Four

Partial Differentiation

CHAPTER OUTLINE

4.1 INTRODUCTION

There are certain functions which depend on two or more variables. The area of a triangle depends on its base and height. Hence, area is the function of two independent variables, i.e., its base and height. When a function of more than one independent input variable changes in one or more of the input variables, it is important to calculate the change in the function itself. The process of finding the rate of change of the function w.r.t. the one variable keeping the remaining variables constant, is called *partial differentiation*. Partial differentiation is used in vector calculus, differential geometry, etc.

4.2 FUNCTIONS OF TWO OR MORE VARIABLES

A function is called a *real - valued function* of two or more variables if there are two or more independent variables. The function $u = f(x, y)$ is a function of two independent variables x and y while u is a dependent variable. u has a definite value for every pair of x and y.

4.2.1 Limit

If $f(x, y)$ is a function of two variables x, y then

$$\lim_{(x, y)\to(a, b)} f(x, y) = l$$

if and only if for any chosen number $\varepsilon > 0$ however small, there exists a number $\delta > 0$ such that

$$|f(x,y)-l|<\varepsilon$$

foı all values of (x, y) for which $|x - a| < \delta$ and $|y - b| < \delta$.

4.2.2 Working Rule for Evaluation of Limit

(i) Evaluate limits

(a) $\lim\limits_{x\to a}\left\{\lim\limits_{y\to b} f(x,y)\right\}$ and (b) $\lim\limits_{y\to b}\left\{\lim\limits_{x\to a} f(x,y)\right\}$

If both the limit values are equal then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.

(ii) If $a = 0$, $b = 0$, evaluate limit along different paths say $y = mx$ or $y = mx^n$, etc.

If all limit values are equal then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.

4.2.3 Theorems on Limit

If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = l$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = m$,

(i) $\lim\limits_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y) = l+m$

(ii) $\lim\limits_{(x,y)\to(a,b)} [f(x,y)-g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) - \lim\limits_{(x,y)\to(a,b)} g(x,y) = l-m$

(iii) $\lim\limits_{(x,y)\to(a,b)} [f(x,y)\cdot g(x,y)] = \lim\limits_{(x,y)\to(a,b)} f(x,y) \cdot \lim\limits_{(x,y)\to(a,b)} g(x,y) = lm$

(iv) $\lim\limits_{(x,y)\to(a,b)} \left[\dfrac{f(x,y)}{g(x,y)}\right] = \dfrac{\lim\limits_{(x,y)\to(a,b)} f(x,y)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)} = \dfrac{l}{m}$, provided $m \neq 0$

4.2.4 Continuity

Let $f(x, y)$ be a function of x and y defined at (a, b) as well as in the neighbourhood of it. The function $f(x, y)$ is continuous at (a, b) if the following three conditions are satisfied:

(i) $f(a, b)$ exists, i.e., $f(x, y)$ is defined at (a, b).

(ii) $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.

(iii) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

A function $f(x, y)$ is continuous in a domain if it is continuous at each point of that domain.

Note If $f(x, y)$ and $g(x, y)$ are continuous at (a, b) then $f+g, f-g, fg, \dfrac{f}{g}$ (provided $g \neq 0$) are continuous at (a, b).

EXAMPLE 4.1 *By considering different paths of approach, show that the function* $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$ *has no limit as* $(x, y) \to (0, 0)$.

Solution:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^4 - y^2}{x^4 + y^2}\right) = \lim_{x\to 0}\frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^4 - y^2}{x^4 + y^2}\right) = \lim_{y\to 0}\left(\frac{-y^2}{y^2}\right) = \lim_{y\to 0}(-1) = -1$$

Since both the limits are different, $f(x, y)$ has no limit as $(x, y) \to (0, 0)$.

EXAMPLE 4.2 *Find* $\displaystyle\lim_{(x, y)\to(0, 0)} \frac{xy}{y^2 - x^2}$.

Solution:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{y^2 - x^2}\right) = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{y^2 - x^2}\right) = \lim_{y\to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{x(mx)}{(mx)^2 - x^2} = \lim_{x\to 0}\frac{m}{m^2 - 1} = \frac{m}{m^2 - 1}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.

EXAMPLE 4.3 *Show that*

$$f(x, y) = 2x^2 + y, \quad (x, y) \neq (1, 2)$$
$$= 0, \quad (x, y) = (1, 2)$$

is discontinuous at $(1, 2)$.

Solution:

$$\lim_{(x, y)\to(1, 2)} f(x, y) = \lim_{(x, y)\to(1, 2)} (2x^2 + y) = 2(1^2) + 2 = 4$$

$$f(1, 2) = 0$$

$$\lim_{(x, y)\to(1, 2)} f(x, y) \neq f(1, 2)$$

Hence, $f(x, y)$ is discontinuous at $(1, 2)$.

EXAMPLE 4.4 *Show that*

$$f(x, y) = \frac{2xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$
$$= 0, \quad (x, y) = (0, 0)$$

is continuous at every point except at the origin.

Solution:

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] = \lim_{x\to 0}\left(\lim_{y\to 0}\frac{2xy}{x^2 + y^2}\right) = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right] = \lim_{y\to 0}\left(\lim_{x\to 0}\frac{2xy}{x^2 + y^2}\right) = \lim_{y\to 0} 0 = 0$$

Putting $y = mx$ and taking limit $x \to 0$,

$$\lim_{x\to 0}\frac{2x(mx)}{x^2+(mx)^2} = \lim_{x\to 0}\frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

Since the last limit depends on m and m is not fixed, the limit does not exist.
Hence, $f(x, y)$ is discontinuous at origin, i.e., (0, 0).
Let $(x, y) = (a, b)$ be an arbitrary point in the xy-plane, where a and b are real numbers.

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)}\frac{2xy}{x^2+y^2} = \frac{2ab}{a^2+b^2}$$

$$f(a, b) = \frac{2ab}{a^2+b^2}$$

which is finite for real values of a and b.

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

This shows that $f(x, y)$ is continuous at (a, b).
Hence, $f(x, y)$ is continuous at every point except at the origin.

EXERCISE 4.1

1. Evaluate the following limits:

   (i) $\displaystyle\lim_{(x,y)\to(1,2)}\frac{3x^2y}{x^2+y^2+5}$

   (ii) $\displaystyle\lim_{(x,y)\to(\infty,2)}\frac{xy+4}{x^2+2y^2}$

   (iii) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x+y}{x+2y}$

   (iv) $\displaystyle\lim_{(x,y)\to(0,1)} e^{-\frac{1}{x^2(y-1)^2}}$

   (v) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{2x-y}{x^2+y^2}$

   [**Ans.:** (i) $\frac{3}{5}$ (ii) 0 (iii) does not exist (iv) 0 (v) does not exist]

2. Show that for $f(x, y) = \dfrac{2x-y}{2x+y}$,
   $\lim_{x\to 0}\left[\lim_{y\to 0} f(x, y)\right] \neq \lim_{y\to 0}\left[\lim_{x\to 0} f(x, y)\right]$.

3. Check the continuity of the following functions:

   (i) $f(x,y) = \dfrac{x}{3x+5y}$ at (0, 0)

   (ii) $f(x,y) = \dfrac{xy}{x^2+y^2}$, $(x, y) \neq (0, 0)$
   $= 0$, $(x, y) = (0, 0)$
   at origin.

   (iii) $f(x,y) = (x^2+y^2)\sin\dfrac{1}{x^2+y^2}$, $(x, y) \neq (0, 0)$
   $= 0$, $(x, y) = (0, 0)$
   at origin.

   (iv) $f(x,y) = \dfrac{x^2y^2}{x^4+y^4}$ at (0, 0)

   [**Ans.:** (i) Discontinuous (ii) Discontinuous (iii) Continuous (iv) Discontinuous]

4.3 PARTIAL DERIVATIVES

A partial derivative of a function of several variables is the ordinary derivative w.r.t. one of the variables, when all the remaining variables are kept constant. Consider a function $u = f(x, y)$. Here, u is the dependent variable and x and y are independent variables. The partial derivative of $u = f(x, y)$ w.r.t. x is the ordinary derivative of u w.r.t. x, keeping y constant. It is denoted by $\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x and is known as first-order partial derivative of u w.r.t. x.

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right]$$

Similarly, the partial derivative of $u = f(x, y)$ w.r.t. y is the ordinary derivative of u w.r.t. y treating x as constant. It is denoted by $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y or f_y and is known as *first-order partial derivative* of u w.r.t. y.

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right]$$

4.3.1 Higher-Order Partial Derivatives

Partial derivatives of higher order, of a function $u = f(x, y)$, are obtained by partial differentiation of first-order partial derivative. Thus, if $u = f(x, y)$ then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

are called second-order partial derivatives. Similarly, other higher-order derivatives can also be obtained.

Notes

(i) If $u = f(x, y)$ possesses continuous second-order partial derivatives $\dfrac{\partial^2 u}{\partial x\, \partial y}$ and $\dfrac{\partial^2 u}{\partial y\, \partial x}$ then $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$. This is called *commutative property*.

(ii) Standard rules for differentiation of sum, difference, product and quotient are also applicable for partial differentiation.

EXAMPLE 4.5 *If* $u = \log(\tan x + \tan y + \tan z)$ *then show that*

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

Solution:

$$u = \log(\tan x + \tan y + \tan z)$$

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 z$$

Hence,
$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2\sin x \cos x \sec^2 x + 2\sin y \cos y \sec^2 y + 2\sin z \cos z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z} = 2$$

EXAMPLE 4.6 *If* $z = x^y + y^x$, *prove that* $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{\partial^2 z}{\partial y\, \partial x}$.

Solution:

$$z = x^y + y^x$$

$$z = e^{\log x^y} + e^{\log y^x} = e^{y \log x} + e^{x \log y}$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y} = e^{y \log x} \cdot \log x + e^{x \log y} \cdot \frac{x}{y}$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=e^{y\log x}\cdot\frac{y}{x}\log x+e^{y\log x}\cdot\frac{1}{x}+e^{x\log y}\cdot\frac{1}{y}+e^{x\log y}\log y\cdot\frac{x}{y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{e^{y\log x}}{x}(y\log x+1)+\frac{e^{x\log y}}{y}(1+x\log y) \quad \ldots(1)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x}=e^{y\log x}\cdot\frac{y}{x}+e^{x\log y}\cdot\log y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{1}{x}(e^{y\log x}+e^{y\log x}y\log x)+e^{x\log y}\cdot\frac{x}{y}\log y+e^{x\log y}\cdot\frac{1}{y}$$

$$\frac{\partial^2 z}{\partial y\,\partial x}=\frac{e^{y\log x}}{x}(1+y\log x)+\frac{e^{x\log y}}{y}(x\log y+1) \quad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial y\,\partial x}=\frac{\partial^2 z}{\partial x\,\partial y}$$

EXAMPLE 4.7 *If* $z=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$, *prove that*

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}=\frac{x^2-y^2}{x^2+y^2}.$$

Solution:

$$z=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating z partially w.r.t. y,

$$\frac{\partial z}{\partial y}=x^2\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)-y^2\frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right)-2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$=\frac{x^3}{x^2+y^2}+\frac{xy^2}{y^2+x^2}-2y\tan^{-1}\left(\frac{x}{y}\right)=x-2y\tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating $\frac{\partial z}{\partial y}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=1-2y\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = 1 - \frac{2y^2}{y^2 + x^2} = \frac{y^2 + x^2 - 2y^2}{y^2 + x^2} = \frac{x^2 - y^2}{x^2 + y^2} \qquad \ldots(1)$$

Differentiating z partially w.r.t. x,

$$\frac{\partial z}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) + x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - \frac{y^2}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2 + y^2} - \frac{y^3}{x^2 + y^2} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = 2x \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1$$

$$\frac{\partial^2 z}{\partial y\,\partial x} = \frac{2x^2}{x^2 + y^2} - 1 = \frac{2x^2 - x^2 - y^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} \qquad \ldots(2)$$

From Eqs (1) and (2),

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

EXAMPLE 4.8 *If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that*

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

Solution:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)v$$

where

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

Differentiating u partially w.r.t. x, y, and z,

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$v = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2) - 3(xy+yz+zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz} \cdot \frac{(x+y+z)}{(x+y+z)}$$

$$= \frac{3(x^3+y^3+z^3-3xyz)}{(x^3+y^3+z^3-3xyz)(x+y+z)} = \frac{3}{x+y+z}$$

Hence,
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}$$

EXAMPLE 4.9 *Find the value of n so that $v = r^n (3\cos^2\theta - 1)$ satisfies the equation*

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = 0.$$

Solution:
$$v = r^n (3\cos^2\theta - 1)$$

Differentiating v partially w.r.t. r,

$$\frac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) = \frac{\partial}{\partial r}[nr^{n+1}(3\cos^2\theta - 1)] = n(n+1)r^n(3\cos^2\theta - 1) \quad \ldots(1)$$

Differentiating v partially w.r.t. θ,

$$\frac{\partial v}{\partial \theta} = r^n \cdot 6\cos\theta(-\sin\theta) = -3r^n \sin 2\theta$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = \frac{\partial}{\partial \theta}(-3r^n \sin\theta \cdot \sin 2\theta) = -3r^n(\cos\theta \sin 2\theta + 2\sin\theta\cos 2\theta)$$

$$= -3r^n[\cos\theta \cdot 2\sin\theta\cos\theta + 2\sin\theta(2\cos^2\theta - 1)]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = -3r^n(2\cos^2\theta + 4\cos^2\theta - 2) = -6r^n(3\cos^2\theta - 1)$$

Given
$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = 0$$

$$n(n+1)\, r^n (3\cos^2\theta - 1) - 6r^n(3\cos^2\theta - 1) = 0$$

$$n(n+1) - 6 = 0$$

$$n^2 + n - 6 = 0$$

$$(n+3)(n-2) = 0$$

$$n = -3, 2$$

EXAMPLE 4.10 *If $x^x y^y z^z = c$, show that at $x = y = z$,*

$$\text{(i)} \quad \frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1} \quad \text{(ii)} \quad \frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \frac{2(x^2-2)}{x(1+\log x)}$$

Solution:

(i)
$$x^x y^y z^z = c$$

Taking logarithm on both the sides,

$$\log x^x + \log y^y + \log z^z = \log c$$

$$x \log x + y \log y + z \log z = \log c \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$x \cdot \frac{1}{x} + \log x + \frac{\partial z}{\partial x} \cdot \log z + z \cdot \frac{1}{z}\frac{\partial z}{\partial x} = 0 \qquad [\because z = f(x, y)]$$

$$\frac{\partial z}{\partial x} = -\frac{1+\log x}{1+\log z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. y,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = -(1+\log x)\left[-\frac{1}{(1+\log z)^2} \cdot \frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$\frac{\partial^2 z}{\partial y \, \partial x} = \frac{(1+\log x)}{z(1+\log z)^2}\left(-\frac{1+\log x}{1+\log z}\right)$$

$$\frac{\partial^2 z}{\partial x \, \partial y} = -\frac{(1+\log x)^2}{z(1+\log z)^3}$$

At $x = y = z$,
$$\frac{\partial^2 z}{\partial x \, \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$$

$$= -[x(\log e + \log x)]^{-1} \qquad [\because \log e = 1]$$

$$= -(x \log ex)^{-1}$$

(ii) Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{1+\log x}{1+\log z}\right) = \frac{(1+\log x)}{(1+\log z)^2} \cdot \frac{1}{z}\frac{\partial z}{\partial x} - \frac{1}{x(1+\log z)}$$

$$= -\frac{(1+\log x)}{z(1+\log z)^2} \cdot \frac{(1+\log x)}{(1+\log z)} - \frac{1}{x(1+\log z)}$$

At $x = y = z$,
$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{x(1+\log x)}$$

Similarly,
$$\frac{\partial^2 z}{\partial y^2} = \frac{-(1+\log y)^2}{z(1+\log z)^3} - \frac{1}{y(1+\log z)}$$

At $x=y=z$, $$\frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)}$$

Hence, $$\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial^2 z}{\partial y^2} = -\frac{2}{x(1+\log x)} - 2xy\left[-\frac{1}{x(1+\log x)}\right] + \left[-\frac{2}{x(1+\log x)}\right]$$

$$= \frac{2(xy-2)}{x(1+\log x)} = \frac{2(x^2-2)}{x(1+\log x)} \qquad [\because x=y=z]$$

EXAMPLE 4.11 *If* $\dfrac{x^2}{a^2+u} + \dfrac{y^2}{b^2+u} + \dfrac{z^2}{c^2+u} = 1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

Solution: $$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1 \qquad \text{...(1)}$$

Differentiating Eq. (1) partially w.r.t. x,

$$\frac{2x}{a^2+u} - \frac{x^2}{(a^2+u)^2}\frac{\partial u}{\partial x} - \frac{y^2}{(b^2+u)^2}\frac{\partial u}{\partial x} - \frac{z^2}{(c^2+u)^2}\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{2x}{a^2+u}$$

$$\frac{\partial u}{\partial x}\cdot p = \frac{2x}{(a^2+u)}$$

where $$p = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{(a^2+u)p}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{2y}{(b^2+u)p} \quad \text{and} \quad \frac{\partial u}{\partial z} = \frac{2z}{(c^2+u)p}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = \frac{4}{p^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{4}{p^2}(p) = \frac{4}{p} \qquad \text{...(2)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{2}{p}\left(\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right) = \frac{2}{p}(1) = \frac{2}{p} \qquad \text{...(3)}$$

From Eqs (2) and (3),

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

EXAMPLE 4.12 *If* $u = \phi(x + ky) + \psi(x - ky)$, *show that* $\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}$.

Solution:

$$u = \phi(x + ky) + \psi(x - ky)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \phi'(x+ky)\cdot 1 + \psi'(x-ky)\cdot 1$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \phi''(x+ky) + \psi''(x-ky) \qquad \text{...(1)}$$

Differentiating u partially w.r.t. y,

$$\frac{\partial u}{\partial y} = \phi'(x+ky)\cdot k + \psi'(x-ky)\cdot(-k)$$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y,

$$\frac{\partial^2 u}{\partial y^2} = \phi''(x+ky)\cdot k^2 + \psi''(x-ky)(-k)^2$$

$$= k^2[\phi''(x+ky) + \psi''(x-ky)] \qquad \text{...(2)}$$

From Eqs (1) and (2),

$$\frac{\partial^2 u}{\partial y^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

EXAMPLE 4.13 *If* $u = f\left(\sqrt{x^2 + y^2}\right)$, *prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\sqrt{x^2+y^2}} \quad f'\left(\sqrt{x^2+y^2}\right) + f''\left(\sqrt{x^2+y^2}\right).$$

Solution: Let

$$\sqrt{x^2 + y^2} = r$$

$$u = f(r)$$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r) = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} = f'(r)\cdot\frac{\partial}{\partial x}\sqrt{x^2+y^2} = f'(r)\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2x$$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{\sqrt{x^2+y^2}} + \frac{f'(r)}{\sqrt{x^2+y^2}} - xf'(r)\frac{1}{2(x^2+y^2)^{\frac{3}{2}}}\cdot 2x$$

$$= f''(r)\frac{x}{\sqrt{x^2+y^2}}\cdot\frac{x}{\sqrt{x^2+y^2}}+\frac{f'(r)}{\sqrt{x^2+y^2}}-\frac{x^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= f''(r)\frac{x^2}{x^2+y^2}+\frac{f'(r)}{\sqrt{x^2+y^2}}-\frac{x^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

Similarly,
$$\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{y^2}{x^2+y^2}+\frac{f'(r)}{\sqrt{x^2+y^2}}-\frac{y^2 f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

Hence,
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f''(r)\frac{(x^2+y^2)}{x^2+y^2}+\frac{2f'(r)}{\sqrt{x^2+y^2}}-\frac{(x^2+y^2)f'(r)}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= f''(r)+\frac{2f'(r)}{\sqrt{x^2+y^2}}-\frac{f'(r)}{\sqrt{x^2+y^2}}$$

$$= f''(r)+\frac{f'(r)}{\sqrt{x^2+y^2}} = f''\left(\sqrt{x^2+y^2}\right)+\frac{f'\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$$

4.3.2 Variables to be Treated as Constants

In some problems, it is difficult to identify which variable is to be treated as constant. In such cases, the variable to be treated as constant is written as the suffix of the bracket. Thus, $\left(\frac{\partial r}{\partial x}\right)_y$ means that r is first to be expressed as a function of x and y and then differentiated w.r.t. x keeping y constant. Similarly, $\left(\frac{\partial x}{\partial r}\right)_\theta$ means that x is first to be expressed as a function of r and θ and then differentiated w.r.t. r keeping θ constant.

EXAMPLE 4.14 *If $x^2 = au + bv$, $y^2 = au - bv$, prove that* $\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u$, *where a, b are constants.*

Solution:

$$x^2 = au + bv \qquad \ldots(1)$$

Differentiating Eq. (1) partially w.r.t. u keeping v constant,

$$2x\left(\frac{\partial x}{\partial u}\right)_v = a$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$$

$$y^2 = au - bv \qquad \dots(2)$$

Differentiating Eq. (2) partially w.r.t. v keeping u constant,

$$2y\left(\frac{\partial y}{\partial v}\right)_u = -b$$

$$\left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$$

Now,

$$x^2 = au + bv,\ y^2 = au - bv$$

$$x^2 + y^2 = 2au$$

$$u = \frac{x^2 + y^2}{2a} \qquad \dots(3)$$

Differentiating Eq. (3) partially w.r.t. x keeping y constant,

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$$

Also,

$$x^2 - y^2 = 2bv$$

$$v = \frac{x^2 - y^2}{2b} \qquad \dots(4)$$

Differentiating Eq. (4) partially w.r.t. y keeping x constant,

$$\left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$$

Hence,

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a}\cdot\frac{a}{2x} = \frac{1}{2}$$

and

$$\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{y}{b}\right)\left(-\frac{b}{2y}\right) = \frac{1}{2}$$

EXAMPLE 4.15 *If $x = r\cos\theta$, $y = r\sin\theta$, prove that*

(i) $r\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r$

(ii) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

Solution:

(i) $x = r\cos\theta$, $y = r\sin\theta$

Differentiating x partially w.r.t. θ keeping r constant,

$$\left(\frac{\partial x}{\partial \theta}\right)_r = -r\sin\theta$$

$$\frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r = -\sin\theta \qquad \ldots(3)$$

Now, $$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Differentiating θ partially w.r.t. x keeping y constant,

$$\left(\frac{\partial \theta}{\partial x}\right)_y = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$r\left(\frac{\partial \theta}{\partial x}\right)_y = -\sin\theta = \frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r \qquad \text{[From Eq. (3)]}$$

(ii) $x^2 + y^2 = r^2$

Differentiating w.r.t. x partially,

$$2x = 2r\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Differentiating $\frac{\partial r}{\partial x}$ partially w.r.t. x,

$$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x^2}{r^3} \qquad \left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}$$

Similarly, $$\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3} \qquad \left[\because \frac{\partial r}{\partial y} = \frac{y}{r}\right]$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3} + \frac{x^2}{r^3} = \frac{1}{r}\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

EXERCISE 4.2

1. If $u=\cos\left(\sqrt{x}+\sqrt{y}\right)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right)=0.$$

2. If $u=2(ax+by)^2-k(x^2+y^2)$ and $a^2+b^2=k$, find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$.
[Ans.: 0]

3. If $e^u=\tan x+\tan y$, show that
$$\sin 2x\frac{\partial u}{\partial x}+\sin 2y\frac{\partial u}{\partial y}=2.$$

4. If $z^3-3yz-3x=0$, show that
(i) $z\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}$
(ii) $z\left[\frac{\partial^2 z}{\partial x\partial y}+\left(\frac{\partial z}{\partial x}\right)^2\right]=\frac{\partial^2 z}{\partial y^2}$

5. If $z(z^2+3x)+3y=0$, prove that
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{2z(x-1)}{(z^2+x)^3}.$$

6. If $u=\log(x^2+y^2)+\tan^{-1}\left(\frac{y}{x}\right)$, show that
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

7. If $u(x,y,z)=\frac{1}{x^2+y^2+z^2}$, find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$.
$$\left[\textbf{Ans.: }\frac{2}{(x^2+y^2+z^2)^2}\right]$$

8. If $x=e^{r\cos\theta}\cos(r\sin\theta)$ and $y=e^{r\cos\theta}\sin(r\sin\theta)$, prove that $\frac{\partial x}{\partial r}=\frac{1}{r}\frac{\partial y}{\partial \theta}$, $\frac{\partial y}{\partial r}=-\frac{1}{r}\frac{\partial x}{\partial \theta}$
Hence, deduce that $\frac{\partial^2 x}{\partial r^2}+\frac{1}{r}\frac{\partial x}{\partial r}+\frac{1}{r^2}\frac{\partial^2 x}{\partial \theta^2}=0.$

9. If $v=(x^2-y^2)f(x,y)$, prove that
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=(x^4-y^4)f''(x,y).$$

10. If $u=f(ax^2+2hxy+by^2)$ and $v=\phi(ax^2+2hxy+by^2)$, show that
$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

11. If $x=\frac{r}{2}(e^\theta+e^{-\theta})$, $y=\frac{r}{2}(e^\theta-e^{-\theta})$, prove that $\left(\frac{\partial x}{\partial r}\right)_\theta=\left(\frac{\partial r}{\partial x}\right)_y$.

12. If $\log_e\theta=r-x$, $r^2=x^2+y^2$, show that
$$\frac{\partial^2\theta}{\partial y^2}=\frac{\theta(x^2+ry^2)}{r^3}.$$

13. If $u=e^{ax}\sin by$, prove that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$.

14. If $u=\tan^{-1}\left(\frac{xy}{\sqrt{1+x^2+y^2}}\right)$, prove that
$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{(1+x^2+y^2)^{\frac{3}{2}}}.$$

15. If $u=\frac{1}{\sqrt{y}}e^{-\frac{(x-a)^2}{4y}}$, prove that
$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}.$$

16. If $u=\tan(y+ax)-(y-ax)^{\frac{3}{2}}$, prove that
$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}.$$

17. If $u=\frac{xy}{2x+y}$, prove that $\frac{\partial^3 u}{\partial y\,\partial z^2}=\frac{\partial^3 u}{\partial z^2\,\partial y}$.

18. If $u=x^m y^n$, prove that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=\frac{\partial^3 u}{\partial y\,\partial x^2}$.

19. Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ for the following functions:

(i) $\sqrt{x+y-1}$

(ii) $\sqrt{1-x^2-y^2}$

(iii) y^x

(iv) $\log_{10}(ax+by)$

(v) $(y-ax)^{\frac{3}{2}}$

$$\left[\begin{array}{l}\textbf{Ans.: (i) } \dfrac{1}{\sqrt{x+y-1}}, \dfrac{1}{\sqrt{x+y-1}} \\ \text{(ii) } \dfrac{-x}{\sqrt{1-x^2-y^2}}, \dfrac{-y}{\sqrt{1-x^2-y^2}} \\ \text{(iii) } y^x \log y, xy^{x-1} \\ \text{(iv) } \dfrac{a}{(\log_e 10)(ax+by)}, \dfrac{b}{(\log_e 10)(ax+by)} \\ \text{(v) } -\dfrac{3a}{2}(y-ax)^{\frac{1}{2}}, \dfrac{3}{2}(y-ax)^{\frac{1}{2}}\end{array}\right]$$

20. If $x^4-xy^2+yz^2-z^4=6$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\left[\textbf{Ans.: } \frac{y^2-4x^3}{2yz-4z^3}, \frac{2xy-z^2}{2yz-4z^3}\right]$$

21. If $z^3+xy-y^2z=6$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at (0, 1, 2).

$$\left[\textbf{Ans.: } -\frac{1}{11}, \frac{4}{11}\right]$$

22. Find the value of n for which $u=t^n e^{-\frac{r^2}{4kt}}$ satisfies the partial differential equation $\dfrac{\partial u}{\partial t}=k\left(\dfrac{\partial^2 u}{\partial r^2}+\dfrac{2}{r}\dfrac{\partial u}{\partial r}\right)$.

$$\left[\textbf{Ans.: } n=-\frac{3}{2}\right]$$

23. If $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$, find $\dfrac{\partial r}{\partial x}, \dfrac{\partial \theta}{\partial x}$ in terms of r, θ, ϕ.

$$\left[\textbf{Ans.: } \sin\theta\cos\phi, \frac{\cos\theta\cos\phi}{r}, \frac{-\sin\phi}{r\sin\theta}\right]$$

24. If $u=x^2(y-z)+y^2(z-x)+z^2(x-y)$, prove that $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=0$.

25. If $u=e^x(x\cos y-y\sin y)$, prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$.

26. If $z(x+y)=x^2+y^2$, prove that

$$x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\,\partial y}=2\frac{\partial z}{\partial x}.$$

27. If $\dfrac{x^2}{a+u}+\dfrac{y^2}{b+u}=1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right).$$

28. If $u=x^y$, prove that $\dfrac{\partial^3 u}{\partial x^2\partial y}=\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial x}$.

29. If $\dfrac{x^2}{2+u}+\dfrac{y^2}{4+u}+\dfrac{z^2}{6+u}=1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=2\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right).$$

30. If $u=(x^2-y^2)f(r)$, where $r=xy$, show that

$$\frac{\partial^2 u}{\partial x\,\partial y}=(x^2-y^2)[3f'(r)+rf''(r)].$$

31. If $z=f(x^2, y)$, prove that $x\dfrac{\partial z}{\partial x}=2y\dfrac{\partial z}{\partial y}$.

32. Prove that $z=\dfrac{1}{r}[f(ct+r)+\phi(ct-r)]$ satisfies the partial differential equation $\dfrac{\partial^2 u}{\partial t^2}=\dfrac{c^2}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$, where c is constant.

33. If $u+iv=f(x+iy)$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

$$\left[\begin{aligned} u &= \frac{1}{2}[f(x+iy)+f(x-iy)], \\ v &= \frac{1}{2i}[f(x+iy)-f(x-iy)] \end{aligned}\right]$$

34. If u, v, w are function of x, y, z given as $x=u+v+w, y=u^2+v^2+w^2, z=u^3+v^3+w^3$, prove that $\dfrac{\partial u}{\partial x}=\dfrac{vw(w-v)}{(u-v)(v-w)(w-u)}$.

35. If $u=(x^2+y^2+z^2)^{\frac{n}{2}}$, find the value of n which satisfies the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

[Ans.: 0, −1]

36. If $u=\log(e^x+e^y)$, show that

$$\left(\frac{\partial^2 u}{\partial x^2}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)-\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2=0.$$

37. If $z=yf(x^2-y^2)$, show that

$$y\left(\frac{\partial z}{\partial x}\right)+x\left(\frac{\partial z}{\partial y}\right)=\frac{xy}{y}.$$

38. If $x^2=a\sqrt{u}+b\sqrt{v}$ and $y^2=a\sqrt{u}-b\sqrt{v}$, where a, b are constant, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v=\frac{1}{2}=\left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u.$$

39. If $u=ax+by, v=bx-ay$, find the value of

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v\left(\frac{\partial y}{\partial v}\right)_x\left(\frac{\partial v}{\partial y}\right)_u.$$

[Ans.: 1]

40. If $z=x\log(x+r)-r$, where $r^2=x^2+y^2$, prove that $\dfrac{\partial^3 z}{\partial x^3}=-\left(\dfrac{x}{r^3}\right)$

4.4 COMPOSITE FUNCTIONS

4.4.1 Chain Rule

Let $z=f(u)$, where u is a function of two variables x and y, i.e., $u=\phi(x, y)$.

From Fig. 4.1,

$$\frac{\partial z}{\partial x}=\frac{dz}{du}\cdot\frac{\partial u}{\partial x} \quad \text{or} \quad \frac{df}{du}\cdot\frac{\partial u}{\partial x} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{dz}{du}\cdot\frac{\partial u}{\partial y} \quad \text{or} \quad \frac{df}{du}\cdot\frac{\partial u}{\partial y} \quad \text{or} \quad f'(u)\frac{\partial u}{\partial y}$$

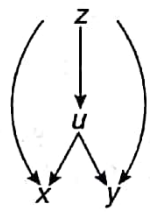


Fig. 4.1 Chain rule

4.4.2 Composite Function of One Variable

If $u=f(x, y)$, where $x=\phi(t), y=\psi(t)$ then u is a function of t and is called the *composite function of a single variable t.*

From Fig. 4.2,

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\dfrac{du}{dt}$ is called the *total derivative* of u.

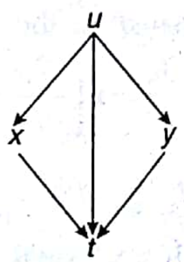


Fig. 4.2 Composite function of one variable

Let $u = f(x, y, z)$ and $x = \phi(t)$, $y = \psi(t)$, $z = \xi(t)$. From Fig. 4.3, total derivative of u is given as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

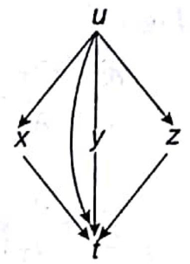


Fig. 4.3 Composite function of one variable

EXAMPLE 4.16 *If* $z = xy^2 + x^2y$, $x = at^2$, $y = 2at$, *find* $\frac{dz}{dt}$.

Solution:

$$z = xy^2 + x^2y,\ x = at^2,\ y = 2at$$

From Fig. 4.4,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = (y^2 + 2xy)2at + (2xy + x^2)2a$$

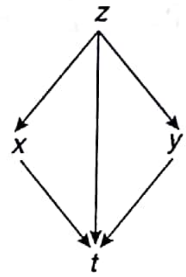


Fig. 4.4

Substituting x, y and z,

$$\frac{dz}{dt} = (4a^2t^2 + 2at^2\cdot 2at)2at + (2at^2\cdot 2at + a^2t^4)2a$$

$$= 4a^2t^2(1+t)2at + a^2t^3(4+t)2a = 8a^3t^3(1+t) + 2a^3t^3(4+t) = 2a^3t^3(8+5t)$$

4.4.3 Composite Function of Two Variables

If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$ then z is a function of u, v and is called the composite function of two variables u and v.

From Fig. 4.5,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

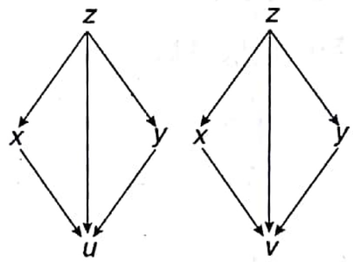


Fig. 4.5 Composite function of two variables

EXAMPLE 4.17 *If* $z = f(x, y)$, $x = u\cosh v$, $y = u\sinh v$, *prove that*

$$\left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2.$$

Solution:

$$z = f(x, y),\quad x = u\cosh v,\quad y = u\sinh v$$

From Fig. 4.6,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cosh v + \frac{\partial z}{\partial y}\sinh v \qquad \text{...(1)}$$

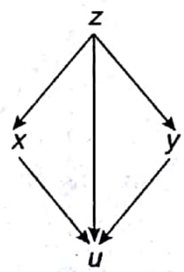


Fig. 4.6

From Fig. 4.7,

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}u\sinh v+\frac{\partial z}{\partial y}u\cosh v$$

$$\frac{1}{u}\cdot\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\sinh v+\frac{\partial z}{\partial y}\cosh v \qquad ...(2)$$

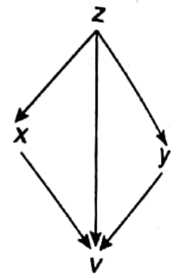


Fig. 4.7

Squaring and subtracting Eq. (2) from Eq. (1),

$$\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2\cosh^2 v+\left(\frac{\partial z}{\partial y}\right)^2\sinh^2 v+2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v-\left(\frac{\partial z}{\partial x}\right)^2\sinh^2 v$$

$$-\left(\frac{\partial z}{\partial y}\right)^2\cosh^2 v-2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\cosh v\sinh v$$

$$=\left(\frac{\partial z}{\partial x}\right)^2(\cosh^2 v-\sinh^2 v)-\left(\frac{\partial z}{\partial y}\right)^2(\cosh^2 v-\sinh^2 v)=\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2$$

EXAMPLE 4.18 *If $z=f(u,v)$, and $u=x^2-y^2, v=2xy$, show that*

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4(u^2+v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right].$$

Solution: $z=f(u,v)$, and $u=x^2-y^2, v=2xy$

From Fig. 4.8,

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=\frac{\partial z}{\partial u}\cdot 2x+\frac{\partial z}{\partial v}\cdot 2y$$

$$=2\left(x\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right) \qquad ...(1)$$

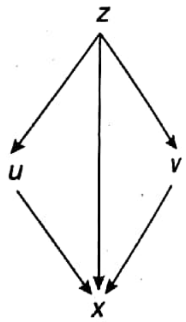


Fig. 4.8

From Fig. 4.9,

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=\frac{\partial z}{\partial u}(-2y)+\frac{\partial z}{\partial v}(2x)$$

$$=2\left(-y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right) \qquad ...(2)$$

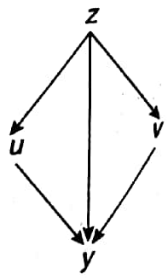


Fig. 4.9

Squaring and adding Eq. (1) and Eq. (2),

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=4\left(x\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}\right)^2+4\left(-y\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right)^2$$

$$= 4\left[x^2\left(\frac{\partial z}{\partial u}\right)^2 + y^2\left(\frac{\partial z}{\partial v}\right)^2 + 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v} + y^2\left(\frac{\partial z}{\partial u}\right)^2 + x^2\left(\frac{\partial z}{\partial v}\right)^2 - 2xy\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}\right]$$

$$= 4(x^2+y^2)\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = 4\left[(x^2+y^2)^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

$$= 4\left[(x^2-y^2)^2 + 4x^2y^2\right]^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] = 4(u^2+v^2)^{\frac{1}{2}}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$

EXAMPLE 4.19 *If* $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, *show that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

Solution: Let

$$l = e^{y-z}, \qquad m = e^{z-x}, \qquad n = e^{x-y}$$

$$\frac{\partial l}{\partial x} = 0, \qquad \frac{\partial m}{\partial x} = -e^{z-x} = -m, \qquad \frac{\partial n}{\partial x} = e^{x-y} = n$$

$$\frac{\partial l}{\partial y} = e^{y-z} = l, \qquad \frac{\partial m}{\partial y} = 0, \qquad \frac{\partial n}{\partial y} = -e^{x-y} = -n$$

$$\frac{\partial l}{\partial z} = -e^{y-z} = -l, \qquad \frac{\partial m}{\partial z} = e^{z-x} = m, \qquad \frac{\partial n}{\partial z} = 0$$

$$u = f(e^{y-z}, e^{z-x}, e^{x-y}) = f(l, m, n)$$

From Fig. 4.10,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$= \frac{\partial u}{\partial l}\cdot 0 + \frac{\partial u}{\partial m}\cdot(-m) + \frac{\partial u}{\partial n}\cdot n$$

$$= -m\frac{\partial u}{\partial m} + n\frac{\partial u}{\partial n} \qquad \text{...(1)}$$

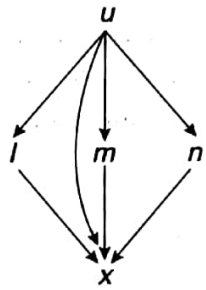


Fig. 4.10

From Fig. 4.11,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$= \frac{\partial u}{\partial l}\cdot l + \frac{\partial u}{\partial m}\cdot 0 + \frac{\partial u}{\partial n}\cdot(-n)$$

$$= l\frac{\partial u}{\partial l} - n\frac{\partial u}{\partial n} \qquad \text{...(2)}$$

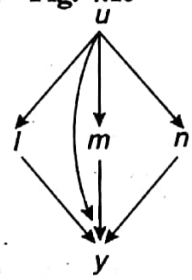


Fig. 4.11

From Fig. 4.12,

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$= \frac{\partial u}{\partial l}(-l) + \frac{\partial u}{\partial m}\cdot m + \frac{\partial u}{\partial n}\cdot 0$$

$$= -l\frac{\partial u}{\partial l} + m\frac{\partial u}{\partial m} \qquad \text{...(3)}$$

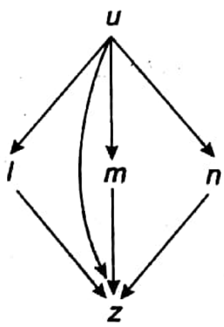


Fig. 4.12

Adding Eqs (1), (2) and (3), $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

EXAMPLE 4.20 *If* $z=f(x,y)$ *where* $x=\log u$, $y=\log v$, *show that*

$$\frac{\partial^2 z}{\partial x\,\partial y}=uv\frac{\partial^2 z}{\partial u\,\partial v}.$$

Solution: $z=f(x,y),\quad x=\log u,\quad y=\log v$

From Fig. 4.13,

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{1}{u}+\frac{\partial z}{\partial y}\cdot 0=\frac{1}{u}\frac{\partial z}{\partial x}$$

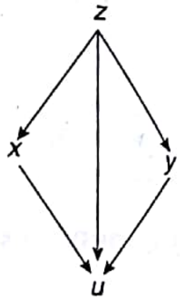


Fig. 4.13

From Fig. 4.14,

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{\partial z}{\partial x}\cdot 0+\frac{\partial z}{\partial y}\frac{1}{v}=\frac{1}{v}\frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial v}\equiv\frac{1}{v}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{1}{v}\frac{\partial}{\partial y}\right)\left(\frac{1}{u}\frac{\partial z}{\partial x}\right)$$

Now,

$$\frac{\partial^2 z}{\partial v\partial u}=\frac{1}{uv}\frac{\partial^2 z}{\partial y\partial x}$$

$$\frac{\partial^2 z}{\partial x\partial y}=uv\frac{\partial^2 z}{\partial u\partial v}$$

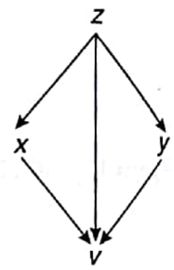


Fig. 4.14

EXAMPLE 4.21 *If* $x+y=2e^{\theta}\cos\phi$ *and* $x-y=2ie^{\theta}\sin\phi$, *prove that*

(i) $\frac{\partial v}{\partial \theta}+i\frac{\partial v}{\partial \phi}=2y\frac{\partial v}{\partial y}$ (ii) $\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial \phi^2}=4xy\frac{\partial^2 v}{\partial x\,\partial y}$

Solution: $x+y=2e^{\theta}\cos\phi,\quad x-y=2ie^{\theta}\sin\phi$

$$2x=2e^{\theta}(\cos\phi+i\sin\phi)$$

$$x=e^{\theta}e^{i\phi}=e^{\theta+i\phi}$$

$$\frac{\partial x}{\partial \theta}=e^{\theta+i\phi}=x$$

$$\frac{\partial x}{\partial \phi}=ie^{\theta+i\phi}=ix$$

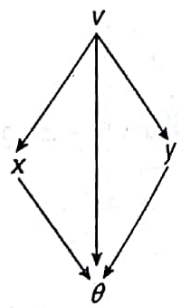


Fig. 4.15

and
$$2y = 2e^{\theta}(\cos\phi - i\sin\phi)$$
$$y = e^{\theta}e^{-i\phi} = e^{\theta-i\phi}$$
$$\frac{\partial y}{\partial \theta} = e^{\theta-i\phi} = y$$
$$\frac{\partial y}{\partial \phi} = -ie^{\theta-i\phi} = -iy$$

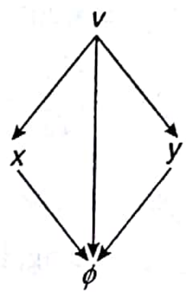


Fig. 4.16

(i) Let
$$v = f(x, y)$$

From Fig. 4.15,
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y \qquad \ldots(1)$$

From Fig. 4.16,
$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \phi} = \frac{\partial v}{\partial x}(ix) + \frac{\partial v}{\partial y}(-iy) = i\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right) \qquad \ldots(2)$$

$$\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + i^2\left(x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y}\right) = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} - x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2y\frac{\partial v}{\partial y}$$

(ii) From Eq. (1),
$$\frac{\partial}{\partial \theta} \equiv x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Now,
$$\frac{\partial^2 v}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta}\right)$$
$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = x\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$
$$= x^2\frac{\partial^2 v}{\partial x^2} + x\frac{\partial v}{\partial x} + xy\frac{\partial^2 v}{\partial x\,\partial y} + xy\frac{\partial^2 v}{\partial y\,\partial x} + y^2\frac{\partial^2 v}{\partial y^2} + y\frac{\partial v}{\partial y}$$
$$= x^2\frac{\partial^2 v}{\partial x^2} + x\frac{\partial v}{\partial x} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} + y\frac{\partial v}{\partial y} \qquad \ldots(3)$$

From Eq. (2),
$$\frac{\partial}{\partial \phi} \equiv ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}$$

Now,
$$\frac{\partial^2 v}{\partial \phi^2} = \frac{\partial}{\partial \phi}\left(\frac{\partial v}{\partial \phi}\right)$$
$$= \left(ix\frac{\partial}{\partial x} - iy\frac{\partial}{\partial y}\right)\left(ix\frac{\partial v}{\partial x} - iy\frac{\partial v}{\partial y}\right) = ix\frac{\partial}{\partial x}\left(ix\frac{\partial v}{\partial x} - iy\frac{\partial v}{\partial y}\right) - iy\frac{\partial}{\partial y}\left(ix\frac{\partial v}{\partial x} - iy\frac{\partial v}{\partial y}\right)$$

$$= -x^2\frac{\partial^2 v}{\partial x^2} - x\frac{\partial v}{\partial x} + xy\frac{\partial^2 v}{\partial x\,\partial y} + xy\frac{\partial^2 v}{\partial y\,\partial x} - y^2\frac{\partial^2 v}{\partial y^2} - y\frac{\partial v}{\partial y}$$

$$= -x^2\frac{\partial^2 v}{\partial x^2} - x\frac{\partial v}{\partial x} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} - y^2\frac{\partial^2 v}{\partial y^2} - y\frac{\partial v}{\partial y} \qquad \ldots(4)$$

Adding, Eqs (3) and (4),

$$\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial \phi^2} = 4xy\frac{\partial^2 v}{\partial x \partial y}$$

EXERCISE 4.3

1. If $z = \tan^{-1}\left(\frac{x}{y}\right)$, where $x = 2t$, $y = 1 - t^2$, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$.

2. If $u = x^3 + y^3$, where $x = a\cos t$, $y = b\sin t$, find $\frac{du}{dt}$.

   [**Ans.:** $-3a^3\cos^2 t\sin t + 3b^2\sin^2 t\cos t$]

3. If $u = xe^y z$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^3 x$, find $\frac{du}{dx}$.

   $\left[\textbf{Ans.:}\ e^y z\left(1 - \frac{x^2}{y} + 3x\cot x\right)\right]$

4. If $u = e^{\frac{r-x}{l}}$, where $r^2 = x^2 + y^2$ and l is a constant, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2}{l}\cdot\frac{\partial u}{\partial x} = \frac{u}{lr}$.

5. If $u = \log r$ and $r = \sqrt{(x-a)^2 + (y-b)^2}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ if a, b are constants.

6. If $u^2(x^2 + y^2 + z^2) = 1$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

7. If $u = r^m$, where $r = \sqrt{x^2 + y^2 + z^2}$, find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

   [**Ans.:** $m(m+1)r^{m-2}$]

8. If $u = f(r)$, where r is given by the relation $x = r\cos\theta$, $y = r\sin\theta$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}$.

9. If $z = f(u, v)$, where $u = x^2 + y^2$, $v = 2xy$ then show that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2 + v^2}\left(\frac{\partial z}{\partial u}\right)$.

10. If $z = f(u, v)$, where $u = x^2 + y^2$, $v = x^2 - y^2$ then show that

    (i) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 4xy\frac{\partial z}{\partial u}$.

    (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4u\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right] + 8v\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}$

11. If $w = (x^2 + y - 2)^4 + (x - y + 2)^3$, where $x = u - 2v + 1$ and $y = 2u + v - 2$, find $\frac{\partial w}{\partial v}$ at $u = 0, v = 0$.

[Ans.: – 882]

12. If $w = x + 2y + z^2, x = \frac{u}{v}, y = u^2 + ev$, $z = 2u$, show that $u\frac{\partial w}{\partial u} + v\frac{\partial w}{\partial v} = 12u^2 + 2ve^v$.

13. If F is a function of x, y, z then show that

$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{dw} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{dz},$$

where $x = u + v + w$, $y = uv + vw + wu$, $z = uvw$.

14. If $z = f(x, y)$, $x = uv$, $y = \frac{u}{v}$, prove that

$$\frac{\partial z}{\partial x} = \frac{1}{2v}\frac{\partial z}{\partial u} + \frac{1}{2u}\frac{\partial z}{\partial u} \text{ and}$$

$$\frac{\partial z}{\partial y} = \frac{v}{2}\frac{\partial z}{\partial u} - \frac{v^2}{2u}\frac{\partial z}{\partial v}.$$

15. If $x = u + v$, $y = uv$ and F is a function of x, y, prove that

$$\frac{\partial^2 F}{\partial u^2} - 2\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial^2 F}{\partial v^2} = (x^2 - 4y)\frac{\partial^2 y}{\partial y^2} - 2\frac{\partial v}{\partial y}.$$

16. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$, prove that

$$\frac{1}{x^{n-1}}\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\frac{\partial u}{\partial z} = 0.$$

17. If $z = f(x, y)$, where $x = u - av$, $y = u + av$ prove that $a^2\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} = 4a^2\frac{\partial^2 z}{\partial x\,\partial y}$.

18. If $z = f(u, v)$, where $u = lx + my$, $v = ly - mx$ prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$

19. If $x = u + av$ *and* $y = u + bv$, transform the equation $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x\,\partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0$ into the equation $\frac{\partial^2 z}{\partial u\,\partial v} = 0$, find the values of a and b.

$$\left[\textbf{Ans.: } a = 1, b = \frac{2}{3}\right]$$

20. If $z = f(x, y)$, $y = e^x$, $v = e^y$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = uv\frac{\partial^2 z}{\partial u\,\partial v}$.

21. If $z = f(x, y)$, $x = \frac{\cos u}{v}$, $y = \frac{\sin u}{v}$, prove that $v\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} = (y - x)\frac{\partial z}{\partial x} - (y + x)\frac{\partial z}{\partial y}$.

22. If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$.

23. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that

$$(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$

24. If $u = f(ax^2 + 2hxy + by^2)$ $v = \phi(ax^2 + 2hxy + by^2)$, show that $\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right)$.

25. If $x + y = 2e^\theta \cos\phi$ and $x - y = 2ie^\theta \sin\phi$, prove that $\frac{\partial v}{\partial \theta} + i\frac{\partial v}{\partial \phi} = 2y\frac{\partial v}{\partial y}$.

26. Find the values of the constants a and b such that $u = x + ay$ and $v = x + by$ transform the equation $9\frac{\partial^2 f}{\partial x^2} - 9\frac{\partial^2 f}{\partial x\,\partial y} + 2\frac{\partial^2 f}{\partial y^2} = 0$ into $\frac{\partial^2 f}{\partial u\,\partial v} = 0$, where f is a function of x and y.

$$\left[\textbf{Ans.: } a = \frac{3}{2}, b = 3\right]$$

27. If $x = r\cosh\theta$, $y = r\sinh\theta$ and $z = f(x, y)$, prove that

(i) $(x-y)\left(\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)=r\dfrac{\partial z}{\partial r}-\dfrac{\partial z}{\partial \theta}$

(ii) $(x^2-y^2)\left(\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}\right)$

$$=r^2\frac{\partial^2 z}{\partial r^2}+r\frac{\partial z}{\partial r}-\frac{\partial^2 z}{\partial \theta^2}$$

28. If $x = e^v \sec u$, $y = e^v \tan u$ and $z = f(x, y)$, prove that $\cos u\left(\dfrac{\partial^2 z}{\partial u\,\partial v}-\dfrac{\partial z}{\partial u}\right)$

$$= xy\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)+(x^2+y^2)\frac{\partial^2 z}{\partial x\,\partial y}.$$

29. If $f(x^2y^3, z-3x)=0$, prove that

$$3x\frac{\partial z}{\partial x}-2y\frac{\partial z}{\partial y}=9x.$$

30. If $f(y+z, x^2+y^2+z^2)=0$, prove that

$$(y-z)\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=x.$$

31. If $f(cx-az, cy-bz)=0$, prove that

$$a\frac{\partial z}{\partial x}-b\frac{\partial z}{\partial y}=c.$$

32. If $x=\dfrac{\cos\theta}{u}$, $y=\dfrac{\sin\theta}{u}$, prove that

$$\left(\frac{\partial x}{\partial u}\right)_\theta\left(\frac{\partial u}{\partial x}\right)_y=\cos^2\theta.$$

33. If $x^2 = au+bv$, $y^2 = au-bv$, prove that

$$\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v=\frac{1}{2}=\left(\frac{\partial v}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u.$$

34. If $u = ax+by$, $v = bx-ay$, prove that

(i) $\left(\dfrac{\partial y}{\partial v}\right)_x\left(\dfrac{\partial v}{\partial y}\right)_u=\dfrac{a^2+b^2}{a^2}$

(ii) $\left(\dfrac{\partial u}{\partial x}\right)_y\left(\dfrac{\partial x}{\partial u}\right)_v=\dfrac{a^2}{a^2+b^2}$

4.5 IMPLICIT FUNCTIONS

Any function of the type $f(x, y) = c$ is called an implicit function, where y is a function of x and c is a constant.

For an implicit function $\quad f(x, y) = c, \quad \dfrac{dy}{dx}=-\dfrac{\left(\dfrac{\partial f}{\partial x}\right)}{\left(\dfrac{\partial f}{\partial y}\right)}$

Proof Let $f(x, y)$ is a function of x and y, where y is a function of x. From Fig. 4.17, total differential coefficient of f w.r.t. x is given by

$$\frac{df}{dx}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

But $\quad f(x, y) = c$

$$\frac{df}{dx}=0$$

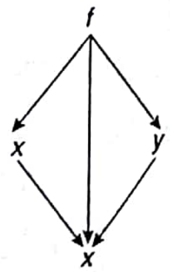


Fig. 4.17 Function of two variables

$$\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

EXAMPLE 4.22 *If* $x^3+y^3=3axy$, *find* $\frac{dy}{dx}$.

Solution: Let

$$f(x,y)=x^3+y^3-3axy$$

$$\frac{dy}{dx}=-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}=-\frac{3x^2-3ay}{3y^2-3ax}=-\frac{x^2-ay}{y^2-ax}=\frac{ay-x^2}{y^2-ax}$$

EXAMPLE 4.23 *If* $f(x,y)=0$, $\phi(x,z)=0$, *show that* $\frac{\partial\phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial\phi}{\partial z}$.

Solution:

$$f(x,y)=0 \text{ and } \phi(x,z)=0$$

$$\frac{dy}{dx}=-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} \text{ and } \frac{dz}{dx}=-\frac{\left(\frac{\partial\phi}{\partial x}\right)}{\left(\frac{\partial\phi}{\partial z}\right)}$$

$$\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}}{-\frac{\left(\frac{\partial\phi}{\partial x}\right)}{\left(\frac{\partial\phi}{\partial z}\right)}}$$

$$\frac{dy}{dz}=\frac{\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}}{\frac{\left(\frac{\partial\phi}{\partial x}\right)}{\left(\frac{\partial\phi}{\partial z}\right)}}$$

Hence, $$\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\frac{dy}{dz}=\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial z}$$

EXERCISE 4.4

1. If $x^3+3x^2+6xy^2+y^3=1$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.}: -\frac{(x^2+2x+2y^2)}{(4xy+y^2)}\right].$$

2. If $x^y=y^x$, prove that $\frac{dy}{dx}=\frac{y(y-x\log y)}{x(x-y\log x)}$.

3. If $f(x, y) = x\sin(x-y)-(x+y)=0$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.}: \frac{[\sin(x-y)](1+x)-1}{x\cos(x-y)+1}\right]$$

4. If $y^{x^y}=\sin x$, find $\frac{dy}{dx}$.

$$\left[\textbf{Ans.}: \frac{-(yx^{y-1}\log y-\cot x)}{x^y\log x\log y+x^y y^{-1}}\right]$$

5. If $x^5+y^5=5a^3x^2$, find $\frac{d^2y}{dx^2}$.

$$\left[\textbf{Ans.}: \frac{6a^3x^2(a^3+x^3)}{y^9}\right]$$

6. If $xy^3-yx^3=6$ is the equation of curve, find the slope of the tangent at the point (1, 2).

$$\left[\textbf{Ans.}: -\frac{2}{11}\right]$$

7. Find $\frac{d^2y}{dx^2}$, *if* $x\sqrt{1-y^2}+y\sqrt{1-x^2}=a$.

$$\left[\textbf{Ans.}: \frac{a}{(1-x^2)^{\frac{3}{2}}}\right]$$

8. If $u = x\log xy$ and $x^2+y^2+3xy-1=0$, find $\frac{du}{dx}$.

$$\left[\textbf{Ans.}: 1+\log xy-\frac{x}{y}\left(\frac{x^2+ay}{y^2+ax}\right)\right]$$

9. If $x^m+y^m=b^m$, show that $\frac{d^2y}{dx^2}=-(m-1)b^m\frac{x^{m-2}}{y^{2m-1}}$.

10. If $x^3+y^3=3ax^2$, find $\frac{d^2y}{dx^2}$.

$$\left[\textbf{Ans.}: -\frac{2a^2x^2}{y^5}\right]$$

4.6 EULER'S THEOREM FOR HOMOGENEOUS FUNCTIONS

A function $u = f(x, y)$ is said to be a homogeneous function of degree n, if

$$u = x^n f\left(\frac{y}{x}\right)$$

where n is a real number.

Note Degree of a homogeneous function $u = f(x, y)$ can be obtained by replacing x by xt and y by yt and if

$$f(xt, yt) = t^n f(x, y) = t^n u$$

then u is a homogeneous function of degree n. Same method can be extended for a function of more than two variables.

HISTORICAL DATA

Leohard Euler (1707–1783) was a Swiss mathematician, physicist, astronomer, logician and engineer who made important and influential discoveries in many branches of mathematics like infinitesimal calculus and graph theory while also making pioneering contributions to several branches such as topology and analytic number theory. He also introduced much of modern mathematical terminology and notations, particularly for mathematical analysis, such as the norton of a mathematical function.

4.6.1 Euler's Theorem for a Function of Two Variables

- **Statement** If u is a homogeneous function of two variables x and y of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof Let $u = f(x, y)$ be a homogeneous function of degree n.

$$f(xt, yt) = t^n f(x, y)$$

Let $X = xt$ and $Y = yt$

$$f(X, Y) = t^n f(x, y) = t^n u$$

$$f(X, Y) = t^n u \qquad \text{...(4.1)}$$

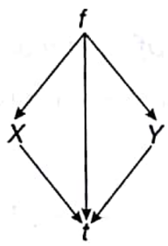


Fig. 4.18 Function of two variables

Differentiating Eq. (4.1) w.r.t. t using composite function,

$$\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial t} = nt^{n-1}u$$

$$\frac{\partial f}{\partial X}\cdot x + \frac{\partial f}{\partial Y}\cdot y = nt^{n-1}u \qquad \text{...(4.2)}$$

At $t = 1$, $X = x$, $Y = y$ and $f(X, Y) = f(x, y) = u$.

Putting $x = 1$ in Eq. (4.2),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

4.6.2 Euler's Theorem for a Function of Three Variables

- **Statement** If u is a homogeneous function of three variables x, y, z of degree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

Proof Let $u=f(x, y, z)$ be a homogeneous function of degree n

$$f(xt, yt, zt)=t^n f(x, y, z)$$

Let $X=xt$, $Y=yt$, $Z=zt$

$$f(X,Y,Z)=t^n f(x, y, z)=t^n u$$

$$f(X,Y,Z)=t^n u \qquad \ldots(4.3)$$

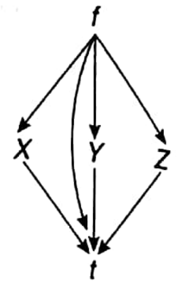


Fig. 4.19 Function of three variables

Differentiating Eq. (4.3) w.r.t. t using composite function,

$$\frac{\partial f}{\partial X}\cdot\frac{\partial X}{\partial t}+\frac{\partial f}{\partial Y}\cdot\frac{\partial Y}{\partial t}+\frac{\partial f}{\partial Z}\cdot\frac{\partial Z}{\partial t}=nt^{n-1}u$$

$$\frac{\partial f}{\partial X}\cdot x+\frac{\partial f}{\partial Y}\cdot y+\frac{\partial f}{\partial Z}\cdot z=nt^{n-1}u \qquad \ldots(4.4)$$

At $t=1$, $X=x$, $Y=y$, $Z=z$ and $f(X, Y, Z)=f(x, y, z)=u$.

Putting $x=1$ in Eq. (4.4),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

4.6.3 Deductions from Euler's Theorem

- **Corollary 1** If u is a homogeneous function of two variables x, y of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

Proof Let u be a homogeneous function of two variables x and y of degree n.

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \qquad \ldots(4.5)$$

Differentiating Eq. (4.5) partially w.r.t x,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}=n\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=(n-1)\frac{\partial u}{\partial x} \qquad \ldots(4.6)$$

Differentiating Eq. (4.5) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial y\,\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=n\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=(n-1)\frac{\partial u}{\partial y} \qquad \ldots(4.7)$$

Multiplying Eq. (4.6) by x and Eq. (4.7) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(n-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=(n-1)nu \quad \text{[Using Eq. (4.5)]}$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

Note If u is a homogeneous function of three variables x, y, z then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+2yz\frac{\partial^2 u}{\partial y\partial z}+z^2\frac{\partial^2 u}{\partial z^2}+2zx\frac{\partial^2 u}{\partial z\partial x}=n(n-1)u$$

EXAMPLE 4.24 *Verify Euler's theorem for* $u=\sqrt{x}+\sqrt{y}+\sqrt{z}$.

Solution: Euler's theorem is

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu \quad \ldots(1)$$

$$u=\sqrt{x}+\sqrt{y}+\sqrt{z}$$

Replacing x by xt, y by yt and z by zt,

$$u=t^{\frac{1}{2}}(\sqrt{x}+\sqrt{y}+\sqrt{z})$$

Hence, u is a homogeneous function of degree $\frac{1}{2}$.

R.H.S. of Eq.(1) $=nu=\frac{1}{2}u$ ... (2)

Differentiating u partially w.r.t. x, y and z,

$$\frac{\partial u}{\partial x}=\frac{1}{2\sqrt{x}},\quad \frac{\partial u}{\partial y}=\frac{1}{2\sqrt{y}},\quad \frac{\partial u}{\partial z}=\frac{1}{2\sqrt{z}}$$

... (3)

L.H.S. of Eq.(1) $= x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)=\frac{1}{2}u$

Hence, from Eqs (2) and (3), the theorem is verified.

EXAMPLE 4.25 *If* $z=\log(x^2+y^2)+\frac{x^2+y^2}{x+y}-2\log(x+y)$, *find the value of* $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$.

Solution: $z=\log(x^2+y^2)+\frac{x^2+y^2}{x+y}-2\log(x+y)=\log(x^2+y^2)+\frac{x^2+y^2}{x+y}-\log(x+y)^2$

$$=\log\frac{x^2+y^2}{(x+y)^2}+\frac{x^2+y^2}{x+y}=u+v$$

where $$u=\log\frac{x^2+y^2}{(x+y)^2},\quad v=\frac{x^2+y^2}{x+y}$$

Replacing x by xt and y by yt in u and v,

$$u = t^0 \log \frac{x^2+y^2}{(x+y)^2}, \quad v = t\left(\frac{x^2+y^2}{x+y}\right)$$

Hence, u is a homogeneous function of degree 0 and v is a homogeneous function of degree 1.
By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \cdot u = 0 \qquad \ldots(1)$$

and

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 1 \cdot v \qquad \ldots(2)$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = 0 + v$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2+y^2}{x+y}$$

EXAMPLE 4.26 *If $v = \frac{1}{r} f(\theta)$, where $x = r\cos\theta$, $y = r\sin\theta$, show that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + v = 0$.*

Solution:

$$x = r\cos\theta, y = r\sin\theta$$

$$r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Now,

$$v = \frac{1}{r} f(\theta) = \frac{1}{\sqrt{x^2+y^2}} f\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Replacing x by xt and y by yt,

$$v = \frac{1}{t\sqrt{x^2+y^2}} f\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = t^{-1}\left[\frac{1}{\sqrt{x^2+y^2}} f\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}\right]$$

Hence, v is a homogeneous function of degree -1.
By Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -1 \cdot v$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + v = 0$$

EXAMPLE 4.27 *If* $u = x^3 \sin^{-1}\left(\frac{y}{x}\right) + x^4 \tan^{-1}\left(\frac{y}{x}\right)$, *find the value of*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \text{ at } x = 1, y = 1.$$

Solution: Let

$$u = v + w$$

where

$$v = x^3 \sin^{-1}\left(\frac{y}{x}\right),\ w = x^4 \tan^{-1}\left(\frac{y}{x}\right)$$

Replacing x by xt and y by yt in v and w,

$$v = t^3\left[x^3 \sin^{-1}\left(\frac{y}{x}\right)\right],\quad w = t^4\left[x^4 \tan^{-1}\left(\frac{y}{x}\right)\right]$$

Hence, v is a homogeneous function of degree 3 and w is a homogeneous function of degree 4. By Euler's theorem and Cor. 1,

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x\, \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3(3-1)v + 3v = 9v \qquad \ldots(1)$$

and

$$x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x\, \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} + x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 4(4-1)w + 4w = 16w \qquad \ldots(2)$$

Adding Eqs (1) and (2),

$$x^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 w}{\partial x^2}\right) + 2xy\left(\frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 w}{\partial x\, \partial y}\right) + y^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right) + x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = 9v + 16w$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9v + 16w$$

At $x = 1, y = 1$,

$$v = \sin^{-1} 1 = \frac{\pi}{2}$$

and

$$w = \tan^{-1} 1 = \frac{\pi}{4}$$

Hence,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x\, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{9\pi}{2} + \frac{16\pi}{4} = \frac{17\pi}{2}$$

EXAMPLE 4.28 *If $f(x, y, z) = 0$, where $f(x, y, z)$ is a homogeneous function of degree n then show that*

$$x^2\frac{\partial^2 z}{\partial x^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} = y^2\frac{\partial^2 z}{\partial y^2}.$$

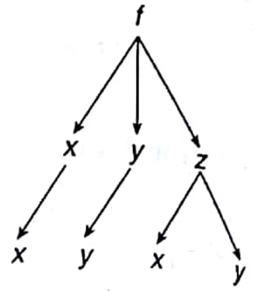


Fig. 4.20

Solution: Here, z is an implicit function of x and y.

$$f(x, y, z) = 0$$

Using composite function,

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \qquad \left[\because \frac{\partial y}{\partial x} = 0\right]$$

$$\frac{\partial f}{\partial x} = -\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} \qquad \ldots(1)$$

Similarly,

$$\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} \qquad \ldots(2)$$

f is a homogeneous function of degree n.
By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf = 0 \qquad [\because f(x, y, z) = 0]$$

Substituting $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ from Eqs (1) and (2),

$$x\left(-\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + y\left(-\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + z\frac{\partial f}{\partial z} = 0$$

$$-x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + z = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \qquad \ldots(3)$$

Differentiating Eq. (3) w.r.t. x,

$$x\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} + y\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2} = -y\frac{\partial^2 z}{\partial x\,\partial y}$$

$$x^2\frac{\partial^2 z}{\partial x^2} = -xy\frac{\partial^2 z}{\partial x\,\partial y} \qquad \ldots(4)$$

Again differentiating Eq. (3) w.r.t. y,

$$x\frac{\partial^2 z}{\partial y\,\partial x}+y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}=\frac{\partial z}{\partial y}$$

$$x\frac{\partial^2 z}{\partial y\,\partial x}=-y\frac{\partial^2 z}{\partial y^2}$$

$$y^2\frac{\partial^2 z}{\partial y^2}=-xy\frac{\partial^2 z}{\partial x\,\partial y} \quad \ldots(5)$$

From Eqs (4) and (5),

$$x^2\frac{\partial^2 z}{\partial x^2}=-xy\frac{\partial^2 z}{\partial x\,\partial y}=y^2\frac{\partial^2 z}{\partial y^2}$$

EXERCISE 4.5

1. Verify Euler's theorem for

   (i) $u=\tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{y}\right)$

   (ii) $u=\log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$

   (iii) $u=\log\left(\dfrac{x^2+y^2}{x^2-y^2}\right)$

   (iv) $u=3x^2yz+5xy^2z+4z^4$

   (v) $u=\dfrac{x^2+y^2+z^2}{x+y+z}$

   (vi) $u=\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}$

2. If $u=\cos\dfrac{xy+yz+zx}{x^2+y^2+z^2}$, find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$.

   [Ans.: 0]

3. If $u=\cos\left(\dfrac{xy+yz}{x^2+y^2+z^2}\right)+\sin\left[\dfrac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{(xy)^{\frac{1}{4}}}\right]$, evaluate $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$.

   [Ans.: 0]

4. If $u=\sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that $\dfrac{\partial u}{\partial x}=-\dfrac{y}{x}\dfrac{\partial u}{\partial y}$.

5. If $z=x^3e^{-\frac{x}{y}}$, find the value of $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}$.

   [Ans.: 6z]

6. If $u=x^2yz-4y^2z^2+2xz^3$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-4u$.

7. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, where u is a homogeneous function in x, y, z of degree n, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2+\left(\dfrac{\partial u}{\partial z}\right)^2=2nu$.

8. If $u=\dfrac{x^3y^3}{x^3+y^3}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3u$.

9. If $u=\dfrac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{3}{2}u$.

10. If $u=\dfrac{xy}{x+y}$, find the value of

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}.$$

[Ans.: 0]

11. If $u=\dfrac{x^3y^3z^3}{x^3+y^3+z^3}+\log\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)$,

prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=6u$.

12. If $u=x\phi\left(\dfrac{y}{x}\right)+\psi\left(\dfrac{y}{x}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

13. If $u=3x^4\cot^{-1}\left(\dfrac{y}{x}\right)+16y^4\cos^{-1}\left(\dfrac{y}{x}\right)$, prove

that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=12u$.

14. If $u=x^3y^2\sin^{-1}\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}$

$+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}+x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=25u$.

15. If $u=x^2\log\left(\dfrac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right)$, prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2x^2\log\left(\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right).$$

16. If $u=f\left(\dfrac{x^2-y^2}{z^2},\dfrac{y^2-z^2}{x^2},\dfrac{z^2-x^2}{y^2}\right)$, prove

that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=0$.

17. If $u=\left(\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}\right)^n$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

18. If $u=x^2\sin^{-1}\left(\dfrac{y}{x}\right)-y^2\cos^{-1}\left(\dfrac{x}{y}\right)$, prove

that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=2u$.

19. If $u=x\sin^{-1}\left(\dfrac{y}{x}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, find the

value of $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\,\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}$.

[Ans.: 0]

20. If $y=x\cos u$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

21. If $u=x^1\left[\tan^{-1}\left(\dfrac{y}{x}\right)+\dfrac{y}{x}e^{-\frac{y}{x}}\right]+y^{-3}$

$\left[\sin^{-1}\left(\dfrac{x}{y}\right)+\dfrac{x}{y}\log\dfrac{x}{y},\right]$ prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+$$

$$y\frac{\partial u}{\partial y}=9u.$$

22. If $z=f(x,y)$ and u, v are homogeneous functions of degree n in x, y then show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=n\left(u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right).$$

23. If $u=(x^2+y^2)^{\frac{2}{3}}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{4}{9}u.$$

■ **Corollary 2** If $z = f(u)$ is a homogeneous function of degree n in variables x and y then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=n\dfrac{f(u)}{f'(u)}$.

Proof By Euler's theorem,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz=nf(u)$$

$$x\frac{\partial}{\partial x}f(u)+y\frac{\partial}{\partial y}f(u)=nf(u)$$

$$xf'(u)\frac{\partial u}{\partial x}+yf'(u)\frac{\partial u}{\partial y}=nf(u)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}$$

Note If $v=f(u)$ is a homogeneous function of degree n in variables x, y and z then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}$$

■ **Corollary 3** If $z = f(u)$ is a homogeneous function of degree n in variables x and y then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where $g(u)=n\dfrac{f(u)}{f'(u)}$.

Proof By Cor. 2,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=g(u) \qquad \text{... (4.8)}$$

Differentiating Eq. (4.8) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}=g'(u)\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\,\partial y}=[g'(u)-1]\frac{\partial u}{\partial x} \qquad \text{... (4.9)}$$

Differentiating Eq. (4.8) partially w.r.t. y,

$$x\frac{\partial^2 u}{\partial y\,\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=g'(u)\frac{\partial u}{\partial y}$$

$$x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=[g'(u)-1]\frac{\partial u}{\partial y} \qquad \text{... (4.10)}$$

Multiplying Eq. (4.9) by x and Eq. (4.10) by y and adding,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[g'(u)-1]\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=[g'(u)-1]g(u)$$ [Using Eq. (4.8)]

where $g(u)=n\dfrac{f(u)}{f'(u)}$.

EXAMPLE 4.29 *If* $u=\log(x^2+y^2+z^2)$, *prove that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=2.$

Solution:

$$u=\log(x^2+y^2+z^2)$$

Replacing x by xt, y by yt, and z by zt,

$$u=\log\left[t^2(x^2+y^2+z^2)\right]$$

u is a nonhomogeneous function. But $e^u=x^2+y^2+z^2$ is a homogeneous function of degree 2.
Let $f(u)=e^u$
By Cor. 2

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f'(u)}=2\frac{e^u}{e^u}=2$$

EXAMPLE 4.30 *If* $u=\cos\left(\dfrac{xy+yz}{x^2+y^2+z^2}\right)+\sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$, *show that*

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right).$$

Solution: Let

$$u=v+w$$

where $v=\cos\left(\dfrac{xy+yz}{x^2+y^2+z^2}\right)$ and $w=\sin\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$

Replacing x by xt, y by yt, and z by zt,

$$v=t^0\cos\left(\frac{xy+yz}{x^2+y^2+z^2}\right),$$

$$w=\sin\left[t^{\frac{1}{2}}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\right]$$

Here v is a homogeneous function of degree 0.

By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=0\cdot v=0 \quad \text{... (1)}$$

w is a nonhomogeneous function. But $\sin^{-1} w=\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)$ is a homogeneous function of degree $\frac{1}{2}$.

Let $f(w)=\sin^{-1} w$

By Cor. 2,

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=n\frac{f(w)}{f'(w)}=\frac{1}{2}\frac{\sin^{-1} w}{\frac{1}{\sqrt{1-w^2}}}=\frac{1}{2}\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\sqrt{1-\sin^2\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}$$

$$=\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2} \quad \text{... (2)}$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2}$$

Hence,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)\cos\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)}{2}$$

EXAMPLE 4.31 *If* $u=\dfrac{x^2y^2z^2}{x^2+y^2+z^2}+\cos^{-1}\left(\dfrac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$ *then find the value of* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$.

Solution: Let

$$u=v+w$$

where
$$v=\frac{x^2y^2z^2}{x^2+y^2+z^2}, \qquad w=\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)$$

Replacing x by xt and y by yt,

$$v=t^4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right), \qquad w=\cos^{-1}\left[t^{\frac{1}{2}}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right]$$

Hence, v is a homogeneous function of degree 4.

By Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}=nv=4v \qquad \dots(1)$$

w is a nonhomogeneous function. But $\cos w$ is a homogeneous function of degree $\frac{1}{2}$.

Let
$$f(w)=\cos w$$

By Cor. 2,

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}+z\frac{\partial w}{\partial z}=n\frac{f(w)}{f'(w)}=\frac{1}{2}\frac{\cos w}{(-\sin w)}=-\frac{1}{2}\cot w \qquad \dots(2)$$

Adding Eqs (1) and (2),

$$x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial z}\right)=4v-\frac{1}{2}\cot w$$

Hence,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=4\left(\frac{x^2y^2z^2}{x^2+y^2+z^2}\right)-\frac{1}{2}\cot\left[\cos^{-1}\left(\frac{x+y+z}{\sqrt{x}+\sqrt{y}+\sqrt{z}}\right)\right]$$

EXAMPLE 4.32 *If* $u=\operatorname{cosec}^{-1}\left(\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right)$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}(13+\tan^2 u).$$

Solution:
$$u=\operatorname{cosec}^{-1}\left(\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right)$$

Replacing x by xt and y by yt,

$$u=\operatorname{cosec}^{-1}\left(t^{\frac{1}{12}}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right)$$

u is a nonhomogeneous function.

But $\operatorname{cosec} u=\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ is a homogeneous function of degree $\frac{1}{12}$.

Let $f(u)=\operatorname{cosec} u$

By Cor. 3,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where
$$g(u)=n\frac{f(u)}{f'(u)}=\frac{1}{12}\frac{\operatorname{cosec} u}{(-\operatorname{cosec} u\cot u)}=-\frac{1}{12}\tan u$$

$$g'(u) = -\frac{1}{12}\sec^2 u = -\frac{1}{12}(1+\tan^2 u)$$

Hence, $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = -\frac{1}{12}\tan u\left[-\frac{1}{12}(1+\tan^2 u)-1\right]$$

$$= -\frac{1}{12}\tan u\left(\frac{-1-\tan^2 u-12}{12}\right) = \frac{\tan u}{144}(13+\tan^2 u)$$

EXERCISE 4.6

1. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

2. If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3\tan\ u.$$

3. If $u = \log\,(x^2+xy+y^2)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2.$$

4. If $(x-y)\tan u = x^3+y^3$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u.$$

5. If $u = \log\,(x^3+y^3-x^2y-xy^2)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 3.$$

6. If $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = 2\tan u.$$

7. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u.$$

8. If $u = \sin^{-1}\left(\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{6}}+y^{\frac{1}{6}}}\right)$, prove that $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{144}\tan u(\tan^2 u-1).$$

9. If $\sqrt{x}+\sqrt{y}\sin^2 u = x^{\frac{1}{3}}+y^{\frac{1}{3}}$, prove that
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right).$$

10. If $u = \cos^{-1}\left(\frac{x^5-2y^5+6z^5}{\sqrt{ax^3+by^3+cz^3}}\right)$, show that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = -\frac{7}{2}\cot u.$$

11. If $\sqrt{x}+\sqrt{y}\cot u - x - y = 0$, prove that $4x\dfrac{\partial u}{\partial x}+4y\dfrac{\partial u}{\partial y}+\sin 2u = 0.$

12. If $u=\sin^{-1}\left(\dfrac{ax+by+cz}{\sqrt{x^n+y^n+z^n}}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=2\tan u.$

13. If $v=\log\sin\left[\dfrac{\pi(2x^2+y^2+xz)^{\frac{1}{2}}}{2(x^2+xy+2yz+z^2)^{\frac{1}{3}}}\right]$, prove that $x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}+z\dfrac{\partial v}{\partial z}$ $=\dfrac{1}{3}e^{-v}\sqrt{1-e^{2v}}\cdot\sin^{-1}(e^{v})$

4.7 JACOBIANS

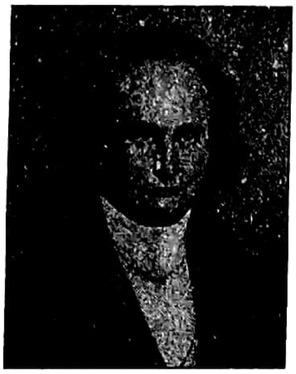
If u and v are continuous and differentiable functions of two independent variables x and y, i.e., $u=f_1(x, y)$ and $v=f_2(x, y)$ then the determinant $\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y}\end{vmatrix}$ is called the Jacobian of u, v with respect to x, y and is denoted as $J=\dfrac{\partial(u,v)}{\partial(x,y)}$.

Similarly, if u, v and w are continuous and differentiable functions of three independent variables x, y, z then the Jacobian of u, v, w with respect to x, y, z is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z}\\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z}\end{vmatrix}$$

The Jacobian is useful in transformation of variables from cartesian to polar, cylindrical and spherical coordinates in multiple integrals.

HISTORICAL DATA



Carl Gustav Jacob Jacobi (1804–1851) was a German mathematician, who made fundamental contributions to elliptic functions, dynamics, differential equations, and number theory. Jacobi was the first Jewish mathematician to be appointed professor at a German university.

One of Jacobi's greatest accomplishments was his theory of elliptic functions and their relation to the elliptic theta function. Theta functions are of great importance in mathematical physics because of their role in the inverse problem for periodic and quasi-periodic flows. The equations of motion are integrable in terms of Jacobi's elliptic functions in the well-known cases of the pendulum, the Euler top, the symmetric Lagrange top in a gravitational field and the Kepler problem (planetary motion in a central gravitational field). He also made fundamental contributions in the study of differential equations and to rational mechanics, notably the Hamilton–Jacobi theory.

4.7.1 Properties of Jacobians

- **Property 1** If u and v are functions of x and y then

$$J \cdot J^* = 1, \text{ where } J = \frac{\partial(u, v)}{\partial(x, y)} \text{ and } J^* = \frac{\partial(x, y)}{\partial(u, v)}$$

Proof Let u and v be two functions of x and y.

$$u = f_1(x, y) \quad \text{and} \quad v = f_2(x, y) \qquad \ldots(4.11)$$

Writing x and y in terms of u and v,

$$x = \phi_1(u, v) \quad \text{and} \quad y = \phi_2(u, v) \qquad \ldots(4.12)$$

Differentiating Eq. (4.11) partially w.r.t. u and v,

$$\frac{\partial u}{\partial u} = \frac{\partial f_1}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f_1}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$1 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \ldots(4.13)$$

$$\frac{\partial u}{\partial v} = \frac{\partial f_1}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f_1}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$0 = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial v} \qquad \ldots(4.14)$$

$$\frac{\partial v}{\partial u} = \frac{\partial f_2}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f_2}{\partial y} \cdot \frac{\partial y}{\partial u},$$

$$0 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial u} \qquad \ldots(4.15)$$

$$\frac{\partial v}{\partial v} = \frac{\partial f_2}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f_2}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$1 = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial v} \qquad \ldots(4.16)$$

$$J \cdot J^* = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad \left[\begin{array}{l}\text{Interchanging rows and columns} \\ \text{of second determinant}\end{array}\right]$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad \text{[Substituting Eqs (4.13), (4.14), (4.15), (4.16)]}$$

$$= 1$$

Note If u, v and w are functions of three variables x, y and z respectively then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(u, v, w)} = 1$$

- **Property 2** If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

Proof Let u, v be functions of r, s and r, s be functions of x, y (Fig. 4.21).

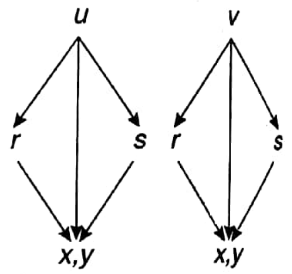


Fig. 4.21

$$\frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial s}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial s}{\partial y} \end{vmatrix} \qquad \begin{bmatrix} \text{Interchanging rows} \\ \text{and columns of} \\ \text{second determinant} \end{bmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u, v)}{\partial(x, y)}$$

Note If u, v, w are functions of r, s, t and r, s, t are functions of x, y, z then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}=\frac{\partial(u, v, w)}{\partial(r, s, t)}\cdot\frac{\partial(r, s, t)}{\partial(x, y, z)}$$

- **Property 3** If functions u, v of two independent variables x, y are dependent then $\frac{\partial(u, v)}{\partial(x, y)}=0.$

Proof If u, v are dependent then there must be a relation $f(u,v)=0$. ...(4.17)

Differentiating Eq. (4.17) partially w.r.t. x and y,

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=0 \quad ...(4.18)$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=0 \quad ...(4.19)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from Eqs (4.18) and (4.19),

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}=0$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=0 \quad \text{[Interchanging rows and columns]}$$

$$\frac{\partial(u, v)}{\partial(x, y)}=0$$

EXAMPLE 4.33

Find the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$ *for each of the following functions:*

(i) $u=x^2-y^2, v=2xy$ (ii) $u=\frac{x+y}{1-xy}, v=\tan^{-1}x+\tan^{-1}y$

Solution:

(i)

$$u=x^2-y^2 \qquad v=2xy$$

$$\frac{\partial u}{\partial x}=2x \qquad \frac{\partial v}{\partial x}=2y$$

$$\frac{\partial u}{\partial y}=-2y \qquad \frac{\partial v}{\partial y}=2x$$

$$\frac{\partial(u, v)}{\partial(x, y)}=\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}=\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}=4(x^2+y^2)$$

(ii) $$u = \frac{x+y}{1-xy} \qquad v = \tan^{-1} x + \tan^{-1} y$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} \qquad \frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} \qquad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$= \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

EXAMPLE 4.34 *If $x = a(u+v)$, $y = b(u-v)$ and $u = r^2 \cos 2\theta$, $v = r^2 \sin 2\theta$, find $\frac{\partial(x, y)}{\partial(r,\theta)}$.*

Solution:

$$x = a(u+v) \qquad y = b(u-v)$$

$$\frac{\partial x}{\partial u} = a \qquad \frac{\partial y}{\partial u} = b$$

$$\frac{\partial x}{\partial v} = a \qquad \frac{\partial y}{\partial v} = -b$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2ab$$

$$u = r^2 \cos 2\theta \qquad v = r^2 \sin 2\theta$$

$$\frac{\partial u}{\partial r} = 2r \cos 2\theta \qquad \frac{\partial v}{\partial r} = 2r \sin 2\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta \qquad \frac{\partial v}{\partial \theta} = 2r^2 \cos 2\theta$$

$$\frac{\partial(u, v)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r \cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix} = 4r^3(\cos^2 2\theta + \sin^2 2\theta) = 4r^3$$

Hence, $$\frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(r, \theta)} = (-2ab)\,(4r^3) = -8abr^3$$

EXAMPLE 4.35 *Verify $J \cdot J^* = 1$ for the following function:*
$x = u$, $y = u \tan v$, $z = w$

Solution:

$$x = u \qquad y = u\tan v \qquad z = w$$

$$\frac{\partial x}{\partial u} = 1 \qquad \frac{\partial y}{\partial u} = \tan v \qquad \frac{\partial z}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = 0 \qquad \frac{\partial y}{\partial v} = u\sec^2 v \qquad \frac{\partial z}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = 0 \qquad \frac{\partial y}{\partial w} = 0 \qquad \frac{\partial z}{\partial w} = 1$$

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \tan v & u\sec^2 v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u\sec^2 v$$

Writing u, v, w in terms of x, y and z,

$$u = x \qquad \tan v = \frac{y}{u} = \frac{y}{x} \qquad w = z$$

$$v = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2} \qquad \frac{\partial w}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2} \qquad \frac{\partial w}{\partial y} = 0$$

$$\frac{\partial u}{\partial z} = 0 \qquad \frac{\partial v}{\partial z} = 0 \qquad \frac{\partial w}{\partial z} = 1$$

$$J^* = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{x}{x^2 + y^2} = \frac{u}{u^2 + u^2\tan^2 v} = \frac{1}{u\sec^2 v}$$

Hence,
$$J \cdot J^* = \frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)} = u\sec^2 v \cdot \frac{1}{u\sec^2 v} = 1$$

EXAMPLE 4.36 *Prove that the functions $u = y+z, v = x+2z^2, w = x-4yz-2y^2$ are functionally dependent and find the relation between them.*

Solution: $u = y+z \qquad v = x+2z^2 \qquad w = x-4yz-2y^2$

$$\frac{\partial u}{\partial x}=0 \qquad \frac{\partial v}{\partial x}=1 \qquad \frac{\partial w}{\partial x}=1$$

$$\frac{\partial u}{\partial y}=1 \qquad \frac{\partial v}{\partial y}=0 \qquad \frac{\partial w}{\partial y}=-4z-4y$$

$$\frac{\partial u}{\partial z}=1 \qquad \frac{\partial v}{\partial z}=4z \qquad \frac{\partial w}{\partial z}=-4y$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4z-4y & -4y \end{vmatrix} = 0-1(-4y-4z)+1(-4z-4y)=0$$

Hence, u, v and w are functionally dependent.

Relation among u, v and w

$$v-2u^2 = (x+2z^2)-2(y+z)^2 = x+2z^2-2y^2-4yz-2z^2 = x-4yz-2y^2 = w$$

4.7.2 Jacobian of Implicit Functions

If u, v are implicit functions of x, y connected by f_1, f_2 such that $f_1(u, v, x, y) = 0$, $f_2(u, v, x, y) = 0$ then

$$\frac{\partial f_1}{\partial x}+\frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x}=0$$

$$\frac{\partial f_1}{\partial y}+\frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y}=0$$

Similarly,

$$\frac{\partial f_2}{\partial x}+\frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x}=0$$

$$\frac{\partial f_2}{\partial y}+\frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y}=0$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} \quad \left[\begin{array}{l}\text{Interchanging rows and} \\ \text{columns of second determinant}\end{array}\right]$$

$$= \begin{vmatrix} \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_1}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\frac{\partial v}{\partial y} \\ \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial f_2}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{\partial f_1}{\partial x} & -\frac{\partial f_1}{\partial y} \\ -\frac{\partial f_2}{\partial x} & -\frac{\partial f_2}{\partial y} \end{vmatrix} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x, y)}$$

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}}$$

Similarly, if u, v, w are implicit functions of x, y, z connected by f_1, f_2, f_3 such that $f_1(u, v, w, x, y, z) = 0$, $f_2(u, v, w, x, y, z), = 0, f_3(u, v, w, x, y, z) = 0$ then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

EXAMPLE 4.37 *If* $x^2 + y^2 + u^2 - v^2 = 0,\ uv + xy = 0,$ *prove that* $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{x^2 - y^2}{u^2 + v^2}$.

Solution: Let

$$f_1 = x^2 + y^2 + u^2 - v^2 \qquad f_2 = uv + xy$$

$$\frac{\partial f_1}{\partial x} = 2x \qquad \frac{\partial f_2}{\partial x} = y$$

$$\frac{\partial f_1}{\partial y} = 2y \qquad \frac{\partial f_2}{\partial y} = x$$

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2 - 2y^2 = 2(x^2 - y^2)$$

$$\frac{\partial f_1}{\partial u} = 2u \qquad \frac{\partial f_2}{\partial u} = v$$

$$\frac{\partial f_1}{\partial v} = -2v \qquad \frac{\partial f_2}{\partial v} = u$$

$$\frac{\partial(f_1, f_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2 = 2(u^2 + v^2)$$

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}} = \frac{2(x^2 - y^2)}{2(u^2 + v^2)} = \frac{x^2 - y^2}{u^2 + v^2}$$

EXAMPLE 4.38 *If* $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$

$u^3 + v^3 + w^3 = x + y + z$

$u + v + w = x^2 + y^2 + z^2$

prove that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = -\dfrac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

Solution: Let $f_1 = u^2 + v^2 + w^2 - x^3 - y^3 - z^3$, $f_2 = u^3 + v^3 + w^3 - x - y - z$, $f_3 = u + v + w - x^2 - y^2 - z^2$

$$\frac{\partial f_1}{\partial x} = -3x^2 \qquad \frac{\partial f_2}{\partial x} = -1 \qquad \frac{\partial f_3}{\partial x} = -2x$$

$$\frac{\partial f_1}{\partial y} = -3y^2 \qquad \frac{\partial f_2}{\partial y} = -1 \qquad \frac{\partial f_3}{\partial y} = -2y$$

$$\frac{\partial f_1}{\partial z} = -3z^2 \qquad \frac{\partial f_2}{\partial z} = -1 \qquad \frac{\partial f_3}{\partial z} = -2z$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -3x^2 & -3y^2 & -3z^2 \\ -1 & -1 & -1 \\ -2x & -2y & -2z \end{vmatrix} = -6 \begin{vmatrix} x^2 & y^2 & z^2 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$$= -6 \begin{vmatrix} x^2 & y^2 - x^2 & z^2 - x^2 \\ 1 & 0 & 0 \\ x & y - x & z - x \end{vmatrix} \quad [\text{By } C_2 - C_1, C_3 - C_1]$$

$$= -6(y-x)(z-x)\begin{vmatrix} x^2 & y+x & z+x \\ 1 & 0 & 0 \\ x & 1 & 1 \end{vmatrix}$$

$$= -6(y-x)(z-x)\begin{vmatrix} x^2 & y+x & z-y \\ 1 & 0 & 0 \\ x & 1 & 0 \end{vmatrix} \quad [\text{By } C_3 - C_2]$$

$$= -6(y-x)(z-x)[(z-y)(1)] = -6(x-y)(y-z)(z-x)$$

$$\frac{\partial f_1}{\partial u} = 2u \qquad \frac{\partial f_2}{\partial u} = 3u^2 \qquad \frac{\partial f_3}{\partial u} = 1$$

$$\frac{\partial f_1}{\partial v} = 2v \qquad \frac{\partial f_2}{\partial v} = 3v^2 \qquad \frac{\partial f_3}{\partial v} = 1$$

$$\frac{\partial f_1}{\partial w} = 2w \qquad \frac{\partial f_2}{\partial w} = 3w^2 \qquad \frac{\partial f_3}{\partial w} = 1$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 2u & 2v & 2w \\ 3u^2 & 3v^2 & 3w^2 \\ 1 & 1 & 1 \end{vmatrix} = 6\begin{vmatrix} u & v & w \\ u^2 & v^2 & w^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} u & v-u & w-u \\ u^2 & v^2-u^2 & w^2-u^2 \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{By } C_2 - C_1, C_3 - C_1]$$

$$= 6(v-u)(w-u)\begin{vmatrix} u & 1 & 1 \\ u^2 & v+u & w+u \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 6(v-u)(w-u)\begin{vmatrix} u & 1 & 0 \\ u^2 & v+u & w-v \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{By } C_3 - C_2]$$

$$= 6(v-u)(w-u)[(-1)(-1)(w-v)] = -6(u-v)(v-w)(w-u)$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}} = (-1)\frac{-6(x-y)(y-z)(z-x)}{-6(u-v)(v-w)(w-u)} = -\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

4.7.3 Partial Derivatives of Implicit Functions

If u, v are implicit functions of x, y connected by f_1, f_2 such that

$$f_1(u, v, x, y) = 0$$

$$f_2(u, v, x, y) = 0$$

then

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{...(4.20)}$$

$$\frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \text{...(4.21)}$$

Similarly,

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{...(4.22)}$$

$$\frac{\partial f_2}{\partial y} + \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial y} = 0 \quad \text{...(4.23)}$$

Solving Eqs (4.20) and (4.22) for $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$,

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}} = -\frac{\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}} \quad \text{...(4.24)}$$

$$\frac{\partial v}{\partial x} = -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}} = -\frac{\frac{\partial(f_1, f_2)}{\partial(u, x)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}} \quad \text{...(4.25)}$$

Similarly, solving Eqs (4.21) and (4.23) for $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$,

$$\frac{\partial u}{\partial y}=-\frac{\frac{\partial(f_1,f_2)}{\partial(y,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\frac{\partial v}{\partial y}=-\frac{\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

Similarly, if u, v, w be implicit functions of x, y, z connected by f_1, f_2, f_3 such that

$$f_1(u,v,w,x,y,z)=0$$
$$f_2(u,v,w,x,y,z)=0$$
$$f_3(u,v,w,x,y,z)=0$$

then

$$\frac{\partial u}{\partial x}=-\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial v}{\partial x}=-\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,x,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial w}{\partial x}=-\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,x)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

and so on.

EXAMPLE 4.39 *If* $x=u+v, y=v^2+w^2, z=w^3+u^3$, *show that* $\frac{\partial u}{\partial x}=\frac{vw}{vw+u^2}$.

Solution: Let

$$f_1=x-u-v$$
$$f_2=y-v^2-w^2$$
$$f_3=z-w^3-u^3$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w}\end{vmatrix}=\begin{vmatrix}1 & -1 & 0\\ 0 & -2v & -2w\\ 0 & 0 & -3w^2\end{vmatrix}=6vw^2$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w}\\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w}\end{vmatrix}=\begin{vmatrix}-1 & -1 & 0\\ 0 & -2v & -2w\\ -3u^2 & 0 & -3w^2\end{vmatrix}$$

$$=(-1)(6vw^2)+(1)(-6u^2w)=-6vw^2-6u^2w$$

$$\frac{\partial u}{\partial x}=-\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}=-\frac{6vw^2}{(-6vw^2-6u^2w)}=\frac{vw}{vw+u^2}$$

EXAMPLE 4.40 *Use Jacobians to show that* $\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v+\frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u=0$, *where* $ux+vy=a,\ \frac{u}{x}+\frac{v}{y}=1.$

Solution: Let

$$f_1=ux+vy-a$$

$$f_2=\frac{u}{x}+\frac{v}{y}-1$$

$$\frac{\partial(f_1,f_2)}{\partial(u,y)}=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y}\end{vmatrix}=\begin{vmatrix}x & v\\ \frac{1}{x} & -\frac{v}{y^2}\end{vmatrix}=-\frac{xv}{y^2}-\frac{v}{x}=-v\left(\frac{x^2+y^2}{xy^2}\right)$$

$$\frac{\partial(f_1,f_2)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{vmatrix}=\begin{vmatrix}u & v\\ -\frac{u}{x^2} & -\frac{v}{y^2}\end{vmatrix}=-\frac{uv}{y^2}+\frac{uv}{x^2}=uv\left(\frac{y^2-x^2}{x^2y^2}\right)$$

$$\frac{\partial(f_1,f_2)}{\partial(x,v)}=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v}\end{vmatrix}=\begin{vmatrix}u & y\\ -\frac{u}{x^2} & \frac{1}{y}\end{vmatrix}=\frac{u}{y}+\frac{uy}{x^2}=\frac{u(x^2+y^2)}{x^2y}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = -\frac{\dfrac{\partial(f_1, f_2)}{\partial(u, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}} = -\frac{-v\left(\dfrac{x^2+y^2}{xy^2}\right)}{uv\left(\dfrac{y^2-x^2}{x^2y^2}\right)} = \frac{x}{u}\left(\frac{x^2+y^2}{y^2-x^2}\right)$$

$$\left(\frac{\partial y}{\partial v}\right)_u = -\frac{\dfrac{\partial(f_1, f_2)}{\partial(x, v)}}{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}} = -\frac{\dfrac{u(x^2+y^2)}{x^2y}}{uv\left(\dfrac{y^2-x^2}{x^2y^2}\right)} = -\frac{y}{v}\left(\frac{x^2+y^2}{y^2-x^2}\right)$$

Hence, $$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = \frac{x^2+y^2}{y^2-x^2} - \frac{x^2+y^2}{y^2-x^2} = 0$$

EXERCISE 4.7

1. Find the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ for each of the following functions:
   (i) $u = x + y,\ v = x - y$
   (ii) $u = x^2,\ v = y^2$
   (iii) $u = 3x + 5y,\ v = 4x - 3y$
   (iv) $u = \dfrac{y-x}{1+xy},\ v = \tan^{-1} y - \tan^{-1} x$
   (v) $u = x \sin y,\ v = y \sin x$.

   [**Ans.:** (i) $\dfrac{-1}{2}$ (ii) $4xy$ (iii) -29 (iv) 0 (v) $\sin x \sin y - xy \cos x \cos y$]

2. Find the Jacobian for each of the following functions:
   (i) $x = e^u \cos v,\quad y = e^u \sin v$
   (ii) $x = u(1 - v),\quad y = uv$
   (iii) $x = uv,\quad y = \dfrac{u+v}{u-v}$

   [**Ans.:** (i) e^{2u} (ii) u (iii) $\dfrac{4uv}{(u-v)^2}$]

3. Find the Jacobian for each of the following functions:
   (i) $u = \dfrac{yz}{x},\ v = \dfrac{zx}{y},\ w = \dfrac{xy}{z}$
   (ii) $u = xyz,\ v = xy + yz + zx,\ w = x + y + z$
   (iii) $u = x^2,\ v = \sin y,\ w = e^{-3z}$
   (iv) $x = \dfrac{1}{2}(u^2 - v^2),\ y = uv,\ z = w$

   [**Ans.:** (i) 4 (ii) $(x-y)(y-z)(z-x)$ (iii) $-6e^{-3z} x \cos y$ (iv) $\dfrac{1}{u^2+v^2}$]

4. Verify that $J \cdot J^* = 1$ for the following functions:
   (i) $u = x + \dfrac{y^2}{x},\ v = \dfrac{y^2}{x}$
   (ii) $x = u(1 - v),\ y = uv$
   (iii) $x = \sin\theta \cos\phi,\ y = \sin\theta \sin\phi$

5. If $u = x + y + z,\ uv = y + z,\ uvw = z$, evaluate $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$.

   [**Ans.:** $u^2 v$]

6. If $u^3 + v^3 = x + y,\ u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{2}\dfrac{(y^2 - x^2)}{uv(u-v)}$.

7. Calculate $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ if

$u=\dfrac{x}{\sqrt{1-r^2}}, v=\dfrac{y}{\sqrt{1-r^2}}, w=\dfrac{z}{\sqrt{1-r^2}}$,

where $r^2 = x^2 + y^2 + z^2$.

$\left[\textbf{Ans.:} (1-r^2)^{-\frac{5}{2}}\right]$

8. If $u = x + y + z$, $u^2v = y + z$, $u^3w = z$, show that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = u^{-5}$.

9. Show that $\dfrac{\partial(u, v)}{\partial(r, \theta)} = 6r^3 \sin 2\theta$, if $u=x^2-2y^2$, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$.

10. Determine whether the following functions are functionally dependent or not. If functionally dependent, find the relation between them.

(i) $u=\dfrac{x-y}{x+y}, v=\dfrac{x+y}{y}$

(ii) $u=\dfrac{x^2-y^2}{x^2+y^2}, v=\dfrac{2xy}{x^2+y^2}$

(iii) $u = \sin x + \sin y$, $v = \sin(x + y)$

(iv) $u=\dfrac{x-y}{x+y}, v=\dfrac{xy}{(x+y)^2}$

(v) $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = x^3 + y^3 + z^3 - 3xyz$

(vi) $u = xe^y \sin z$, $v = xe^y \cos z$, $w = x^2 e^{2y}$

(vii) $u=\dfrac{3x^2}{2(y+z)}, v=\dfrac{2(y+z)}{3(x-y)^2}$, $w=\dfrac{x-y}{x}$.

Ans.: (i) Dependent, $u=\dfrac{2-v}{v}$
(ii) Dependent, $u^2+v^2=1$
(iii) Independent
(iv) Dependent, $4v=1-u^2$
(v) Dependent, $2w=u(3v-u^2)$
(vi) Dependent, $u^2+v^2=w$
(vii) Dependent, $uvw^2=1$

11. If $u^3+v^3=x+y$, $u^2+v^2=x^3+y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)}=\dfrac{1}{2}\dfrac{y^2-x^2}{uv(u-v)}$

12. If $u_1=x_1+x_2+x_3+x_4$, $u_1u_2=x_2+x_3+x_4$, $u_1u_2u_3=x_3+x_4$, $u_1u_2u_3u_4=x_4$, show that $\dfrac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1, u_2, u_3, u_4)}=u_1^3u_2^2u_3$.

13. If $u^2+xv^2-uxy=0$, $v^2-xy^2+2uv+u^2=0$ find $\dfrac{\partial u}{\partial x}$.

$\left[\textbf{Ans.:} -\dfrac{(v^2-uy)(2v+2u)+2xvy^2}{2(u+v)(2u-xy-2xv)}\right]$

14. If $x=u+v+w$, $y=u^2+v^2+w^2$, $z=u^3+v^3+w^3$ then show that

$\dfrac{\partial u}{\partial x}=\dfrac{vw}{(u-v)(u-w)}$.

15. If $u^2+xv^2=x+y$, $v^2+yu^2=x-y$, find $\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial y}$.

$\left[\textbf{Ans.:} \dfrac{\partial u}{\partial x}=\dfrac{1-x-v^2}{2u(1-xy)}, \dfrac{\partial v}{\partial y}=-\dfrac{1+y+u^2}{2v(1-xy)}\right]$

16. If $u^2+xv^2-xy=0$, $u^2+xyv+v^2=0$, find $\dfrac{\partial u}{\partial x}$.

$\left[\textbf{Ans.:} \dfrac{xyv^2+xy^2+2vy-2v^2}{2u(xy+2v-2xy)}\right]$

4.8 TAYLOR'S SERIES FOR FUNCTIONS OF TWO VARIABLES

- **Statement** If $f(x+h, y+k)$ is a given function which can be expanded into a series of positive ascending powers of h and k then

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \cdots$$

Proof By *Taylor's expansion for function of a single variable, expanding $f(x+h, y+k)$ as a function of x,

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x} f(x, y+k) + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2} f(x, y+k)$$

$$+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3} f(x, y+k) + \cdots \qquad \ldots(4.26)$$

Again, expanding $f(x, y+k)$ as a function of y,

$$f(x, y+k) = f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y)$$

$$+\frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f(x, y) + \cdots \qquad \ldots(4.27)$$

Differentiating Eq. (4.27) partially w.r.t. x,

$$\frac{\partial}{\partial x} f(x, y+k) = \frac{\partial}{\partial x} f(x, y) + k\frac{\partial^2}{\partial x\,\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^3}{\partial x\,\partial y^2} f(x, y) + \cdots\cdots \qquad \ldots(4.28)$$

Differentiating Eq. (4.28) partially w.r.t. x,

$$\frac{\partial^2}{\partial x^2} f(x, y+k) = \frac{\partial^2}{\partial x^2} f(x, y) + k\frac{\partial^3}{\partial x^2\partial y} f(x, y) + \cdots\cdots \qquad \ldots(4.29)$$

Differentiating Eq. (4.29) partially w.r.t. x,

$$\frac{\partial^3}{\partial x^3} f(x, y+k) = \frac{\partial^3}{\partial x^3} f(x, y) + \cdots\cdots \qquad \ldots(4.30)$$

Substituting Eqs (4.27), (4.28), (4.29) and (4.30) in Eq. (4.26),

$$f(x+h, y+k) = \left[f(x, y) + k\frac{\partial}{\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2} f(x, y) + \frac{k^3}{3!}\frac{\partial^3}{\partial y^3} f(x, y) + \cdots\right]$$

$$+h\left[\frac{\partial}{\partial x} f(x, y) + k\frac{\partial^2}{\partial x\,\partial y} f(x, y) + \frac{k^2}{2!}\frac{\partial^3}{\partial x\,\partial y^2} f(x, y) + \cdots\right]$$

$$+\frac{h^2}{2!}\left[\frac{\partial^2}{\partial x^2} f(x, y) + k\frac{\partial^3}{\partial x^2\partial y} f(x, y) + \ldots\right] + \frac{h^3}{3!}\left[\frac{\partial^3}{\partial x^3} f(x, y) + \cdots\right] + \cdots$$

* Refer Chapter 3 for Historical Data of Taylor.

$$= f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x, y) + \frac{1}{2!}\left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\partial y} + k^2\frac{\partial^2}{\partial y^2}\right)f(x, y)$$

$$+\frac{1}{3!}\left(h^3\frac{\partial^3}{\partial x^3} + 3h^2k\frac{\partial^3}{\partial x^2\partial y} + 3hk^2\frac{\partial^3}{\partial x\partial y^2} + k^3\frac{\partial^3}{\partial y^3}\right)f(x, y) + \cdots \quad \ldots(4.31)$$

$$= f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(x, y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y)$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x, y) + \cdots$$

This is known as Taylor's series expansion of $f(x + h, y + k)$ in powers of h and k.
Putting $x = a$ and $y = b$ in Eq. (4.31),

$$f(a+h, b+k) = f(a, b) + \left[h f_x(a, b) + k f_y(a, b)\right]$$

$$+\frac{1}{2!}\left[h^2 f_{xx}(a, b) + 2hk f_{xy}(a, b) + k^2 f_{yy}(a, b)\right]$$

$$+\frac{1}{3!}\left[h^3 f_{xxx}(a, b) + 3h^2k f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)\right] + \cdots$$

where $f_x = \dfrac{\partial f}{\partial x}, f_y = \dfrac{\partial f}{\partial y}, f_{xx} = \dfrac{\partial^2 f}{\partial x^2}, f_{yy} = \dfrac{\partial^2 f}{\partial y^2}, f_{xy} = \dfrac{\partial^2 f}{\partial x\partial y}$, etc.

Putting $a + h = x$ and $b + k = y$,

$$f(x, y) = f(a, b) + \left[(x-a)f_x(a, b) + (y-b)f_y(a, b)\right]$$

$$+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\right] + \cdots$$

This expansion is called the expansion of $f(x, y)$ in powers of $(x - a)$ and $(y - b)$.
Putting $a = 0$ and $b = 0$,

$$f(x,y) = f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\right] + \cdots$$

This is known as **Maclaurin's** series expansion of $f(x,y)$.

EXAMPLE 4.41 *Expand e^{x+y} in power of $(x - 1)$ and $(y + 1)$ up to first-degree terms.*

Solution: Let

$$f(x, y) = e^{x+y}$$

By Taylor's expansion,

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] + \cdots \quad \ldots(1)$$

Here, $a = 1$ and $b = -1$.

$$f(x, y) = e^{x+y} \qquad f(1, -1) = 1$$

$$f_x(x, y) = e^{x+y} \qquad f_x(1, -1) = 1$$

$$f_y(x, y) = e^{x+y} \qquad f_y(1, -1) = 1$$

Substituting these values in Eq. (1),

$$f(x, y) = 1 + [(x-1)(1) + (y+1)(1)] + \cdots$$
$$e^{x+y} = 1 + (x-1) + (y+1) + \cdots$$

EXAMPLE 4.42 *Expand* $\tan^{-1}\left(\dfrac{y}{x}\right)$ *in Taylor series at* (1, 1) *up to second-degree terms.*

Solution: Let $$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

By Taylor's expansion,

$$f(x, y) = f(a, b) + [(x-a)f_x(a, b) + (y-b)f_y(a, b)] + \frac{1}{2!}\Big[(x-a)^2 f_{xx}(a, b)$$
$$+ 2(x-a)(y-b)f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b)\Big] + \cdots \qquad \ldots(1)$$

Here, $a = 1$ and $b = 1$.

$$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right) \qquad f(1, 1) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$f_x(x, y) = \frac{1}{1 + \dfrac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2} \qquad f_x(1, 1) = -\frac{1}{2}$$

$$f_y(x, y) = \frac{1}{1 + \dfrac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2} \qquad f_y(1, 1) = \frac{1}{2}$$

$$f_{xx}(x, y) = \frac{y(2x)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2} \qquad f_{xx}(1, 1) = \frac{1}{2}$$

$$f_{xy}(x, y) = \frac{(x^2 + y^2)1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad f_{xy}(1, 1) = 0$$

$$f_{yy}(x, y) = -\frac{x \cdot 2y}{(x^2 + y^2)^2} = -\frac{2xy}{(x^2 + y^2)^2} \qquad f_{yy}(1, 1) = -\frac{1}{2}$$

Substituting these values in Eq. (1),

$$f(x, y) = \frac{\pi}{4} + \left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right] + \cdots$$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \cdots$$

EXAMPLE 4.43 *Expand* $\frac{(x+h)(y+k)}{x+h+y+k}$ *in powers of h and k up to second-degree terms.*

Solution: Let

$$f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k} \qquad \ldots(1)$$

By Taylor's expansion,

$$f(x+h, y+k) = f(x, y) + \left[hf_x(x, y) + kf_y(x, y)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(x, y) + 2hk f_{xy}(x, y) + k^2 f_{yy}(x, y)\right] + \cdots \qquad \ldots(2)$$

Putting $h = 0$ and $k = 0$ in Eq. (1),

$$f(x, y) = \frac{xy}{x+y}$$

$$f_x(x, y) = \frac{(x+y)y - xy}{(x+y)^2} = \frac{y^2}{(x+y)^2}$$

$$f_y(x, y) = \frac{(x+y)x - xy}{(x+y)^2} = \frac{x^2}{(x+y)^2}$$

$$f_{xx}(x, y) = -\frac{2y^2}{(x+y)^3}$$

$$f_{xy}(x, y) = \frac{(x+y)^2 \cdot 2x - x^2 \cdot 2(x+y)}{(x+y)^4} = \frac{2x(x+y)(x+y-x)}{(x+y)^4} = \frac{2xy}{(x+y)^3}$$

$$f_{yy}(x, y) = -\frac{2x^2}{(x+y)^3}$$

Substituting these values in Eq. (2),

$$f(x+h, y+k) = \frac{xy}{x+y} + \left[h\frac{y^2}{(x+y)^2} + k\frac{x^2}{(x+y)^2}\right]$$

$$+\frac{1}{2!}\left[h^2\left\{-\frac{2y^2}{(x+y)^3}\right\} + 2hk\frac{2xy}{(x+y)^3} + k^2\left\{-\frac{2x^2}{(x+y)^3}\right\}\right]+\cdots$$

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \frac{hy^2+kx^2}{(x+y)^2} - \frac{h^2y^2 - 2hkxy + k^2x^2}{(x+y)^3} + \cdots$$

EXAMPLE 4.44 *Expand $e^x \cos y$ in powers of x and y as far as the terms of the third degree.*

Solution: Let

$$f(x, y) = e^x \cos y$$

By Taylor's expansion,

$$f(x, y) = f(0, 0) + [x\, f_x(0, 0) + y\, f_y(0, 0)]$$

$$+\frac{1}{2!}\left[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)\right]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0, 0) + 3x^2y\, f_{xxy}(0, 0) + 3xy^2\, f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)\right]+\cdots \quad \ldots(1)$$

$$\begin{aligned}
f(x, y) &= e^x \cos y & f(0, 0) &= 1\\
f_x(x, y) &= e^x \cos y & f_x(0, 0) &= 1\\
f_y(x, y) &= -e^x \sin y & f_y(0, 0) &= 0\\
f_{xx}(x, y) &= e^x \cos y & f_{xx}(0, 0) &= 1\\
f_{xy}(x, y) &= -e^x \sin y & f_{xy}(0, 0) &= 0\\
f_{yy}(x, y) &= -e^x \cos y & f_{yy}(0, 0) &= -1\\
f_{xxx}(x, y) &= e^x \cos y & f_{xxx}(0, 0) &= 1\\
f_{xxy}(x, y) &= -e^x \sin y & f_{xxy}(0, 0) &= 0\\
f_{xyy}(x, y) &= -e^x \cos y & f_{xyy}(0, 0) &= -1\\
f_{yyy}(x, y) &= e^x \sin y & f_{yyy}(0, 0) &= 0
\end{aligned}$$

Substituting these values in Eq. (1),

$$f(x, y) = 1 + [x(1) + y(0)] + \frac{1}{2!}\left[x^2(1) + 2xy(0) + y^2(-1)\right]$$

$$+\frac{1}{3!}\left[x^3(1) + 3x^2y(0) + 3xy^2(-1) + y^3(0)\right]+\cdots$$

$$e^x \cos y = 1 + x + \frac{1}{2}(x^2 - y^2) + \frac{1}{6}(x^3 - 3xy^2) + \cdots$$

EXERCISE 4.8

1. Expand $e^x \sin y$ in powers of x and y up to third-degree terms.

$$\left[\textbf{Ans.: } y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \cdots\right]$$

2. Expand $xy^2 + \sin xy$ at the point $\left(1, \frac{\pi}{2}\right)$ up to terms of second degree.

$$\left[\begin{array}{l}\textbf{Ans.:}\\ 1+\frac{\pi^2}{4}+\frac{\pi^2}{4}(x-1)+\pi\left(y-\frac{\pi}{2}\right)-\frac{\pi^2}{8}\\ (x-1)^2+\frac{\pi}{2}(x-1)\left(y-\frac{\pi}{2}\right)+\frac{1}{2}\left(y-\frac{\pi}{2}\right)+\cdots\end{array}\right]$$

3. Expand $e^x \cos y$ about $\left(1, \frac{\pi}{4}\right)$ up to the terms of second degree.

$$\left[\textbf{Ans.: } \left\{\frac{e}{\sqrt{2}}1+(x-1)-\left(y-\frac{\pi}{4}\right)+\frac{1}{2}(x-1)^2 -(x-1)\left(y-\frac{\pi}{4}\right)-\frac{1}{2}\left(y-\frac{\pi}{4}\right)^2+\cdots\right\}\right]$$

4. Obtain terms up to the third degree in the Taylor's series expansion of $e^x \sin y$ around the point $\left(1, \frac{\pi}{2}\right)$.

$$\left[\begin{array}{l}\textbf{Ans.:}\\ e+(x-1)e+\frac{1}{2!}\left[(x-1)^2e-\left(y-\frac{\pi}{2}\right)^2e\right]\\ +\frac{1}{3!}\left[(x-1)^3e-3e(x-1)\left(y-\frac{\pi}{2}\right)^2\right]+\cdots\end{array}\right]$$

5. Expand x^2+xy+y^2 in powers of $(x-1)$ and $(y-2)$ up to second-degree terms.

$$\left[\begin{array}{l}\textbf{Ans.: } (x-1)^2+(y-2)^2\\ +(x-1)(y-2)+4(x-1)+5(y-2)+7\end{array}\right]$$

6. Expand $\sin(x+h)(y+k)$ by Taylor's theorem.

$$\left[\begin{array}{l}\textbf{Ans.: } \sin xy+(hy+kx)\cos xy\\ \qquad +hk\cos xy-\frac{1}{2}(hy+kx)^2\sin xy+\cdots\end{array}\right]$$

7. Expand x^y about (1, 1) up to second-degree terms.

$$\left[\begin{array}{l}\textbf{Ans.: } 1+(x-1)+(x-1)(y-1)+\frac{1}{2!}[(x-1)^2\\ \qquad +2(x-1)(y-1)]+\cdots\end{array}\right]$$

8. Using Taylor's series, verify that

(i) $\cos(x+y) = 1-\frac{1}{2!}(x+y)^2 + \frac{1}{4!}(x+y)^4 - \cdots$

(ii) $\tan^{-1}(x+y) = (x+y) + \frac{1}{3}(x+y)^3 + \cdots$

4.9 ERRORS AND APPROXIMATIONS

Let $u=f(x,y)$ be a continuous function of x and y. If δx and δy are small increments in x and y respectively and δu is corresponding increment in u then

$$u+\delta u = f(x+\delta x, y+\delta y)$$

$$\delta u = f(x+\delta x, y+\delta y) - f(x,y)$$

$$= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$ [Expanding by Taylor's theorem and ignoring higher powers and products of δx and δy.]

or
$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$$

For a function $u = f(x, y, z)$ of three variables,

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

If δx is the error in x then

(i) δx is known as *absolute error* in x.

(ii) $\frac{\delta x}{x}$ is known as *relative error* in x.

(iii) $\frac{\delta x}{x} \times 100$ is known as *percentage error* in x.

EXAMPLE 4.45 *The focal length of a mirror is found from the formula $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the percentage error in f if u and v are both in error by 2% each.*

Solution:

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

$$-\frac{2}{f^2}\delta f = -\frac{1}{v^2}\delta v + \frac{1}{u^2}\delta u$$

$$-\frac{2}{f}\frac{\delta f}{f} \times 100 = -\frac{1}{v}\frac{\delta v}{v} \times 100 + \frac{1}{u}\frac{\delta u}{u} \times 100$$

Putting $\frac{\delta u}{u} \times 100 = 2$, $\frac{\delta v}{v} \times 100 = 2$

$$-\frac{2}{f}\frac{\delta f}{f} \times 100 = -\frac{1}{v}(2) + \frac{1}{u}(2) = -2\left(\frac{1}{v} - \frac{1}{u}\right) = -2\left(\frac{2}{f}\right)$$

$$\frac{\delta f}{f} \times 100 = 2$$

Hence, percentage error in $f = 2\%$.

EXAMPLE 4.46 *If the measurement of radius of the base and height of a right circular cone are incorrect by –1% and 2% respectively, prove that there will be no error in the volume.*

Solution: Let r and h be the radius of base and height of the right circular cone and V be its volume.

$$V = \frac{1}{3}\pi r^2 h$$

Taking logarithm on both sides,

$$\log V = \log\frac{\pi}{3} + 2\log r + \log h$$

$$\frac{1}{V}\delta V = 0 + \frac{2}{r}\delta r + \frac{1}{h}\delta h$$

$$\frac{\delta V}{V}\times 100 = 2\frac{\delta r}{r}\times 100 + \frac{\delta h}{h}\times 100$$

Putting $\frac{\delta r}{r}\times 100 = -1$, $\frac{\delta h}{h}\times 100 = 2$,

$$\frac{\delta V}{V}\times 100 = 2(-1) + 2 = 0$$

Hence, there will be no error in the volume.

EXAMPLE 4.47 *A balloon is in the form of a right circular cylinder of 1.5 m radius and 4 m height and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the height by 0.05 m, find the percentage change in the volume of the balloon.*

Solution: Let r and h be the radius and height of the cylindrical balloon respectively and V be its volume (Fig. 4.22).

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\delta V = \pi(2rh\,\delta r + r^2\delta h) + \frac{4}{3}\pi(3r^2\delta r)$$

Putting $r = 1.5$ m, $h = 4$ m, $\delta r = 0.01$ m, $\delta h = 0.05$ m,

$$\delta V = \pi[2\times 1.5\times 4\times 0.01 + (1.5)^2(0.05)] + 4\pi(1.5)^2(0.01)$$

$$= \pi(0.12 + 0.1225 + 0.09) = 3.225\pi$$

and

$$V = \pi\left((1.5)^2 4 + \frac{4}{3}(1.5)^3\right) = 13.5\pi$$

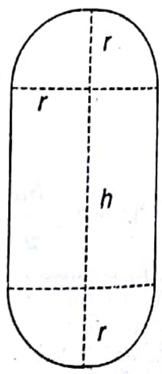


Fig. 4.22

Percentage change in the volume $= \frac{\delta V}{V}\times 100 = \frac{3.225}{13.5}\times 100 = 2.389\%$

EXAMPLE 4.48 *At a distance of 120 feet from the foot of a tower, the elevation of its top is 60°. If the possible error in measuring the distance and elevation are 1 inch and 1 minute respectively, find the approximate error in the calculated height of the tower.*

Solution: Let h, x and θ be the height, horizontal distance and angle of elevation of the tower respectively (Fig. 4.23).

$$\tan\theta = \frac{h}{x}$$

$$h = x\tan\theta$$

Taking logarithm on both sides,

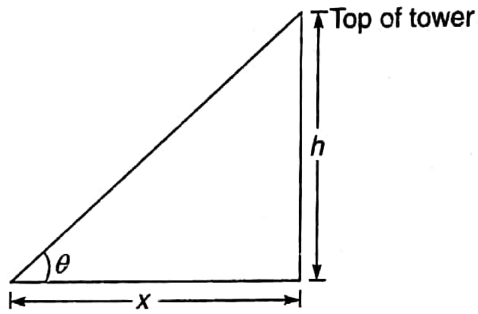


Fig. 4.23

$$\log h = \log x + \log\tan\theta$$

$$\frac{1}{h}\delta h = \frac{1}{x}\delta x + \frac{1}{\tan\theta}\sec^2\theta\,\delta\theta$$

Putting $x = 120$ ft., $\theta = 60°$,

$$\delta x = 1 \text{ inch} = \frac{1}{12} ft., \ \delta\theta = 1\text{minute} = \frac{1}{60}\cdot\frac{\pi}{180} \text{ radians}$$

$$h = x\tan\theta = 120\tan 60° = 120\sqrt{3},$$

and $$\frac{1}{h}\delta h = \frac{1}{120}\cdot\frac{1}{12} + \frac{1}{\sqrt{3}}\cdot 4\cdot\frac{1}{60}\cdot\frac{\pi}{180}$$

$$\delta h = 0.284 \text{ ft.}$$

Hence, error in the height of the tower = 0.284 ft.

EXAMPLE 4.49 *If Δ be the area of the triangle, prove that the error in Δ resulting from a small error in side c is given by*

$$\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right)\delta c$$

where $$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Solution: $$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\log\Delta = \frac{1}{2}[\log s + \log(s-a) + \log(s-b) + \log(s-c)]$$

$$\frac{1}{\Delta}\delta\Delta = \frac{1}{2}\left[\frac{1}{s}\delta s + \frac{1}{s-a}\delta\,(s-a) + \frac{1}{s-b}\delta\,(s-b) + \frac{1}{s-c}\delta\,(s-c)\right]$$

$$= \frac{1}{2}\left[\frac{\delta s}{s} + \frac{\delta s - \delta a}{s-a} + \frac{\delta s - \delta b}{s-b} + \frac{\delta s - \delta c}{s-c}\right]$$

$$s=\frac{1}{2}(a+b+c), \text{ where } a \text{ and } b \text{ are constant}$$

$$\therefore \quad \delta s=\frac{\delta c}{2} \qquad [\because \delta a=0, \delta b=0]$$

Hence,
$$\delta\Delta=\frac{\Delta}{2}\left[\frac{\delta c}{2s}+\frac{\delta c}{2(s-a)}+\frac{\delta c}{2(s-b)}+\frac{\frac{\delta c}{2}-\delta c}{s-c}\right]=\frac{\Delta}{4}\left[\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right]\delta c$$

EXAMPLE 4.50 *Evaluate* $(1.99)^2(3.01)^3(0.98)^{\frac{1}{10}}$ *using approximation.*

Solution: Let
$$u=x^2y^3z^{\frac{1}{10}}$$

$$\log u=2\log x+3\log y+\frac{1}{10}\log z$$

$$\frac{1}{u}\delta u=2\frac{1}{x}\delta x+3\frac{1}{y}\delta y+\frac{1}{10}\frac{1}{z}\delta z$$

Putting $x=2, \quad y=3, \quad z=1,$

$$\delta x=1.99-2=-0.01,$$
$$\delta y=3.01-3=0.01,$$
$$\delta z=0.98-1=-0.02$$
$$u=2^2\ 3^3\ 1^{\frac{1}{10}}=108$$

and
$$\frac{1}{108}\delta u=2\cdot\left(\frac{1}{2}\right)(-0.01)+3\left(\frac{1}{3}\right)(0.01)+\frac{1}{10}\left(\frac{1}{1}\right)(-0.02)$$
$$\delta u=-0.216$$

Approximate value $=u+\delta u=108-0.216=107.784.$

EXERCISE 4.9

1. In calculating the volume of right circular cone, errors of 2.75% and 1.25% are made in height and radius of the base. Find the percentage error in volume.

[Ans.: 5.25%**]**

2. The height of a cone is $H=30$ cm, the radius of base $R=10$ cm. How will the volume of the cone change if H is increasing by 3 mm while R is decreasing by 1 mm?

[Ans.: decreased by 10π cm^3**]**

3. How is the relative change in $V = \pi r^2 h$ related to relative change in r and h? How are percentage changes related?

[**Ans.:** Relative change $\frac{\delta V}{V} = \frac{2}{r}\delta r + \frac{1}{h}\delta h$ and percentage change in volume = 2% change in radius + 1% change in height.]

4. In calculating the total surface area of a cylinder, error of 1% each are made in measuring the height and the base radius. Find the percentage error in calculating the total surface area.

[**Ans.:** 2%]

5. Find the percentage error in calculating the area of a rectangle when an error of 2% is made in measuring each of its sides.

[**Ans.:** 4%]

6. If R_1 and R_2 are two resistances in parallel, their resistance R is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ If there is an error of 2% in both R_1 and R_2, find percentage error in R.

[**Ans.:** 2%]

7. One side of a rectangle is $a = 10$ cm and the other side $b = 24$ cm. How will the diagonal l of the rectangle change if a is increased by 4 mm and b is decreased by 1 mm?

$\left[\textbf{Ans.:} \frac{4}{65}\text{cm}\right]$

8. The resistance R of circuit was found by using the formula $I = \frac{E}{R}$. If there is an error of 0.1 ampere in reading I and 0.5 volts in reading E, find the corresponding percentage error in R when $I = 15$ amperes and $E = 100$ volts.

[**Ans.:** – 0.167%]

9. The voltage V across a resistor is measured with error h, and the resistance R is measured with an error R. Show that the error in calculating the power $W = \frac{V^2}{R}$ is $\frac{V}{R^2}(2Rh - VR)$. If V can be measured to an accuracy of 0.5% and to an accuracy of 1%, what is the approximate possible percentage error in W?

[**Ans.:** 0%]

10. The radius and height of a cone are 4 cm and 6 cm respectively. What is the error in its volume if the scale used in taking the measurement is short by 0.01 cm per cm?

[**Ans.:** 0.96π cm^3]

11. Show that the error in calculating the time period of a pendulum at any place is zero if an error of μ% is made in measuring its length and gravity at that place.

12. At a distance of 20 metres from the foot of a tower, the elevation of its top is 60°. If the possible error in measuring distance and elevation are 1 cm and 1 minute, find the approximate error in calculating height.

[**Ans.:** 0.040]

13. The diameter and the altitude of a right circular cylinder are measured as 24 cm and 30 cm respectively. There is an error of 0.1 cm in each measurement. Find the possible error in the volume of the cylinder.

[**Ans.:** 50.4π cm]

14. If the measurements of base radius and height of a right circular cone are changed by −1% and 2%, show that there will be no error in the volume.

15. If $f = x^2 y^3 z^{\frac{1}{10}}$, find the approximate value of f when $x = 1.99$, $y = 3.01$ and $z = 0.98$.

[Ans.: 107.784**]**

16. If $f = x^3 y^2 z^4$, find the approximate value of f when $x = 1.99$, $y = 3.01$, $z = 0.99$.

[Ans.: 68.5202**]**

17. If $f = (160 - x^3 - y^3)^{\frac{1}{3}}$, find the approximate value of $f(2.1, 2.9) - f(2, 3)$

[Ans.: 0.016**]**

18. If $f = e^{xyz}$, find the approximate value of f when $x = 0.01$, $y = 1.01$, $z = 2.01$.

[Ans.: 1.02**]**

19. Find $[(2.92)^3 + (5.87)^3]^{\frac{1}{5}}$ approximately by using the theory of approximation.

[Ans.: 2.96**]**

20. Find $[(11.99)^2 + (5.01)^2]^{\frac{1}{2}}$ approximately by using the theory of approximation.

[Ans.: 12.99**]**

21. Find $(1.04)^{3.01}$ by using theory of approximation.

[Ans.: 1.1253**]**

22. If $f(x, y) = (50 - x^2 - y^2)^{\frac{1}{2}}$ find the approximate value of $[f(3, 4) - f(3.1, 3.9)]$

[Ans.: -0.018**]**

23. Find $\log\left[\sqrt[3]{1.04} + \sqrt[4]{0.97} - 1\right]$ approximately by using the theory of approximation.

[Ans.: 0.0058**]**

24. The diameter and altitude of a can in the shape of a right circular cylinder are measured as 40 cm and 60 cm respectively. The possible error in each measurement is $\pm$ 5%. Find approximately the maximum possible percentage error in the computed values of the volume and the lateral surface.

[Ans.: ± 15 cm^3, ± 10 cm^2**]**

4.10 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Let $u = f(x, y)$ be a continuous function of x and y. u will be maximum at $x = a$, $y = b$, if $f(a, b) > f(a + h, b + k)$ and will be minimum at $x = a$, $y = b$, if $f(a, b) < f(a + h, b + k)$ for small positive or negative values of h and k.

The point at which function $f(x, y)$ is either maximum or minimum is known as *stationary point.* The value of the function at stationary point is known as extreme (maximum or minimum) value of the function $f(x, y)$.

Working Rule to Determine the Maxima and Minima (Extreme Values) of a Function $f(x, y)$

(i) Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ simultaneously for x and y.

(ii) Obtain the values of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x\, \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$.

(iii) (a) If $rt - s^2 > 0$ and $r < 0$ (or $t < 0$) at (a, b) then $f(x, y)$ is maximum at (a, b) and the maximum value of the function is $f(a, b)$.

(b) If $rt - s^2 > 0$ and $r > 0$ (or $t > 0$) at (a, b) then $f(x, y)$ is minimum at (a, b) and the minimum value of the function is $f(a, b)$.

(c) If $rt - s^2 < 0$ at (a, b) then $f(x, y)$ is neither maximum nor minimum at (a, b). Such a point is known as *saddle point.*

(d) If $rt - s^2 = 0$ at (a, b) then no conclusion can be made about the extreme values of $f(x, y)$ and further investigation is required.

EXAMPLE 4.51 *Discuss the maxima and minima of the function $x^2 + y^2 + 6x + 12$.*

Solution: Let

$$f(x, y) = x^2 + y^2 + 6x + 12$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$2x + 6 = 0$$

$$2(x + 3) = 0$$

$$x + 3 = 0$$

$$x = -3$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$2y = 0$$

$$y = 0$$

Stationary point is $(-3, 0)$.

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III: At $(-3, 0)$

$$rt - s^2 = 2(2) - 0 = 4 > 0 \quad \text{and } r > 0$$

Hence, $f(x, y)$ is minimum at $(-3, 0)$.

$$f_{\min} = (-3)^2 + 0 + 6(-3) + 12 = 3$$

EXAMPLE 4.52 *Find the extreme values of the function $x^3 + y^3 - 63(x + y) + 12xy$.*

Solution Let

$$f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$3x^2 - 63 + 12y = 0$$

$$3x^2 + 12y = 63 \qquad \ldots(1)$$

$$x^2 + 4y = 21$$

$$\frac{\partial f}{\partial y} = 0$$

$$3y^2 - 63 + 12x = 0$$

and
$$12x + 3y^2 = 63 \quad \cdots(2)$$

$$4x + y^2 = 21$$

Equating Eqs (1) and (2),

$$x^2 + 4y = 4x + y^2$$

$$x^2 - y^2 - 4(x - y) = 0$$

$$(x + y)(x - y) - 4(x - y) = 0$$

$$(x - y)(x + y - 4) = 0$$

$$x - y = 0, x + y - 4 = 0$$

$$y = x, y = 4 - x$$

Putting $y = x$ in Eq. (1),

$$x^2 + 4x - 21 = 0,$$

$$(x + 7)(x - 3) = 0$$

$$x = -7, 3$$

$$\therefore y = -7, 3$$

Stationary points are $(-7, -7)$, $(3, 3)$.

Putting $y = 4 - x$ in Eq. (1),

$$x^2 + 4(4 - x) = 21$$

$$x^2 - 4x - 5 = 0,$$

$$(x + 1)(x - 5) = 0$$

$$x = -1, 5$$

$$\therefore y = 5, -1$$

Stationary points are $(-1, 5)$, $(5, -1)$.

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 6x$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = 12$$

$$t = \frac{\partial^2 f}{\partial y^2} = 6y$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
$(-7, -7)$	-42	12	-42	$1620 > 0$ and $r < 0$	maximum
$(3, 3)$	18	12	18	$180 > 0$ and $r > 0$	minimum
$(-1, 5)$	-6	12	30	$-324 < 0$	neither maximum nor minimum
$(5, -1)$	30	12	-6	$-324 < 0$	neither maximum nor minimum

Hence, $f(x, y)$ is maximum at $(-7, -7)$.

$$f_{\max} = (-7)^3 + (-7)^3 - 63(-7-7) + 12(-7)(-7) = 784$$

and $f(x, y)$ is minimum at $(3, 3)$.

$$f_{\min} = 3^3 + 3^3 - 63(3+3) + 12(3)(3) = -216$$

EXAMPLE 4.53 *Examine the function $x^3 y^2(12 - 3x - 4y)$ for extreme values.*

Solution: Let

$$f(x, y) = 12x^3y^2 - 3x^4y^2 - 4x^3y^3$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$36x^2y^2 - 12x^3y^2 - 12x^2y^3 = 0$$

$$12x^2y^2(3 - x - y) = 0$$

$$x = 0,\ y = 0,\ x + y = 3 \qquad \ldots(1)$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$24x^3y - 6x^4y - 12x^3y^2 = 0$$

$$6x^3y(4 - x - 2y) = 0$$

$$x = 0,\ y = 0,\ x + 2y = 4 \qquad \ldots(2)$$

Considering six pairs of equations of Eqs (1) and (2),

$$\begin{array}{rr}
x = 0 & y = 0 \\
x = 0 & x + 2y = 4 \\
y = 0 & x + 2y = 4 \\
x + y = 3 & x = 0 \\
x + y = 3 & y = 0 \\
x + y = 3 & x + 2y = 4
\end{array}$$

Solving these equations, the following pairs of stationary points are obtained:
(0, 0), (0, 2), (4, 0), (0, 3), (3, 0), (2, 1)

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 72xy^2 - 36x^2y^2 - 24xy^3 = 12xy^2(6-3x-2y)$$

$$s = \frac{\partial^2 f}{\partial x\,\partial y} = 72x^2y - 24x^3y - 36x^2y^2 = 12x^2y(6-2x-3y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 24x^3 - 6x^4 - 24x^3y = 6x^3(4-x-4y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	*Conclusion*
(0, 0)	0	0	0	0	no conclusion
(0, 2)	0	0	0	0	no conclusion
(4, 0)	0	0	0	0	no conclusion
(0, 3)	0	0	0	0	no conclusion
(3, 0)	0	0	162	0	no conclusion
(2, 1)	−48	−48	−96	$2304 > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at (2, 1).

$$f_{\max} = (2^3)(1^2)(12-6-4) = 16$$

EXAMPLE 4.54 *Find the extreme values of* $\sin x \sin y \sin (x + y)$.

Solution: Let

$$f(x, y) = \sin x \sin y \sin (x + y)$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$\sin y\,[\cos x \sin(x+y) + \sin x \cos(x+y)] = 0$$

$$\sin y \sin(2x+y) = 0$$

$$\frac{1}{2}[\cos 2x - \cos(2x+2y)] = 0$$

$$\cos 2x - \cos(2x+2y) = 0 \qquad \text{...(1)}$$

and

$$\frac{\partial f}{\partial y} = 0$$

$$\sin x\,[\cos y \sin(x+y) + \sin y \cos(x+y)] = 0$$

$$\sin x \sin(x+2y) = 0$$

$$\frac{1}{2}[\cos 2y - \cos(2x+2y)] = 0$$

$$\cos 2y - \cos(2x+2y) = 0 \qquad \text{...(2)}$$

Equating Eqs (1) and (2),

$$\cos 2x = \cos 2y$$

$$x = y$$

Substituting $x = y$ in Eq. (1),

$$\cos 2x - \cos(2x+2x) = 0$$

$$\cos 2x = \cos 4x = 2\cos^2 2x - 1$$

$$2\cos^2 2x - \cos 2x - 1 = 0$$

$$\cos 2x = \frac{1 \pm \sqrt{1+8}}{4} = 1, -\frac{1}{2}$$

$$\cos 2x = 1 = \cos 0, \qquad \cos 2x = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

$$x = 0, \qquad x = \frac{\pi}{3}$$

$$y = 0, \qquad y = \frac{\pi}{3}$$

Stationary points are $(0,0), \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin 2x + \sin 2(x+y) = 2\sin y \cos(2x+y)$$

$$s = \frac{\partial^2 f}{\partial x\, \partial y} = \sin 2(x+y)$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin 2y + \sin 2(x+y) = 2\sin x \cos(x+2y)$$

Step III:

(x, y)	r	s	t	$rt - s^2$	Conclusion
(0, 0)	0	0	0	0	no conclusion
$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$	$\frac{9}{4} > 0$ and $r < 0$	maximum

Hence, $f(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

$$f_{\max} = \sin\frac{\pi}{3}\sin\frac{\pi}{3}\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

EXAMPLE 4.55 *Find the points on the surface $z^2 = xy + 1$ nearest to the origin. Also, find that distance.*

Solution: Let $P(x, y, z)$ be any point on the surface $z^2 = xy + 1$.

Its distance from the origin is given by

$$d = \sqrt{(x^2 + y^2 + z^2)}$$

$$d^2 = x^2 + y^2 + z^2$$

Since P lies on the surface $z^2 = xy + 1$,

$$d^2 = x^2 + y^2 + xy + 1$$

Let

$$f(x, y) = x^2 + y^2 + xy + 1$$

Step I: For extreme values,

$$\frac{\partial f}{\partial x} = 0$$

$$2x + y = 0 \quad \ldots(1)$$

$$\frac{\partial f}{\partial y} = 0$$

$$2y + x = 0 \quad \ldots(2)$$

Solving Eqs (1) and (2),

$$x = 0, y = 0$$

Stationary point is (0, 0).

Step II:

$$r = \frac{\partial^2 f}{\partial x^2} = 2$$

$$s = \frac{\partial^2 f}{\partial x \, \partial y} = 1$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2$$

Step III: At (0, 0), $r = 2, t = 2, s = 1$

$$rt - s^2 = (2)(2) - (1)^2 = 3 > 0$$

Also,

$$r = 2 > 0$$

$f(x, y)$, i.e., d^2 is minimum at (0, 0) and hence, d is minimum at (0, 0).

At (0, 0),

$$z^2 = xy + 1 = 1$$

$$z = \pm 1$$

Hence, d is minimum at (0, 0, 1) and (0, 0, –1).
The points (0, 0, 1) and (0, 0, –1) on the surface $z^2 = xy + 1$ are nearest to the origin.

$$\text{Minimum distance} = \sqrt{0 + 0 + 1} = 1$$

EXERCISE 4.10

1. Examine maxima and minima of the following functions and find their extreme values:
   (i) $2+2x+2y-x^2-y^2$
   (ii) $x^2y^2-5x^2-8xy-5y^2$
   (iii) $x^2+y^2+xy+x-4y+5$
   (iv) $x^2+y^2+6x=12$
   (v) $x^3y^2(1-x-y)$
   (vi) $xy(3a-x-y)$
   (vii) $x^3+3xy^2-3x^2-3y^2+4$
   (viii) $x^4+y^4-2(x-y)^2$
   (ix) $x^4+x^2y+y^2$
   (x) $x^4+y^4-4a^2xy$
   (xi) $y^4-x^4+2(x^2-y^2)$
   (xii) $x^3+3x^2+y^2+4xy$
   (xiii) $x^2y-3x^2-2y^2-4y+3$
   (xiv) $x^4-y^4-x^2-y^2+1.$

[**Ans.:** (i) Maximum at (1, 1);4
(ii) Maximum at (0, 0);0
(iii) Minimum at (–2, 3);–2
(iv) Minimum at (–3, 0);3
(v) Maximum at $\left(\frac{1}{2},\frac{1}{3}\right);\frac{1}{432}$
(vi) Maximum at $(a,a);a^3$
(vii) Maximum at (0, 0);4
(viii) Minimum at $\left(\sqrt{2},-\sqrt{2}\right)$ and $\left(-\sqrt{2},\sqrt{2}\right);-8$
(ix) Minimum at (0, 0);0
(x) Minimum at (a, a) and $(-a, a);a^4$
(xi) No extreme values
(xii) No extreme values
(xiii) Maximum at (0, –1);5
(xiv) Maximum at(0, 0);1, minimum at $\left(\pm\frac{1}{\sqrt{2}},\pm\sqrt{\frac{1}{\sqrt{2}}}\right);\frac{1}{2}$]

2. A rectangular box, open at the top, is to have a volume of 32 cc. Find the dimensions of the box requiring least materials for its construction.
   [**Ans.:** 4, 4, 2]

3. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.
   [**Ans.:** 40, 40, 40]

4. The sum of three positive numbers is 'a'. Determine the maximum value of their product.
   $\left[\textbf{Ans.: } \frac{a^3}{27} \text{ at } \left(\frac{a}{3},\frac{a}{3},\frac{a}{3}\right)\right]$

5. Find the volume of the largest rectangular parallelepiped that can be inscribed in an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$
   $\left[\textbf{Ans.: } \frac{8abc}{3\sqrt{3}}\right]$

6. Prove that area of a triangle with constant perimeter is maximum when the triangle is equilateral.

7. Find the shortest distance from the origin to the surface $xyz^2=2$.
   [**Ans.:** 2]

8. Find the shortest distance from the origin to the plane $x-2y-2z=3$.
   [**Ans.:** 1]

9. Find the shortest distance between the lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$.
   [**Ans.:** $2\sqrt{29}$]

10. Find the maximum value of cos A cos B cos C, where A, B, C are angles of a triangle.
   $\left[\textbf{Ans.: } \text{maximum at } \left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right);\frac{1}{8}\right]$

4.11 LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Let $f(x, y, z)$ be a function of three variables x, y, z, and the variables be connected by the relation

$$\phi(x, y, z) = 0 \quad \ldots(4.32)$$

Let

$$f(x, y, z) + \lambda\phi(x, y, z) = 0 \quad \ldots(4.33)$$

be an auxiliary equation.

Differentiating Eq. (4.33) partially w.r.t. x, y, z,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad \ldots(4.34)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad \ldots(4.35)$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad \ldots(4.36)$$

Eliminating λ from Eqs (4.34), (4.35) and (4.36), the values of x, y, and z are obtained for which $f(x, y, z)$ has stationary value. This method of obtaining stationary values of $f(x, y, z)$ is called Lagrange's method of undetermined multipliers, and equations (4.34), (4.35) and (4.36) are called *Lagrange's equations*. The term λ is called *undetermined multiplier.*

EXAMPLE 4.56 *Find the maximum value of $f = x^2y^3z^4$, subject to the condition $x + y + z = 5$.*

Solution: Let

$$f(x, y, z) = x^2y^3z^4 \quad \ldots(1)$$

$$x + y + z = 5 \quad \ldots(2)$$

Let

$$\phi(x, y, z) = x + y + z - 5 = 0$$

Let the auxiliary equation be

$$x^2y^3z^4 + \lambda(x + y + z - 5) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t x,

$$2xy^3z^4 + \lambda = 0 \quad \ldots(4)$$

$$\lambda = -2xy^3z^4$$

Differentiating Eq. (3) partially w.r.t y,

$$3x^2y^2z^4 + \lambda = 0 \quad \ldots(5)$$

$$\lambda = -3x^2y^2z^4$$

Differentiating Eq. (3) partially w.r.t z,

$$4x^2y^3z^3 + \lambda = 0 \quad \ldots(6)$$

$$\lambda = -4x^2y^3z^3$$

From Eqs (4), (5), and (6),

$$2xy^3z^4 = 3x^2y^2z^4 = 4x^2y^3z^3$$
$$2yz = 3xz = 4xy$$
$$\therefore\ y = \frac{3}{2}x \quad \text{and} \quad z = 2x$$

Substituting y and z in Eq. (2),

$$x + \frac{3}{2}x + 2x = 5$$
$$9x = 10$$
$$x = \frac{10}{9}$$
$$\therefore\ y = \frac{3}{2}x = \frac{3}{2}\left(\frac{10}{9}\right) = \frac{5}{3}$$

and

$$z = 2x = 2\left(\frac{10}{9}\right) = \frac{20}{9}$$

Maximum value of $x^2y^3z^4 = \left(\frac{10}{9}\right)^2\left(\frac{5}{3}\right)^3\left(\frac{20}{9}\right)^4 = \frac{(2^{10})(5^9)}{3^{15}}.$

EXAMPLE 4.57 *Find the minimum and maximum distance from the point* (1, 2, 2) *to the sphere* $x^2 + y^2 + z^2 = 36$.

Solution: Let (x, y, z) be any point on the sphere. Its distance D from the point (1, 2, 2) is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z-2)^2}$$

Let

$$D^2 = f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2 \quad \text{...(1)}$$
$$x^2 + y^2 + z^2 = 36 \quad \text{...(2)}$$

Let

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 36$$

Let the auxiliary equation be

$$\left[(x-1)^2 + (y-2)^2 + (z-2)^2\right] + \lambda(x^2 + y^2 + z^2 - 36) = 0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t x,

$$2(x-1) + \lambda(2x) = 0$$
$$\lambda = -\frac{x-1}{x} = -1 + \frac{1}{x} \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t y,

$$2(y-2) + \lambda(2y) = 0$$
$$\lambda = -\frac{y-2}{y} = -1 + \frac{2}{y} \quad \text{...(5)}$$

Differentiating Eq. (3) partially, w.r.t. z,

$$2(z-2)+\lambda(2z)=0$$

$$\lambda=-\frac{z-2}{z}=-1+\frac{2}{z} \quad \ldots(6)$$

From Eqs (4), (5) and (6),

$$-1+\frac{1}{x}=-1+\frac{2}{y}=-1+\frac{2}{z}$$

$$\frac{1}{x}=\frac{2}{y}=\frac{2}{z}$$

$$y=2x \quad \text{and} \quad z=2x$$

Substituting y and z in Eq. (2),

$$x^2+4x^2+4x^2=36$$

$$9x^2=36$$

$$x^2=4$$

$$x=\pm 2$$

$$\therefore y=\pm 4$$

and

$$z=\pm 4$$

Minimum distance $=\sqrt{(2-1)^2+(4-2)^2+(4-2)^2}=\sqrt{1+4+4}=3$

Maximum distance $=\sqrt{(-2-1)^2+(-4-2)^2+(-4-2)^2}=\sqrt{9+36+36}=9$

EXAMPLE 4.58 *A wire of length b is cut into two parts which are bent in the form of a square and circle respectively. Find the least value of the sum of the areas so found.*

Solution: Let the piece of length x be bent in the form of a square so that each side is $\frac{x}{4}$.

The area of the square, $A_1=\frac{x}{4}\cdot\frac{x}{4}=\frac{x^2}{16}$.

Suppose a piece of length y is bent in the form of a circle of radius r; so perimeter of the circle is y.

$$2\pi r=y$$

$$r=\frac{y}{2\pi}$$

The area of the circle, $A_2=\pi\left(\frac{y}{2\pi}\right)^2=\frac{y^2}{4\pi}$.

Let sum of the areas be given as

$$f(x,y)=\frac{x^2}{16}+\frac{y^2}{4\pi} \quad \text{...(1)}$$

Also,

$$x+y=b \quad \text{...(2)}$$

Let

$$\phi(x,y)=x+y-b$$

Let the auxiliary equation be

$$\left(\frac{x^2}{16}+\frac{y^2}{4\pi}\right)+\lambda(x+y-b)=0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$\frac{2x}{16}+\lambda=0$$

$$\lambda=-\frac{x}{8} \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$\frac{2y}{4\pi}+\lambda=0$$

$$\lambda=-\frac{y}{2\pi} \quad \text{...(5)}$$

From Eqs (4) and (5),

$$\frac{x}{8}=\frac{y}{2\pi}$$

$$y=\frac{\pi}{4}x$$

Substituting y in Eq. (2),

$$x+\frac{\pi}{4}x=b$$

$$x=\frac{4b}{4+\pi}$$

$$\therefore \quad y=\frac{\pi b}{4+\pi}$$

Hence, the least value of the sum of the areas is

$$\frac{x^2}{16}+\frac{y^2}{4\pi}=\frac{1}{16}\left(\frac{4b}{4+\pi}\right)^2+\frac{1}{4\pi}\left(\frac{\pi b}{4+\pi}\right)^2=\frac{b^2}{(4+\pi)^2}\left(1+\frac{\pi^2}{4\pi}\right)=\frac{b^2\pi(4+\pi)}{4\pi(4+\pi)^2}=\frac{b^2}{4(\pi+4)}$$

EXAMPLE 4.59 *Divide 24 into three parts such that the continued product of the first, square of the second and cube of the third may be maximum.*

Solution: Let x, y and z be three parts of 24.

$$f(x,y,z)=xy^2z^3 \quad \text{...(1)}$$

$$x+y+z=24 \quad \text{...(2)}$$

Let

$$\phi(x, y, z)=x+y+z-24=0$$

Let the auxiliary equation be

$$xy^2 z^3+\lambda (x+y+z-24)=0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. x,

$$y^2z^3+\lambda=0$$

$$\lambda=-y^2z^3 \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. y,

$$2xyz^3+\lambda=0$$

$$\lambda=-2xyz^3 \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. z,

$$3xy^2z^2+\lambda=0$$

$$\lambda=-3xy^2z^2 \quad \text{...(6)}$$

From Eqs (4), (5), (6),

$$y^2z^3=2xyz^3=3xy^2z^2$$

Dividing by xy^2z^3,

$$\frac{1}{x}=\frac{2}{y}=\frac{3}{z}$$

$$y=2x,\ z=3x$$

Substituting y, z in Eq. (2)

$$x+2x+3x=24$$

$$6x=24$$

$$x=4$$

$$\therefore \quad y=8,\ z=12$$

Hence, 4, 8 and 12 are three parts of 24.

EXAMPLE 4.60 *A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters the earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the surface of the probe $T(x, y, z)=8x^2+4yz-16z+600$. Find the hottest points on the probe's surface.*

Solution: Let

$$f(x,y,z)=8x^2+4yz-16z+600 \quad \text{...(1)}$$

$$4x^2+y^2+4z^2=16 \quad \text{...(2)}$$

Let $$\phi(x, y, z) = 4x^2 + y^2 + 4z^2 - 16 = 0$$

Let the auxiliary equation be

$$(8x^2 + 4yz - 16z + 600) + \lambda(4x^2 + y^2 + 4z^2 - 16) = 0 \quad \ldots(3)$$

Differentiating Eq. (3) partially w.r.t. x,

$$16x + \lambda(8x) = 0$$

$$\lambda = -2 \quad \ldots(4)$$

Differentiating Eq. (3) partially w.r.t. y,

$$4z + \lambda(2y) = 0$$

$$\lambda = -\frac{2z}{y} \quad \ldots(5)$$

Differentiating Eq. (3) partially w.r.t. z,

$$4y - 16 + \lambda(8z) = 0$$

$$\lambda = \frac{16 - 4y}{8z} = \frac{4 - y}{2z} \quad \ldots(6)$$

From Eqs (4) and (5),

$$-2 = -\frac{2z}{y}$$

$$y = z \quad \ldots(7)$$

From Eqs (4) and (6),

$$-2 = \frac{4 - y}{2y}$$

$$-4y = 4 - y$$

$$-3y = 4$$

$$y = -\frac{4}{3}$$

$$z = -\frac{4}{3}$$

Substituting in Eq. (2),

$$4x^2 + \frac{16}{9} + \frac{64}{9} = 16, \quad 4x^2 = \frac{64}{9}$$

$$x^2 = \frac{16}{9}, \qquad x = \pm\frac{4}{3}$$

The hottest points on the probe's surface are $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$.

EXAMPLE 4.61 *As the dimensions of a triangle ABC are varied, show that the maximum value of* $\cos A \cos B \cos C$ *is obtained when the triangle is equilateral.*

Solution: Let

$$f(A, B, C) = \cos A \cos B \cos C \quad \text{...(1)}$$

In a triangle ABC,

$$A + B + C = 180° \quad \text{...(2)}$$

Let

$$\phi(A, B, C) = A + B + C - 180°$$

Let the auxiliary equation be

$$\cos A \cos B \cos C + \lambda\,(A + B + C - 180°) = 0 \quad \text{...(3)}$$

Differentiating Eq. (3) partially w.r.t. A,

$$-\sin A \cos B \cos C + \lambda = 0$$

$$\lambda = \sin A \cos B \cos C \quad \text{...(4)}$$

Differentiating Eq. (3) partially w.r.t. B,

$$-\cos A \sin B \cos C + \lambda = 0$$

$$\lambda = \cos A \sin B \cos C \quad \text{...(5)}$$

Differentiating Eq. (3) partially w.r.t. C,

$$-\cos A \cos B \sin C + \lambda = 0$$

$$\lambda = \cos A \cos B \sin C \quad \text{...(6)}$$

From Eqs (4), (5) and (6),

$$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$$

Dividing by $\cos A \cos B \cos C$,

$$\tan A = \tan B = \tan C$$

$$A = B = C$$

Hence, the triangle ABC is equilateral.

EXERCISE 4.11

1. Find stationary values of the function $f(x, y, z) = x^2 + y^2 + z^2$, given that $z^2 = xy + 1$.

   [**Ans.:** (0, 0, −1), (0, 0, 1)]

2. Find the stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ subject to the fulfillment of the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$, given a, b, c are not zero.

   $$\left[\textbf{Ans.:}\ x=\frac{1}{a}(a+b+c),\quad y=\frac{1}{b}(a+b+c),\quad z=\frac{1}{c}(a+b+c)\right]$$

3. Find the largest product of the numbers x, y and z when $x + y + z^2 = 16$.

   $$\left[\textbf{Ans.:}\ \frac{4096}{25\sqrt{5}}\right]$$

4. Find the largest product of the numbers x, y and z when $x^2 + y^2 + z^2 = 9$.

   $$\left[\textbf{Ans.:}\ 3\sqrt{3}\right]$$

5. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point (1, 1, 1).

   $$\left[\textbf{Ans.:}\ \left(\frac{3}{2}, 2, \frac{5}{2}\right)\right]$$

6. Find the shortest distance from the point (1, 2, 2) to the sphere $x^2 + y^2 + z^2 = 36$.

   [**Ans.:** 3]

7. Find the maximum distance from the origin (0, 0) to the curve $3x^2 + 3y^2 + 4xy - 2 = 0$.

   $$\left[\textbf{Ans.:}\ \sqrt{2}\right]$$

8. Decompose a positive number a into three parts so that their product is maximum.

   $$\left[\textbf{Ans.:}\ \left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)\right]$$

9. Find the maximum value of $x^m y^n z^p$ when $x + y + z = a$.

   $$\left[\textbf{Ans.:}\ \frac{a^{m+n+p} m^m n^n p^p}{(m+n+p)^{m+n+p}}\right]$$

10. Find the dimensions of a rectangular box of maximum capacity whose surface area is given when

    (i) box is open at the top
    (ii) box is closed

    $$\left[\textbf{Ans.:}\ (i)\ \sqrt{\frac{s}{3}}, \sqrt{\frac{s}{3}}, \frac{1}{2}\sqrt{\frac{s}{3}}\quad (ii)\ \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}, \sqrt{\frac{s}{6}}\right]$$

11. Determine the perpendicular distance of the point (a, b, c) from the plane $lx + my + nz = 0$.

    $$\left[\textbf{Ans.:}\ \frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}}\right]$$

12. Find the length and breadth of a rectangle of maximum area that can be inscribed in the ellipse $4x^2 + y^2 = 36$.

    $$\left[\textbf{Ans.:}\ \frac{3\sqrt{2}}{2}, \sqrt{2}, Area = 12\right]$$

13. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid of revolution $4x^2 + 4y^2 + 9z^2 = 36$.

    $$\left[\textbf{Ans.:}\ 16\sqrt{3}\right]$$

14. Find the extreme volume of $x^2 + y^2 + z^2 + xy + xz + yz$ subject to the conditions $x + y + z = 1$ and $x + 2y + 3z = 3$.

$$\left[\textbf{Ans.:} \frac{1}{6}, \frac{1}{3}, \frac{5}{6}\right]$$

15. Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.

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