Ten

Differential Equations

CHAPTER OUTLINE

- Introduction
- Differential Equations
- Ordinary Differential Equations of First Order and First Degree
- Ordinary Differential Equations of First Order and Higher Degree
- Homogeneous Linear Differential Equations of Higher Order with Constant Coefficients
- Nonhomogeneous Linear Differential Equations of Higher Order with Constant Coefficients
- Higher Order Linear Differential Equations with Variable Coefficients
- Method of Variation of Parameters
- Method of Undetermined Coefficients
- Simultaneous Linear Differential Equations with Constant Coefficients
- Applications of Ordinary Differential Equations of First Order and First Degree
- Applications of Higher Order Linear Differential Equations

10.1 INTRODUCTION

Differential equations are very important in engineering mathematics. A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and of its derivatives of various orders. It provides the medium for the interaction between mathematics and various branches of science and engineering. Most common differential equations are radioactive decay, chemical reactions, Newton's law of cooling, series *RL*, *RC*, and *RLC* circuits, simple harmonic motions, etc.

10.2 DIFFERENTIAL EQUATIONS

A differential equation is an equation which involves variables (dependent and independent) and their derivatives, e.g.,

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^x \qquad \ldots (10.1)$$

$$\left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3 = 0 \qquad \cdots (10.2)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \cdots (10.3)$$

Equations (10.1) and (10.2) involve ordinary derivatives and are, hence, called *ordinary differential equations*, whereas Eq. (10.3) involves partial derivatives and is, hence, called *partial differential equation*.

10.2.1 Order

Order of a differential equation is the order of the highest derivative present in the equation, e.g., the order of Eqs (10.1) and (10.2) is 2.

10.2.2 Degree

Degree of a differential equation is the power of the highest order derivative after clearing the radical sign and fraction, e.g., the degree of Eq. (10.1) is 1 and the degree of Eq. (10.2) is 2.

10.2.3 Solution or Primitive

Solution of a differential equation is a relation between the dependent and independent variables (excluding derivatives), which satisfies the equation. The solution of a differential equation is not always unique. It may have more than one solution or sometimes no solution.

The general solution of a differential equation of order n contains n arbitrary constants.

The particular solution of a differential equation is obtained from the general solution by giving particular values to the arbitrary constants.

10.3 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation which contains the first-order and first-degree derivative of y (dependent variable) and known functions of x (independent variable) and y is known as the ordinary differential equation of first order and first degree. The general form of this equation can be written as

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

or in explicit form as

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y)dx + N(x, y)dy = 0$$

The solution of the differential equation can be obtained by classifying them as follows:

(i) Variables separable
(ii) Homogeneous differential equations
(iii) Nonhomogeneous differential equations
(iv) Exact differential equations
(v) Non-exact differential equations reducible to exact form
(vi) Linear differential equations
(vii) Nonlinear differential equations reducible to linear form

10.3.1 Variables Separable

A differential equation of the form

$$M(x)dx + N(y)dy = 0 \quad \ldots (10.4)$$

where M (x) is the function of x only and N (y) is the function of y only, is called a differential equation with variables separable as in Eq. (10.4), where the function of x and the function of y can be separated easily.

Integrating Eq. (10.4), the solution is

$$\int M(x)dx + \int N(y)dy = c$$

where c is the arbitrary constant.

EXAMPLE 10.1 *Solve* $y(1+x^2)^{\frac{1}{2}}dy + x\sqrt{1+y^2}\,dx = 0.$

Solution: $y(1+x^2)^{\frac{1}{2}}dy = -x\sqrt{1+y^2}\,dx$

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx + c$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)dx + c$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

EXAMPLE 10.2 *Solve* $(4x+y)^2\dfrac{dx}{dy} = 1.$

Solution: $\dfrac{dy}{dx} = (4x+y)^2 \quad \ldots (1)$

Let $4x + y = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

$$\left(\frac{d^2y}{dx^2}\right)^2 - \left[\left(\frac{dy}{dx}\right)^2 + 1\right]^3 = 0 \qquad \cdots (10.2)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x + y \qquad \cdots (10.3)$$

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Solution: $y(1+x^2)^{\frac{1}{2}}dy = -x\sqrt{1+y^2}\,dx$

$$\int \frac{y}{\sqrt{1+y^2}}dy = -\int \frac{x}{\sqrt{1+x^2}}dx + c$$

$$\frac{1}{2}\int (1+y^2)^{-\frac{1}{2}}(2y)dy = -\frac{1}{2}\int (1+x^2)^{-\frac{1}{2}}(2x)dx + c$$

$$\frac{1}{2}\cdot\frac{(1+y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\cdot\frac{(1+x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c \qquad \left[\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

EXAMPLE 10.2 *Solve* $(4x+y)^2\dfrac{dx}{dy} = 1.$

Solution: $\dfrac{dy}{dx} = (4x+y)^2 \qquad \ldots (1)$

Let $4x + y = t$

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\int \frac{dt}{t^2+4} = \int dx + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y}{2}\right) = x + c$$

EXAMPLE 10.3 *Solve* $\left(x\frac{dy}{dx} - y\right)\cos\left(\frac{y}{x}\right) + x = 0.$

Solution: Let $\frac{y}{x} = t$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{dt}{dx}$$

$$x\frac{dy}{dx} - y = x^2\frac{dt}{dx}$$

Substituting in the given equation,

$$x^2\frac{dt}{dx}\cdot\cos t + x = 0$$

$$\cos t\, dt = -\frac{dx}{x}$$

Integrating both the sides,

$$\int \cos t\, dt = -\int \frac{dx}{x} + c$$

$$\sin t = -\log x + c$$

$$\sin\left(\frac{y}{x}\right) = -\log x + c$$

EXAMPLE 10.4 *Solve* $(x\log x)\frac{dy}{dx} = 2y,\ y(2) = (\log 2)^2.$

Solution: $\frac{dy}{2y} = \frac{dx}{x\log x}$

Integrating both the sides,

$$\int \frac{dy}{2y} = \int \frac{1}{\log x} \cdot \frac{1}{x} dx$$

$$\frac{1}{2} \log y = \log(\log x) + \log c \qquad \left[\because \int \frac{f'(x)}{f(x)} dx = \log f(x) + c\right]$$

$$\log y^{\frac{1}{2}} = \log(c \log x)$$

$$y^{\frac{1}{2}} = c \log x \qquad \text{...(1)}$$

Given $\quad y(2) = (\log 2)^2$

Putting $x = 2, y = (\log 2)^2$ in Eq. (1),

$$(\log 2) = c \log 2$$

$$c = 1$$

Hence, the solution is

$$y^{\frac{1}{2}} = \log x$$

$$y = (\log x)^2$$

EXERCISE 10.1

Solve the following differential equations:

1. $y^2 \dfrac{dy}{dx} + x^2 = 0$

   [**Ans.:** $x^3 + y^3 = c$]

2. $(e^y + 1) \cos x \, dx + e^y \sin x \, dy = 0$

   [**Ans.:** $(e^y + 1) \sin x = c$]

3. $y \dfrac{dy}{dx} = xe^{-x} \sqrt{1 - y^2}$

   [**Ans.:** $\sqrt{1 - y^2} = (x + 1)e^{-x} + c$]

4. $x(e^{4y} - 1) \dfrac{dy}{dx} + (x^2 - 1)e^{2y} = 0, x > 0$

   [**Ans.:** $\cosh(2y) = \log x - \dfrac{x^2}{2} + c$]

5. $\dfrac{dy}{dx} = \dfrac{\sin x + \dfrac{\log x}{x}}{\cos y - \sec^2 y}$

   [**Ans.:** $\sin y - \tan y = -\cos x + \dfrac{1}{2}(\log x)^2 + c$]

6. $(x+1)\left(\dfrac{dy}{dx}-1\right)=2(y-x)$

$\left[\textbf{Ans.:}\ y-x=c(x+1)^2\right]$

7. $\dfrac{dy}{dx}=\dfrac{y-x}{y-x+2}$

$\left[\textbf{Ans.:}\ (y-x)^2=c-4y\right]$

8. $x\dfrac{dy}{dx}=y+x^2\tan\left(\dfrac{y}{x}\right)$

$\left[\textbf{Ans.:}\ \sin\left(\dfrac{y}{x}\right)=ce^x\right]$

9. $x\dfrac{dy}{dx}=e^{-xy}-y$

$\left[\textbf{Ans.:}\ e^{xy}=x+c\right]$

10. $(1+x^3)dy-x^2y\,dx=0,\ y(1)=2$

$\left[\textbf{Ans.:}\ y^3=4(1+x^3)\right]$

11. $\dfrac{dy}{dx}+2y=x^2y,\ y(0)=1$

$\left[\textbf{Ans.:}\ y=e^{\frac{x^3}{3}-2x}\right]$

12. $e^y\left(\dfrac{dy}{dx}+1\right)=1,\ y(0)=1$

$\left[\textbf{Ans.:}\ e^y=1-(1-e)e^{-x}\right]$

13. $\cos y\,dx+(1+e^{-x})\sin y\,dy=0,$

$y(0)=\dfrac{\pi}{4}$

$\left[\textbf{Ans.:}\ (1+e^x)\sec y=2\sqrt{2}\right]$

10.3.2 Homogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx}=\frac{M(x,y)}{N(x,y)} \qquad \dots (10.5)$$

is called a *homogeneous equation* if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree, i.e., degree of the RHS of Eq. (10.5) is zero.

Equation (10.5) can be reduced to variable-separable form by putting $y = vx$.

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Equation (10.5) reduces to

$$v+x\frac{dv}{dx}=\frac{M(x,vx)}{N(x,vx)}=g(v)$$

$$x\frac{dv}{dx}=g(v)-v$$

$$\frac{dv}{g(v)-v}=\frac{dx}{x}$$

The above equation is in variable-separable form and can be solved by integrating

$$\int \frac{dv}{g(v)-v} = \int \frac{dx}{x} + c$$

After integrating and replacing v by $\frac{y}{x}$, the solution of Eq. (10.5) is obtained.

Note *Homogeneous functions*: A function $f(x, y, z)$ is said to be a homogeneous function of degree n, if for any positive number t,

$$f(xt, yt, zt) = t^n f(x, y, z),$$

where n is a real number.

EXAMPLE 10.5 *Solve* $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$.

Solution:

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} = \frac{M(x, y)}{N(x, y)} \quad \ldots (1)$$

The equation is homogeneous since M and N are of the same degree 1.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x} = v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both the sides,

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\log\left(v + \sqrt{v^2+1}\right) = \log x + \log c = \log cx$$

$$v + \sqrt{v^2+1} = cx$$

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} = cx$$

$$y + \sqrt{y^2 + x^2} = cx^2$$

EXAMPLE 10.6 *Solve* $\frac{y}{x}\cos\frac{y}{x}dx - \left(\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}\right)dy = 0.$

Solution:

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\frac{y}{x}}{\frac{x}{y}\sin\frac{y}{x} + \cos\frac{y}{x}} = \frac{M(x,y)}{N(x,y)} \quad \ldots(1)$$

The equation is homogeneous since M and N are of the same degree 0.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v + x\frac{dv}{dx} = \frac{v\cos v}{\frac{1}{v}\sin v + \cos v}$$

$$x\frac{dv}{dx} = \frac{v\cos v}{\frac{1}{v}\sin v + \cos v} - v = \frac{-\sin v \cdot v}{\sin v + v\cos v}$$

$$\left(\frac{\sin v + v\cos v}{-v\sin v}\right)dv = \frac{dx}{x}$$

$$\left(\frac{1}{v} + \cot v\right)dv = -\frac{dx}{x}$$

Integrating both the sides,

$$\int\left(\frac{1}{v} + \cot v\right)dv = -\int\frac{dx}{x}$$

$$\log v + \log\sin v = -\log x + \log c$$

$$v\sin v = \frac{c}{x}$$

$$\frac{y}{x}\sin\frac{y}{x} = \frac{c}{x}$$

$$y\sin\frac{y}{x} = c$$

EXAMPLE 10.7 *Solve* $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0,\ y(1) = 0.$

Solution:

$$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \quad \ldots(1)$$

The equation is homogeneous since the degree of each term is same.

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting in Eq. (1),

$$v+x\frac{dv}{dx}=v-\text{cosec } v$$

$$x\frac{dv}{dx}=-\text{cosec } v$$

$$\sin v\, dv=-\frac{dx}{x}$$

Integrating both the sides,

$$\int \sin v\, dv=-\int\frac{dx}{x}$$

$$-\cos v=-\log x+c$$

$$\log x-\cos v=c$$

$$\log x-\cos\left(\frac{y}{x}\right)=c \qquad \ldots (2)$$

Given, $y(1)=0$
Putting $x=1, y=0$ in Eq. (2),

$$\log 1-\cos 0=c$$

$$c=-1$$

Hence, the solution is

$$\log x-\cos\left(\frac{y}{x}\right)=-1$$

EXERCISE 10.2

Solve the following differential equations:

1. $x(y-x)\frac{dy}{dx}=y(y+x)$

$\left[\textbf{Ans.: } \frac{y}{x}-\log xy=c\right]$

2. $x\frac{dy}{dx}=y(\log y-\log x+1)$

$\left[\textbf{Ans.: } \log\frac{y}{x}=cx\right]$

3. $y\,dx+x\log\frac{y}{x}dy-2x\,dy=0$

$\left[\textbf{Ans.: } y=c\left(1+\log\frac{x}{y}\right)\right]$

4. $\left(xe^{\frac{y}{x}}-y\sin\frac{y}{x}\right)dx+x\sin\frac{y}{x}dy=0$

$\left[\textbf{Ans.: } \log x^2-e^{-\frac{y}{x}}\left(\sin\frac{y}{x}+\cos\frac{y}{x}\right)=c\right]$

5. $x\dfrac{dy}{dx} = y + x\sec\left(\dfrac{y}{x}\right)$

$\left[\textbf{Ans.: } \sin\dfrac{y}{x} = \log(cx)\right]$

6. $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\dfrac{x}{y}\right)dy = 0$

$\left[\textbf{Ans.: } x + ye^{\frac{x}{y}} = c\right]$

7. $(3xy + y^2)dx + (x^2 + xy)dy = 0$

$y(1) = 1$

$\left[\textbf{Ans.: } x^2y\,(2x+y) = 3\right]$

8. $3x\dfrac{dy}{dx} - 3y + (x^2 - y^2)^{\frac{1}{2}} = 0,\ y\,(1) = 1$

$\left[\textbf{Ans.: } 3\cos^{-1}\left(\dfrac{y}{x}\right) - \log x = 0\right]$

9. $xy\log\dfrac{x}{y}dx + \left(y^2 - x^2\log\dfrac{x}{y}\right)dy = 0,$

$y\,(1) = e$

$\left[\textbf{Ans.: } \dfrac{x^2}{2y^2}\log\dfrac{x}{y} - \dfrac{x^2}{4y^2} + \log y = 1 - \dfrac{3}{4e^2}\right]$

10.3.3 Nonhomogeneous Differential Equations

A differential equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \ldots (10.6)$$

is called a nonhomogeneous equation, where $a_1, b_1, c_1, a_2, b_2, c_2$ are all constants. These equations are classified into two parts and can be solved by the following methods:

Case I If

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = m$$

$$a_1 = a_2m,\ b_1 = b_2m$$

then Eq. (10.6) reduces to

$$\frac{dy}{dx} = \frac{m(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2} \qquad \ldots (10.7)$$

Putting $a_2x + b_2y = t$, $a_2 + b_2\dfrac{dy}{dx} = \dfrac{dt}{dx}$, Eq. (10.7) reduces to variable-separable form and can be solved using the method of variable-separable equation.

Case II If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ then substituting $x = X + h,\ y = Y + k$ in Eq. (10.6),

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2} = \frac{(a_1X + b_1Y) + (a_1h + b_1k + c_1)}{(a_2X + b_2Y) + (a_2h + b_2k + c_2)} \qquad \ldots (10.8)$$

Choosing h, k such that

$$a_1h + b_1k + c_1 = 0,\ a_2h + b_2k + c_2 = 0$$

then Eq. (10.8) reduces to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is a homogeneous equation and can be solved using the method of homogeneous equations. Finally, substituting $X = x - h$, $Y = y - k$, the solution of Eq. (10.6) is obtained.

Problems Based on Case I $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

EXAMPLE 10.8 *Solve* $(x + y - 1)dx + (2x + 2y - 3)dy = 0.$

Solution:

$$\frac{dy}{dx} = -\frac{x+y-1}{2x+2y-3} = \frac{-x-y+1}{2x+2y-3} \quad \dots (1)$$

The equation is nonhomogeneous and $\frac{a_1}{a_2} = \frac{b_1}{b_2} = -\frac{1}{2}$

Let $x + y = t$

$$1 + \frac{dy}{dx} = \frac{dt}{dx}, \qquad \frac{dy}{dx} = \frac{dt}{dx} - 1$$

Substituting in Eq. (1),

$$\frac{dt}{dx} - 1 = \frac{-t+1}{2t-3}$$

$$\frac{dt}{dx} = \frac{-t+1}{2t-3} + 1 = \frac{-t+1+2t-3}{2t-3} = \frac{t-2}{2t-3}$$

$$\left(\frac{2t-3}{t-2}\right)dt = dx$$

$$\left(2 + \frac{1}{t-2}\right)dt = dx$$

Integrating both the sides,

$$\int\left(2 + \frac{1}{t-2}\right)dt = \int dx$$

$$2t + \log(t-2) = x + c$$

$$2(x+y) + \log(x+y-2) = x + c$$

$$x + 2y + \log(x+y-2) = c$$

Problems Based on Case II $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

EXAMPLE 10.9 *Solve* $(x+2y)\,dx+(y-1)\,dy=0.$

Solution:
$$\frac{dy}{dx}=\frac{-x-2y}{y-1} \qquad \cdots(1)$$

The equation is nonhomogeneous and $\frac{-1}{0} \neq \frac{-2}{1}$

Let $x = X+h, \qquad y = Y+k$

$dx = dX, \qquad dy = dY$

$$\frac{dy}{dx}=\frac{dY}{dX}$$

Substituting in Eq. (1),

$$\frac{dY}{dX}=\frac{-(X+h)-2(Y+k)}{(Y+k)-1}=\frac{(-X-2Y)+(-h-2k)}{Y+(k-1)} \qquad \cdots(2)$$

Choosing h, k such that

$$-h-2k=0,\ k-1=0 \qquad \cdots(3)$$

Solving these equations,

$$k=1,\ h=-2$$

Substituting Eq. (3) in Eq. (2),

$$\frac{dY}{dX}=\frac{-X-2Y}{Y} \qquad \cdots(4)$$

which is a homogeneous equation.
Let $Y = vX$

$$\frac{dY}{dX}=v+X\frac{dv}{dX}$$

Substituting in Eq. (4),

$$v+X\frac{dv}{dX}=\frac{-X-2vX}{vX}=\frac{-1-2v}{v}$$

$$X\frac{dv}{dX}=\frac{-1-2v}{v}-v=\frac{-1-2v-v^2}{v}=\frac{-(v+1)^2}{v}$$

$$\frac{v}{(v+1)^2}\,dv=-\frac{dX}{X}$$

$$\left[\frac{1}{v+1}-\frac{1}{(v+1)^2}\right]dv=-\frac{dX}{X}$$

Integrating both the sides,

$$\int \frac{1}{v+1}\mathrm{d}v - \int \frac{1}{(v+1)^2}\mathrm{d}v = -\int \frac{\mathrm{d}X}{X}$$

$$\log(v+1) + \frac{1}{v+1} = -\log X + c$$

$$\log\left(\frac{Y}{X}+1\right) + \frac{1}{\frac{Y}{X}+1} = -\log X + c$$

$$\log\left(\frac{Y+X}{X}\right) + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) - \log X + \frac{X}{Y+X} = -\log X + c$$

$$\log(Y+X) + \frac{X}{Y+X} = c$$

Now,

$$X = x - h = x + 2$$
$$Y = y - k = y - 1$$

Hence, the solution is

$$\log(x+y+1) + \left(\frac{x+2}{x+y+1}\right) = c$$

EXERCISE 10.3

Solve the following differential equations:

1. $(x+2y)\mathrm{d}x + (3x+6y+3)\mathrm{d}y = 0$

   [**Ans.:** $x+3y-3\log|x+2y+3| = c$]

2. $(6x-4y+1)\mathrm{d}y - (3x-2y+1)\mathrm{d}x = 0$

   [**Ans.:** $4x-8y-\log(12x-8y+1) = c$]

3. $(x+y+3)\mathrm{d}y = (x+y-3)\mathrm{d}x$

   [**Ans.:** $-x+y-3\log(x+y) = c$]

4. $(x+y+3)\mathrm{d}x - (2x+2y-1)\mathrm{d}y = 0$

   [**Ans.:** $-3x+6y-7\log|3x+3y+2| = c$]

5. $(y-x+2)\mathrm{d}y = (y-x)\mathrm{d}x$

   [**Ans.:** $(y-x)^2+4y = c$]

6. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{2x-y+1}{x+y}$

   $\left[\textbf{Ans.: } \log\left[2\left(x+\frac{1}{3}\right)^2 + \left(y-\frac{1}{3}\right)^2\right] + \sqrt{2}\tan^{-1}\left[\frac{3y-1}{\sqrt{2}(3x+1)}\right] = c\right]$

7. $(x+y-1)\mathrm{d}x - (x-y-1)\mathrm{d}y = 0$

   $\left[\textbf{Ans.: } \log[(x-1)^2+y^2] - 2\tan^{-1}\left(\frac{y}{x-1}\right) = c\right]$

8. $(x-y-1)dx+(4y+x-1)dy=0$

$$\left[\textbf{Ans.: } \log[4y^2+(x-1)^2] + \tan^{-1}\left(\frac{2y}{x-1}\right)=c\right]$$

9. $(y-x+2)dx+(x+y+6)dy=0$

$$\left[\textbf{Ans.: } (y+4)^2+2(x+2)(y+4) -(x+2)^2=c\right]$$

10. $\dfrac{dy}{dx}=\dfrac{2x+9y-20}{6x+2y-10}$

$$[\textbf{Ans.: } (2x-y)^2=c(x+2y-5)]$$

11. $(3x+2y+3)dx-(x+2y-1)dy=0,$ $y(-2)=1$

$$[\textbf{Ans.: } (2x+2y+1)(3x-2y+9)^4=-1]$$

12. $(x+y+2)dx-(x-y-4)dy=0,$ $y(1)=0$

$$\left[\textbf{Ans.: } \log\left[(x-1)^2+(y+3)^2\right] +2\tan^{-1}\left(\frac{x-1}{y+3}\right)=2\log 3\right]$$

10.3.4 Exact Differential Equations

Any first-order differential equation which is obtained by differentiation of its general solution without any elimination or reduction of terms is known as an *exact differential equation*.

If $f(x, y) = c$ is the general solution

then
$$df=0$$
$$\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0$$
$$M(x,y)dx+N(x,y)dy=0 \qquad \dots (10.9)$$

represents an exact differential equation

where
$$M(x,y)=\frac{\partial f}{\partial x}, \qquad N(x,y)=\frac{\partial f}{\partial y}$$
$$\frac{\partial M}{\partial y}=\frac{\partial^2 f}{\partial y\partial x}, \qquad \frac{\partial N}{\partial x}=\frac{\partial^2 f}{\partial x\partial y}$$

But,
$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$$

Hence,
$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Thus, the necessary condition for a differential equation to be an exact differential equation is
$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

The solution of Eq. (10.9) can be written as

$$\int_{y\ \text{constant}} M(x, y)dx + \int(\text{terms of N not containing } x)dy = c$$

Sometimes, integration of M w.r.t. x is tedious, whereas N can be integrated easily w.r.t. y. In this case, the solution can be written as

$$\int(\text{terms of M not containing } y)dx + \int_{x\ \text{constant}} N(x, y)dy = c$$

EXAMPLE 10.10 *Solve* $\left(x\sqrt{1-x^2y^2}-y\right)dy+\left(x+y\sqrt{1-x^2y^2}\right)dx=0.$

Solution: $N = x\sqrt{1-x^2y^2}-y,\qquad M = x+y\sqrt{1-x^2y^2}$

$$\frac{\partial N}{\partial x} = \sqrt{1-x^2y^2}+x\left[\frac{-2xy^2}{2\sqrt{1-x^2y^2}}\right], \qquad \frac{\partial M}{\partial y} = \sqrt{1-x^2y^2}+y\left[\frac{-2x^2y}{2\sqrt{1-x^2y^2}}\right]$$

$$= \sqrt{1-x^2y^2}-\frac{x^2y^2}{\sqrt{1-x^2y^2}} \qquad = \sqrt{1-x^2y^2}-\frac{x^2y^2}{\sqrt{1-x^2y^2}}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int_{y\ \text{constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int\left(x+y\sqrt{1-x^2y^2}\right)dx+\int(-y)\,dy=c$$

$$\frac{x^2}{2}+y^2\int\left(\sqrt{\frac{1}{y^2}-x^2}\right)dx-\frac{y^2}{2}=c$$

$$\frac{x^2}{2}+y^2\left[\frac{x}{2}\sqrt{\frac{1}{y^2}-x^2}+\frac{1}{2y^2}\sin^{-1}\left(\frac{x}{\frac{1}{y}}\right)\right]-\frac{y^2}{2}=c$$

$$\frac{x^2-y^2}{2}+\frac{xy}{2}\sqrt{1-x^2y^2}+\frac{1}{2}\sin^{-1}(xy)=c$$

$$x^2-y^2+xy\sqrt{1-x^2y^2}+\sin^{-1}(xy)=2c=k$$

EXAMPLE 10.11 *Solve* $\left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0,\ y(0) = 4.$

Solution: $M = 1+e^{\frac{x}{y}}, \qquad N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$

$$\frac{\partial M}{\partial y} = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right), \qquad \frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(\frac{1}{y}\right)\left(1-\frac{x}{y}\right)+e^{\frac{x}{y}}\left(-\frac{1}{y}\right)$$

$$= -\frac{x}{y^2}e^{\frac{x}{y}}, \qquad = -\frac{x}{y^2}e^{\frac{x}{y}}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int\limits_{y\text{ constant}} M\,dx + \int\limits_{\text{terms not containing }x} N\,dy = c$$

$$\int\left(1+e^{\frac{x}{y}}\right)dx + \int 0\,dy = c$$

$$x + \frac{e^{\frac{x}{y}}}{\frac{1}{y}} = c$$

$$x + ye^{\frac{x}{y}} = c \qquad \dots (1)$$

Given, $y(0) = 4$

Substituting in Eq. (1),

$$0 + 4e^0 = c$$

$$4 = c$$

Hence, the solution is

$$x + ye^{\frac{x}{y}} = 4$$

EXAMPLE 10.12 *Solve* $\left[\log(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right]dx + \frac{2xy}{x^2+y^2}dy = 0.$

Solution: $M = \left[\log(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right], \qquad N = \frac{2xy}{x^2+y^2}$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y - \frac{2x^2}{(x^2+y^2)^2}\cdot 2y, \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2+y^2} - \frac{2xy}{(x^2+y^2)^2}\cdot 2x$$

$$= \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2}, \qquad = \frac{2y}{x^2+y^2} - \frac{4x^2y}{(x^2+y^2)^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int\limits_{\text{terms not containing } y} M\,dx + \int\limits_{x \text{ constant}} N\,dy = c$$

$$\int 0\,dx + \int \frac{2xy}{x^2 + y^2}\,dy = c$$

$$x\log(x^2 + y^2) = c$$

EXAMPLE 10.13 *For what values of a and b is the differential equation* $(y + x^3)dx + (ax + by^3)dy = 0$ *exact? Also, find the solution of the equation.*

Solution:

$$M = y + x^3, \qquad N = ax + by^3$$

$$\frac{\partial M}{\partial y} = 1, \qquad \frac{\partial N}{\partial x} = a$$

The equation will be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = a$$

Hence, the equation is exact for $a = 1$ and for all values of b.
Substituting $a = 1$ in the equation, $(y + x^3)\,dx + (x + by^3)\,dy = 0$, which is exact.
Hence, the solution is

$$\int\limits_{y \text{ constant}} M\,dx + \int\limits_{\text{terms not containing } x} N\,dy = c$$

$$\int (y + x^3)\,dx + \int by^3\,dy = c$$

$$xy + \frac{x^4}{4} + \frac{by^4}{4} = c$$

EXAMPLE 10.14 *Solve* $(\cos x + y\sin x)\,dx = (\cos x)dy,\ y(\pi) = 0.$

Solution: $(\cos x + y\sin x)\,dx - (\cos x)\,dy = 0$

$$M = \cos x + y\sin x, \qquad N = -\cos x$$

$$\frac{\partial M}{\partial y} = \sin x, \qquad \frac{\partial N}{\partial x} = \sin x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int_{y\text{ constant}} M\,dx + \int_{\text{terms not containing } x} N\,dy = c$$

$$\int (\cos x + y\sin x)\,dx + \int 0\,dy = c$$

$$\sin x - y\cos x = c \qquad \dots(1)$$

Given, $y(\pi) = 0$

Substituting $x = \pi$, $y = 0$ in Eq. (1),

$$\sin\pi - 0 = c$$

$$0 = c$$

Hence, the solution is

$$\sin x - y\cos x = 0$$

$$y = \tan x$$

EXERCISE 10.4

Solve the following differential equations:

1. $(2x^3 + 3y)\,dx + (3x + y - 1)\,dy = 0$

$\left[\textbf{Ans.: } x^4 + 6xy + y^2 - 2y = c\right]$

2. $\sinh x \cos y\,dx - \cosh x \sin y\,dy = 0$

$\left[\textbf{Ans.: } \cosh x \cos y = c\right]$

3. $2x(ye^{x^2} - 1)\,dx + e^{x^2}\,dy = 0$

$\left[\textbf{Ans.: } ye^{x^2} - x^2 = c\right]$

4. $\left(1 + x^2\sqrt{y}\right)y\,dx + \left(x^2\sqrt{y} + 2\right)x\,dy = 0$

$\left[\textbf{Ans.: } 2xy + \frac{2}{3}x^3y^{\frac{3}{2}} = c\right]$

5. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$\left[\textbf{Ans.: } \sin x(e^y + 1) = c\right]$

6. $\dfrac{dy}{dx} = \dfrac{y+1}{(y+2)e^y - x}$

$\left[\textbf{Ans.: } (y+1)(x - e^y) = c\right]$

7. $(x - y\cos x)\,dx - \sin x\,dy = 0$, $y\left(\frac{\pi}{2}\right) = 1$

$\left[\textbf{Ans.: } x^2 - 2y\sin x = \frac{\pi^2}{4} - 2\right]$

8. $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0$, $y(1) = 1$

$\left[\textbf{Ans.: } x^2y + xe^y = e + 1\right]$

9. $xe^{x^2+y^2}\,dx + y(1 + e^{x^2+y^2})\,dy = 0$, $y(0) = 0$

$\left[\textbf{Ans.: } y^2 + e^{x^2+y^2} = 1\right]$

10. $\left(4x^3y^3 + \frac{1}{x}\right)dx + \left(3x^4y^2 - \frac{1}{y}\right)dy = 0$, $y(1) = 1$

$\left[\textbf{Ans.: } x^4y^3 + \log\left(\frac{x}{y}\right) = 1\right]$

10.3.5 Non-Exact Differential Equations Reducible to Exact Form

Sometimes, a differential equation is not exact but can be made exact by multiplying with a suitable function. This function is known as *Integrating Factor* (IF). There may exist more than one integrating factor to a differential equation.

Here, different methods are discussed to find an IF to a non-exact differential equation,

$$M\,dx + N\,dy = 0$$

Case I If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ (function of x alone) then $\text{IF} = e^{\int f(x)dx}$

After multiplication with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

EXAMPLE 10.15 *Solve* $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0.$

Solution:

$$M = xy^2 - e^{\frac{1}{x^3}}, \qquad N = -x^2y$$

$$\frac{\partial M}{\partial y} = 2xy, \qquad \frac{\partial N}{\partial x} = -2xy$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy-(-2xy)}{-x^2y} = -\frac{4}{x}$$

$$\text{IF} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4} = \frac{1}{x^4}$$

Multiplying the DE by the IF,

$$\frac{1}{x^4}(xy^2 - e^{\frac{1}{x^3}})dx - \frac{1}{x^4}(x^2y)dy = 0$$

$$\left(\frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}\right)dx - \frac{y}{x^2}dy = 0$$

$$M_1 = \frac{y^2}{x^3} - \frac{e^{\frac{1}{x^3}}}{x^4}, \qquad N_1 = -\frac{y}{x^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{2y}{x^3}, \qquad \frac{\partial N_1}{\partial x} = \frac{2y}{x^3}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int\limits_{y\text{ constant}} M_1 dx + \int\limits_{\text{terms not containing } x} N_1 dy = c$$

$$\int\left(\frac{y^2}{x^3}-\frac{e^{\frac{1}{x^3}}}{x^4}\right)dx+\int 0\,dy=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}\int e^{\frac{1}{x^3}}\left(-\frac{3}{x^4}\right)dx=c$$

$$-\frac{y^2}{2x^2}+\frac{1}{3}e^{\frac{1}{x^3}}=c \qquad \left[\because \int e^{f(x)}f'(x)dx=e^{f(x)}+c\right]$$

EXAMPLE 10.16 *Solve* $x\sin x\frac{dy}{dx}+y(x\cos x-\sin x)=2.$

Solution:

$$x\sin x\,dy+(xy\cos x-y\sin x-2)dx=0$$

$$(xy\cos x-y\sin x-2)dx+x\sin x\,dy=0$$

$$M=xy\cos x-y\sin x-2 \qquad N=x\sin x$$

$$\frac{\partial M}{\partial y}=x\cos x-\sin x \qquad \frac{\partial N}{\partial x}=\sin x+x\cos x$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{(x\cos x-\sin x)-(\sin x+x\cos x)}{x\sin x}=-\frac{2\sin x}{x\sin x}=-\frac{2}{x}$$

$$IF=e^{\int -\frac{2}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=x^{-2}=\frac{1}{x^2}$$

Multiplying the DE by the IF,

$$\frac{1}{x^2}(xy\cos x-y\sin x-2)dx+\frac{1}{x^2}(x\sin x)dy=0$$

$$\left(\frac{y}{x}\cos x-\frac{y}{x^2}\sin x-\frac{2}{x^2}\right)dx+\frac{1}{x}\sin x\,dy=0$$

$$M_1=\frac{y}{x}\cos x-\frac{y}{x^2}\sin x-\frac{2}{x^2}, \qquad N_1=\frac{\sin x}{x}$$

$$\frac{\partial M_1}{\partial y}=\frac{\cos x}{x}-\frac{\sin x}{x^2}, \qquad \frac{\partial N_1}{\partial x}=\frac{\cos x}{x}-\frac{\sin x}{x^2}$$

Since $\dfrac{\partial M_1}{\partial y} = \dfrac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the solution is

$$\underset{\text{terms not containing } y}{\int M_1\, dx} + \underset{x \text{ constant}}{\int N_1\, dy} = c$$

$$\int -\frac{2}{x^2}\, dx + \int \frac{\sin x}{x}\, dy = c$$

$$\frac{2}{x} + \left(\frac{\sin x}{x}\right) y = c$$

$$\frac{2}{x} + \frac{y \sin x}{x} = c$$

Case II If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = f(y)$ (function of y alone) then $IF = e^{\int f(y)dy}$

After multiplying with the IF, the equation becomes exact and can be solved using the method of exact differential equations.

EXAMPLE 10.17 *Solve* $\left(\dfrac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)dy = 0.$

Solution:

$$M = \frac{y}{x}\sec y - \tan y, \qquad N = \sec y \log x - x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x}\sec y + \frac{y}{x}\sec y \tan y - \sec^2 y, \qquad \frac{\partial N}{\partial x} = \frac{\sec y}{x} - 1$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the equation is not exact.

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{\dfrac{\sec y}{x} - 1 - \dfrac{\sec y}{x} - \dfrac{y}{x}\sec y \tan y + \sec^2 y}{\dfrac{y}{x}\sec y - \tan y} = \frac{-\dfrac{y}{x}\sec y \tan y + \tan^2 y}{\dfrac{y}{x}\sec y - \tan y} = -\tan y$$

$$IF = e^{\int -\tan y\, dy} = e^{-\log \sec y} = e^{\log(\sec y)^{-1}} = (\sec y)^{-1} = \cos y$$

Multiplying the DE by the IF,

$$\cos y\left(\frac{y}{x}\sec y - \tan y\right)dx + \cos y(\sec y \log x - x)dy = 0$$

$$\left(\frac{y}{x} - \sin y\right)dx + (\log x - x\cos y)dy = 0$$

$$\frac{3}{7x}dx+\frac{4}{7y}dy=0$$

$$M_1=\frac{3}{7x},\qquad N_1=\frac{4}{7y}$$

$$\frac{\partial M_1}{\partial y}=0,\qquad \frac{\partial N_1}{\partial x}=0$$

Since $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int_{y\text{ constant}} M_1 dx+\int_{\text{terms not containing }x} N_1 dy=c$$

$$\int\frac{3}{7x}dx+\int\frac{4}{7y}dy=\log c$$

$$\frac{3}{7}\log x+\frac{4}{7}\log y=\log c$$

$$\log x^{\frac{3}{7}}+\log y^{\frac{4}{7}}=\log c$$

$$\log\left(x^{\frac{3}{7}}y^{\frac{4}{7}}\right)=\log c$$

$$x^{\frac{3}{7}}y^{\frac{4}{7}}=c \qquad \text{...(1)}$$

Given, $y(1)=1$

Substituting $x=1$, $y=1$ in Eq. (1),

$$(1)^{\frac{3}{7}}\cdot(1)^{\frac{4}{7}}=c,\qquad 1=c$$

Hence, the solution is

$$x^{\frac{3}{7}}y^{\frac{4}{7}}=1$$

Case V If the differential equation is of the type

$$x^{m_1}y^{n_1}(a_1y\,dx+b_1x\,dy)+x^{m_2}y^{n_2}(a_2y\,dx+b_2x\,dy)=0$$

then IF $=x^h y^k$

where $$\frac{m_1+h+1}{a_1}=\frac{n_1+k+1}{b_1}$$

and $$\frac{m_2+h+1}{a_2}=\frac{n_2+k+1}{b_2}$$

Solving these two equations, the values of h and k are obtained.

EXAMPLE 10.20 *Solve* $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0.$

Solution: $M = x^7y^2 + 3y, \qquad N = 3x^8y - x$

$$\frac{\partial M}{\partial y} = 2x^7y + 3, \qquad \frac{\partial N}{\partial x} = 24x^7y - 1$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, the equation is not exact.

Rewriting the equation,

$$x^7y^2\,dx + 3x^8y\,dy + 3y\,dx - x\,dy = 0$$

$$x^7y(y\,dx + 3x\,dy) + (3y\,dx - x\,dy) = 0$$

$$m_1 = 7, n_1 = 1, a_1 = 1, b_1 = 3, m_2 = 0, n_2 = 0, a_2 = 3, \; b_2 = -1$$

$$\frac{m_1 + h + 1}{a_1} = \frac{n_1 + k + 1}{b_1}$$

$$\frac{7 + h + 1}{1} = \frac{1 + k + 1}{3}$$

$$3h + 24 = k + 2$$

$$3h - k = -22 \qquad \dots (1)$$

and

$$\frac{m_2 + h + 1}{a_2} = \frac{n_2 + k + 1}{b_2}$$

$$\frac{0 + h + 1}{3} = \frac{0 + k + 1}{-1}$$

$$-h - 1 = 3k + 3 \qquad \dots (2)$$

$$h + 3k = -4$$

Solving Eqs (1) and (2),

$$h = -7, k = 1$$

$$\text{IF} = x^{-7}y$$

Multiplying the DE by the IF,

$$x^{-7}y(x^7y^2 + 3y)dx + x^{-7}y(3x^8y - x)dy = 0$$

$$(y^3 + 3x^{-7}y^2)dx + (3xy^2 - x^{-6}y)dy = 0$$

$$M_1 = y^3 + 3x^{-7}y^2, \qquad N_1 = 3xy^2 - x^{-6}y$$

$$\frac{\partial M_1}{\partial y} = 3y^2 + 6x^{-7}y, \qquad \frac{\partial N_1}{\partial x} = 3y^2 + 6x^{-7}y$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int_{y\ \text{constant}} M_1 dx + \int_{\text{terms not containing } x} N_1 dy = c$$

$$\int (y^3 + 3x^{-7}y^2)dx + \int 0\,dy = c$$

$$xy^3 + \frac{3x^{-6}y^2}{-6} = c$$

$$xy^3 - \frac{x^{-6}y^2}{2} = c$$

Case VI *Integrating Factors by Inspection*

Sometimes the integrating factor can be identified by regrouping the terms of the differential equation. The following table helps in identifying the IF after regrouping the terms.

S. No.	Group of Terms	Integrating Factor	Exact Differential Equation
1.	$dx \pm dy$	$\frac{1}{x \pm y}$	$\frac{dx \pm dy}{x \pm y} = d[\log(x \pm y)]$
2.	$y\,dx + x\,dy$	$\frac{1}{2xy}$	$y\,dx + x\,dy = d(xy)$ $2x^2y\,dy + 2xy^2dx = d(x^2y^2)$
		$\frac{1}{xy}$	$\frac{y\,dx + x\,dy}{xy} = d[\log(xy)]$
		$\frac{1}{(xy)^n}$	$\frac{y\,dx + x\,dy}{(xy)^n} = d\left[\frac{(xy)^{1-n}}{1-n}\right], n \neq 1$
3.	$y\,dx - x\,dy$	$\frac{1}{y^2}$	$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$
		$\frac{1}{x^2 + y^2}$	$\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$
		$\frac{1}{x^2}$	$\frac{y\,dx - x\,dy}{x^2} = d\left(-\frac{y}{x}\right)$
		$\frac{1}{xy}$	$\frac{y\,dx - x\,dy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right]$

S. No.	Group of Terms	Integrating Factor	Exact Differential Equation
4.	$x\,dx \pm y\,dy$	2	$2x\,dx \pm 2y\,dy = d(x^2 \pm y^2)$
		$\dfrac{1}{(x^2 \pm y^2)}$	$\dfrac{2x\,dx \pm 2y\,dy}{x^2 \pm y^2} = d[\log(x^2 \pm y^2)]$
		$\dfrac{1}{(x^2 \pm y^2)^n}$	$\dfrac{2x\,dx \pm 2y\,dy}{(x^2 \pm y^2)^n} = d\left[\dfrac{(x^2 \pm y^2)^{1-n}}{2(1-n)}\right]$
5.	$2y\,dx + x\,dy$	x	$2xy\,dx + x^2\,dy = d(x^2y)$
6.	$y\,dx + 2x\,dy$	y	$y^2\,dx + 2xy\,dy = d(xy^2)$
7.	$2y\,dx - x\,dy$	$\dfrac{x}{y^2}$	$\dfrac{2xy\,dx - x^2dy}{y^2} = d\left(\dfrac{x^2}{y}\right)$
8.	$2x\,dy - y\,dx$	$\dfrac{y}{x^2}$	$\dfrac{2xy\,dy - y^2dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$

EXAMPLE 10.21 *Solve* $(1+xy)y\,dx + (1-xy)x\,dy = 0$.

Solution: $y\,dx + xy^2\,dx + x\,dy - x^2y\,dy = 0$

Regrouping the terms,

$$(y\,dx + x\,dy) + (xy^2dx - x^2y\,dy) = 0$$

Dividing the equation by x^2y^2,

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

$$d\left(-\frac{1}{xy}\right) + \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating both the sides,

$$-\frac{1}{xy} + \log x - \log y = c$$

$$-\frac{1}{xy} + \log\frac{x}{y} = c$$

EXAMPLE 10.22 *Solve* $y(x^3e^{xy} - y)dx + x(y + x^3e^{xy})dy = 0.$

Solution: $x^3ye^{xy}dx - y^2dx + xy\,dy + x^4e^{xy}dy = 0$

Regrouping the terms,

$$x^3ye^{xy}dx + x^4e^{xy}dy - y^2dx + xy\,dy = 0$$

Dividing the equation by x^3,

$$ye^{xy}dx + xe^{xy}dy - \frac{1}{2}\left(\frac{y^2\cdot 2x\,dx - x^2\cdot 2y\,dy}{x^4}\right) = 0$$

$$d(e^{xy}) + \frac{1}{2}d\left(\frac{y^2}{x^2}\right) = 0$$

Integrating both the sides,

$$e^{xy} + \frac{1}{2}\frac{y^2}{x^2} = c$$

EXAMPLE 10.23 *If x^n is an integrating factor of $(y - 2x^3)dx - x(1 - xy)dy = 0$ then find n and solve the equation.*

Solution: If x^n is an IF then after multiplication with x^n, the equation becomes exact.

$$(x^ny - 2x^{n+3})dx - x^{n+1}(1 - xy)dy = 0 \text{ is an exact DE}$$

where $$M = x^ny - 2x^{n+3},\ N = -x^{n+1} + x^{n+2}y$$

and $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$x^n = -(n+1)x^n + (n+2)x^{n+1}y$$

$$(n+2)x^n(1 + xy) = 0$$

$$n + 2 = 0$$

$$n = -2$$

Putting $n = -2$ in the equation,

$$(x^{-2}y - 2x)dx - x^{-1}(1 - xy)dy = 0$$

$$\left(\frac{y}{x^2} - 2x\right)dx - \left(\frac{1}{x} - y\right)dy = 0$$

$$M = \frac{y}{x^2} - 2x, \qquad N = -\frac{1}{x} + y$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2}, \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, the solution is

$$\int\limits_{y\text{ constant}} M\,dx + \int\limits_{\text{terms not containing } x} N\,dy = c$$

$$\int \left(\frac{y}{x^2} - 2x\right) dx + \int y\,dy = c$$

$$-\frac{y}{x} - x^2 + \frac{y^2}{2} = c$$

EXERCISE 10.5

Solve the following differential equations:

1. $(x^2 + y^2 + x)\,dx + xy\,dy = 0$

   $\left[\textbf{Ans.:}\ 3x^4 + 4x^3 + 6x^2y^2 = c\right]$

2. $(y - 2x^3)\,dx - (x - x^2y)\,dy = 0$

   $\left[\textbf{Ans.:}\ xy^2 - 2y - 2x^3 = cx\right]$

3. $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

   $\left[\textbf{Ans.:}\ x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c\right]$

4. $\left(2x\sinh\dfrac{y}{x} + 3y\cosh\dfrac{y}{x}\right)dx - \left(3x\cosh\dfrac{y}{x}\right)dy = 0$

   $\left[\textbf{Ans.:}\ -3\sinh\dfrac{y}{x} = cx^{\frac{2}{3}}\right]$

5. $(x\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)\,dx$

   $\left[\textbf{Ans.:}\ \dfrac{\tan y}{x} + x^3 - \sin y = c\right]$

6. $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

   $\left[\textbf{Ans.:}\ e^x(x^2 + y^2) = c\right]$

7. $y(xy + e^x)\,dx - e^x\,dy = 0$

   $\left[\textbf{Ans.:}\ \dfrac{x^2}{2} + \dfrac{e^x}{y} = c\right]$

8. $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$

   $\left[\textbf{Ans.:}\ x^3e^y + x + \dfrac{x}{y} = c\right]$

9. $y(x^2y + e^x)\,dx - e^x\,dy = 0$

   $\left[\textbf{Ans.:}\ \dfrac{x^3}{3} + \dfrac{e^x}{y} = c\right]$

10. $(2x^2y + e^x)y\,dx - (e^x + y^3)\,dy = 0$

    $\left[\textbf{Ans.:}\ 4x^3y - 3y^3 + 6e^x = cy\right]$

11. $y\log y\,dx + (x - \log y)\,dy = 0$

    $\left[\textbf{Ans.:}\ 2x\log y = c + (\log y)^2\right]$

12. $2xy\,dx + (y^2 - x^2)\,dy = 0$

$$\left[\textbf{Ans.: } x^2 + y^2 = cy\right]$$

13. $\dfrac{dy}{dx} = -\dfrac{x^2y^3 + 2y}{2x - 2x^3y^2}$

$$\left[\textbf{Ans.: } \frac{1}{3}\log\frac{x}{y^2} - \frac{1}{3x^2y^2} = c\right]$$

14. $y(x+y)\,dx - x(y-x)\,dy = 0$

$$\left[\textbf{Ans.: } \log\sqrt{xy} - \frac{y}{2x} = c\right]$$

15. $3y\,dx + 2x\,dy = 0,\ y(1) = 1$

$$\left[\textbf{Ans.: } yx^{\frac{3}{2}} = 1\right]$$

16. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

$$\left[\textbf{Ans.: } -\frac{2}{3}x^{-\frac{3}{2}}y^{\frac{3}{2}} + 4x^{\frac{1}{2}}y^{\frac{1}{2}} = c\right]$$

17. $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$

$$\left[\textbf{Ans.: } \frac{7}{5}x^{\frac{10}{7}}y^{-\frac{5}{7}} - \frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}} = c\right]$$

10.3.6 Linear Differential Equations (Leibnitz's Linear Equations)

If each term in a differential equation, including the derivative, is linear in terms of the dependent variable then the equation is called *linear*.

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (10.10)}$$

where P and Q are functions of x, is called a linear differential equation and is linear in y.

To solve Eq. (10.10), obtain the integrating factor (IF) as

$$\text{IF} = e^{\int P dx}$$

Multiplying Eq. (10.10) by the IF,

$$e^{\int P dx}\frac{dy}{dx} + Pe^{\int P dx}y = Qe^{\int P dx}$$

$$\frac{d}{dx}\left[e^{\int P dx}y\right] = Qe^{\int P dx}$$

Integrating w.r.t. x,

$$e^{\int P dx}y = \int Qe^{\int P dx}dx + c$$

or

$$(\text{IF})\,y = \int (\text{IF})Q + c \qquad \text{... (10.11)}$$

Equation (10.11) is the solution of the differential equation (10.10).

The standard form of a linear equation of the first order is also commonly known as Leibnitz's linear equation.

EXAMPLE 10.24 *Solve* $x(x-1)\frac{dy}{dx}-(x-2)y = x^3(2x-1)$.

Solution: $\frac{dy}{dx}-\frac{(x-2)}{x(x-1)}y = \frac{x^2(2x-1)}{(x-1)}$

The equation is linear in y.

$$P = -\frac{x-2}{x(x-1)}, \qquad Q = \frac{x^2(2x-1)}{x-1}$$

$$= -\left(\frac{2}{x}-\frac{1}{x-1}\right) \qquad \text{[Using partial fraction]}$$

$$\text{IF} = e^{\int\left(-\frac{2}{x}+\frac{1}{x-1}\right)dx} = e^{-2\log x+\log(x-1)} = e^{\log\left(\frac{x-1}{x^2}\right)} = \frac{x-1}{x^2}$$

Hence, the solution is

$$\left(\frac{x-1}{x^2}\right)\cdot y = \int\left(\frac{x-1}{x^2}\right)\cdot x^2\left(\frac{2x-1}{x-1}\right)dx + c = x^2 - x + c$$

$$y = \frac{x^3(x-1)}{x-1}+\frac{cx^2}{x-1}$$

$$y = x^3+\frac{cx^2}{x-1}$$

EXAMPLE 10.25 *Solve* $y\log y\,dx+(x-\log y)\,dy = 0$.

Solution: Rewriting the equation,

$$y\log y\frac{dx}{dy}+x-\log y = 0$$

$$\frac{dx}{dy}+\left(\frac{1}{y\log y}\right)x = \frac{1}{y}$$

The equation is linear in x.

$$P = \frac{1}{y\log y}, \quad Q = \frac{1}{y}$$

$$\text{IF} = e^{\int\frac{1}{y\log y}dy} = e^{\log(\log y)} \qquad \left[\because \int\frac{f'(y)}{f(y)}dy = \log f(y)\right]$$

$$= \log y$$

Hence, the solution is

$$(\log y)x = \int (\log y)\frac{1}{y}\,dy + c$$

$$x\log y = \frac{(\log y)^2}{2} + c$$

EXAMPLE 10.26 *Solve* $(1+\sin y)\,dx = (2y\cos y - x\sec y - x\tan y)\,dy$.

Solution: Rewriting the equation,

$$(1+\sin y)\frac{dx}{dy} = 2y\cos y - (\sec y + \tan y)x$$

$$(1+\sin y)\frac{dx}{dy} + \left(\frac{1+\sin y}{\cos y}\right)x = 2y\cos y$$

$$\frac{dx}{dy} + \left(\frac{1}{\cos y}\right)x = \frac{2y\cos y}{1+\sin y}$$

The equation is linear in x.

$$P = \frac{1}{\cos y},\quad Q = \frac{2y\cos y}{1+\sin y}$$

$$\text{IF} = e^{\int \frac{1}{\cos y}dy} = e^{\int \sec y\,dy} = e^{\log(\sec y + \tan y)} = \sec y + \tan y$$

Hence, the solution is

$$(\sec y + \tan y)x = \int (\sec y + \tan y)\left(\frac{2y\cos y}{1+\sin y}\right)dy + c$$

$$= 2\int \left(\frac{1+\sin y}{\cos y}\right)\left(\frac{y\cos y}{1+\sin y}\right)dy + c = 2\int y\,dy + c$$

$$(\sec y + \tan y)x = y^2 + c$$

EXAMPLE 10.27 *Solve* $L\frac{di}{dt} + iR = \sin\omega t,\quad t \geq 0,\quad i(0) = 0$, *where* R, ω, *and* L *are constants.*

Solution: Rewriting the equation,

$$\frac{di}{dt} + \frac{R}{L}i = \frac{\sin\omega t}{L}$$

The equation is linear in i.

$$P = \frac{R}{L}, \quad Q = \frac{\sin \omega t}{L}$$

$$IF = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

The solution is

$$e^{\frac{R}{L}t} \cdot i = \int e^{\frac{R}{L}t} \cdot \frac{\sin \omega t}{L} dt + c$$

$$e^{\frac{R}{L}t} \cdot i = \frac{1}{L}\left[\frac{e^{\frac{R}{L}t}}{\frac{R^2}{L^2}+\omega^2}\left(\frac{R}{L}\sin \omega t - \omega \cos \omega t\right)\right] + c = \frac{e^{\frac{R}{L}t} L}{R^2+\omega^2 L^2}\left(\frac{R}{L}\sin \omega t - \omega \cos \omega t\right) + c$$

$$i = \frac{1}{R^2+\omega^2 L^2}(R\sin \omega t - \omega L \cos \omega t) + ce^{-\frac{R}{L}t} \qquad \ldots (1)$$

Given, $i(0) = 0$

Putting $i = 0, t = 0$ in Eq. (1),

$$0 = \frac{1}{R^2+\omega^2 L^2}(0 - \omega L) + ce^0$$

$$c = \frac{\omega L}{R^2+\omega^2 L^2}$$

Hence, the solution is

$$i = \frac{1}{R^2+\omega^2 L^2}(R\sin \omega t - \omega L \cos \omega t) + \frac{\omega L}{R^2+\omega^2 L^2} e^{-\frac{R}{L}t}$$

EXAMPLE 10.28 *If* $\frac{dy}{dx} + y\tan x = \sin 2x$, $y(0) = 0$, *show that the maximum value of* y *is* $\frac{1}{2}$.

Solution: The equation is linear in y.

$$P = \tan x, \ Q = \sin 2x$$

$$IF = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$$

Hence, the solution is

$$(\sec x)y = \int \sec x \cdot \sin 2x \, dx + c = \int \sec x \cdot 2\sin x \cos x \, dx + c = 2\int \sin x \, dx + c$$

$$y \sec x = -2\cos x + c$$

$$y = -2\cos^2 x + c\cos x \qquad \ldots (1)$$

Given, $y(0) = 0$

Putting $x = 0, y = 0$ in Eq. (1),

$$0 = -2\cos 0 + c\cos 0 = -2 + c$$

$$c = 2$$

Hence, the solution is

$$y = -2\cos^2 x + 2\cos x$$

For maximum or minimum value,

$$\frac{dy}{dx} = 0$$

$$-4\cos x(-\sin x) - 2\sin x = 0$$

$$2\sin x(2\cos x - 1) = 0$$

$\sin x = 0,\ x = 0$ and $2\cos x - 1 = 0,\ \cos x = \frac{1}{2},\ x = \frac{\pi}{3}$

$x = 0$ and $x = \frac{\pi}{3}$ are the points of extreme values.

Now,

$$\frac{dy}{dx} = 2\sin 2x - 2\sin x$$

$$\frac{d^2y}{dx^2} = 4\cos 2x - 2\cos x$$

When $x = 0,\ \frac{d^2y}{dx^2} = 2 > 0,\ y$ is minimum at $x = 0$.

When $x = \frac{\pi}{3},\ \frac{d^2y}{dx^2} = 4\cos\frac{2\pi}{3} - 2\cos\frac{\pi}{3} = 4\left(-\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) = -3 < 0$, y is maximum at $x = \frac{\pi}{3}$.

Putting $x = \frac{\pi}{3}$ in Eq. (2), the maximum value of y is obtained.

$$y_{max} = -2\cos^2\frac{\pi}{3} + 2\cos\frac{\pi}{3} = -\frac{1}{2} + 1 = \frac{1}{2}$$

EXERCISE 10.6

Solve the following differential equations:

1. $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$

$$\left[\textbf{Ans.: } y = \frac{c}{x^2} + x + \frac{1}{x}\right]$$

2. $(x+1)\frac{dy}{dx} - 2y = (x+1)^4$

$$\left[\textbf{Ans.: } y = \left(\frac{x^2}{2} + x + c\right)(x+1)^2\right]$$

3. $\frac{1}{x}\frac{dy}{dx} + 2y = e^{-x^2}$

$$\left[\textbf{Ans.: } ye^{x^2} = \frac{x^2}{2} + c\right]$$

4. $(y+1)dx + [x - (y+2)e^y]dy = 0$

$$\left[\textbf{Ans.: } (y+1)(x - e^y) = c\right]$$

5. $dx + x\,dy = e^{-y}\sec^2 y\,dy$

$\left[\textbf{Ans.:}\ xe^{y} - \tan y + c\right]$

6. $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1$

$\left[\textbf{Ans.:}\ ye^{2\sqrt{x}} = 2\sqrt{x} + c\right]$

7. $\cos^2 x\dfrac{dy}{dx} + y = \tan x$

$\left[\textbf{Ans.:}\ y = \tan x - 1 + ce^{-\tan x}\right]$

8. $\sqrt{a^2 + x^2}\dfrac{dy}{dx} + y = \sqrt{a^2 + x^2} - x$

$\left[\textbf{Ans.:}\ \left(x + \sqrt{x^2 + a^2}\right)y = a^2 x + c\right]$

9. $\dfrac{dy}{dx} = \dfrac{1}{x + e^{y}}$

$\left[\textbf{Ans.:}\ xe^{-y} = c + y\right]$

10. $\dfrac{dy}{dx} - \left(\dfrac{3}{x}\right)y = x^3,\quad y(1) = 4$

$\left[\textbf{Ans.:}\ y = x^3(x + 3)\right]$

11. $(1 + x^2)\dfrac{dy}{dx} - 2xy = 2x(1 + x^2), y(0) = 1$

$\left[\textbf{Ans.:}\ y = (1 + x^2)[1 + \log(1 + x^2)]\right]$

12. $x\dfrac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x^2,$
$y(\pi) = \pi^3 e^{\pi} + 2\pi^2$

$\left[\textbf{Ans.:}\ y = 2x^2 + (e^x + \sin x)x^3\right]$

13. $\dfrac{dy}{dx} + \dfrac{y}{x} = \log x,\ y(1) = 1$

$\left[\textbf{Ans.:}\ y = \dfrac{x\log x}{2} - \dfrac{x}{4} + \dfrac{5}{4x}\right]$

10.3.7 Nonlinear Differential Equations Reducible to Linear Form

Type 1 *Bernoulli's Equation*

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n \qquad \dots (10.12)$$

where P and Q are functions of x or constants is a nonlinear equation known as *Bernoulli's equation*. This equation can be made linear using the following method:
Dividing Eq. (10.12) by y^n,

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q \qquad \dots (10.13)$$

Let $\dfrac{1}{y^{n-1}} = v$

$$\frac{(1-n)}{y^n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx}=\frac{1}{(1-n)}\cdot\frac{dv}{dx}$$

Substituting in Eq. (10.13),

$$\frac{1}{1-n}\cdot\frac{dv}{dx}+Pv=Q$$

$$\frac{dv}{dx}+(1-n)Pv=Q$$

The equation is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v=\frac{1}{y^{n-1}}$, the solution of Eq. (10.12) is obtained.

EXAMPLE 10.29 *Solve* $\frac{dy}{dx}+\frac{2y}{x}=y^2x^2$.

Solution: The equation is in Bernoulli's form.

Dividing the equation by y^2,

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{y}\cdot\frac{2}{x}=x^2 \qquad \text{... (1)}$$

Let $\frac{1}{y}=v$, $\quad -\frac{1}{y^2}\frac{dy}{dx}=\frac{dv}{dx}$

Substituting in Eq. (1),

$$-\frac{dv}{dx}+\left(\frac{2}{x}\right)v=x^2$$

$$\frac{dv}{dx}-\left(\frac{2}{x}\right)v=-x^2 \qquad \text{... (2)}$$

The equation is linear in v.

$$P=-\frac{2}{x},\ Q=-x^2$$

$$IF=e^{\int-\frac{2}{x}dx}=e^{-2\log x}=e^{\log x^{-2}}=x^{-2}=\frac{1}{x^2}$$

The solution of Eq. (2) is

$$\frac{1}{x^2}v=\int\frac{1}{x^2}(-x^2)dx+c=\int-dx+c=-x+c$$

$$v=-x^3+cx^2$$

Hence,

$$\frac{1}{y}=-x^3+cx^2$$

EXAMPLE 10.30 *Solve* $xy(1+xy^2)\dfrac{dy}{dx}=1.$

Solution: Rewriting the equation,

$$\frac{dx}{dy}=xy+x^2y^3$$

$$\frac{dx}{dy}-xy=x^2y^3$$

The equation is in Bernoulli's form, where x is a dependent variable.
Dividing the equation by x^2,

$$\frac{1}{x^2}\frac{dx}{dy}-\left(\frac{1}{x}\right)y=y^3 \qquad \text{... (1)}$$

Let $-\dfrac{1}{x}=v,\ \dfrac{1}{x^2}\dfrac{dx}{dy}=\dfrac{dv}{dy}$

Substituting in Eq. (1),

$$\frac{dv}{dy}+vy=y^3 \qquad \text{... (2)}$$

The equation is linear in v.

$$P=y,\quad Q=y^3$$

$$IF=e^{\int y\,dy}=e^{\frac{y^2}{2}}$$

The solution of Eq. (2) is

$$e^{\frac{y^2}{2}}\cdot v=\int e^{\frac{y^2}{2}}y^3dy+c$$

Putting $\dfrac{y^2}{2}=t,\quad y\,dy=dt$

$$e^{\frac{y^2}{2}}\cdot v=\int e^t\cdot 2t\,dt+c=2(e^tt-e^t)+c=2e^t(t-1)+c=2e^{\frac{y^2}{2}}\left(\frac{y^2}{2}-1\right)+c$$

$$v=y^2-2+ce^{-\frac{y^2}{2}}$$

Hence, $$-\frac{1}{x}=y^2-2+ce^{\frac{-y^2}{2}}$$

EXAMPLE 10.31 *Solve* $\dfrac{dr}{d\theta}=r\tan\theta-\dfrac{r^2}{\cos\theta}.$

Solution: Rewriting the equation,

$$\frac{dr}{d\theta} - r\tan\theta = -\frac{r^2}{\cos\theta}$$

The equation is in Bernoulli's form, where r is a dependent variable.
Dividing the equation by r^2,

$$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{\tan\theta}{r} = -\frac{1}{\cos\theta} \qquad \text{...(1)}$$

Let $-\frac{1}{r} = v$,

$$\frac{1}{r^2}\frac{dr}{d\theta} = \frac{dv}{d\theta}$$

Substituting in Eq. (1),

$$\frac{dv}{d\theta} + v\tan\theta = -\frac{1}{\cos\theta} \qquad \text{...(2)}$$

The equation is linear in v.

$$P = \tan\theta,\ Q = -\frac{1}{\cos\theta}$$

$$IF = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

The solution of Eq. (2) is

$$\sec\theta \cdot v = \int \sec\theta\left(-\frac{1}{\cos\theta}\right)d\theta + c = \int -\sec^2\theta\, d\theta + c = -\tan\theta + c$$

Hence,
$$\sec\theta\left(-\frac{1}{r}\right) = -\tan\theta + c$$

$$\frac{\sec\theta}{r} = \tan\theta - c$$

Type 2 *Equations of the form* $f'(y)\frac{dy}{dx} + Pf(y) = Q$ ... (10.14)

where P and Q are functions of x or constants can be reduced to the linear form by putting $f(y) = v,\ f'(y)\frac{dy}{dx} = \frac{dv}{dx}$ in Eq. (10.14)

$$\frac{dv}{dx} + Pv = Q \qquad \text{... (10.15)}$$

Equation (10.15) is linear in v and can be solved using the method of linear differential equations. Finally, substituting $v = f(y)$, the solution of Eq. (10.14) is obtained.

EXAMPLE 10.32 *Solve* $x\dfrac{dy}{dx}+y\log y = xye^x$.

Solution: Dividing the equation by xy,

$$\frac{1}{y}\frac{dy}{dx}+\frac{\log y}{x}=e^x \qquad \ldots (1)$$

Let $\log y = v, \dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx}+\frac{v}{x}=e^x \qquad \ldots (2)$$

The equation is linear in v.

$$P=\frac{1}{x}, Q=e^x$$

$$IF = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

The solution of Eq. (2) is

$$xv = \int xe^x dx + c = xe^x - e^x + c$$

$$xv = e^x(x-1)+c$$

Hence, $$x\log y = e^x(x-1)+c.$$

EXAMPLE 10.33 *Solve* $\dfrac{dy}{dx}=\dfrac{y^3}{e^{2x}+y^2}$.

Solution:

Rewriting the equation, $\dfrac{dx}{dy}=\dfrac{e^{2x}}{y^3}+\dfrac{1}{y}$

$$e^{-2x}\frac{dx}{dy}-\frac{e^{-2x}}{y}=\frac{1}{y^3} \qquad \ldots (1)$$

Let $e^{-2x}=v, -2e^{-2x}\dfrac{dx}{dy}=\dfrac{dv}{dy}$,

$$e^{-2x}\frac{dx}{dy}=-\frac{1}{2}\frac{dv}{dy}$$

Substituting in Eq. (1),

$$-\frac{1}{2}\frac{dv}{dy}-\frac{v}{y}=\frac{1}{y^3}$$

$$\frac{dv}{dy}+\frac{2}{y}\cdot v=\frac{-2}{y^3}$$

The equation is linear in v.

$$P=\frac{2}{y}, Q=-\frac{2}{y^3}$$

$$IF=e^{\int \frac{2}{y}dy}=e^{2\log y}=e^{\log y^2}=y^2$$

The solution of Eq. (2) is

$$y^2\cdot v=\int y^2\left(-\frac{2}{y^3}\right)dy+c$$

$$y^2\cdot v=-2\int \frac{1}{y}dy+c=-2\log y+c$$

Hence, $$y^2e^{-2x}=-2\log y+c$$

EXAMPLE 10.34 *Solve* $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x^2}(\log z)^2.$

Solution: Rewriting the equation,

$$\frac{1}{z(\log z)^2}\frac{dz}{dx}+\frac{1}{\log z}\cdot\frac{1}{x}=\frac{1}{x^2} \quad ...(1)$$

Let $\frac{-1}{\log z}=v$, $\quad \frac{1}{(\log z)^2}\cdot\frac{1}{z}\frac{dz}{dx}=\frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{dv}{dx}-\frac{v}{x}=\frac{1}{x^2} \quad ...(2)$$

The equation is linear in v.

$$P=-\frac{1}{x}, Q=\frac{1}{x^2}$$

$$IF=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=x^{-1}=\frac{1}{x}$$

The solution of Eq. (2) is

$$\frac{1}{x}\cdot v=\int\frac{1}{x}\cdot\frac{1}{x^2}dx+c=\int x^{-3}dx+c=\frac{x^{-2}}{-2}+c$$

Hence,
$$\frac{1}{x}\left(-\frac{1}{\log z}\right)=-\frac{1}{2x^2}+c$$

$$\frac{1}{x\log z}=\frac{1}{2x^2}-c$$

EXAMPLE 10.35 *Solve* $\frac{dy}{dx}+x(x+y)=x^3(x+y)^3-1.$

Solution: $\frac{dy}{dx}+x(x+y)=x^3(x+y)^3-1$ ... (1)

Let $x+y=z, 1+\frac{dy}{dx}=\frac{dz}{dx}, \frac{dy}{dx}=\frac{dz}{dx}-1$

Substituting in Eq. (1),

$$\frac{dz}{dx}-1+xz=x^3z^3-1$$

$$\frac{dz}{dx}+xz=x^3z^3 \quad ... (2)$$

Dividing Eq. (2) by z^3,

$$\frac{1}{z^3}\frac{dz}{dx}+\frac{x}{z^2}=x^3 \quad ... (3)$$

Let $\frac{1}{z^2}=v, -\frac{2}{z^3}\frac{dz}{dx}=\frac{dv}{dx}, \frac{1}{z^3}\frac{dz}{dx}=-\frac{1}{2}\frac{dv}{dx}$

Substituting in Eq. (3),

$$-\frac{1}{2}\frac{dv}{dx}+xv=x^3$$

$$\frac{dv}{dx}-2xv=-2x^3 \quad ... (4)$$

The equation is linear in v.

$$P=-2x, Q=-2x^3$$

$$IF=e^{\int-2x\,dx}=e^{-x^2}$$

The solution of Eq. (4) is

$$e^{-x^2}\cdot v=\int e^{-x^2}(-2x^3)dx+c$$

Let $x^2 = t$, $2x\,dx = dt$

$$e^{-x^2} \cdot v = -\int te^{-t}dt + c = te^{-t} + e^{-t} + c = (x^2+1)e^{-x^2} + c$$

$$v = (x^2+1) + ce^{x^2}$$

Substituting the value of v,

$$\frac{1}{z^2} = (x^2+1) + ce^{x^2}$$

Hence, $$\frac{1}{(x+y)^2} = (x^2+1) + ce^{x^2}$$

EXAMPLE 10.36 *Solve* $2xy\left(\frac{dy}{dx}\right) = (y^2+6) + x^{\frac{3}{2}}(y^2+6)^4$.

Solution: Let $y^2 + 6 = z$, $2y\frac{dy}{dx} = \frac{dz}{dx}$

Substituting in the given equation,

$$x\frac{dz}{dx} = z + x^{\frac{3}{2}}z^4$$

$$\frac{1}{z^4}\frac{dz}{dx} = \frac{1}{xz^3} + x^{\frac{1}{2}}$$

$$\frac{1}{z^4}\frac{dz}{dx} - \frac{1}{xz^3} = x^{\frac{1}{2}} \quad \ldots (1)$$

Let $-\frac{1}{z^3} = v$, $\frac{3}{z^4}\frac{dz}{dx} = \frac{dv}{dx}$, $\frac{1}{z^4}\frac{dz}{dx} = \frac{1}{3}\frac{dv}{dx}$

Substituting in Eq. (1),

$$\frac{1}{3}\frac{dv}{dx} + \frac{v}{x} = x^{\frac{1}{2}}$$

$$\frac{dv}{dx} + \frac{3v}{x} = 3x^{\frac{1}{2}} \quad \ldots (2)$$

The equation is linear in v.

$$P = \frac{3}{x}, \quad Q = 3x^{\frac{1}{2}}$$

$$\text{IF} = e^{\int \frac{3}{x}dx} = e^{3\log x} = e^{\log x^3} = x^3$$

The solution Eq. (2) is

$$x^3 v = \int x^3 \cdot 3x^{\frac{1}{2}}\,dx + c = 3\int x^{\frac{7}{2}}\,dx + c = 3\cdot\frac{2}{9}x^{\frac{9}{2}} + c$$

$$x^3 v = \frac{2}{3}x^{\frac{9}{2}} + c,$$

$$x^3\left(-\frac{1}{z^3}\right) = \frac{2}{3}x^{\frac{9}{2}} + c$$

Hence, $$-\left(\frac{x}{y^2+6}\right)^3 = \frac{2}{3}x^{\frac{9}{2}} + c.$$

EXERCISE 10.7

Solve the following differential equations:

1. $\dfrac{dy}{dx} = x^3y^3 - xy$

   $\left[\textbf{Ans.:}\ \dfrac{1}{y^2} = x^2 + 1 + ce^{x^2}\right]$

2. $x^2y - x^3\dfrac{dy}{dx} = y^4\cos x$

   $\left[\textbf{Ans.:}\ x^3 = y^3(3\sin x - c)\right]$

3. $x(3x + 2y^2)\,dx + 2y(1 + x^2)\,dy = 0$

   $\left[\textbf{Ans.:}\ y^2(1 + x^2) = -x^3 + c\right]$

4. $x\,dy - [y + xy^3(1 + \log x)]\,dx = 0$

   $\left[\textbf{Ans.:}\ x^2 = -\dfrac{2}{3}x^3y^2\left(\dfrac{2}{3} + \log x\right) + cy^2\right]$

5. $\dfrac{dy}{dx} + y = y^2e^x$

   $\left[\textbf{Ans.:}\ -\dfrac{e^{-x}}{y} = x + c\right]$

6. $x\,dy + y\,dx = x^3y^6\,dx$

   $\left[\textbf{Ans.:}\ \dfrac{2}{y^5} = 5x^3 + cx^5\right]$

7. $x\dfrac{dy}{dx} + y = y^3x^{n+1}$

   $\left[\textbf{Ans.:}\ \dfrac{n-1}{y^2} = cx^2 - 2x^{n+1}\right]$

8. $xy(1 + x^2y^2)\dfrac{dy}{dx} = 1$

   $\left[\textbf{Ans.:}\ \dfrac{1}{x^2} = ce^{-y^2} - y^2 + 1\right]$

9. $x^2y^3\,dx + (x^3y - 2)\,dy = 0$

   $\left[\textbf{Ans.:}\ x^3 = \dfrac{2}{y} + \dfrac{2}{3} + ce^{\frac{3}{y}}\right]$

10. $y\dfrac{dx}{dy} = x - yx^2\cos y$

    $\left[\textbf{Ans.:}\ \dfrac{y}{x} = y\sin y + \cos y + c\right]$

11. $\dfrac{dy}{dx}=\dfrac{e^y}{x^2}-\dfrac{1}{x}$

$\left[\textbf{Ans.: } 2xe^{-y}=1+2cx^2\right]$

12. $\dfrac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$

$\left[\textbf{Ans.: } 2\tan^{-1}y=(x^2-1)+ce^{-x^2}\right]$

13. $(y+e^y-e^{-x})dx+(1+e^y)dy=0$

$\left[\textbf{Ans.: } y+e^y=(x+c)e^{-x}\right]$

14. $x^2\cos y\dfrac{dy}{dx}=2x\sin y-1$

$\left[\textbf{Ans.: } 3x\sin y=cx^3+1\right]$

15. $\dfrac{dy}{dx}+\dfrac{1}{x}\tan y=\dfrac{1}{x^2}\tan y\sin y$

$\left[\textbf{Ans.: } \operatorname{cosec} y=1+cx\right]$

16. $x^2\dfrac{dy}{dx}=\sin^2 y-(\sin y\cos y)x$

$\left[\textbf{Ans.: } \cot y=\dfrac{1}{2x}+cx\right]$

17. $\dfrac{dr}{d\theta}=\dfrac{r\sin\theta-r^2}{\cos\theta}$

$\left[\textbf{Ans.: } \dfrac{1}{r}=c\cos\theta+\sin\theta\right]$

18. $\cos x\dfrac{dy}{dx}+4y\sin x=4\sqrt{y}\sec x$

$\left[\textbf{Ans.: } \sqrt{y}\sec^2 x=2\left(\tan x+\dfrac{\tan^3 x}{3}\right)+c\right]$

19. $\sin y\dfrac{dy}{dx}=\cos x(2\cos y-\sin^2 x)$

$\left[\textbf{Ans.: } 4\cos y=2\sin^2 x-2\sin x+1-4ce^{-2\sin x}\right]$

20. $e^y\left(\dfrac{dy}{dx}+1\right)=e^x$

$\left[\textbf{Ans.: } e^{x+y}=\dfrac{e^{2x}}{2}+c\right]$

21. $4x^2y\dfrac{dy}{dx}=3x(3y^2+2)+2(3y^2+2)^3$

$\left[\textbf{Ans.: } 4x^9=(3y^2+2)^2(-3x^8+c)\right]$

10.4 ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND HIGHER DEGREE

If the degree of $\dfrac{dy}{dx}$ in a differential equation of first order is higher than 1, it is convenient to denote $\dfrac{dy}{dx}$ by p. Hence, a differential equation of first order and higher degree can be written as

$$f(x, y, p)=0$$

There are three cases of such equations, viz.,

(i) Equation solvable for p
(ii) Equation solvable for y
(iii) Equation solvable for x

Case I *Equations solvable for p*

Let $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0$ ...(10.16)

be an ordinary differential equation of first order and n^{th} degree, where $p = \frac{dy}{dx}$ and $P_1, P_2, \ldots, P_n$ are functions of x and y.

If Eq. (10.16) is solvable for p, its LHS can be resolved into n rational factors of the first degree. Equation (10.16) can be written as

$$\left[p - f_1(x, y)\right]\left[p - f_2(x, y)\right] \ldots \left[p - f_n(x, y)\right] = 0$$

Equating each factor of the LHS to zero,

$$p = f_1(x, y), \quad p = f_2(x, y), \ldots, p = f_n(x, y)$$

Let the general solution of these equations be respectively,

$$F_1(x, y, c) = 0, \quad F_2(x, y, c) = 0, \ldots, F_n(x, y, c) = 0$$

Since the given equation is of the first order, its general solution will have only one arbitrary constant c. These n solutions constitute the general solution of Eq. (10.16).

The general solution of Eq. (10.16) is

$$F_1(x, y, c) F_2(x, y, c) \ldots F_n(x, y, c) = 0$$

EXAMPLE 10.37 *Solve* $x^2 p^2 + 3xyp + 2y^2 = 0$.

Solution:

$$x^2 p^2 + 3xyp + 2y^2 = 0$$

$$(xp + y)(xp + 2y) = 0$$

$$(xp + y) = 0, \quad (xp + 2y) = 0$$

$$x\frac{dy}{dx} + y = 0, \quad x\frac{dy}{dx} + 2y = 0$$

$$\frac{dy}{y} + \frac{dx}{x} = 0, \quad \frac{dy}{y} + \frac{2}{x}dx = 0$$

Integrating both the sides,

$$\int \frac{dy}{y} + \int \frac{dx}{x} = 0, \quad \int \frac{dy}{y} + \int \frac{2}{x}dx = 0$$

$$\log y + \log x = \log c, \quad \log y + 2\log x = \log c$$

$$xy = c, \qquad x^2 y = c$$

$$(xy - c) = 0, \qquad (x^2 y - c) = 0$$

Hence, the general solution is

$$(xy - c)(x^2 y - c) = 0$$

EXAMPLE 10.38 *Solve* $p^2 + 2py\cot x - y^2 = 0.$

Solution: The given equation is quadratic in p.

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2} = -y\cot x \pm y\operatorname{cosec} x$$

$$p = y(-\cot x + \operatorname{cosec} x), \qquad p = y(-\cot x - \operatorname{cosec} x)$$

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x), \qquad \frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$

$$\frac{dy}{y} + (\cot x - \operatorname{cosec} x)dx = 0, \qquad \frac{dy}{y} + (\cot x + \operatorname{cosec} x)dx = 0$$

Integrating both the sides,

$$\int \frac{dy}{y} + \int (\cot x + \operatorname{cosec} x)dx = 0, \qquad \int \frac{dy}{y} + \int (\cot x - \operatorname{cosec} x)dx = 0$$

$$\log y + \log\sin x + \log\tan\frac{x}{2} = \log c, \qquad \log y + \log\sin x - \log\tan\frac{x}{2} = \log c$$

$$\log\left(y\sin x\tan\frac{x}{2}\right) = \log c, \qquad \log\frac{y\sin x}{\tan\frac{x}{2}} = \log c$$

$$y\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = c, \qquad \frac{y\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)} = c$$

$$y\left(2\sin^2\frac{x}{2}\right) = c, \qquad y\left(2\cos^2\frac{x}{2}\right) = c$$

$$y(1 - \cos x) = c, \qquad y(1 + \cos x) = c$$

$$y(1 - \cos x) - c = 0, \qquad y(1 + \cos x) - c = 0$$

Hence, the general solution is

$$[y(1 - \cos x) - c][y(1 + \cos x) - c] = 0$$

EXAMPLE 10.39 Solve $p^2 - 2p\cosh x + 1 = 0$.

Solution:

$$p^2 - 2p\cosh x + 1 = 0$$

$$p^2 - p\left(e^x + e^{-x}\right) + 1 = 0$$

$$p\left(p - e^x\right) - e^{-x}\left(p - e^x\right) = 0$$

$$\left(p - e^x\right)\left(p - e^{-x}\right) = 0$$

$$p - e^x = 0, \qquad p - e^{-x} = 0$$

$$\frac{dy}{dx} - e^x = 0, \qquad \frac{dy}{dx} - e^{-x} = 0$$

$$dy - e^x dx = 0, \qquad dy - e^{-x} dx = 0$$

Integrating both the sides,

$$\int dy - \int e^x dx = 0, \qquad \int dy - \int e^{-x} dx = 0$$

$$y - e^x = c, \qquad y + e^{-x} = c$$

$$y - e^x - c = 0, \qquad y + e^{-x} - c = 0$$

Hence, the general solution is

$$\left(y - e^x - c\right)\left(y + e^{-x} - c\right) = 0$$

EXERCISE 10.8

Solve the following differential equations:

1. $p^2 - 9p + 18 = 0$

   $\left[\textbf{Ans.:} \left(y - 3x + c\right)\left(y - 6x + c\right) = 0\right]$

2. $\left(p - xy\right)\left(p - x^2\right)\left(p - y^2\right) = 0$

   $\left[\textbf{Ans.:} \left(\log y - \frac{1}{2}x^2 - c\right)\left(y - \frac{1}{3}x^3 - c\right)\left(x + \frac{1}{y} + c\right) = 0\right]$

3. $\left(p + y + x\right)\left(xp + x + y\right)\left(p + 2x\right) = 0$

   $\left[\textbf{Ans.:} \left(1 - x - y + ce^{-x}\right)\left(2xy + x^2 - c\right)\left(y + x^2 - c\right) = 0\right]$

4. $p^2 - p\left(e^x + e^{-x}\right) + 1 = 0$

   $\left[\textbf{Ans.:} \left(y - e^{2x} - c\right)\left(y - e^{-2x} - c\right) = 0\right]$

5. $p^2 + \left(x - e^x\right)p - xe^x = 0$

   $\left[\textbf{Ans.:} \left(y - e^x + c\right)\left(y + \frac{1}{2}x^2 + c\right) = 0\right]$

6. $p(p-y)=x(x+y)$

$\left[\textbf{Ans.:}\left(x+y+1-ce^{-x}\right)\left(2y+x^2-c\right)=0\right]$

7. $p^3+2p^2x-p^2y^2-2pxy^2=0$

$\left[\textbf{Ans.:}(y-c)\left(y+x^2-c\right)(1+xy-cy)=0\right]$

8. $p^3+3xp^2-y^3p^2-3xy^3p=0$

$\left[\textbf{Ans.:}(y-c)\left(y+\frac{3}{2}x^2-c\right)\left(2xy^2+1-2cy^2\right)=0\right]$

9. $xyp^2-(x+y)p+1=0$

$\left[\textbf{Ans.:}\left(x+\frac{1}{2}y^2-c\right)(y+\log x-c)=0\right]$

10. $p^3(x+2y)+3p^2(x+y)+(y+2x)p=0$

$\left[\textbf{Ans.:}\left(x+\frac{1}{2}y^2-c\right)(y+\log x-c)=0\right]$

Case II *Equations solvable for y*

If the given ordinary differential equation is solvable for y, it can be put in the form

$$y=f(x,p) \qquad \text{...(10.17)}$$

Differentiating Eq. (10.17) w.r.t. x,

$$p=\frac{dy}{dx}=\phi\left(x,p,\frac{dp}{dx}\right) \qquad \text{...(10.18)}$$

Equation (10.18) is a differential equations in p and x.

Let the general solution of Eq. (10.18) be

$$F(x,p,c)=0 \qquad \text{...(10.19)}$$

where c is an arbitrary constant. The elimination of p from Eqs (10.17) and (10.19) gives the general solution.

Notes

(i) If the elimination of p is not possible, Eqs (10.17) and (10.19) are solved for x and y in terms of p. The two parametric equations

$$x=F_1(p,c)$$

and

$$y=F_2(p,c)$$

taken together constitute the general solution of Eq. (10.17), where p is the parameter.

(ii) If Eqs (10.17) and (10.19) are easily not solvable for x and y and then Eqs (10.17) and (10.19) taken together represent the general solution of Eq. (10.19).

(iii) If any factor of Eq. (10.18) does not contain the term $\frac{dp}{dx}$, the elimination of p from that factor equation and Eq. (10.17) gives a solution which does not contain any arbitrary constants. Such a solution is called a *singular solution*.

In this section, only general solutions are obtained.

EXAMPLE 10.40 *Solve* $y = 2px - xp^2$.

Solution:

$$y = 2px - xp^2 \quad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$p = 2p + 2x\frac{dp}{dx} - p^2 - 2xp\frac{dp}{dx} \qquad \left[\because \frac{dy}{dx} = p\right]$$

$$p + 2x\frac{dp}{dx}(1-p) - p^2 = 0$$

$$p(1-p) + 2x\frac{dp}{dx}(1-p) = 0$$

$$\left(p + 2x\frac{dp}{dx}\right)(1-p) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dx}$,

$$p + 2x\frac{dp}{dx} = 0$$

$$2\frac{dp}{p} + \frac{dx}{x} = 0$$

Integrating both the sides,

$$\int 2\frac{dp}{p} + \int \frac{dx}{x} = 0$$

$$2\log p + \log x = \log c$$

$$p^2 x = c$$

$$p^2 = \frac{c}{x} \quad \text{...(2)}$$

From Eq. (1),

$$y + xp^2 = 2px$$

$$\left(y + xp^2\right)^2 = 4p^2x^2 \quad \text{...(3)}$$

Eliminating p from Eqs (2) and (3),

$$(y+c)^2 = 4\left(\frac{c}{x}\right)x^2 = 4cx$$

which is the general solution.

EXAMPLE 10.41 *Solve* $y = 2px + p^n$.

Solution:
$$y = 2px + p^n \qquad \text{...(1)}$$

Differentiating Eq. (1) w.r.t. x,

$$p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \qquad \left[\because \frac{dy}{dx} = p\right]$$

$$p + \left(2x + np^{n-1}\right)\frac{dp}{dx} = 0$$

$$p\frac{dx}{dp} + 2x + np^{n-1} = 0$$

$$\frac{dx}{dp} + \frac{2}{p}x = -np^{n-2} \qquad \text{...(2)}$$

which is a linear equation in x and p.

$$\text{IF} = e^{\int \frac{2}{p}dp} = e^{2\log p} = p^2$$

The general solution of Eq. (2) is

$$p^2 . x = \int p^2\left(-np^{n-2}\right)dp + c$$

$$xp^2 = -n\int p^n\, dp + c = -\frac{np^{n+1}}{n+1} + c$$

$$x = -\frac{np^{n-1}}{n+1} + \frac{c}{p^2} \qquad \text{...(3)}$$

Substituting the value of x in Eq. (1),

$$y = -\frac{2n}{n+1}p^n + \frac{2c}{p} + p^n = \frac{2c}{p} - \frac{n-1}{n+1}p^n \qquad \text{...(4)}$$

Equations (3) and (4) taken together, with the parameter p, constitute the general solution.

EXAMPLE 10.42 *Solve* $y = x + a\tan^{-1}p$.

Solution:
$$y = x + a\tan^{-1}p \qquad \text{...(1)}$$

Differentiating Eq.(1) w.r.t. x,

$$p = 1 + a\frac{1}{1+p^2}\frac{dp}{dx} \qquad \left[\because \frac{dy}{dx} = p\right]$$

$$\frac{a}{1+p^2}\frac{dp}{dx} = p - 1$$

$$dx = \frac{a}{(p-1)(p^2+1)}dp = \frac{a}{2}\left(\frac{1}{p-1} - \frac{p+1}{p^2+1}\right)dp$$

Integrating both the sides,

$$\int dx = \int \frac{a}{2}\left(\frac{1}{p-1} - \frac{p+1}{p^2+1}\right)dp$$

$$x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1} p\right] + c = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} - \tan^{-1} p\right] + c \qquad \ldots(2)$$

Substituting the value of x in Eq. (1),

$$y = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{p^2+1}} + \tan^{-1} p\right] + c \qquad \ldots(3)$$

Equations (2) and (3) taken together constitute the general solution.

EXERCISE 10.9

Solve the following differential equations:

1. $y = -px + x^4 p^2$

   $\left[\textbf{Ans.: } xy = -c + c^2 x\right]$

2. $y = \dfrac{x}{p} - ap$

   $\left[\begin{aligned}\textbf{Ans.: } x &= \frac{p}{\sqrt{1-p^2}}\left(c + a\sin^{-1} p\right)\\ y &= -ap + \frac{1}{\sqrt{1-p^2}}\left(c + a\sin^{-1} p\right)\end{aligned}\right]$

3. $y = \sin p - p\cos p$

   $\left[\textbf{Ans.: } x = c - \cos\left\{\dfrac{\sqrt{1-(c-x)^2} - y}{c - x}\right\}\right]$

4. $y = 3x + a\log p$

   $\left[\textbf{Ans.: } y = 3x - a\log\left(\frac{1}{3} - ce^{\frac{3x}{a}}\right)\right]$

5. $y = 2px + p^2$

   $\left[\textbf{Ans.: } 4\left(y^2 - 3cx\right)\left(x^2 + y\right) = \left(xy + 3c\right)^2\right]$

6. $y=(1+p)x+p^2$

$$\left[\begin{aligned}\textbf{Ans.: } x&=2(1-p)+ce^{-p}\\ y&=2-p^2+c(1+p)e^{-p}\end{aligned}\right]$$

7. $y=2xp-p^3$

$$\left[\textbf{Ans.: } x=\frac{3}{4}p^2+\frac{c}{p^2},\ y=\frac{1}{2}p^3+\frac{2c}{p}\right]$$

8. $xp^2-2yp+ax=0$

$$\left[\textbf{Ans.: } 2y=cx^2+\frac{a}{c}\right]$$

9. $yp^2-2xp+y=0$

$$\left[\textbf{Ans.: } y^2=2cx-c^2\right]$$

10. $y=(1+p)x+ap^2$

$$\left[\begin{aligned}\textbf{Ans.: } x&=ce^{-p}-2a(p-1)\\ y&=(1+p)ce^{-p}-2a(p-1)+ap^2\end{aligned}\right]$$

Case III *Equations solvable for x*

If the given ordinary differential equation is solvable for x, it can be put in the form

$$x=f(y,p) \qquad \text{...(10.20)}$$

Differentiating Eq. (10.20) w.r.t. y,

$$\frac{1}{p}=\frac{dx}{dy}=\phi\left(y,p,\frac{dp}{dy}\right) \qquad \text{...(10.21)}$$

Equation (10.21) is a differential equation in p and y
Let the general solution of Eq. (10.21) be

$$F(y,p,c)=0 \qquad \text{...(10.22)}$$

where c is an arbitrary constant. The elimination of p from Eqs (10.20) and (10.21) gives the general solution.

Notes

(i) If the elimination of p is not possible, Eqs. (10.20) and (10.22) are solved for x and y in terms of p.
The two parametric equations

$$x=F_1(p,c)$$

and

$$y=F_2(p,c)$$

taken together constitute the general solution of Eq. (10.20), where p is the parameter.

(ii) If Eqs (10.20) and (10.22) are not easily solvable for x and y then Eqs (10.20) and (10.22) taken together represent the general solution of Eq. (10.20).

(iii) If any factor of Eq. (10.21) does not contain the term $\frac{dp}{dy}$, the elimination of p from that factor equation and Eq. (10.20) gives a solution which does not contain any arbitrary constants. Such a solution is called a *singular solution*.
In this section, only general solutions are obtained.

EXAMPLE 10.43 Solve $yp^2 - 2xp + y = 0$.

Solution:

$$yp^2 - 2xp + y = 0 \qquad \text{...(1)}$$

$$2x = yp + \frac{y}{p} \qquad \text{...(2)}$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{2}{p} = p + y\frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

$$p + y\frac{dp}{dy} - \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} = 0$$

$$y\frac{dp}{dy}\left(1 - \frac{1}{p^2}\right) + p\left(1 - \frac{1}{p^2}\right) = 0$$

$$\left(p + y\frac{dp}{dy}\right)\left(1 - \frac{1}{p^2}\right) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dy}$,

$$p + y\frac{dp}{dy} = 0$$

$$\frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating both the sides,

$$\int \frac{dp}{p} + \int \frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = \frac{c}{y}.$$

Substituting the value of p in Eq. (1),

$$y\frac{c^2}{y^2} - 2x\frac{c}{y} + y = 0$$

$$y^2 = 2cx - c^2$$

which is the general solution.

EXAMPLE 10.44 Solve $y = 2px + y^2 p^3$.

Solution:

$$y = 2px + y^2 p^3 \quad \ldots(1)$$

$$x = \frac{y}{2p} - \frac{y^2 p^2}{2} \quad \ldots(2)$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - y^2 p \frac{dp}{dy}$$

$$\frac{1}{p} - \frac{1}{2p} + yp^2 = -y\left(\frac{1}{2p^2} + py\right)\frac{dp}{dy}$$

$$\left(1 + 2yp^3\right)p = -y\left(1 + 2yp^3\right)\frac{dp}{dy}$$

$$\left(1 + 2yp^3\right)p + \left(1 + 2yp^3\right)y\frac{dp}{dy} = 0$$

$$\left(p + y\frac{dp}{dy}\right)\left(1 + 2yp^3\right) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dy}$,

$$p + y\frac{dp}{dy} = 0$$

$$\frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating both the sides,

$$\int \frac{dp}{p} + \int \frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = \frac{c}{y}$$

Substituting

EXERCISE 10.10

Solve the following differential equations

1. $x = y + a\log p$

$$\left[\textbf{Ans.}: x = c + a\log\frac{p}{p-1}, y = c - a\log(p-1)\right]$$

2. $x = y + p^2$

$$\left[\begin{aligned}\textbf{Ans.}: x &= c - \{2p + 2\log(p-1)\}\\ y &= c - \{p^2 + 2p + 2\log(p-1)\}\end{aligned}\right]$$

3. $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$

$$\left[\textbf{Ans.}: y = -\frac{1}{1+p^2} + c\right]$$

4. $x = p^3 - p + 2$

$$\left[\textbf{Ans.}: x = p^3 - p + 2, y = \frac{3}{4}p^4 - \frac{p^2}{2} + c\right]$$

5. $y - 2xp + ayp^2 = 0$

$$\left[\textbf{Ans.}: 2cx = y^2 + ac^2\right]$$

6. $p^3 - p(y+3) + x = 0$

$$\left[\begin{aligned}\textbf{Ans.}: x &= cp(1-p^2)^{-\frac{1}{2}} + 2p\\ y &= c(1-p^2)^{-\frac{1}{2}}\end{aligned}\right]$$

7. $y - xp = a(y^2 + p)$

$$\left[\textbf{Ans.}: c(x+a)(ay-1) + y = 0\right]$$

8. $xp^3 = a + bp$

$$\left[\textbf{Ans.}: x = \frac{a}{p^3} + \frac{b}{p^2}, y = \frac{3a}{2p^2} + \frac{2b}{p} + c\right]$$

9. $ayp^2 + (2x - b)p - y = 0$

$$\left[\textbf{Ans.}: ac^2 + (2x-b)c - y^2 = 0\right]$$

10. $p^2y + 2px - y = 0$

$$\left[\textbf{Ans.}: 2cx = y^2 - c^2\right]$$

10.4.1 Clairaut's Equation

An equation of the form

$$y = px + f(p) \qquad \ldots(10.23)$$

is known as Clairaut's equation.

Differentiating Eq. (10.23) w.r.t. x,

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0$$

EXAMPLE 10.44 Solve $y = 2px + y^2p^3$.

Solution:

$$y = 2px + y^2p^3 \quad \dots(1)$$

$$x = \frac{y}{2p} - \frac{y^2p^2}{2} \quad \dots(2)$$

Differentiating Eq. (2) w.r.t. y,

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - y^2p\frac{dp}{dy}$$

$$\frac{1}{p} - \frac{1}{2p} + yp^2 = -y\left(\frac{1}{2p^2} + py\right)\frac{dp}{dy}$$

$$\left(1+2yp^3\right)p = -y\left(1+2yp^3\right)\frac{dp}{dy}$$

$$\left(1+2yp^3\right)p + \left(1+2yp^3\right)y\frac{dp}{dy} = 0$$

$$\left(p + y\frac{dp}{dy}\right)\left(1+2yp^3\right) = 0$$

Neglecting the second factor which does not contain $\frac{dp}{dy}$,

$$p + y\frac{dp}{dy} = 0$$

$$\frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating both the sides,

$$\int\frac{dp}{p} + \int\frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$py = c$$

$$p = \frac{c}{y}$$

Substituting the value of p in Eq. (1),

$$y = 2\frac{c}{y}x + y^2\frac{c^3}{y^3}$$

$$y^2 = 2cx + c^3$$

which is the general solution.

EXERCISE 10.10

Solve the following differential equations:

1. $x = y + a\log p$

$\left[\textbf{Ans.:} x = c + a\log\frac{p}{p-1}, y = c - a\log(p-1)\right]$

2. $x = y + p^2$

$\left[\begin{array}{l}\textbf{Ans.:} x = c - \{2p + 2\log(p-1)\} \\ y = c - \{p^2 + 2p + 2\log(p-1)\}\end{array}\right]$

3. $p = \tan\left(x - \frac{p}{1+p^2}\right)$

$\left[\textbf{Ans.:} y = -\frac{1}{1+p^2} + c\right]$

4. $x = p^3 - p + 2$

$\left[\textbf{Ans.:} x = p^3 - p + 2, y = \frac{3}{4}p^4 - \frac{p^2}{2} + c\right]$

5. $y - 2xp + apy^2 = 0$

$\left[\textbf{Ans.:} 2cx = y^2 + ac^2\right]$

6. $p^3 - p(y+3) + x = 0$

$\left[\begin{array}{l}\textbf{Ans.:} x = cp(1-p^2)^{-\frac{1}{2}} + 2p \\ y = c(1-p^2)^{-\frac{1}{2}}\end{array}\right]$

7. $y - xp = a(y^2 + p)$

$\left[\textbf{Ans.:} c(x+a)(ay-1) + y = 0\right]$

8. $xp^3 = a + bp$

$\left[\textbf{Ans.:} x = \frac{a}{p^3} + \frac{b}{p^2}, y = \frac{3a}{2p^2} + \frac{2b}{p} + c\right]$

9. $ayp^2 + (2x - b)p - y = 0$

$\left[\textbf{Ans.:} ac^2 + (2x-b)c - y^2 = 0\right]$

10. $p^2y + 2px - y = 0$

$\left[\textbf{Ans.:} 2cx = y^2 - c^2\right]$

10.4.1 Clairaut's Equation

An equation of the form

$$y = px + f(p) \qquad \text{...(10.23)}$$

is known as Clairaut's equation.

Differentiating Eq. (10.23) w.r.t. x,

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

$$x + f'(p) = 0 \text{ or } \frac{dp}{dx} = 0$$

The solution of the equation $\frac{dp}{dx} = 0$ is

$$p = c \quad \text{...(10.24)}$$

Eliminating p from Eqs (10.23) and (10.24),

$$y = cx + f(c) \quad \text{...(10.25)}$$

which is the general solution of Eq. (10.23). Hence, the solution of Clairaut's equation is obtained on replacing p by c.

Note Elimination of p from the equation $x + f'(p) = 0$ gives a solution which does not contain any arbitrary constants. Such a solution is called a singular solution of Eq. (10.23) which gives the envelope of the family of straight lines represented by Eq. (10.25).
In this section, only general solutions are obtained.

HISTORICAL DATA

Alexis Claude Clairaut (1713–1765) was a prominent French mathematician, astronomer, geophysicist, and intellectual. Clairaut was born in Paris, France, where his father taught mathematics. He was a prodigy—at the age of twelve, he wrote a memoir on four geometrical curves and under his father's tutelage, he made such rapid progress in the subject that in his thirteenth year he read before the *Académie française* an account of the properties of four curves which he had discovered. When only sixteen, he finished a treatise on tortuous curves, which, on its publication in 1731, procured his admission into the French Academy of Sciences, although he was below the legal age as he was only eighteen.

In 1736, together with Pierre Louis Maupertuis, he took part in the expedition to Lapland, which was undertaken for the purpose of estimating a degree of the meridian arc. After his return he published his treatise *Théorie de la figure de la terre* (1743). In this work, he promulgated the theorem, known as Clairaut's theorem, which connects the gravity at points on the surface of a rotating ellipsoid with the compression and the centrifugal force at the equator. In 1849, Stokes showed that Clairaut's result was true whatever the interior constitution or density of the Earth, provided the surface was a spheroid of equilibrium of small ellipticity.

Clairaut subsequently wrote various papers on the orbit of the Moon, and on the motion of comets as affected by the perturbation of the planets, particularly on the path of Halley's comet.

10.4.2 Lagrange's Equation*

An equation of the form

$$y = xf_1(p) + f_2(p) \qquad ...(10.26)$$

is known as Lagrange's equation.

This is a generalization of Clairaut's equation. Putting $f_1(p) = p$, Clairaut's equation is obtained. Differentiating Eq. (10.26) w.r.t. x,

$$p = f_1(p) + \left[xf_1'(p) + f_2'(p)\right]\frac{dp}{dx}$$

$$\left[p - f_1(p)\right]\frac{dx}{dp} - xf_1'(p) = f_2'(p)$$

which is a linear differential equation in x, and, hence, it can be solved.

EXAMPLE 10.45 *Solve* $e^{3x}(p-1) + p^3e^{2y} = 0.$

Solution: Let $e^x = u$ and $e^y = v$

$$e^x dx = du \text{ and } e^y dy = dv$$

$$\frac{e^y}{e^x}\frac{dy}{dx} = \frac{dv}{du}$$

$$p = \frac{dy}{dx} = \frac{e^x}{e^y}\cdot\frac{dv}{du} = \frac{u}{v}\cdot\frac{dv}{du} = \frac{u}{v}P, \text{ where } P = \frac{dv}{du}$$

Substituting the value of u, v, and p in the given equation,

$$u^3\left(\frac{u}{v}P - 1\right) + \left(\frac{u}{v}P\right)^3 v^2 = 0$$

$$\frac{u^3}{v}\left(uP - v + P^3\right) = 0$$

$$uP - v + P^3 = 0$$

$$v = uP + P^3$$

which is a differential equation in Clairaut's form. Hence, the general solution is

$$v = uc + c^3$$

$$e^y = ce^x + c^3$$

where c is an arbitrary constant.

EXAMPLE 10.46 *Solve* $(px - y)(py + x) = a^2p.$

Solution: Let $x^2 = u$ and $y^2 = v$

* Refer Chapter 3 for Historical Data on Lagrange.

$2xdx = du$ and $2ydy = dv$

$$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$$

$$p = \frac{x}{y}\frac{dv}{du} = \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du} = \frac{\sqrt{u}}{\sqrt{v}}P, \text{ where } P = \frac{dv}{du}$$

Substituting the value of u, v, and p in the given equation,

$$\left(\frac{\sqrt{u}}{\sqrt{v}}P\sqrt{u} - \sqrt{v}\right)\left(\frac{\sqrt{u}}{\sqrt{v}}P\sqrt{v} + \sqrt{u}\right) = a^2\frac{\sqrt{u}}{\sqrt{v}}P$$

$$\frac{1}{\sqrt{v}}(uP - v)\sqrt{u}(P+1) = a^2\frac{\sqrt{u}}{\sqrt{v}}P$$

$$(uP - v)(P+1) = a^2P$$

$$uP - v = \frac{a^2P}{P+1}$$

$$v = uP - \frac{a^2P}{P+1}$$

which is a differential equation in Clairaut's form.
Hence, the general solution is

$$v = uc - \frac{a^2c}{c+1}$$

$$y^2 = cx^2 - \frac{a^2c}{c+1}$$

EXAMPLE 10.47 *Solve* $x^2p^2 + y(2x+y)p + y^2 = 0.$

Solution: Let $y = u$ and $xy = v$

$dy = du$ and $xdy + ydx = dv$

$$\frac{xdy + ydx}{dy} = \frac{dv}{du}$$

$$x + y\frac{dx}{dy} = \frac{dv}{du}$$

$$\frac{v}{u} + u\frac{1}{p} = P, \quad \text{where } P \equiv \frac{dv}{du}$$

$$p = \frac{u^2}{uP - v}$$

Substituting the value of u, v, and p in the given equation,

$$\frac{v^2}{u^2}\frac{u^4}{(uP-v)^2}+u\left(2\frac{v}{u}+u\right)\frac{u^2}{uP-v}+u^2=0$$

$$v^2u^2+\left(2v+u^2\right)(uP-v)u^2+u^2(uP-v)^2=0$$

$$v^2u^2+\left(2v+u^2\right)u^3P-\left(2v+u^2\right)u^2v+u^2\left(u^2P^2-2uvP+v^2\right)=0$$

$$-vu^4+\left(2vu^3+u^5-2u^3v\right)P+u^4P^2=0$$

$$-vu^4+u^5P+u^4P^2=0$$

$$\left(-v+uP+P^2\right)u^4=0$$

$$v=uP+P^2$$

which is a differential equation in Clairaut's form.
Hence, the general solution is

$$v=uc+c^2$$
$$xy=yc+c^2$$

EXERCISE 10.11

Solve the following differential equations:

1. $y=px+p-p^2$

 $\left[\textbf{Ans.: } y=cx+c-c^2\right]$

2. $y=px+\left(1+p^2\right)^{\frac{1}{2}}$

 $\left[\textbf{Ans.: } y=cx+\left(1+c^2\right)^{\frac{1}{2}}\right]$

3. $p=\tan(px-y)$

 $\left[\textbf{Ans.: } c=\tan(cx-y)\right]$

4. $\dfrac{(y-px)^2}{1+p^2}=a^2$

 $\left[\textbf{Ans.: } (y-cx)^2=a^2\left(1+c^2\right)\right]$

5. $xp^3-(y+3)p^2+4=0$

 $\left[\textbf{Ans.: } y=cx+\dfrac{4}{c^2}-3\right]$

6. $p=e^{(y-px)}$

 $\left[\textbf{Ans.: } c=e^{(y-cx)}\right]$

7. $(xp-y)^2=p^2-1$

 $\left[\textbf{Ans.: } (cx-y)^2=c^2-1\right]$

8. $(y-px)^2\left(1+p^2\right)=a^2p^2$

 $\left[\textbf{Ans.: } (y-cx)^2\left(1+c^2\right)=a^2c^2\right]$

9. $(x-a)p^2+(x-y)p-y=0$

 $\left[\textbf{Ans.: } y=cx-\dfrac{ac^2}{c+1}\right]$

10. $y=px+\sin^{-1}p$

 $\left[\textbf{Ans.: } y=cx+\sin^{-1}c\right]$

11. $xyp^2 - (x^2 + y^2 - 1)p + xy = 1$

$$\left[\textbf{Ans.}: c^2x^2 - c(x^2 + y^2 + 1) + y^2 = 0\right]$$

12. $y = px + \dfrac{p}{x}$

$$\left[\textbf{Ans.}: y = cx^3 + c\right]$$

13. $xy(y - px) = x + py$

$$\left[\textbf{Ans.}: y^2 = cx^2 + (1 + c)\right]$$

14. $xp^2 - 2yp + x + 2y = 0$

$$\left[\textbf{Ans.}: 2c^2x^2 - 2c(y - x) + 1 = 0\right]$$

15. $y = 2xp + \tan^{-1}(xp^2)$

$$\left[\textbf{Ans.}: y = c\sqrt{x} + \tan^{-1}\left(\frac{c^2}{4}\right)\right]$$

16. $(px^2 + y^2)(px + y) = (p + 1)^2$

$$\left[\textbf{Ans.}: c^2(x + y) - cxy - 1 = 0\right]$$

17. $(2x^2 + 1)p^2 + (x^2 + y^2 + 2xy + 2)p + 2y^2 + 1 = 0$

$$\left[\textbf{Ans.}: xy - 1 = c(x + y) + c^2\right]$$

18. $y^2 \log y = xyp + p^2$

$$\left[\textbf{Ans.}: \log y = c(c + x)\right]$$

10.5 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = 0 \qquad \ldots (10.27)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants, is known as a homogeneous linear differential equation of order n with constant coefficients. This equation is known as linear since the degree of the dependent variable y and all its differential coefficients is one.

Equation (10.27) can also be written as

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$$

$$f(D)y = 0$$

where $f(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n$.

Here, $D \equiv \dfrac{d}{dx}$ is known as the *differential operator*.

The operator D obeys the laws of algebra.

General Solution of a Homogeneous Linear Differential Equation

The homogeneous equation

$$f(D)y = 0 \qquad \ldots (10.28)$$

can be solved by replacing D by m in $f(D)$ and solving the *Auxiliary Equation* (*AE*).

$$f(m) = 0 \qquad \ldots (10.29)$$

The general solution of Eq. (10.28) depends upon the nature of the roots of the auxiliary equation (10.29). If $m_1, m_2, m_3, \ldots, m_n$ are n roots of the auxiliary equation, the following cases arise:

Case I *Real and Distinct Roots*

If the roots $m_1, m_2, m_3, \ldots, m_n$ are real and distinct then the solution of Eq. (10.27) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

Case II *Real and Repeated Roots*

If two roots m_1, m_2 are real and equal and the remaining $(n-2)$ roots $m_3, m_4, \ldots, m_n$ are all real and distinct then the solution of Eq. (10.27) is

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Note If, however, r roots $m_1, m_2, m_3, \ldots, m_r$ are equal and the remaining $(n-r)$ roots $m_{r+1}, m_{r+2}, \ldots, m_n$ are all real and distinct then the solution of Eq. (10.27) is

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots + c_r x^{r-1})e^{m_1 x} + c_{r+1} e^{m_{r+1} x} + \ldots + c_n e^{m_n x}$$

Case III *Complex Roots*

If two roots m_1, m_2 are complex, say, $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ (conjugate pair) and the remaining $(n-2)$ roots $m_3, m_4, \ldots, m_n$ are real and distinct then the solution of Eq. (10.27) is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Here, α is the real part and β is the imaginary part of the conjugate pair of complex roots.

Note If, however, two pairs of complex roots m_1, m_2 and m_3, m_4 are equal, say, $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$ and the remaining $(n-4)$ roots $m_5, m_6, \ldots, m_n$ are real and distinct then the solution of Eq. (10.27) is

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots + c_n e^{m_n x}$$

Remarks

(i) In all the above cases, $c_1, c_2, \ldots, c_n$ are arbitrary constants.
(ii) In the general solution of a homogeneous equation, the number of arbitrary constants is always equal to the order of that homogeneous equation.

EXAMPLE 10.48 *Solve* $2D^2 y - 2Dy - y = 0$.

Solution: The equation can be written as

$$(2D^2 - 2D - 1)y = 0$$

The auxiliary equation is

$$2m^2-2m-1=0$$

$$m=\frac{2\pm\sqrt{4+8}}{4}=\frac{2\pm2\sqrt{3}}{4}=\frac{1\pm\sqrt{3}}{2}$$

The roots are real and distinct.
Hence, the general solution is

$$y=c_1e^{\left(\frac{1+\sqrt{3}}{2}\right)x}+c_2e^{\left(\frac{1-\sqrt{3}}{2}\right)x}$$

EXAMPLE 10.49 *Solve* $(D^4-6D^3+12D^2-8D)y=0$.

Solution: The auxiliary equation is

$$m^4-6m^3+12m^2-8m=0$$
$$m(m^3-6m^2+12m-8)=0$$
$$m(m-2)(m^2-4m+4)=0$$
$$m(m-2)(m-2)^2=0$$
$$m=0,\ 2,\ 2,\ 2$$

The root $m=2$ is repeated thrice.
Hence, the general solution is

$$y=c_1e^{0x}+(c_2+c_3x+c_4x^2)e^{2x}=c_1+(c_2+c_3x+c_4x^2)e^{2x}$$

EXAMPLE 10.50 *Solve* $\left(D^3+1\right)y=0$.

Solution: The auxiliary equation is

$$m^3+1=0$$
$$(m+1)(m^2-m+1)=0$$
$$m+1=0, m^2-m+1=0$$
$$m=-1, m=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

The root $m=-1$ is real and two roots are complex with $\alpha=\frac{1}{2}$, $\beta=\frac{\sqrt{3}}{2}$.
Hence, the general solution is

$$y=c_1e^{-x}+e^{\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)$$

EXAMPLE 10.51 *Solve* $(D^4 - 1)y = 0$.

Solution: The auxiliary equation is

$$m^4 - 1 = 0$$
$$m^4 = 1$$
$$m^2 = 1, \quad m^2 = -1$$
$$m = \pm 1, \quad m = \pm i$$

The roots $m = 1$ and $m = -1$ are real and distinct, and the two roots are complex with $\alpha = 0, \beta = 1$.
Hence, the general solution is

$$y = c_1e^x + c_2e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x) = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$$

EXAMPLE 10.52 *Solve* $(D^4 + 8D^2 + 16)y = 0$.

Solution: The auxiliary equation is

$$m^4 + 8m^2 + 16 = 0$$
$$(m^2 + 4)^2 = 0$$
$$m = \pm 2i, \ \pm 2i$$

The pair of roots are complex and repeated twice with $\alpha = 0, \ \beta = 2$.
Hence, the general solution is

$$y = e^{0x}[(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x] = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$$

EXERCISE 10.12

Solve the following differential equations:

1. $(D^2 + D - 2)y = 0$

   $\left[\textbf{Ans.:}\ y = c_1e^{-2x} + c_2e^x\right]$

2. $(D^2 + 4D + 1)y = 0$

   $\left[\textbf{Ans.:}\ y = c_1e^{(-2+\sqrt{3})x} + c_2e^{(-2-\sqrt{3})x}\right]$

3. $(D^2 + 2\pi D + \pi^2)y = 0$

   $\left[\textbf{Ans.:}\ y = (c_1 + c_2x)e^{-\pi x}\right]$

4. $(D^2 + 6D + 11)y = 0$

   $\left[\textbf{Ans.:}\ y = e^{-3x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)\right]$

5. $[D^2 - 2aD + (a^2 + b^2)y] = 0$

   $[\textbf{Ans.:}\ y = e^{ax}(c_1 \cos bx + c_2 \sin bx)]$

6. $(D^3 + 5D^2 + 8D + 6)y = 0$

   $\left[\textbf{Ans.:}\ y = c_1e^{-3x} + e^{-x}(c_2 \cos x + c_3 \sin x)\right]$

7. $(D^4 - 3D^3 + 3D^2 - D)y = 0$

$\left[\textbf{Ans.:}\ y = c_1 + (c_2 + c_3x + c_4x^2)e^x\right]$

8. $(D^4 + 8D^2 - 9)y = 0$

$\left[\textbf{Ans.:}\ y = c_1e^{x} + c_2e^{-x} + c_3\cos 3x + c_4\sin 3x\right]$

9. $(D^4 + 2D^3 - 9D^2 - 10D + 50)y = 0$

$\left[\begin{aligned}\textbf{Ans.:}\ y = e^{2x}&(c_1\cos x + c_2\sin x)\\ + e^{-3x}&(c_3\cos x + c_4\sin x)\end{aligned}\right]$

10. $(D^4 + 18D^3 + 81)y = 0$

$\left[\textbf{Ans.:}\ y = (c_1 + c_2x)\cos 3x + (c_3 + c_4x)\sin 3x\right]$

10.6 NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANT COEFFICIENTS

An ordinary differential equation of the form

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_ny = Q(x) \quad \ldots (10.30)$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and Q is a function of x, is known as a *nonhomogeneous linear differential equation with constant coefficients.*

Equation (10.30) can also be written as

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n)y = Q(x) \quad \ldots (10.31)$$

$$f(D)y = Q(x)$$

where $f(D) = a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots + a_n$

10.6.1 General Solution of a Nonhomogeneous Linear Differential Equation

A general solution of Eq. (10.30) is obtained in two parts as

General solution = Complementary Function + Particular Integral

$$y = CF + PI$$

The Complementary Function (CF) is the general solution of the homogeneous equation obtained by putting $Q(x) = 0$ in Eq. (10.30).

The Particular Integral (PI) is any particular solution of the nonhomogeneous equation (10.30) and contains no arbitrary constants.

Inverse Operator and Particular Integral $f(D)$ is known as the *differential operator* and $\dfrac{1}{f(D)}$ is known as the *inverse differential operator.*

$$f(D)\left[\frac{1}{f(D)}Q(x)\right] = Q(x)$$

This shows that $\frac{1}{f(D)}Q(x)$ satisfies the equation $f(D)y = Q(x)$ and since $\frac{1}{f(D)}Q(x)$ does not contain any arbitrary constants, gives the PI of the equation $f(D)y = Q(x)$.

Hence,
$$\text{PI} = \frac{1}{f(D)}Q(x)$$

(i) If $f(D) = D$ then $\text{PI} = \frac{1}{D}Q(x) = \int Q(x)\,dx$

(ii) If $f(D) = D - a$ then the equation $f(D)y = Q(x)$ becomes

$$(D-a)y = Q(x)$$

$$\frac{dy}{dx} - ay = Q(x) \text{ is a first-order linear differential equation.}$$

$$\text{IF} = e^{\int -a\,dx} = e^{-ax}$$

The solution is

$$ye^{-ax} = \int e^{-ax}Q(x)\,dx + c$$

$$y = e^{ax}\int Q(x)e^{-ax}\,dx + ce^{ax}$$

Here, ce^{ax} is the complementary function since it contains the arbitrary constant c and $e^{ax}\int Q(x)e^{-ax}\,dx$ is the particular integral.

Hence,
$$\text{PI} = \frac{1}{D-a}Q(x) = e^{ax}\int Q(x)e^{-ax}\,dx$$

10.6.2 Direct (Short-cut) Method of Obtaining Particular Integral (PI)

This method depends on the nature of $Q(x)$ in Eq. (10.30). The particular integral by this method can be obtained when $Q(x)$ has the following forms:

(i) $Q(x) = e^{ax+b}$

(ii) $Q(x) = \sin(ax+b)$ or $\cos(ax+b)$

(iii) $Q(x) = x^m$ or polynomial in x

(iv) $Q(x) = e^{ax}v(x)$

(v) $Q(x) = xv(x)$

Case I $Q(x) = e^{ax+b}$

$$f(D)y = e^{ax+b}$$

Now, $D(e^{ax+b}) = ae^{ax+b}$, $D^2(e^{ax+b}) = a^2e^{ax+b}$, , $D^n e^{ax+b} = a^n e^{ax+b}$

Consider $f(D)(e^{ax+b}) = (a_0D^n + a_1D^{n-1} + \ldots + a_n)e^{ax+b} = (a_0a^n + a_1a^{n-1} + \ldots + a_n)e^{ax+b} = f(a)e^{ax}$

Operating both the sides with $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)\left(e^{ax+b}\right)\right]=\frac{1}{f(D)}\left[f(a)e^{ax+b}\right]$$

$$e^{ax+b}=f(a)\frac{1}{f(D)}e^{ax+b}$$

$$\frac{1}{f(a)}e^{ax+b}=\frac{1}{f(D)}e^{ax+b},\quad f(a)\neq 0$$

$$\frac{1}{f(D)}e^{ax+b}=\frac{1}{f(a)}e^{ax+b},\quad f(a)\neq 0$$

Hence, PI $=\frac{1}{f(a)}e^{ax+b}$ if $f(a)\neq 0$

Note If $f(a)=0$ then $(D-a)$ is a factor of $f(D)$ and, hence, the above rule fails.

Let $f(D)=(D-a)\phi(D)$, where $\phi(a)\neq 0$

$$\text{PI}=\frac{1}{f(D)}e^{ax+b}=\frac{1}{(D-a)\phi(D)}e^{ax+b}=\frac{1}{\phi(a)}\cdot\frac{1}{(D-a)}e^{ax+b}$$

$$=\frac{1}{\phi(a)}\cdot e^{ax}\int e^{-ax}e^{ax+b}dx=\frac{1}{\phi(a)}\cdot e^{ax}\cdot xe^{b}=x\frac{1}{\phi(a)}e^{ax+b} \quad \cdots (10.32)$$

Since
$$f(D)=(D-a)\phi(D)$$
$$f'(D)=(D-a)\phi'(D)+\phi(D)$$
$$f'(a)=\phi(a)$$

Substituting in Eq. (10.32),

$$\frac{1}{f(D)}e^{ax+b}=x\cdot\frac{1}{f'(a)}e^{ax+b},\text{ where } f'(a)\neq 0$$

If $f'(a)=0$ then repeating the above process,

$$\frac{1}{f(D)}e^{ax+b}=x\left[x\cdot\frac{1}{f''(a)}e^{ax+b}\right]=x^2\frac{1}{f''(a)}e^{ax+b},\qquad \text{where } f''(a)\neq 0$$

In general, if $(D-a)^r$ is a factor of $f(D)$ then

$$\frac{1}{f(D)}e^{ax}=x^r\frac{1}{f^{(r)}(a)}e^{ax+b}$$

Hence,
$$\text{PI}=x^r\frac{1}{f^{(r)}(a)}e^{ax+b}.$$

EXAMPLE 10.53 *Solve* $(D^2-4)y=e^{2x}+e^{-4x}$.

Solution: The auxiliary equation is

$$m^2-4=0$$

$$(m-2)(m+2)=0$$

$$m=2,-2 \quad \text{(real and distinct)}$$

$$\text{CF}=c_1e^{2x}+c_2e^{-2x}$$

$$\text{PI}=\frac{1}{D^2-4}\left(e^{2x}+e^{-4x}\right)=\frac{1}{D^2-4}e^{2x}+\frac{1}{D^2-4}e^{-4x}$$

$$=x\cdot\frac{1}{2D}e^{2x}+\frac{1}{(-4)^2-4}e^{-4x}=x\cdot\frac{1}{2(2)}e^{2x}+\frac{1}{12}e^{-4x}$$

$$=\frac{x}{4}e^{2x}+\frac{1}{12}e^{-4x}$$

Hence, the general solution is

$$y=c_1e^{2x}+c_2e^{-2x}+\frac{x}{4}e^{2x}+\frac{1}{12}e^{-4x}$$

EXAMPLE 10.54 *Solve* $(D^6-64)y=e^x\cosh 2x$.

Solution: The auxiliary equation is

$$m^6-64=0$$

$$(m^3)^2-(8)^2=0$$

$$(m^3+8)(m^3-8)=0$$

$$(m+2)(m^2-2m+4)(m-2)(m^2+2m+4)=0$$

$$m+2=0,\ m^2-2m+4=0,$$
$$m-2=0,\ m^2+2m+4=0$$

$$m=-2,\quad m=1\pm i\sqrt{3},\qquad m=2,\quad m=-1\pm i\sqrt{3}$$

Two roots are real and the two pairs of roots are complex.

$$\text{CF}=c_1e^{-2x}+c_2e^{2x}+e^{x}\left(c_3\cos\sqrt{3}x+c_4\sin\sqrt{3}x\right)+e^{-x}\left(c_5\cos\sqrt{3}x+c_6\sin\sqrt{3}x\right)$$

$$\text{PI}=\frac{1}{D^6-64}e^x\cosh 2x=\frac{1}{D^6-64}\left[e^x\left(\frac{e^{2x}+e^{-2x}}{2}\right)\right]$$

$$=\frac{1}{D^6-64}\left[\frac{1}{2}(e^{3x}+e^{-x})\right]=\frac{1}{2}\left[\frac{1}{3^6-64}e^{3x}+\frac{1}{(-1)^6-64}e^{-x}\right]=\frac{1}{2}\left(\frac{e^{3x}}{665}-\frac{e^{-x}}{63}\right)$$

Hence, the general solution is

$$y=c_1e^{-2x}+c_2e^{2x}+e^x(c_3\cos\sqrt{3}x+c_4\sin\sqrt{3}x)+e^{-x}(c_5\cos\sqrt{3}x+c_6\sin\sqrt{3}x)+\frac{1}{2}\left(\frac{e^{3x}}{665}-\frac{e^{-x}}{63}\right)$$

EXAMPLE 10.55 Solve $(D^2+6D+9)y = 5^x - \log 2$.

Solution: The auxiliary equation is

$$m^2+6m+9=0$$
$$(m+3)^2=0$$
$$m=-3,-3 \quad \text{(real and repeated)}$$
$$\text{CF} = (c_1+c_2x)e^{-3x}$$
$$\text{PI} = \frac{1}{D^2+6D+9}(5^x-\log 2) = \frac{1}{(D+3)^2}(e^{x\log 5}) - \frac{1}{(D+3)^2}(\log 2)e^{0\cdot x}$$
$$= \frac{1}{(\log 5+3)^2}e^{x\log 5} - \log 2\cdot\frac{1}{(0+3)^2}e^{0\cdot x} = \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Hence, the general solution is

$$y = (c_1+c_2x)e^{-3x} + \frac{5^x}{(\log 5+3)^2} - \frac{\log 2}{9}$$

Case II $Q(x) = \sin(ax+b)$ or $\cos(ax+b)$

(i) If $Q(x) = \sin(ax+b)$ then Eq. (10.31) reduces to

$$f(D)y = \sin(ax+b)$$

Now,

$$D[\sin(ax+b)] = a\cos(ax+b)$$
$$D^2[\sin(ax+b)] = (-a^2)\sin(ax+b)$$
$$D^3[\sin(ax+b)] = -a^3\cos(ax+b)$$
$$D^4[\sin(ax+b)] = a^4\sin(ax+b)$$
$$(D^2)^2[\sin(ax+b)] = (-a^2)^2\sin(ax+b)$$

In general,

$$(D^2)^r[\sin(ax+b)] = (-a^2)^r\sin(ax+b)$$

This shows that

$$\phi(D^2)\sin(ax+b) = \phi(-a^2)\sin(ax+b)$$

Operating both the sides with $\dfrac{1}{\phi(D^2)}$,

$$\frac{1}{\phi(D^2)}\left[\phi(D^2)\sin(ax+b)\right] = \frac{1}{\phi(D^2)}\left[\phi(-a^2)\sin(ax+b)\right]$$

$$\sin(ax+b) = \phi(-a^2)\frac{1}{\phi(D^2)}\sin(ax+b)$$

$$\frac{1}{\phi(-a^2)}\sin(ax+b) = \frac{1}{\phi(D^2)}\sin(ax+b)$$

$$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b)$$

If $f(D) = \phi(D^2)$ then

$$PI = \frac{1}{f(D)}\sin(ax+b) = \frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b), \text{ if } \phi(-a^2) \neq 0$$

If $\phi(-a^2) = 0$ then $(D^2 + a^2)$ is a factor of $\phi(D^2)$ and, hence, the above rule fails.

$$PI = \frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(D^2)}\left[\text{IP of } e^{i(ax+b)}\right]$$

$$= \text{IP of } \frac{1}{\phi(D^2)}e^{i(ax+b)}$$

$$= \text{IP of } x\cdot\frac{1}{\phi'(D^2)}e^{i(ax+b)}$$

$$= \text{IP of } x\cdot\frac{1}{\phi'(i^2a^2)}e^{i(ax+b)} = \text{IP of } x\cdot\frac{1}{\phi'(-a^2)}e^{i(ax+b)}$$

$$= x\cdot\frac{1}{\phi'(-a^2)}\sin(ax+b)$$

$$\left[\because \phi(i^2a^2) = \phi(-a^2) = 0\right]$$

If $\phi'(-a^2) = 0$ then

$$\frac{1}{\phi(D^2)}\sin(ax+b) = x^2\cdot\frac{1}{\phi''(-a^2)}\sin(ax+b), \quad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2) = 0$ then

$$PI = \frac{1}{\phi(D^2)}\sin(ax+b) = x^{(r+1)}\frac{1}{\phi^{(r+1)}(-a^2)}\sin(ax+b), \quad \text{where } \phi^{(r+1)}(-a^2) \neq 0$$

(ii) Similarly, if $Q(x) = \cos(ax+b)$

$$PI = \frac{1}{\phi(D^2)}\cos(ax+b) = \frac{1}{\phi(-a^2)}\cos(ax+b), \; \phi(-a^2) \neq 0$$

If $\phi(-a^2) = 0$ then

$$PI = \frac{1}{\phi(D^2)}\cos(ax+b) = x\cdot\frac{1}{\phi'(-a^2)}\cos(ax+b)$$

If $\phi'(-a^2) = 0$ then

$$PI = \frac{1}{\phi(D^2)}\cos(ax+b) = x^2\cdot\frac{1}{\phi''(-a^2)}\cos(ax+b), \quad \text{where } \phi''(-a^2) \neq 0$$

In general, if $\phi^{(r)}(-a^2) = 0$ then

$$\text{PI} = \frac{1}{\phi(D^2)}\cos(ax+b) = x^{r+1}\frac{1}{\phi^{(r+1)}(-a^2)}\cos(ax+b), \quad \text{where } \phi^{(r+1)}(-a^2) \neq 0$$

Note If after replacing D^2 by $-a^2$, $f(D)$ contains terms of D then the denominator is rationalized to obtain the even powers of D.

EXAMPLE 10.56 *Solve* $(D^2+9)y = \sin 4x$.

Solution: The auxiliary equation is

$$m^2+9=0$$

$$m = \pm 3i \text{ (complex)}$$

$$\text{CF} = c_1\cos 3x + c_2 \sin 3x$$

$$\text{PI} = \frac{1}{D^2+9}\sin 4x = \frac{1}{-4^2+9}\sin 4x = -\frac{1}{7}\sin 4x$$

Hence, the general solution is

$$y = c_1\cos 3x + c_2\sin 3x - \frac{1}{7}\sin 4x$$

EXAMPLE 10.57 *Find the particular integral of* $(D^2+1)y = \sin x \sin 2x$.

Solution:
$$\text{PI} = \frac{1}{D^2+1}\sin x\sin 2x = \frac{1}{D^2+1}\left(\frac{\cos x - \cos 3x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cos x - \frac{1}{D^2+1}\cos 3x\right] = \frac{1}{2}\left[x\cdot\frac{1}{2D}\cos x - \frac{1}{-3^2+1}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2}\int\cos x\,dx + \frac{1}{8}\cos 3x\right] = \frac{1}{2}\left[\frac{x}{2}\sin x + \frac{1}{8}\cos 3x\right] = \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

EXAMPLE 10.58 *Solve* $(D^2+1)y = \sin^2 x$.

Solution: The auxiliary equation is

$$m^2+1=0$$

$$m = \pm i \text{ (complex)}$$

$$\text{CF} = c_1\cos x + c_2\sin x$$

$$\text{PI} = \frac{1}{D^2+1}\sin^2 x = \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{D^2+1}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{D^2+1}\cos 2x = \frac{1}{2}\cdot\frac{1}{D^2+1}e^{0x} - \frac{1}{2}\cdot\frac{1}{-2^2+1}\cos 2x$$

$$= \frac{1}{2} \cdot \frac{1}{0+1} e^{0x} - \frac{1}{2} \cdot \frac{1}{-3} \cos 2x = \frac{1}{2} + \frac{1}{6} \cos 2x$$

Hence, the general solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6} \cos 2x$$

EXAMPLE 10.59 ✓*Solve* $(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^x$.

Solution: The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 2, 1 \quad \text{(real and distinct)}$$

$$\text{CF} = c_1 e^x + c_2 e^{2x}$$

$$\text{PI} = \frac{1}{D^2 - 3D + 2}\left[2\cos(2x+3) + 2e^x\right] = 2\frac{1}{D^2 - 3D + 2}\cos(2x+3) + 2\frac{1}{D^2 - 3D + 2} e^x$$

$$= 2\frac{1}{-2^2 - 3D + 2}\cos(2x+3) + 2\frac{1}{(D-1)(D-2)} e^x = 2\frac{1}{-2 - 3D}\cos(2x+3) + 2\frac{1}{D-1}\frac{1}{(1-2)} e^x$$

$$= -2\frac{3D-2}{9D^2 - 4}\cos(2x+3) - 2x\frac{1}{1}e^x = \frac{-2\left[3D\cos(2x+3) - 2\cos(2x+3)\right]}{9(-2^2) - 4} - 2xe^x$$

$$= \frac{-2\left[-3\sin(2x+3)(2) - 2\cos(2x+3)\right]}{-36 - 4} - 2xe^x = \frac{1}{20}\left[-6\sin(2x+3) - 2\cos(2x+3)\right] - 2xe^x$$

$$= -\frac{1}{10}\left[3\sin(2x+3) + \cos(2x+3)\right] - 2xe^x$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{2x} - \frac{1}{10}\left[3\sin(2x+3) + \cos(2x+3)\right] - 2xe^x$$

Case III $Q(x) = x^m$

In this case, Eq. (10.31) reduces to $f(D)y = x^m$.

Hence, $$\text{PI} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m = 1 + \phi(D)^{-1} x^m$$

where $f(D)$ is expressed as $1 + \phi(D)$.

Expanding in ascending powers of D up to D^m using *binomial expansion*, since $D^n x^m = 0$ when $n > m$,

$$\text{PI} = (a_0 + a_1 D + a_2 D^2 + \ldots + a_m D^m) x^m$$

EXAMPLE 10.60 Solve $(D^4 - 2D^3 + D^2)y = x^3$.

Solution: The auxiliary equation is

$$m^4 - 2m^3 + m^2 = 0$$
$$m^2\left(m^2 - 2m + 1\right) = 0$$
$$m^2\left(m - 1\right)^2 = 0$$
$$m = 0, 0, 1, 1 \text{ (real and repeated)}$$

Both the roots are real and repeated twice.

$$CF = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x)e^x = c_1 + c_2 x + (c_3 + c_4 x)e^x$$

$$PI = \frac{1}{D^4 - 2D^3 + D^2} x^3 = \frac{1}{D^2(D^2 - 2D + 1)} x^3$$

$$= \frac{1}{D^2(1-D)^2} \cdot x^3 = \frac{1}{D^2}(1-D)^{-2} x^3 = \frac{1}{D^2}(1 + 2D + 3D^2 + 4D^3 + 5D^4 + \ldots)x^3$$

$$= \frac{1}{D^2}(x^3 + 2Dx^3 + 3D^2x^3 + 4D^3x^3 + 5D^4x^3 + \ldots)$$

$$= \frac{1}{D^2}(x^3 + 2\cdot 3x^2 + 3\cdot 6x + 4\cdot 6 + 0) = \frac{1}{D^2}(x^3 + 6x^2 + 18x + 24)$$

$$= \int\left[\int (x^3 + 6x^2 + 18x + 24)dx\right]dx = \int\left(\frac{x^4}{4} + 6\frac{x^3}{3} + 18\frac{x^2}{2} + 24x\right)dx$$

$$= \frac{x^5}{20} + 2\frac{x^4}{4} + 9\frac{x^3}{3} + 24\frac{x^2}{2} = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

Hence, the general solution is

$$y = c_1 + c_2 x + (c_3 + c_4 x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

EXAMPLE 10.61 Solve $(D^2 + 2)y = x^3 + x^2 + e^{-2x} + \cos 3x$.

Solution: The auxiliary equation is

$$m^2 + 2 = 0,$$
$$m = \pm i\sqrt{2} \text{ (complex)}$$
$$CF = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$$
$$PI = \frac{1}{D^2 + 2}(x^3 + x^2 + e^{-2x} + \cos 3x)$$

$$= \frac{1}{2\left(1+\frac{D^2}{2}\right)}(x^3+x^2) + \frac{1}{D^2+2}e^{-2x} + \frac{1}{D^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1+\frac{D^2}{2}\right)^{-1}(x^3+x^2) + \frac{1}{4+2}e^{-2x} + \frac{1}{-3^2+2}\cos 3x$$

$$= \frac{1}{2}\left(1-\frac{D^2}{2}+\frac{D^4}{4}-\ldots\right)(x^3+x^2) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}D^2(x^3+x^2) + \frac{D^4}{8}(x^3+x^2) - \cdots\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

$$= \left[\frac{1}{2}(x^3+x^2) - \frac{1}{4}(6x+2) + 0\right] + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Hence, the general solution is

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + \frac{1}{2}(x^3+x^2-3x-1) + \frac{e^{-2x}}{6} - \frac{\cos 3x}{7}$$

Case IV $Q = e^{ax}V$, where V is a function of x.

In this case, Eq. (10.31) reduces to $f(D)y = e^{ax}V$.
Let u be a function of x.

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u = e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = D\left[e^{ax}(D+a)u\right] = ae^{ax}(D+a)u + e^{ax}(D^2+aD)u = e^{ax}(D^2+2aD+a^2)u = e^{ax}(D+a)^2u$$

In general,

$$D^r(e^{ax}u) = e^{ax}(D+a)^r u$$

Let $D^r = f(D)$, $\quad (D+a)^r = f(D+a)$

$$f(D)(e^{ax}u) = e^{ax}f(D+a)u$$

Operating both the sides with $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)(e^{ax}u)\right] = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

$$e^{ax}u = \frac{1}{f(D)}\left[e^{ax}f(D+a)u\right]$$

Putting $f(D+a)u = V$, $u = \frac{1}{f(D+a)}V$

$$e^{ax}\cdot\frac{1}{f(D+a)}V=\frac{1}{f(D)}(e^{ax}V)$$

Hence,
$$PI=\frac{1}{f(D)}\cdot e^{ax}V=e^{ax}\cdot\frac{1}{f(D+a)}V$$

EXAMPLE 10.62 *Solve* $(D^4-1)y=\cosh x\cos x$.

Solution: The auxiliary equation is

$$m^4-1=0$$
$$(m^2-1)(m^2+1)=0$$
$$m=\pm1 \text{ (real and distinct)},\quad m=\pm i \text{ (complex)}$$
$$CF=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x$$

$$PI=\frac{1}{D^4-1}\cosh x\cos x=\frac{1}{(D^2+1)(D^2-1)}\left(\frac{e^x+e^{-x}}{2}\right)\cos x$$
$$=\frac{1}{(D^2+1)(D^2-1)}\left(\frac{e^x\cos x}{2}\right)+\frac{1}{(D^2+1)(D^2-1)}\left(\frac{e^{-x}\cos x}{2}\right)$$
$$=\frac{e^x}{2}\frac{1}{[(D+1)^2+1][(D+1)^2-1]}\cos x+\frac{e^{-x}}{2}\frac{1}{[(D-1)^2+1][(D-1)^2-1]}\cos x$$
$$=\frac{e^x}{2}\frac{1}{(D^2+2D+2)(D^2+2D)}\cos x+\frac{e^{-x}}{2}\frac{1}{(D^2-2D+2)(D^2-2D)}\cos x$$
$$=\frac{e^x}{2}\frac{1}{(-1^2+2D+2)(-1^2+2D)}\cos x+\frac{e^{-x}}{2}\frac{1}{(-1^2-2D+2)(-1^2-2D)}\cos x$$
$$=\frac{e^x}{2}\frac{1}{(2D+1)(2D-1)}\cos x+\frac{e^{-x}}{2}\frac{1}{(1-2D)[-(1+2D)]}\cos x$$
$$=\frac{e^x}{2}\frac{1}{4D^2-1}\cos x-\frac{e^{-x}}{2}\frac{1}{(1-4D^2)}\cos x=\frac{e^x}{2}\frac{1}{[4(-1^2)-1]}\cos x-\frac{e^{-x}}{2}\frac{1}{[1-4(-1^2)]}\cos x$$
$$=\frac{e^x}{2}\left(\frac{1}{-5}\right)\cos x-\frac{e^{-x}}{2}\frac{1}{5}\cos x=-\frac{1}{5}\cos x\left(\frac{e^x+e^{-x}}{2}\right)=-\frac{1}{5}\cos x\cosh x$$

Hence, the general solution is

$$y=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x-\frac{1}{5}\cos x\cosh x$$

EXAMPLE 10.63 *Solve* $(D^2-1)y = e^x(\sin x + 1 + x^2)$.

Solution: The auxiliary equation is

$$m^2 - 1 = 0$$

$$m = 1, -1 \quad \text{(real and distinct)}$$

$$CF = c_1 e^x + c_2 e^{-x}$$

$$PI = \frac{1}{D^2-1} e^x (\sin x + 1 + x^2) = e^x \frac{1}{(D+1)^2 - 1}(\sin x + 1 + x^2)$$

$$= e^x \left[\frac{1}{D^2+2D} \sin x + \frac{1}{D^2+2D} e^{0x} + \frac{1}{D^2+2D} x^2 \right]$$

$$= e^x \left[\frac{1}{-1^2+2D} \sin x + x \frac{1}{2D+2} e^{0x} + \frac{1}{2D\left(1+\frac{D}{2}\right)} x^2 \right] \left[\begin{array}{l} \because D^2+2D=0 \\ \text{at } D=0 \text{ for } e^{0x} \end{array} \right]$$

$$= e^x \left[\frac{2D+1}{(2D-1)(2D+1)} \sin x + x \frac{1}{2(0)+2} e^{0x} + \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1} x^2 \right]$$

$$= e^x \left[\frac{2D+1}{4D^2-1} \sin x + \frac{x}{2} + \frac{1}{2D}\left(1 - \frac{D}{2} + \frac{D^2}{4} - \frac{D^3}{8} + \ldots\right) x^2 \right]$$

$$= e^x \left[\frac{2D+1}{4(-1^2)-1} \sin x + \frac{x}{2} + \frac{1}{2D}\left(x^2 - \frac{1}{2}Dx^2 + \frac{1}{4}D^2x^2 - \frac{1}{8}D^3x^2 + \cdots\right) \right]$$

$$= e^x \left[\frac{2D\sin x + \sin x}{-5} + \frac{x}{2} + \frac{1}{2D}\left(x^2 - \frac{2x}{2} + \frac{2}{4} - 0\right) \right]$$

$$= e^x \left[\frac{2\cos x + \sin x}{-5} + \frac{x}{2} + \frac{1}{2}\int\left(x^2 - x + \frac{1}{2}\right) dx \right]$$

$$= e^x \left[\frac{2\cos x + \sin x}{-5} + \frac{x}{2} + \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right) \right]$$

$$= e^x \left[\frac{2\cos x + \sin x}{-5} + \frac{x^3}{6} - \frac{x^2}{4} + \frac{3x}{4} \right]$$

Hence, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + e^x \left[\frac{2\cos x + \sin x}{-5} + \frac{x^3}{6} - \frac{x^2}{4} + \frac{3x}{4} \right]$$

EXAMPLE 10.64 Solve $\left(D^2+2D+1\right)y=\dfrac{e^{-x}}{x^2}$.

Solution: The auxiliary equation is

$$m^2+2m+1=0$$
$$(m+1)^2=0$$
$$m=-1,-1 \quad \text{(real and repeated)}$$
$$\text{CF}=(c_1+c_2x)e^{-x}$$
$$\text{PI}=\frac{1}{D^2+2D+1}\left(\frac{e^{-x}}{x^2}\right)=\frac{1}{(D+1)^2}\left(\frac{e^{-x}}{x^2}\right)$$
$$=e^{-x}\frac{1}{(D-1+1)^2}\left(\frac{1}{x^2}\right)=e^{-x}\frac{1}{D^2}x^{-2}$$
$$=e^{-x}\frac{1}{D}\int x^{-2}dx=e^{-x}\frac{1}{D}\left(\frac{x^{-2+1}}{-2+1}\right)$$
$$=e^{-x}\frac{1}{D}x^{-1}=-e^{-x}\int\frac{dx}{x}=-e^{-x}\log x$$

Hence, the general solution is

$$y=(c_1+c_2x)e^{-x}-e^{-x}\log x$$
$$=e^{-x}(c_1+c_2x-\log x)$$

Case V $Q=xV$, where V is a function of x.

In this case, Eq. (10.31) reduces to $f(D)y=xV$.

Let u be a function of x.

$$D(xu)=xDu+u$$
$$D^2(xu)=D(xDu+u)=xD^2u+Du+Du=xD^2u+2Du$$
$$D^3(xu)=D(xD^2u+2Du)=xD^3u+D^2u+2D^2u=xD^3u+3D^2u$$

In general,

$$D^r(xu)=xD^ru+rD^{r-1}u=xD^ru+\left[\frac{d}{dD}(D^r)\right]u$$

Let $D^r=f(D)$

$$f(D)(xu)=x\,f(D)u+\left[\frac{d}{dD}f(D)\right]u=xf(D)u+f'(D)u$$

Putting $f(D)u = V$, $u = \frac{1}{f(D)}V$ in the above equation,

$$f(D)\left[x\frac{1}{f(D)}V\right] = xV + f'(D)\left[\frac{1}{f(D)}V\right]$$

$$xV = f(D)\left[x\frac{1}{f(D)}V\right] - f'(D)\left[\frac{1}{f(D)}V\right]$$

Operating both the sides with $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}xV = \frac{1}{f(D)}\left[f(D)\left(x\frac{1}{f(D)}V\right)\right] - \frac{1}{f(D)}\left[f'(D)\left(\frac{1}{f(D)}\right)V\right] = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

Hence, $$\text{PI} = \frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

EXAMPLE 10.65 *Solve* $\left(D^2 + 2D + 1\right)y = xe^{-x}\cos x$.

Solution: The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1 \quad \text{(real and repeated)}$$

$$\text{CF} = (c_1 + c_2x)e^{-x}$$

$$\text{PI} = \frac{1}{D^2 + 2D + 1}xe^{-x}\cos x = \frac{1}{(D+1)^2}xe^{-x}\cos x$$

$$= e^{-x}\frac{1}{(D-1+1)^2}x\cos x = e^{-x}\frac{1}{D^2}x\cos x$$

$$= e^{-x}\left[x\frac{1}{D^2}\cos x - \frac{2D}{(D^2)^2}\cos x\right] = e^{-x}\cdot\left(x\frac{1}{-1^2}\cos x - \frac{2D}{(-1^2)^2}\cos x\right)$$

$$= e^{-x}(-x\cos x - 2D\cos x) = e^{-x}(-x\cos x + 2\sin x)$$

Hence, the general solution is

$$y = (c_1 + c_2x)e^{-x} + e^{-x}(-x\cos x + 2\sin x)$$

EXAMPLE 10.66 Solve $(D^2-1)y = x\sin x + e^x + x^2e^x$.

Solution: The auxiliary equation is

$$m^2-1=0$$

$$m = 1, -1 \quad \text{(real and distinct)}$$

$$CF = c_1e^x + c_2e^{-x}$$

$$PI = \frac{1}{D^2-1}\left(x\sin x + e^x + x^2e^x\right) = \frac{1}{D^2-1}x\sin x + \frac{1}{D^2-1}e^x + \frac{1}{D^2-1}x^2e^x$$

$$= \left[x\frac{1}{D^2-1}\sin x - \frac{2D}{\left(D^2-1\right)^2}\sin x\right] + x\frac{1}{2D}e^x + e^x\frac{1}{(D+1)^2-1}x^2$$

$$= \left[x\frac{1}{-1^2-1}\sin x - \frac{2D}{\left(-1^2-1\right)^2}\sin x\right] + \frac{x}{2(1)}e^x + e^x\frac{1}{D^2+2D}x^2$$

$$= \left[-\frac{x\sin x}{2} - \frac{2D\sin x}{4}\right] + \frac{xe^x}{2} + e^x\frac{1}{2D\left(1+\frac{D}{2}\right)}x^2$$

$$= \left[-\frac{x\sin x}{2} - \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}x^2$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x\frac{1}{2D}\left(1-\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}+\cdots\right)x^2$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x\frac{1}{2D}\left(x^2-\frac{2x}{2}+\frac{2}{4}-0\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + e^x\frac{1}{2}\int\left(x^2-x+\frac{1}{2}\right)dx$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{xe^x}{2} + \frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{1}{2}x\right)$$

$$= -\left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{4}\right)$$

Hence, the general solution is

$$y = c_1e^x + c_2e^{-x} - \left[\frac{x\sin x}{2} + \frac{\cos x}{2}\right] + \frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{4}\right)$$