e^{-t}+e^{-2t}(\sin 3t-\cos 3t)\right]$

15. $\dfrac{2s-1}{s^4+s^2+1}$

$$\left[\begin{aligned}\textbf{Ans.: } &\frac{1}{2}e^{\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{2}e^{\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t\\ &-\frac{1}{2}e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2}t-\frac{5}{2\sqrt{3}}e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t\end{aligned}\right]$$

16. $\dfrac{s}{s^4+4a^4}$ $\left[\textbf{Ans.:}\ \dfrac{1}{2a^2}\sin at \sinh at\right]$

17. $\dfrac{s^2}{s^4+4a^4}$ $\left[\textbf{Ans.:}\ \dfrac{1}{2a}[\sinh at \cos at + \cosh at \sin at]\right]$

15.10 CONVOLUTION THEOREM

If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\mathrm{d}u$

where $$\int_0^t f_1(u) f_2(t-u)\mathrm{d}u = f_1(t) * f_2(t)$$

$f_1(t)*f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$.

Proof $F_1(s)\cdot F_2(s) = L\{f_1(t)\}\cdot L\{f_2(t)\} = \int_0^\infty e^{-su} f_1(u)\mathrm{d}u \cdot \int_0^\infty e^{-sv} f_2(v)\mathrm{d}v$

$$= \int_0^\infty\int_0^\infty e^{-s(u+v)} f_1(u) f_2(v)\mathrm{d}u\,\mathrm{d}v = \int_0^\infty f_1(u)\left[\int_0^\infty e^{-s(u+v)} f_2(v)\mathrm{d}v\right]\mathrm{d}u$$

Putting $u+v=t,\quad \mathrm{d}v = \mathrm{d}t$

When $v=0,\quad t=u$

When $v\to\infty,\quad t\to\infty$

$$F_1(s)\cdot F_2(s) = \int_0^\infty f_1(u)\left[\int_u^\infty e^{-st} f_2(t-u)\mathrm{d}t\right]\mathrm{d}u = \int_0^\infty\int_u^\infty e^{-st} f_1(u) f_2(t-u)\mathrm{d}t\,\mathrm{d}u$$

Limits of t: $t=u$ to $t=\infty$

Limits of u: $u=0$ to $u=\infty$

The region of integration is bounded by the lines $u=0$ and $u=t$. To change the order of integration, draw a vertical strip which starts from the line $u=0$ and terminates on the line $u=t$. Hence, u varies from 0 to t and t varies from 0 to ∞ (Fig. 15.9).

$$F_1(s)\cdot F_2(s) = \int_0^\infty e^{-st}\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\,\mathrm{d}t$$

$$= L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\}$$

Hence, $L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u) f_2(t-u)\mathrm{d}u$

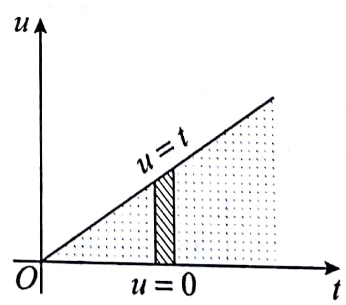


Fig. 15.9 Illustration of Convolution Theorem

Note Convolution operation is commutative, i.e.,

$$L\left\{\int_0^t f_1(u) f_2(t-u)\mathrm{d}u\right\} = L\left\{\int_0^t f_1(t-u) f_2(u)\mathrm{d}u\right\}$$

EXAMPLE 15.71 *Find the inverse Laplace transform of* $\dfrac{1}{s^2(s+5)}$.

Solution: Let $F(s)=\dfrac{1}{s^2(s+5)}$

Let $F_1(s)=\dfrac{1}{s^2}$, $F_2(s)=\dfrac{1}{s+5}$

$f_1(t)=t$, $f_2(t)=e^{-5t}$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t u\,e^{-5(t-u)}\mathrm{d}u=\int_0^t u\,e^{-5t+5u}\mathrm{d}u$$

$$=e^{-5t}\int_0^t u\,e^{5u}\mathrm{d}u=e^{-5t}\left[\left|u\frac{e^{5u}}{5}-(1)\frac{e^{5u}}{25}\right|_0^t\right]$$

$$=e^{-5t}\left[\left(t\frac{e^{5t}}{5}-\frac{e^{5t}}{25}\right)-\left(0-\frac{1}{25}\right)\right]=e^{-5t}\left[t\frac{e^{5t}}{5}-\frac{e^{5t}}{25}+\frac{1}{25}\right]$$

$$=\frac{t}{5}-\frac{1}{25}+\frac{e^{-5t}}{25}=\frac{1}{25}(e^{-5t}+5t-1)$$

EXAMPLE 15.72 *Find the inverse Laplace transform of* $\dfrac{1}{(s-2)^4(s+3)}$.

Solution: Let $F(s)=\dfrac{1}{(s-2)^4(s+3)}$

Let $F_1(s)=\dfrac{1}{(s-2)^4}$, $F_2(s)=\dfrac{1}{s+3}$

$f_1(t)=e^{2t}\dfrac{t^3}{6}$, $f_2(t)=e^{-3t}$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{2u}\frac{u^3}{6}e^{-3(t-u)}\mathrm{d}u=\frac{e^{-3t}}{6}\int_0^t u^3e^{5u}\mathrm{d}u=\frac{e^{-3t}}{6}\left|u^3\frac{e^{5u}}{5}-3u^2\frac{e^{5u}}{25}+6u\frac{e^{5u}}{125}-6\frac{e^{5u}}{625}\right|_0^t$$

$$=\frac{e^{-3t}}{6}\left[t^3\frac{e^{5t}}{5}-3t^2\frac{e^{5t}}{25}+6t\frac{e^{5t}}{125}-6\frac{e^{5t}}{625}+\frac{6}{625}\right]=\frac{e^{-3t}}{625}+\frac{e^{2t}}{6}\left[\frac{t^3}{5}-\frac{3t^2}{25}+\frac{6t}{125}-\frac{6}{625}\right]$$

EXAMPLE 15.73 *Find the inverse Laplace transform of* $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$.

Solution: Let $F(s)=\dfrac{1}{(s^2+a^2)(s^2+b^2)}$

Let $F_1(s)=\dfrac{1}{s^2+a^2}$, $\quad F_2(s)=\dfrac{1}{s^2+b^2}$

$f_1(t)=\dfrac{1}{a}\sin at$, $\quad f_2(t)=\dfrac{1}{b}\sin bt$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t \frac{1}{a}\sin au\cdot\frac{1}{b}\sin b(t-u)du=\frac{1}{ab}\int_0^t \sin au\sin b(t-u)du$$

$$=-\frac{1}{2ab}\int_0^t\left[\cos\{(a-b)u+bt\}-\cos\{(a+b)u-bt\}\right]du$$

$$=-\frac{1}{2ab}\left|\frac{\sin\{(a-b)u+bt\}}{a-b}-\frac{\sin\{(a+b)u-bt\}}{a+b}\right|_0^t$$

$$=-\frac{1}{2ab}\left[\frac{\sin at}{a-b}-\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right]$$

$$=-\frac{1}{2ab}\left[2b\frac{\sin at}{a^2-b^2}-2a\frac{\sin bt}{a^2-b^2}\right]=\frac{a\sin bt-b\sin at}{ab(a^2-b^2)}$$

EXAMPLE 15.74 *Find the inverse Laplace transform of* $\dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$.

Solution: Let $F(s)=\dfrac{s(s+1)}{(s^2+1)(s^2+2s+2)}$

Let $F_1(s)=\dfrac{s+1}{s^2+2s+2}$, $\quad F_2(s)=\dfrac{s}{s^2+1}$

$=\dfrac{s+1}{(s+1)^2+1}$,

$f_1(t)=e^{-t}\cos t$, $\quad f_2(t)=\cos t$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-u}\cos u\cos(t-u)\,du=\frac{1}{2}\int_0^t e^{-u}\left[\cos t+\cos(2u-t)\right]du$$

$$=\frac{1}{2}\left|-e^{-u}\cos t+\frac{e^{-u}}{5}\{-\cos(2u-t)+2\sin(2u-t)\}\right|_0^t$$

$$=\frac{1}{2}\left[-e^{-t}\cos t+\frac{e^{-t}}{5}(-\cos t+2\sin t)-\frac{1}{5}(-\cos t-2\sin t)+\cos t\right]$$

$$=\frac{1}{10}\left[e^{-t}(2\sin t-6\cos t)+(2\sin t+6\cos t)\right]$$

EXAMPLE 15.75 *Find the inverse Laplace transform of* $\frac{1}{(s+3)(s^2+2s+2)}$.

Solution: Let $F(s)=\frac{1}{(s+3)(s^2+2s+2)}$

Let
$$F_1(s)=\frac{1}{s^2+2s+2},\qquad F_2(s)=\frac{1}{s+3}$$
$$=\frac{1}{(s+1)^2+1},$$
$$f_1(t)=e^{-t}\sin t\qquad f_2(t)=e^{-3t}$$

By the convolution theorem,

$$L^{-1}\{F(s)\}=\int_0^t e^{-u}\sin u\, e^{-3(t-u)}du=e^{-3t}\int_0^t e^{2u}\sin u\, du$$
$$=e^{-3t}\left|\frac{e^{2u}}{5}(2\sin u-\cos u)\right|_0^t=\frac{e^{-3t}}{5}\left[e^{2t}(2\sin t-\cos t)+1\right]$$
$$=\frac{1}{5}\left[e^{-t}(2\sin t-\cos t)+e^{-3t}\right]$$

EXAMPLE 15.76 *Find the inverse Laplace transform of* $\frac{1}{(s^2+4)(s+1)^2}$.

Solution: Let $F(s)=\frac{1}{(s^2+4)(s+1)^2}$

Considering $F(s)$ as a product of three functions,

$$F(s)=\frac{1}{(s^2+4)}\cdot\frac{1}{s+1}\cdot\frac{1}{s+1}$$

Let
$$F_1(s)=\frac{1}{s^2+4},\qquad F_2(s)=\frac{1}{s+1},\qquad F_3(s)=\frac{1}{s+1}$$
$$f_1(t)=\frac{1}{2}\sin 2t,\qquad f_2(t)=e^{-t},\qquad f_3(t)=e^{-t}$$

By the convolution theorem,

$$L^{-1}\{F_1(s)\cdot F_2(s)\}=\int_0^t \frac{1}{2}\sin 2u\, e^{-(t-u)}du=\frac{e^{-t}}{2}\left|\frac{e^u}{5}(\sin 2u-2\cos 2u)\right|_0^t$$

$$=\frac{e^{-t}}{10}\left[e^t(\sin 2t-2\cos 2t)+2\right]=\frac{\sin 2t-2\cos 2t}{10}+\frac{e^{-t}}{5}$$

$$L^{-1}\{F_1(s)F_2(s)F_3(s)\}=\int_0^t\left[\frac{\sin 2u-2\cos 2u}{10}+\frac{e^{-u}}{5}\right]e^{-(t-u)}du$$

$$=\frac{e^{-t}}{10}\int_0^t\left[e^u(\sin 2u-2\cos 2u)+2\right]du$$

$$=\frac{e^{-t}}{10}\left|\frac{e^u}{5}\{(\sin 2u-2\cos 2u)-2(\cos 2u+2\sin 2u)\}+2u\right|_0^t$$

$$=\frac{e^{-t}}{10}\left[\frac{e^t}{5}(-3\sin 2t-4\cos 2t)+2t+\frac{4}{5}\right]$$

$$=\frac{2}{25}e^{-t}+\frac{te^{-t}}{5}-\frac{1}{50}(3\sin 2t+4\cos 2t)$$

EXERCISE 15.24

Find the inverse Laplace transforms of the following functions:

1. $\dfrac{1}{(s+3)(s-1)}$

$\left[\textbf{Ans.: } \dfrac{e^t}{4}(1-e^{-4t})\right]$

2. $\dfrac{1}{s(s^2+4)}$

$\left[\textbf{Ans.: } \dfrac{1}{4}(1-\cos 2t)\right]$

3. $\dfrac{1}{(s-3)(s+3)^2}$

$\left[\textbf{Ans.: } \dfrac{1}{36}(e^{3t}-e^{-3t}-6te^{-3t})\right]$

4. $\dfrac{s}{(s^2+4)^2}$

$\left[\textbf{Ans.: } \dfrac{1}{4}t\sin 2t\right]$

5. $\dfrac{s^2}{(s^2-a^2)^2}$

$\left[\textbf{Ans.: } \dfrac{1}{2}(\sinh at+at\cosh at)\right]$

6. $\dfrac{1}{s(s^2-a^2)}$

$\left[\textbf{Ans.: } \dfrac{1}{a^2}(\cosh at-1)\right]$

7. $\dfrac{1}{s^3(s^2+1)}$

$\left[\textbf{Ans.: } \dfrac{t^2}{2}+\cos t-1\right]$

8. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

$\left[\textbf{Ans .:} \dfrac{1}{3}(2\sin 2t - \sin t)\right]$

9. $\dfrac{s}{(s^2-a^2)^2}$

$\left[\textbf{Ans .:} \dfrac{1}{2a}(at\cosh at + \sinh at)\right]$

10. $\dfrac{s}{(s^2+a^2)^3}$

$\left[\textbf{Ans .:} \dfrac{t}{8a^3}(\sin at - at\cos at)\right]$

11. $\dfrac{s+3}{(s^2+6s+13)^2}$

$\left[\textbf{Ans .:} \dfrac{1}{4}e^{-3t}\ t\sin 2t\right]$

12. $\dfrac{s}{s^4+8s^2+16}$

$\left[\textbf{Ans .:} \dfrac{1}{4}t\sin 2t\right]$

13. $\dfrac{(s+3)^2}{(s^4+6s+5)^2}$

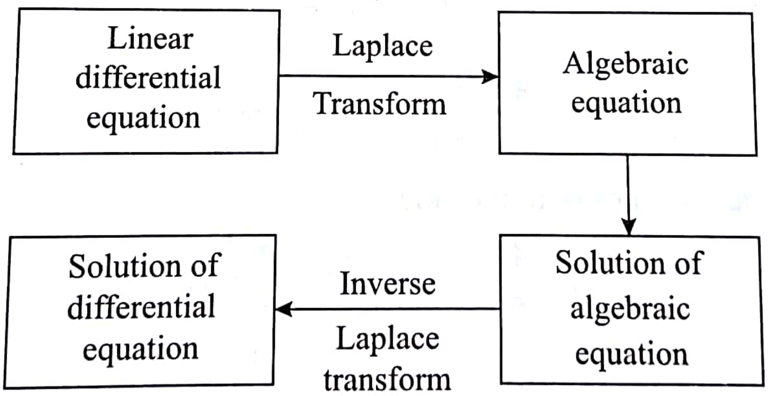
$\left[\textbf{Ans .:} \dfrac{1}{4}(2t\cosh 2t + \sinh 2t)\right]$

14. $\dfrac{1}{s(s+1)(s+2)}$

$\left[\textbf{Ans .:} \dfrac{1}{2} + \dfrac{1}{2}e^{-2t} - e^{-t}\right]$

15.11 APPLICATIONS OF LAPLACE TRANSFORM TO DIFFERENTIAL EQUATIONS OF SECOND ORDER WITH CONSTANT COEFFICIENTS

The Laplace transform is useful in solving linear differential equations with given initial conditions by using algebraic methods. Initial conditions are included from the very beginning of the solution.



EXAMPLE 15.77 *Solve* $\dfrac{dy}{dt} + y = \cos 2t,\ y(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$sY(s) - y(0) + Y(s) = \frac{s}{s^2+4}$$

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$$sY(s)-1+Y(s)=\frac{s}{s^2+4} \qquad [\because y(0)=1]$$

$$(s+1)\,Y(s)=\frac{s}{s^2+4}+1=\frac{s^2+s+4}{(s^2+4)}$$

$$Y(s)=\frac{s^2+s+4}{(s+1)(s^2+4)}$$

By partial fraction expansion,

$$Y(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+4}$$

$$s^2+s+4=A(s^2+4)+(Bs+C)\,(s+1) \qquad \ldots (1)$$

Putting $s=-1$ in Eq. (1),

$$4=5A \qquad \ldots (2)$$

Putting $s=0$ in Eq. (1),

$$4=4A+C \qquad \ldots (3)$$

Putting $s=1$ in Eq. (1),

$$6=5A+2B+2C \qquad \ldots (4)$$

Solving Eqs (2), (3), and (4),

$$A=\frac{4}{5},\ B=\frac{1}{5},\ C=\frac{4}{5}$$

$$Y(s)=\frac{4}{5}\cdot\frac{1}{s+1}+\frac{1}{5}\cdot\frac{s}{s^2+4}+\frac{4}{5}\cdot\frac{1}{s^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{4}{5}e^{-t}+\frac{1}{5}\cos 2t+\frac{2}{5}\sin 2t$$

EXAMPLE 15.78 *Solve* $(D^2+9)y=18t,\ \ y(0)=0,\ \ y\left(\frac{\pi}{2}\right)=1.$

Solution: Taking Laplace transform of both the sides,

$$\left[s^2\,Y(s)-sy(0)-y'(0)\right]+9Y(s)=\frac{18}{s^2}$$

Let $\quad y'(0)=A$

$$s^2\,Y(s)-A+9Y(s)=\frac{18}{s^2} \qquad [\because y(0)=0]$$

$$(s^2+9)\,Y(s)=\frac{18}{s^2}+A$$

$$Y(s)=\frac{18}{s^2(s^2+9)}+\frac{A}{s^2+9}=\frac{18}{9}\left(\frac{1}{s^2}-\frac{1}{s^2+9}\right)+\frac{A}{s^2+9}=\frac{2}{s^2}+\frac{A-2}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=2t+\frac{A-2}{3}\sin 3t$$

Putting $t=\dfrac{\pi}{2}$ and $y\left(\dfrac{\pi}{2}\right)=1$,

$$1=2\cdot\frac{\pi}{2}+\frac{A-2}{3}\sin\frac{3\pi}{2}=\pi-\frac{A-2}{3}$$

$$3=3\pi-A+2$$

$$A=3\pi-1$$

Hence, $y(t)=2t+\dfrac{3\pi-1-2}{3}\sin 3t=2t+(\pi-1)\sin 3t$

EXAMPLE 15.79 *Solve* $y''+9y=\cos 2t,\ y(0)=1,\ y\left(\dfrac{\pi}{2}\right)=-1.$

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s)-s\,y(0)-y'(0)]+9Y(s)=\frac{s}{s^2+4}$$

Let $y'(0)=A$

$$s^2Y(s)-s-A+9Y(s)=\frac{s}{s^2+4} \qquad [\because y(0)=1]$$

$$(s^2+9)\,Y(s)=\frac{s}{s^2+4}+s+A$$

$$Y(s)=\frac{s}{(s^2+4)(s^2+9)}+\frac{s}{s^2+9}+\frac{A}{s^2+9}=\frac{s}{5}\left[\frac{(s^2+9)-(s^2+4)}{(s^2+4)(s^2+9)}\right]+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

$$=\frac{1}{5}\cdot\frac{s}{s^2+4}-\frac{1}{5}\cdot\frac{s}{s^2+9}+\frac{s}{s^2+9}+\frac{A}{s^2+9}=\frac{1}{5}\cdot\frac{s}{s^2+4}+\frac{4}{5}\cdot\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

Taking inverse Laplace transform of both the sides,

$$y(t)=\frac{1}{5}\cos 2t+\frac{4}{5}\cos 3t+\frac{A}{3}\sin 3t$$

Putting $t = \frac{\pi}{2}$ and $y\left(\frac{\pi}{2}\right) = -1$,

$$-1 = -\frac{1}{5} - \frac{A}{3}$$

$$A = \frac{12}{5}$$

Hence, $y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$

EXAMPLE 15.80 *Solve* $(D^2 + 2D + 5)y = e^{-t}\sin t,\ y(0) = 0,\ y'(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) = \frac{1}{(s+1)^2+1}$$

$$s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{s^2+2s+2} \qquad [\because y(0) = 0, y'(0) = 1]$$

$$(s^2+2s+5)\,Y(s) = \frac{1}{s^2+2s+2} + 1 = \frac{s^2+2s+3}{s^2+2s+2}$$

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

By partial fraction expansion,

$$Y(s) = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{(s^2+2s+5)}$$

$$s^2+2s+3 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

$$= (A+C)\,s^3 + (2A+B+2C+D)s^2 + (5A+2B+2C+2D)s + (5B+2D)$$

Equating the coefficients of s^3, s^2, s and s^0,

$$A + C = 0$$
$$2A + B + 2C + D = 1$$
$$5A + 2B + 2C + 2D = 2$$
$$5B + 2D = 3$$

Solving these equations,

$$A = 0,\ B = \frac{1}{3},\ C = 0,\ D = \frac{2}{3}$$

$$Y(s) = \frac{1}{3}\cdot\frac{1}{s^2+2s+2} + \frac{2}{3}\cdot\frac{1}{s^2+2s+5} = \frac{1}{3}\cdot\frac{1}{(s+1)^2+1} + \frac{2}{3}\cdot\frac{1}{(s+1)^2+4}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t = \frac{e^{-t}}{3}(\sin t + \sin 2t)$$

EXAMPLE 15.81 *Solve* $\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t,\ y(0) = 1.$

Solution: Taking Laplace transform of both the sides,

$$sY(s) - y(0) + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1}$$

$$sY(s) - 1 + 2Y(s) + \frac{1}{s}Y(s) = \frac{1}{s^2+1} \qquad [\because y(0) = 1]$$

$$\left(s + 2 + \frac{1}{s}\right)Y(s) = \frac{1}{s^2+1} + 1 = \frac{s^2+2}{s^2+1}$$

$$\frac{s^2+2s+1}{s}Y(s) = \frac{s^2+2}{s^2+1}$$

$$Y(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+2s+1)} = \frac{s(s^2+2)}{(s^2+1)(s+1)^2}$$

By partial fraction expansion,

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+1}$$

$$s(s^2+2) = A(s+1)(s^2+1) + B(s^2+1) + (Cs+D)(s+1)^2$$

$$s^3 + 2s = (A + C)s^3 + (A + B + 2C + D)s^2 + (A + C + 2D)s + A + B + D$$

Equating the coefficients of s^3,

$$1 = A + C \qquad \dots (1)$$

Equating the coefficients of s^2,

$$0 = A + B + 2C + D \qquad \dots (2)$$

Equating the coefficients of s^1,

$$2 = A + C + 2D \qquad \dots (3)$$

Equating the coefficients of s^0,

$$0 = A + B + D \qquad \dots (4)$$

Solving Eqs (1), (2), (3), and (4),

$$A = 1, B = -\frac{3}{2}\ C = 0, D = \frac{1}{2}$$

$$Y(s) = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform of both the sides,

$$y(t) = e^{-t} - \frac{3}{2}e^{-t}t + \frac{1}{2}\sin t$$

EXERCISE 15.25

Using Laplace transform, solve the following differential equations:

1. $y'+4y=1; y(0)=-3$

$$\left[\textbf{Ans.}: y(t)=\frac{1}{4}-\frac{13}{4}e^{-4t}\right]$$

2. $y'+4y=\cos t; y(0)=0$

$$\left[\textbf{Ans.}: y(t)=-\frac{4}{17}e^{-4t}+\frac{4}{17}\cos t+\frac{1}{17}\sin t\right]$$

3. $y'+3y=10\sin t; y(0)=0$

$$\left[\textbf{Ans.}: y(t)=e^{-3t}-\cos t+3\sin t\right]$$

4. $y'+0.2y=0.01t;\ y(0)=-0.25$

[Ans. : $y(t)=0.05\,t-0.25$]

5. $y''+5y'+4y=0,\ y(0)=1, y'(0)=-1$

$$\left[\textbf{Ans.}: y(t)=\frac{4}{3}e^{-t}-\frac{1}{3}e^{-4t}\right]$$

6. $y''+2y'-3y=6e^{-2t},\ y(0)=2,\ y'(0)=-14$

$$\left[\textbf{Ans.}: y(t)=-2e^{-2t}+\frac{11}{2}e^{-3t}-\frac{3}{2}e^{t}\right]$$

7. $y''-3y'+2y=4t+e^{3t},\ y(0)=1,\ y'(0)=1$

$$\left[\textbf{Ans.}: y(t)=-\frac{1}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}+2t+3\right]$$

8. $y''+2y'+y=3te^{-t},\ y(0)=4,\ y'(0)=2$

$$\left[\textbf{Ans.}: y(t)=4e^{-t}+6te^{-t}+\frac{t^3}{2}e^{-t}\right]$$

9. $y''+y=\sin t\cdot\sin 2t,\ y(0)=1,\ y'(0)=0$

$$\left[\textbf{Ans.}: y(t)=\frac{15}{16}\cos t+\frac{t}{4}\sin t+\frac{1}{16}\cos 3t\right]$$

10. $y''+y=e^{-2t}\sin t,\ y(0)=0, y'(0)=0$

$$\left[\begin{aligned}&\textbf{Ans.}: y(t)=\frac{1}{8}\sin t-\frac{1}{8}\cos t\\ &\qquad +\frac{1}{8}e^{-2t}\sin t+\frac{1}{8}e^{-2t}\cos t\end{aligned}\right]$$

11. $y''+y=t\cos 2t,\ y(0)=0, y'(0)=0$

$$\left[\textbf{Ans.}: y(t)=\frac{4}{9}\sin 2t-\frac{5}{9}\sin t-\frac{1}{3}t\cos 2t\right]$$

12. $y'+y-2\int_0^t y\,dt=\frac{t^2}{2},\ y(0)=1, y'(0)=-2.$

$$\left[\textbf{Ans.}: y(t)=\frac{1}{3}e^{t}+\frac{11}{12}e^{-2t}-\frac{1}{2}t-\frac{1}{4}\right]$$

15.12 APPLICATIONS OF LAPLACE TRANSFORM TO A SYSTEM OF SIMULTANEOUS DIFFERENTIAL EQUATIONS

The Laplace transform can also be used to solve two or more simultaneous differential equations. The Laplace transform method transforms the differential equations into algebraic equations.

EXAMPLE 15.82

Solve $\dfrac{dx}{dt}-y=e^{t}$ *and* $\dfrac{dy}{dt}+x=\sin t$

where $x(0)=1$ *and* $y(0)=0.$

Solution: Taking Laplace transform of both the equations,

$$sX(s)-x(0)-Y(s)=\frac{1}{s-1}$$

$$sX(s)-Y(s)=\frac{1}{s-1}+1=\frac{s}{s-1} \quad \ldots (1)$$

and

$$s\,Y(s)-y(0)+X(s)=\frac{1}{s^2+1}$$

$$s\,Y(s)+X(s)=\frac{1}{s^2+1} \quad \ldots (2)$$

Multiplying Eq. (1) by s,

$$s^2X(s)-s\,Y(s)=\frac{s^2}{s-1} \quad \ldots (3)$$

Adding Eqs (2) and (3),

$$(s^2+1)\,X(s)=\frac{1}{s^2+1}+\frac{s^2}{s-1}$$

$$X(s)=\frac{1}{(s^2+1)^2}+\frac{s^2}{(s-1)(s^2+1)}=\frac{1}{(s^2+1)^2}+\frac{1}{2}\left(\frac{1}{s-1}+\frac{s}{s^2+1}+\frac{1}{s^2+1}\right) \quad \ldots (4)$$

Substituting $X(s)$ in Eq. (1),

$$Y(s)=s\,X(s)-\frac{s}{s-1}=\frac{s}{(s^2+1)^2}+\frac{s^3}{(s-1)(s^2+1)}-\frac{s}{s-1}$$

$$Y(s)=\frac{s}{(s^2+1)^2}-\frac{s}{(s-1)\,(s^2+1)}=\frac{s}{(s^2+1)^2}-\frac{1}{2}\left(\frac{1}{s-1}-\frac{s}{s^2+1}+\frac{1}{s^2+1}\right) \quad \ldots (5)$$

Taking the inverse Laplace transform of Eqs (4) and (5),

$$x(t)=\frac{1}{2}(\sin t-t\cos t)+\frac{1}{2}(e^t+\cos t+\sin t)=\frac{1}{2}(e^t+\cos t+2\sin t-t\cos t)$$

and $$y(t)=\frac{1}{2}t\sin t-\frac{1}{2}(e^t-\cos t+\sin t)=\frac{1}{2}(t\sin t-e^t+\cos t-\sin t)$$

EXAMPLE 15.83 *Solve* $\dfrac{d^2x}{dt^2}-\dfrac{dy}{dt}=te^{-t}-2e^{-t}-3$ *and* $\dfrac{dx}{dt}-2y-x=-2te^{-t}+e^{-t}-6t$ *where* $x(0)=0$, $x'(0)=1$ *and* $y(0)=0$.

Solution: Taking Laplace transform of both the equations,

$$[s^2X(s)-sx(0)-x'(0)]-[sY(s)-y(0)]=\frac{1}{(s+1)^2}-\frac{2}{s+1}-\frac{3}{s}$$

$$s^2 X(s) - s\,Y(s) = 1+\frac{1}{(s+1)^2}-\frac{2}{s+1}-\frac{3}{s}$$

$$s^2 X(s) - s\,Y(s) = \frac{s^2}{(s+1)^2}-\frac{3}{s} \qquad \dots (1)$$

and
$$s\,X(s) - x(0) - 2\,Y(s) - X(s) = -\frac{2}{(s+1)^2}+\frac{1}{s+1}-\frac{6}{s^2}$$

$$(s-1)\,X(s) - 2\,Y(s) = \frac{s-1}{(s+1)^2}-\frac{6}{s^2} \qquad \dots (2)$$

Multiplying Eq. (2) by $\frac{s}{2}$,

$$\frac{s(s-1)}{2}X(s)-sY(s)=\frac{s(s-1)}{2(s+1)^2}-\frac{3}{s} \qquad \dots (3)$$

Subtracting Eq. (3) from Eq. (1),

$$\frac{(s^2+s)}{2}X(s)=\frac{s^2+s}{2(s+1)^2}$$

$$X(s)=\frac{1}{(s+1)^2} \qquad \dots (4)$$

Substituting $X(s)$ in Eq. (1),

$$\frac{s^2}{(s+1)^2}-sY(s)=\frac{s^2}{(s+1)^2}-\frac{3}{s}$$

$$Y(s)=\frac{3}{s^2} \qquad \dots (5)$$

Taking inverse Laplace transform of Eqs (4) and (5),

$$x(t) = t\,e^{-t}$$

and
$$y(t) = 3t$$

EXERCISE 15.26

Solve the following simultaneous equations:

1. $\frac{dx}{dt}+\frac{dy}{dt}+x=e^{-t}$

   $\frac{dx}{dt}+2\frac{dy}{dt}+2x+2y=0$

   where $x(0)=-1, y(0)=1$.

   [**Ans.:** $x(t)=-e^{-t}(\cos t+\sin t)$, $y(t)=e^{-t}(1+\sin t)$]

2. $\dfrac{dx}{dt} = 2x - 3y$

$\dfrac{dy}{dt} = y - 2x$

where $x(0) = 8, y(0) = 3$.

$\left[\textbf{Ans.: } \begin{aligned} x(t) &= 5e^{-t} + 8e^{4t}, \\ y(t) &= 5e^{-t} - 2e^{4t} \end{aligned}\right]$

3. $\dfrac{dx}{dt} - \dfrac{dy}{dt} + 2y = \cos 2t$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x = \sin 2t$

where $x(0) = 0, y(0) = -1$.

$\left[\begin{aligned} &\textbf{Ans.:} \\ &x(t) = \frac{1}{2}e^{t}(\cos t + \sin t) - \frac{1}{2}\cos 2t, \\ &y(t) = -e^{t}(\cos t - \sin t) - \sin 2t \end{aligned}\right]$

4. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = e^{-t}$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = e^{-t}$

where $x(0) = 2,\ y(0) = 1$.

$\left[\textbf{Ans.: } \begin{aligned} x(t) &= 2\cos t + 8\sin t, \\ y(t) &= \cos t - 13\sin t + \sinh t \end{aligned}\right]$

5. $\dfrac{d^2x}{dt^2} + y = -5\cos 2t$

$\dfrac{d^2y}{dt^2} + x = 5\cos 2t$

where $x(0) = 1, x'(0) = 1,$
$y'(0) = 1, y(0) = -1$.

$\left[\textbf{Ans.: } \begin{aligned} x(t) &= \sin t + \cos 2t, \\ y(t) &= \sin t - \cos 2t \end{aligned}\right]$

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