Chapter-2

Free, Damped and Forced Vibration

2.1. Introduction

In the previous chapter, we have discussed free simple harmonic motion where the amplitude of vibration is constant. Now, we will observe, how this amplitude of free vibration will change with time due to, **(i) viscous resistive force of the medium** or due to **(ii) applied periodic force**. Hence, we want to study damped and forced vibration. Before discussing damped and forced vibration, let us recapitulate free vibration.

2.2. Free Vibration

If a body is made to vibrate by disturbing it from its state of equilibrium and leaving to vibrate itself, the body vibrates with its own definite natural frequency. Such a vibration, which takes place only under the influence of its own elastic force is called natural or free vibration. The frequency of vibration depends upon mass, shape and elastic properties of the body.

When a body of mass m is displaced by a distance x from the position of rest, the equation of free vibration as discussed earlier can be written as,

$$m\frac{d^2x}{dt^2} = -ax, \quad a = \text{constant} = \text{restoring force per unit displacement.}$$

i.e.

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}, \text{ where } \omega = \sqrt{\frac{a}{m}} = \text{natural angular frequency.} \quad \cdots(2.1)$$

This is the differential equation of S.H.M. and its solution is,

$$x = A\sin(\omega t + \delta) \quad \cdots(2.2)$$

Here, A is the amplitude of vibration and it is constant with time [Fig. 1(a)]. The natural cyclic frequency of vibration of the body is, $f_0 = \frac{\omega}{2\pi}$. Consequently, the total energy for free vibration remains constant.

Ideally, the body vibrates with its natural frequency for an indefinite time with a constant amplitude. In practice, the free vibration of a body can never be realized as the amplitude decays with time due to the viscosity of the medium or other frictional forces (external or internal) and finally the body comes to rest at its mean position. Such vibrations of decaying amplitude are known as *damped (or resisted) vibrations.*

2.3. Damped Vibration and Its Analytical Treatment

When a body or a particle vibrates, it faces a number of frictional forces (external or internal). These forces decrease the amplitude of vibration with time and the motion of the body ultimately stops altogether. Such vibration of decaying amplitude is called damped vibration.

- ***Example :*** The motion of simple pendulum whose amplitude gradually decreases with time and the bob finally comes to rest is an example of damped simple harmonic motion.

The time displacement curve of a damped vibration is shown in the Fig. 1(b).

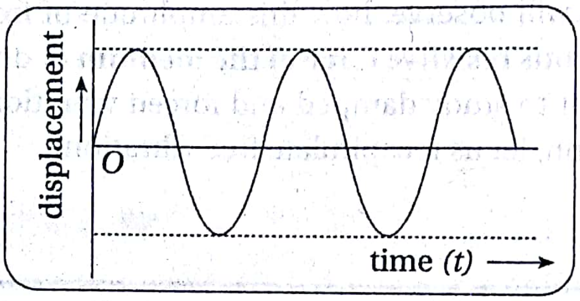

Fig. 1 ▷ (a) Undamped vibration

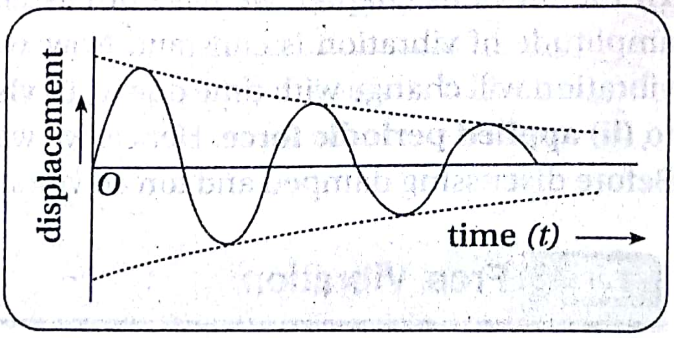

Fig. 1 ▷ (b) Damped vibration

Differential equation of damped vibration In case of *damped harmonic vibration*, there are *two types of forces* acting on the vibrating body. These are,

① the **restoring force**, which is proportional to the displacement and tends to bring the particle back to its initial position. If x be the displacement from the equilibrium position, the restoring force $= -ax$, where a is a constant, representing **the restoring force per unit displacement.**

② the **damping** or **retarding force** is proportional to velocity and is given by $-b\frac{dx}{dt}$, where the constant b is the resistive force, caused by friction per unit velocity *i.e.* **damping coefficient**[1] and it is defined as **damping force per unit velocity.**

Therefore, *the equation of motion of the damped vibration* is,

$$m\frac{d^2x}{dt^2} = -ax - b\frac{dx}{dt}, \text{ where } m \text{ is the mass of the body} \quad \cdots(2.3)$$

or, $$\frac{d^2x}{dt^2} + \frac{b}{m}\cdot\frac{dx}{dt} + \frac{a}{m}x = 0$$

or, $$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0 \quad \cdots(2.4)$$

This is the differential equation of motion of a damped harmonic oscillator.

Here, $k = \frac{b}{2m}$, is called **damping constant** and $\omega = \sqrt{\frac{a}{m}}$ is called **restoring constant.**

[1] Physical significance of damping coefficients : It measures the effect of resistive force of the medium on the motion of the system. Its unit is s^{-1}.

The equation (2.4) is a 2nd order differential equation. To solve this equation, we take a trial solution as, $x = A'e^{\alpha t}$.

$$\therefore \quad \frac{dx}{dt} = A'\alpha e^{\alpha t} = \alpha x \text{ and } \frac{d^2x}{dt^2} = A'\alpha^2 e^{\alpha t} = \alpha^2 x$$

Substituting in equation (2.4) we get,

$$\alpha^2 x + 2k\alpha x + \omega^2 x = 0 \qquad \cdots(2.5)$$

$$\text{or,} \quad \alpha^2 + 2k\alpha + \omega^2 = 0$$

$$\therefore \quad \alpha = \frac{-2k \pm \sqrt{4k^2 - 4\omega^2}}{2} = -k \pm \sqrt{k^2 - \omega^2}$$

So, the **general solution** of equation (2.4) is,

$$x = Ae^{(-k+\sqrt{k^2-\omega^2})t} + Be^{(-k-\sqrt{k^2-\omega^2})t} \qquad \cdots(2.6)$$

$$\text{or,} \quad x = e^{-kt}(Ae^{\sqrt{k^2-\omega^2}t} + Be^{-\sqrt{k^2-\omega^2}t}) \qquad \cdots(2.7)$$

where A and B are two arbitrary constants and can be computed from the initial conditions.

The following three cases of motion may occur in the solution :

Case 1 **Heavy damping or overdamped or dead beat motion**

When k (*damping constant*) $> \omega$ (*restoring constant*), then $\sqrt{k^2 - \omega^2} < k$. So, both the exponents of equation (2.6) become decreasing. Hence, $x \to 0$ as $e^{-\infty} = 0$ for $t \to \infty$. So the displacement x of the body gradually decreases exponentially with time and it comes to its equilibrium position at $t \to \infty$, *without performing any oscillation* [Fig. 2]. This type of motion is called *overdamped or aperiodic or dead beat.*

The motion of pendulum in a viscous liquid and discharge of capacitor through a resistor (e.g. dead beat galvanometer) are the *examples of this type of motion.*

Case 2 **Critical damping**

When $k \to \omega$, the equation 2.7 can be written as $x = e^{-kt}(Ae^{\beta t} + Be^{-\beta t})$.

where $\beta = \sqrt{k^2 - \omega^2}$ is a very small quantity.

Hence, neglecting smaller quantities we have

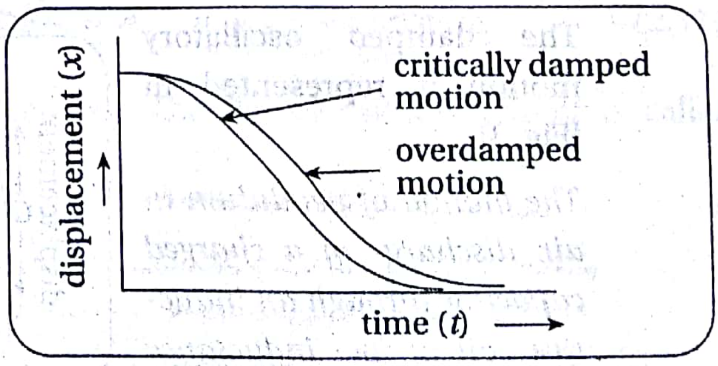

Fig. 2 ▷ Graphical representation of overdamped and critically damped motion

$$x = e^{-kt}\{A(1+\beta t) + B(1-\beta t)\}$$

$$= e^{-kt}\{(A+B) + (A-B)\beta t\} = e^{-kt}(P + Qt) \qquad \cdots(2.8)$$

where $P = A + B$ and $Q = (A - B)\beta$

This type of motion is known as critically damped motion [Fig. 2]. In this case, damped oscillatory motion suddenly changes into a dead beat motion but the particle tends to move to equilibrium much more rapidly than the over damped motion. It is also a *non-oscillatory motion.*

Case 3 Light damping or underdamped motion

When $k < \omega$, then $\sqrt{k^2 - \omega^2} = i\sqrt{\omega^2 - k^2} = i\omega'$

where $\omega' = \sqrt{\omega^2 - k^2}$ is a real quantity.

So, from equation (2.7) we get,

$$x = e^{-kt}(Ae^{i\omega' t} + Be^{-i\omega' t})$$

$$= e^{-kt}[(A+B)\cos\omega' t + i(A-B)\sin\omega' t] \quad \cdots(2.9)$$

Taking, $A + B = C\sin\phi$ and $i(A - B) = C\cos\phi$, we have

$$x = e^{-kt}[C\sin\omega' t\cos\phi + C\cos\omega' t\sin\phi]$$

or, $$\boxed{x = Ce^{-kt}\sin(\omega' t + \phi)} \quad \cdots(2.10)$$

Equation **(2.10) represents a damped oscillatory motion of frequency** (f),

$$f = \frac{\omega'}{2\pi} = \frac{\sqrt{\omega^2 - k^2}}{2\pi} = \frac{\omega}{2\pi}\sqrt{1 - \frac{k^2}{\omega^2}}$$

or, $$f = f_0\sqrt{1 - \frac{k^2}{\omega^2}}$$

and ***amplitude is proportional to*** e^{-kt}. Here, f_0 **is the natural frequency** (*i.e.* the frequency without damping). The term e^{-kt} **is called the damping factor.**

So, the **damping has two effects on the oscillatory motion**—

(i) ***the amplitude*** (Ce^{-kt}) ***decreases exponentially with time.***

(ii) *the decrease of frequency* (f) *of vibration of the body with increase of the period of oscillation.*

The damped oscillatory motion is represented in [Fig. 3].

The motion of pendulum in air, discharge of a charged capacitor through an inductive coil of low inductance (i.e. the electric oscillations of LCR circuit) and oscillation of coil in ballistic galvanometer are the examples of lightly damped oscillatory motion.

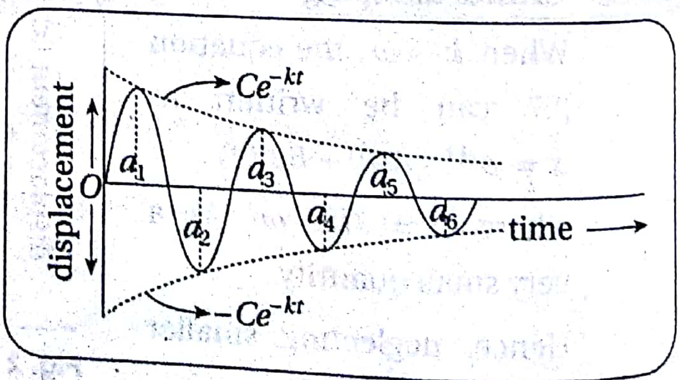

Fig. 3 ▷ Underdamped oscillatory motion (when the body is projected from the mean position with a certain velocity)

Attenuation is a general term that refers to any reduction in the strength of signal.

2.4. Attenuation Coefficients of a Vibrating System

In damped oscillatory motion, the displacement of the vibrating body at any time is given by,

$$\boxed{x = Ce^{-kt}\sin(\omega' t + \phi)}$$

The amplitude of vibration at time t, is Ce^{-kt}. So, the energy will decay as square of amplitude *i.e.* $(e^{-kt})^2 = e^{-2kt}$ with time. $E = \frac{1}{2} m A^2 \omega^2$ A = amplitude

The following three parameters give the attenuation of a vibrating system completely

① Logarithmic decrement,

② Relaxation time and

③ Quality factor

These are discussed below.

① **Logarithmic decrement** *It measures the rate at which the amplitude of damped oscillatory motion dies away (or falls off).*

Let $a_1, a_2, a_3, a_4, \cdots$ be the successive amplitudes [Fig. 3] at time $t = \frac{T}{4}, \frac{3T}{4}, \frac{5T}{4}, \frac{7T}{4}, \cdots$ respectively, where T is the time period of oscillation.

We know, the general equation of damped oscillatory motion of a body [from equation (2.10)] is,

$$x = Ce^{-kt}\sin(\omega' t + \delta)$$

$$\therefore \quad a_1 = Ce^{-\frac{kT}{4}},\ a_2 = Ce^{-\frac{3kT}{4}},\ a_3 = Ce^{-\frac{5kT}{4}},\ a_4 = Ce^{-\frac{7kT}{4}} \text{ etc.,}$$

So, we can write

$$\frac{a_1}{a_2} = \frac{a_2}{a_3} = \cdots = \frac{a_{n-1}}{a_n} = e^{\frac{kT}{2}} = d, \quad d = e^{\frac{kT}{2}} = \text{constant} \qquad \cdots(2.11)$$

and it (d) is called its decrement. The log of this decrement is called logarithmic decrement (λ).

$$\therefore \quad \lambda = \ln d = \frac{kT}{2}\ln e$$

$$\therefore \quad \lambda = \frac{kT}{2}$$

$$\text{So,} \quad \lambda\left(=\frac{kT}{2}\right) = \ln\frac{a_1}{a_2} = \ln\frac{a_2}{a_3} = \cdots \qquad \cdots(2.12)$$

Thus *logarithmic decrement is defined as the natural logarithm of the ratio of two successive amplitudes of damped oscillations.*

Special Note :

Sometimes, logarithmic decrement can also be defined by the natural logarithm of the ratio of two arbitrary alternate amplitudes of damped oscillatory motion separated by one time period. In that case,

$$\frac{a_1}{a_3} = \frac{a_3}{a_5} = \cdots = e^{kT}$$

$\therefore\ \lambda = \ln(e^{kT}) = kT \qquad [\because\ a_1 = Ce^{-kT},\ a_2 = Ce^{-2kT},\ a_3 = Ce^{-3kT}, \cdots]$

② **Relaxation time (τ)** We are now interested to find how rapidly the motion is damped by frictional resistance.

The energy of a particle of mass m for damped oscillatory vibration can be written as,

$$E = \text{kinetic energy} + \text{potential energy}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}ax^2 \qquad \cdots(2.13)$$

Now, the general equation of damped oscillatory motion,

$$x = Ce^{-kt}\sin(\omega' t + \delta) \qquad \cdots(2.14)$$

where $\omega' = \sqrt{\omega^2 - k^2}$ = angular frequency.

So, velocity $v = \dfrac{dx}{dt}$

$$= Ce^{-kt}\omega'\cos(\omega' t + \delta) - Ce^{-kt}k\sin(\omega' t + \delta) \qquad \cdots(2.15)$$

If the body is excited by giving a velocity v_0 suddenly in the mean position *i.e.* at $t = 0$, then $\dfrac{dx}{dt} = v_0$ and $\delta = 0$.

So, we get from the equation (2.15)

$$v_0 = C\omega' \quad \text{or,} \quad C = \frac{v_0}{\omega'}$$

The general equation of damped oscillatory motion

$$x = \frac{v_0}{\omega'}e^{-kt}\sin\omega' t \qquad \cdots(2.16)$$

and $v = \dfrac{v_0}{\omega'}e^{-kt}\omega'\cos\omega' t - \dfrac{v_0}{\omega'}e^{-kt}k\sin\omega' t$

$$= v_0 e^{-kt}\cos\omega' t \quad [\because\ k \text{ is very small}]$$

$$\therefore\quad E = \frac{1}{2}mv^2 + \frac{1}{2}ax^2 \qquad \cdots(2.17)$$

$$= \frac{1}{2}mv_0^2 e^{-2kt}\cos^2\omega' t + \frac{1}{2}m\omega^2\frac{v_0^2}{\omega'^2}e^{-2kt}\sin^2\omega' t \quad [\because\ a = \omega^2 m]$$

In case of low damping,

$$\omega' = \sqrt{\omega^2 - k^2} \simeq \omega = \text{natural angular frequency.} \quad [\because k \approx 0]$$

$$\therefore \quad E = \frac{1}{2}mv_0^2\, e^{-2kt}\cos^2\omega t + \frac{1}{2}mv_0^2\, e^{-2kt}\sin^2\omega t$$

$$= \frac{1}{2}mv_0^2\, e^{-2kt}(\cos^2\omega t + \sin^2\omega t)$$

$$= \frac{1}{2}mv_0^2\, e^{-2kt} \qquad \cdots(2.19)$$

Initially, at $t = 0$, $E = \frac{1}{2}mv_0^2 = E_0$ (say)

$$\therefore \quad E = E_0\, e^{-2kt} \qquad \cdots(2.20)$$

If $\quad t = \frac{1}{2k} = \tau$ (say),

$$E = \frac{E_0}{e}$$

This time $\tau = \frac{1}{2k}$ is known as *relaxation time*. So, **the relaxation time is defined as the time taken to fall the energy of the damped oscillatory motion to $\frac{1}{e}$ (about 37%) of its initial energy.**

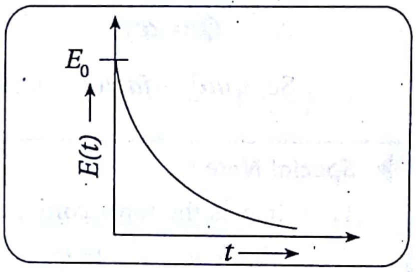

Fig. 4 ▷ Energy of an underdamped motion

Hence, $E = E_0\, e^{-\frac{t}{\tau}}$ and $\tau = \frac{1}{2k}$

③ **Quality factor** The quality factor of a damped harmonic oscillator is used to differentiate between light and heavy damping. *The ratio of maximum value of restoring force to the maximum value of the damping force is called quality factor.* So, for less value of damping, quality factor will be higher and thus quality will be better.

If A be the amplitude of vibration of the system, the maximum value of the restoring force is aA and damping force is $b\left(\frac{dx}{dt}\right)_{\max}$. Here, a is restoring force per unit displacement and b is the damping force per unit velocity.

For damped oscillation, the velocity for small damping can be written from equation (2.17) as,

$$v = v_0\, e^{-kt}\cos\omega' t, \text{ where } v_0 = C\omega' \qquad \cdots(2.21)$$

$$\therefore \quad v_{\max} = v_0\, e^{-kt}$$

$$= C\omega' e^{-kt}$$

$$= A\omega', \text{ where } A = \text{amplitude} = C e^{-kt} \qquad \cdots(2.22)$$

In case of low damping, maximum restoring force,

$$= aA = m\omega^2 A \quad \left[\because \omega' \simeq \omega = \sqrt{\frac{a}{m}} \text{ for low value of } k\right] \qquad \cdots(2.23)$$

Maximum damping force,

$$= bv_{\text{max}} = bA\omega'$$

$$= bA\omega \quad [\because \text{ for small } k,\ \omega' = \omega]$$

$$= (2km)A\omega \quad [\because b = 2km] \qquad \cdots(2.24)$$

$\therefore$ Quality factor, $Q = \dfrac{\text{maximum value of restoring force}}{\text{maximum value of damping force}}$

$$= \frac{aA}{b\left(\frac{dx}{dt}\right)_{\text{max}}} = \frac{m\omega^2 A}{2mk\omega A}$$

$$= \frac{\omega}{2k} = \omega\tau \quad \left[\because \text{relaxation time } (\tau) = \frac{1}{2k}\right]$$

$\therefore$ $\boxed{Q = \omega\tau}$ $\cdots(2.25)$

So, *quality factor = natural angular frequency × relaxation time.*

Special Note :

1. If a is the force constant of an oscillating body of mass m, then natural angular frequency $\omega(=2\pi f_0) = \sqrt{\dfrac{a}{m}}$.

$\therefore$ $Q = \tau\sqrt{\dfrac{a}{m}}$ $\cdots(1)$

2. We may find out Q factor by comparing the total energy $\left(\frac{1}{2}m\omega^2 A^2\right)$ of the oscillating body with the energy lost (*i.e.* dissipated energy) to overcome frictional forces over a quarter of a time period by the body.

Hence, the work done by the frictional force to cover the distance of amplitude (A) in quarter of a period (T)

$$W = \text{average value of frictional force} \times A$$

$$= \left(\frac{0 + 2kmA\omega}{2}\right)A = kmA^2\omega$$

$[\because$ maximum value of frictional force $= 2km\omega A$ from equation (2.24)$]$ $\cdots(2)$

Again, energy of the oscillating body, $E = \frac{1}{2}m\omega^2 A^2$

Hence, $\dfrac{E}{W} = \dfrac{\frac{1}{2}m\omega^2 A^2}{kmA^2\omega} = \dfrac{\omega}{2k} = Q$ $\cdots(3)$

So, the quality factor of an oscillating body may also be defined as the ratio of its total energy to the energy lost due to frictional force over a quarter of a time period. $\cdots(4)$

2.5. Dissipation of Power in Damped Harmonic Motion

We know that when a particle vibrates, it has to do work against the damping force. So, a certain fraction of energy is always dissipated in the form of energy for damped oscillatory motion.

We have already obtained the energy of a particle of mass m for damped oscillatory motion as,

$$E = E_0 e^{-2kt}$$

So, after one full time period T, the energy,

$$E_1 = E_0 e^{-2k(t+T)} = E_0 e^{-2kt} \cdot e^{-2kT}$$

$$= E e^{-2kT} \quad [\because E = E_0 e^{-2kt}] \qquad \cdots(2.26)$$

So, the loss of energy in one time period (T)

$$= E - E_1 = E - E e^{-2kT}$$

$$= E - E\left\{1 - 2kT + \frac{4k^2T^2}{2!} - \cdots\right\}$$

$$= E - E(1 - 2kT) \quad \text{[neglecting the higher order terms of } k]$$

$$= 2EkT \qquad \cdots(2.27)$$

So, the rate of loss of energy over a time period $= 2Ek = \frac{bE}{m}$ $\quad [\because 2k = \frac{b}{m}]$

unit of damping constante

Problem 1

The amplitude of damped oscillatory motion of frequency 300 s^{-1} decays to $\frac{1}{10}$ of its initial value after 1800 cycles. Find its

i **damping constant** **ii** **quality factor** **iii** **relaxation time**

iv **the time in which its energy is reduced to $\frac{1}{e}$ of its initial energy.**

v **the time in which its energy is reduced to $\frac{1}{2}$ of its initial value.**

Solution

i The amplitude of damped oscillatory motion at any instant t is

$A = Ce^{-kt}$, where C = constant, k = **damping constant of the system**

Now, consider for $t = 0$, $C = A_0$ (initial amplitude)

$$\therefore \quad A = A_0 e^{-kt} \qquad \cdots(1)$$

Now, after time $t = \frac{1800}{300} = 6$ s, its amplitude $A = \frac{A_0}{10}$

Hence, we get from equation (1)

$$\therefore \frac{A_0}{10} = A_0 e^{-6k} \quad \text{or,} \quad e^{6k} = 10$$

$$\text{or,} \quad 6k = \ln 10 \quad \text{or,} \quad k = \frac{2.3}{6} = 0.383 \approx 0.38 \text{ s}^{-1}$$

ii Quality factor $Q = \dfrac{\omega}{2k} = \dfrac{2\pi n}{2k}$, n = cyclic frequency = 300

$$= \frac{3.14 \times 300}{0.38} = 2478.95$$

iii Relaxation time $\tau = \dfrac{1}{2k} = \dfrac{1}{2 \times 0.38} = 1.32$ s

iv From the definition of relaxation time, we know it is the time taken to fall the energy of damped oscillatory motion to $\frac{1}{e}$ of its initial energy.

Hence, required relaxation time = τ = 1.32 s

v Let, t be the required time in which its energy is reduced to $\frac{1}{2}$ of its initial value. Therefore,

$$E = E_0 e^{-2kt} \quad \text{or,} \quad \frac{E}{E_0} = e^{-2kt} \quad \text{or,} \quad \frac{1}{2} = e^{-2 \times 0.38t} = e^{-0.76t}$$

$\therefore$ $t = 0.91$ s

Problem 2

A system of unit mass whose natural angular frequency in the absence of damping is $4\text{rad}\cdot\text{s}^{-1}$ is subjected to a small damping force which is proportional to the velocity of the system. The constant of proportionality is 10s^{-1}. Show that the system is overdamped and that the general solution for displacement is $x = Ae^{-2t} + Be^{-8t}$.

Solution Mass of the system, $m = 1$ unit, natural angular frequency $\omega_0\left(=\sqrt{\frac{a}{m}}\right) = 4\text{rad}\cdot\text{s}^{-1}$, damping coefficient $b = 10\text{s}^{-1}$, damping constant $k = \dfrac{b}{2m} = \dfrac{10}{2 \times 1} = 5\text{s}^{-1}$

As $k > \omega$, the motion is overdamped

Now, $k^2 = 25$, $\omega^2 = 16$ $\therefore$ $\sqrt{k^2 - \omega^2} = \sqrt{25 - 16} = 3$

Hence, the general solution for overdamped motion is

$$x = Ae^{(-k + \sqrt{k^2 - \omega^2})t} + Be^{(-k - \sqrt{k^2 - \omega^2})t}$$

$$\text{or,} \quad x = Ae^{(-5+3)t} + Be^{(-5-3)t}$$

$$\text{or,} \quad x = Ae^{-2t} + Be^{-8t}$$

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Problem 3

The natural angular frequency of a simple harmonic oscillator of mass 2g is 0.8rad $\cdot$ s^{-1}. It undergoes critically damped motion when taken to a viscous medium. Find the damping force on the oscillator when its speed is 0.2cm $\cdot$ s^{-1}.

Solution The damping force $= b\dfrac{dx}{dt}$,

Now, b = damping coefficient $= 2mk$, where k = damping constant, m = mass of the oscillator = 2g

At critical damping, damping constant (k) = restoring constant (ω)

$\therefore$ The damping force $= (2m\omega)\dfrac{dx}{dt} = 2\cdot(2)\cdot(0.8)\cdot(0.2)$ dyne $= 0.64$ dyne.

Problem 4

If the damping force acting on a body is of constant magnitude, show that the frequency of vibration of a damped oscillator is not affected by the magnitude of damping.

Solution Let F_0 be the damping force/velocity acting on the body.

$\therefore$ The equation of motion of damped vibration can be written as

$$m\frac{d^2x}{dt^2} + ax + F_0 = 0 \qquad \text{or,} \quad m\frac{d^2x}{dt^2} + a\left(x + \frac{F_0}{a}\right) = 0 \qquad \cdots 1$$

Let introducing a new variable $y = x + \dfrac{F_0}{a}$, we get from eqn. (1)

$$m\frac{d^2y}{dt^2} + ay = 0$$

or, $$\frac{d^2y}{dt^2} + \frac{a}{m}y = 0$$

or, $$\frac{d^2y}{dt^2} + \omega^2 y = 0$$, where $\omega = \sqrt{a/m}$ = natural undamped angular frequency

Thus, the frequency of damped vibration is equal to the natural (undamped) angular frequency. So, in that case, the frequency of vibration of a damped oscillator is not affected by the magnitude of damping.

2.6. Forced Vibration

The amplitude of oscillations for a damped oscillatory motion of a body goes on decreasing with time due to loss of energy to overcome the resistive forces. *If an external periodic force is supplied to the system to make up for the losses, the amplitude of vibration does not decay with time and the body will vibrate (i.e. oscillate) regularly under such periodic force. This vibration of the body is called* ***forced vibration***. So, we have to apply external periodic force to maintain the oscillation.

Suppose, a particle of mass m executing damped harmonic motion is subjected to an external periodic force $F_0 e^{ipt}$[2], where p is the cyclic frequency of periodic force. Let x be the instantaneous displacement from the rest. So, the equation of forced vibration of such a system is given by,

$$m\frac{d^2x}{dt^2} = -ax - b\frac{dx}{dt} + F_0 e^{ipt}$$

or, $$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = fe^{ipt} \qquad \cdots(2.28)$$

where $k = \frac{b}{2m}$ = damping constant, $\omega = \sqrt{\frac{a}{m}}$ = *natural angular frequency* and $f = \frac{F_0}{m}$ *i.e. amplitude of driving force per unit mass.*

This is a 2nd order first degree differential equation of forced vibration. Its solution consists of two parts. These are—

① particular integral

② complementary function.

① **For particular integral**

The solution is,

$$x = A'e^{ipt} \qquad \cdots(2.29) \qquad \frac{dx}{dt} = ipA'e^{ipt} \qquad \cdots(2.30)$$

$$\frac{d^2x}{dt^2} = -p^2 A' e^{ipt} \qquad \cdots(2.31)$$

Substituting these values in equation (2.28), we get,

$$-p^2A'e^{ipt} + 2kA'ipe^{ipt} + \omega^2 A'e^{ipt} = fe^{ipt}$$

or, $$A'[-p^2 + 2ikp + \omega^2] = f$$

or, $$A' = \frac{f}{(\omega^2 - p^2) + i2kp} \qquad \cdots(2.32)$$

Let, $\omega^2 - p^2 = B\cos\phi$ and $2kp = B\sin\phi$

$\therefore$ $$B^2 = 4k^2p^2 + (\omega^2 - p^2)^2 \quad \text{or,} \quad B = \sqrt{4k^2p^2 + (\omega^2-p^2)^2} \qquad \cdots(2.33)$$

and $$\tan\phi = \frac{2kp}{\omega^2 - p^2} \qquad \cdots(2.34)$$

$\therefore$ $$A' = \frac{f}{B\{\cos\phi + i\sin\phi\}} \quad \text{[from the equation (2.32)]}$$

$$= \frac{f}{Be^{i\phi}} = \frac{f}{\sqrt{4k^2p^2 + (\omega^2-p^2)^2}}e^{-i\phi} \qquad \cdots(2.35)$$

● Periodic force may be considered either $F_0 \cos pt$ or $F_0 \sin pt$. Here, we have considered the periodic force by a complex quantity *i.e.* $F_0 \cos pt + iF_0 \sin pt \; (= F_0 e^{ipt})$

Hence, we get the particular integral from the equation (2.29)

$$x = A' e^{ipt} = \frac{f}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}} e^{i(pt-\phi)} \quad \cdots (2.36)$$

Hence, amplitude,

$$A = \frac{f}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}} \quad \cdots (2.37)$$

② For complementary function

The complementary function can be obtained by putting the right hand side of equation (2.28) equal to zero, *viz,*

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0 \quad \cdots (2.38)$$

When damping constant k is small, *the solution of this equation is the equation of damped oscillatory motion* and it is given by,

$$x = Ce^{-kt}\sin(\omega' t + \delta)$$

C and δ are two constants depending on the initial condition and $\omega' = \sqrt{\omega^2 - k^2}$ = angular frequency.

So, *the general solution of equation* (2.28) is,

$$x = \textit{complementary function} + \textit{particular integral}$$

So, $$\boldsymbol{x = Ce^{-kt}\sin(\omega' t + \delta) + \frac{f}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}} e^{i(pt-\phi)}} \quad \cdots (2.39)$$

The first part (*i.e.,* transient term) of the solution for x represents damped simple harmonic motion. After a few time, the vibration becomes negligible as its amplitude diminishes exponentially with time. Thus after a lapse of time, the second part (steady state term) represents the forced sustained oscillation whose frequency $\left(\frac{p}{2\pi}\right)$ will be the frequency of the applied force. The steady state term describes the behaviour of oscillator after the transient has died away.

Hence, **the solution of the forced vibration** (*i.e.* vibration of the body) can be written as,

$$\boldsymbol{x = \frac{f}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}} e^{i(pt-\phi)}} \quad \cdots (2.40)$$

Here, **amplitude** $A = \frac{f}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}}$ $\cdots$ (2.41)

frequency $= \frac{p}{2\pi}$ and its phase $\phi = \tan^{-1}\left(\frac{2kp}{\omega^2 - p^2}\right)$ $\cdots$ (2.42)

If the physical excitations are $F_0\cos pt$ or $F_0\sin pt$, the physical solutions are extracted from the complex solution by taking its real or imaginary part respectively.

For an example, when the driving force is $F_0 \sin pt$, the steady state solution of the forced vibration is,

$$x = \frac{f}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}} \sin(pt - \phi), \text{ where } f = \frac{F_0}{m} \quad \cdots (2.43)$$

or $\quad x = A\sin(pt - \phi)$

The equation shows that the phase difference between the displacement of driven oscillator and driving force depends upon the frequency of the applied force and damping coefficient of the motion. The above equation shows that the displacement x of driven oscillator lags behind the driving force, $F = F_0 \sin pt$ by an angle ϕ.

The velocity of the body executing forced vibration is,

$$v = \frac{dx}{dt} = Ap\cos(pt - \phi) = Ap\sin\left(pt - \phi + \frac{\pi}{2}\right)$$
$$= Ap\sin(pt - \psi) \quad \cdots (2.44)$$

This equation indicates that the phase of the velocity of the driven oscillator is $\left(pt - \phi + \frac{\pi}{2}\right)$. *So in steady state condition, the velocity of the driven oscillator at any instant of time leads the displacement by* $\frac{\pi}{2}$.

Special Note :

If the periodic force is $F_0 \sin pt$, then the equation of motion is,

$$m\frac{d^2x}{dt^2} = -ax - b\frac{dx}{dt} + F_0 \sin pt$$

or, $$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = f\sin pt; \left[f = \frac{F_0}{m}\right]$$

There are several ways to find particular integral of this non-homogeneous 2nd order differential equation. We shall find it convenient to use complex number.

To get the equation of motion in complex form, let x_1 be the displacement for the driving force $F_0 \cos pt$ and x_2 be the corresponding displacement by the other driving force $F_0 \sin pt$. The equations of motion for the two cases are respectively,

$$\frac{d^2x_1}{dt^2} + 2k\frac{dx_1}{dt} + \omega^2 x_1 = f\cos pt \quad \ldots (1)$$

and $$\frac{d^2x_2}{dt^2} + 2k\frac{dx_2}{dt} + \omega^2 x_2 = f\sin pt \quad \ldots (2)$$

Multiplying equation (2) by $i = \sqrt{-1}$ and adding with equation (1), we get,

$$\left(\frac{d^2x_1}{dt^2} + i\frac{d^2x_2}{dt^2}\right) + 2k\left(\frac{dx_1}{dt} + i\frac{dx_2}{dt}\right) + \omega^2(x_1 + ix_2) = f(\cos pt + i\sin pt)$$

If we write $x = x_1 + ix_2$, then the above equation reduces to,

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = fe^{ipt} \quad \ldots (3)$$

This equation is similar to equation (2.28).

So, we can find out particular integral *i.e.* the solution of this equation in similar way that we have done earlier.

2.6.1. Phase Difference between Driven Oscillator and Driving Force

In steady forced vibration, the displacement x of driven oscillator is given by $x = A\sin(pt - \phi)$ where $\tan\phi = \frac{2kp}{\omega^2 - p^2}$. Suppose the angular frequency p of applied periodic force is gradually increased from 0 to ∞. Let us find how ϕ changes with p.

① When $p = 0$, $\tan\phi = 0$ and hence $\phi = 0$. Thus, *there is no difference of phase difference between the driven system and the driver.*

② When $p = \omega$, $\tan\phi = \infty$ and hence $\phi = \frac{\pi}{2}$. Thus, *at resonance the driven system lags behind the driver by an angle $\frac{\pi}{2}$.*

③ When $p < \omega$, $\tan\phi = +\text{ve}$, it means that difference of phase has a value in between 0 and $\frac{\pi}{2}$.

④ When $p > \omega$, $\tan\phi = -\text{ve}$ which indicates that $\frac{\pi}{2} < \phi < \pi$.

⑤ When $p \to \infty$, $\tan\phi \to 0$ that indicates $\phi \to \pi$.

Thus for all values of p, ϕ lies in between 0 and π and $\phi = \frac{\pi}{2}$ at resonance.

2.7. Amplitude and Velocity Resonance

The displacement amplitude and velocity amplitude of forced vibration depend on the frequency of the applied periodic force. **When the displacement amplitude of forced vibration is maximum for a particular frequency (resonance frequency) of the applied periodic force, the phenomena is called amplitude resonance. Similarly when the velocity amplitude attains a maximum value for a certain frequency of the applied periodic force, the phenomena is known as velocity resonance.**

2.7.1. Amplitude Resonance

The amplitude of forced vibration is given by,

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}} \quad \cdots(2.45)$$

So, amplitude (A) varies with the frequency (p) of the applied force.

Now, A is maximum for a particular value of p, for which the denominator of equation (2.45) is minimum.

$$\therefore \quad \frac{d}{dp}[(\omega^2 - p^2)^2 + 4k^2p^2] = 0 \quad \text{or,} \quad 2(\omega^2 - p^2)(-2p) + 8k^2p = 0$$

$$\text{or,} \quad -4p[(\omega^2 - p^2) - 2k^2] = 0 \quad \text{or,} \quad (\omega^2 - p^2) - 2k^2 = 0 \quad [\because p \neq 0]$$

$$\text{or,} \quad p^2 = \omega^2 - 2k^2$$

$$\text{or,} \quad p = \sqrt{\omega^2 - 2k^2} = p_r \text{ (say), provided } \omega^2 > 2k^2. \quad \cdots(2.46)$$

Hence, the cyclic resonant frequency is given by, $f_r = \frac{\sqrt{\omega^2 - 2k^2}}{2\pi}$, at which amplitude $A = A_{max}$.

Note that, the angular frequency for amplitude resonance $\sqrt{\omega^2 - 2k^2}$ is less than both natural angular frequency (ω) for undamped motion and that frequency $\sqrt{\omega^2 - k^2}$ *i.e.* ω' in presence of damping.

From the equation (2.45) we get,

$$A_{max} = \frac{f}{\sqrt{(\omega^2 - \omega^2 + 2k^2)^2 + 4k^2(\omega^2 - 2k^2)}}$$

[$\because$ at resonance condition, frequency $p = \sqrt{\omega^2 - 2k^2}$]

$$= \frac{f}{\sqrt{4k^2\omega^2 - 4k^4}} = \frac{f}{2k\sqrt{\omega^2 - k^2}}$$

$$= \frac{f}{2k\sqrt{\omega^2 - 2k^2 + k^2}}$$

$$= \frac{f}{2k\sqrt{p^2 + k^2}} \quad [\because p = \sqrt{\omega^2 - 2k^2}] \qquad \cdots (2.47)$$

This shows that maximum amplitude (A_{max}) depends upon the damping factor The variation of amplitude A with the frequency of applied force p. for different values of damping constant k is shown in the following Fig. 5. The following cases are observed—

Case 1 From equations (2.45) and (2.47), we can conclude that the maximum amplitude is greater for the lower value of damping factor k except $p = \omega$ *i.e.* for $p > \omega$ or $p < \omega$

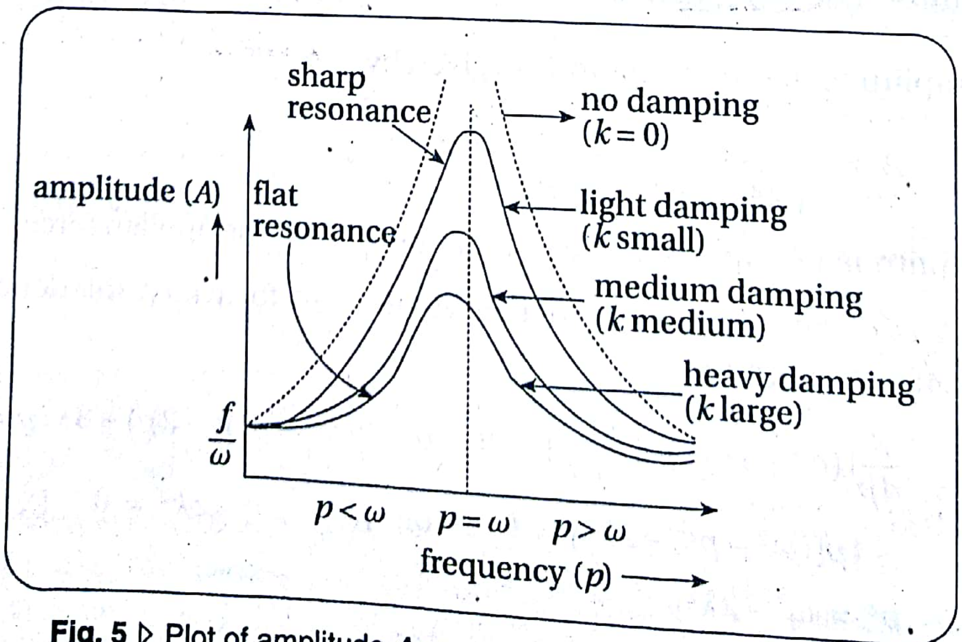

Fig. 5 ▷ Plot of amplitude A as a function of applied frequency p

Case 2 For low damping (k), when frequency (p) of the applied force = natural frequency (ω), then from equations (2.41) and (2.42) we can write, the amplitude at resonant frequency

$$A_{max} = \frac{f}{2kp} \qquad \cdots(2.48)$$

and phase $\phi = \tan^{-1}\left(\frac{2kp}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$ $\cdots(2.49)$

So, for undamped motion (*i.e.*, $k = 0$), when $\omega = p$, then the amplitude of oscillation becomes very large and theoretically infinite (*i.e.* $A_{max} \to \infty$). The displacement lags behind the applied force by $\frac{\pi}{2}$.

2.7.2. A Amplitude Resonance: Sharpness of Resonance

The amplitude of forced vibration,

$$A = \frac{f}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}$$

We have seen that when the frequency of the impressed force has a value to satisfy the condition of resonance (*i.e.* $p = \sqrt{\omega^2 - 2k^2}$), the amplitude of forced vibration is maximum [equation (2.47)] and $A_{max} = \frac{f}{2k\sqrt{p^2 + k^2}}$.

If the frequency of the applied force changes from the resonant frequency ($p = \sqrt{\omega^2 - 2k^2}$), the amplitude falls and this reduction in amplitude is also more if the damping factor (k) is small, [as the effect of the term $4k^2p^2$ in the denominator of equation (2.45) is smaller compared with the variation of $(\omega^2 - p^2)$].

When the fall in amplitude is large for a small deviation of the frequency from the resonant frequency, the resonance is termed as sharp, otherwise it is said to be flat.

So, the **sharpness of resonance may be defined as the rate of fall in amplitude (or reduction in amplitude) with the change of the frequency of the applied periodic force on each side of resonant frequency.**

The sharpness of resonance depends on damping constant k.

1. For low damping constant k and $p = \omega$ (natural frequency of the vibration), amplitude of the forced vibration at resonance $A_{max} = \frac{f}{2kp}$.

   So, when $k \to 0$, $A_{max} \to \infty$ at $p = \omega$.

2. For smaller value of damping factor k, the resonance curve is sharper for the change of frequency of the applied periodic force from the resonant frequency.

3. For larger value of the damping factor k, the resonance curve is flatter for the change of frequency of the periodic force on each side of the resonant frequency.

The plot of the displacement amplitude as a function of frequency of applied force for different values of damping is shown in Fig. 6.

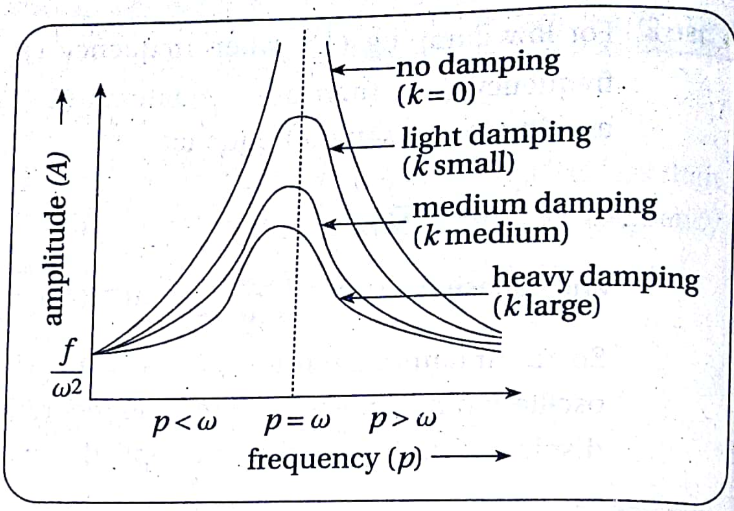

Fig. 6 ▷ Variation of amplitude with applied frequency for different damping constant

As the damping decreases, the angular frequency for which the amplitude (A) is maximum, moves towards the frequency (ω) of undamped motion.

The sharpness of resonance can be understood more clearly with the help of half width of the resonance curve.

We consider a resonance curve for the amplitude resonance [Fig.7]. The amplitude is maximum at angular frequency $p_r (= \sqrt{\omega^2 - 2k^2})$ of the applied force and becomes $\frac{A_{max}}{2}$ at frequency $p_h = |p_r \pm \Delta|$, where $\Delta = |p_r - p_h|$

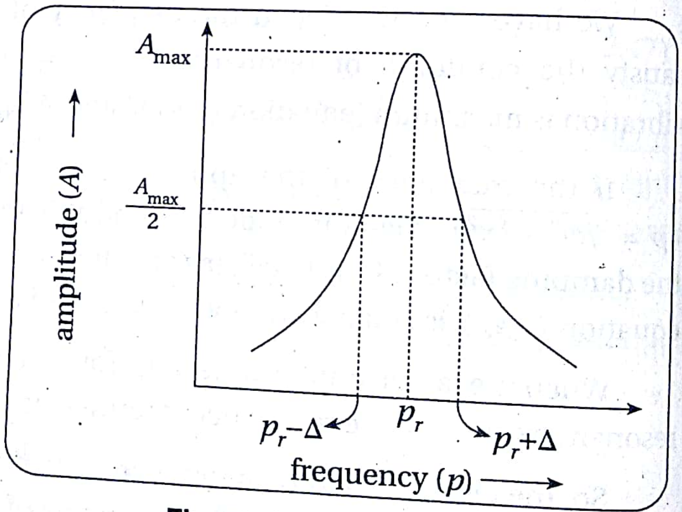

Fig. 7 ▷ Resonance curve

This $\Delta (= \sqrt{3}k)$, which can be used to measure the sharpness of resonance, is called the **half width** of the resonance curve. This shows that for small value of k (damping constant) p_h is close to p_r. Hence, the resonance curve will be more sharp for smaller value of half width Δ of the resonance curve.

2.7.2. B Velocity Resonance : Sharpness of Velocity Resonance

When the amplitude of velocity of the forced vibration under the action of impressed periodic force is maximum, then the phenomenon is known as velocity resonance with the applied force.

If the impressed periodic force on the oscillating body is $F_0 \sin pt$, the equation of forced vibration is,

$$x = A \sin(pt - \phi) \quad \text{[from the equation (2.43)]}$$

Hence, the velocity of the particle undergoing forced vibration,

$$v = \frac{dx}{dt} = Ap\cos(pt - \phi)$$

So, **the peak value of velocity amplitude,**

$$V = Ap = \frac{fp}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}, \text{ where } f = \frac{F_0}{m}$$

$$= \frac{f}{\sqrt{\left(\frac{\omega^2 - p^2}{p}\right)^2 + 4k^2}} = \frac{f}{\sqrt{\left(\frac{\omega^2}{p} - p\right)^2 + 4k^2}} \quad \cdots (2.50)$$

As p is varied, V is also changed and when $\omega = p$, the denominator of the above expression is minimum and V is maximum. *Maximum value of* V is $V_{max} = \frac{f}{2k}$.

So, *velocity resonance occurs when the natural frequency of the body is equal to the frequency of the applied periodic force.*

The variation of the velocity amplitude with the angular frequency of the driver for different values of the damping constant is shown in Fig 8.

We shall observe the following variations of velocity amplitude for different frequencies of the applied force.

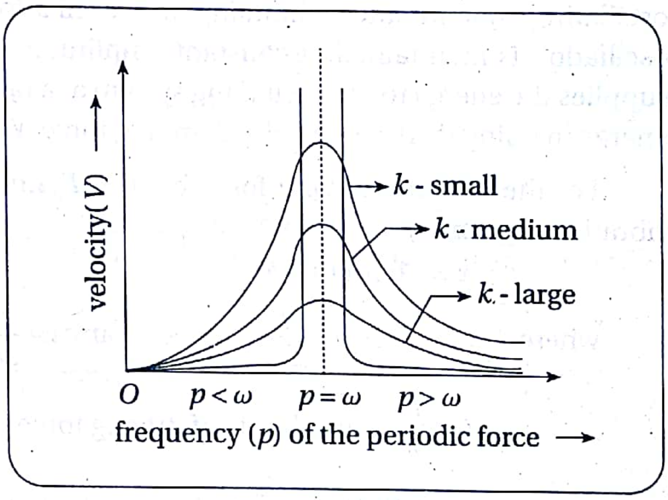

Fig. 8 ▷ Variation of velocity amplitude as a function of frequency of applied force

Case 1 When $p = 0$, $V = 0$.

Case 2 When $p = \omega$, $V = V_{max} = \frac{f}{2k}$

This is the condition of velocity resonance. It occurs when the frequency of the applied force is equal to the natural angular frequency (ω). So, the maximum value of V occurs at $p = \omega$ for all values of k. In that case the driven system gains maximum kinetic energy.

Case 3 When damping (k) is zero, the velocity amplitude V is infinite at $p = \omega$.

So, greater the maximum velocity amplitude, lower the value of the damping constant.

Thus, the sharpness of velocity resonance may be defined as the rate of fall in velocity amplitude (or reduction in velocity amplitude) with the change of the frequency of the applied periodic force on each side of resonant frequency. The variation of velocity amplitude with the frequency of the applied force is shown in Fig. 8.

Special Note :

As the kinetic energy of the forced vibrating system is $\frac{1}{2}mv^2$, the velocity resonance is seen simultaneously for the maximum kinetic energy of the driven system with a variation of frequency of the driver. This is why velocity resonance is sometimes referred as the **energy resonance**. The maximum kinetic energy of forced system,

$$E_m = \frac{1}{2}mV_{\max}^2 = \frac{1}{2}m\frac{f^2}{4k^2} \quad \left[\because V_{\max} = \frac{f}{2k}, f = \frac{F_0}{m}\right]$$

2.8. Rate of Dissipation of Energy and Power Supply in Forced Vibration

In forced vibration, the external periodic driving force supplies energy to the oscillating system and maintains the oscillation. Hence, in steady state, when oscillation is maintained at constant amplitude, it is expected that the driving force supplies the energy to the oscillating system at a rate equal to that at which it dissipates energy in doing work against the damping force. We shall prove this statement.

Let the external driving force be $F = F_0 \sin pt$. So, the displacement of forced vibration is given by,

$$x = A\sin(pt-\phi)$$

where, $A = \dfrac{f}{\sqrt{(\omega^2-p^2)^2+4k^2p^2}}$, $\tan\phi = \dfrac{2kp}{\omega^2-p^2}$,

$f = \dfrac{F_0}{m}$ = amplitude of driving force per unit mass of the body

$\therefore \quad v$ = velocity $= \dfrac{dx}{dt} = Ap\cos(pt-\phi) = V\cos(pt-\phi)$

where $V = Ap$ = peak value of velocity

So, the instantaneous power supplied to the oscillator,

P = instantaneous force × velocity

$$= F\cdot\frac{dx}{dt} = F_0 \sin pt \times V\cos(pt-\phi)$$

[$\because$ From equation (2.43), $x = A\sin(pt-\phi)$]

$$= F_0 \sin pt \times V[\cos pt \cos\phi + \sin pt \sin\phi]$$

$$= VF_0 \cos\phi \sin pt \cos pt + VF_0 \sin\phi \sin^2 pt \quad \cdots (2.51)$$

Average power supply Average power supplied over one complete cycle,

$$\overline{P} = VF_0 \cos\phi \frac{1}{T}\int_0^T \sin pt \cos pt\, dt + VF_0 \sin\phi \frac{1}{T}\int_0^T \sin^2 pt\, dt \quad \cdots (2.52)$$

$\left[\text{where } T = \text{time period of driving periodic force} = \dfrac{2\pi}{p}\right]$

But, we know that the integral of the product of 'cos' and 'sin' within the interval of its periodicity is zero whereas the integral of $\sin^2$ (or $\cos^2$) within the interval of its periodicity is $\frac{1}{2}$ × periodicity $= \frac{1}{2}\times\frac{2\pi}{p}$.

Hence, we get from the expression of $\overline{P}$,

$$\overline{P} = VF_0 \sin\phi \frac{\frac{1}{2} \times \frac{2\pi}{p}}{\frac{2\pi}{p}} \quad \left[\because T = \frac{2\pi}{p}\right]$$

$$= \frac{1}{2} VF_0 \sin\phi = \frac{1}{2}(Ap)F_0 \sin\phi \quad [\because V = Ap] \qquad \cdots(2.53)$$

Again,

$$\tan\phi = \frac{2kp}{\omega^2 - p^2} = \frac{2\left(\frac{b}{2m}\right)p}{\frac{a}{m} - p^2} \quad \left[\because k = \frac{b}{2m} \text{ and } \omega = \sqrt{\frac{a}{m}}\right]$$

$$= \frac{bp}{a - mp^2} = \frac{b}{\frac{a}{p} - mp} \qquad \cdots(2.54)$$

$$\therefore \quad \sin\phi = \frac{b}{\sqrt{b^2 + \left(\frac{a}{p} - mp\right)^2}} = \frac{b}{\sqrt{b^2 + \left(mp - \frac{a}{p}\right)^2}} \qquad \cdots(2.55)$$

and

$$V = Ap = \frac{\left(\frac{F_0}{m}\right)p}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}} \quad \left[\text{putting } A = \frac{\frac{F_0}{m}}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}}\right]$$

$$= \frac{\left(\frac{F_0}{m}\right)p}{\sqrt{\left(\frac{a}{m} - p^2\right)^2 + 4\frac{b^2}{4m^2}p^2}} \quad \left[\because \omega^2 = \frac{a}{m} \text{ and } k = \frac{b}{2m}\right]$$

$$= \frac{F_0 p}{(a - mp^2)^2 + p^2b^2}$$

$$= \frac{F_0 p}{p\sqrt{b^2 + \left(mp - \frac{a}{p}\right)^2}} = \frac{F_0}{\sqrt{b^2 + \left(mp - \frac{a}{p}\right)^2}} \qquad \cdots(2.56)$$

Substituting the values of Ap and $\sin\phi$ in the expression of $\overline{P}$ [*i.e.* in Eqn. (2.53)]

$$\overline{P} = \frac{1}{2}(Ap)F_0 \sin\phi = \frac{1}{2}\frac{F_0}{\sqrt{b^2 + \left(mp - \frac{a}{p}\right)^2}} \cdot F_0 \frac{b}{\sqrt{b^2 + \left(mp - \frac{a}{p}\right)^2}}$$

$$= \frac{1}{2} b \frac{F_0^{\,2}}{b^2 + \left(mp - \frac{a}{p}\right)^2}$$

$$= \frac{1}{2} bV^2 \quad \left[\because \text{From equation (2.56), } V = \frac{F_0}{\sqrt{b^2 + \left(mp - \frac{a}{p}\right)^2}}\right] \qquad \cdots(2.57)$$

Rate of dissipation of energy Similarly, the rate of dissipation of energy (D) is given by the instantaneous rate of doing work against frictional force.

$\therefore$ D = instantaneous frictional force × instantaneous velocity

$$= \left(b\frac{dx}{dt}\right)\left(\frac{dx}{dt}\right) = b\left(\frac{dx}{dt}\right)^2, \text{ where } b = 2km$$

$$= bV^2\cos^2(pt-\phi) \qquad \cdots(2.58)$$

Hence, mean rate of dissipation of energy over one complete cycle (*i.e.* one time period)

$$\overline{D} = bV^2\frac{1}{T}\int_0^T \cos^2(pt-\phi)dt = bV^2\frac{p}{2\pi}\int_0^{\frac{2\pi}{p}} \cos^2(pt-\phi)dt$$

$$= bV^2\left(\frac{p}{2\pi}\right)\left(\frac{1}{2}\frac{2\pi}{p}\right) = \frac{1}{2}bV^2 \qquad \cdots(2.59)$$

Comparing equations (2.57) and (2.59), we can say that the power supply to the system is exactly equal to the rate of dissipation of energy in forced vibration.

Condition for maximum power intake Actually, the power supplied by the driver (external periodic supply) is equal to the power intake by the oscillating system.

So, power intake by the oscillating system,

$$= \frac{1}{2}bV^2 \quad \text{[from equation (2.57)]}$$

$$= \frac{1}{2}b\frac{F_0^2}{b^2+\left(mp-\frac{a}{p}\right)^2} \quad \text{[from equation (2.56)]}$$

So, when denominator is minimum, the power intake by the oscillator is maximum.

$\therefore$ The power intake is maximum, when

$$mp-\frac{a}{p} = 0 \quad \text{or,} \quad p^2 = \frac{a}{m}$$

or, $p^2 = \omega^2$ $\left[\because \text{natural angular frequency } (\omega) \text{ of the oscillator } = \sqrt{\frac{a}{m}}\right]$

or, $\boxed{p = \omega}$ $\cdots(2.60)$

which is the condition for velocity resonance. So, we may conclude that the maximum power intake by the oscillator is observed at velocity resonance.

2.9. Mechanical and Electrical Analogy of Forced Vibration

The emf equation of the adjoining series LCR circuit (Fig. 9) is given by,

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E_0e^{ipt} \qquad \cdots(2.61)$$

here E_0e^{ipt} (in complex form) is the applied periodic emf and p is the cyclic requency.

Again, the equation of forced vibration of a mechanical system of mass m subjected to an external periodic force $F_0 e^{ipt}$ is,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + ax = F_0 e^{ipt} \quad \cdots (2.62)$$

Hence, the equations (2.61) and (2.62) are similar in nature and we can make an analogy between an electrical and mechanical system.

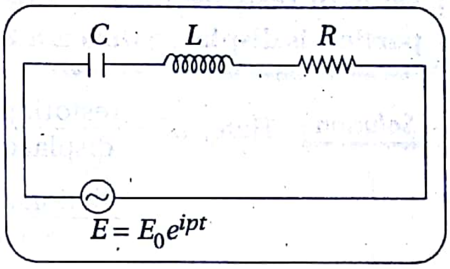

Fig. 9 ▷ Series LCR circuit

Here, the mass m is analogous to self inductance L, the resistive force per unit velocity b (mechanical resistance) to electrical resistance R, restoring force per unit displacement a is analogous to $\frac{1}{C}$ (C = capacitance) and displacement x to charge q flowing through the electrical circuit. The periodic force $F_0 e^{ipt}$ applied on the mechanical system is comparable to applied emf $E_0 e^{ipt}$ impressed on the circuit. Similarly, $\frac{dq}{dt}$ is analogous to the velocity $\frac{dx}{dt}$.

2.9.1. Relation between Mechanical and Electrical Impedance

The solution of equation (2.62) as discussed earlier can be written as,

$$x = \frac{\frac{F_0}{m}}{\sqrt{\left(\frac{a}{m} - p^2\right)^2 + 4\cdot\frac{b^2}{4m^2}p^2}} e^{i(pt-\phi)} \quad \text{or,} \quad x = \frac{F_0 e^{i(pt-\phi)}}{\sqrt{(a-mp^2)^2 + b^2p^2}}$$

$$\text{or,} \quad x = \frac{F_0 e^{i(pt-\phi)}}{p\sqrt{b^2 + \left(\frac{a}{p} - mp\right)^2}} \quad \text{or,} \quad x = \frac{F_0 e^{i(pt-\phi)}}{pZ_m} \quad \cdots (2.63)$$

where 'Z_m' is the *mechanical impedance* $= \sqrt{b^2 + \left(pm - \frac{a}{p}\right)^2}$

and $\quad \phi = \text{phase} = \tan^{-1}\left(\frac{b}{\frac{a}{p} - pm}\right)$

Similarly, the solution of equation (2.61) for electrical circuit can be written as,

$$q = \frac{E_0 e^{i(pt-\phi)}}{p\sqrt{R^2 + \left(pL - \frac{1}{pC}\right)^2}} = \frac{E_0 e^{i(pt-\phi)}}{pZ_e} \quad \cdots (2.64)$$

where Z_e is the *electrical impedance* $= \sqrt{R^2 + \left(pL - \frac{1}{pC}\right)^2}$

and $\quad \phi = \text{phase} = \tan^{-1}\left(\frac{R}{\frac{1}{pC} - pL}\right) \quad \cdots (2.65)$

Therefore, *comparing equations* (2.63) *and* (2.64) *we can say that the mechanical impedance Z_m is equivalent to electrical impedance Z_e.*

Problem 1

A particle of mass 0.2 kg is acted on by a restoring force per unit displacement $10 \times 10^{-3}\,\mathrm{N \cdot m^{-1}}$ and a frictional force per unit velocity of $2 \times 10^{-3}\,\mathrm{N \cdot m^{-1} \cdot s}$. (i) Find whether the motion is oscillatory or non-oscillatory? (ii) For what value of resistive (or frictional) force, the damping will be critical? (iii) If the particle is displaced through 2 cm and then released, find its period.

Solution Here, $a = \dfrac{\text{restoring force}}{\text{displacement}} = 10 \times 10^{-3}\,\mathrm{N \cdot m^{-1}}$

$$b = \frac{\text{frictional force}}{\text{velocity}} = 2 \times 10^{-3}\,\mathrm{N \cdot m^{-1} \cdot s}$$

$$m = 0.2\ \text{kg}$$

(i) $\therefore\ k = \dfrac{b}{2m} = \dfrac{2 \times 10^{-3}}{2 \times 0.2} = \dfrac{0.001}{0.2} = 0.005\ \mathrm{s^{-1}}$

$$\therefore\ \omega = \sqrt{\frac{a}{m}} = \sqrt{\frac{10 \times 10^{-3}}{0.2}} = \sqrt{\frac{10^{-2}}{0.2}} = 0.22\ \text{cycles} \cdot \mathrm{s^{-1}}$$

Since, $k < \omega$ the motion is oscillatory.

(ii) For critical damping $k = \omega$ or, $\dfrac{b}{2m} = \sqrt{\dfrac{a}{m}}$ or, $\dfrac{b^2}{4m^2} = \dfrac{a}{m}$

or, $b = \sqrt{4ma}$

$$= \sqrt{4 \times 0.2 \times 10 \times 10^{-3}} = \sqrt{8 \times 10^{-3}}$$

$$= 8.944 \times 10^{-2}\,\mathrm{N \cdot m^{-1} \cdot s}$$

(iii) Time period

$$T = \frac{2\pi}{\sqrt{\omega^2 - k^2}} = \frac{2\pi}{\sqrt{(0.22)^2 - (0.005)^2}} = 28.57\ \text{s}$$

Problem 2

The equation for displacement of a point of a damped oscillator is given by,

$$x = 5e^{-0.25t}\sin\left(\frac{\pi}{2}\right)t\ \text{m}$$

Find the velocity of the oscillating point at $t = \frac{T}{4}$ and T, where T is the time period of oscillator.

[I.A.S. 1987]

Solution We know that the equation for displacement of damped oscillator is given by,

$x = Ce^{-kt}\sin(\omega t + \delta)$, where k is damping constant and C = constant

Here, $x = 5e^{-0.25t}\sin\left(\dfrac{\pi}{2}\right)t$ ...(1)

This is similar to equation, $x = Ce^{-kt}\sin\omega t$ [when $\delta = 0$] ...(2)

Comparing equations (1) and (2), we get $\omega = \frac{\pi}{2}$

So, time period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4\text{ s}$

Now, $\frac{dx}{dt} = 5(-0.25)e^{-0.25t}\sin\left(\frac{\pi}{2}\right)t + 5\left(\frac{\pi}{2}\right)e^{-0.25t}\cos\left(\frac{\pi}{2}\right)t$

[from equation (1)]

i For $t = \frac{T}{4}$ i.e. $t = \frac{4}{4} = 1\text{ s}$

$$\text{Velocity} = \frac{dx}{dt} = -1.25e^{-0.25}\sin\left(\frac{\pi}{2}\right) + \left(\frac{5\pi}{2}\right)e^{-0.25}\cos\left(\frac{\pi}{2}\right)$$

$$= -1.25e^{-0.25} = -1.25 \times (0.368)^{0.25}$$

$$= -1.25 \times 0.779 = -0.974\text{ m}\cdot\text{s}^{-1}$$

So, the velocity is in opposite direction.

ii For $t = T = 4\text{ s}$

$$\text{Velocity} = \frac{dx}{dt} = -(1.25)e^{-0.25\times 4}\sin\left(\frac{\pi}{2}\right)4 + \left(\frac{5\pi}{2}\right)e^{-0.25\times 4}\cos\left(\frac{\pi}{2}\right)4$$

$$= \left(\frac{5\pi}{2}\right)e^{-1} = 2.89\text{ m}\cdot\text{s}^{-1}$$

Problem 3

A body of mass 10 g is acted upon by a restoring force per unit displacement of 10^7 dyn · cm^{-1}, a frictional force per unit velocity of 4×10^3 dyn · cm^{-1} · s and a driving force of $10^5 \cos pt$ dyn. Find the value of maximum amplitude.

Solution Here the mass of the body = 10g, the driving force, $F\cos pt = 10^5 \cos pt$

$\therefore$ $F = 10^5$ dyne and acceleration $f = \frac{F}{m}$

The amplitude of forced vibration is given by,

$$A = \frac{\frac{F}{m}}{\sqrt{(\omega^2 - p^2)^2 + 4k^2p^2}},$$

So, when $p = \omega$, $A = A_{max} = \frac{\frac{F}{m}}{2kp}$

Now, restoring force constant $a = 10^7$ dyn · cm^{-1}

Here, $p = \omega = \sqrt{\frac{a}{m}} = \sqrt{\frac{10^7}{10}} = 10^3\text{ s}^{-1}$

Again, $k = \frac{b}{2m} = \frac{4\times 10^3}{2\times 10}$ or, $2k = 4\times 10^2\text{ s}^{-1}$

$$\therefore \quad A_{max} = \frac{\frac{10^5}{10}}{4\times 10^2 \times 10^3} = \frac{1}{40} = 0.025\text{ cm}$$

Problem 4

In one-dimensional motion of a mass of 10 g, it is acted upon by a restoring force per unit displacement of $10\text{ dyn}\cdot\text{cm}^{-1}$ and a resisting force per unit velocity of $2\text{ dyn}\cdot\text{cm}^{-1}\cdot\text{s}$.

i **Find whether the motion is aperiodic or oscillatory.**

ii **Find the value of the resisting force which will make the motion critically damped.**

iii **Find the value of mass for which the given forces will make the motion critically damped.** **[C.U. 1990]**

Solution Here, $a = \dfrac{\text{restoring force}}{\text{displacement}} = 10\text{ dyn}\cdot\text{cm}^{-1}$

$$b = \frac{\text{frictional force}}{\text{velocity}} = 2\text{ dyn}\cdot\text{cm}^{-1}\cdot\text{s}$$

$$\text{mass} = m = 10\text{ g}$$

i $\therefore \quad k = \dfrac{b}{2m} = \dfrac{2}{20} = 0.1\text{ s}^{-1}$

$$\omega = \sqrt{\frac{a}{m}} = \sqrt{\frac{10}{10}} = 1\text{ cycles}\cdot\text{s}^{-1}$$

$\because \quad \omega > k$, the motion is oscillatory.

ii For critical damping, $\omega = k$

or, $\sqrt{\dfrac{a}{m}} = \dfrac{b}{2m}$ or, $\dfrac{b^2}{4m^2} = \dfrac{a}{m}$

or, $b = \sqrt{4ma} = \sqrt{4\times 10\times 10} = 20\text{ dyn}\cdot\text{cm}^{-1}\cdot\text{s}$

iii From case (ii), we can write for critical damping corresponding to $b = 20\text{ dyn}\cdot\text{cm}^{-1}\cdot\text{s}$, corresponding mass $m = \dfrac{b^2}{4a} = \dfrac{400}{4\times 10} = 10\text{ g}$

Problem 5

If a harmonic oscillator of mass m and natural frequency ω_0 is driven by a force $F_0 \sin\omega t$ and the damping is proportional to 2μ times the velocity of the oscillator then the displacement is given by

$$x = \frac{\frac{F_0}{m}}{\sqrt{(\omega_0^2-\omega^2)^2+4\mu^2\omega^2}}\sin(\omega t-\theta) \quad \text{where } \tan\theta = \frac{2\mu\omega}{\omega_0^2-\omega^2}$$

Using the above relation (you need not deduce the relation) show that at velocity resonance velocity is in phase with the driving force. **[W.B.U.T. 2007]**

Solution $x = \dfrac{\frac{F_0}{m}}{\sqrt{(\omega_0^2-\omega^2)^2+4\mu^2\omega^2}}\sin(\omega t-\theta)$

$= A\sin(\omega t-\theta)$, where $A = \dfrac{\frac{F_0}{m}}{\sqrt{(\omega_0^2-\omega^2)^2+4\mu^2\omega^2}}$

Now velocity $v = \frac{dx}{dt} = A\omega\cos(\omega t - \theta)$

At velocity resonance $\omega_0 = \omega$

$\therefore$ Phase $\theta = \tan^{-1}\frac{2\mu\omega}{0} = \tan^{-1}\infty = 90°$ [since $\tan\alpha = \tan 90°$]

$\therefore \quad v = A\omega\cos(\omega t - 90°) = A\omega\sin\omega t$

But the driving force $F = F_0\sin\omega t$.

$\therefore$ At velocity resonance velocity is in phase with driving force.

Problem 6

Find out the value of the driving frequency at which the voltage across the capacitor is maximum. **[W.B.U.T. 2007]**

Solution $V_C = \frac{q}{C} = \frac{E_0 e^{i(pt-\phi)}}{pC\sqrt{R^2 + \left(pL - \frac{1}{pC}\right)^2}}$ [see equation 2.64]

If $pL = \frac{1}{pC}$, the voltage across the capacitor is maximum and so the driving frequency will be $p = \frac{1}{\sqrt{LC}}$

Exercise

Multiple Choice Questions

1. If k is the damping constant (in damped vibration) and ω is the restoring constant, then the frequency for forced vibration is—

Ⓐ $\frac{1}{2\pi}\sqrt{\omega^2 - 2k^2}$ Ⓑ $\frac{1}{2\pi}\sqrt{\omega^2 - k^2}$ Ⓒ $\frac{\omega}{2\pi}$ **Ans.** Ⓑ

2. If the logarithmic decrement (λ) is defined as the log of the ratio of the amplitudes half cycle apart, then it is related to the damping constant k and time period T as—

Ⓐ $\lambda = kT$ Ⓑ $\lambda = \frac{kT}{2}$ Ⓒ $\lambda = \sqrt{kT}$ **Ans.** Ⓑ

3. If a is the force constant of an oscillating body of mass m, the Q-factor is—

Ⓐ $Q = \tau\sqrt{am}$ Ⓑ $Q = \tau\sqrt{\frac{m}{a}}$ Ⓒ $\tau\sqrt{\frac{a}{m}}$ **[W.B.U.T. 2009]**

where τ = relaxation time **Ans.** Ⓒ

4. If the natural frequency of a body is ω_0 and the damping constant is k, then the quality factor is defined as—

Ⓐ $Q = \frac{\omega_0}{k}$ Ⓑ $Q = \frac{\omega_0}{2k}$ Ⓒ $Q = \frac{2k}{\omega_0}$ **Ans.** Ⓑ

5. The maximum amplitude in forced vibration is—

Ⓐ $\frac{f}{2k\sqrt{p^2+k^2}}$ Ⓑ $\frac{f}{\sqrt{p^2+k^2}}$ Ⓒ $\frac{f}{2k\sqrt{p^2-k^2}}$ **Ans.** Ⓐ

6. When the frequency (p) of applied force and the natural frequency of vibration (ω) are equal then the value of maximum amplitude is—

Ⓐ $\frac{f}{\sqrt{2kp}}$ Ⓑ $\frac{f}{2kp}$ Ⓒ $\frac{f}{2k}$ **Ans.** Ⓑ

7. If the damping constant k tends to zero, then the maximum amplitude at resonance—

Ⓐ tends to infinity Ⓑ tends to one Ⓒ tends to $\frac{1}{\omega}$ **Ans.** Ⓐ

8. For velocity resonance, the frequency of applied force (p) and that of natural frequency (ω) are equal and in this case the maximum velocity amplitude is equal to—

Ⓐ $V_{max} = \frac{f}{2\omega}$ Ⓑ $V_{max} = \frac{f}{k}$ Ⓒ $V_{max} = \frac{f}{2k}$ $\left[\text{where } f = \frac{F_0}{m}\right]$

Ans. Ⓒ

9. When the frequency (p) of applied force equals to zero then the velocity amplitude V equals to—

Ⓐ 0 Ⓑ 1 Ⓒ $\frac{1}{2}$ **Ans.** Ⓐ

10. At velocity resonance, when the damping force constant k equals to zero then at $p = \omega$, velocity amplitude V equals to—

Ⓐ ∞ Ⓑ 0 Ⓒ 1 **Ans.** Ⓐ

11. Maximum power intake by an oscillator is observed at—

Ⓐ velocity resonance

Ⓑ amplitude resonance

Ⓒ none of these

Ans. Ⓐ

12. The amount of power supplied to a system is equal to the rate of dissipation of energy in—

Ⓐ forced vibration Ⓑ damped vibration Ⓒ none of these **Ans.** Ⓐ

13. For heavy damping the motion is—

Ⓐ oscillatory Ⓑ non-oscillatory Ⓒ none of these **Ans.** Ⓑ

14. For critical damping, the damping constant is equal to the restoring constant and in this case the motion will be—

Ⓐ oscillatory Ⓑ non-oscillatory Ⓒ none of these **Ans.** Ⓑ

15. For light damping, the damping constant is less than the restoring constant and for this case the motion will be—

Ⓐ non-oscillatory Ⓑ oscillatory Ⓒ harmonic **Ans.** Ⓑ

16. For forced vibration the amplitude decays—

Ⓐ exponentially Ⓑ linearly Ⓒ none of these **Ans.** Ⓐ

17. At the relaxation time (τ) the energy dissipates to—

Ⓐ 0.37% of the initial value

Ⓑ 0.63% of the initial value

Ⓒ remains same **Ans.** Ⓐ

18. The relaxation time (τ) is related to damping constant (k) as—

Ⓐ $\tau = \frac{1}{k}$ Ⓑ $\tau = k$ Ⓒ $\tau = \frac{1}{2k}$ **Ans.** Ⓒ

19. If ω is the restoring constant and relaxation time is τ, then the Q-factor is defined in terms of ω and τ as—

Ⓐ $Q = \omega\tau$ Ⓑ $Q = \frac{\omega}{\tau}$ Ⓒ none **Ans.** Ⓐ

20. For large value of restoring constants, the Q-factor—

Ⓐ decreases Ⓑ increases Ⓒ remains same **Ans.** Ⓑ

21. For small value of damping constant k, the Q-factor—

Ⓐ increases Ⓑ decreases Ⓒ remains same **Ans.** Ⓐ

22. For large value of damping constant k, the Q-factor—

Ⓐ increases Ⓑ decreases Ⓒ remains same **Ans.** Ⓑ

23. For small value of restoring constant, the Q-factor—

Ⓐ decreases Ⓑ remains same Ⓒ increases **Ans.** Ⓐ

24. The displacement (x) in acoustic circuit is related to—

Ⓐ charge Ⓑ inductance Ⓒ capacitance

in electric circuit **Ans.** Ⓐ

25. The inductance (L) in electrical circuit is equivalent to—

Ⓐ mass m Ⓑ displacement x Ⓒ velocity v

in acoustic circuit **Ans.** Ⓐ

26. The restoring force per unit length (l) of an acoustic circuit is equivalent to—

Ⓐ $\frac{1}{C}$ Ⓑ C Ⓒ LC

where C = capacitance, L = inductance **Ans.** Ⓐ

27. For large value of damping constant k the resonance curve will be—

Ⓐ flatter Ⓑ sharper Ⓒ unchanged **Ans.** Ⓐ

28. If the frequency of the applied periodic force is p, the damping constant is k and the restoring constant is ω, then for amplitude resonance—

Ⓐ $\omega = p$ Ⓑ $p = \sqrt{\omega^2 - k^2}$ Ⓒ $p = \sqrt{\omega^2 - 2k^2}$ **Ans.** Ⓒ

29. If the frequency of the applied periodic force is p, the damping constant is k and the restoring constant ω is then for velocity resonance—

Ⓐ $\omega = p$ Ⓑ $p = \sqrt{\omega^2 - 2k^2}$ Ⓒ $p = \sqrt{\omega^2 + k^2}$ **Ans.** Ⓐ

30. For low damping, when the frequency of applied periodic force is equal to the natural angular frequency of the body then the phase difference between the displacement and applied force is given by—

Ⓐ π Ⓑ 2π Ⓒ $\frac{\pi}{2}$ **Ans.** Ⓒ

31. The difference in phase between the driver and driven system at velocity resonance is—

Ⓐ $\frac{\pi}{2}$ Ⓑ 0 Ⓒ 2π **Ans.** Ⓐ

32. The difference in phase between the driven system and driver at amplitude resonance is—

Ⓐ 0 Ⓑ 2π Ⓒ $\frac{\pi}{2}$ **Ans.** Ⓒ

33. Example of weakly damped harmonic oscillator is—

Ⓐ dead-beat galvanometer
Ⓑ tangent galvanometer
Ⓒ ballistic galvanometer **Ans.** Ⓒ

[W.B.U.T. 2007]

34. In damped vibration the resisting force or retarding force is proportional to—

Ⓐ displacement
Ⓑ momentum
Ⓒ instantaneous velocity **Ans.** Ⓒ

Short Answer Type Questions

1. Define damped vibration. Write down the differential equation for damped vibratory motion explaining the physical significance of each term in the equation. [See Article 2.3] **[W.B.U.T. 2005]**
2. Obtain the condition of amplitude resonance in a forced vibration subject to a damping force proportional to the velocity. [See Article 2.7.1] **[C.U.(Hons) 2001]**
3. Write down the differential equation for damped vibration and solve the equation for critical damping. [See Article 2.3 and 2.3.1 : Case 2]
4. What is forced vibration? Write down the differential equation for forced vibration and explain each of the terms appearing in the equation.

[See Article 2.6]

5. Derive the condition of velocity resonance for forced vibrational differential equation.
[See Article 2.7.2B]

6. What is sharpness of resonance? Obtain an expression for the measurement of sharpness of resonance.
[See Article 2.7.2A]

7. Discuss the analogy between acoustic and electric system. [See Article 2.9]

8. Give the relation between mechanical and electrical impedance.
[See Article 2.9.1]

9. What is resonance? Distinguish between amplitude and velocity resonance. What is the power factor at velocity resonance?
[B.U.(H) 2002, C.U. (Hons) '92, '94, '99]

Hint: If the applied force is $F = F_0 e^{ipt} = F_0 \cos pt$ (say), then the instantaneous power,

$$P = \text{instantaneous force} \times \text{velocity}$$

$$= F_0 \cos pt \times \frac{d}{dt}[A \sin(pt - \phi)]$$

Since, $x = A\sin(pt-\phi)$ and $A = \dfrac{F_0}{m\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}}$

$$= F_0 \cos pt \times Ap\cos(pt - \phi)$$

$$= F_0 V\left[\cos^2 pt \cos\phi + \frac{1}{2}\sin 2pt \sin\phi\right] \text{ where, } V = Ap$$

The average of $\cos^2 pt$ for a complete cycle (T) is $\frac{1}{2}$, while that of $\sin 2pt$ is zero. Hence the average power,

$$\overline{P} = \frac{F_0 V}{2}\cos\phi = \frac{F_0}{\sqrt{2}} \times \frac{V}{\sqrt{2}}\cos\phi$$

$$= \text{rms force} \times \text{rms velocity} \times \cos\phi$$

where $\cos\phi$ is called the power factor.

Long Answer Type Questions

1. [a] Establish the differential equation of damped harmonic motion.
[See Article 2.3] **[W.B.U.T. 2005, 2009]**

[b] Solve the equation for light damping and prove that the amplitude of vibration decreases exponentially with time. [See Article 2.3] **[W.B.U.T. 2009]**

[c] Solve the equation for underdamped motion and prove that the amplitude of vibration decreases exponentially with time.
[See Article 2.3.1 : case 3] **[W.B.U.T. 2005]**

[d] Give a graphical comparison among the following four types of harmonic motion.

(i) simple harmonic motion (ii) underdamped harmonic motion (iii) overdamped harmonic motion (iv) critically damped harmonic motion. [See Article 2.3] **[W.B.U.T. 2005, 2009]**

2. [a] Write down the differential equation for damped harmonic vibration. Solve the equation for overdamped and critically damped motion.
[See Article 2.3 and Article 2.3.1 case 1 and 2]

[b] What are the similarities and dissimilarities of critically damped and over-damped motion? [See Article 2.3.1 : case 1 and case 2 and Fig 2]

[c] Show that the ratio of successive amplitudes of damped oscillatory (or underdamped motion) is constant.
[See Article 2.4 logarithmic decrement]

3. [a] Write short notes on logarithmic decrement, relaxation time and quality factor. [See Article 2.4]

[b] Prove that the rate of loss of energy in one time period is $\frac{bE}{m}$, where b is damping coefficient *i.e.* damping force per unit velocity and E is the energy of a damped oscillatory motion of a particle of mass m. [See Article 2.5]

4. State the algebraic relation by which the displacement is related to time in case of a damped harmonic motion. Derive the relation between the damping constant and logarithmic decrement. Derive the relation between the first throw of a ballistic galvanometer with the amplitude of underdamped oscillation in terms of the logarithmic decrement.
[See Article 2.3.1 and Article 2.4 : case 1] **[W.B.U.T. 2006]**

5. [a] Define forced vibration. Give the theory of forced vibration. [See Article 2.6]

[b] In steady state forced vibration describe how the phase of driven system changes with frequency of the driving force. [See Article 2.6]

[b] Deduce the condition for velocity resonance and explain sharpness of velocity resonance. [See Article 2.7.2B]

6. [a] Distinguish between free and forced vibration.

[b] Write down the equation of forced vibration and solve the equation of motion. Explain what you understand by Q-factor.
[See Article 2.6 and Article 2.4] **[B.U.(H) '98, 2004, 2005, C.U. (Hons) '98]**

[c] Establish the condition for amplitude resonance and explain the sharpness of amplitude resonance. [See Article 2.7.1 and 2.7.2 A] **[B.U.(H) 2005]**

[d] Distinguish between amplitude and velocity resonsance. **[C.U.(H) '96]**

7. [a] What is resonance? Distinguish between amplitude resonance and velocity resonance. [See Article 2.7] **[C.U. (Hons) '99]**

[b] Draw the variation of velocity amplitude with the applied frequency p for different damping constant. [See Article 2.7.2B Fig. 8]

[c] Prove that the maximum power intake by the oscillator is observed at velocity resonance.

[See Article 2.8]

8. [a] Define sharpness of resonance. [See Article 2.7.2A] **[W.B.U.T. 2003]**

[b] Find the energy dissipated per unit time per unit mass for forced vibration.

[See Article 2.8] **[W.B.U.T. 2003]**

or, Calculate the average power spend by the driving force.

[See Article 2.8] **[C.U. (Hons) '98]**

or, Show that in forced vibration, supplied power is equal to the rate of dissipation of energy. [See Article 2.8] **[C.U.(H) '96, B.U.(H) 2005]**

9. Write down the differential equation of a series LCR circuit driven by a sinusoidal voltage. Identify the natural frequency of the circuit. Find out the condition that this circuit will show an oscillatory decay and find out the relaxation time.

[W.B.U.T. 2007]

> **Hint:** The differential equation is $L\cdot\dfrac{d^2q}{dt^2}+R\cdot\dfrac{dq}{dt}+\dfrac{q}{C}=E_0e^{ipt}$
>
> **Natural frequency:** $\dfrac{1}{\sqrt{LC}}$
>
> **For oscillatory decay:** $k<\omega$ *i.e.,* $\dfrac{R}{2L}<\dfrac{1}{\sqrt{LC}}$
>
> **Relaxation time:** $\tau=\dfrac{1}{2k}=\dfrac{L}{R}$

Numerical Problems

1. The amplitude of an oscillator of frequency 200 cycles $\cdot\, s^{-1}$ falls to $\frac{1}{10}$ of its initial value after 2000 cycles. Calculate *(i)* its relaxation time *(ii)* its quality factor *(iii)* time in which its energy falls to $\frac{1}{10}$ of its initial value *(iv)* damping constant. **[Rajasthan university 1986]**

[Ans : *(i)* $\tau = 2.17\text{s}$, *(ii)* $Q = 435$, *(iii)* $t = 5\text{s}$ *(iv)* $k = 0.23\ s^{-1}$

2. A mass of 10g is acted on by a restoring force 5 dyne $\cdot$ cm^{-1} and resisting force 2 dyne $\cdot$ cm^{-1} $\cdot$ s. Find whether the motion is aperiodic or oscillatory. Find the value of resisting force for which the motion is critically damped. **[C.U 1999]**

[Ans : *(i)* motion is oscillatory *(ii)* resistive force/velocity = $b = 10\sqrt{2}$ dyne $\cdot$ cm^{-1} $\cdot$ s]

[**Hint:**

$\therefore$ *(i)* $k\left(=\dfrac{b}{2m}\right)<\omega\left(=\sqrt{\dfrac{a}{m}}\right) \Rightarrow$ motion is oscillatory

(ii) For critical damping, $k=\omega$ or, $b=2\sqrt{ma}$]

3. A vibrator of mass 1 g is acted upon by a restoring force per unit displacement of 10^4 N $\cdot$ m^{-1}, retarding force per unit velocity of 4N $\cdot$ m^{-1} $\cdot$ s and a driving

force of $\cos\omega t$ N. Find the value of maximum possible amplitude. **[C.U. 2001]**
[Ans. 1.02×10^{-4}m]

[**Hint:** At resonant frequency $p(=p_r) = \sqrt{\omega^2 - 2k^2}$,

the maximum amplitude, $A_{max} = \dfrac{f}{2k\sqrt{\omega^2 - k^2}}$ [from eqn. 2.47]

Here, $f = \dfrac{F_0}{m} = \dfrac{1}{10^{-3}} = 10^3 \text{N} \cdot \text{kg}^{-1}$; $\omega^2 = \dfrac{a}{m} = \dfrac{10^4}{10^{-3}}$ $(\text{rad} \cdot \text{s}^{-1})^2$ and damping constant $k = \dfrac{b}{2m} = \dfrac{4}{2 \times 10^{-3}} = 2 \times 10^3 \text{N} \cdot \text{kg}^{-1} \cdot \text{m}^{-1} \cdot \text{s}$]

4. Two forced harmonic oscillations have same displacement amplitudes at the frequencies $\omega_1 = 400 \text{ rad} \cdot \text{s}^{-1}$ and $\omega_2 = 800 \text{ rad} \cdot \text{s}^{-1}$. Calculate the resonant frequency at which the displacement is maximum.

[BPUT (2nd sems) 2007] [Ans : $200\sqrt{10}\text{s}^{-1}$]

[**Hint:** $x = \dfrac{f\sin(pt - \phi)}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}}$

$\because$ The amplitudes at $p = \omega_1$ and $p = \omega_2$ are same.

$\therefore\ 4k^2\omega_1^2 + (\omega^2 - \omega_1^2)^2 = 4k^2\omega_2^2 + (\omega^2 - \omega_2^2)^2$

or, $4k^2 = 2\omega^2 - \omega_1^2 - \omega_2^2$

Now resonant frequency $= \sqrt{\omega^2 - 2k^2} = \sqrt{\dfrac{\omega_1^2 + \omega_2^2}{2}}$]

5. Evaluate the Q-factor of a damped oscillator with resonant frequency 500Hz and damping co-efficient 0.5 per second. [Ans : 1000π]

6. A mass of 0.1 kg is suspended by a light spring of spring constant $120 \text{N} \cdot \text{m}^{-1}$. If this spring oscillates under a damping force of damping coefficient $2\text{N} \cdot \text{m}^{-1} \cdot \text{s}$ and a driving force of $F = 4\cos 50t$ N find the amplitude of oscillation and phase relative to the driving force in steady state.

[Ans : $A = 0.028$m $\delta = 37.56°$]

[**Hint:** $A = \dfrac{f}{\sqrt{4k^2p^2 + (\omega^2 - p^2)^2}}$ where $\omega^2 = \dfrac{a}{m} = \dfrac{120}{0.1} = 1200$

$$2k = \frac{b}{m} = \frac{2}{0.1} = 20$$

$$p = 50$$

$$f = \frac{F_0}{m} = \frac{4}{0.1} = 40$$

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$$\tan\delta = \frac{2kp}{\omega^2 - p^2} = \frac{20 \times 50}{1200 - 2500} = -0.7692 \text{ or, } \delta = 37.56°$$

∴ The oscillator lags behind the driving force by an angle 37.56°]

7. A ballistic galvanometer has a time period 2s in its underdamped condition.

8. When a transient current passes through it, the first deflection on the lamp and scale arrangement is 30cm. After 5 complete oscillations, the deflection reduces to 8cm. Calculate the logarithmic decrement. [Ans : $\lambda = 1.321$]

[**Hint :** For first deflection at time $t = \frac{T}{4}$, amplitude

$$\therefore A_1 = A_0 e^{-k\frac{T}{4}}, \; T = 2$$

After 5 oscillations, the amplitude number is 6

$$\therefore A_6 = A_0 e^{-k\left(5T + \frac{T}{4}\right)} \quad \therefore \frac{A_1}{A_6} = \frac{30}{8} = e^{5kT} = e^{10k} \;\; \therefore k = 0.132$$

logarithmic decrement : $\lambda = \frac{kT}{2} = \frac{0.132 \times 2}{2}$]

7. A particle of mass 2 g is subjected to an elastic force per unit displacement $0.03 \text{ N} \cdot \text{m}^{-1}$ and frictional force per unit velocity $0.005 \text{ N} \cdot \text{m}^{-1} \cdot \text{s}$. If it is displaced through 2 cm and then released, find whether the resulting motion is oscillatory or not. If so, find its period.

[**Hint :** Damping coefficient = frictional force per unit velocity
$0.005 \text{ N} \cdot \text{m}^{-1} \cdot \text{s}$
Restoring coefficient = restoring force per unit displacement
$0.03 \text{ N} \cdot \text{m}^{-1}$, mass $m = 2$ g

∴ Damping constant $k = \frac{b}{2m}$

Restoring constant $\omega = \sqrt{\frac{a}{m}}$

If $\omega > k$ then the motion will be oscillatory and $T = \frac{2\pi}{\sqrt{\omega^2 - k^2}}$]