EXERCISE 10.13

Solve the following differential equations:

1. $(D^2+D+2)y = e^{\frac{x}{2}}$

$$\left[\textbf{Ans.: } y = e^{-\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{7}x}{2}\right) + c_2\sin\left(\frac{\sqrt{7}x}{2}\right)\right] + -\frac{4}{11}e^x + \frac{1}{4}xe^{\frac{x}{2}}\right]$$

2. $(D^2-4)y = (1+e^x)^2$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}\right]$$

3. $(D^2+4D+5)y = -2\cosh x + 2^x$

$$\left[\textbf{Ans.: } y = e^{-2x}(c_1\cos x + c_2\sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2} + \frac{2^x}{(\log 2)^2 + 4(\log 2) + 5}\right]$$

4. $(D^3+D^2+D+1)y = \sin 2x$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2\cos x + c_3\sin x + \frac{1}{15}(2\cos 2x - \sin 2x)\right]$$

5. $(D^3-2D^2+4D)y = e^{2x} + \sin 2x$

$$\left[\textbf{Ans.: } y = c_1 + e^x\left(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x\right) + \frac{1}{8}(e^{2x} + \sin 2x)\right]$$

6. $(D^3+2D^2+D)y = \sin^2 x$

$$\left[\textbf{Ans.: } y = c_1 + (c_2 + c_3x)e^{-x} + \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)\right]$$

7. $(D^2+D-6)y = e^{2x}$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-3x} + \frac{xe^{2x}}{5}\right]$$

8. $(9D^2+6D+1)y = e^{-\frac{x}{3}}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{-\frac{x}{3}} + \frac{x^2}{18}e^{-\frac{x}{3}}\right]$$

9. $(D^2-4)y = x^2$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)\right]$$

10. $(D^2+D)y = x^2 + 2x + 4$

$$\left[\textbf{Ans.: } y = c_1 + c_2e^{-x} + \frac{x^3}{3} + 4x\right]$$

11. $(D^2+1)y = e^{2x} + \cosh 2x + x^3$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + \frac{e^{2x}}{5} + \frac{1}{5}\cosh 2x + x^3 - 6x\right]$$

12. $(D-1)^2(D+1)^2 y = \sin^2\frac{x}{2} + e^x + x$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} + \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}e^x + x\right]$$

13. $(D^2-3D+2)y = 2e^x\cos\frac{x}{2}$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} - \frac{8}{5}e^x\left(2\sin\frac{x}{2} + \cos\frac{x}{2}\right)\right]$$

14. $(4D^3 - 12D^2 + 13D - 10)y = 16e^{\frac{x}{2}} \cos x$

$$\left[\textbf{Ans.:}\ y = c_1 e^{2x} + e^{\frac{x}{2}}(c_2 \cos x + c_3 \sin x) - \frac{4xe^{\frac{x}{2}}}{13}(2\cos x + 3\sin x)\right]$$

15. $(D^2 + 4D + 8)y = 12e^{-2x} \sin x \sin 3x$

$$\left[\textbf{Ans.:}\ y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{2}e^{-2x}(3x \sin 2x + \cos 4x)\right]$$

16. $(4D^2 + 9D + 2)y = xe^{-2x}$

$$\left[\textbf{Ans.:}\ y = c_1 e^{-2x} + c_2 e^{-\frac{x}{4}} - \frac{1}{98}(7x^2 + 8x)e^{-2x}\right]$$

17. $(D^2 + 4)y = x \sin x$

$$\left[\textbf{Ans.:}\ y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{9}(3x \sin x - 2\cos x)\right]$$

18. $(D^2 + 9)y = xe^{2x} \cos x$

$$\left[\textbf{Ans.:}\ y = c_1 \cos 3x + c_2 \sin 3x + \frac{e^{2x}}{400}[(30x - 11)\cos x + (10x - 2)\sin x]\right]$$

19. $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

$$\left[\textbf{Ans.:}\ y = (c_1 + c_2 x)e^{2x} - e^{2x}[4x \cos 2x + (2x^2 - 3)\sin 2x]\right]$$

20. $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$

$$\left[\textbf{Ans.:}\ y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 9)\right]$$

10.6.3 General Method of Obtaining Particular Integrals (PI)

In a linear differential equation

$$f(D)y = Q(x)$$

if $Q(x)$ is not in any of the standard forms discussed in the previous section then the particular integral is obtained using the general method described below.

$$\text{PI} = \frac{1}{f(D)}Q(x) = \frac{1}{(D - m_1)(D - m_2)\ldots(D - m_n)}Q(x)$$

$$= \left(\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right)Q(x) \qquad \text{[Using partial fraction expansion]}$$

$$= A_1 \cdot \frac{1}{D - m_1}Q(x) + A_2 \cdot \frac{1}{D - m_2}Q(x) + \ldots + A_n \cdot \frac{1}{D - m_n}Q(x)$$

$$= A_1 e^{m_1 x}\int Q(x) \cdot e^{-m_1 x}dx + A_2 e^{m_2 x}\int Q(x) \cdot e^{-m_2 x}dx + \ldots + A_n e^{m_n x}\int Q(x)e^{-m_n x}dx$$

This method can be applied for any form of $Q(x)$. But sometimes, integration of the terms become complicated and lengthy. Hence, direct (short-cut) methods are preferred to find the PI and the general method is used only if the direct method cannot be applied.

EXAMPLE 10.67 *Solve* $(D^2+3D+2)y = e^{e^x}$.

Solution: The auxiliary equation is

$$m^2+3m+2=0$$
$$(m+1)(m+2)=0$$
$$m=-1,-2 \text{ (real and distinict)}$$
$$\text{CF} = c_1e^{-x}+c_2e^{-2x}$$
$$\text{PI} = \frac{1}{D^2+3D+2}e^{e^x} = \frac{1}{(D+2)(D+1)}e^{e^x} = \frac{1}{(D+2)}\left(e^{-x}\int e^{e^x}e^x\,dx\right)$$
$$= \frac{1}{D+2}\left(e^{-x}e^{e^x}\right) \qquad \left[\because \int e^{f(x)}f'(x)\,dx = e^{f(x)}\right]$$
$$= e^{-2x}\int e^{-x}e^{e^x}e^{2x}\,dx = e^{-2x}\int e^{e^x}e^x\,dx = e^{-2x}e^{e^x}$$

Hence, the general solution is

$$y = c_1e^{-x}+c_2e^{-2x}+e^{-2x}e^{e^x}$$

EXAMPLE 10.68 *Solve* $(D^2-1)y=(1+e^{-x})^{-2}$.

Solution: The auxiliary equation is

$$m^2-1=0$$
$$m=\pm 1 \text{ (real and distinct)}$$
$$\text{CF} = c_1e^x+c_2e^{-x}$$
$$\text{PI} = \frac{1}{D^2-1}(1+e^{-x})^{-2} = \frac{1}{(D-1)}\cdot\frac{1}{(D+1)}\frac{1}{(1+e^{-x})^2}$$
$$= \left[\frac{(D+1)-(D-1)}{2(D-1)(D+1)}\right]\frac{1}{(1+e^{-x})^2} = \frac{1}{2}\left(\frac{1}{D-1}-\frac{1}{D+1}\right)\frac{1}{(1+e^{-x})^2}$$
$$= \frac{1}{2}\left[\frac{1}{D-1}\cdot\frac{1}{(1+e^{-x})^2} - \frac{1}{D+1}\cdot\frac{1}{(1+e^{-x})^2}\right]$$

Also, $\dfrac{1}{(D-1)}\cdot\dfrac{1}{(1+e^{-x})^2} = e^x\displaystyle\int\frac{1}{(1+e^{-x})^2}\cdot e^{-x}\,dx$

Let $1+e^{-x}=t,\ -e^{-x}dx=dt$

$$\frac{1}{(D-1)}\cdot\frac{1}{(1+e^{-x})^2} = e^x\int\frac{1}{t^2}(-dt) = e^x\left(\frac{1}{t}\right) = \frac{e^x}{1+e^{-x}}$$

Now, $\dfrac{1}{D+1}\dfrac{1}{\left(1+e^{-x}\right)^2} = e^{-x}\displaystyle\int \frac{1}{(1+e^{-x})^2}\cdot e^x dx = e^{-x}\int \frac{e^{2x}}{(e^x+1)^2}\, e^x dx$

Let $1+e^x = t,\ e^x dx = dt$

$$\frac{1}{(D+1)}\cdot\frac{1}{(1+e^{-x})^2} = e^{-x}\int \frac{(t-1)^2}{t^2}dt = e^{-x}\int \frac{\left(t^2-2t+1\right)}{t^2}dt$$

$$= e^{-x}\int\left(1-\frac{2}{t}+\frac{1}{t^2}\right)dt = e^{-x}\left(t-2\log t-\frac{1}{t}\right)$$

$$= e^{-x}\left[1+e^x-2\log(1+e^x)-\frac{1}{1+e^x}\right]$$

$$= e^{-x}+1-2e^{-x}\log(1+e^x)-\frac{e^{-x}}{1+e^x}$$

$$\therefore \text{PI} = \frac{1}{2}\left[\frac{e^x}{1+e^{-x}}-e^{-x}-1+2e^{-x}\log(1+e^x)+\frac{e^{-x}}{1+e^x}\right]$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-x}+\frac{1}{2}\left[\frac{e^x}{1+e^{-x}}-e^{-x}-1+2e^{-x}\log(1+e^x)+\frac{e^{-x}}{1+e^x}\right]$$

EXAMPLE 10.69 *Solve* $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$.

Solution: The auxiliary equation is

$$m^2+5m+6=0$$

$$(m+2)(m+3)=0$$

$$m=-2,-3 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^{-2x}+c_2e^{-3x}$$

$$\text{PI} = \frac{1}{D^2+5D+6}e^{-2x}\sec^2 x(1+2\tan x)$$

$$= \frac{1}{(D+3)(D+2)}e^{-2x}\sec^2 x(1+2\tan x)$$

$$= \frac{(D+3)-(D+2)}{(D+3)(D+2)}e^{-2x}\sec^2 x(1+2\tan x)$$

$$= \left(\frac{1}{D+2}-\frac{1}{D+3}\right)e^{-2x}\sec^2 x(1+2\tan x)$$

Now, $\frac{1}{D+2}e^{-2x}\sec^2 x(1+2\tan x) = e^{-2x}\int e^{-2x}\sec^2 x(1+2\tan x)\cdot e^{2x}dx = e^{-2x}\int \sec^2 x(1+2\tan x)dx$

$$= \frac{e^{-2x}}{2}\cdot\frac{(1+2\tan x)^2}{2}\left[\because \int f(x)\cdot f'(x)dx = \frac{\{f(x)\}^2}{2}\right]$$

$$= \frac{e^{-2x}}{4}(1+2\tan x)^2$$

Also, $\frac{1}{D+3}e^{-2x}\sec^2 x(1+2\tan x) = e^{-3x}\int e^{-2x}\sec^2 x(1+2\tan x)\cdot e^{3x}dx = e^{-3x}\int e^{x}\sec^2 x(1+2\tan x)dx$

$$= e^{-3x}\left[\int e^x\sec^2 x\,dx + \int e^x\sec^2 x\cdot 2\tan x\,dx\right]$$

$$= e^{-3x}\left[e^x\sec^2 x - \int e^x\cdot 2\sec x\cdot\sec x\tan x\,dx + \int e^x\sec^2 x\cdot 2\tan x\,dx\right]$$

$$= e^{-2x}\sec^2 x$$

$$\therefore \quad \text{PI} = \frac{e^{-2x}}{4}(1+2\tan x)^2 - e^{-2x}\sec^2 x$$

$$= \frac{e^{-2x}}{4}(1+4\tan^2 x+4\tan x) - e^{-2x}(1+\tan^2 x) = \frac{e^{-2x}}{4}(4\tan x - 3)$$

Hence, the general solution is

$$y = c_1e^{-2x} + c_2e^{-3x} + \frac{e^{-2x}}{4}(4\tan x - 3)$$

EXERCISE 10.14

Solve the following differential equations:

1. $(D^2+3D+2)y = \sin e^x$

$\left[\textbf{Ans.:}\ y = c_1e^{-x} + c_2e^{-2x} - e^{-2x}\sin e^x\right]$

2. $(D^2+1)y = \operatorname{cosec} x$

$\left[\textbf{Ans.:}\ y = c_1\cos x + c_2\sin x - x\cos x + \sin x\log(\sin x)\right]$

3. $(D^2+4)y = \tan 2x$

$\left[\textbf{Ans.:}\ y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)\right]$

4. $(D^2+1)y = x - \cot x$

$\left[\textbf{Ans.:}\ y = c_1\cos x + c_2\sin x - x\cos^2 x + x\sin^2 x - \sin x\log(\operatorname{cosec} x - \cot x)\right]$

5. $(D^2+D)y=\dfrac{1}{1+e^x}$

$$\left[\text{Ans.: } y=c_1+c_2e^{-x} -e^{-x}[e^x\log(e^{-x}+1) +\log(e^x+1)]\right]$$

6. $(D^2-2D+2)y=e^x\tan x$

$$\left[\text{Ans.: } y=e^x(c_1\cos x+c_2\sin x) -e^x\cos x\log(\sec x+\tan x)\right]$$

7. $(D^2-4D+4)y=8x^2e^{2x}\sin 2x$

$$\left[\text{Ans.: } y=(c_1+c_2x)e^{2x} -e^{2x}(2x^2\sin 2x +4x\cos 2x-3\sin 2\right.$$

8. $(D^2+2D+1)y=e^{-x}\log x$

$$\left[\text{Ans.: } y=(c_1+c_2x)e^{-x} +\frac{x^2}{2}e^{-x}\left(\log x-\frac{3}{2}\right.\right.$$

10.7 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In this section, two types of differential equations with variable coefficients are discussed. These d ferential equations have variable coefficients and can be solved by reducing them to linear differenti equations with constant-coefficient form.

10.7.1 Cauchy's Linear Equation

An equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}x\frac{dy}{dx}+a_ny=Q(x) \quad \ldots (10.33$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called Cauchy's linear equation.

To solve Eq. (10.33),

Let $x=e^z$, $1=e^z\dfrac{dz}{dx}$, $\dfrac{dz}{dx}=\dfrac{1}{e^z}=\dfrac{1}{x}$

Now,
$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{dy}{dz}\cdot\frac{1}{x}=\frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx}=\frac{dy}{dz},\ xDy=\mathcal{D}y,\quad \text{where } \mathcal{D}\equiv\frac{d}{dz} \text{ and } D\equiv\frac{d}{dx}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x}$$

$$x^2\frac{d^2y}{dx^2}=\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right) \text{ or } x^2D^2y=\mathcal{D}(\mathcal{D}-1)y$$

Similarly,

$$x^3D^3y = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$$

....

....

$$x^nD^ny = \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (10.33),

$$[a_0\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1)+a_1\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)+\ldots+a_{n-1}\mathcal{D}+a_n]y = Q(e^z)$$

which is a linear differential equation with constant coefficients and can be solved by usual methods described in the previous section.

Note An equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\ldots\ldots\ldots+a_{n-1}x\frac{dy}{dx}+a_ny=0$$

where $a_0, a_1, a_2 \ldots, a_{n-1}, a_n$ are constants, is called a *Cauchy–Euler equation*. It is a linear homogeneous differential equation with variable coefficients.

EXAMPLE 10.70 *Solve* $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y = x^2+\frac{1}{x^2}$.

Solution:

$$(x^2D^2+4xD+2)y = x^2+\frac{1}{x^2}$$

Putting $x = e^z$,

$$[\mathcal{D}(\mathcal{D}-1)+4\mathcal{D}+2]y = e^{2z}+\frac{1}{e^{2z}}, \quad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(\mathcal{D}^2+3\mathcal{D}+2)y = e^{2z}+e^{-2z}$$

The auxiliary equation is

$$m^2+3m+2=0$$

$$(m+2)(m+1)=0$$

$$m = -2, -1 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^{-2z}+c_2e^{-z} = \frac{c_1}{x^2}+\frac{c_2}{x}$$

$$\text{PI} = \frac{1}{\mathcal{D}^2+3\mathcal{D}+2}\left(e^{2z}+e^{-2z}\right) = \frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{2z} + \frac{1}{\mathcal{D}^2+3\mathcal{D}+2}e^{-2z}$$

$$= \frac{1}{4+3(2)+2}e^{2z} + \frac{1}{(\mathcal{D}+2)(\mathcal{D}+1)}e^{-2z} = \frac{e^{2z}}{12} + \frac{1}{(\mathcal{D}+2)}\left[\frac{1}{-2+1}\right]e^{-2z}$$

$$= \frac{e^{2z}}{12} - \frac{1}{(\mathcal{D}+2)}e^{-2z} = \frac{e^{2z}}{12} - z\cdot\frac{1}{1}e^{-2z} = \frac{x^2}{12} - (\log x)\frac{1}{x^2}$$

Hence, the general solution is

$$y = \frac{c_1}{x^2} + \frac{c_2}{x} + \frac{x^2}{12} - \frac{1}{x^2}\log x$$

EXAMPLE 10.71 *Solve* $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$.

Solution: $(x^2D^2 - xD + 1)y = \left(\frac{\log x}{x}\right)^2$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1) - \mathcal{D} + 1\right]y = \left(\frac{z}{e^z}\right)^2, \quad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(\mathcal{D}^2 - 2\mathcal{D} + 1)y = z^2e^{-2z}$$

$$(\mathcal{D}-1)^2 y = z^2e^{-2z}$$

The auxiliary equation is

$$(m-1)^2 = 0$$

$$m = 1, 1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2 z)e^z = (c_1 + c_2\log x)x$$

$$\text{PI} = \frac{1}{(\mathcal{D}-1)^2}\left(z^2e^{-2z}\right) = e^{-2z}\frac{1}{(\mathcal{D}-2-1)^2}z^2$$

$$= e^{-2z}\frac{1}{(\mathcal{D}-3)^2}z^2 = \frac{e^{-2z}}{9}\frac{1}{\left(1-\frac{\mathcal{D}}{3}\right)^2}z^2$$

$$= \frac{e^{-2z}}{9}\left(1-\frac{\mathcal{D}}{3}\right)^{-2}z^2 = \frac{e^{-2z}}{9}\left[1+\frac{2\mathcal{D}}{3}+3\frac{\mathcal{D}^2}{9}+3\frac{\mathcal{D}^3}{27}+\dots\right]z^2$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}\mathcal{D}z^2+\frac{1}{3}\mathcal{D}^2z^2+\frac{1}{9}\mathcal{D}^3z^2+\dots\right]$$

$$= \frac{e^{-2z}}{9}\left[z^2+\frac{2}{3}(2z)+\frac{1}{3}(2)+0\right] = \frac{e^{-2z}}{9}\left[z^2+\frac{4}{3}z+\frac{2}{3}\right]$$

$$= \frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

Hence, the general solution is

$$y=(c_1+c_2\log x)x+\frac{1}{9x^2}\left[(\log x)^2+\frac{4}{3}\log x+\frac{2}{3}\right]$$

EXAMPLE 10.72 *Solve* $(4x^2D^2+1)y=19\cos(\log x)+22\sin(\log x)$.

Solution: $(4x^2D^2+1)y=19\cos(\log x)+22\sin(\log x)$

Putting $x=e^z$,

$$[4\mathcal{D}(\mathcal{D}-1)+1]\,y=19\cos z+22\sin z,\qquad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(4\mathcal{D}^2-4\mathcal{D}+1)y=19\cos z+22\sin z$$

The auxiliary equation is

$$4m^2-4m+1=0$$

$$(2m-1)^2=0$$

$$m=\frac{1}{2},\frac{1}{2}\ \text{(real and repeated)}$$

$$\text{CF}=(c_1+c_2z)e^{\frac{1}{2}z}=(c_1+c_2\log x)x^{\frac{1}{2}}$$

$$\text{PI}=\frac{1}{4\mathcal{D}^2-4\mathcal{D}+1}(19\cos z+22\sin z)$$

$$=\frac{1}{4(-1^2)-4\mathcal{D}+1}(19\cos z+22\sin z)$$

$$=\frac{1}{-(4\mathcal{D}+3)}\cdot\frac{(4\mathcal{D}-3)}{(4\mathcal{D}-3)}(19\cos z+22\sin z)$$

$$=\frac{4\mathcal{D}-3}{-(16\mathcal{D}^2-9)}(19\cos z+22\sin z)$$

$$=\frac{4\mathcal{D}-3}{-[16(-1^2)-9]}(19\cos z+22\sin z)$$

$$=\frac{1}{25}[4(-19\sin z+22\cos z)-3(19\cos z+22\sin z)]$$

$$=\frac{1}{25}(31\cos z-142\sin z)$$

$$=\frac{1}{25}[31\cos(\log x)-142\sin(\log x)]$$

Hence, the general solution is

$$y=(c_1+c_2\log x)x^{\frac{1}{2}}+\frac{1}{25}[31\cos(\log x)-142\sin(\log x)]$$

EXAMPLE 10.73 Solve $\left(D+\frac{1}{x}\right)^2 y=\frac{1}{x^4}$.

Solution:
$$\left(D+\frac{1}{x}\right)^2 y=\left(\frac{d}{dx}+\frac{1}{x}\right)^2 y=\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{d}{dx}+\frac{1}{x}\right)y$$
$$=\left(\frac{d}{dx}+\frac{1}{x}\right)\left(\frac{dy}{dx}+\frac{y}{x}\right)=\frac{d^2y}{dx^2}+\frac{d}{dx}\left(\frac{y}{x}\right)+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}$$
$$=\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}+\frac{1}{x}\frac{dy}{dx}+\frac{y}{x^2}=\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}$$

Substituting in the given equation,
$$\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}=\frac{1}{x^4}$$
$$\left(D^2+\frac{2}{x}D\right)y=\frac{1}{x^4}$$

Multiplying the equation by x^2,
$$(x^2D^2+2xD)y=\frac{1}{x^2}$$

Putting $x=e^z$,
$$[\mathcal{D}(\mathcal{D}-1)+2\mathcal{D}]y=\frac{1}{e^{2z}},\quad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$
$$(\mathcal{D}^2+\mathcal{D})y=e^{-2z}$$

The auxiliary equation is
$$m^2+m=0$$
$$m(m+1)=0$$
$$m=0,-1 \text{ (real and distinct)}$$
$$\text{CF}=c_1e^{0z}+c_2e^{-z}=c_1+c_2x^{-1}=c_1+\frac{c_2}{x}$$
$$\text{PI}=\frac{1}{\mathcal{D}^2+\mathcal{D}}e^{-2z}=\frac{1}{4-2}e^{-2z}=\frac{1}{2}e^{-2z}=\frac{1}{2}(x)^{-2}=\frac{1}{2x^2}$$

Hence, the general solution is
$$y=c_1+\frac{c_2}{x}+\frac{1}{2x^2}$$

EXAMPLE 10.74 *Solve* $(x^2D^2-4xD+6)y=-x^4\sin x$.

Solution: $(x^2D^2-4xD+6)y=-x^4\sin x$

Putting $x = e^z$,

$$\left[\mathcal{D}(\mathcal{D}-1)-4\mathcal{D}+6\right]y = -e^{4z}\sin e^z, \quad \text{where } \mathcal{D} \equiv \frac{d}{dz}$$

$$(D^2-5D+6)y = -e^{4z}\sin e^z$$

The auxiliary equation is

$$m^2-5m+6=0$$
$$(m-2)(m-3)=0$$
$$m = 2, 3 \text{ (real and distinct)}$$
$$\text{CF} = c_1e^{2z}+c_2e^{3z} = c_1x^2+c_2x^3$$
$$\text{PI} = \frac{1}{\mathcal{D}^2-5\mathcal{D}+6}(-e^{4z}\sin e^z) = \frac{1}{(\mathcal{D}-2)(\mathcal{D}-3)}(-e^{4z}\sin e^z)$$
$$= \frac{(\mathcal{D}-2)-(\mathcal{D}-3)}{(\mathcal{D}-2)(\mathcal{D}-3)}(-e^{4z}\sin e^z) = \left(\frac{1}{\mathcal{D}-3}-\frac{1}{\mathcal{D}-2}\right)(-e^{4z}\sin e^z)$$
$$= \frac{1}{\mathcal{D}-2}(e^{4z}\sin e^z)-\frac{1}{\mathcal{D}-3}(e^{4z}\sin e^z)$$
$$= e^{2z}\int e^{4z}\sin e^z\cdot e^{-2z}dz - e^{3z}\int e^{4z}\sin e^z\cdot e^{-3z}dz$$
$$= e^{2z}\int \sin e^z\cdot e^{2z}dz - e^{3z}\int \sin e^z\cdot e^z dz$$

Putting $e^z = t$, $e^z dz = dt$

$$\text{PI} = e^{2z}\int \sin t\cdot t\,dt - e^{3z}\int \sin t\,dt = e^{2z}(-t\cos t+\sin t)-e^{3z}(-\cos t)$$
$$= e^{2z}(-e^z\cos e^z+\sin e^z)+e^{3z}\cos e^z = e^{2z}\sin e^z = x^2\sin x$$

Hence, the general solution is

$$y = c_1x^2+c_2x^3+x^2\sin x$$

EXERCISE 10.15

Solve the following differential equations:

1. $(x^2D^2+xD-1)y=0$

$$\left[\textbf{Ans.:}\ y = c_1x+\frac{c_2}{x}\right]$$

2. $(9x^2D^2+3xD+10)y=0$

$$\left[\textbf{Ans.:}\ y = x^{\frac{1}{3}}\left[c_1\cos(\log x)+c_2\sin(\log x)\right]\right]$$

3. $(x^3D^3+3x^2D^2+14xD+34)y=0$

$$\left[\textbf{Ans.:}\ \frac{c_1}{x^2}+x[c_2\cos(4\log x)+c_3\sin(4\log x)]\right]$$

4. $(x^3D^3+6x^2D^2-12)y=\frac{12}{x^2}$

$$\left[\textbf{Ans.:}\ y = c_1x^2+\frac{c_2}{x^2}+\frac{c_3}{x^3}-\frac{3}{x^2}\log x\right]$$

5. $(4x^3D^3 + 12x^2D^2 + xD + 1)y = 50\sin(\log x)$

$$\left[\text{Ans.}: y = (c_1 + c_2 \log x)x^{\frac{1}{2}} + \frac{c_3}{x} + \sin(\log x) + 7\cos(\log x)\right]$$

6. $(x^2D^2 - 3xD + 3)y = 2 + 3\log x$

$$\left[\text{Ans.}: y = c_1x + c_2x^3 + \log x + 2\right]$$

7. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\sin(\log x) + 1}{x}$

$$\left[\text{Ans.}: y = x^2\left[c_1\cosh\left(\sqrt{3}\log x\right) + c_2\sinh\left(\sqrt{3}\log x\right)\right] + \frac{1}{6x} + \frac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]\right]$$

8. $(x^2D^3 + 3xD^2 + D)y = x^2\log x$

$$\left[\text{Ans.}: c_1 + c_2\log x + c_3(\log x)^2 + \frac{x^3}{27}(\log x - 1)\right]$$

9. $(x^2D^2 - 2xD + 2)y = (\log x)^2 - \log x^2$

$$\left[\text{Ans.}: y = c_1x + c_2x^2 + \frac{1}{2}[(\log x)^2 + \log x] + \frac{1}{4}\right]$$

10. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1-x)^2}$

$$\left[\text{Ans.}: y = \frac{1}{x}(c_1 + c_2\log x) + \frac{1}{x}\log\frac{x}{x-1}\right]$$

10.7.2 Legendre's Linear Equation

An equation of the form

$$a_0(a+bx)^n\frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots$$

$$\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_ny = Q(x) \quad \ldots (10.34)$$

where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants, is called *Legendre's linear equation.*

Let $(a+bx) = e^z$

$$b = e^z\frac{dz}{dx}, \qquad \frac{dz}{dx} = \frac{b}{e^z} = \frac{b}{a+bx}$$

Now,
$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\cdot\frac{b}{(a+bx)}$$

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz}$$

$$(a+bx)Dy = b\mathcal{D}y, \qquad \text{where } D \equiv \frac{d}{dx} \text{ and } \mathcal{D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\cdot\frac{dy}{dz}\right) = -\frac{b}{(a+bx)^2}\cdot b\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = -\frac{b^2}{(a+bx)^2}\cdot\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d^2y}{dz^2}\left(\frac{b}{a+bx}\right)$$

$$(a+bx)^2 \frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$(a+bx)^2 D^2 y = b^2(\mathcal{D}^2 - \mathcal{D})y = b^2\mathcal{D}(\mathcal{D}-1)y$$

Similarly, $(a+bx)^3 D^3 y = b^3\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y$

....

....

$$(a+bx)^n D^n y = b^n\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)\ldots[\mathcal{D}-(n-1)]y$$

Substituting these derivatives in Eq. (10.34),

$$\left[\left\{a_0 b^n \mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+1)\right\} + \left\{a_1 b^{n-1}\mathcal{D}(\mathcal{D}-1)\ldots(\mathcal{D}-n+2)\right\} + \ldots + a_{n-1}\mathcal{D} + a_n\right]y = Q\left(\frac{e^z - a}{b}\right)$$

which is a linear differential equation with constant coefficients and can be solved by usual methods described in the previous section.

HISTORICAL DATA

Adrien-Marie Legendre (1752–1833) was a French mathematician. Legendre made numerous contributions to mathematics. Well-known and important concepts such as the Legendre polynomials and Legendre transformation are named after him.

Most of his work was brought to perfection by others: his work on roots of polynomials inspired Galois theory; Abel's work on elliptic functions was built on Legendre's; some of Gauss' work in statistics and number theory completed that of Legendre. He developed the least squares method, which has broad application in linear regression, signal processing, statistics, and curve fitting.

In 1830, he gave a proof of Fermat's last theorem for exponent $n = 5$, which was also proven by Lejeune Dirichlet in 1828.

In number theory, he conjectured the quadratic reciprocity law, subsequently proved by Gauss; in connection to this, the Legendre symbol is named after him. He also did pioneering work on the distribution of primes, and on the application of analysis to number theory. His 1798 conjecture of the Prime number theorem was rigorously proved by Hadamard and de la Vallée-Poussin in 1896.

He is known for the Legendre transformation, which is used to go from the Lagrangian to the Hamiltonian formulation of classical mechanics. In thermodynamics, it is also used to obtain the enthalpy and the Helmholtz and Gibbs (free) energies from the internal energy. He is also the namegiver of the Legendre polynomials, solutions to Legendre's differential equation, which occur frequently in physics and engineering applications, e.g., electrostatics.

Legendre is best known as the author of *Éléments de géométrie*, which was published in 1794 and was the leading elementary text on the topic for around 100 years. This text greatly rearranged and simplified many of the propositions from Euclid's Elements to create a more effective textbook.

No portrait of him is known.

EXAMPLE 10.75 *Solve* $(2x+3)^2\frac{d^2y}{dx^2}-(2x+3)\frac{dy}{dx}-12y=6x.$

Solution:

$$\left[(2x+3)^2D^2-(2x+3)D-12\right]y=6x$$

Putting $2x+3=e^z$,

$$\left[4\mathcal{D}(\mathcal{D}-1)-2\mathcal{D}-12\right]y=6\left(\frac{e^z-3}{2}\right),\quad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$\left(4\mathcal{D}^2-6\mathcal{D}-12\right)y=3e^z-9$$

The auxiliary equation is

$$4m^2-6m-12=0$$

$$2m^2-3m-6=0$$

$$m=\frac{3\pm\sqrt{57}}{4}\quad\text{(real and distinct)}$$

$$\text{CF}=c_1e^{\left(\frac{3+\sqrt{57}}{4}\right)z}+c_2e^{\left(\frac{3-\sqrt{57}}{4}\right)z}=c_1(2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)}+c_2(2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)}$$

$$\text{PI}=\frac{1}{4\mathcal{D}^2-6\mathcal{D}-12}\left(3e^z-9\right)=3\frac{1}{4\mathcal{D}^2-6\mathcal{D}-12}e^z-9\frac{1}{4\mathcal{D}^2-6\mathcal{D}-12}e^{0z}$$

$$=\frac{3}{4(1)-6(1)-12}e^z-\frac{9}{-12}e^{0z}=-\frac{3}{14}e^z+\frac{3}{4}=-\frac{3}{14}(2x+3)+\frac{3}{4}$$

Hence, the general solution is

$$y=c_1(2x+3)^{\left(\frac{3+\sqrt{57}}{4}\right)}+c_2(2x+3)^{\left(\frac{3-\sqrt{57}}{4}\right)}-\frac{3}{14}(2x+3)+\frac{3}{4}$$

EXAMPLE 10.76 *Solve* $(x-1)^3\frac{d^3y}{dx^3}+2(x-1)^2\frac{d^2y}{dx^2}-4(x-1)\frac{dy}{dx}+4y=4\log(x-1).$

Solution:

$$(x-1)^3\frac{d^3y}{dx^3}+2(x-1)^2\frac{d^2y}{dx^2}-4(x-1)\frac{dy}{dx}+4y=4\log(x-1)$$

Putting $(x-1)=e^z$,

$$[\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)+2\mathcal{D}(\mathcal{D}-1)-4\mathcal{D}+4]y=4z,\quad \text{where } \mathcal{D}\equiv\frac{d}{dz}$$

$$(\mathcal{D}^3-\mathcal{D}^2-4\mathcal{D}+4)y=4z$$

The auxiliary equation is

$$m^3 - m^2 - 4m + 4 = 0$$

$$(m^2 - 4)(m - 1) = 0$$

$$m = \pm 2, 1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^z + c_2 e^{2z} + c_3 e^{-2z} = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2}$$

$$\text{PI} = \frac{1}{\mathcal{D}^3 - \mathcal{D}^2 - 4\mathcal{D} + 4} \cdot 4z = \frac{1}{4\left(1 - \dfrac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)} \cdot 4z$$

$$= \left(1 - \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^{-1} z = \left[1 + \frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4} + \left(\frac{4\mathcal{D} + \mathcal{D}^2 - \mathcal{D}^3}{4}\right)^2 + \ldots\right] z$$

$$= z + \mathcal{D}(z) + (\text{Higher powers of } \mathcal{D})z = z + 1 + 0 = \log(x-1) + 1$$

Hence, the general solution is

$$y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1$$

EXERCISE 10.16

Solve the following differential equations:

1. $[(1+x)^2 D^2 + (1+x)D + 1]y = 2\sin\log(x+1)$

$$\left[\textbf{Ans.: } y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \log(1+x)\cos\log(1+x)\right]$$

2. $[(x+2)^2 D^2 - (x+2)D + 1]y = 3x + 4$

$$\left[\textbf{Ans.: } y = [c_1 + c_2\log(x+2)](x+2) + \frac{3}{2}[\log(x+2)]^2 (x+2) - 2\right]$$

3. $[(x-1)^3 D^3 + 2(x-1)^2 D^2 - 4(x-1)D + 4]y = 4\log(x-1)$

$$\left[\textbf{Ans.: } y = c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^{-2} + \log(x-1) + 1\right]$$

4. $[(2x+1)^2 D^2 - 2(2x+1)D - 12]y = 6x$

$$\left[\textbf{Ans.: } y = c_1(2x+1)^{-1} + c_2(2x+1)^3 - \frac{3}{8}x + \frac{1}{16}\right]$$

5. $[(2+3x)^2 D^2 + 5(2+3x)D - 3]y = x^2 + x + 1$

$$\left[\textbf{Ans.: } c_1(2+3x)^{\frac{1}{3}} + c_2(2+3x)^{-1} + \frac{1}{27}\left[\frac{1}{15}(2+3x)^2 + \frac{1}{4}(2+3x) - 7\right]\right]$$

6. $[(2x-1)^3 D^3 + (2x-1)D - 2]y = 0$

$$\left[\textbf{Ans.: } y = c_1(2x-1) + (2x-1)\left[c_2(2x-1)^{\frac{\sqrt{3}}{2}} + c_3(2x-1)^{-\frac{\sqrt{3}}{2}}\right]\right]$$

10.8 METHOD OF VARIATION OF PARAMETERS

This method is used to find the particular integral if the complementary function is known. In this method, the particular integral is obtained by varying the arbitrary constants of the complementary function and, hence, is known as *variation-of-parameters method.*

Consider a linear nonhomogeneous differential equation of second order with constant coefficients:

$$\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=Q(x) \qquad \cdots (10.35)$$

Let the complementary function be

$$\text{CF}=c_1y_1+c_2y_2 \qquad \cdots (10.36)$$

where y_1, y_2 are the solutions of

$$\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=0 \qquad \cdots (10.37)$$

Let the particular integral be

$$y=u(x)y_1+v(x)y_2 \qquad \cdots (10.38)$$

where u and v are unknown functions of x.

Differentiating Eq. (10.38) w.r.t. x,

$$y'=uy_1'+vy_2'+u'y_1+v'y_2$$

Let u, v satisfy the equation

$$u'y_1+v'y_2=0 \qquad \cdots (10.39)$$

Then
$$y'=uy_1'+vy_2'$$

Differentiating y' again w.r.t. x,

$$y''=uy_1''+vy_2''+u'y_1'+v'y_2'$$

Substituting y'', y', and y in Eq. (10.35),

$$uy_1''+vy_2''+u'y_1'+v'y_2'+a_1(uy_1'+vy_2')+a_2(uy_1+vy_2)=Q(x)$$
$$u(y_1''+a_1y_1'+a_2y_1)+v(y_2''+a_1y_2'+a_2y_2)+u'y_1'+v'y_2'=Q(x)$$

Since y_1 and y_2 satisfy Eq. (10.37),

$$u'y_1'+v'y_2'=Q \qquad \cdots (10.40)$$

Solving Eqs (10.39) and (10.40) by using Cramer's rule,

$$u'=\frac{\begin{vmatrix}0 & y_2\\ Q & y_2'\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}}=\frac{-y_2Q}{y_1y_2'-y_1'y_2}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & Q \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 Q}{y_1 y_2' - y_1' y_2}$$

$$u = \int -\frac{y_2 Q}{y_1 y_2' - y_1' y_2}\,dx = \int -\frac{y_2 Q}{W}\,dx \qquad \text{... (10.41)}$$

$$v = \int \frac{y_1 Q}{y_1 y_2' - y_1' y_2}\,dx = \int \frac{y_1 Q}{W}\,dx \qquad \text{... (10.42)}$$

where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is known as the *Wronskian* of y_1, y_2.

Hence, the required general solution of Eq. (10.35) is

$$y = \text{CF} + \text{PI} = c_1 y_1 + c_2 y_2 + u y_1 + v y_2$$

where u, v are obtained using equations (10.41) and (10.42).

Note The above method can also be extended for third-order differential equations. Let the complementary function of a third-order differential equation be

$$\text{CF} = c_1 y_1 + c_2 y_2 + c_3 y_3$$

Let $\text{PI} = u(x) y_1 + v(x) y_2 + w(x) y_3$

where
$$u(x) = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\,dx$$

$$v(x) = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\,dx$$

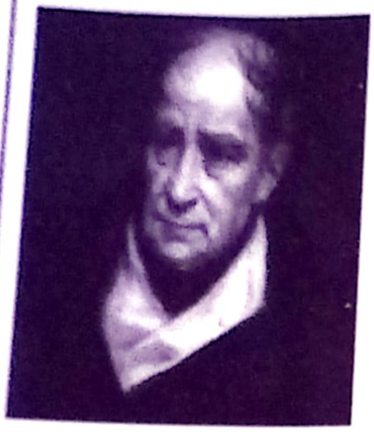
$$w(x) = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W}\,dx$$

Wronskian
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Working Rule for solving differential equations

1. Find the complementary function as $\text{CF} = c_1 y_1 + c_2 y_2$.
2. Find the Wronskian of y_1, y_2 as $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
3. Assume the particular integral as $\text{PI} = u(x) y_1 + v(x) y_2$.
4. Find u and v by evaluating the integrals $u = \int \frac{-y_2 Q}{W}\,dx$, $v = \int \frac{y_1 Q}{W}\,dx$.
5. Substitute u and v in PI and write the general solution as $y = \text{CF} + \text{PI}$.

HISTORICAL DATA



Józef Maria Hoene-Wronski (1776–1853) was a Polish Messianist philosopher who worked in many fields of knowledge, not only as philosopher but also as mathematician, physicist, inventor, lawyer, and economist.

Though during his lifetime nearly all his work was dismissed as nonsense, some of it has come in later years to be seen in a more favorable light. Although nearly all his grandiose claims were in fact unfounded, his mathematical work contains flashes of deep insight and many important intermediary results. Most significant was his work on series. He had strongly criticized Lagrange's use of infinite series, introducing instead a novel series expansion for a function. His criticisms of Lagrange were for the most part unfounded, but the coefficients in Wronski's new series were found to be important after his death, forming the determinants now known as the Wronskians (the name was given them by Thomas Muir in 1882).

EXAMPLE 10.77 *Solve* $(D^2+1)y = \operatorname{cosec} x$.

Solution: The auxiliary equation is

$$m^2 = -1$$

$$m = \pm i \quad \text{(complex)}$$

$$CF = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

$$\text{Wronskian } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Let $$PI = u\cos x + v\sin x \qquad \ldots(1)$$

where $$u = \int \frac{-y_2 Q}{W}dx = -\int \frac{\sin x \operatorname{cosec} x}{1}dx = -\int dx = -x$$

and $$v = \int \frac{y_1 Q}{W}dx = \int \frac{\cos x \operatorname{cosec} x}{1}dx = \int \cot x\, dx = \log \sin x$$

Substituting u and v in Eq. (1),

$$PI = -x\cos x + (\log \sin x)\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \cdot \log \sin x$$

EXAMPLE 10.78 *Solve* $(D^2 + 4)y = \cot 2x$.

Solution: The auxiliary equation is

$$m^2 + 4 = 0$$
$$m = \pm 2i \quad \text{(complex)}$$
$$\text{CF} = c_1 \cos 2x + c_2 \sin 2x$$
$$y_1 = \cos 2x, \quad y_2 = \sin 2x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

Let $\text{PI} = u\cos 2x + v\sin 2x$ ...(1)

where $u = \int -\frac{y_2 Q}{W}\,dx = \int -\frac{\sin 2x \cot 2x}{2}\,dx = -\frac{1}{2}\int \sin 2x\left(\frac{\cos 2x}{\sin 2x}\right)dx$

$$= -\frac{1}{2}\int \cos 2x\,dx = -\frac{1}{2}\frac{\sin 2x}{2} = -\frac{1}{4}\sin 2x$$

$$v = \int \frac{y_1 Q}{W}\,dx = \int \frac{\cos 2x \cot 2x}{2}\,dx = \frac{1}{2}\int \frac{\cos^2 2x}{\sin 2x}\,dx$$

$$= \frac{1}{2}\int \frac{1-\sin^2 2x}{\sin 2x}\,dx = \frac{1}{2}\int (\operatorname{cosec} 2x - \sin 2x)\,dx$$

$$= \frac{1}{2}\left[\frac{\log(\operatorname{cosec} 2x - \cot 2x)}{2} + \frac{\cos 2x}{2}\right] = \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]$$

Substituting u and v in Eq.(1),

$$\text{PI} = -\frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]\sin 2x$$

Hence, the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}\sin 2x \cos 2x + \frac{1}{4}\left[\log(\operatorname{cosec} 2x - \cot 2x) + \cos 2x\right]\sin 2x$$

EXAMPLE 10.79 *Solve* $(D^2 - 1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$.

Solution: The auxiliary equation is

$$m^2 - 1 = 0$$
$$m = \pm 1 \quad \text{(real and distinct)}$$

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$$\text{CF} = c_1e^x + c_2e^{-x}$$

$$y_1 = e^x, \quad y_2 = e^{-x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^0 - e^0 = -2$

Let
$$\text{PI} = ue^x + ve^{-x} \qquad \ldots(1)$$

where
$$u = \int \frac{-y_2 Q}{W}\,dx = -\int \frac{e^{-x}\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2}\,dx$$

Let $e^{-x} = t, \quad -e^{-x}dx = dt$

$$u = -\frac{1}{2}\int (t\sin t + \cos t)dt = -\frac{1}{2}\left[t(-\cos t) - (-\sin t) + \sin t\right] = \frac{1}{2}t\cos t - \sin t$$

$$= \frac{1}{2}e^{-x}\cos(e^{-x}) - \sin(e^{-x})$$

and
$$v = \int \frac{y_1 Q}{W}\,dx = \int \frac{e^x\left[e^{-x}\sin(e^{-x}) + \cos(e^{-x})\right]}{-2}\,dx = \int \frac{e^x\left[\cos(e^{-x}) + e^{-x}\sin(e^{-x})\right]}{-2}\,dx$$

$$= -\frac{1}{2}e^x\cos(e^{-x}) \qquad \left[\because \int e^x\{f(x) + f'(x)\}dx = e^x f(x) \quad \text{Here } f(x) = \cos e^{-x}\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{2}\cos(e^{-x}) - e^x\sin(e^{-x}) - \frac{1}{2}\cos(e^{-x}) = -e^x\sin(e^{-x})$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-x} - e^x\sin(e^{-x})$$

EXAMPLE 10.80 *Solve* $(D^2 + 5D + 6)y = e^{-2x}\sec^2 x(1 + 2\tan x)$.

Solution: The auxiliary equation is

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, -3 \text{ (real and distinct)}$$

$$\text{CF} = c_1e^{-2x} + c_2e^{-3x}$$

$$y_1 = e^{-2x}, \qquad y_2 = e^{-3x}$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix} = -3e^{-5x} + 2e^{-5x} = -e^{-5x}$

Let
$$\text{PI} = ue^{-2x} + ve^{-3x} \qquad \ldots(1)$$

where $$u = \int \frac{-y_2 Q}{W} dx = -\int \frac{e^{-3x} e^{-2x} \sec^2 x(1+2\tan x)}{-e^{-5x}} dx = \int (1+2\tan x)\frac{2\sec^2 x}{2} dx$$

[Multiplying and dividing by 2]

$$= \frac{1}{2}\frac{(1+2\tan x)^2}{2} \qquad \left[\because \int f(x).f'(x)dx = \frac{\{f(x)\}^2}{2} \quad \text{Here, } f(x) = (1+2\tan x)\right]$$

$$= \frac{1}{4}(1+2\tan x)^2$$

and $$v = \int \frac{y_1 Q}{W} dx = \int \frac{e^{-2x} e^{-2x} \sec^2 x(1+2\tan x)}{-e^{-5x}} dx$$

$$= -\int e^x \sec^2 x(1+2\tan x)dx = -\int e^x(\sec^2 x + 2\sec^2 x \tan x)dx$$

$$= -e^x \sec^2 x \qquad \left[\because \int e^x\{f(x)+f'(x)\}dx = e^x f(x) \quad \text{Here, } f(x) = \sec^2 x\right]$$

Substituting u and v in Eq. (1),

$$\text{PI} = \frac{1}{4}(1+2\tan x)^2 e^{-2x} + \left(-e^x \sec^2 x\right)e^{-3x}$$

Hence, the general solution is

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{4}(1+2\tan x)^2 e^{-2x} - e^{-2x}\sec^2 x$$

EXAMPLE 10.81 *Solve* $(D^2+1)y = \dfrac{1}{1+\sin x}$.

Solution: The auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$y_1 = \cos x, \qquad y_2 = \sin x$$

Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

Let $$\text{PI} = u\cos x + v\sin x \qquad \ldots(1)$$

where $$u = \int \frac{-y_2 Q}{W} dx = \int -\frac{\sin x}{1} \cdot \frac{1}{1+\sin x} dx$$

$$= -\int \frac{\sin x}{1+\sin x} \cdot \frac{(1-\sin x)}{(1-\sin x)} dx = -\int \frac{\sin x - \sin^2 x}{1-\sin^2 x} dx$$

$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x}\,dx = -\int \left(\tan x \sec x - \tan^2 x\right)dx$$

$$= -\int \left(\tan x \sec x - \sec^2 x + 1\right)dx = -\left(\sec x - \tan x + x\right)$$

and
$$v = \int \frac{y_1 Q}{W}\,dx = \int \frac{\cos x}{1} \cdot \frac{1}{1+\sin x}\,dx = \int \frac{\cos x}{1+\sin x}\,dx$$

$$= \log\left(1+\sin x\right) \qquad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log\{f(x)\} \quad \text{Here } f(x) = 1+\sin x\right.$$

Substituting u and v in Eq. (1),

$$\text{PI} = -\left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x - \left(\sec x - \tan x + x\right)\cos x + \left[\log(1+\sin x)\right]\sin x$$

EXAMPLE 10.82 *Solve* $(D^3 + D)y = \operatorname{cosec} x.$

Solution: The auxiliary equation is

$$m^3 + m = 0$$

$$m\left(m^2 + 1\right) = 0$$

$$m = 0 \text{ (real)}, \quad m = \pm i \text{ (complex)}$$

$$\text{CF} = c_1 + c_2 \cos x + c_3 \sin x$$

$$y_1 = 1, \quad y_2 = \cos x, \quad y_3 = \sin x$$

Wronskian
$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$= 1\left(\sin^2 x + \cos^2 x\right) - \cos x.(0-0) + \sin x(0-0) = 1$$

Let
$$\text{PI} = u \cdot 1 + v\cos x + w\sin x \qquad \ldots(1)$$

where
$$u = \int \frac{(y_2 y_3' - y_3 y_2')Q}{W}\,dx = \int \frac{\left[\cos x \cos x - \sin x(-\sin x)\right]\operatorname{cosec} x}{1}\,dx$$

$$= \int \left(\cos^2 x + \sin^2 x\right)\operatorname{cosec} x\,dx = \int \operatorname{cosec} x\,dx = \log\left(\operatorname{cosec} x - \cot x\right)$$

$$v = \int \frac{(y_3 y_1' - y_1 y_3')Q}{W}\,dx = \int \frac{[\sin x.0 - 1.\cos x]\operatorname{cosec} x}{1}\,dx$$
$$= \int(-\cos x)\operatorname{cosec} x\,dx = -\int \cot x\,dx = -\log\sin x$$

$$w = \int \frac{(y_1 y_2' - y_2 y_1')Q}{W}\,dx = \int \frac{[1.(-\sin x) - \cos x.0]\operatorname{cosec} x}{1}\,dx = \int -dx = -x$$

Substituting u, v, and w in Eq. (1),

$$\text{PI} = \log(\operatorname{cosec} x - \cot x).1 + (-\log\sin x)\cos x + (-x)\sin x$$
$$= \log(\operatorname{cosec} x - \cot x) - \cos x\log\sin x - x\sin x$$

Hence, the general solution is

$$y = c_1 + c_2\cos x + c_3\sin x + \log(\operatorname{cosec} x - \cot x) - \cos x\log\sin x - x\sin x$$

EXERCISE 10.17

Solve the following differential equations:

1. $(D^2 + 3D + 2)y = \sin e^x$

   [**Ans.:** $y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x}\sin e^x$]

2. $(D^2 + 1)y = \operatorname{cosec} x$

   [**Ans.:** $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\,\log(\sin x)$]

3. $(D^2 + 4)y = \tan 2x$

   [**Ans.:** $y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}\cos 2x\log(\sec 2x + \tan 2x)$]

4. $(D^2 + 1)y = x - \cot x$

   [**Ans.:** $y = c_1\cos x + c_2\sin x - x\cos^2 x + x\sin^2 x - \sin x\log(\operatorname{cosec} x - \cot x)$]

5. $(D^2 + D)y = \dfrac{1}{1 + e^x}$

   [**Ans.:** $y = c_1 + c_2 e^{-x} - e^{-x}[e^x\log(e^{-x} + 1) + \log(e^x + 1)]$]

6. $(D^2 - 2D + 2)y = e^x\tan x$

   [**Ans.:** $y = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\log(\sec x + \tan x)$]

7. $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

$$\left[\textbf{Ans.: } y=(c_1+c_2x)e^{2x} - e^{2x}(2x^2\sin 2x + 4x\cos 2x - 3\sin 2x)\right]$$

8. $(D^2+2D+1)y = e^{-x}\log x$

$$\left[\textbf{Ans.: } y=(c_1+c_2x)e^{-x} + \frac{x^2}{2}e^{-x}\left(\log x - \frac{3}{2}\right)\right]$$

10.9 METHOD OF UNDETERMINED COEFFICIENTS

This method can be used to find the particular integral only if linearly independent derivatives of $Q(x)$ are finite in number. This restriction implies that $Q(x)$ can only have the terms such as k, x^n, e^{ax}, $\sin ax$, $\cos ax$ and combinations of such terms, where k, a are constants and n is a positive integer. However, when $Q(x) = \frac{1}{x}$ or $\tan x$ or $\sec x$, etc., this method fails, since each function has an infinite number of linearly independent derivatives.

In this method, the particular integral is assumed as a linear combination of the terms in $Q(x)$ and all its linearly independent derivatives. Some of the choices of particular integral are given below.

S. No.	Q(x)	Particular Integral
1.	ke^{ax}	Ae^{ax}
2.	$k\sin(ax+b)$ or $k\cos(ax+b)$	$A\sin(ax+b)+B\cos(ax+b)$
3.	$ke^{ax}\sin(bx+c)$ or $ke^{ax}\cos(bx+c)$	$Ae^{ax}\sin(bx+c)+Be^{ax}\cos(bx+c)$
4.	kx^n, $n=0,1,2,\ldots$	$A_nx^n+A_{n-1}x^{n-1}+\ldots\ldots+A_2x^2+A_1x+A_0$
5.	kx^ne^{ax}, $n=0,1,2,\ldots.$	$e^{ax}(A_nx^n+A_{n-1}x^{n-1}+\ldots\ldots+A_2x^2+A_1x+A_0)$
6.	$kx^n\sin(ax+b)$ or $kx^n\cos(ax+b)$	$x^n[A_n\sin(ax+b)+B_n\cos(ax+b)]+x^{n-1}[A_{n-1}\sin(ax+b)+B_{n-1}\cos(ax+b)]+\ldots.. +x[A_1\sin(ax+b)+B_1\cos(ax+b)] +[A_0\sin(ax+b)+B_0\cos(ax+b)]$
7.	$kx^ne^{ax}\sin(bx+c)$ or $kx^ne^{ax}\cos(bx+c)$	$e^{ax}[x^n\{A_n\sin(ax+b)+B_n\cos(ax+b)\} +x^{n-1}\{A_{n-1}\sin(ax+b)+B_{n-1}\cos(ax+b)\} +\ldots..+x\{A_1\sin(ax+b)+B_1\cos(ax+b)\} +\{A_0\sin(ax+b)+B_0\cos(ax+b)\}]$

In the table, $A_0, A_1, A_2, \ldots, A_n$ are coefficients to be determined. To obtain the values of these coefficients, the fact that the particular integral satisfies the given differential equation is used.

However, before assuming the particular integral, it is necessary to compare the terms of $Q(x)$ with the complementary function. While comparing the terms, the following different cases arise.

Case I If no terms of $Q(x)$ occur in the complementary function then the particular integral is assumed from the table depending on the nature of $Q(x)$.

Case II If a term u of $Q(x)$ is also a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to u should be multiplied by x^r.

Case III If $x^s u$ is a term of $Q(x)$ and only u is a term of the complementary function corresponding to an r-fold root then the assumed particular integral corresponding to $x^s u$ should be multiplied by x^r.

Note In cases (ii) and (iii), initially similar types of terms appear in the complementary function and in the assumed particular integral. After multiplication by x^r, the terms of the particular integral change. Hence, this method avoids the repetition of similar terms in the complementary function and particular integral.

EXAMPLE 10.83 *Solve* $(D^2 - 9)y = x + e^{2x} - \sin 2x$.

Solution: The auxiliary equation is

$$m^2 - 9 = 0$$

$$m = \pm 3 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^{3x} + c_2 e^{-3x}$$

$$Q = x + e^{2x} - \sin 2x$$

Let the particular integral be

$$y = A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x$$

$$Dy = A_1 + 2A_3 e^{2x} + 2A_4 \cos 2x - 2A_5 \sin 2x$$

$$D^2 y = 4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x$$

Substituting these derivatives in the given equation,

$$4A_3 e^{2x} - 4A_4 \sin 2x - 4A_5 \cos 2x - 9(A_1 x + A_2 + A_3 e^{2x} + A_4 \sin 2x + A_5 \cos 2x)$$
$$= x + e^{2x} - \sin 2x$$

$$(-5A_3)e^{2x} - 9A_1 x - 9A_2 + \sin 2x(-13A_4) + \cos 2x(-13A_5) = x + e^{2x} - \sin 2x$$

Comparing coefficients on both the sides,

$$-5A_3 = 1, \qquad A_3 = -\frac{1}{5}$$

$$-9A_1 = 1, \qquad A_1 = -\frac{1}{9}$$

$$-9A_2 = 0, \qquad A_2 = 0$$

$$-13A_4 = -1, \qquad A_4 = \frac{1}{13}$$

$$-13A_5 = 0, \qquad A_5 = 0$$

$$\text{PI} = -\frac{1}{9}x - \frac{1}{5}e^{2x} + \frac{1}{13}\sin 2x$$

Hence, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-3x} - \frac{x}{9} - \frac{e^{2x}}{5} + \frac{\sin 2x}{13}$$

EXAMPLE 10.84 *Solve* $(D^2 - 2D)y = e^x \sin x$.

Solution: The auxiliary equation is

$$m^2 - 2m = 0$$

$$m = 0,\ -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 + c_2 e^{2x}$$

$$\text{Q} = e^x \sin x$$

Let the particular integral be

$$y = \text{A}_1 e^x \sin x + \text{A}_2 e^x \cos x$$

$$\text{D}y = \text{A}_1(e^x \sin x + e^x \cos x) + \text{A}_2(e^x \cos x - e^x \sin x)$$

$$= (\text{A}_1 - \text{A}_2)e^x \sin x + (\text{A}_1 + \text{A}_2)e^x \cos x$$

$$\text{D}^2 y = (\text{A}_1 - \text{A}_2)(e^x \sin x + e^x \cos x) + (\text{A}_1 + \text{A}_2)(e^x \cos x - e^x \sin x)$$

$$= -2\text{A}_2 e^x \sin x + 2\text{A}_1 e^x \cos x$$

Substituting these derivatives in the given equation,

$$-2\text{A}_2 e^x \sin x + 2\text{A}_1 e^x \cos x - 2(\text{A}_1 - \text{A}_2)e^x \sin x - 2(\text{A}_1 + \text{A}_2)e^x \cos x = e^x \sin x$$

$$-2\text{A}_1 e^x \sin x - 2\text{A}_2 e^x \cos x = e^x \sin x$$

Comparing coefficients on both the sides,

$$-2\text{A}_1 = 1, \qquad \text{A}_1 = -\frac{1}{2}$$

$$2\text{A}_2 = 0, \qquad \text{A}_2 = 0$$

$$\text{PI} = -\frac{1}{2} e^x \sin x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x$$

EXAMPLE 10.85 *Solve* $(D^3 + 3D^2 + 2D)y = x^2 + 4x + 8$.

Solution: The auxiliary equation is

$$m^3 + 3m^2 + 2m = 0$$

$$m(m+1)(m+2) = 0$$

$$m = 0, -1, -2 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 + c_2 e^{-x} + c_3 e^{-2x}$$

$$\text{Q} = x^2 + 4x + 8$$

Let the particular integral be

$$y = \text{A}_1 x^2 + \text{A}_2 x + \text{A}_3$$

Since the constant occurs in $\text{Q}(x)$ and is also a part of the CF corresponding to the 1-fold root $m = 0$, multiplying the assumed particular integral by x,

$$y = \text{A}_1 x^3 + \text{A}_2 x^2 + \text{A}_3 x$$

$$\text{D}y = 3\text{A}_1 x^2 + 2\text{A}_2 x + \text{A}_3$$

$$\text{D}^2 y = 6\text{A}_1 x + 2\text{A}_2$$

$$\text{D}^3 y = 6\text{A}_1$$

Substituting these derivatives in the given equation,

$$6\text{A}_1 + 3(6\text{A}_1 x + 2\text{A}_2) + 2(3\text{A}_1 x^2 + 2\text{A}_2 x + \text{A}_3) = x^2 + 4x + 8$$

$$6\text{A}_1 x^2 + (18\text{A}_1 + 4\text{A}_2)x + (6\text{A}_1 + 6\text{A}_2 + 2\text{A}_3) = x^2 + 4x + 8$$

Comparing coefficients on both the sides,

$$6\text{A}_1 = 1, \qquad \text{A}_1 = \frac{1}{6}$$

$$18\text{A}_1 + 4\text{A}_2 = 4, \qquad \text{A}_2 = \frac{1}{4}(4-3) = \frac{1}{4}$$

$$6\text{A}_1 + 6\text{A}_2 + 2\text{A}_3 = 8, \qquad \text{A}_3 = \frac{1}{2}(8 - 6\text{A}_1 - 6\text{A}_2) = \frac{1}{2}\left(8 - 1 - \frac{3}{2}\right) = \frac{11}{4}$$

$$\text{PI} = \frac{1}{6}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x$$

Hence, the general solution is

$$y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4}$$

EXAMPLE 10.86 *Solve* $(D^2 + 1)y = 4x\cos x - 2\sin x$.

Solution: Tha auxiliary equation is

$$m^2 + 1 = 0$$

$$m = \pm i \text{ (complex)}$$

$$\text{CF} = c_1 \cos x + c_2 \sin x$$

$$\text{Q} = 4x \cos x - 2 \sin x$$

Let the particular integral be

$$y = \text{A}_1 x \sin x + \text{A}_2 x \cos x + \text{A}_3 \sin x + \text{A}_4 \cos x$$

Since $x \cos x$ and its derivatives occur in Q(x) and $\cos x$ is a part of the CF corresponding to the 1-4 pair of complex roots $m = \pm i$, multiplying the assumed particular integral by x,

$$y = A_1x^2 \sin x + A_2x^2 \cos x + A_3x \sin x + A_4x \cos x$$

$$Dy = A_1x^2 \cos x + 2A_1x \sin x - A_2x^2 \sin x + 2A_2x \cos x + A_3x \cos x + A_3 \sin x - A_4x \sin x + A_4 \cos x$$

$$= (A_1 \cos x - A_2 \sin x)x^2 + (2A_1 - A_4)x \sin x + (2A_2 + A_3)x \cos x + A_3 \sin x + A_4 \cos x$$

$$D^2y = (-A_1 \sin x - A_2 \cos x)x^2 + (A_1 \cos x - A_2 \sin x)(2x) + (2A_1 - A_4)\sin x + (2A_1 - A_4)x \cos x + (2A_2 + A_3)\cos x - (2A_2 + A_3)x \sin x + A_3 \cos x - A_4 \sin x$$

$$= -A_1x^2 \sin x - A_2x^2 \cos x + (4A_1 - A_4)x \cos x - (4A_2 + A_3)x \sin x + 2(A_1 - A_4)\sin x + 2(A_2 + A_3)\cos x$$

Substituting these derivatives in the given equation,

$$-A_1x^2 \sin x - A_2x^2 \cos x + (4A_1 - A_4)x \cos x - (4A_2 + A_3)x \sin x + 2(A_1 - A_4)\sin x + 2(A_2 + A_3)\cos x + A_1x^2 \sin x + A_2x^2 \cos x + A_3x \sin x + A_4x \cos x = 4x \cos x - 2\sin x$$

$$4A_1x \cos x - 4A_2x \sin x + 2(A_1 - A_4)\sin x + 2(A_2 + A_3)\cos x = 4x \cos x - 2\sin x$$

Comparing coefficients on both the sides,

$$4A_1 = 4, \qquad A_1 = 1$$
$$-4A_2 = 0, \qquad A_2 = 0$$
$$2(A_1 - A_4) = -2, \qquad A_4 = A_1 + 1 = 2$$
$$2(A_2 + A_3) = 0, \qquad A_3 = 0$$

$$\text{PI} = x^2 \sin x + 2x \cos x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + x^2 \sin x + 2x \cos x$$

EXERCISE 10.18

Solve the following differential equations using the method of undetermined coefficients:

1. $(D^2 + 6D + 8)y = e^{-3x} + e^x$

   $\left[\textbf{Ans.:}\ y = c_1e^{-2x} + c_2e^{-4x} - e^{-3x} + \dfrac{e^x}{15}\right]$

2. $(D^2 + 2D + 5)y = 6\sin 2x + 7\cos 2x$

   $\left[\textbf{Ans.:}\ y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + 2\sin 2x - \cos 2x\right]$

3. $(D^3 - D^2 + D - 1)y = 6\cos 2x$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2\cos x + c_3\sin x + \frac{2}{5}(\cos 2x - 2\sin 2x)\right]$$

4. $(2D^2 - D - 3)y = x^3 + x + 1$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{\frac{3x}{2}} - \frac{1}{27}(9x^3 - 9x^2 + 51x - 20)\right]$$

5. $(D^2 - 2D + 3)y = x^2 + \sin x$

$$\left[\textbf{Ans.: } y = e^x\left(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x\right) + \frac{1}{27}(9x^2 + 6x - 8) + \frac{1}{4}(\sin x + \cos x)\right]$$

6. $(D^4 - 1)y = x^4 + 1$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - x^4 - 25\right]$$

7. $(D^2 - 1)y = e^{3x}\cos 2x - e^{2x}\sin 3x$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)\right]$$

8. $(D^2 + 3D + 2)y = 12e^{-x}\sin^3 x$

$$\left[\textbf{Ans.: } y = c_1e^{-x} + c_2e^{-2x} + \frac{e^{-x}}{10}[(\cos 3x + 3\sin 3x) - 45(\cos x + \sin x)]\right]$$

9. $(D^2 - D - 6)y = 5e^{-2x} + 10e^{3x}$

$$\left[\textbf{Ans.: } y = c_1e^{3x} + c_2e^{-2x} + 2xe^{3x} - xe^{-2x}\right]$$

10. $(D^2 + 16)y = 16\sin 4x$

$$\left[\textbf{Ans.: } y = c_1\cos 4x + c_2\sin 4x - 2x\cos 4x\right]$$

11. $(D^3 - 2D^2 + 4D - 8)y = 8(x^2 + \cos 2x)$

$$\left[\textbf{Ans.: } y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x - (x^2 + x) - \frac{x}{2}(\cos 2x + \sin 2x)\right]$$

12. $(D^2 - 6D + 13)y = 6e^{3x}\sin x\cos x$

$$\left[\textbf{Ans.: } y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - \frac{3x}{4}e^{3x}\cos 2x\right]$$

13. $(D^2 - 4D + 4)y = x^3e^{2x} + xe^{2x}$

$$\left[\textbf{Ans.: } y = (c_1 + c_2x)e^{2x} + \left(\frac{x^5}{20} + \frac{x^3}{6}\right)e^{2x}\right]$$

14. $(D^2 + 1)y = \sin^3 x$

$$\left[\textbf{Ans.: } y = c_1\cos x + c_2\sin x + \frac{1}{32}\sin 3x - \frac{3}{8}x\cos x\right]$$

15. $(D^3 - D^2 - 4D + 4)y = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5xe^{2x} + e^{2x}$

$$\left[\textbf{Ans.: } y = c_1e^x + c_2e^{2x} + c_3e^{-2x} + \frac{x^2}{2} + \frac{x^3}{6}e^{2x}\right]$$

16. $(D^2 - 5D + 6)y = e^x(2x - 3)$,
$y(0) = 1,\ y'(0) = 3$

$\left[\textbf{Ans.:}\ y = e^{2x} + xe^x\right]$

17. $(D^3 - D)y = 4e^{-x} + 3e^{2x}$,
$y(0) = 0,\ y'(0) = -1,\ y''(0) = 2$

$\left[\begin{array}{l}\textbf{Ans.:}\ y = c_1 + c_2e^x + c_3e^{-x} \\ \quad + 2xe^{-x} + \dfrac{1}{2}e^{2x}\end{array}\right]$

10.10 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Linear differential equations with more than one dependent variable and a single independent variable are called simultaneous linear differential equations with constant coefficients and can be solved by eliminating one of the dependent variables. This method is known as the *elimination method*. These equations can also be solved by using the Laplace transform method, matrices method, or short-cut operator method. Here, the elimination method is discussed.

Note The total number of arbitrary constants in the general solution is equal to the order of the differential equation of that dependent variable which is obtained first. If the total number of arbitrary constants are more than the order of the differential equation (degree of auxiliary equation) then arbitrary constants are obtained by putting the dependent variables and their derivatives (as required) in the given simultaneous equation.

EXAMPLE 10.87 *Solve* $\frac{dx}{dt} - 3x - 6y = t^2,\ \frac{dy}{dt} + \frac{dx}{dt} - 3y = e^t$.

Solution: Putting $\frac{d}{dt} \equiv D$, the equations reduce to

$$(D-3)x - 6y = t^2 \qquad \text{... (1)}$$

$$Dx + (D-3)y = e^t \qquad \text{... (2)}$$

Eliminating y from Eqs (1) and (2) by operating Eq. (1) by (D – 3) and multiplying Eq. (2) by 6 and then adding,

$$(D-3)^2x + 6Dx = (D-3)t^2 + 6e^t$$

$$(D^2+9)x = 2t - 3t^2 + 6e^t$$

The auxiliary equation is

$$m^2 + 9 = 0$$

$$m = \pm 3i \text{ (complex)}$$

$$\text{CF} = c_1\cos 3t + c_2 \sin 3t$$

$$\text{PI} = \frac{1}{D^2+9}(2t - 3t^2) + \frac{1}{D^2+9}.6e^t = \frac{1}{9}\left(1 + \frac{D^2}{9}\right)^{-1}(2t - 3t^2) + \frac{6e^t}{10}$$

$$= \frac{1}{9}\left(1 - \frac{D^2}{9} + \frac{D^4}{81} - \cdots\right)(2t - 3t^2) + \frac{3}{5}e^t = \frac{1}{9}\left[(2t - 3t^2) - \frac{1}{9}(0-6) + 0\right] + \frac{3}{5}e^t$$

$$= -\frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$

Hence,
$$x = c_1\cos 3t + c_2\sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$

$$Dx = -3c_1\sin 3t + 3c_2\cos 3t - \frac{2t}{3} + \frac{2}{9} + \frac{3}{5}e^t$$

Putting the value of x and Dx in Eq. (1),

$$-3c_1\sin 3t + 3c_2\cos 3t - \frac{2t}{3} + \frac{2}{9} + \frac{3}{5}e^t - 3\left(c_1\cos 3t + c_2\sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t\right) - t^2 = 6y$$

$$y = -\frac{1}{2}(c_1 + c_2)\sin 3t + \frac{1}{2}(c_2 - c_1)\cos 3t - \frac{2t}{9} - \frac{1}{5}e^t$$

Hence, the general solution is

$$x = c_1\cos 3t + c_2\sin 3t - \frac{t^2}{3} + \frac{2t}{9} + \frac{2}{27} + \frac{3}{5}e^t$$

$$y = -\frac{1}{2}(c_1 + c_2)\sin 3t + \frac{1}{2}(c_2 - c_1)\cos 3t - \frac{2t}{9} - \frac{1}{5}e^t$$

EXAMPLE 10.88 *Solve* $D^2y = x - 2,\ D^2x = y + 2$.

Solution:
$$D^2y - x = -2 \quad \ldots (1)$$

$$-y + D^2x = 2 \quad \ldots (2)$$

Eliminating y from Eqs (1) and (2) by operating Eq. (2) by D^2 and then adding,

$$-x + D^4x = -2 + D^2(2)$$

$$(D^4 - 1)x = -2$$

The auxiliary equation is

$$m^4 - 1 = 0$$

$$m = 1,\ -1,\ i,\ -i$$

$$CF = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t$$

$$PI = \frac{1}{D^4 - 1}(-2) = \frac{1}{D^4 - 1}(-2e^{0t}) = 2$$

Hence, $x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + 2$

$$Dx = c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t$$

$$D^2x = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$$

Putting D^2x in Eq. (2),

$$y = D^2x - 2 = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t - 2$$

Hence, the general solution is

$$x = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t + 2$$
$$y = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t - 2$$

EXAMPLE 10.89 *Solve* $(D^2 + D + 1)x + (D^2 + 1)y = e^t$, $(D^2 + D)x + D^2y = e^{-t}$.

Solution:

$$(D^2 + D + 1)x + (D^2 + 1)y = e^t \quad \ldots(1)$$
$$(D^2 + D)x + D^2y = e^{-t} \quad \ldots(2)$$

Eliminating y from Eqs (1) and (2) by operating Eq. (1) by D^2 and Eq. (2) by $(D^2 + 1)$ and then subtracting,

$$D^2(D^2 + D + 1)x - (D^2 + 1)(D^2 + D)x = D^2e^t - (D^2 + 1)e^{-t}$$
$$-Dx = e^t - e^{-t} - e^{-t}$$
$$Dx = -e^t + 2e^{-t} \quad \ldots(3)$$

Integrating w.r.t. t,

$$x = -e^t - 2e^{-t} + c_1$$
$$D^2x = -e^t - 2e^{-t}$$

Putting Dx, D^2x in Eq. (2),

$$D^2y = e^{-t} - D^2x - Dx = e^{-t} + e^t + 2e^{-t} + e^t - 2e^{-t}$$
$$D^2y = e^{-t} + 2e^t$$

Integrating w.r.t. t,

$$Dy = -e^{-t} + 2e^t + k_1$$

Integrating again w.r.t. t,

$$y = e^{-t} + 2e^t + k_1t + k_2$$

Since the order of Eq. (3) is one, there should be only one arbitrary constant in the general solution. Putting x, Dx, D^2x, y, D^2y in Eq. (1),

$$(D^2 + D + 1)x + (D^2 + 1)y = e^t$$
$$(-e^t - 2e^{-t} - e^t + 2e^{-t} - e^t - 2e^{-t} + c_1) + (e^{-t} + 2e^t + e^{-t} + 2e^t + k_1t + k_2) = e^t$$
$$e^t + c_1 + k_1t + k_2 = e^t$$
$$k_1t + k_2 = -c_1$$

Therefore, $y = e^{-t} + 2e^{t} - c_1$

Hence, the general solution is

$$x = -e^{t} - 2e^{-t} + c_1$$
$$y = 2e^{t} + e^{-t} - c_1$$

EXERCISE 10.19

Solve the following differential equations:

1. $\dfrac{dx}{dt} = 3x + 8y, \quad \dfrac{dy}{dt} = -x - 3y$

$\left[\textbf{Ans.: } x = -4c_1e^{t} - 2c_2e^{-t},\ y = c_1e^{t} - c_2e^{-t}\right]$

2. $\dfrac{dx}{dt} = 2y - 1, \quad \dfrac{dy}{dt} = 1 + 2x$

$\left[\textbf{Ans.: } x = c_1e^{2t} + c_2e^{-2t} - \dfrac{1}{2},\ y = c_1e^{2t} - c_2e^{-2t} + \dfrac{1}{2}\right]$

3. $(D + 6)y - Dx = 0$, $(3 - D)x - 2Dy = 0$ with $x = 2, y = 3$ at $t = 0$.

$\left[\textbf{Ans.: } x = 4e^{2t} - 2e^{-3t},\ y = e^{2t} + 2e^{-3t}\right]$

4. $\dfrac{dx}{dt} + y - 1 = \sin t, \quad \dfrac{dy}{dt} + x = \cos t$

$\left[\textbf{Ans.: } x = c_1e^{t} + c_2e^{-t},\ y = 1 + \sin t - c_1e^{t} + c_2e^{-t}\right]$

5. $(D + 5)x + (D + 7)y = 2e^{t}$, $(2D + 1)x + (3D + 1)y = e^{t}$

$\left[\textbf{Ans.: } x = \dfrac{1}{1+5t}\left\{(2 - 8c_2)e^{t} + \dfrac{5}{2}c_1e^{-2t}\right\},\ y = c_1e^{-2t} + c_2e^{t}\right]$

6. $\dfrac{d^2x}{dt^2} + y = \sin t, \quad \dfrac{d^2y}{dt^2} + x = \cos t$

$\left[\textbf{Ans.: } x = c_1e^{t} + c_2e^{-t} + c_3\cos t + c_4\sin t - \dfrac{t}{4}\cos t + \dfrac{t}{4}\sin t\right.$

$\left. y = -c_1e^{t} - c_2e^{-t} + c_3\cos t + c_4\sin t + \dfrac{1}{4}(2 + t)(\sin t - \cos t)\right]$

7. $D^2x + 3x - 2y = 0$, $D^2x + D^2y - 3x + 5y = 0$ with $x = 0, y = 0, Dx = 3, Dy = 2$ when $t = 0$

$\left[\textbf{Ans.: } x = \dfrac{1}{4}\left(11\sin t + \dfrac{1}{3}\sin 3t\right),\ y = \dfrac{1}{4}(11\sin t - \sin 3t)\right]$

10.11 APPLICATIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

10.11.1 Orthogonal Trajectories

Two families of curves are called orthogonal trajectories of each other if every curve of one family cuts each curve of the other family at right angles.

Working Rule to find the orthogonal trajectories

(a) Cartesian curve $f(x, y, c) = 0$

(i) Obtain the differential equation $F\left(x, y, \frac{dy}{dx}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(x, y, -\frac{dx}{dy}\right) = 0$.

(iii) Solve the differential equation $F\left(x, y, -\frac{dx}{dy}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

(b) Polar curve $f(r, \theta, c) = 0$

(i) Obtain the differential equation $F\left(r, \theta, \frac{dr}{d\theta}\right) = 0$ by differentiating and eliminating c from the equation of the family of curves.

(ii) Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in the above differential equation to obtain the differential equation of the family of orthogonal trajectories as $F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$.

(iii) Solve the differential equation $F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$ to obtain the equation of the family of orthogonal trajectories.

EXAMPLE 10.90 *Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$.*

Solution: The equation of the family of curves is

$$ay^2 = x^3 \qquad \dots (1)$$

Differentiating Eq. (1) w.r.t. x,

$$a \cdot 2y \frac{dy}{dx} = 3x^2$$

Substituting $a = \frac{x^3}{y^2}$ from Eq. (1),

$$\frac{x^3}{y^2} \cdot 2y \frac{dy}{dx} = 3x^2$$

$$\frac{2x}{y} \frac{dy}{dx} = 3 \qquad \dots (2)$$

This is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (2),

$$\frac{-2x}{y}\frac{dx}{dy} = 3 \quad \text{... (3)}$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int -2x\,dx = \int 3y\,dy$$

$$-x^2 = \frac{3y^2}{2} + c$$

$$-2x^2 = 3y^2 + 2c$$

$$2x^2 + 3y^2 + 2c = 0$$

which is the equation of the required orthogonal trajectories.

EXAMPLE 10.91 *Find the equation of the family of all orthogonal trajectories of the family of circles, which pass through the origin (0, 0) and have centres on the y-axis.*

Solution: The equation of the family of circles passing through (0, 0) and having centres on the y-axis is

$$x^2 + y^2 + 2fy = 0 \quad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. x,

$$2x + 2y\frac{dy}{dx} + 2f\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y+f} \quad \text{... (2)}$$

From Eq. (1),

$$f = -\frac{x^2+y^2}{2y}$$

$$y + f = y - \frac{x^2+y^2}{2y} = \frac{y^2-x^2}{2y}$$

Substituting in Eq. (2),

$$\frac{dy}{dx} = \frac{-2xy}{y^2-x^2} \quad \text{... (3)}$$

This is the differential equation of the given family of circles.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (3),

$$\frac{dx}{dy} = \frac{2xy}{y^2-x^2}$$

This is the differential equation of the family of orthogonal trajectories.

$$(y^2 - x^2)\,dx - 2xy\,dy = 0$$

$$M = y^2 - x^2, \qquad N = -2xy$$

$$\frac{\partial M}{\partial y} = 2y, \qquad \frac{\partial N}{\partial x} = -2y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The equation is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{4y}{-2xy} = -\frac{2}{x}$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

Multiplying Eq. (4) by $\frac{1}{x^2}$,

$$\left(\frac{y^2}{x^2} - 1\right)dx - \frac{2y}{x}dy = 0$$

$$M_1 = \frac{y^2}{x^2} - 1, \qquad N_1 = -\frac{2y}{x}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} = \frac{2y}{x^2},$$

The equation is exact.
Hence, the solution is

$$\int_{y\ \text{constant}} \left(\frac{y^2}{x^2} - 1\right)dx - \int 0\,dy = c$$

$$\frac{-y^2}{x} - x = c$$

$$x^2 + y^2 + cx = 0$$

which is the equation of the required orthogonal trajectories representing the equation of the family of the circles with centre on the x-axis and passing through the origin.

EXAMPLE 10.92 *Show that the family of confocal conics $\frac{x^2}{a} + \frac{y^2}{a-b} = 1$ is self-orthogonal, where a is the parameter and b is the constant.*

Solution: The equation of the family of curves is

$$\frac{x^2}{a} + \frac{y^2}{a-b} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$\frac{2x}{a}+\frac{2y}{a-b}\frac{dy}{dx}=0$$

$$\frac{yy'}{a-b}=-\frac{x}{a}, \qquad \text{where } y'=\frac{dy}{dx}$$

$$ayy'=-ax+bx$$

$$a(x+yy')=bx$$

$$a=\frac{bx}{x+yy'}$$

Putting the value of a in Eq. (1),

$$\frac{x^2(x+yy')}{bx}+\frac{y^2}{\frac{bx}{x+yy'}-b}=1$$

$$\frac{x(x+yy')}{b}+\frac{y^2(x+yy')}{-byy'}=1$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'} \qquad \text{... (2)}$$

This is the differential equation of the given family of curves.

Replacing y' by $-\frac{1}{y'}$ in Eq. (2),

$$\frac{-\frac{x}{y'}-y}{-\frac{1}{y'}}=\frac{b}{x+\left(-\frac{y}{y'}\right)}$$

$$x+yy'=\frac{by'}{xy'-y}$$

$$\frac{xy'-y}{y'}=\frac{b}{x+yy'}$$

which is the same as Eq. (2). Therefore, the differential equation of the family of orthogonal trajectories is the same as the differential equation of the family of curves. Hence, the given family of curves is self-orthogonal.

EXAMPLE 10.93 *Find the orthogonal trajectories of the family of the curves $r^n \sin n\theta = a^n$.*

Solution: The family of the curves is given by the equation

$$r^n \sin n\theta = a^n \qquad \text{... (1)}$$

Differentiating Eq. (1) w.r.t. θ,

$$nr^{n-1}\frac{dr}{d\theta}\cdot\sin n\theta + r^n n\cos n\theta = 0$$

$$\frac{dr}{d\theta} = -r\cot n\theta$$

This is the differential equation of the given family of curves.

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in Eq. (2),

$$-r^2\frac{d\theta}{dr} = -r\cot n\theta$$

$$r\frac{d\theta}{dr} = \cot n\theta$$

This is the differential equation of the family of orthogonal trajectories.
Separating the variables and integrating Eq. (3),

$$\int \tan n\theta \, d\theta = \int \frac{dr}{r}$$

$$\frac{\log\sec n\theta}{n} = \log r + \log c$$

$$\log\sec n\theta = n\log rc = \log(rc)^n$$

$$\sec n\theta = c^n r^n$$

$$r^n\cos n\theta = k, \text{ where } k = \frac{1}{c^n}$$

which is the equation of the required orthogonal trajectories.

EXERCISE 10.20

1. Find the orthogonal trajectories of the families of the following curves:
   (i) $y^2 = 4ax$
   (ii) $x^2 - y^2 = ax$
   (iii) $y^2 = \frac{x^3}{a-x}$
   (iv) $x^2 + y^2 + 2ay + b = 2$
   (v) $(a+x)y^2 = x^2(3a-x)$

Ans.: (i) $2x^2 + y^2 = c$
(ii) $y(y^2 + 3x^2) = c$
(iii) $(x^2+y^2)^2 = c(2x^2+y^2)$
(iv) $x^2 + y^2 + 2cx - b = 0$
(v) $(x^2+y^2)^5 = cy^3(5x^2+y^2)$

2. Show that the family of confocal conics $\frac{x^2}{a^2+c} + \frac{y^2}{b^2+c} = 1$ is self-orthogonal.

where a and b are constants and c is the parameter.

3. Find the value of the constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of the ellipses $x^2 + 2y^2 - y = c_2$.

$$\left[\textbf{Ans.: } d = \frac{1}{4}\right]$$

4. Find the orthogonal trajectories of the families of the following curves:

(i) $r = a(1 + \cos\theta)$

(ii) $r = \dfrac{2a}{1+\cos\theta}$

(iii) $r^2 = a\sin^2\theta$

(iv) $r^n = a^n \cos n\theta$

(v) $r = a(\sec\theta + \tan\theta)$

(vi) $r = ae^{\theta}$

Ans.: (i) $r = c(1-\cos\theta)$

(ii) $r = \dfrac{c}{1-\cos\theta}$

(iii) $r^2 = c^2\cos 2\theta$

(iv) $r^n = c^n \sin n\theta$

(v) $\log r = -\sin\theta + c$

(vi) $r = ce^{-\theta}$

10.11.2 Electrical Circuits

A simple electric circuit consists of a voltage source, resistor, inductor, and capacitor. To find current, voltage, or change in an electric circuit, a differential equation is formed using Kirchhoff's Voltage Law (KVL) which states that the algebraic sum of all the voltages in a closed loop or circuit is zero. The voltage across the resistor, inductor, and capacitor are given by

$$v_R = Ri$$

$$v_L = L\frac{di}{dt}$$

$$v_C = \frac{1}{C}\int i\, dt$$

R-L Circuit

Figure 10.1 shows a simple *R-L* circuit. Applying Kirchhoff's voltage law to the circuit,

$$Ri + L\frac{di}{dt} = e(t)$$

The differential equation is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{e(t)}{L}$$

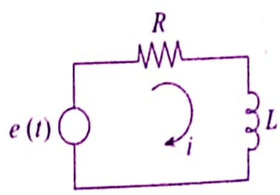

Fig. 10.1 *R-L* circuit

R-C Circuit

Figure 10.2 shows a simple R-C circuit. Applying Kirchhoff's voltage law to the circuit,

$$Ri + \frac{1}{C}\int i\,dt = e(t)$$

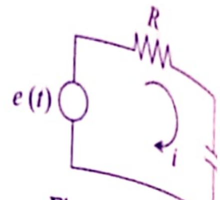

Fig. 10.2 R-C circuit

Differentiating the equation,

$$R\frac{di}{dt} + \frac{i}{C} = \frac{d}{dt}e(t)$$

The differential equation is

$$\frac{di}{dt} + \frac{1}{RC}i = \frac{de(t)}{dt}$$

EXAMPLE 10.94 *A circuit consisting of a resistance R and inductance L is connected in series with a voltage E. (i) Find the value of the current at any time t. Given that i = 0 at t = 0. (ii) Show that the current builds up to half its maximum value in* $\frac{L}{R}\log 2$ seconds.

Solution: (i) Applying Kirchhoff's law to the series R-L circuit,

$$Ri + L\frac{di}{dt} = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

The equation is linear in i.

$$P = \frac{R}{L},\ Q = \frac{E}{L}$$

$$IF = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

The solution is

$$i.e^{\frac{R}{L}t} = \int \frac{E}{L}e^{\frac{R}{L}t}dt + c = \frac{E}{L}\cdot\frac{L}{R}e^{\frac{R}{L}t} + c$$

$$i = \frac{E}{R} + c\,e^{-\frac{R}{L}t}$$

At $t = 0, i = 0$

$$0 = \frac{E}{R} + c$$

$$c = -\frac{E}{R}$$

Hence,

$$i = \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

(ii) The current reaches its maximum value as $t \to \infty$

$$i(\infty) = \frac{E}{R} = I_{max}$$

When
$$i = \frac{I_{max}}{2} = \frac{E}{2R}$$

$$\frac{E}{2R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

$$\frac{1}{2} = 1 - e^{-\frac{R}{L}t}$$

$$e^{-\frac{R}{L}t} = \frac{1}{2}$$

$$e^{\frac{R}{L}t} = 2$$

$$\frac{R}{L}t = \log 2$$

$$t = \frac{L}{R}\log 2$$

EXAMPLE 10.95 *A circuit consisting of a resistance R and a condenser of capacity C is connected in series with a voltage E. Assuming that there is no charge on the condenser at t = 0, find the value of the current i, voltage, and charge q at any time t.*

Solution: Applying Kirchhoff's law to the series R-C circuit,

$$Ri + \frac{1}{C}\int i \, dt = E$$

But
$$i = \frac{dq}{dt}$$

$$R\frac{dq}{dt} + \frac{q}{C} = E$$

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$$

The equation is linear in q.

$$P = \frac{1}{RC}, \quad Q = \frac{E}{R}$$

$$IF = e^{\int \frac{1}{RC}dt} = e^{\frac{t}{RC}}$$

The solution is

$$q\, e^{\frac{t}{RC}} = \int e^{\frac{t}{RC}} \frac{E}{R} dt + k$$

$$q e^{\frac{t}{RC}} = \frac{E}{R}\frac{e^{\frac{t}{RC}}}{\frac{1}{RC}} + k = CE\, e^{\frac{t}{RC}} + k$$

$$q = CE + ke^{-\frac{t}{RC}}$$

At $t = 0,\ q = 0$

$$0 = CE + k$$
$$k = -CE$$

Hence,
$$q = CE - CE\, e^{-\frac{t}{RC}} = CE\left(1 - e^{-\frac{t}{RC}}\right) = CE\left(1 - e^{-\frac{t}{RC}}\right)$$

$$i = \frac{dq}{dt} = \frac{d}{dt}\left(CE - CE\, e^{-\frac{t}{RC}}\right) = CE\frac{d}{dt}\left(1 - e^{-\frac{t}{RC}}\right) = \frac{CE}{RC}e^{-\frac{t}{RC}} = \frac{E}{R}e^{-\frac{t}{RC}}$$

$$e = \frac{1}{C}\int i\, dt = \frac{1}{C}\int \frac{E}{R}e^{-\frac{t}{RC}} = -E\, e^{-\frac{t}{RC}} + k$$

At $t = 0,\ e = 0$

$$0 = -E + k$$
$$k = E$$
$$e = -Ee^{-\frac{t}{RC}} + E = E\left(1 - e^{-\frac{t}{RC}}\right)$$

EXERCISE 10.21

1. A coil having a resistance of 15 ohms and an inductance of 10 henries is connected to a 90-volt supply. Determine the value of the current after 2 seconds.

   [**Ans.**: 5.985 amp]

2. If a voltage of $20 \cos 5t$ is applied to a series circuit consisting of a 10-ohm resistor and a 2-henry inductor, determine the current at any time $t > 0$.

   [**Ans.**: $i = \cos 5t + \sin 5t - e^{-5t}$]

3. A capacitor of C farads with voltage v_0 is discharged through a resistance of R ohms. Show that if q coulombs is the charge on the capacitor, i amperes is the current and v volts is the voltage at time t, show that

   $$v = v_0 e^{-\frac{t}{RC}}.$$

4. Find the current in a series R–C circuit with $R = 10\ \Omega$, $C = 0.1\ F$, $e(t) = 110 \sin 314\, t$, $i(0) = 0$.

   $$\left[\textbf{Ans.}: i(t) = 0.035(\cos 314\, t + 314 \sin 314\, t - e^{-t})\right]$$

5. Determine the charge and current at any time t in a series R–C circuit with $R = 10\,\Omega$, $C = 2 \times 10^{-4} F$ and $E = 100\ V$. Given that $q(0) = 0$.

   $$\left[\textbf{Ans.}: q(t) = \frac{1 - e^{-500t}}{50},\ i(t) = 10e^{-500t}\right]$$

10.11.3 Mechanical Systems

If a body moves in a straight line starting from a fixed point O and covers a distance x at any instant t then the velocity of the body is given by

$$v = \frac{dx}{dt}$$

and acceleration of the body is given by

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

or

$$a = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v = v\frac{dv}{dx}$$

If the mass of the body is m and the force acting on it is F then by Newton's second law of motion,

$$F = ma = m\frac{dv}{dt}$$

or

$$F = mv\frac{dv}{dx}$$

This is the equation of the motion of the particle.

EXAMPLE 10.96 *A particle of mass m moves in a horizontal straight line with acceleration $\frac{mk}{x^3}$ directed towards the origin at a distance x from the origin. If initially the particle was at rest at a distance a from the origin, show that it will be at a distance $\frac{a}{2}$ from the origin at $t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$.*

Solution: Since the acceleration is directed towards the origin, the equation of motion is given by

$$mv\frac{dv}{dx} = -\frac{mk}{x^3}$$

$$v\frac{dv}{dx} = -\frac{k}{x^3}$$

Separating the variables and integrating,

$$\int v\,dv = \int -\frac{k}{x^3}\,dx$$

$$\frac{v^2}{2} = \frac{k}{2x^2} + c$$

Initially, when $v = 0$, $x = a$

$$0 = \frac{k}{2a^2} + c$$

$$c = -\frac{k}{2a^2}$$

Hence,

$$\frac{v^2}{2} = \frac{k}{2x^2} - \frac{k}{2a^2}$$

$$v^2 = \frac{k(a^2 - x^2)}{a^2x^2}$$

$$v = \pm\sqrt{k}\,\frac{\sqrt{a^2 - x^2}}{ax}$$

$$\frac{dx}{dt} = -\sqrt{k}\,\frac{\sqrt{a^2 - x^2}}{ax}$$

[Negative sign is taken since x is decreasing with time]

Separating the variables and integrating,

$$\int \frac{ax}{\sqrt{a^2 - x^2}}\,dx = -\int \sqrt{k}\,dt$$

$$-\frac{a}{2}\int (a^2 - x^2)^{-\frac{1}{2}}(-2x)\,dx = -\sqrt{k}\int dt$$

$$-\frac{a}{2}\cdot 2(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t + c' \qquad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$-a(a^2 - x^2)^{\frac{1}{2}} = -\sqrt{k}t + c'$$

At $t = 0$, $x = a$

$$c' = 0$$

Hence,

$$a(a^2 - x^2)^{\frac{1}{2}} = \sqrt{k}t$$

When $x = \frac{a}{2}$,

$$a\left(a^2 - \frac{a^2}{4}\right)^{\frac{1}{2}} = \sqrt{k}t$$

$$t = \frac{a^2}{2}\sqrt{\frac{3}{k}}$$

EXAMPLE 10.97 *A particle is projected with velocity v_0 along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance covered by the particle in time t is $\frac{1}{\mu v_0}\left[\sqrt{1 + \mu v_0^2 t} - 1\right]$.*

Solution: Resistance per unit mass $= \mu v^3$, where v is the velocity at any instant t. By Newton's second law,

$$v\frac{dv}{dx} = -\mu v^3$$

$$\frac{dv}{v^2} = -\mu\, dx$$

Separating the variables and integrating,

$$\int \frac{dv}{v^2} = \int -\mu\, dx$$

$$-\frac{1}{v} = -\mu x + c$$

Initially, $v = v_0$, $x = 0$

$$c = -\frac{1}{v_0}$$

Hence,

$$-\frac{1}{v} = -\mu x - \frac{1}{v_0}$$

$$\frac{1}{v} = \frac{\mu v_0 x + 1}{v_0}$$

$$v = \frac{v_0}{\mu v_0 x + 1}$$

$$\frac{dx}{dt} = \frac{v_0}{\mu v_0 x + 1}$$

where x is the distance travelled at any instant t.
Separating the variables and integrating,

$$\int (\mu v_0 x + 1) dx = \int v_0\, dt$$

$$\mu v_0 \frac{x^2}{2} + x = v_0 t + k$$

At $t = 0$, $x = 0$,

$$k = 0$$

Hence, $\mu v_0 x^2 + 2x - 2v_0 t = 0$

$$x = \frac{-2 \pm \sqrt{4 + 8\mu v_0^2 t}}{2\mu v_0} = \frac{-1 \pm \sqrt{1 + 2\mu v_0^2 t}}{\mu v_0}$$

But distance is always positive.

Hence,

$$x = \frac{-1 + \sqrt{1 + 2\mu v_0^2\, t}}{\mu v_0}$$

EXAMPLE 10.98 *A body of mass m, falling from rest, is subjected to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that* $\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}$, *where* $mg = ka^2$.

Solution: The forces acting on the body are

(i) its weight mg acting downwards

(ii) air resistance kv^2 acting upwards

Net force acting upon the body $= mg - kv^2 = ka^2 - kv^2 = k(a^2 - v^2)$

By Newton's second law,

$$mv\frac{dv}{dx} = k(a^2 - v^2)$$

$$\frac{v}{a^2 - v^2}dv = \frac{k}{m}dx$$

Integrating both the sides,

$$-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x + c$$

At $x = 0$, $v = 0$

$$-\frac{1}{2}\log a^2 = c$$

Hence,

$$-\frac{1}{2}\log(a^2 - v^2) = \frac{k}{m}x - \frac{1}{2}\log a^2$$

$$\frac{2\,kx}{m} = \log a^2 - \log(a^2 - v^2) = \log\frac{a^2}{a^2 - v^2}$$

EXERCISE 10.22

1. A moving body is opposed by a force per unit mass of value Cx and resistance per unit mass of value bv^2, where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.

$$\left[\textbf{Ans.: } v = \frac{1}{b}\sqrt{\frac{C}{2}(1 - 2bx - e^{-2bx})}\right]$$

2. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand tank with velocity v_0?

$$\left[\textbf{Ans.: } t = \frac{2}{k}\sqrt{v_0}\right]$$

3. A particle of mass m is projected vertically with a velocity v. If the air resistance is directly proportional to the velocity then show that the particle will reach maximum height in time $\frac{m}{k}\log\left(1 + \frac{kv^2}{mg}\right)$.

4. A body of mass m falls from rest under gravity in a fluid whose resistance to motion at any instant is mk times the velocity, where k is a constant. Find the terminal velocity of the body and also the time required to attain one half of its terminal velocity.

Hint: Terminal velocity is velocity at $t \to \infty$.

$$\left[\textbf{Ans.: } v = \frac{g}{k},\ t = \frac{1}{k}\log 2\right]$$

5. A particle is moving in a straight line with acceleration $k\left(x+\frac{a^4}{x^3}\right)$ directed towards the origin. If it starts from rest at a distance a from the origin, show that it will reach the origin at the end of time $\frac{\pi}{4\sqrt{k}}$.

6. A vehicle starts from rest and its acceleration is given by $k\left(1-\frac{t}{T}\right)$, where k and T are constants. Find the maximum speed and the distance travelled when the maximum speed is attained.

$$\left[\textbf{Ans.: } v_{max}=\frac{kT}{2},\ x=\frac{kT^2}{3}\right]$$

7. The distance x descended by a parachuter satisfies the differential equation $\left(\frac{dx}{dt}\right)^2=k^2\left[1-e^{\frac{-2gx}{k^2}}\right]$, where k and g are constants. Show that $x=\frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$ if $x=0$ at $t=0$.

10.11.4 Rate of Growth or Decay

If the rate of change of a quantity y at any instant t is directly proportional to the quantity present at that time then

(i) the differential equation of growth is $\frac{dy}{dt}=ky$

(ii) the differential equation of decay is $\frac{dy}{dt}=-ky$

EXAMPLE 10.99

In a culture of yeast, at each instant, the time rate of change of active ferment is proportional to the amount present. If the active ferment doubles in two hours, how much can be expected at the end of 8 hours at the same rate of growth? Find also, how much time will elapse before the active ferment grows to eight times its initial value.

Solution: Let y quantity of active ferment be present at any time t.

The equation of fermentation of yeast is

$$\frac{dy}{dt}=ky,\quad \text{where } k \text{ is a constant}$$

Separating the variables and integrating,

$$\int\frac{dy}{y}=\int k\,dt$$

$$\log y=kt+c$$

Let at $t=0$, $\quad y=y_0$

Hence

Therefore, at $t = 2$, $y = 2y_0$

$$\log\left(\frac{2y_0}{y_0}\right) = k(2)$$

$$k = \frac{1}{2}\log 2$$

Substituting in Eq. (1),

$$\log\left(\frac{y}{y_0}\right) = \frac{t}{2}\log 2$$

$$y = y_0 e^{\frac{t}{2}\log 2}$$

(i) When $t = 8$,

$$y = y_0 e^{4\log 2} = y_0 e^{\log 2^4} = y_0 \cdot 2^4$$

$$y = 16 y_0$$

Hence, the active ferment grows 16 times its initial value at the end of 8 hours.

(ii) When $y = 8y_0$,

$$8y_0 = y_0 e^{\frac{t}{2}\log 2}$$

$$\log 8 = \frac{t}{2}\log 2$$

$$\log 2^3 = \frac{t}{2}\log 2$$

$$3\log 2 = \frac{t}{2}\log 2$$

$$t = 6 \text{ hours}$$

Hence, the active ferment grows 8 times its initial value at the end of 6 hours.

EXAMPLE 10.100 *Find the half-life of uranium, which disintegrates at a rate proportional to the amount present at any instant. Given that m_1 and m_2 grams of uranium are present at times t_1 and t_2 respectively.*

Solution: Let m grams of uranium be present at any time t. The equation of disintegration of uranium is

$$\frac{dm}{dt} = -km, \qquad \text{where } k \text{ is a constant}$$

$$\frac{dm}{m} = -k\,dt$$

Integrating both the sides,

$$\log m = -kt + c$$

At $t = 0$, $m = m_0$

$$\log m_0 = c$$

Hence,

$$\log m = -kt + \log m_0$$

$$kt = \log m_0 - \log m \quad \ldots (1)$$

At $t = t_1$, $m = m_1$ and at $t = t_2$, $m = m_2$

$$kt_1 = \log m_0 - \log m_1 \quad \ldots (2)$$

$$kt_2 = \log m_0 - \log m_2 \quad \ldots (3)$$

Subtracting Eq. (2) from Eq. (3),

$$k(t_2 - t_1) = \log m_1 - \log m_2$$

$$k = \frac{\log\left(\frac{m_1}{m_2}\right)}{t_2 - t_1}$$

Let T be the half-life of uranium, i.e., at $t = T$, $m = \frac{1}{2}m_0$

From Eq. (1),

$$kT = \log m_0 - \log\frac{m_0}{2} = \log 2$$

$$T = \frac{\log 2}{k} = \frac{(t_2 - t_1)\log 2}{\log\left(\frac{m_1}{m_2}\right)}$$

EXERCISE 10.23

1. If the population of a country doubles in 50 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants?

   [Ans. : 79 years]

2. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours?

   [Ans. : 605]

3. A radioactive substance disintegrates at a rate proportional to its mass. When the mass is 10 mg, the rate of disintegration is

0.051 mg per day. How long will it take for the mass to be reduced from 10 mg to 5 mg?

[**Ans.** : 136 days]

4. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years?

$\left[\textbf{Ans.}: \left(1-\frac{1}{p}\right)^{21} \text{ times the initial amount}\right]$

5. Find the time required for the sum of money to double itself at 5% per annum compounded continuously.

[**Ans.** : 13.9 years]

10.11.5 Newton's Law of Cooling

Newton's law of cooling states that rate of change of temperature of a body is directly proportional to the difference between the temperature of the body and that of the surrounding medium.

If T is the temperature of the body and T_0 is the temperature of the surrounding medium at any time t, then its differential equation is

$$\frac{dT}{dt} = -k(T - T_0), \qquad \text{where } k \text{ is a constant}$$

EXAMPLE 10.101 *According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 40°C and the substance cools from 80°C to 60°C in 20 minutes, what will be the temperature of the substance after 40 minutes?*

Solution: Let T be the temperature of the substance at the time t.

$$\frac{dT}{dt} = -k(T - 40)$$

Separating the variables and integrating,

$$\int \frac{dT}{T-40} = \int -k\,dt$$

$$\log(T-40) = -kt + c$$

At $t = 0$, $T = 80$

$$\log 40 = c$$

Hence,

$$kt = \log 40 - \log(T-40)$$

At $t = 20$, $T = 60$

$$20k = \log 40 - \log 20 = \log 2$$

$$k = \frac{1}{20}\log 2$$

Hence,

$$t \cdot \frac{1}{20}\log 2 = \log 40 - \log(T-40)$$

At $t = 40$,

$$40 \cdot \frac{1}{20} \log 2 = \log 40 - \log(T - 40)$$

$$2 \log 2 = \log \frac{40}{T - 40}$$

$$4 = \frac{40}{T - 40}$$

$$T = 50°\text{C}$$

EXERCISE 10.24

1. Water at a temperature of 100°C cools in 10 minutes to 88°C in a room at a temperature of 25°C. Find the temperature of water after 20 minutes.

   [Ans. : 77.9°C]

2. If the temperature of the air is 30°C and a substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C.

   [Ans. : 52.5 minutes]

10.12 APPLICATIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

10.12.1 Simple Harmonic Motion

If a particle moves in a straight line with an acceleration directly proportional to its displacement from a fixed point O and is always directed towards O then the motion is said to be simple harmonic motion.

Let the displacement of a particle from a fixed point O at some instant t is x (Fig. 10.3).

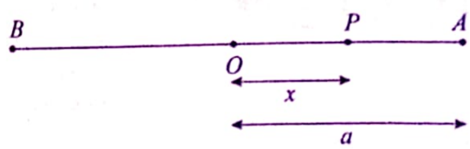

Fig. 10.3 Simple harmonic motion

Then $$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$(D^2 + \omega^2)x = 0 \qquad \text{... (10.43)}$$

where $$D \equiv \frac{d}{dt}$$

The solution of the differential equation (10.43) is

$$x = c_1 \cos \omega t + c_2 \sin \omega t \qquad \text{... (10.44)}$$

Velocity of the particle at the point P is

$$v = \frac{dx}{dt} = -\omega c_1 \sin \omega t + \omega c_2 \cos \omega t \qquad \text{... (10.45)}$$

Let the particle start from rest at the distance a from the fixed point O.

Then at $t = 0$, $x = a$, $v = 0$

From Eq. (10.44),

$$a = c_1$$

From Eq. (10.45),

$$0 = \omega c_2$$

$$c_2 = 0$$

Hence, displacement $\quad x = a \cos \omega t$

and velocity, $v = -a\omega \sin \omega t = -a\omega\sqrt{1-\cos^2 \omega t} = -a\omega\sqrt{1-\dfrac{x^2}{a^2}}$

$$v = -\omega\sqrt{a^2 - x^2}$$

Nature of Motion The particle starts from rest and moves towards O and attains its maximum velocity at O.

Hence, $\quad |v_{\max}| = a\omega$

At O, the acceleration is zero but velocity is maximum. Hence, the particle moves further and comes to rest at B such that $OA = OB$. Then it retraces its path and oscillates between A and B.

(i) The amplitude (maximum displacement from O) = a

(ii) The time period (time for a complete oscillation) = $\dfrac{2\pi}{\omega}$

(iii) The frequency (number of oscillations per second) = $\dfrac{1}{\text{Time period}} = \dfrac{\omega}{2\pi}$.

EXAMPLE 10.102

A particle is executing simple harmonic motion with an amplitude of 5 metres and a time of 4 seconds. Find the time required by the particle in passing between points which are at distances 4 and 2 metres from the centre of force and are on the same side of it.

Solution: Amplitude, $a = 5$ metres

Time period, $\quad \dfrac{2\pi}{\omega} = 4 \text{ seconds}$

$$\omega = \frac{\pi}{2}$$

Let the particle be at distances 4 and 2 metres from the centre at times t_1 and t_2 seconds respectively.

Since $x = a\cos\omega t$,

$$4 = 5\cos\left(\frac{\pi}{2}t_1\right)$$

$$t_1 = \frac{2}{\pi}\cos^{-1}\frac{4}{5} = 23.47 \text{ seconds}$$

and
$$2 = 5\cos\left(\frac{\pi}{2}t_2\right)$$
$$t_2 = \frac{2}{\pi}\cos^{-1}\frac{2}{5} = 42.29 \text{ seconds}$$

Time required in passing between the points at distances 4 and 2 metres $= t_2 - t_1 = 18.82$ seconds.

EXAMPLE 10.103

A particle of 4 g mass executing SHM has velocities 8 cm/s and 6 cm/s when it is at distances 3 cm and 4 cm from the centre of its path. Find its period and amplitude. Also, find the force acting on the particle when it is at a distance of 1 cm from the centre.

Solution: Velocity of the particle when it is at a distance x from the centre is
$$v^2 = \omega^2(a^2 - x^2)$$

At $x = 3$, $v = 8$ and at $x = 4$, $v = 6$
$$(8)^2 = \omega^2\left[a^2 - (3)^2\right]$$
$$64 = \omega^2(a^2 - 9) \qquad \text{... (1)}$$

and
$$(6)^2 = \omega^2\left[a^2 - (4)^2\right]$$
$$36 = \omega^2(a^2 - 16) \qquad \text{... (2)}$$

Dividing Eq. (1) by Eq. (2),
$$\frac{64}{36} = \frac{a^2 - 9}{a^2 - 16}$$
$$a^2 = 25$$
$$a = 5$$

Hence, amplitude = 5 cm
Putting $a = 5$ in Eq. (1),
$$64 = \omega^2(25 - 9)$$
$$\omega^2 = 4$$
$$\omega = 2$$

Hence, time period $= \dfrac{2\pi}{\omega} = \dfrac{2\pi}{2} = \pi$ seconds

Acceleration $= -\omega^2 x$
At $x = 1$, acceleration $= -\omega^2 = -4$
Force = Mass × Acceleration = 4 (– 4) = –16 dynes
Negative sign indicates that the force is acting towards the centre.

EXERCISE 10.25

1. A particle is executing simple harmonic motion with a 20 cm amplitude and a time of 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are the same side of it.

   [**Ans.:** 0.38 second]

2. A particle of 4 g mass vibrates through one cm on each side of the centre making 330 complete vibrations per minute. Assuming its motion to be SHM, show that the maximum force upon the particle is $484\ \pi^2$ dynes.

3. Find the time of a complete oscillation in a simple harmonic motion if $x = x_1$, $x = x_2$ and $x = x_3$ when $t = 1$ s, $t = 2$ s, $t = 3$ s respectively.

   $\left[\textbf{Ans.:}\ \frac{2\pi}{\theta},\ \text{where}\ \cos\theta = \frac{x_1 + x_3}{2x_2}\right]$

10.12.2 Simple Pendulum

A simple pendulum consists of a heavy mass m, called a *bob*, attached to one end of a light inextensible string with the other end fixed. The mass of the string is negligible as compared to the mass m (bob) (Fig. 10.4).

Let the pendulum be suspended from a fixed point O. Let l be the length of the light string and m be the mass of the bob. Let P be the position of the bob at any instant t. Let the arc $AP = s$ and θ be the angle which OP makes with the vertical line OA, then $s = l\theta$.

The equation of motion of the bob along the tangent is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta$$

$$\frac{d^2}{dt^2}(l\theta) = -g\sin\theta$$

$$l\frac{d^2\theta}{dt^2} = -g\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

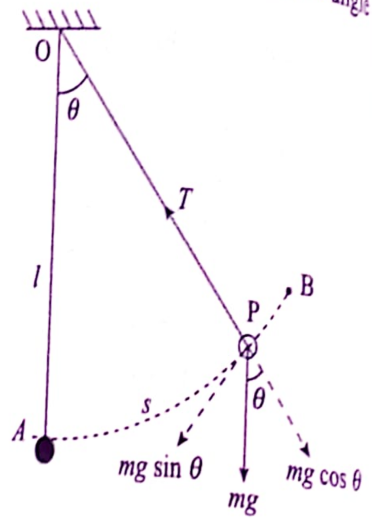

Fig. 10.4 Simple pendulum

For sufficiently small θ, higher powers of θ can be neglected.

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0, \quad \text{where } \omega^2 = \frac{g}{l}$$

This shows that the motion of the bob is a simple harmonic motion.

$$\text{Time period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

The motion of the bob from one extreme position to another extreme position completes half an oscillation and is called a *beat* or a *swing*.

Hence, time of one beat $= \pi\sqrt{\frac{l}{g}}$

Change in the Number of Beats If a simple pendulum of length l makes n beats in time t then

$$t = n\pi\sqrt{\frac{l}{g}}$$

$$n = \frac{t}{\pi}\sqrt{\frac{g}{l}}$$

$$\log n = \log\frac{t}{\pi} + \frac{1}{2}(\log g - \log l)$$

Differentiating both sides,

$$\frac{dn}{n} = \frac{1}{2}\left(\frac{dg}{g} - \frac{dl}{l}\right)$$

This gives the change in number of beats as g and (or) l changes.

(i) If l is constant and g changes,

$$\frac{dn}{n} = \frac{1}{2}\frac{dg}{g}$$

(ii) If l changes and g is constant,

$$\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l}$$

EXAMPLE 10.104 *A clock with a seconds pendulum is gaining 2 minutes a day. Prove that the length of the pendulum must be decreased by 0.0028 of its original length to make it go correctly.*

Solution: Total number of beats per day,

$$n = 24 \times 60 \times 60 = 86400 \text{ seconds}$$

Gain per day, $dn = 2$ minutes $= 120$ seconds

Let l be the original length and dl be the change in length.

Assuming g to be constant,

$$\frac{dn}{n} = -\frac{1}{2}\frac{dl}{l}$$

$$\frac{dl}{l} = -\frac{2 \times 120}{86400} = -0.0028$$

$$dl = -0.0028\, l$$

Hence, the length must be decreased by 0.0028 of its original length.

EXAMPLE 10.105 *The differential equation of a simple pendulum is $\frac{d^2x}{dt^2}+\omega^2x = F \sin nt$ where ω and F are constants. If at $t = 0$, $x = 0$, $\frac{dx}{dt} = 0$, determine the motion when $n = \omega$.*

Solution: The differential equation is

$$\frac{d^2x}{dt^2}+\omega^2x = F\sin nt \quad \ldots(1)$$

$$(D^2+\omega^2)x = F\sin nt$$

The auxiliary equation is

$$m^2+\omega^2 = 0$$

$$m = \pm i\omega \text{ (complex)}$$

$$\text{CF} = c_1\cos\omega t + c_2\sin\omega t$$

If $n = \omega$,

$$\text{PI} = \frac{1}{D^2+\omega^2}F\sin nt$$

$$\text{PI} = \frac{1}{D^2+\omega^2}F\sin\omega t = Ft\cdot\frac{1}{2D}\sin\omega t = \frac{Ft}{2}\int\sin\omega t\, dt = -\frac{Ft}{2\omega}\cos\omega t$$

Hence, the general solution of Eq. (1) is

$$x = \text{CF}+\text{PI} = c_1\cos\omega t + c_2\sin\omega t - \frac{Ft}{2\omega}\cos\omega t$$

$$\frac{dx}{dt} = -c_1\omega\sin\omega t + c_2\omega\cos\omega t - \frac{F}{2\omega}(\cos\omega t - t\omega\sin\omega t)$$

At $t=0, x=0$ and $\frac{dx}{dt}=0$

$$0 = c_1$$

and

$$0 = c_2\omega - \frac{F}{2\omega}$$

$$c_2 = \frac{F}{2\omega^2}$$

Hence, the equation of motion is

$$x = \frac{F}{2\omega^2}\sin\omega t - \frac{Ft}{2\omega}\cos\omega t$$

EXERCISE 10.26

1. A clock loses five seconds a day. Find the alteration required in the length of its pendulum in order to keep correct time.

   $\left[\textbf{Ans.:} \text{ Shortened by } \dfrac{1}{8640} \text{ of its original length}\right]$

2. A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another. Compare the acceleration due to gravity at the two places.

   $\left[\textbf{Ans.:} \dfrac{4321}{4319}\right]$

3. If a pendulum clock loses 9 minutes per week, what change is required in the length of the pendulum in order to keep correct time?

   [**Ans.:** 1.7 mm]

10.12.3 Oscillation of a Spring

Consider a spring suspended vertically from a fixed point support. Let a mass m attached to the lower end A of the spring stretch the spring by a length e, called *elongation,* and come to rest at B. This position is called *static equilibrium*.

Now, the mass is set in motion from the equilibrium position. Let at any time t, the mass is at P such that $BP = x$ (Fig. 10.5). The mass m experiences the following forces:

(i) Gravitational force mg acting downwards

(ii) Restoring force $k\,(e + x)$ due to displacement of spring acting upwards

(iii) Damping (frictional or resistance) force $c\dfrac{dx}{dt}$ of the medium opposing the motion (acting upward)

(iv) External force $F\,(t)$ considering downward direction as positive; by Newton's second law, the differential equation of motion of the mass m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - c\frac{dx}{dt} + F(t)$$

where k is known as *spring constant*, c is known as *damping constant*, and g is *gravitational constant*.

At the equilibrium position B,

$$mg = ke$$

Hence,
$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt} + F(t)$$

$$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = F(t)$$

Let $\dfrac{c}{m} = 2\lambda$ and $\dfrac{k}{m} = \omega^2$

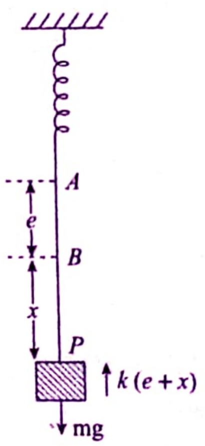

Fig. 10.5 Stretched spring

$$\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=F(t) \quad \cdots(10.46)$$

which represents the equation of motion and its solution gives the displacement x of the mass m at any instant t.

Let us consider the different cases of motion.

1. Free Oscillation If the external force $F(t)$ is absent and damping force is negligible then Eq. (10.46) reduces to

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

which represents the equation of simple harmonic motion.

Hence, the motion of the mass m is SHM.

$$\text{Time period}=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{k}}$$

$$\text{Frequency}=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

2. Free Damped Oscillation If the external force $F(t)$ is absent and damping is present then Eq. (10.46) reduces to

$$\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=0$$

3. Forced Undamped Oscillation If an external periodic force $F(t)=Q\cos nt$ is applied to the support of the spring and damping force is negligible then Eq. (10.46) reduces to

$$\frac{d^2x}{dt^2}+\omega^2x=Q\cos nt$$

$$(D^2+\omega^2)x=Q\cos nt \quad \cdots(10.47)$$

$$CF=c_1\cos\omega t+c_2\sin\omega t$$

$$PI=\frac{1}{D^2+\omega^2}Q\cos nt$$

Hence, the general solution of Eq. (10.47) is

$$x=CF+PI$$

If frequency of the external force $\frac{n}{2\pi}$ and the natural frequency $\frac{\omega}{2\pi}$ are same, i.e., $\omega=n$ then resonance occurs.

4. Forced Damped Oscillation If an external periodic force $F(t)=Q\cos nt$ is applied to the support of the spring and damping force is present then Eq. (10.46) reduces to

$$\frac{d^2x}{dt^2}+2\lambda\frac{dx}{dt}+\omega^2x=Q\cos nt$$

The auxiliary equation is $p^2 + 2\lambda p + \omega^2 = 0$ ...(10.48)

The general solution is

$$x = \text{CF} + \text{PI} = x_c + x_p$$

The x_c is known as *transient term* and tends to zero as $t \to \infty$. Thus, the term represents damped oscillations. The x_p is known as *steady-state term*. This term represents simple harmonic motion of period $\dfrac{2\pi}{n}$.

$$p = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

The motion of the mass depends on the nature of the roots of Eq. (10.48), i.e., on discriminant $\lambda^2 - \omega^2$.

Case I If $\lambda^2 - \omega^2 > 0$ then the roots of Eq. (10.48) are real and distinct.

$$x_c = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2 - \omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}\,t}\right).$$

$$x_c \to 0 \text{ as } t \to \infty.$$

This shows that in this case, damping is so large that no oscillation can occur. Hence, the motion is called *overdamped* or *dead-beat motion*.

Case II If $\lambda^2 - \omega^2 = 0$. then roots of Eq. (10.47) are equal and real.

$$x_c = (c_1 + c_2 t)e^{-\lambda t}$$

$$x_c \to 0 \text{ as } t \to \infty.$$

In this case, damping is just enough to prevent oscillation. Hence, the motion is called *critically damped*.

Case III If $\lambda^2 - \omega^2 < 0$ then the roots of Eq. (10.47) are imaginary.

$$p = -\lambda \pm i\sqrt{\omega^2 - \lambda^2}$$

Hence,
$$x_c = e^{-\lambda t}\left[c_1 \cos\left(\sqrt{\omega^2 - \lambda^2}\right)t + c_2 \sin\left(\sqrt{\omega^2 - \lambda^2}\right)t\right]$$

In this case, motion is oscillatory due to the presence of the trigonometric factor. Such a motion is called *damped oscillatory motion*.

Free Oscillation

EXAMPLE 10.106 *A body weighing 20 kg is hung from a spring. A pull of 40 kg weight will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of* *the body from its equilibrium position at time t seconds, the maximum velocity and the period of oscillation.*

Solution: Since a pull of 40 kg weight stretches the spring to 10 cm, i.e., 0·1 m,

$$40 = k \times 0 \cdot 1$$
$$k = 400 \text{ kg/m}$$

Weight of the body, $W = 20$ kg

$$m = \frac{W}{g} = \frac{20}{9.8}$$

The equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \text{ where } \omega^2 = \frac{k}{m} = \frac{400}{\frac{20}{9.8}} = 196$$

$$\frac{d^2x}{dt^2} + 196x = 0$$

$$(D^2 + 196)x = 0 \qquad \cdots (1)$$

The auxiliary equation is

$$m^2 + 196 = 0$$
$$m = \pm 14i \text{ (complex)}$$

Hence, the general solution of Eq. (1) is

$$x = c_1 \cos 14t + c_2 \sin 14t$$

$$\frac{dx}{dt} = -14c_1 \sin 14t + 14c_2 \cos 14t$$

At $t = 0$, $x = 20$ cm $= 0.2$ m, $v = \frac{dx}{dt} = 0$,

$$0 \cdot 2 = c_1$$
$$\text{and} \quad 0 = 14c_2$$
$$c_2 = 0$$

(i) Hence, displacement of the body from its equilibrium position at time t is given by

$$x = 0 \cdot 2 \cos 14t$$

(ii) Amplitude = 20 cm = 0·2 m

Maximum velocity $= \omega \times \text{Amplitude} = 14 \times 0 \cdot 2 = 2 \cdot 8$ ms

(iii) Period of oscillation $= \frac{2\pi}{\omega} = \frac{2\pi}{14} = 0.45$ s

Free Damped Oscillation

EXAMPLE 10.107 *A 3 lb weight on a spring stretches it to 6 inches. Suppose a damping force λv is present ($\lambda > 0$). Show that the motion is (i) critically damped if $\lambda = 1.5$, (ii) overdamped if $\lambda > 1.5$, and (iii) oscillatory if $\lambda < 1.5$.*

Solution: A 3 lb weight stretches the spring to 6 inches, i.e., 0.5 ft.

$$W = 3\text{lb}, e = 0.5\,\text{ft}$$

$$W = ke$$

$$3 = k \times 0.5$$

$$k = 6\,\text{lb/ft}$$

$$\text{Mass} = \frac{W}{g} = \frac{3}{32} \qquad [\because g = 32\,\text{ft/s}^2]$$

$$\text{Damping force} = \lambda v, \qquad \text{where } \lambda > 0$$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + \lambda\frac{dx}{dt} = 0$$

$$\frac{3}{32}D^2x + 6x + \lambda Dx = 0$$

$$\left(D^2 + \frac{32}{3}\lambda D + 64\right)x = 0, \qquad \text{where } D = \frac{d}{dt}$$

The auxiliary equation is

$$m^2 + \frac{32}{3}\lambda m + 64 = 0 \qquad \text{...(1)}$$

$$m = \frac{-\frac{32}{3}\lambda \pm \sqrt{\left(\frac{32}{3}\lambda\right)^2 - 256}}{2} = \frac{-32\lambda \pm \sqrt{1024\lambda^2 - 2304}}{6}$$

(a) The motion is critically damped when roots of Eq. (1) are equal, i.e., $1024\lambda^2 - 2304 = 0$.

$$\lambda = 1{\cdot}5.$$

(b) The motion is overdamped when roots of Eq. (1) are real and distinct, i.e., $1024\lambda^2 - 2304 > 0$.

$$\lambda > 1{\cdot}5.$$

(c) The motion is oscillatory when roots of Eq. (1) are imaginary, i.e., $1024\lambda^2 - 2304 < 0$.

$$\lambda < 1{\cdot}5.$$

Forced Undamped Oscillation

EXAMPLE 10.108 *Determine whether resonance occurs in a system consisting of a 32 lb weight attached to a spring with constant $k = 4$ lb/ft and an external force of 16 sin 2t and no damping force present. Initially, $x = \frac{1}{2}$ and $\frac{dx}{dt} = -4$.*

Solution: $W = 32$ lb, $\quad k = 4$ lb/ft

$$m = \frac{W}{g} = \frac{32}{32} = 1 \qquad [\because g = 32 \text{ ft/s}^2]$$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = 16\sin 2t$$

$$\frac{d^2x}{dt^2} + 4x = 16\sin 2t$$

$$(D^2+4)x = 16\sin 2t \qquad \ldots(1)$$

The auxiliary equation is

$$m^2 + 4 = 0$$

$$m = \pm 2i \text{ (complex)}$$

$$\text{CF} = c_1\cos 2t + c_2\sin 2t$$

$$\text{PI} = \frac{1}{D^2+4}16\sin 2t = 16t\frac{1}{2D}\sin 2t = 8t\int \sin 2t\, dt = 8t\left(-\frac{\cos 2t}{2}\right) = -4t\cos 2t$$

Hence, the general solution of Eq. (1) is

$$x = c_1\cos 2t + c_2\sin 2t - 4t\cos 2t$$

$$\frac{dx}{dt} = -2c_1\sin 2t + 2c_2\cos 2t - 4\cos 2t + 8t\sin 2t$$

Initially, at $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = -4$

$$\frac{1}{2} = c_1$$

and

$$-4 = 2c_2 - 4$$

$$c_2 = 0$$

Hence, $x = \frac{1}{2}\cos 2t - 4t\cos 2t$

$$\omega^2 = \frac{k}{m} = \frac{4}{1}$$

$$\omega = 2$$

Also, $n = 2$

Frequency of the external force $= \dfrac{n}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$ cycles/second

Natural frequency $= \dfrac{\omega}{2\pi} = \dfrac{2}{2\pi} = \dfrac{1}{\pi}$ cycles/second

Since both the frequencies are same, resonance occurs in the system.

Forced Damped Oscillations

EXAMPLE 10.109 *Determine the transient and steady-state solutions of a mechanical system with 6 lb weight, 12 lb/ft stiffness constant, damping force of 1.5 times the instantaneous velocity, external force of 24 cos 8t, and initial conditions $x = \frac{1}{3}$ ft, $\frac{dx}{dt} = 0$.*

Solution: $W = 6$ lb, $k = 12$ lb/ft

$$m = \frac{W}{g} = \frac{6}{32} \quad [\because g = 32\ ft/s^2]$$

Damping force $= 1.5\dfrac{dx}{dt}$

The equation of motion is

$$m\frac{d^2x}{dt^2} + kx + 1.5\frac{dx}{dt} = 24\cos 8t$$

$$\frac{6}{32}\frac{d^2x}{dt^2} + 12x + 1.5\frac{dx}{dt} = 24\cos 8t$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 64x = 128\cos 8t$$

$$(D^2 + 8D + 64)x = 128\cos 8t \qquad \dots (1)$$

The auxiliary equation is

$$m^2 + 8m + 64 = 0$$

$$m = \frac{-8 \pm \sqrt{64 - 256}}{2} = -4 \pm i4\sqrt{3} \text{ (complex)}$$

$$\text{CF} = e^{-4t}(c_1\cos 4\sqrt{3}t + c_2\sin 4\sqrt{3}t) = x_c, \text{ say}$$

which gives the transient solution.

$$\text{PI} = \frac{1}{D^2+8D+64}\cdot 128\cos 8t = 128\cdot\frac{1}{-64+8D+64}\cos 8t = 16\int \cos 8t\, dt = 16\frac{\sin 8t}{8} = 2\sin 8t$$

which gives the steady-state solution.

Hence, the general solution of Eq. (1) is

$$x = e^{-4t}(c_1\cos 4\sqrt{3}t + c_2\sin 4\sqrt{3}t) + 2\sin 8t$$

$$\frac{dx}{dt} = -4e^{-4t}(c_1\cos 4\sqrt{3}t + c_2\sin 4\sqrt{3}t)$$
$$+e^{-4t}(-4\sqrt{3}c_1\sin 4\sqrt{3}t + 4\sqrt{3}c_2\cos 4\sqrt{3}t) + 16\cos 8t$$

Initially, at $t = 0$, $x = \frac{1}{3}$ and $\frac{dx}{dt} = 0$

$$\frac{1}{3} = c_1$$

and $0 = -4c_1 + 4\sqrt{3}c_2 + 16$

$$c_2 = -\frac{11\sqrt{3}}{9}$$

Hence, the transient solution is

$$x_c = e^{-4t}\left(\frac{1}{3}\cos 4\sqrt{3}t - \frac{11\sqrt{3}}{9}\sin 4\sqrt{3}t\right)$$
$$= \frac{e^{-4t}}{9}(3\cos 4\sqrt{3}t - 11\sqrt{3}\sin 4\sqrt{3}t)$$

and the steady-state solution is

$$x_p = 2\sin 8t$$

EXERCISE 10.27

1. A body weighing 4·9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity, and the period of oscillation.

[Ans.: 0.06 cos $20t$, 1.2 m/s, 0.314 s]

2. A mass of 200 g is tied at the end of a spring which extends to 4 cm under a force of 196, 000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after release, if there be a damping

force of 2000 dynes per cm per second. What should be the damping force for the dead-beat motion?

$$\left[\begin{array}{l}\textbf{Ans.:}\\ e^{-5t}\left(5\cos\sqrt{220}t+\dfrac{25}{\sqrt{220}}\sin\sqrt{220}t\right),\\ 6261\end{array}\right]$$

3. A spring of negligible weight which stretches 1 inch under a tension of 2 lb is fixed at one end and is attached to a weight of W lb at the other. It is found that resonance occurs when an axial periodic force of $2\cos 2t$ lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by $x = ct\sin 2t$, and find the values of W and c.

$$\left[\textbf{Ans.:}\ W = 6g, c = \frac{1}{12}\right]$$

4. Find the steady-state and transient oscillations of the mechanical system corresponding to the differential equation $x'' + 2x' + 2x = \sin 2t - 2\cos 2t$, $x(0) = x'(0) = 0$.

[**Ans.** : $-0.5\sin 2t$, $e^{-t}\sin t$]

5. If weight $W = 16$ lb, spring constant $k = 10$ lb/ft, damping force $= 2\dfrac{d\dot{x}}{dt}$, external force $F(t)$ is $5\cos 2t$, find the motion of the weight given $x(0) = \dot{x}(0) = 0$. Write the transient and steady-state solutions.

$$\left[\begin{array}{l}\textbf{Ans.:}\ x(t) = e^{-2t}\left(-\dfrac{3}{8}\sin 4t - \dfrac{1}{2}\cos 4t\right)\\ \qquad +\dfrac{1}{2}\cos 2t + \dfrac{1}{4}\sin 2t\\ \text{Transient solution}: \dfrac{5e^{-2t}}{8}\cos(4t - 0.64)\\ \text{Transient solution}: \dfrac{\sqrt{5}}{4}\cos(2t - 0.46)\end{array}\right]$$

10.12.4 Deflection of Beams

When a beam made up of fibres is bent, the fibres of the upper half are compressed and those of the lower half are stretched. In between, there is a region, where the fibres are neither compressed nor stretched. This region is called the *neutral surface* of the beam, and the curve of any particular fibre on neutral surface is called the *elastic curve* or *deflection curve* of the beam. The line at which any plane section of the beam cuts the neutral surface is called the *neutral axis* of that section.

The bending moment M created by the forces acting above and below the neutral surface in opposite direction is

$$M = \frac{EI}{R}$$

where E = Modulus of elasticity of the beam

I = Moment of inertia of the section about the neutral axis

R = Radius of curvature of the elastic curve at any point $P(x, y)$

$$= \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}},$$ where y represents deflection at a distance x from one end

Assuming deflection to be very small, $\left(\frac{dy}{dx}\right)^2$ can be neglected.

$$R = \frac{1}{\frac{d^2y}{dx^2}}$$

$$M = EI\frac{d^2y}{dx^2} \qquad \text{...(10.49)}$$

which is the differential equation of the elastic curve.

Boundary Conditions

The arbitrary constants in the solution of Eq. (10.49) can be found using the following boundary conditions:

(i) End freely supported: $y = 0,\ \frac{d^2y}{dx^2} = 0$

(ii) End fixed horizontally: $y = 0,\ \frac{dy}{dx} = 0$

(iii) End perfectly free: $\frac{d^2y}{dx^2} = 0,\ \frac{d^3y}{dx^3} = 0$

EXAMPLE 10.110 *A light horizontal strut AB is freely pinned at A and B. It is under the action of equal and opposite compressive forces P at its ends and it carries a load W at its centre. Then for* $0 < x < \frac{l}{2}$, $EI\frac{d^2y}{dx^2} + Py + \frac{1}{2}Wx = 0$. *Also,* $y = 0$ *at* $x = 0$ *and* $\frac{dy}{dx} = 0$ *at* $x = \frac{l}{2}$. *Prove that* $y = \frac{W}{2P}\left(\frac{\sin nx}{n\cos\frac{nl}{2}} - x\right)$ *where* $n^2 = \frac{P}{EI}$.

Solution: The equation of action of forces is

$$EI\frac{d^2y}{dx^2} + Py + \frac{1}{2}Wx = 0$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y + \frac{W}{2EI}x = 0$$

$$(D^2 + n^2)y = -\frac{W}{2EI}x, \text{ where } \frac{P}{EI} = n^2 \text{ and } D = \frac{d}{dx}$$

$$(D^2 + n^2)y = -\frac{n^2W}{2P}x \qquad \text{...(1)}$$

The auxiliary equation is

$$m^2 + n^2 = 0$$

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$$m = \pm in \text{ (complex)}$$
$$\text{CF} = c_1 \cos nx + c_2 \sin nx$$

$$\text{PI} = \frac{1}{D^2+n^2}\left(-\frac{n^2W}{2P}x\right) = -\frac{n^2W}{2P}\cdot\frac{1}{n^2}\left(1+\frac{D^2}{n^2}\right)^{-1}x$$
$$= -\frac{W}{2P}\left(1-\frac{D^2}{n^2}+\frac{D^4}{n^4}-\ldots\right)x = -\frac{Wx}{2P}$$

Hence, the general solution of Eq. (1) is

$$y = c_1\cos nx + c_2\sin nx - \frac{Wx}{2P}$$
$$\frac{dy}{dx} = -c_1 n\sin nx + c_2 n\cos nx - \frac{W}{2P}$$

At $x = 0, \quad y = 0$

$$0 = c_1$$

At $x = \frac{l}{2}, \ \frac{dy}{dx} = 0$

$$0 = c_2 n\cos\frac{nl}{2} - \frac{W}{2P}$$
$$c_2 = \frac{W}{2P}\cdot\frac{1}{n\cos\frac{nl}{2}}$$

Hence, $$y = \frac{W}{2P}\left(\frac{\sin nx}{n\cos\frac{nl}{2}} - x\right), \quad \text{where } n^2 = \frac{P}{EI}$$

EXAMPLE 10.111 *Find the equation of the elastic curve and its maximum deflection for the simply supported beam of length 2l, having a uniformly distributed load W per unit length.*

Solution: Consider the segment $AP = x$ (Fig. 10.6).

The forces acting on the segment AP are

(i) The upward thrust Wl at A
(ii) The load Wx at the midpoint of AP

$$\text{Moment } M = Wlx - Wx\frac{x}{2} = Wlx - \frac{Wx^2}{2} \qquad \ldots (1)$$

The equation of the elastic curve is

$$M = EI\frac{d^2y}{dx^2} \qquad \ldots (2)$$

Equating Eqs (1) and (2),

$$EI\frac{d^2y}{dx^2} = Wlx - \frac{Wx^2}{2}$$

Integrating w.r.t. x,

$$EI\frac{dy}{dx} = Wl\frac{x^2}{2} - \frac{Wx^3}{6} + c_1$$

Integrating again w.r.t. x,

$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} + c_1x + c_2$$

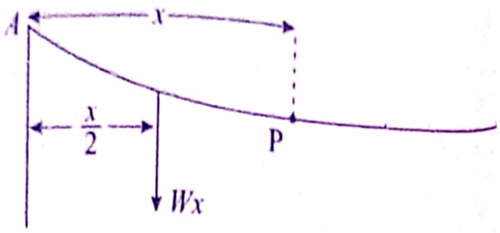

Fig. 10.6

Since ends A and B are freely supported, at A, $x = 0$, $y = 0$ and at B, $x = 2l$, $y = 0$

Putting in Eq. (3),

$$0 = c_2$$

and

$$0 = \frac{Wl}{6}(2l)^3 - \frac{W}{24}(2l)^4 + c_1(2l)$$

$$c_1 = \frac{-Wl^3}{3}$$

Hence,

$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} - \frac{Wl^3x}{3}$$

$$y = \frac{Wx}{EI}\left(\frac{lx^2}{6} - \frac{x^3}{24} - \frac{l^3}{3}\right)$$

Deflection of the beam is maximum at $x = l$ (midpoint)

$$y_{\max} = \frac{Wl}{EI}\left(\frac{l^3}{6} - \frac{l^3}{24} - \frac{l^3}{3}\right) = -\frac{5Wl^4}{24EI}$$

EXERCISE 10.28

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\dfrac{d^2y}{dx^2} = Py - \dfrac{W}{2}x$. With conditions $x = 0$, $y = 0$ and $x = l$, $\dfrac{dy}{dx} = 0$, prove that the deflection δ and the bending moment M at the centre $(x = l)$ are given by $\delta = \dfrac{W}{2Pn}(nl - \tanh nl)$ and $M = -\dfrac{W}{2n}\tanh nl$, where $n^2EI = P$.

2. The shape of a strut of length l subjected to an end thrust P and lateral load W per unit length, when the ends are built in, is given by $EI\dfrac{d^2y}{dx^2} + Py = \dfrac{Wx^2}{2} - \dfrac{Wlx}{2} + M$, where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $\dfrac{dy}{dx} = 0$ at $x = 0$ and $\dfrac{dy}{dx} = 0$ at $x = \dfrac{l}{2}$.

$$\left[\text{Ans.: } y = \frac{Wl}{2Pn}\operatorname{cosec}\frac{nl}{2}\cos\left(nx - \frac{nl}{2}\right) - \frac{Wl}{2Pn}\cot\frac{nl}{2} + \frac{W}{2P}(x^2 - lx)\right]$$

3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is P, prove that the maximum deflection is $\frac{W}{Pa^2}\left(\sec\frac{al}{2} - 1\right) - \frac{Wl^2}{8P}$, where $\frac{P}{EI} = a^2$. Prove also that the maximum bending moment is of magnitude $\frac{W}{a^2}\left(\sec\frac{al}{2} - 1\right)$.

4. A horizontal tie-rod is freely pined at each end. It carries a uniform load W per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment taking the origin at one of its ends.

$$\left[\text{Ans.: } \frac{W}{Pa^2}\left(\operatorname{sech}\frac{al}{2} - 1\right) + \frac{Wl^2}{8P},\ \frac{W}{a^2}\left(\operatorname{sech}\frac{al}{2} - 1\right)\right]$$

10.12.5 Electrical Circuits

A second-order electrical circuit consists of a resistor, an inductor, and a capacitor in series with an emf $e(t)$ (Fig 10.7).

Applying Kirchhoff's voltage law to the circuit,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t) \qquad \dots (10.50)$$

But $\quad i = \frac{dq}{dt}$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad \dots (10.51)$$

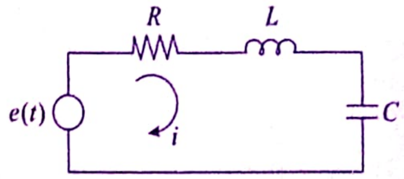

Fig. 10.7 Electrical circuit

Differentiating Eq. (10.50) w.r.t. t,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de(t)}{dt} \qquad \dots (10.52)$$

Equations (10.51) and (10.52) are both second-order linear nonhomogeneous ordinary differential equations.

EXAMPLE 10.112 *A circuit consists of an inductance of 2 henries, a resistance of 4 ohms, and a capacitance of 0.05 farad. If $q = i = 0$ at $t = 0$, (i) find $q(t)$ and $i(t)$ when there is a constant emf of 100 volts. (ii) Find steady-state solutions.*

Solution: (i) The differential equation of the *RLC* circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e(t)$$

$$2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + \frac{q}{0.05} = 100$$

$$\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 10q = 50$$

Equating Eqs (1) and (2),

$$EI\frac{d^2y}{dx^2} = Wlx - \frac{Wx^2}{2}$$

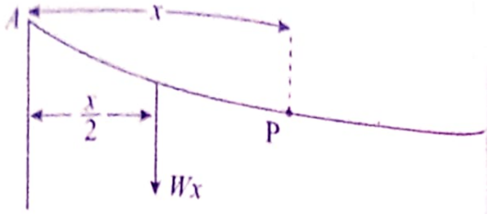

Fig. 10.6

Integrating w.r.t. x,

$$EI\frac{dy}{dx} = Wl\frac{x^2}{2} - \frac{Wx^3}{6} + c_1$$

Integrating again w.r.t. x,

$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} + c_1x + c_2$$

Since ends A and B are freely supported, at A, $x = 0$, $y = 0$ and at B, $x = 2l$, $y = 0$

Putting in Eq. (3),

$$0 = c_2$$

and

$$0 = \frac{Wl}{6}(2l)^3 - \frac{W}{24}(2l)^4 + c_1(2l)$$

$$c_1 = \frac{-Wl^3}{3}$$

Hence,

$$EI\,y = \frac{Wlx^3}{6} - \frac{Wx^4}{24} - \frac{Wl^3x}{3}$$

$$y = \frac{Wx}{EI}\left(\frac{lx^2}{6} - \frac{x^3}{24} - \frac{l^3}{3}\right)$$

Deflection of the beam is maximum at $x = l$ (midpoint)

$$y_{\max} = \frac{Wl}{EI}\left(\frac{l^3}{6} - \frac{l^3}{24} - \frac{l^3}{3}\right) = -\frac{5Wl^4}{24EI}$$

EXERCISE 10.28

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\frac{d^2y}{dx^2} = Py - \frac{W}{2}x$. With conditions $x = 0$, $y = 0$ and $x = l$, $\frac{dy}{dx} = 0$, prove that the deflection δ and the bending moment M at the centre $(x = l)$ are given by $\delta = \frac{W}{2Pn}(nl - \tanh nl)$ and $M = -\frac{W}{2n}\tanh nl$, where $n^2EI = P$.

2. The shape of a strut of length l subjected to an end thrust P and lateral load W per unit length, when the ends are built in, is given by $EI\frac{d^2y}{dx^2} + Py = \frac{Wx^2}{2} - \frac{Wlx}{2} + M$, where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $\frac{dy}{dx} = 0$ at $x = 0$ and $\frac{dy}{dx} = 0$ at $x = \frac{l}{2}$.

$$\left[\textbf{Ans.:}\ y = \frac{Wl}{2Pn}\operatorname{cosec}\frac{nl}{2}\cos\left(nx - \frac{nl}{2}\right) - \frac{Wl}{2Pn}\cot\frac{nl}{2} + \frac{W}{2P}(x^2 - lx)\right]$$

3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is P, prove that the maximum deflection is $\frac{W}{Pa^2}\left(\sec\frac{al}{2} - 1\right) - \frac{Wl^2}{8P}$, where $\frac{P}{EI} = a^2$. Prove also that the maximum bending moment is of magnitude $\frac{W}{a^2}\left(\sec\frac{al}{2} - 1\right)$.

4. A horizontal tie-rod is freely pined at each end. It carries a uniform load W per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment taking the origin at one of its ends.

$$\left[\textbf{Ans.:}\ \frac{W}{Pa^2}\left(\operatorname{sech}\frac{al}{2} - 1\right) + \frac{Wl^2}{8P},\ \frac{W}{a^2}\left(\operatorname{sech}\frac{al}{2} - 1\right)\right]$$

10.12.5 Electrical Circuits

A second-order electrical circuit consists of a resistor, an inductor, and a capacitor in series with an emf $e(t)$ (Fig 10.7).

Applying Kirchhoff's voltage law to the circuit,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e(t) \qquad \text{... (10.50)}$$

But

$$i = \frac{dq}{dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e(t) \qquad \text{... (10.51)}$$

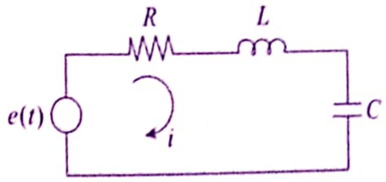

Fig. 10.7 Electrical circuit

Differentiating Eq. (10.50) w.r.t. t,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de(t)}{dt} \qquad \text{... (10.52)}$$

Equations (10.51) and (10.52) are both second-order linear nonhomogeneous ordinary differential equations.

EXAMPLE 10.112 *A circuit consists of an inductance of 2 henries, a resistance of 4 ohms, and a capacitance of 0.05 farad. If $q = i = 0$ at $t = 0$, (i) find $q(t)$ and $i(t)$ when there is a constant emf of 100 volts. (ii) Find steady-state solutions.*

Solution: (i) The differential equation of the RLC circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e(t)$$

$$2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + \frac{q}{0.05} = 100$$

$$\frac{d^2q}{dt^2} + 2\frac{dq}{dt} + 10q = 50$$

$$(D^2+2D+10)q = 50$$

The auxiliary equation is

$$m^2+2m+10=0$$
$$m = -1 \pm 3i \text{ (complex)}$$
$$\text{CF} = e^{-t}(c_1\cos 3t + c_2 \sin 3t)$$
$$\text{PI} = \frac{1}{D^2+2D+10}(50e^{0t}) = \frac{1}{10}\cdot 50 = 5$$

The general solution is

$$q = e^{-t}(c_1\cos 3t + c_2\sin 3t) + 5$$

At $t=0$, $q=0$

$$0 = c_1 + 5$$
$$c_1 = -5$$

Differentiating Eq. (1) w.r.t. t,

$$i = \frac{dq}{dt} = e^{-t}(-3c_1\sin 3t + 3c_2\cos 3t) - e^{-t}(c_1\cos 3t + c_2\sin 3t)$$

At $t=0, i=0$

$$0 = 3c_2 - c_1$$
$$3c_2 = c_1$$
$$c_2 = -\frac{5}{3}$$

Hence, $q(t) = 5 + e^{-t}\left(-5\cos 3t - \frac{5}{3}\sin 3t\right) = 5 - \frac{5}{3}e^{-t}(3\cos 3t + \sin 3t)$

and $i(t) = e^{-t}(15\sin 3t - 5\cos 3t) + e^{-t}\left(5\cos 3t + \frac{5}{3}\sin 3t\right) = \frac{50}{3}e^{-t}\sin 3t$

(ii) The steady-state solution is obtained by putting $t=\infty$.

$$q(t) = 5$$
$$i(t) = 0$$

EXAMPLE 10.113 *(i) Determine q and i in an RLC circuit with L = 0.5 H, R = 6 Ω, C = 0.02 F, e = 24 sin 10t and initial conditions q = i = 0 at t = 0. (ii) Find steady-state and transient solutions.*

Solution: The differential equation of the *RLC* circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e$$
$$0.5\frac{d^2q}{dt^2} + 6\frac{dq}{dt} + \frac{q}{0.02} = 24\sin 10t$$

$$\frac{d^2q}{dt^2} + 12\frac{dq}{dt} + 100q = 48\sin 10t$$

$$(D^2 + 12D + 100)q = 48\sin 10t$$

The auxiliary solution is

$$m^2 + 12m + 100 = 0$$

$$m = -6 \pm 8i \text{ (complex)}$$

$$\text{CF} = e^{-6t}(c_1\cos 8t + c_2\sin 8t)$$

$$\text{PI} = \frac{1}{D^2 + 12D + 100}48\sin 10t = 48\cdot\frac{1}{-10^2 + 12D + 100}\sin 10t$$

$$= \frac{48}{12}\int \sin 10t\, dt = 4\left(-\frac{\cos 10t}{10}\right) = -\frac{2}{5}\cos 10t$$

The general solution is

$$q = e^{-6t}(c_1\cos 8t + c_2\sin 8t) - \frac{2}{5}\cos 10t \quad \dots (1)$$

Differentiating Eq. (1) w.r.t. t,

$$i = \frac{dq}{dt} = -6e^{-6t}(c_1\cos 8t + c_2\sin 8t) + e^{-6t}(-8c_1\sin 8t + 8c_2\cos 8t) + \frac{2}{5}\cdot 10\sin 10t$$

$$= e^{-6t}[(-6c_1 + 8c_2)\cos 8t - (6c_2 + 8c_1)\sin 8t] + 4\sin 10t$$

At $t = 0$, $q = 0$, $i = 0$

$$0 = c_1 - \frac{2}{5}$$

$$c_1 = \frac{2}{5}$$

and

$$0 = -6c_1 + 8c_2$$

$$c_2 = \frac{6c_1}{8} = \frac{3}{10}$$

Hence,

$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right) - \frac{2}{5}\cos 10t$$

and

$$i(t) = e^{-6t}(-5\sin 8t) + 4\sin 10t$$

The steady-state solution is obtained by putting $t = \infty$.

$$q(t) = --\cos 10t$$

$$i(t) \quad 4\sin 10t$$

The transient solution is

$$q(t) = e^{-6t}\left(\frac{2}{5}\cos 8t + \frac{3}{10}\sin 8t\right)$$

$$i(t) = e^{-6t}(-5\sin 8t)$$

EXERCISE 10.29

1. A circuit consists of a resistance of 5 ohms, an inductance of 0.05 henries, and a capacitance of 4×10^{-4} farads. If $q(0) = 0$, $i(0) = 0$, find $q(t)$ and $i(t)$ when an emf of 110 volts is applied.

$$\left[\begin{array}{l}\textbf{Ans.: } q(t) = e^{-50t}\left(-\frac{11}{250}\cos 50\sqrt{19}t - \frac{11\sqrt{19}}{4750}\sin 50\sqrt{19}t\right) + \frac{11}{250}, \\ i(t) = \frac{44}{\sqrt{19}}e^{-50t}\sin 50\sqrt{19}t\end{array}\right]$$

2. Determine the charge on the capacitor at any time t in a series circuit having a resistor of 2 Ω, inductor of 0.1 H, capacitor of $\frac{1}{260}$ F, and $e(t) = 100 \sin 60t$. If the initial current and initial charge on the capacitor are both zero, find the steady-state solution.

$$\left[\begin{array}{l}\textbf{Ans.: } q(t) = \frac{6e^{-10t}}{61}(6\sin 50t + 5\cos 50t) - \frac{5}{\sqrt{61}}(5\sin 60t + 6\cos 60t), \text{ Steady-state solution: } \\ q(t) = -\frac{5}{61}(5\sin 60t + 6\cos 60t).\end{array}\right]$$

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