3 GAMMA FUNCTION

Gamma function is defined by the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$, $n > 0$ and is denoted by Γn.

Hence,

$$\Gamma n = \int_0^\infty e^{-x}x^{n-1}dx,\ n > 0$$

Alternate Form of Gamma Function

$$\Gamma n = 2\int_0^\infty e^{-x^2}x^{2n-1}dx$$

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Proof By definition,

$$\overline{|n} = \int_0^\infty e^{-x} x^{n-1} \mathrm{d}x$$

Let $x = t^2$, $\quad \mathrm{d}x = 2t\,\mathrm{d}t$

$$\overline{|n} = \int_0^\infty e^{-t^2} \cdot t^{2n-2} \cdot 2t\,\mathrm{d}t = 2\int_0^\infty e^{-t^2} \cdot t^{2n-1}\,\mathrm{d}t$$

Changing the variable t to x,

$$\overline{|n} = 2\int_0^\infty e^{-x^2} \cdot x^{2n-1} \mathrm{d}x$$

6.4 PROPERTIES OF GAMMA FUNCTIONS

(1) $\overline{|n+1} = n\,\overline{|n}$

Proof $\overline{|n+1} = \int_0^\infty e^{-x} x^n \mathrm{d}x$

Integrating by parts,

$$\overline{|n+1} = \left|-e^{-x}x^n\right|_0^\infty - \int_0^\infty (-e^{-x})\, n x^{n-1} \mathrm{d}x = n\int_0^\infty e^{-x} x^{n-1} \mathrm{d}x = n\,\overline{|n}$$

Hence, $\quad \overline{|n+1} = n\,\overline{|n}$

This is known as *recurrence* or *reduction formula* for Gamma function.

Notes

(i) $\overline{|n+1} = n!$ $\quad$ if n is a positive integer

(ii) $\overline{|n+1} = n\,\overline{|n}$ $\quad$ if n is a positive real number

(iii) $\overline{|n} = \dfrac{\overline{|n+1}}{n}$, $\quad$ if n is a negative fraction

(iv) $\overline{|n}\,\overline{|1-n} = \dfrac{\pi}{\sin n\pi}$

(2) $\overline{\left|\dfrac{1}{2}\right.} = \sqrt{\pi}$

Proof By alternate form of Gamma function,

$$\overline{\left|\frac{1}{2}\right.} = 2\int_0^\infty e^{-x^2} x^{2\left(\frac{1}{2}\right)-1} \mathrm{d}x = 2\int_0^\infty e^{-x^2} \mathrm{d}x$$

$$\overline{\left|\frac{1}{2}\right.} \cdot \overline{\left|\frac{1}{2}\right.} = 2\int_0^\infty e^{-x^2} \mathrm{d}x \cdot 2\int_0^\infty e^{-y^2} \mathrm{d}y = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\,\mathrm{d}y$$

Changing to polar coordinates, $x = r\cos\theta$, $\quad y = r\sin\theta$

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

Limits of x $\quad x = 0$ to $x \to \infty$

Limits of y $\quad y = 0$ to $y \to \infty$

This shows that the region of integration is the first quadrant.
Draw an elementary radius vector in the region which starts from the pole and extends up to ∞ (Fig. 6.1).

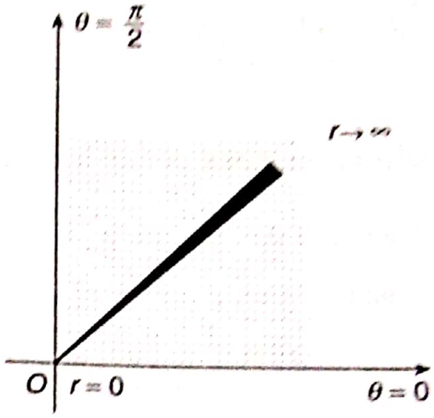

Fig. 6.1 Region of integration

Limits of r $\quad r = 0 \quad$ to $\quad r \to \infty$

Limits of θ $\quad \theta = 0 \quad$ to $\quad \theta = \frac{\pi}{2}$

$$\left\lceil\frac{1}{2}\cdot\right\lceil\frac{1}{2} = 4\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}\cdot r\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{\infty}\left(-\frac{1}{2}\right)e^{-r^2}(-2r)\,dr$$

$$= \frac{4}{-2}\left|\theta\right|_0^{\frac{\pi}{2}}\left|e^{-r^2}\right|_0^{\infty} \qquad \left[\because \int e^{f(r)}\cdot f'(r)dr = e^{f(r)}\right]$$

$$= -2\cdot\frac{\pi}{2}(0-1) = \pi$$

$$\left\lceil\frac{1}{2} = \sqrt{\pi}\right.$$

EXAMPLE 6.5 *Given* $\left\lceil\frac{8}{5}\right. = 0.8935$, *find the value of* $\left\lceil -\frac{12}{5}\right.$.

Solution: $\lceil n = \dfrac{\lceil n+1}{n}$

$$\left\lceil -\frac{12}{5}\right. = \frac{\left\lceil -\frac{12}{5}+1\right.}{-\frac{12}{5}} = -\frac{5}{12}\cdot\frac{\left\lceil -\frac{7}{5}+1\right.}{-\frac{7}{5}} = \frac{25}{84}\cdot\frac{\left\lceil -\frac{2}{5}+1\right.}{-\frac{2}{5}} = -\frac{125}{168}\cdot\frac{\left\lceil \frac{3}{5}+1\right.}{\frac{3}{5}} = -\frac{625}{504}\left\lceil\frac{8}{5}\right. = -\frac{625}{504}(0.8935) = -1.108$$

EXAMPLE 6.6 *Evaluate* $\int_0^{\infty} e^{-\sqrt{x}}x^{\frac{1}{4}}\,dx$.

Solution: Let $\sqrt{x} = t$, $x = t^2$, $dx = 2t\,dt$

When $x = 0, \qquad t = 0$

When $x \to \infty, \qquad t \to \infty$

$$\int_0^{\infty} e^{-\sqrt{x}}x^{\frac{1}{4}}\,dx = \int_0^{\infty} e^{-t}(t^2)^{\frac{1}{4}}\,2t\,dt = 2\int_0^{\infty} e^{-t}t^{\frac{3}{2}}\,dt = 2\left\lceil\frac{5}{2}\right. = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right. = \frac{3}{2}\sqrt{\pi}$$

EXAMPLE 6.7 *Evaluate* $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\,dx \cdot \int_0^\infty x^4 e^{-x^6}\,dx.$

Solution: Let $x^3 = t$, $x = t^{\frac{1}{3}}$, $dx = \frac{1}{3}t^{-\frac{2}{3}}dt$

When $x = 0$, $\quad t = 0$

When $x \to \infty$, $\quad t \to \infty$

$$\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\,dx \cdot \int_0^\infty x^4 e^{-x^6}\,dx = \int_0^\infty \frac{e^{-t}}{t^{\frac{1}{6}}}\cdot\frac{1}{3}t^{-\frac{2}{3}}dt\cdot\int_0^\infty t^{\frac{4}{3}}e^{-t^2}\cdot\frac{1}{3}t^{-\frac{2}{3}}dt$$

$$= \frac{1}{9}\int_0^\infty e^{-t}t^{-\frac{5}{6}}dt\cdot\int_0^\infty e^{-t^2}t^{\frac{2}{3}}dt = \frac{1}{9}\Gamma\frac{1}{6}\cdot\frac{1}{2}\cdot 2\int_0^\infty e^{-t^2}t^{2\left(\frac{5}{6}\right)-1}dt$$

$$= \frac{1}{9}\Gamma\frac{1}{6}\cdot\frac{1}{2}\Gamma\frac{5}{6} \qquad \left[\because 2\int_0^\infty e^{-x^2}x^{2n-1}dx = \Gamma n\right]$$

$$= \frac{1}{18}\Gamma\frac{1}{6}\,\Gamma\left(1-\frac{1}{6}\right) = \frac{1}{18}\cdot\frac{\pi}{\sin\frac{\pi}{6}} = \frac{\pi}{9}$$

EXAMPLE 6.8 *Evaluate* $\int_0^1 \frac{dx}{\sqrt{x\log\left(\frac{1}{x}\right)}}.$

Solution: $$\int_0^1 \frac{dx}{\sqrt{x\log\left(\frac{1}{x}\right)}} = \int_0^1 x^{-\frac{1}{2}}\left[\log\left(\frac{1}{x}\right)\right]^{-\frac{1}{2}}dx \qquad \ldots (1)$$

Let $\log\left(\frac{1}{x}\right) = t$, $\frac{1}{x} = e^t$, $x = e^{-t}$, $dx = -e^{-t}\,dt$

When $x = 0$, $\quad t \to \infty$

When $x = 1$, $\quad t = 0$

$$\int_0^1 \frac{1}{\sqrt{x\log\left(\frac{1}{x}\right)}}dx = \int_\infty^0 (e^{-t})^{-\frac{1}{2}}\cdot t^{-\frac{1}{2}}(-e^{-t})\,dt = \int_0^\infty e^{-\frac{t}{2}}t^{\frac{1}{2}-1}dt$$

$$= \frac{\Gamma\frac{1}{2}}{\left(\frac{1}{2}\right)^{\frac{1}{2}}} = \sqrt{2\pi} \qquad \left[\because \int_0^\infty e^{-kx}x^{n-1}dx = \frac{\Gamma n}{k^n}\right]$$

...MPLE 6.9 Evaluate $\int_0^\infty \frac{x^a}{a^x}\,dx$.

...n: Let $a^x = e^t$, $x \log a = t$, $dx = \frac{1}{\log a}dt$

$x = 0$, $t = 0$

$x \to \infty$, $t \to \infty$

$$\int_0^\infty \frac{x^a}{a^x}dx = \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot \frac{1}{e^t}\cdot\frac{1}{\log a}dt = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}t^a dt = \frac{1}{(\log a)^{a+1}}\overline{|a+1} = \frac{\overline{|a+1}}{(\log a)^{a+1}}$$

...MPLE 6.10 Evaluate $\int_0^\infty 3^{-4x^2}dx$.

...ion: Let $3^{-4x^2} = e^{-t}$, $-4x^2\log 3 = -t\log e$, $4x^2\log 3 = t$

$$x = \frac{\sqrt{t}}{2\sqrt{\log 3}},\quad dx = \frac{1}{2\sqrt{\log 3}}\cdot\frac{1}{2\sqrt{t}}dt$$

$x = 0$, $t = 0$

$x \to \infty$, $t \to \infty$

$$\int_0^\infty 3^{-4x^2}dx = \int_0^\infty e^{-t}\cdot\frac{1}{4\sqrt{\log 3}}\cdot\frac{1}{\sqrt{t}}dt = \frac{1}{4\sqrt{\log 3}}\int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = \frac{1}{4\sqrt{\log 3}}\overline{\left|\frac{1}{2}\right.} = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

...XAMPLE 6.11 *Prove that* $\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$.

...tion:

$$\int_0^\infty xe^{-ax}\sin bx\,dx = \int_0^\infty xe^{-ax}\,[\text{Imaginary part of } e^{ibx}]dx$$

$$= \text{Im. part}\int_0^\infty xe^{-ax}\cdot e^{ibx}dx = \text{Im. part}\int_0^\infty e^{-(a-ib)x}\cdot x\,dx$$

$$= \text{Im. part}\frac{\overline{|2}}{(a-ib)^2}\qquad\left[\because \int_0^\infty e^{-kx}x^{n-1}dx = \frac{\overline{|n}}{k^n}\right]$$

$$= \text{Im. part}\frac{1}{(a^2-b^2)-2iab} = \text{Im. part}\left[\frac{(a^2-b^2)+2iab}{(a^2-b^2)^2+4a^2b^2}\right] = \frac{2ab}{(a^2+b^2)^2}$$

EXERCISE 6.3

1. Evaluate the following integrals:

(i) $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$

(ii) $\int_0^\infty e^{-\frac{x^2}{4}}\, dx$

(iii) $\int_0^\infty \frac{e^{-\sqrt{x}}}{x^{\frac{7}{4}}}\, dx$

(iv) $\int_0^1 (x \log x)^4\, dx$

(v) $\int_0^1 \sqrt{\log \frac{1}{x}}\, dx$

(vi) $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

(vii) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx$

(viii) $\int_0^1 \sqrt[3]{x \log \frac{1}{x}}\, dx$

(ix) $\int_0^\infty 5^{-4x^2}\, dx$

Ans.: (i) $\frac{315}{16}\sqrt{\pi}$ (ii) $\sqrt{\pi}$ (iii) $\frac{8}{3}\sqrt{\pi}$ (iv) $\frac{4!}{5^5}$ (v) $\frac{\sqrt{\pi}}{2}$ (vi) $\sqrt{\pi}$ (vii) $\frac{6}{625}$ (viii) $\left(\frac{3}{4}\right)^{\frac{4}{3}} \Gamma\left(\frac{4}{3}\right)$ (ix) $\frac{\sqrt{\pi}}{4\sqrt{\log 5}}$

2. Prove that

$$\int_0^\infty x^m e^{-ax^n}\, dx = \frac{1}{n a^{\frac{(m+1)}{n}}} \Gamma\left(\frac{m+1}{n}\right)$$

3. Prove that

$$\int_0^\infty x^2 e^{-x^4}\, dx \cdot \int_0^\infty e^{-x^4}\, dx = \frac{\pi}{8\sqrt{2}}$$

4. Prove that

$$\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx \cdot \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}}\, dx = \frac{\pi}{2\sqrt{2}}$$

5. Prove that

$$\int_0^\infty x e^{-x^8}\, dx \cdot \int_0^\infty x^2 e^{-x^4}\, dx = \frac{\pi}{16\sqrt{2}}$$

6. Prove that

$$\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}}$$

7. Prove that

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$$

8. Prove that

$$\int_0^\infty x^{m-1} \cos ax\, dx = \frac{\Gamma(m)}{a^m} \cos\left(\frac{m\pi}{2}\right)$$

9. Prove that $\int_0^\infty x^{n-1} e^{-ax} \sin bx\, dx$

$$= \frac{\Gamma(n)}{(a^2+b^2)^{\frac{n}{2}}} \sin\left(n \tan^{-1} \frac{b}{a}\right)$$

5 BETA FUNCTION

ta function $B(m, n)$ is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m > 0,\ n > 0.$$

$m, n)$ is also known as Euler's integral of the first kind.

igonometric Form of Beta Function

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}x \cos^{2n-1}x\,dx$$

oof $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

t $x = \sin^2\theta$, $\quad dx = 2\sin\theta\cos\theta\,d\theta$

hen $x = 0$, $\quad \theta = 0$

hen $x = 1$, $\quad \theta = \dfrac{\pi}{2}$

$$B(m, n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

placing θ by x,

$$B(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}x \cos^{2n-1}x\,dx$$

Corollary Putting $2m-1 = p$, $\quad 2n-1 = q$

$$m = \frac{p+1}{2}, \qquad n = \frac{q+1}{2}$$

$$B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x\,dx$$

6 PROPERTIES OF BETA FUNCTIONS

Symmetry $B(m, n) = B(n, m)$

oof $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

t $1 - x = t$, $\quad -dx = dt$

hen $x = 0$, $\quad t = 1$

hen $x = 1$, $\quad t = 0$

$$B(m, n) = \int_1^0 (1-t)^{m-1} t^{n-1}(-dt) = \int_0^1 t^{n-1}(1-t)^{m-1}\,dt = B(n, m)$$

Relation between Beta and Gamma Functions

$$B(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

Proof By alternate form of Gamma function,

$$\boxed{m}\,\boxed{n} = 2\int_0^\infty e^{-x^2} x^{2m-1}\mathrm{d}x \cdot 2\int_0^\infty e^{-y^2} y^{2n-1}\mathrm{d}y$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\mathrm{d}x\,\mathrm{d}y$$

Changing to polar coordinates, $x = r\cos\theta,\ y = r\sin\theta$

$$\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$$

Limits of x $\quad x = 0 \quad$ to $\quad x \to \infty$

Limits of y $\quad y = 0 \quad$ to $\quad y \to \infty$

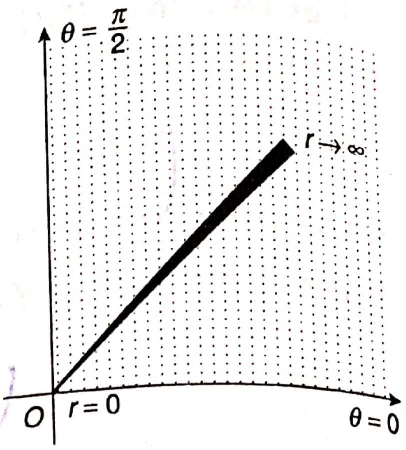

Fig. 6.2 Region of integration

This shows that the region of integration is the first quadrant.

Draw an elementary radius vector in the region which starts from the pole and extends up to ∞ (Fig. 6.2).

Limits of r $\quad r = 0 \quad$ to $\quad r \to \infty$

Limits of θ $\quad \theta = 0 \quad$ to $\quad \theta = \frac{\pi}{2}$

$$\boxed{m}\,\boxed{n} = 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= 4\int_0^{\frac{\pi}{2}}(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}\mathrm{d}\theta \cdot \int_0^{\frac{\pi}{2}} e^{-r^2} r^{2(m+n)-1}\mathrm{d}r = 4\cdot\frac{1}{2}B(m,n)\cdot\frac{1}{2}\boxed{m+n}$$

$$B(m, n) = \frac{\boxed{m}\,\boxed{n}}{\boxed{m+n}}$$

3. Duplication Formula

$$\boxed{m}\,\boxed{m+\frac{1}{2}} = \frac{\sqrt{\pi}\,\boxed{2m}}{2^{2m-1}}$$

Proof $B(m, n) = 2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,\mathrm{d}\theta$

Putting $n = m$,

$$B(m, m) = 2\int_0^{\frac{\pi}{2}}(\sin\theta\cos\theta)^{2m-1}\,\mathrm{d}\theta$$

$$\frac{\boxed{m}\,\boxed{m}}{\boxed{2m}} = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}(\sin 2\theta)^{2m-1}\,\mathrm{d}\theta$$

Let $2\theta = t, \quad \mathrm{d}\theta = \frac{1}{2}\mathrm{d}t$

When $\theta = 0, \quad t = 0$

When $\theta = \frac{\pi}{2}, \quad t = \pi$

$$\frac{\Gamma m \cdot \Gamma m}{\Gamma 2m} = \frac{2}{2^{2m-1}}\int_0^{\pi}(\sin t)^{2m-1}\cdot\frac{1}{2}\,dt$$

$$= \frac{1}{2^{2m-1}}\cdot 2\int_0^{\frac{\pi}{2}}(\sin t)^{2m-1}(\cos t)^0\,dt \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ if } f(2a-x)=f(x)\right]$$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}}(\sin t)^{2m-1}(\cos t)^{2\left(\frac{1}{2}\right)-1}\,dt = \frac{1}{2^{2m-1}}B\left(m,\frac{1}{2}\right) = \frac{1}{2^{2m-1}}\frac{\Gamma m\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}}\frac{\Gamma m\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma 2m}{2^{2m-1}}$$

EXAMPLE 6.12 *Prove that* $B(n, n) = \frac{\sqrt{\pi}}{2^{2n-1}}\cdot\frac{\Gamma n}{\Gamma\left(n+\frac{1}{2}\right)}$.

Solution: $$B(n, n) = \frac{\Gamma n\,\Gamma n}{\Gamma 2n} = \frac{\Gamma n}{\Gamma 2n}\cdot\frac{\Gamma n\,\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)} = \frac{\Gamma n}{\Gamma 2n}\cdot\frac{1}{\Gamma\left(n+\frac{1}{2}\right)}\cdot\frac{\sqrt{\pi}\,\Gamma 2n}{2^{2n-1}} \quad \text{[By Duplication formula]}$$

$$= \frac{\sqrt{\pi}}{2^{2n-1}}\cdot\frac{\Gamma n}{\Gamma\left(n+\frac{1}{2}\right)}$$

EXAMPLE 6.13 *Prove that* $\Gamma n\,\Gamma\left(\frac{1-n}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\frac{n}{2}}{2^{1-n}\cos\frac{n\pi}{2}}$.

Solution: $$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Replacing n by $\dfrac{n+1}{2}$,

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\frac{n+1}{2}\right)=\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi}$$

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)}$$

$$\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi\,\Gamma\left(\frac{n}{2}\right)}{\cos\frac{n\pi}{2}}$$

$$\frac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}}\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi\,\Gamma\left(\frac{n}{2}\right)}{\cos\frac{n\pi}{2}}$$

$$\Gamma(n)\,\Gamma\left(\frac{1-n}{2}\right)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}{2^{1-n}\cos\frac{n\pi}{2}}$$

EXERCISE 6.4

1. Find the value of

   (i) $B\left(\frac{5}{2},\frac{3}{2}\right)$ (ii) $B\left(\frac{1}{2},\frac{2}{3}\right)$

   $\left[\textbf{Ans.:}\ \text{(i)}\ \frac{\pi}{16}\quad \text{(ii)}\ \frac{2\pi}{\sqrt{3}}\right]$

2. If $B(n, 2)=\dfrac{1}{42}$ and n is a positive integer, find the value of n.

   [Ans. : $n = 6$**]**

3. Prove that

   $$B\left(n+\frac{1}{2},\ n+\frac{1}{2}\right)=\frac{1}{2^{2n}}\cdot\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\sqrt{\pi}.$$

4. Prove that
   $B(m, n) = B(m, n + 1) + B(m + 1, n)$.

5. Prove that

   $$\Gamma\left(\frac{3}{2}-n\right)\Gamma\left(\frac{3}{2}+n\right)=\left(\frac{1}{4}-n^2\right)\pi\sec n\pi,$$
   $(-1 < 2n < 1)$.

...ems Based on Definition of Beta Functions

...MPLE 6.14 Evaluate $\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx$.

$$\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx = \int_0^{2a} x^{\frac{5}{2}}\sqrt{2a-x}\,dx$$

...on:
$x = 2at$, $dx = 2a\,dt$
$x = 0$, $t = 0$
$x = 2a$, $t = 1$

$$\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx = \int_0^1 (2at)^{\frac{5}{2}}\sqrt{2a-2at}\cdot 2a\,dt = 16a^4\int_0^1 t^{\frac{5}{2}}(1-t)^{\frac{1}{2}}\,dt = 16a^4\,B\left(\frac{7}{2},\frac{3}{2}\right)$$

$$= 16a^4\frac{\sqrt{\frac{7}{2}}\sqrt{\frac{3}{2}}}{\sqrt{5}} = \frac{16a^4}{24}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\frac{1}{2}}\cdot\frac{1}{2}\sqrt{\frac{1}{2}} = \frac{15\pi a^4}{24}$$

...MPLE 6.15 Prove that $\int_0^a \frac{dx}{(a^n-x^n)^{\frac{1}{n}}} = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$.

...tion: Let $x^n = a^n t$, $x = at^{\frac{1}{n}}$, $dx = \frac{a}{n}t^{\frac{1}{n}-1}dt$

...en $x=0$, $t = 0$
...en $x=a$, $t = 1$

$$\int_0^a \frac{dx}{(a^n-x^n)^{\frac{1}{n}}} = \int_0^1 \frac{1}{(a^n-a^nt)^{\frac{1}{n}}}\cdot\frac{a}{n}t^{\frac{1}{n}-1}dt = \frac{1}{n}\int_0^1 (1-t)^{-\frac{1}{n}}t^{\frac{1}{n}-1}dt = \frac{1}{n}B\left(-\frac{1}{n}+1,\frac{1}{n}\right)$$

$$= \frac{1}{n}\cdot\frac{\sqrt{1-\frac{1}{n}}\sqrt{\frac{1}{n}}}{\sqrt{1}} = \frac{1}{n}\cdot\frac{\pi}{\sin\frac{\pi}{n}} \qquad \left[\because \sqrt{1-n}\sqrt{n} = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

...XAMPLE 6.16 Prove that $\int_a^b (x-a)^m(b-x)^n\,dx = (b-a)^{m+n+1}B(m+1,n+1)$ *and hence, deduce that* $\int_5^9 \sqrt[4]{(x-5)(9-x)}\,dx = \frac{2\left(\sqrt{\frac{1}{4}}\right)^2}{3\sqrt{\pi}}$.

...ution: Let $(x-a) = (b-a)\,t$, $\quad dx = (b-a)\,dt$

When $x = a$, $t = 0$
When $x = b$, $t = 1$

$$\int_a^b (x-a)^m (b-x)^n \, dx = \int_0^1 [(b-a)t]^m [b - \{a + (b-a)t\}]^n (b-a)dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n \, dt$$

$$= (b-a)^{m+n+1} B(m+1, \ n+1)$$

Putting $a = 5$, $b = 9$, $m = \dfrac{1}{4}$, $n = \dfrac{1}{4}$ in the above integral,

$$\int_5^9 (x-5)^{\frac{1}{4}} (9-x)^{\frac{1}{4}} dx = (9-5)^{\frac{1}{4}+\frac{1}{4}+1} B\left(\frac{1}{4}+1, \ \frac{1}{4}+1\right)$$

$$= 2^3 \frac{\left|\frac{5}{4}\right. \left|\frac{5}{4}\right.}{\left|\frac{5}{2}\right.} = 8 \frac{\left(\frac{1}{4}\left|\frac{1}{4}\right.\right)^2}{\frac{3}{2} \cdot \frac{1}{2} \left|\frac{1}{2}\right.} = \frac{2\left(\left|\frac{1}{4}\right.\right)^2}{3\sqrt{\pi}}$$

EXAMPLE 6.17 *Prove that* $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \frac{B(m,n)}{(a+b)^m \, a^n}$ *and hence, evaluate* $\displaystyle\int_0^1 \frac{x^2 - 2x^3 + x^4}{(1+x)^6} dx.$

Solution: Let $x = \dfrac{at}{a+b-bt}$, $\quad dx = \dfrac{a(a+b-bt) - at(-b)}{(a+b-bt)^2} dt = \dfrac{a(a+b)}{(a+b-bt)^2} dt$

When $x = 0$, $t = 0$
When $x = 1$, $t = 1$

Also,
$$1 - x = 1 - \frac{at}{a+b-bt} = \frac{(a+b)(1-t)}{a+b-bt}$$

$$a + bx = a + \frac{bat}{a+b-bt} = \frac{a(a+b)}{a+b-bt}$$

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}} dx = \int_0^1 \frac{\left(\dfrac{at}{a+b-bt}\right)^{m-1} \left[\dfrac{(a+b)(1-t)}{a+b-bt}\right]^{n-1}}{\left[\dfrac{a(a+b)}{a+b-bt}\right]^{m+n}} \cdot \frac{a(a+b)}{(a+b-bt)^2} dt$$

$$= \frac{1}{(a+b)^m \, a^n} \int_0^1 t^{m-1}(1-t)^{n-1} dt = \frac{1}{(a+b)^m \, a^n} B(m, \ n)$$

$\ldots= 1, b = 1, m = 3, n = 3$ in the above integral,

$$\int_0^1 \frac{x^2(1-x)^2}{(1+x)^6}\,dx = \frac{1}{(1+1)^3 \cdot 1^3} B(3, 3)$$

$$\int_0^1 \frac{x^2-2x^3+x^4}{(1+x)^6}\,dx = \frac{1}{8}\frac{\Gamma 3\,\Gamma 3}{\Gamma 6} = \frac{1}{8}\cdot\frac{4}{120} = \frac{1}{240}$$

…CISE 6.5

…luate the following integrals:

(i) $\int_0^1 \sqrt{1-x^m}\,dx$

(ii) $\int_0^1 \frac{dx}{\sqrt{1-x^6}}$

(iii) $\int_0^1 \left(1-x^{\frac{1}{4}}\right)^{\frac{2}{3}} dx$

(iv) $\int_0^2 x^2(2-x)^{-\frac{1}{2}}dx$

(v) $\int_0^a x^4\sqrt{a^2-x^2}\,dx$

(vi) $\int_0^{\frac{1}{2}} x^3\sqrt{1-4x^2}\,dx$

[**Ans.:** (i) $\frac{1}{m}B\left(\frac{1}{m}, \frac{3}{2}\right)$ (ii) $\frac{1}{8}B\left(\frac{1}{8}, \frac{1}{2}\right)$ (iii) $\frac{128}{1155}$ (iv) $\frac{64\sqrt{2}}{15}$ (v) $\frac{\pi a^6}{32}$ (vi) $\frac{1}{120}$]

2. Prove that

(i) $\int_3^7 \sqrt[4]{(x-3)(7-x)}\,dx = \frac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$

(ii) $\int_5^6 (x-5)^5(6-x)^6\,dx = \frac{5!\,6!}{12!}$.

3. Prove that $\int_0^1 \frac{x^{-\frac{1}{3}}(1-x)^{-\frac{2}{3}}}{(1+2x)}\,dx = \frac{\pi}{3^{\frac{7}{6}}}$.

4. Prove that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(1+x)^{m+n}}\,dx = \frac{B(m, n)}{2^m}$,

and hence, evaluate $\int_0^1 \frac{x^3-2x^2+x}{(1+x)^5}\,dx$.

[**Ans.:** $\frac{1}{48}$]

5. Prove that $\int_0^1 \frac{x^{n-1}}{(1+cx)(1-x)^n}\,dx$

$$= \frac{1}{(1+c)^n}\cdot\frac{\pi}{\sin n\pi},\quad 0 < n < 1.$$

…s Based on Trigonometric Form of Beta Functions

…E 6.18 *Evaluate* $\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta\, d\theta$.

$$\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta\, d\theta = \int_0^{\frac{\pi}{4}} \cos^3 2\theta (2\sin 2\theta \cos 2\theta)^4\, d\theta$$

$$= 16\int_0^{\frac{\pi}{4}} \cos^7 2\theta \sin^4 2\theta\, d\theta$$

Let $2\theta = t, \qquad d\theta = \frac{1}{2}dt$

When $\theta = 0, \qquad t = 0$

When $\theta = \frac{\pi}{4}, \qquad t = \frac{\pi}{2}$

$$\int_0^{\frac{\pi}{4}} \cos^3 2\theta \sin^4 4\theta \, d\theta = 16\int_0^{\frac{\pi}{2}} \sin^4 t \cdot \cos^7 t \cdot \frac{1}{2}dt = 8 \cdot \frac{1}{2} B\left(\frac{5}{2}, 4\right) = 4\frac{\left\lceil\frac{5}{2}\right. \left\lceil 4\right.}{\left\lceil\frac{13}{2}\right.}$$

$$= 4 \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right. \cdot 3!}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.} = \frac{128}{1155}$$

EXAMPLE 6.19 *Evaluate* $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} d\theta.$

Solution: $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\right]^{\frac{1}{3}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}}\left(\sin\frac{\pi}{4}\cos\theta + \cos\frac{\pi}{4}\sin\theta\right)^{\frac{1}{3}} d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2^{\frac{1}{6}}\left[\sin\left(\frac{\pi}{4}+\theta\right)\right]^{\frac{1}{3}} d\theta \cdot$$

Let $\frac{\pi}{4} + \theta = t, \qquad d\theta = dt$

When $\theta = -\frac{\pi}{4}, \qquad t = 0$

When $\theta = \frac{\pi}{4}, \qquad t = \frac{\pi}{2}$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\frac{1}{3}} d\theta = 2^{\frac{1}{6}}\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{1}{3}} dt = 2^{\frac{1}{6}}\int_0^{\frac{\pi}{2}} (\sin t)^{\frac{1}{3}}(\cos t)^0 dt = \frac{2^{\frac{1}{6}}}{2} B\left(\frac{4}{6}, \frac{1}{2}\right)$$

$$= \frac{1}{2^{\frac{5}{6}}} \cdot \frac{\left\lceil\frac{2}{3}\right. \left\lceil\frac{1}{2}\right.}{\left\lceil\frac{7}{6}\right.} = \frac{1}{2^{\frac{5}{6}}} \frac{\left\lceil\frac{2}{3}\right. \sqrt{\pi}}{\frac{1}{6}\left\lceil\frac{1}{6}\right.} = \frac{6\sqrt{\pi}}{2^{\frac{5}{6}}} \frac{\left\lceil\frac{2}{3}\right.}{\left\lceil\frac{1}{6}\right.}$$

EXAMPLE 6.20 *Evaluate* $\int_0^{\pi} x \sin^7 x \cos^4 x \, dx.$

Solution: $$\int_0^{\pi} x \sin^7 x \cos^4 x \, dx = \int_0^{\pi} (\pi - x)\sin^7(\pi - x)\cos^4(\pi - x) dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \pi\int_0^{\pi} \sin^7 x \cos^4 x \, dx - \int_0^{\pi} x\sin^7 x \cos^4 x \, dx$$

$$\int_0^{\pi} x\sin^7 x \cos^4 x \, dx = \pi\int_0^{\pi} \sin^7 x \cos^4 x \, dx$$

$$= \pi\left[\int_0^{\frac{\pi}{2}} \sin^7 x \cos^4 x \, dx + \int_0^{\frac{\pi}{2}} \sin^7(\pi - x)\cos^4(\pi - x)dx\right]$$

$$\left[\because \int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx\right]$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \sin^7 x \cos^4 x \, dx = 2\pi \cdot \frac{1}{2} B\left(4, \frac{5}{2}\right) = \pi \frac{\Gamma(4)\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{13}{2}\right)} = \pi \frac{3!\,\Gamma\left(\frac{5}{2}\right)}{\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}$$

$$\int_0^{\pi} x\sin^7 x \cos^4 x \, dx = \frac{16\pi}{1155}$$

ERCISE 6.6

1. Evaluate the following integrals:

(i) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$

(ii) $\int_0^{\frac{\pi}{6}} \cos^6 3\theta \sin^2 6\theta \, d\theta$

(iii) $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\sqrt{3}\sin\theta + \cos\theta\right)^{\frac{1}{4}} d\theta$

(iv) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\theta\,(1+\sin\theta)^2 d\theta$

(v) $\int_0^{2\pi} \sin^2\theta\,(1+\cos\theta)^4 d\theta$

(vi) $\int_0^{\pi} x \sin^5 x \cos^6 x \, dx$

Ans.: (i) $\frac{\pi}{\sqrt{2}}$ (ii) $\frac{7\pi}{384}$ (iii) $2^{-\frac{3}{4}}\sqrt{\pi}\,\frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{9}{8}\right)}$ (iv) $\frac{8}{5}$ (v) $\frac{21\pi}{8}$ (vi) $\frac{8\pi}{693}$

2. Prove that

$$\int_0^{\frac{\pi}{2}} (\sin x)^{2n} dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n!)}\cdot\frac{\pi}{2}.$$

6.7 BETA FUNCTION AS IMPROPER INTEGRAL

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Proof Let $x = \tan^2\theta$, $dx = 2\tan\theta\sec^2\theta\,d\theta$

When $x = 0$, $\theta = 0$

When $x \to \infty$, $\theta = \frac{\pi}{2}$

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^{\frac{\pi}{2}} \frac{(\tan^2\theta)^{m-1}}{(1+\tan^2\theta)^{m+n}} \cdot 2\tan\theta\sec^2\theta\,d\theta = 2\int_0^{\frac{\pi}{2}} \frac{(\tan\theta)^{2m-1}\sec^2\theta}{(\sec\theta)^{2m+2n}}\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta = B(m, n)$$

EXAMPLE 6.21 *Prove that* $\int_0^\infty \frac{x^2}{(1+x^4)^3}\,dx = \frac{5\pi\sqrt{2}}{128}$.

Solution: Let $x^4 = t$, $x = t^{\frac{1}{4}}$, $dx = \frac{1}{4}t^{-\frac{3}{4}}dt$

When $x = 0$, $t = 0$

When $x \to \infty$, $t \to \infty$

$$\int_0^\infty \frac{x^2}{(1+x^4)^3}\,dx = \int_0^\infty \frac{t^{\frac{1}{2}}}{(1+t)^3}\cdot\frac{1}{4}t^{-\frac{3}{4}}dt = \frac{1}{4}\int_0^\infty \frac{t^{-\frac{1}{4}}}{(1+t)^3}\,dt = \frac{1}{4}\int_0^\infty \frac{t^{\frac{3}{4}-1}}{(1+t)^{\frac{3}{4}+\frac{9}{4}}}\,dt = \frac{1}{4}B\left(\frac{3}{4}, \frac{9}{4}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{9}{4}\right)}{\Gamma(3)}$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\cdot\frac{5}{4}\cdot\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}{2!} = \frac{5}{128}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

$$= \frac{5}{128}\cdot\frac{\pi}{\sin\frac{\pi}{4}} \qquad \left[\because \Gamma(1-n)\,\Gamma(n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{5\pi\sqrt{2}}{128}$$

EXAMPLE 6.22 *Prove that* $\int_0^\infty \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\,dx = \frac{1}{2}B(n+m, n-m)$, $n > m$.

Solution:
$$\int_0^\infty \frac{e^{2mx}+e^{-2mx}}{(e^x+e^{-x})^{2n}}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{(e^{2mx}+e^{-2mx})\,e^{2nx}}{(e^{2x}+1)^{2n}}\,dx \qquad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(-x) = f(x)\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{2(m+n)x} + e^{2(n-m)x}}{(1+e^{2x})^{2n}}\,dx$$

$2e^{2x}\,dx = dt,\qquad dx = \frac{1}{2t}\,dt$

$t = 0$

$t \to \infty,$

$$\int_0^{\infty} \frac{e^{2mx} + e^{-2mx}}{(e^x + e^{-x})^{2n}}\,dx = \frac{1}{2}\int_0^{\infty} \frac{t^{m+n} + t^{n-m}}{(1+t)^{2n}} \cdot \frac{1}{2t}\,dt$$

$$= \frac{1}{4}\left[\int_0^{\infty} \frac{t^{(m+n)-1}}{(1+t)^{(m+n)+(n-m)}}\,dt + \int_0^{\infty} \frac{t^{(n-m)-1}}{(1+t)^{(n-m)+(n+m)}}\,dt\right] = \frac{1}{4}[B(m+n, n-m) + B(n-m, n+m)]$$

$$= \frac{1}{2}B(n+m, n-m) \qquad [\because B(m,n) = B(n,m)]$$

EXAMPLE 6.23 *Prove that* $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx.$

Solution: $B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \qquad \ldots (1)$

Consider

$$I = \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Putting $x = \frac{1}{y}$, $\qquad dx = -\frac{1}{y^2}\,dy$

When $x = 1$, $\qquad y = 1$

When $x \to \infty$, $\qquad y = 0$

$$I = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy = \int_0^1 \frac{y^{n-1}}{(y+1)^{m+n}}\,dy$$

Substituting in Eq. (1),

$$B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

Replacing y by x,

$$B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx$$

EXAMPLE 6.24 *Prove that* $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{\left(\overline{|\frac{1}{4}}\right)^2}{4\sqrt{\pi}}$.

Solution: Let $\tan\frac{\theta}{2} = t,\ \frac{\theta}{2} = \tan^{-1} t,\ d\theta = \frac{2}{1+t^2}dt$

$$\sin\theta = \frac{2t}{1+t^2}$$

When $\theta = 0$, $t = 0$

When $\theta = \frac{\pi}{2}$, $t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \int_0^1 \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{2t}{1+t^2}\right)^2}} \cdot \frac{2}{1+t^2}dt = 2\int_0^1 \frac{1}{(1+t^4)^{\frac{1}{2}}}dt$$

Let $t^4 = u,\ t = u^{\frac{1}{4}},\ dt = \frac{1}{4}u^{-\frac{3}{4}}du$

When $t = 0$, $u = 0$

When $t = 1$, $u = 1$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = 2\int_0^1 \frac{1}{(1+u)^{\frac{1}{2}}} \cdot \frac{1}{4}u^{-\frac{3}{4}}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{u^{\frac{1}{4}-1}}{(1+u)^{\frac{1}{2}}}du = \frac{1}{4}\int_0^1 \frac{u^{\frac{1}{4}-1}+u^{\frac{1}{4}-1}}{(1+u)^{\frac{1}{4}+\frac{1}{4}}}du$$

$$= \frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{\overline{|\frac{1}{4}}\,\overline{|\frac{1}{4}}}{\overline{|\frac{1}{2}}} = \frac{\left(\overline{|\frac{1}{4}}\right)^2}{4\sqrt{\pi}}$$

where $\delta A_r = \delta x_r \cdot \delta y_r$.

If the number of elementary rectangles is increased then the area of each rectangle decreases. Hence, as $n \to \infty$, $\delta A_r \to 0$. The limit of the sum given by Eq. (7.1), if it exists, is called the double integral of $f(x, y)$ over the region R and is denoted by $\iint_R f(x, y)\mathrm{d}A$.

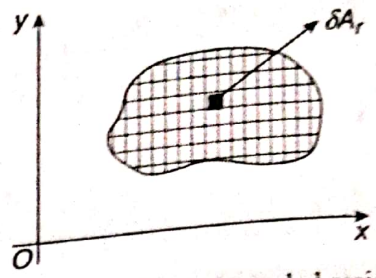

Fig. 7.1 Closed and bounded region R

Hence,

$$\iint_R f(x, y)\mathrm{d}A = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$

7.2.1 Evaluation of Double Integration

Double integral of a function $f(x, y)$ over a region R can be evaluated by two successive integrations. There are two different methods to evaluate a double integral.

Method I Let the region R, i.e., $PQRS$ be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$ (Fig. 7.2).

In the region $PQRS$, draw a vertical strip AB. Along the strip AB, y varies from y_1 to y_2 and x is fixed. Therefore, the double integral is integrated first w.r.t. y between the limits y_1 and y_2 treating x as constant.

Now, move the strip AB horizontally from PS (i.e., $x = a$) to QR (i.e., $x = b$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. x between the limits a and b.

Hence,

$$\iint_R f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y\right]\mathrm{d}x$$

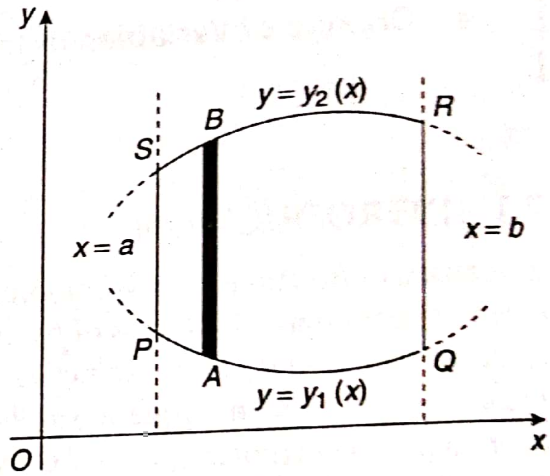

Fig. 7.2 Illustration of Method I

Method II Let the region R, i.e., $PQRS$ be bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and the lines $y = c$, $y = d$ (Fig. 7.3).

In the region $PQRS$, draw a horizontal strip AB. Along the strip AB, x varies from x_1 to x_2 and y is fixed. Therefore, the double integral is integrated first w.r.t. x between the limits x_1 and x_2 treating y as constant.

Now, move the strip AB vertically from PQ (i.e., $y = c$) to RS (i.e., $y = d$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. y between the limits c and d.

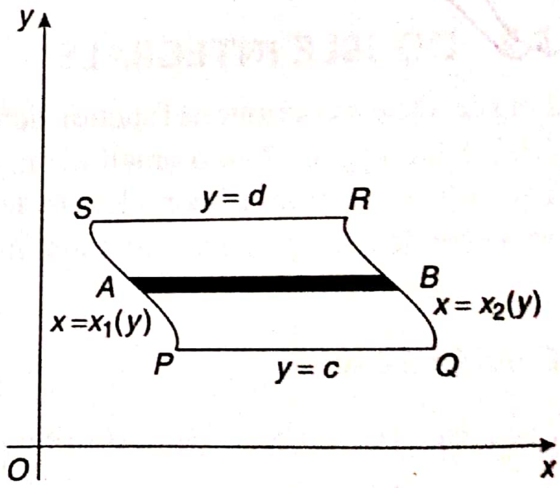

Fig. 7.3 Illustration of Method II

$$\iint f(x, y)dx\,dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)dx\right]dy$$

If all the four limits are constant then the function $f(x, y)$ can be integrated w.r.t. any variable first. But if $f(x, y)$ is implicit and is discontinuous within or on the boundary of the region of integration then the change of the order of integration will affect the result.

If all the four limits are constant and $f(x, y)$ is explicit then double integral can be written as product of two single integrals.

If inner limits depend on x then the function $f(x, y)$ is integrated first w.r.t. y and vice versa.

EXAMPLE 7.1 Evaluate $\int_2^a \int_2^b \frac{dx\,dy}{xy}$.

Solution:

$$\int_2^a \int_2^b \frac{dx\,dy}{xy} = \int_2^a \left(\int_2^b \frac{dx}{x}\right)\frac{dy}{y} = \int_2^a \left|\log x\right|_2^b \frac{1}{y}dy = (\log b - \log 2)\int_2^a \frac{1}{y}dy$$

$$= \log\left(\frac{b}{2}\right)\cdot\left|\log y\right|_2^a = \log\left(\frac{b}{2}\right)\cdot(\log a - \log 2) = \log\left(\frac{b}{2}\right)\cdot\log\left(\frac{a}{2}\right)$$

Other Method Since both the limits are constant and integrand (function) is explicit in x and y, integral can be written as

$$\int_2^a \int_2^b \frac{dx\,dy}{xy} = \int_2^a \frac{dy}{y}\cdot\int_2^b \frac{dx}{x} = \left|\log y\right|_2^a \cdot \left|\log x\right|_2^b = (\log a - \log 2)(\log b - \log 2) = \log\left(\frac{a}{2}\right)\cdot\log\left(\frac{b}{2}\right)$$

$$= \log\left(\frac{b}{2}\right)\cdot\log\left(\frac{a}{2}\right)$$

EXAMPLE 7.2 Evaluate $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{1-x^2-y^2}$

Solution:

$$\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \int_0^1 \left[\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx}{\sqrt{(1-y^2)-x^2}}\right]dy = \int_0^1 \left|\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right|_0^{\sqrt{\frac{1-y^2}{2}}} dy$$

$$= \int_0^1 \left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0\right)dy = \frac{\pi}{4}\left|y\right|_0^1 = \frac{\pi}{4}$$

EXERCISE 7.1

Evaluate the following integrals:

1. $\int_1^2\int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2\,dx\,dy$ $\left[\text{Ans.: } \frac{856}{945}\right]$

2. $\int_0^1\int_0^y xye^{x-2}\,dx\,dy$ $\left[\text{Ans.: } \frac{1}{4e}\right]$

3. $\int_0^1\int_0^x e^{x+y}\,dx\,dy$ $\left[\text{Ans.: } \frac{1}{2}(e-1)^2\right]$

4. $\int_{10}^1\int_0^{\frac{1}{y}} ye^{xy}\,dx\,dy$ [Ans.: $9(1-e)$]

5. $\int_1^{\log 8}\int_0^{\log y} e^{x+y}\,dx\,dy$ [Ans.: $8(\log 8-1)$]

6. $\int_0^1\int_{y^2}^y (1+xy^2)\,dx\,dy$ $\left[\text{Ans.: } \frac{41}{210}\right]$

7. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\,dy\,dx$ $\left[\text{Ans.: } \frac{2a^4}{3}\right]$

7.2.2 Working Rule for Evaluation of Double Integration over a Given Region

1. If the region is bounded by more than one curve then find the points of intersection of all the curves.
2. Draw all the curves and mark their point of intersection.
3. Identify the region of integration.
4. Draw a vertical or horizontal strip in the region whichever makes the integration easier.
5. The vertical strip starts from the lowest part of the region and terminates on the highest part of the region.
6. **For vertical strip**
   (i) The lower limit of y is obtained from the curve, where the vertical strip starts and the upper limit of y is obtained from the curve, where it terminates.
   (ii) The lower limit of x is the x-coordinate of the leftmost point of the region and the upper limit of x is the x-coordinate of the rightmost point of the region.
7. The horizontal strip starts from the left part of the region and terminates on the right part of the region.
8. **For horizontal strip**
   (i) The lower limit of x is obtained from the curve, where the horizontal strip starts and upper limit is obtained from the curve, where it terminates.
   (ii) The lower limit of y is the y-coordinate of the lowest point of the region and the upper limit of y is the y-coordinate of the highest point of the region.
9. If variation along the strip changes within the region then the region is divided into parts.

EXAMPLE 7.3 *Evaluate* $\iint (a-x)^2\,dx\,dy$, *over the right half of the circle* $x^2+y^2=a^2$.

Solution:

1. The region of integration is PQR (Fig. 7.4).
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to the y-axis which starts from the part of the circle $x^2+y^2=a^2$ below x-axis and terminates on the part of the circle $x^2+y^2=a^2$ above the x-axis.
3. Limits of

$y: y=-\sqrt{a^2-x^2}$ to $y=\sqrt{a^2-x^2}$

Limits of $x: x=0$ to $x=a$

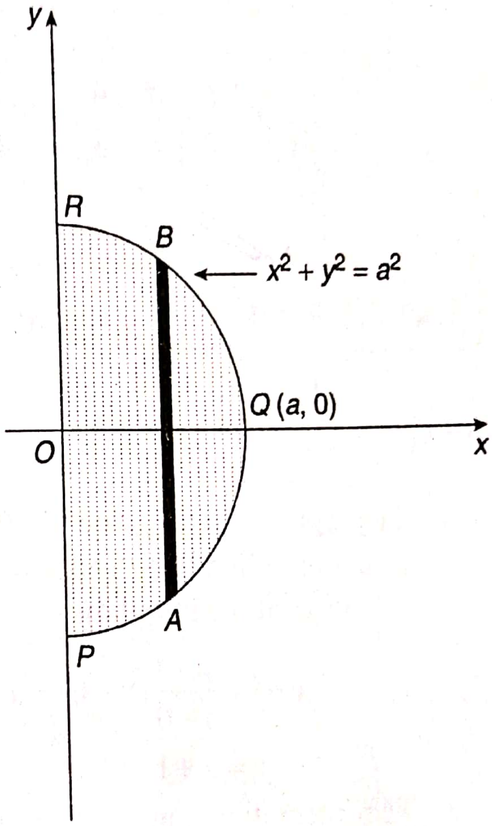

Fig. 7.4

$$I=\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(a-x)^2\,dx\,dy$$

$$=\int_0^a(a-x)^2\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dy\,dx$$

$$=\int_0^a(a-x)^2\,\left|y\right|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\,dx$$

$$=\int_0^a(a^2+x^2-2ax)\cdot 2\sqrt{a^2-x^2}\,dx$$

$$=2\int_0^a(a^2+x^2-2ax)\sqrt{a^2-x^2}\,dx$$

Putting $x=a\sin\theta,\quad dx=a\cos\theta\,d\theta$

Then $x=0,\qquad \theta=0$

Then $x=a,\qquad \theta=\dfrac{\pi}{2}$

$$I=2\int_0^{\frac{\pi}{2}}(a^2+a^2\sin^2\theta-2a^2\sin\theta)\cdot a\cos\theta\cdot a\cos\theta\,d\theta$$

$$=2a^4\int_0^{\frac{\pi}{2}}(\cos^2\theta+\sin^2\theta\cos^2\theta-2\sin\theta\cos^2\theta)d\theta$$

$$=a^4\left[B\left(\frac{3}{2},\frac{1}{2}\right)+B\left(\frac{3}{2},\frac{3}{2}\right)-2B\left(1,\frac{3}{2}\right)\right]$$

$$=a^4\left[\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}+\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}-2\frac{\Gamma(1)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}\right]$$

$$= a^4\left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{1}+\frac{\left(\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)^2}{2!}-2\frac{\Gamma\left(\frac{3}{2}\right)}{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}\right]$$

$$= a^4\left[\frac{\pi}{2}+\frac{\pi}{8}-\frac{4}{3}\right]= a^4\left[\frac{5\pi}{8}-\frac{4}{3}\right]$$

EXAMPLE 7.4

Evaluate $\iint (x^2 - y^2)\mathrm{d}x\,\mathrm{d}y$ *over the triangle with vertices* (0, 1), (1, 1), (1, 2).

Solution:

1. The region of integration is ΔPQR (Fig. 7.5).
2. Equation of the line PQ is $y = 1$.
   Equation of the line PR is
   $$y-1=\frac{2-1}{1-0}(x-0)=x$$
   $$y=x+1$$
3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to the y-axis which starts from the line $y = 1$ and terminates on the line $y = x + 1$.
4. Limits of y: $y = 1$ to $y = x + 1$
   Limits of x: $x = 0$ to $x = 1$

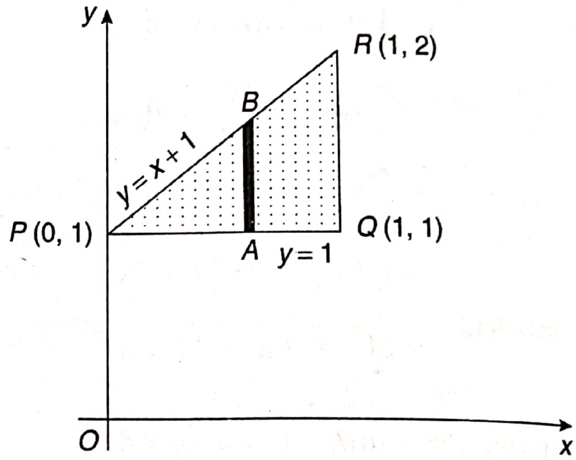

Fig. 7.5

$$I=\int_0^1\int_1^{x+1}(x^2-y^2)\mathrm{d}y\,\mathrm{d}x=\int_0^1\left|x^2y-\frac{y^3}{3}\right|_1^{x+1}\mathrm{d}x$$

$$=\int_0^1\left[x^2(x+1)-\frac{(x+1)^3}{3}-x^2+\frac{1}{3}\right]\mathrm{d}x$$

$$=\left|\frac{x^4}{4}+\frac{x^3}{3}-\frac{(x+1)^4}{12}-\frac{x^3}{3}+\frac{x}{3}\right|_0^1=\frac{1}{4}+\frac{1}{3}-\frac{16}{12}+\frac{1}{12}=-\frac{2}{3}$$

EXAMPLE 7.5

Evaluate $\iint (x^2 + y^2)\mathrm{d}x\,\mathrm{d}y$, *over the region bounded by the lines* $y = 4x$, $x + y = 3$, $y = 0$, $y = 2$.

olution:

1. The region of integration is $OPQR$ (Fig. 7.6).
2. The integration can be done w.r.t. any variable first. But in case of a vertical strip, the region is divided into three parts. Therefore, draw a horizontal strip AB parallel to the x-axis which starts from the line $y = 4x$ and terminates on the line $x + y = 3$.

Limits of x: $x = \frac{y}{4}$ to $x = 3 - y$

Limits of y: $y = 0$ to $y = 2$

$$I = \int_0^2 \int_{\frac{y}{4}}^{3-y} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^2 \left| \frac{x^3}{3} + xy^2 \right|_{\frac{y}{4}}^{3-y} dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + (3-y)y^2 - \frac{1}{3} \cdot \frac{y^3}{64} - \frac{y^3}{4} \right] dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + 3y^2 - \frac{241}{192} y^3 \right] dy$$

$$= \left| \frac{1}{3} \cdot \frac{(3-y)^4}{-4} + 3 \cdot \frac{y^3}{3} - \frac{241}{192} \cdot \frac{y^4}{4} \right|_0^2$$

$$= -\frac{1}{12} + 8 - \frac{241}{192} \cdot 4 - \left(-\frac{27}{4} \right) = \frac{463}{48}$$

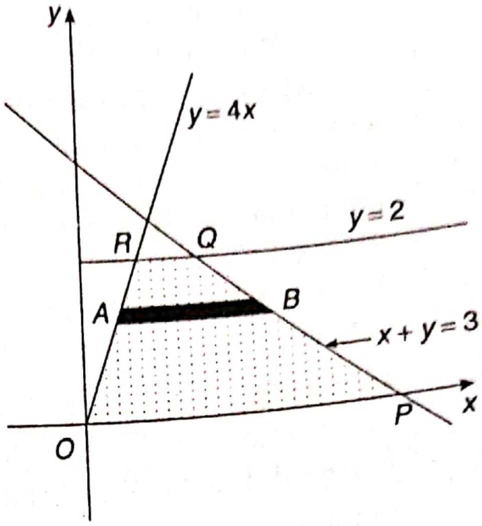

Fig. 7.6

EXAMPLE 7.6 *Evaluate $\iint xy\, dx\, dy$, over the region enclosed by the circle $x^2 + y^2 - 2x = 0$, the parabola $y^2 = 2x$ and the line $y = x$.*

Solution:

1. The region of integration is $OPQRO$ (Fig. 7.7).
2. (i) The points of intersection of the circle $x^2 + y^2 - 2x = 0$ and the line $y = x$ are obtained as

$$x^2 + x^2 - 2x = 0$$
$$x = 0, 1$$
$$\therefore y = 0, 1$$

The points of intersection are O (0, 0) and P (1, 1).

(ii) The point of intersection of the circle $x^2 + y^2 - 2x = 0$ and the parabola $y^2 = 2x$ is obtained as

$$x^2 + 2x - 2x = 0$$
$$x = 0$$
$$\therefore y = 0$$

The point of intersection is O (0, 0).

(iii) The points of intersection of the parabola $y^2 = 2x$ and the line $y = x$ are obtained as $x^2 = 2x$

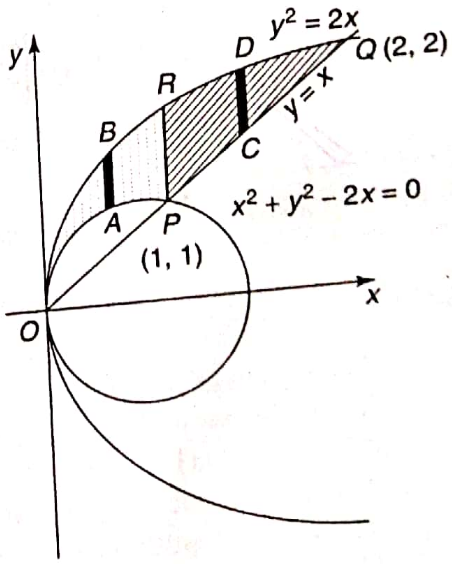

Fig. 7.7

$x = 0, 2$

$\therefore y = 0, 2$

The points of intersection are $O(0, 0)$ and $Q(2, 2)$.

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. y, first a vertical strip is drawn in the region. But one vertical strip does not cover the entire region. Hence, divide the region $OPQRO$ into two subregions OPR and RPQ and draw one vertical strip in each subregion.
4. In the subregion OPR, the strip starts from the circle $x^2 + y^2 - 2x = 0$ and terminates on the parabola $y^2 = 2x$.

   Limits of y : $y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$

   Limits of x : $x = 0$ to $x = 1$
5. In the subregion RPQ, the strip starts from the line $y = x$ and terminates on the parabola $y^2 = 2x$.

   Limits of y : $y = x$ to $y = \sqrt{2x}$

   Limits of x : $x = 1$ to $x = 2$

$$I = \iint xy\,dx\,dy = \iint_{OPR} xy\,dx\,dy + \iint_{RPQ} xy\,dx\,dy$$

$$= \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,dy\,dx + \int_1^2 \int_x^{\sqrt{2x}} xy\,dy\,dx$$

$$= \int_0^1 x \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\,dy\,dx + \int_1^2 x \int_x^{\sqrt{2x}} y\,dy\,dx$$

$$= \int_0^1 x \left|\frac{y^2}{2}\right|_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x \left|\frac{y^2}{2}\right|_x^{\sqrt{2x}} dx$$

$$= \frac{1}{2}\int_0^1 x(2x - 2x + x^2)dx + \frac{1}{2}\int_1^2 x(2x - x^2)dx$$

$$= \frac{1}{2}\left|\frac{x^4}{4}\right|_0^1 + \frac{1}{2}\left|\frac{2x^3}{3} - \frac{x^4}{4}\right|_1^2 = \frac{1}{8} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{8} = \frac{7}{12}$$

EXAMPLE 7.7 *Evaluate* $\iint \frac{dx\,dy}{x^4 + y^2}$, *over the region bounded by the* $y \geq x^2$, $x \geq 1$.

Solution:

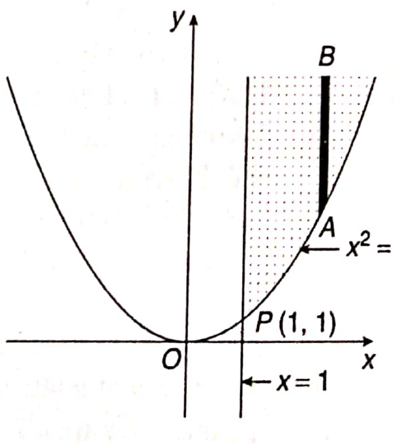

Fig. 7.8

1. The region of integration is bounded by $y \geq x^2$ (the region inside the parabola $x^2 = y$) and $x \geq 1$ (the region on the right of line $x = 1$) (Fig. 7.8).
2. The point of intersection of $x^2 = y$ and $x = 1$ is obtained as $1 = y$. The point of intersection is $P(1, 1)$.
3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to the y-axis in the region which starts from the parabola $x^2 = y$ and extends up to infinity.
4. Limits of y : $y = x^2$ to $y \to \infty$

   Limits of x : $x = 1$ to $x \to \infty$

$$I = \int_1^\infty \int_{x^2}^\infty \frac{1}{x^4+y^2}\,dy\,dx = \int_1^\infty \left|\frac{1}{x^2}\tan^{-1}\frac{y}{x^2}\right|_{x^2}^\infty dx = \int_1^\infty \frac{1}{x^2}(\tan^{-1}\infty - \tan^{-1}1)dx$$

$$= \int_1^\infty \frac{1}{x^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)dx = \frac{\pi}{4}\left|-\frac{1}{x}\right|_1^\infty = \frac{\pi}{4}$$

...RCISE 7.2

...the following integrals:

1. $\iint \frac{1}{xy}\,dx\,dy$, over the rectangle $1 \le x \le 2$, $1 \le y \le 2$ [Ans.: $(\log 2)^2$]

2. $\iint \sin\pi(ax+by)\,dx\,dy$, over the triangle bounded by the lines $x=0$, $y=0$ and $ax+by=1$ $\left[\text{Ans.: } \frac{1}{\pi ab}\right]$

3. $\iint e^{3x+4y}\,dx\,dy$, over the triangle bounded by the lines $x=0$, $y=0$, and $x+y=1$ $\left[\text{Ans.: } \frac{1}{12}(3e^4-4e^3+1)\right]$

4. $\iint xy\sqrt{1-x-y}\,dx\,dy$, over the triangle bounded by $x=0$, $y=0$ and $x+y=1$ $\left[\text{Ans.: } \frac{16}{945}\right]$

5. $\iint \sqrt{xy-y^2}\,dx\,dy$, over the triangle having vertices (0, 0), (10, 1), (1, 1) [Ans.: 6]

6. $\iint (x+y+a)\,dx\,dy$, over the region bounded by the circle $x^2+y^2=a^2$ [Ans.: πa^3]

7. $\iint xy\,dx\,dy$, over the region bounded by the x-axis, the line $y = 2x$ and the parabola $y=\frac{x^2}{4a}$ $\left[\text{Ans.: } \frac{2048}{3}a^4\right]$

8. $\iint 5-2x-y(dx)\,dy$, over the region bounded by the x-axis, the line $x+2y=3$ and the parabola $y^2=x$ $\left[\text{Ans.: } \frac{217}{60}\right]$

9. $\iint (4x^2-y^2)^{\frac{1}{2}}\,dx\,dy$, over the triangle bounded by the x-axis, the line $y=x$ and $x=1$ $\left[\text{Ans.: } \frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)\right]$

10. $\iint xy(x+y)\,dx\,dy$, over the region bounded by the parabola $y^2=x$, $x^2=y$ $\left[\text{Ans.: } \frac{3}{28}\right]$

11. $\iint xy(x+y)\,dx\,dy$, over the region bounded by the curve $x^2=y$ and the line $x=y$ $\left[\text{Ans.: } \frac{3}{56}\right]$

12. $\iint xy(x-1)\,dx\,dy$, over the region bounded by the rectangular hyperbola $xy = 4$, the lines $y=0$, $x=1$, $x=4$ and the x-axis [Ans.: $8(3-\log 4)$]

7.3 CHANGE OF ORDER OF INTEGRATION

Sometimes, evaluation of double integrals becomes easier by changing the order of integration. To change the order of integration, first the region of integration is drawn with the help of the given limits. Then a vertical or horizontal strip is drawn as per the required order of integration. This change of order also changes the limits of integration.

Type I Change of Order of Integration

EXAMPLE 7.8 *Change the order of integration of* $\int_0^\infty \int_x^\infty f(x, y)\,dx\,dy$.

Solution:

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.

   The correct form of the integral $= \int_0^\infty \int_x^\infty f(x, y)\,dy\,dx$

2. Limits of y : $y = x$ to $y = \infty$, along vertical strip [Fig. 7.9]
   Limits of x : $x = 0$ to $x = \infty$

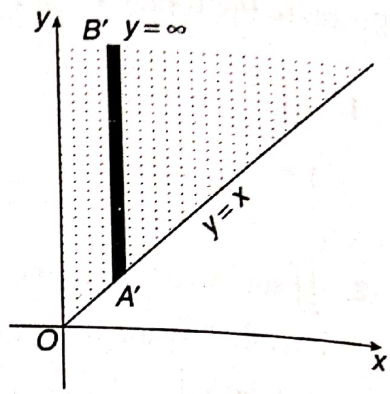

Fig. 7.9

3. The region is bounded by the lines $y = x$ and $x = 0$.
4. Here, the only point of intersection is the origin O.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to the x-axis which starts from the line $x = 0$ and terminates on the line $y = x$ [Fig. 7.10].
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = \infty$
   Hence, the given integral after change of order is

$$\int_0^\infty \int_x^\infty f(x, y)\,dy\,dx = \int_0^\infty \int_0^y f(x, y)\,dx\,dy$$

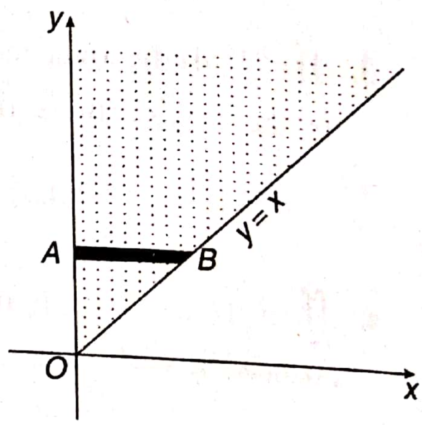

Fig. 7.10

EXAMPLE 7.9 *Change the order of integration of* $\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,dx\,dy$.

Solution:

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = \dfrac{y-8}{4}$ to $x = \dfrac{y}{4}$
   Limits of y : $y = 0$ to $y = 8$

...region is bounded by the line $y = 4x + 8$, $y = 4x$, $y = 8$ and ...axis ($y = 0$) [Fig. 7.11].

...point of intersection of $y = 4x$ and $y = 8$ is obtained as ...$4x$, i.e., $x = 2$.

...point of intersection is $P(2, 8)$.

...change the order of integration, i.e., to integrate first ...y, divide the region $OPQR$ into two subregions OQR ...OPQ. Draw a vertical strip parallel to the ...axis in each subregion [Fig. 7.12].

(i) In the subregion OQR, the strip AB starts from x-axis and terminates on the line $y = 4x + 8$.

Limits of $y : y = 0$ to $y = 4x + 8$

Limits of $x : x = -2$ to $x = 0$

(ii) In the subregion OPQ, the strip CD starts from the line $y = 4x$ and terminates on the line $y = 8$.

Limits of $y : y = 4x$ to $y = 8$

Limits of $x : x = 0$ to $x = 2$

...the given integral after change of order is

$$\iint f(x, y)\,dx\,dy = \int_{-2}^{0}\int_{0}^{4x+8} f(x, y)\,dy\,dx + \int_{0}^{2}\int_{4x}^{8} f(x, y)\,dy\,dx$$

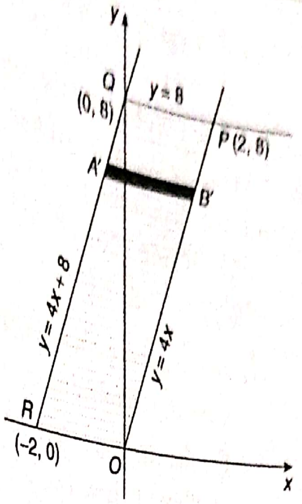

Fig. 7.11

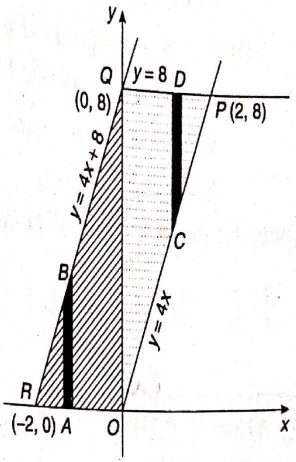

Fig. 7.12

...AMPLE 7.10

Change the order of integration of $\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x, y)\,dx\,dy$.

...tion:

1. The function is integrated first w.r.t. x and then w.r.t. y.

2. Limits of $x : x = 0$ to $x = \dfrac{y^2}{a}$

   Limits of $y : y = -a$ to $y = a$

3. The region is bounded by the y-axis, the parabola $y^2 = ax$, and the line $y = -a$, and $y = a$ [Fig. 7.13].

4. (i) The point of intersection of $y^2 = ax$ and $y = -a$ is obtained as

   $a^2 = ax$, i.e., $x = a$.

   The point of intersection is $R(a, -a)$.

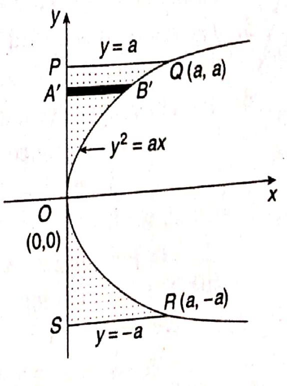

Fig. 7.13