Seven

Multiple Integrals

CHAPTER OUTLINE

- Introduction
- Double Integrals
- Change of Order of Integration
- Double Integrals in Polar Coordinates
- Change of Variables of Integrals
- Area Enclosed by Plane Curves Using Double Integrals
- Area of a Curved Surface Using Double Integrals
- Triple Integrals
- Volume of Solids Using Triple Integrals

7.1 INTRODUCTION

Integration of functions of two or more variables is normally called multiple integration. The particular case of integration of functions of two variables is called *double integration* and that of three variables is called *triple integration*. Sometimes, the variables have to be changed to simplify the integrand while evaluating the multiple integrals. Variables can be changed by substitution or by changing the coordinate system (polar, spherical or cylindrical coordinates). Multiple integrals are useful in evaluating plane area, mass and volume of solid regions, etc.

7.2 DOUBLE INTEGRALS

Let $f(x, y)$ be a continuous function defined in a closed and bounded region R in the xy-plane (Fig. 7.1). Divide the region R into small elementary rectangles by drawing lines parallel to coordinate axes. Let the total number of complete rectangles which lie inside the region R be n. Let δA_r be the area of r^{th} rectangle and (x_r, y_r) be any point in this rectangle.

Consider the sum $$S = \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r \qquad \text{...(7.1)}$$

where $\delta A_r = \delta x_r \cdot \delta y_r$.

If the number of elementary rectangles is increased then the area of each rectangle decreases. Hence, as $n \to \infty$, $\delta A_r \to 0$. The limit of the sum given by Eq. (7.1), if it exists, is called the double integral of $f(x, y)$ over the region R and is denoted by $\iint_R f(x, y)\mathrm{d}A$.

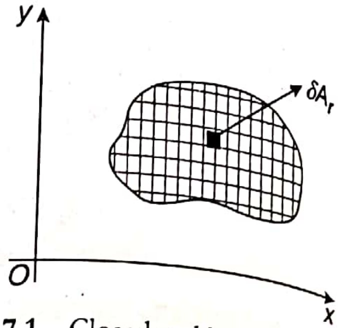

Fig. 7.1 Closed and bounded region R

Hence, $$\iint_R f(x, y)\mathrm{d}A = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$

7.2.1 Evaluation of Double Integration

Double integral of a function $f(x, y)$ over a region R can be evaluated by two successive integrations. There are two different methods to evaluate a double integral.

Method I Let the region R, i.e., $PQRS$ be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$ (Fig. 7.2).

In the region $PQRS$, draw a vertical strip AB. Along the strip AB, y varies from y_1 to y_2 and x is fixed. Therefore, the double integral is integrated first w.r.t. y between the limits y_1 and y_2 treating x as constant.

Now, move the strip AB horizontally from PS (i.e., $x = a$) to QR (i.e., $x = b$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. x between the limits a and b. Hence,

$$\iint_R f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\mathrm{d}y\right]\mathrm{d}x$$

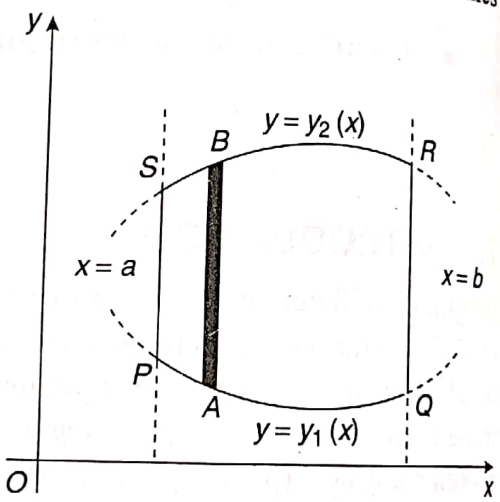

Fig. 7.2 Illustration of Method I

Method II Let the region R, i.e., $PQRS$ be bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and the lines $y = c$, $y = d$ (Fig. 7.3).

In the region $PQRS$, draw a horizontal strip AB. Along the strip AB, x varies from x_1 to x_2 and y is fixed. Therefore, the double integral is integrated first w.r.t. x between the limits x_1 and x_2 treating y as constant.

Now, move the strip AB vertically from PQ (i.e., $y = c$) to RS (i.e., $y = d$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. y between the limits c and d.

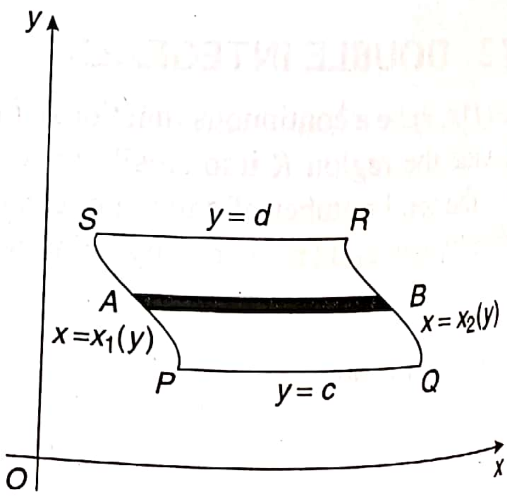

Fig. 7.3 Illustration of Method II

Hence, $$\iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right]dy$$

Notes

(i) If all the four limits are constant then the function $f(x, y)$ can be integrated w.r.t. any variable first. But if $f(x, y)$ is implicit and is discontinuous within or on the boundary of the region of integration then the change of the order of integration will affect the result.

(ii) If all the four limits are constant and $f(x, y)$ is explicit then double integral can be written as product of two single integrals.

(iii) If inner limits depend on x then the function $f(x, y)$ is integrated first w.r.t. y and vice versa.

EXAMPLE 7.1 *Evaluate* $\int_2^a \int_2^b \frac{dx\,dy}{xy}$.

Solution:

$$\int_2^a \int_2^b \frac{dx\,dy}{xy} = \int_2^a \left(\int_2^b \frac{dx}{x}\right)\frac{dy}{y} = \int_2^a \left|\log x\right|_2^b \frac{1}{y}dy = (\log b - \log 2)\int_2^a \frac{1}{y}dy$$

$$= \log\left(\frac{b}{2}\right)\cdot\left|\log y\right|_2^a = \log\left(\frac{b}{2}\right)\cdot(\log a - \log 2) = \log\left(\frac{b}{2}\right)\cdot\log\left(\frac{a}{2}\right)$$

Another Method Since both the limits are constant and integrand (function) is explicit in x and y, the integral can be written as

$$\int_2^a \int_2^b \frac{dx\,dy}{xy} = \int_2^a \frac{dy}{y}\cdot\int_2^b \frac{dx}{x} = \left|\log y\right|_2^a\cdot\left|\log x\right|_2^b = (\log a - \log 2)(\log b - \log 2) = \log\left(\frac{a}{2}\right)\cdot\log\left(\frac{b}{2}\right)$$

$$= \log\left(\frac{b}{2}\right)\cdot\log\left(\frac{a}{2}\right)$$

EXAMPLE 7.2 *Evaluate* $\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{1-x^2-y^2}$

Solution:

$$\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \int_0^1 \left[\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx}{\sqrt{(1-y^2)-x^2}}\right]dy = \int_0^1 \left|\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right|_0^{\sqrt{\frac{1-y^2}{2}}} dy$$

$$= \int_0^1 \left(\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0\right)dy = \frac{\pi}{4}\left|y\right|_0^1 = \frac{\pi}{4}$$

EXERCISE 7.1

Evaluate the following integrals:

1. $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2 dx\, dy$ $\left[\textbf{Ans.}: \frac{856}{945}\right]$

2. $\int_0^1 \int_0^y xye^{x-2} dx\, dy$ $\left[\textbf{Ans.}: \frac{1}{4e}\right]$

3. $\int_0^1 \int_0^x e^{x+y} dx dy$ $\left[\textbf{Ans.}: \frac{1}{2}(e-1)^2\right]$

4. $\int_{10}^1 \int_0^{\frac{1}{y}} ye^{xy} dx\, dy$ [**Ans.:** $9(1-e)$]

5. $\int_1^{\log 8} \int_0^{\log y} e^{x+y} dx\, dy$ [**Ans.:** $8(\log 8 - 1)$]

6. $\int_0^1 \int_{y^2}^y (1+xy^2) dx\, dy$ $\left[\textbf{Ans.}: \frac{41}{210}\right]$

7. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} xy\, dy\, dx$ $\left[\textbf{Ans.}: \frac{2a^4}{3}\right]$

7.2.2 Working Rule for Evaluation of Double Integration over a Given Region

1. If the region is bounded by more than one curve then find the points of intersection of all the curves.
2. Draw all the curves and mark their point of intersection.
3. Identify the region of integration.
4. Draw a vertical or horizontal strip in the region whichever makes the integration easier.
5. The vertical strip starts from the lowest part of the region and terminates on the highest part of the region.
6. **For vertical strip**
   (i) The lower limit of y is obtained from the curve, where the vertical strip starts and the upper limit of y is obtained from the curve, where it terminates.
   (ii) The lower limit of x is the x-coordinate of the leftmost point of the region and the upper limit of x is the x-coordinate of the rightmost point of the region.
7. The horizontal strip starts from the left part of the region and terminates on the right part of the region.
8. **For horizontal strip**
   (i) The lower limit of x is obtained from the curve, where the horizontal strip starts and upper limit is obtained from the curve, where it terminates.
   (ii) The lower limit of y is the y-coordinate of the lowest point of the region and the upper limit of y is the y-coordinate of the highest point of the region.
9. If variation along the strip changes within the region then the region is divided into parts.

EXAMPLE 7.3 *Evaluate* $\iint (a-x)^2 \, dx \, dy$, *over the right half of the circle* $x^2+y^2=a^2$.

Solution:

1. The region of integration is PQR (Fig. 7.4).
2. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to the y-axis which starts from the part of the circle $x^2+y^2=a^2$ below x-axis and terminates on the part of the circle $x^2+y^2=a^2$ above the x-axis.

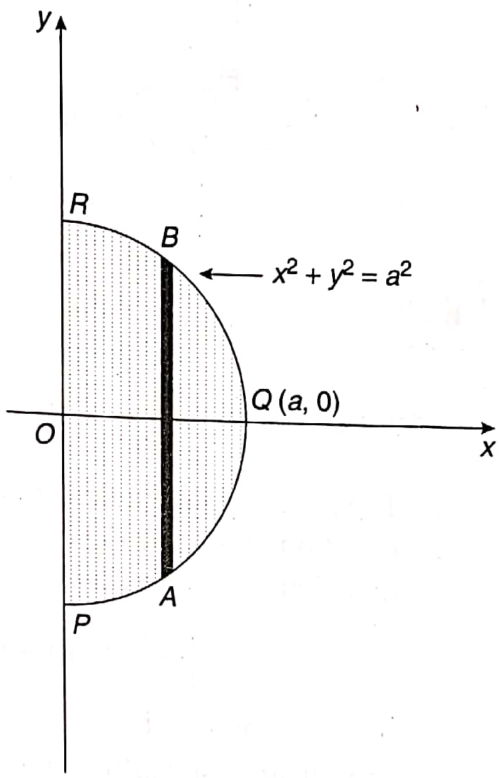

Fig. 7.4

3. Limits of
$y: y=-\sqrt{a^2-x^2}$ to $y=\sqrt{a^2-x^2}$
Limits of $x: x=0$ to $x=a$

$$I=\int_0^a\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(a-x)^2\,dx\,dy$$
$$=\int_0^a(a-x)^2\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dy\,dx$$
$$=\int_0^a(a-x)^2\left|y\right|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$$
$$=\int_0^a(a^2+x^2-2ax)\cdot 2\sqrt{a^2-x^2}\,dx$$
$$=2\int_0^a(a^2+x^2-2ax)\sqrt{a^2-x^2}\,dx$$

Putting $x=a\sin\theta$, $dx=a\cos\theta\,d\theta$

When $x=0$, $\theta=0$

When $x=a$, $\theta=\frac{\pi}{2}$

$$I=2\int_0^{\frac{\pi}{2}}(a^2+a^2\sin^2\theta-2a^2\sin\theta)\cdot a\cos\theta\cdot a\cos\theta\,d\theta$$
$$=2a^4\int_0^{\frac{\pi}{2}}(\cos^2\theta+\sin^2\theta\cos^2\theta-2\sin\theta\cos^2\theta)d\theta$$
$$=a^4\left[B\left(\frac{3}{2},\frac{1}{2}\right)+B\left(\frac{3}{2},\frac{3}{2}\right)-2B\left(1,\frac{3}{2}\right)\right]$$
$$=a^4\left[\frac{\boxed{\frac{3}{2}}\boxed{\frac{1}{2}}}{\boxed{2}}+\frac{\boxed{\frac{3}{2}}\boxed{\frac{3}{2}}}{\boxed{3}}-2\frac{\boxed{1}\boxed{\frac{3}{2}}}{\boxed{\frac{5}{2}}}\right]$$

$$= a^4\left[\frac{\frac{1}{2}\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{1} + \frac{\left(\frac{1}{2}\Gamma\frac{1}{2}\right)^2}{2!} - 2\frac{\Gamma\frac{3}{2}}{\frac{3}{2}\Gamma\frac{3}{2}}\right]$$

$$= a^4\left[\frac{\pi}{2} + \frac{\pi}{8} - \frac{4}{3}\right] = a^4\left[\frac{5\pi}{8} - \frac{4}{3}\right]$$

EXAMPLE 7.4 *Evaluate* $\iint (x^2 - y^2)\,dx\,dy$ *over the triangle with vertices* (0, 1), (1, 1), (1, 2).

Solution:

1. The region of integration is ΔPQR (Fig. 7.5).
2. Equation of the line PQ is $y = 1$.
   Equation of the line PR is
   $$y - 1 = \frac{2-1}{1-0}(x - 0) = x$$
   $$y = x + 1$$
3. The integration can be done w.r.t. any variable first. Draw a vertical strip AB parallel to the y-axis which starts from the line $y = 1$ and terminates on the line $y = x + 1$.
4. Limits of y: $y = 1$ to $y = x + 1$
   Limits of x: $x = 0$ to $x = 1$

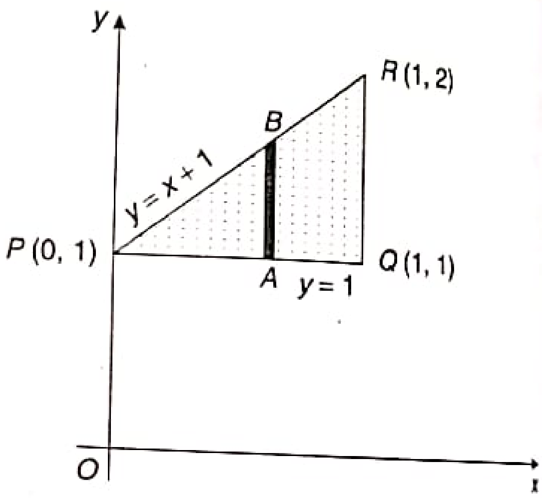

Fig. 7.5

$$I = \int_0^1 \int_1^{x+1} (x^2 - y^2)\,dy\,dx = \int_0^1 \left| x^2 y - \frac{y^3}{3} \right|_1^{x+1} dx$$

$$= \int_0^1 \left[x^2(x+1) - \frac{(x+1)^3}{3} - x^2 + \frac{1}{3} \right] dx$$

$$= \left| \frac{x^4}{4} + \frac{x^3}{3} - \frac{(x+1)^4}{12} - \frac{x^3}{3} + \frac{x}{3} \right|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{16}{12} + \frac{1}{12} = -\frac{2}{3}$$

EXAMPLE 7.5 *Evaluate* $\iint (x^2 + y^2)\,dx\,dy$, *over the region bounded by the lines* $y = 4x$, $x + y = 3$, $y = 0$, $y = 2$.

Solution:

1. The region of integration is $OPQR$ (Fig. 7.6).
2. The integration can be done w.r.t. any variable first. But in case of a vertical strip, the region is divided into three parts. Therefore, draw a horizontal strip AB parallel to the x-axis which starts from the line $y = 4x$ and terminates on the line $x + y = 3$.

3. Limits of $x: x = \frac{y}{4}$ to $x = 3 - y$

Limits of $y: y = 0$ to $y = 2$

$$I = \int_0^2 \int_{\frac{y}{4}}^{3-y} (x^2 + y^2) \, dx \, dy$$

$$= \int_0^2 \left| \frac{x^3}{3} + xy^2 \right|_{\frac{y}{4}}^{3-y} dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + (3-y)y^2 - \frac{1}{3} \cdot \frac{y^3}{64} - \frac{y^3}{4} \right] dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + 3y^2 - \frac{241}{192} y^3 \right] dy$$

$$= \left| \frac{1}{3} \cdot \frac{(3-y)^4}{-4} + 3 \cdot \frac{y^3}{3} - \frac{241}{192} \cdot \frac{y^4}{4} \right|_0^2$$

$$= -\frac{1}{12} + 8 - \frac{241}{192} \cdot 4 - \left(-\frac{27}{4} \right) = \frac{463}{48}$$

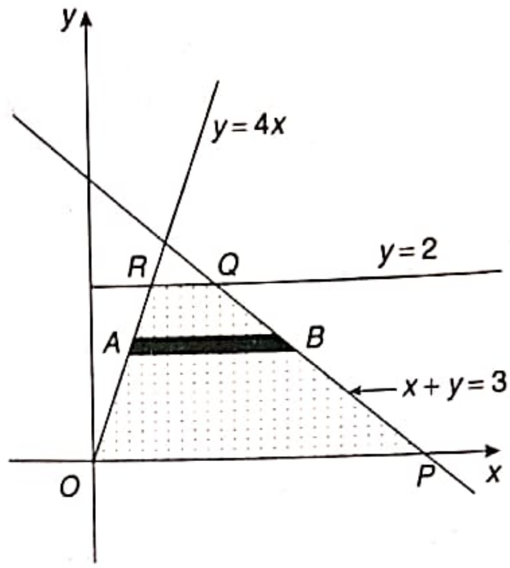

Fig. 7.6

EXAMPLE 7.6 *Evaluate* $\iint xy \, dx \, dy$, *over the region enclosed by the circle* $x^2 + y^2 - 2x = 0$, *the parabola* $y^2 = 2x$ *and the line* $y = x$.

Solution:

1. The region of integration is $OPQRO$ (Fig. 7.7).
2. (i) The points of intersection of the circle $x^2 + y^2 - 2x = 0$ and the line $y = x$ are obtained as

$$x^2 + x^2 - 2x = 0$$
$$x = 0, 1$$
$$\therefore y = 0, 1$$

The points of intersection are $O(0, 0)$ and $P(1, 1)$.

(ii) The point of intersection of the circle $x^2 + y^2 - 2x = 0$ and the parabola $y^2 = 2x$ is obtained as

$$x^2 + 2x - 2x = 0$$
$$x = 0$$
$$\therefore y = 0$$

The point of intersection is $O(0, 0)$.

(iii) The points of intersection of the parabola $y^2 = 2x$ and the line $y = x$ are obtained as $x^2 = 2x$

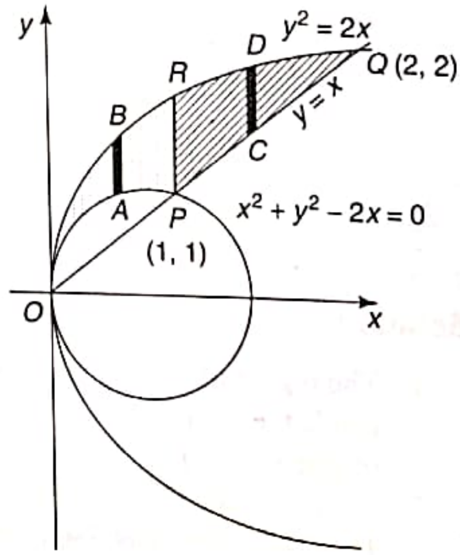

Fig. 7.7

$x = 0, 2$

$\therefore y = 0, 2$

The points of intersection are O (0, 0) and Q (2, 2).

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. y, first a vertical strip is drawn in the region. But one vertical strip does not cover the entire region. Hence, divide the region $OPQRO$ into two subregions OPR and RPQ and draw one vertical strip in each subregion.
4. In the subregion OPR, the strip starts from the circle $x^2 + y^2 - 2x = 0$ and terminates on the parabola $y^2 = 2x$.

   Limits of y: $y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$

   Limits of x: $x = 0$ to $x = 1$
5. In the subregion RPQ, the strip starts from the line $y = x$ and terminates on the parabola $y^2 = 2x$.

   Limits of y: $y = x$ to $y = \sqrt{2x}$

   Limits of x: $x = 1$ to $x = 2$

$$I = \int\int xy\,dx\,dy = \iint_{OPR} xy\,dx\,dy + \iint_{RPQ} xy\,dx\,dy$$

$$= \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,dy\,dx + \int_1^2 \int_x^{\sqrt{2x}} xy\,dy\,dx$$

$$= \int_0^1 x \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\,dy\,dx + \int_1^2 x \int_x^{\sqrt{2x}} y\,dy\,dx$$

$$= \int_0^1 x \left| \frac{y^2}{2} \right|_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x \left| \frac{y^2}{2} \right|_x^{\sqrt{2x}} dx$$

$$= \frac{1}{2} \int_0^1 x(2x - 2x + x^2)\,dx + \frac{1}{2} \int_1^2 x(2x - x^2)\,dx$$

$$= \frac{1}{2} \left| \frac{x^4}{4} \right|_0^1 + \frac{1}{2} \left| \frac{2x^3}{3} - \frac{x^4}{4} \right|_1^2 = \frac{1}{8} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{8} = \frac{7}{12}$$

EXAMPLE 7.7 *Evaluate* $\iint \frac{dx\,dy}{x^4 + y^2}$, *over the region bounded by the* $y \geq x^2$, $x \geq 1$.

Solution:

1. The region of integration is bounded by $y \geq x^2$ (the region inside the parabola $x^2 = y$) and $x \geq 1$ (the region on the right of line $x = 1$) (Fig. 7.8).
2. The point of intersection of $x^2 = y$ and $x = 1$ is obtained as $1 = y$. The point of intersection is $P(1, 1)$.
3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to the y-axis in the region which starts from the parabola $x^2 = y$ and extends up to infinity.
4. Limits of y : $y = x^2$ to $y \to \infty$

   Limits of x : $x = 1$ to $x \to \infty$

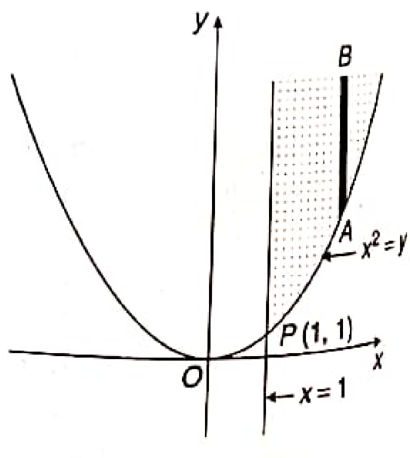

Fig. 7.8

$$I=\int_1^\infty\int_{x^2}^\infty \frac{1}{x^4+y^2}\,dydx=\int_1^\infty \left|\frac{1}{x^2}\tan^{-1}\frac{y}{x^2}\right|_{x^2}^\infty dx=\int_1^\infty \frac{1}{x^2}(\tan^{-1}\infty-\tan^{-1}1)dx$$

$$=\int_1^\infty \frac{1}{x^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)dx=\frac{\pi}{4}\left|-\frac{1}{x}\right|_1^\infty=\frac{\pi}{4}$$

EXERCISE 7.2

Evaluate the following integrals:

1. $\iint \frac{1}{xy}\,dx\,dy$, over the rectangle $1 \le x \le 2$, $1 \le y \le 2$ **[Ans.:** $(\log 2)^2$**]**

2. $\iint \sin\pi(ax+by)\,dx\,dy$, over the triangle bounded by the lines $x=0$, $y=0$ and $ax+by=1$ $\left[\textbf{Ans.: } \frac{1}{\pi ab}\right]$

3. $\iint e^{3x+4y}\,dx\,dy$, over the triangle bounded by the lines $x=0$, $y=0$, and $x+y=1$ $\left[\textbf{Ans.: } \frac{1}{12}(3e^4-4e^3+1)\right]$

4. $\iint xy\sqrt{1-x-y}\,dx\,dy$, over the triangle bounded by $x=0$, $y=0$ and $x+y=1$ $\left[\textbf{Ans.: } \frac{16}{945}\right]$

5. $\iint \sqrt{xy-y^2}\,dx\,dy$, over the triangle having vertices (0, 0), (10, 1), (1, 1) **[Ans.: 6]**

6. $\iint (x+y+a)\,dx\,dy$, over the region bounded by the circle $x^2+y^2=a^2$ **[Ans.:** πa^3**]**

7. $\iint xy\,dx\,dy$, over the region bounded by the x-axis, the line $y=2x$ and the parabola $y=\frac{x^2}{4a}$ $\left[\textbf{Ans.: } \frac{2048}{3}a^4\right]$

8. $\iint 5-2x-y\,(dx)\,dy$, over the region bounded by the x-axis, the line $x+2y=3$ and the parabola $y^2=x$ $\left[\textbf{Ans.: } \frac{217}{60}\right]$

9. $\iint (4x^2-y^2)^{\frac{1}{2}}\,dx\,dy$, over the triangle bounded by the x-axis, the line $y=x$ and $x=1$ $\left[\textbf{Ans.: } \frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)\right]$

10. $\iint xy(x+y)\,dx\,dy$, over the region bounded by the parabola $y^2=x$, $x^2=y$ $\left[\textbf{Ans.: } \frac{3}{28}\right]$

11. $\iint xy(x+y)\,dx\,dy$, over the region bounded by the curve $x^2=y$ and the line $x=y$ $\left[\textbf{Ans.: } \frac{3}{56}\right]$

12. $\iint xy(x-1)\,dx\,dy$, over the region bounded by the rectangular hyperbola $xy=4$, the lines $y=0$, $x=1$, $x=4$ and the x-axis **[Ans.: 8(3 − log 4)]**

7.3 CHANGE OF ORDER OF INTEGRATION

Sometimes, evaluation of double integrals becomes easier by changing the order of integration. To change the order of integration, first the region of integration is drawn with the help of the given limits. Then a vertical or horizontal strip is drawn as per the required order of integration. This change of order also changes the limits of integration.

Type I Change of Order of Integration

EXAMPLE 7.8 *Change the order of integration of* $\int_0^\infty \int_x^\infty f(x, y)\,dx\,dy$.

Solution:

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.

   The correct form of the integral $= \int_0^\infty \int_x^\infty f(x, y)\,dy\,dx$

2. Limits of $y: y = x$ to $y = \infty$, along vertical strip [Fig. 7.9]
   Limits of $x: x = 0$ to $x = \infty$
3. The region is bounded by the lines $y = x$ and $x = 0$.
4. Here, the only point of intersection is the origin O.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to the x-axis which starts from the line $x = 0$ and terminates on the line $y = x$ [Fig. 7.10].
   Limits of x: $x = 0$ to $x = y$
   Limits of y: $y = 0$ to $y = \infty$
   Hence, the given integral after change of order is

$$\int_0^\infty \int_x^\infty f(x, y)\,dy\,dx = \int_0^\infty \int_0^y f(x, y)\,dx\,dy$$

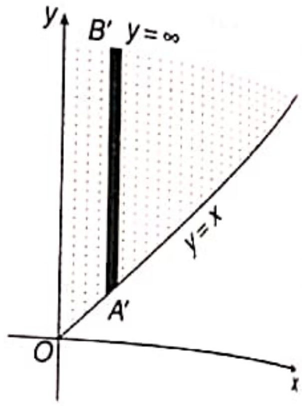

Fig. 7.9

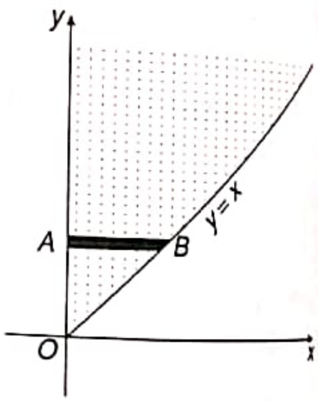

Fig. 7.10

EXAMPLE 7.9 *Change the order of integration of* $\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,dx\,dy$.

Solution:

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x: $x = \frac{y-8}{4}$ to $x = \frac{y}{4}$
   Limits of y: $y = 0$ to $y = 8$

3. The region is bounded by the line $y = 4x + 8$, $y = 4x$, $y = 8$ and the x-axis ($y = 0$) [Fig. 7.11].
4. The point of intersection of $y = 4x$ and $y = 8$ is obtained as $8 = 4x$, i.e., $x = 2$.
   The point of intersection is P (2, 8).
5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region $OPQR$ into two subregions OQR and OPQ. Draw a vertical strip parallel to the y-axis in each subregion [Fig. 7.12].
   (i) In the subregion OQR, the strip AB starts from x-axis and terminates on the line $y = 4x + 8$.
   Limits of $y : y = 0$ to $y = 4x + 8$
   Limits of $x : x = -2$ to $x = 0$
   (ii) In the subregion OPQ, the strip CD starts from the line $y = 4x$ and terminates on the line $y = 8$.
   Limits of $y : y = 4x$ to $y = 8$
   Limits of $x : x = 0$ to $x = 2$

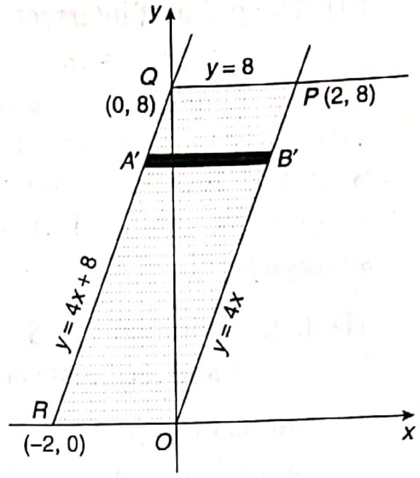

Fig. 7.11

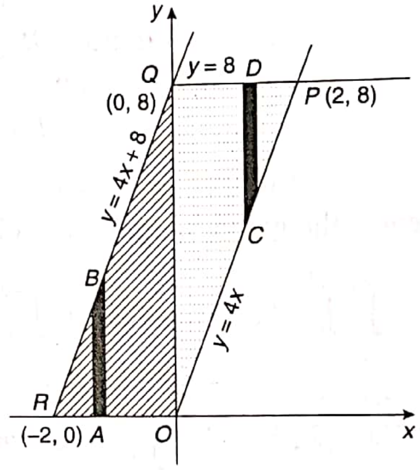

Fig. 7.12

Hence, the given integral after change of order is

$$\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_{-2}^{0}\int_0^{4x+8} f(x, y)\mathrm{d}y\,\mathrm{d}x + \int_0^2\int_{4x}^{8} f(x, y)\mathrm{d}y\,\mathrm{d}x$$

EXAMPLE 7.10 *Change the order of integration of* $\int_{-a}^{a}\int_0^{\frac{y^2}{a}} f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Solution:

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of $x : x = 0$ to $x = \dfrac{y^2}{a}$
   Limits of $y : y = -a$ to $y = a$
3. The region is bounded by the y-axis, the parabola $y^2 = ax$, and the line $y = -a$, and $y = a$ [Fig. 7.13].
4. (i) The point of intersection of $y^2 = ax$ and $y = -a$ is obtained as
   $a^2 = ax$, i.e., $x = a$.
   The point of intersection is R $(a, -a)$.

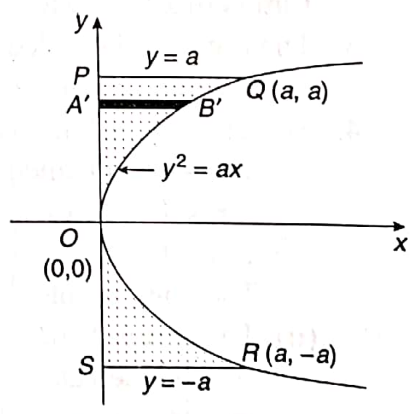

Fig. 7.13

(ii) The point of intersection of $y^2 = ax$ and $y = a$ is obtained as
$a^2 = ax$, i.e., $x = a$.
The point of intersection is $Q\,(a, a)$.

5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions ORS and OPQ. Draw a vertical strip parallel to the y-axis in each subregion [Fig. 7.14].

   (i) In the subregion ORS, the strip AB starts from the line $y = -a$ and terminates on the parabola $y^2 = ax$.

   Limits of y : $y = -a$ to $y = -\sqrt{ax}$ (part of the parabola below x-axis)

   Limits of x : $x = 0$ to $x = a$

   (ii) In the subregion OPQ, the strip CD starts from the parabola $y^2 = ax$ and terminates on the line $y = a$.

   Limits of y : $y = \sqrt{ax}$ to $y = a$

   Limits of x : $x = 0$ to $x = a$

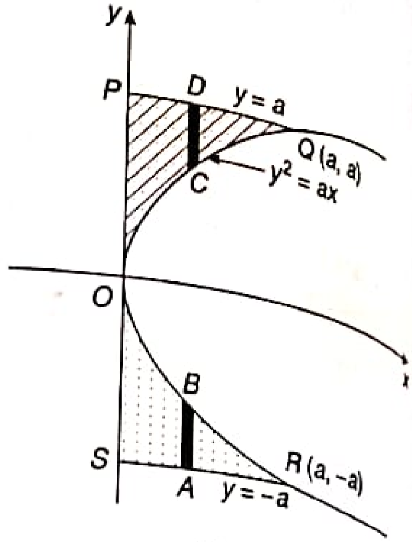

Fig. 7.14

Hence, the given integral after change of order is

$$\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x, y)\,dx\,dy = \int_{0}^{a}\int_{-a}^{-\sqrt{ax}} f(x, y)\,dy\,dx + \int_{0}^{a}\int_{\sqrt{ax}}^{a} f(x, y)\,dy\,dx$$

EXAMPLE 7.11 *Change the order of integration of* $\int_{0}^{4}\int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x, y)\,dy\,dx$.

Solution:

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4x - x^2}$ to $y = \sqrt{4x}$
   Limits of x : $x = 0$ to $x = 4$.
3. The region is bounded by the circle $x^2 + y^2 - 4x = 0$, the parabola $y^2 = 4x$ and the line $x = 4$ [Fig. 7.15].
4. (i) The point of intersection of $x^2 + y^2 - 4x = 0$ and $y^2 = 4x$ is obtained as
   $x^2 = 0$, i.e., $x = 0$.
   $\therefore\ y = 0$
   The point of intersection is $O\,(0, 0)$.

   (ii) The points of intersection of $y^2 = 4x$ and $x = 4$ are obtained as
   $y^2 = 16$, i.e., $y = \pm 4$.
   The points of intersection are $Q\,(4, 4)$ and $Q'(4, -4)$.

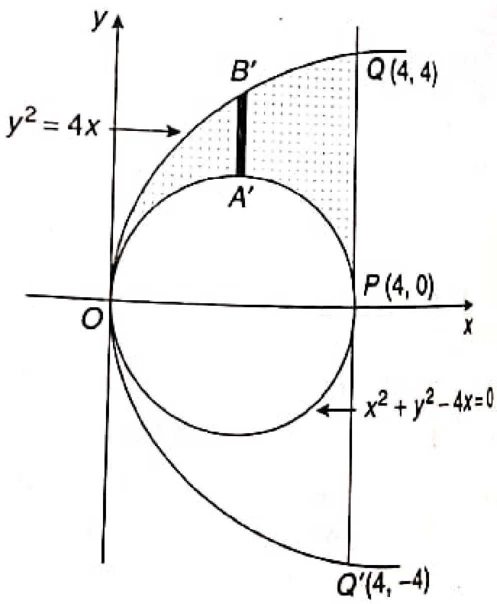

Fig. 7.15

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions ORT, TPS and RSQ. Draw a horizontal strip parallel to the x-axis in each subregion [Fig. 7.16].

(i) In the subregion ORT, the strip AB starts from the parabola $y^2 = 4x$ and terminates on the circle $x^2 + y^2 - 4x = 0$.

Limits of x : $x = \frac{y^2}{4}$ to $x = 2 - \sqrt{4 - y^2}$

(Part of the circle, where $x < 2$)

Limits of y : $y = 0$ to $y = 2$

(ii) In the subregion TPS, the strip CD starts from the circle $x^2 + y^2 - 4x = 0$ and terminates on the line $x = 4$.

Limits of x : $x = 2 + \sqrt{4 - y^2}$ (Part of circle where $x > 2$) to $x = 4$

Limits of y : $y = 0$ to $y = 2$

(iii) In the subregion RSQ, the strip EF starts from the parabola $y^2 = 4x$ and terminates on the line $x = 4$.

Limits of x : $x = \frac{y^2}{4}$ to $x = 4$

Limits of y : $y = 2$ to $y = 4$

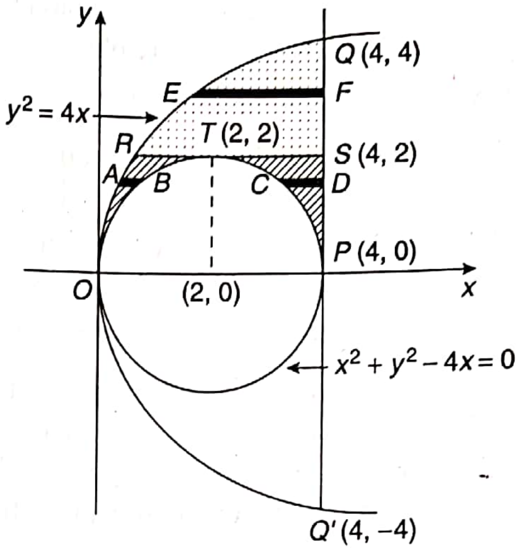

Fig. 7.16

Hence, the given integral after change of order is

$$\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x, y)\,dy\,dx = \int_0^2 \int_{\frac{y^2}{4}}^{2-\sqrt{4-y^2}} f(x, y)\,dx\,dy + \int_0^2 \int_{2+\sqrt{4-y^2}}^{4} f(x, y)\,dx\,dy + \int_2^4 \int_{\frac{y^2}{4}}^{4} f(x, y)\,dx\,dy$$

EXAMPLE 7.12 *Change the order of integration of* $\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,dy\,dx$.

Solution:

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4-x}$ to $y = (4-x)^2$.
   Limits of x : $x = 0$ to $x = 2$
3. The region is enclosed by the parabolas $y^2 = 4 - x$, $y = (4-x)^2$, the lines $x = 0$ and $x = 2$ [Fig. 7.17]
4. (i) The points of intersection of $x = 2$ and $y^2 = 4 - x$ are obtained as

   $y^2 = (4 - 2)$, i.e., $y = \pm\sqrt{2}$.

   The points of intersection are

   $Q(2, \sqrt{2})$ and $Q'(2, -\sqrt{2})$.

   (ii) The point of intersection of $x = 2$ and $y = (4 - x)^2$ is obtained as

   $y = (4 - 2)^2 = 4$.

   The point of intersection is $S(2, 4)$.

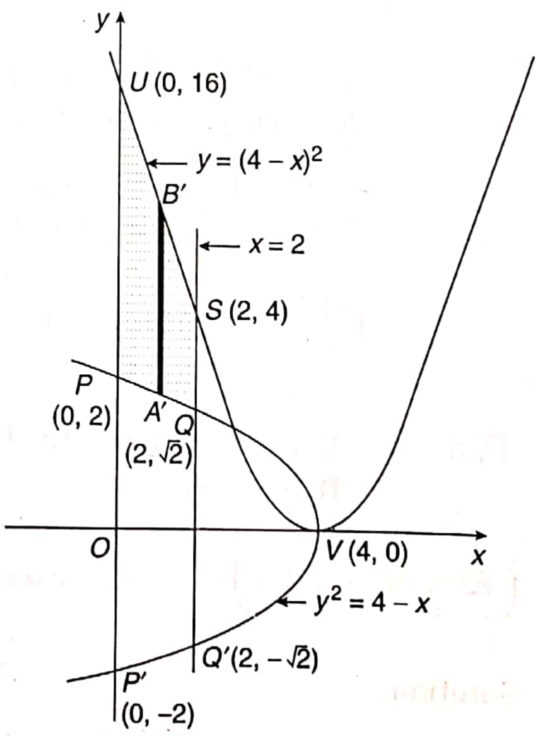

Fig. 7.17

(iii) The points of intersection of $x = 0$ and $y^2 = 4 - x$ are obtained as

$y^2 = 4$, i.e., $y = \pm 2$.

The points of intersection are $P(0, 2)$ and $P'(0, -2)$.

(iv) The point of intersection of $x = 0$ and $y = (4 - x)^2$ is obtained as $y = 16$. The point of intersection is $U(0, 16)$.

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions PQR, $PRST$ and STU. Draw a horizontal strip in each subregion [Fig. 7.18].

(i) In the subregion PQR, the strip AB starts from the parabola $y^2 = 4 - x$ and terminates on the line $x = 2$.

Limits of x : $x = 4 - y^2$ to $x = 2$

Limits of y : $y = \sqrt{2}$ to $y = 2$

(ii) In the subregion $PRST$, the strip CD starts from y-axis and terminates on the line $x = 2$.

Limits of x : $x = 0$ to $x = 2$

Limits of y : $y = 2$ to $y = 4$

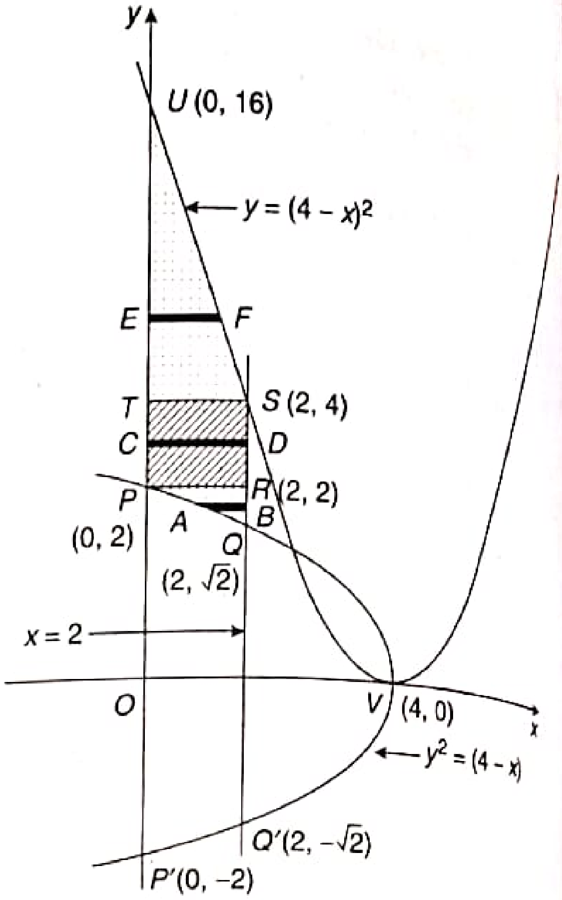

Fig. 7.18

(iii) In the subregion STU, the strip EF starts from y-axis and terminates on the parabola $y = (4 - x)^2$.

Limits of x : $x = 0$ to $x = 4 - \sqrt{y}$

(Part of the parabola, where $x < 4$)

Limits of y : $y = 4$ to $y = 16$

Hence, the given integral after change of order is

$$\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,dy\,dx = \int_{\sqrt{2}}^{2} \int_{4-y^2}^{2} f(x, y)\,dx\,dy + \int_2^4 \int_0^2 f(x, y)\,dx\,dy + \int_4^{16} \int_0^{4-\sqrt{y}} f(x, y)\,dx\,dy$$

Type II Evaluation of Double Integrals by Changing the Order of Integration

EXAMPLE 7.13 *Change the order of integration and evaluate* $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy.$

Solution:

1. Since inner limits depend on y, the function is integrated first w.r.t. x.

2. Limits of $x: x = a - \sqrt{a^2 - y^2}$ to $x = a + \sqrt{a^2 - y^2}$, along the horizontal strip $A'B'$
   Limits of $y: y = 0$ to $y = a$
3. The region is bounded by the circle $(x - a)^2 + y^2 = a^2$ and the line $y = 0$. Since limits of y are positive, the region is the part of the circle $(x - a)^2 + y^2 = a^2$ above the x-axis [Fig. 7.19].

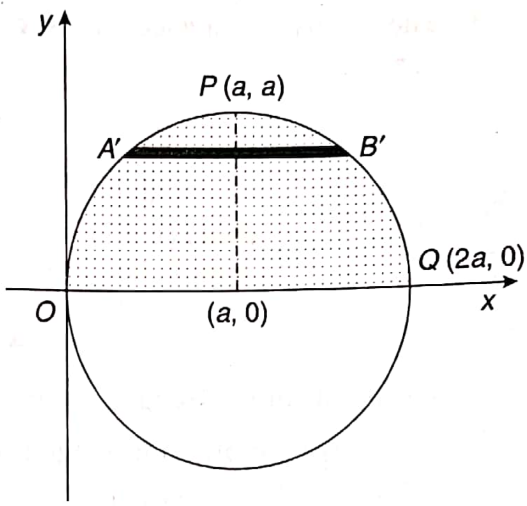

Fig. 7.19

4. The points of intersection of the circle with the x-axis are $O(0, 0)$ and $Q(2a, 0)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to the y-axis which starts from the x-axis and terminates on the circle [Fig. 7.20].

$(x-a)^2 + y^2 = a^2$

or $x^2 + y^2 - 2ax = 0$

Limits of $y: y = 0$ to $y = \sqrt{2ax - x^2}$

Limits of $x: x = 0$ to $x = 2a$

Hence, the given integral after change of order is

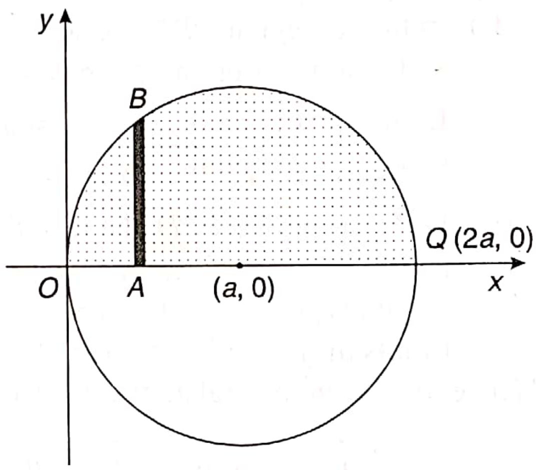

Fig. 7.20

$$\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \mathrm{d}x\, \mathrm{d}y = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \mathrm{d}y\, \mathrm{d}x$$

$$= \int_0^{2a} \left| y \right|_0^{\sqrt{2ax-x^2}} \mathrm{d}x = \int_0^{2a} \sqrt{2ax - x^2}\, \mathrm{d}x$$

$$= \int_0^{2a} \sqrt{a^2 - (x-a)^2}$$

$$= \left| \frac{(x-a)}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right) \right|_0^{2a}$$

$$= \frac{a}{2}\sqrt{0} + \frac{a^2}{2}\sin^{-1}1 - \frac{(0-a)}{2}\sqrt{0} - \frac{a^2}{2}\sin^{-1}(-1)$$

$$= a^2 \sin^{-1}1 \quad [\because \sin^{-1}(-1) = -\sin^{-1}(1)] = a^2\frac{\pi}{2} = \frac{\pi a^2}{2}$$

EXAMPLE 7.14 *Change the order of integration and evaluate* $\int_0^1 \int_{x^2}^{2-x} xy\, \mathrm{d}x\, \mathrm{d}y$.

Solution:

1. Since inner limits depend on x, the function is integrated first w.r.t. y.

   The correct form of the integral $= \int_0^1 \int_{x^2}^{2-x} xy\, \mathrm{d}y\, \mathrm{d}x$
2. Limits of $y: y = x^2$ to $y = 2 - x$, along the vertical strip $A'B'$
   Limits of $x: x = 0$ to $x = 1$

3. The region is bounded by the y-axis, the line $x + y = 2$ and the parabola $x^2 = y$. Since given limits of x and y are positive, the region lies in the first quadrant. [Fig. 7.21].
4. The points of intersection of $x + y = 2$ and $x^2 = y$ are obtained as

$$x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$(x-1)(x+2) = 0$$
$$x = 1, -2$$
$$y = 1, 4$$

The points of intersection are $P(1, 1)$ and $P'(-2, 4)$.

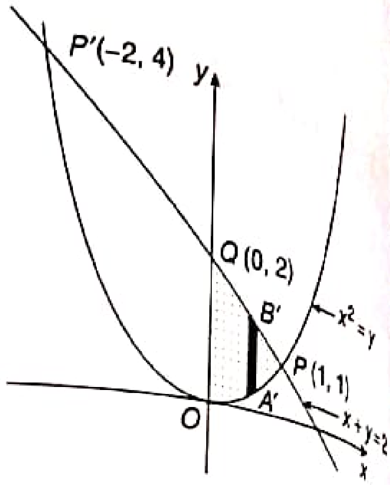

Fig. 7.21

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and RPQ. Draw a horizontal strip parallel to the x-axis in each subregion [Fig. 7.22].

(i) In the subregion OPR, the strip AB starts from the y-axis and terminates on the parabola $x^2 = y$.

Limits of x : $x = 0$ to $x = \sqrt{y}$

Limits of y : $y = 0$ to $y = 1$

(ii) In the subregion RPQ, the strip CD starts from the y-axis and terminates on the line $x + y = 2$.

Limits of x : $x = 0$ to $x = 2 - y$

Limits of y : $y = 1$ to $y = 2$

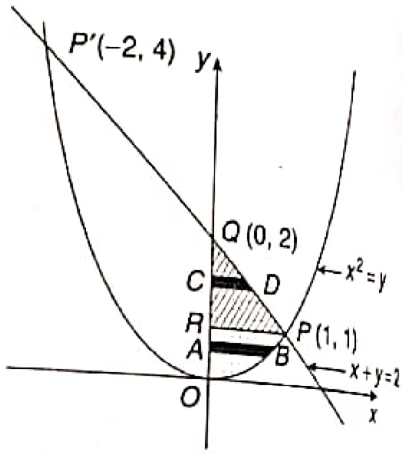

Fig. 7.22

Hence, the given integral after change of order is

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 \left|\frac{x^2}{2}\right|_0^{\sqrt{y}} y\,dy + \int_1^2 \left|\frac{x^2}{2}\right|_0^{2-y} y\,dy$$

$$= \frac{1}{2}\int_0^1 (y)y\,dy + \frac{1}{2}\int_1^2 (2-y)^2 y\,dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)dy$$

$$= \frac{1}{2}\left|\frac{y^3}{3}\right|_0^1 + \frac{1}{2}\left|4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4}\right|_1^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right)$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8}$$

EXAMPLE 7.15 *Change the order of integration and evaluate* $\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy.$

Solution:

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x: x=0$ to $x = a-\sqrt{a^2-y^2}$
   Limits of $y: y=0$ to $y=a$
3. The region is bounded by the circle $(x-a)^2+y^2=a^2$, the lines $y=a$ and $x=0$ [Fig. 7.23].

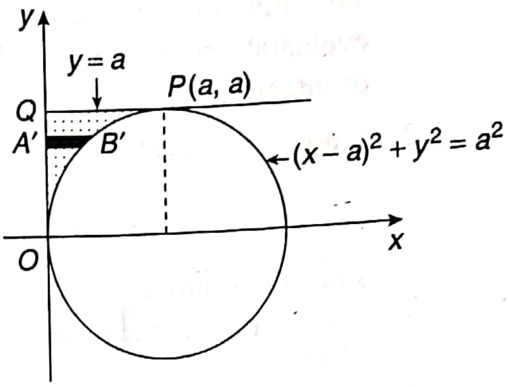

Fig. 7.23

4. The point of intersection of
   $(x-a)^2+y^2=a^2$ and $y=a$ is obtained as
   $(x-a)^2+a^2=a^2$, i.e., $x=a$
   The point of intersection is $P(a, a)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to the y-axis which starts from the circle $(x-a)^2+y^2=a^2$ and terminates on the line $y=a$ [Fig. 7.24].

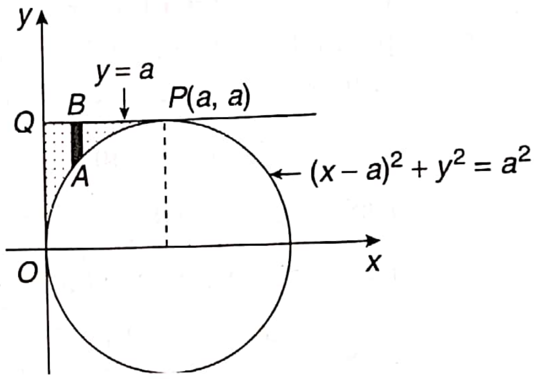

Fig. 7.24

Limits of $y: y=\sqrt{2ax-x^2}$ to $y=a$

Limits of $x: x=0$ to $x=a$

Hence, the given integral after change of order is

$$\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy = \int_0^a \int_{\sqrt{2ax-x^2}}^a \frac{x\log(x+a)}{(x-a)^2}\,y\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left|\frac{y^2}{2}\right|_{\sqrt{2ax-x^2}}^a dx = \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left(\frac{a^2-2ax+x^2}{2}\right)dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx = \frac{1}{2}\left[\left|\frac{x^2}{2}\log(x+a)\right|_0^a - \int_0^a \frac{x^2}{2}\cdot\frac{1}{x+a}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\int_0^a \left\{(x-a)+\frac{a^2}{x+a}\right\}dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\left|\frac{x^2}{2} - ax + a^2\log(x+a)\right|_0^a\right]$$

$$= \frac{1}{4}\left(a^2\log 2a - \frac{a^2}{2} + a^2 - a^2\log 2a + a^2\log a\right)$$

$$= \frac{1}{4}\left(\frac{a^2}{2} + a^2\log a\right) = \frac{a^2}{8}(1+2\log a)$$

EXAMPLE 7.16 *Change the order of integration and evaluate* $\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}dx\,dy.$

Solution:

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x: x=0$ to $x=\sqrt{1-4y^2}$

   Limits of $y: y=0$ to $y=\frac{1}{2}$
3. Since the limits of x and y are positive, the region is the part of the ellipse in the first quadrant [Fig. 7.25].

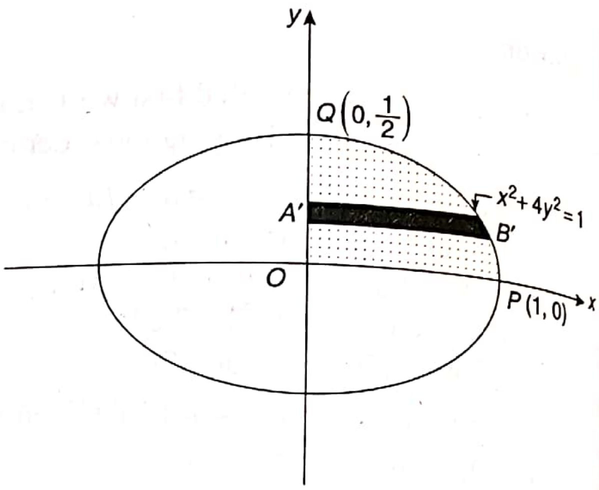

Fig. 7.25

4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the x-axis and terminates on the ellipse $x^2+4y^2=1$ [Fig. 7.26].

   Limits of $y: y=0$ to $y=\frac{1}{2}\sqrt{1-x^2}$

   Limits of $x: x=0$ to $x=1$

   Hence, the given integral after change of order is

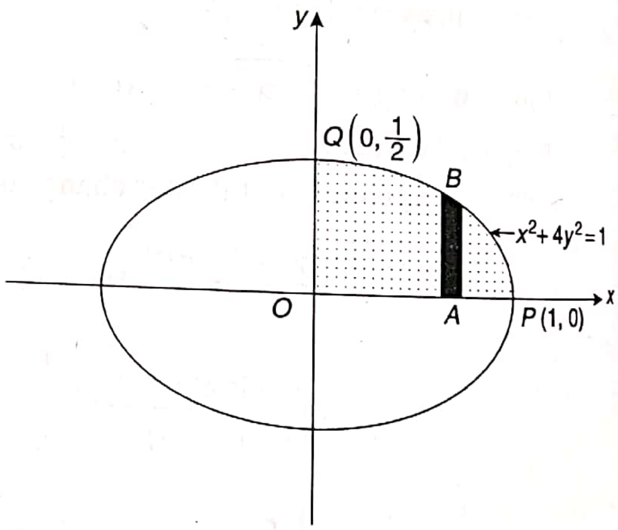

Fig. 7.26

$$\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}dx\,dy$$

$$=\int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\int_0^{\frac{1}{2}\sqrt{1-x^2}} \frac{1}{\sqrt{(1-x^2)-y^2}}dy\,dx$$

$$=\int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\left|\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right|_0^{\frac{1}{2}\sqrt{1-x^2}}dx$$

$$=\int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\left(\sin^{-1}\frac{1}{2}-\sin^{-1}0\right)dx$$

$$=\int_0^1 \frac{2-(1-x^2)}{\sqrt{1-x^2}}\cdot\frac{\pi}{6}dx$$

$$=\frac{\pi}{6}\int_0^1\left(\frac{2}{\sqrt{1-x^2}}-\sqrt{1-x^2}\right)dx$$

$$=\frac{\pi}{6}\left|2\sin^{-1}x-\frac{x}{2}\sqrt{1-x^2}-\frac{1}{2}\sin^{-1}x\right|_0^1 \qquad \left[\because \int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=\frac{\pi}{6}\left(\frac{3}{2}\sin^{-1}1\right)=\frac{\pi}{4}\cdot\frac{\pi}{2}=\frac{\pi^2}{8}$$

EXAMPLE 7.17 *Change the order of integration and evaluate* $\int_0^1\int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx.$

Solution:

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of $y: y=x$ to $y=\frac{1}{x}$

   Limits of $x: x=0$ to $x=1$
3. The region is bounded by the rectangular hyperbola $xy=1$, the line $y=x$ and the y-axis in the first quadrant [Fig. 7.27].

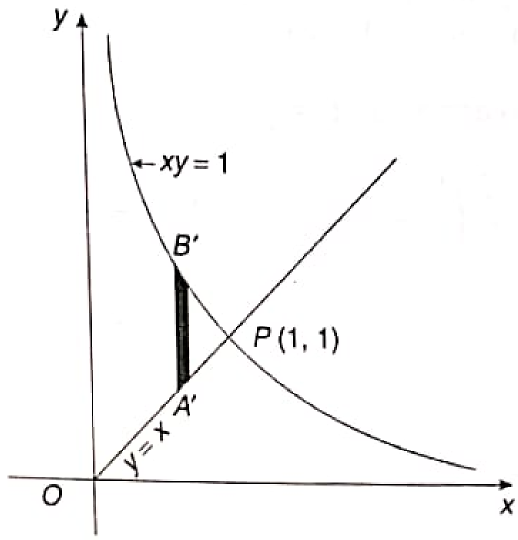

Fig. 7.27

4. The point of intersection of $xy = 1$ and $y = x$ in the first quadrant is obtained as

   $x^2 = 1$, i.e., $x = 1$

   $\therefore y = 1$

   The point of intersection is $P(1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPQ and QPR. Draw a horizontal strip parallel to the x-axis in each subregion [Fig. 7.28].

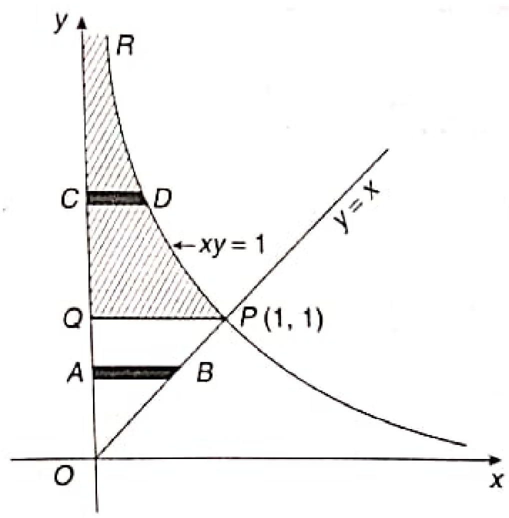

Fig. 7.28

(i) In the subregion OPQ, the strip AB starts from the y-axis and terminates on the line $y=x$.

Limits of $x: x=0$ to $x=y$

Limits of $y: y=0$ to $y=1$

(ii) In the subregion QPR, the strip CD starts from the y-axis and terminates on the rectangular hyperbola $xy=1$.

Limits of $x: x=0$ to $x=\frac{1}{y}$

Limits of $y: y=1$ to $y\to\infty$.

Hence, the given integral after change of order is

$$\int_0^1\int_x^{\frac{1}{x}}\frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx = \int_0^1\int_0^y \frac{y}{1+y^2}\frac{1}{(1+xy)^2}\,dx\,dy + \int_1^\infty\int_0^{\frac{1}{y}}\frac{y}{1+y^2}\frac{1}{(1+xy)^2}\,dx\,dy$$

$$= \int_0^1 \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^y dy + \int_1^\infty \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^{\frac{1}{y}} dy$$

$$= -\int_0^1 \frac{1}{1+y^2}\left(\frac{1}{1+y^2}-1\right)dy - \int_1^\infty \frac{1}{1+y^2}\left(\frac{1}{2}-1\right)dy$$

$$= -\int_0^1 \left[\frac{1}{(1+y^2)^2} - \frac{1}{1+y^2} \right] dy + \frac{1}{2}\int_1^\infty \frac{1}{1+y^2}\, dy$$

Putting $y = \tan\theta$ in the first term of the first integral, $dy = \sec^2\theta\, d\theta$,
When $y = 0$, $\theta = 0$

When $y = 1$, $\theta = \frac{\pi}{4}$

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\, dy\, dx = -\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} + \left|\tan^{-1} y\right|_0^1 + \frac{1}{2}\left|\tan^{-1} y\right|_1^\infty$$

$$= -\int_0^{\frac{\pi}{4}} \frac{(1+\cos 2\theta)}{2}\, d\theta + (\tan^{-1} 1 - \tan^{-1} 0) + \frac{1}{2}(\tan^{-1}\infty - \tan^{-1} 1)$$

$$= -\frac{1}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}} + \frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= -\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + \frac{3\pi}{8} = \frac{\pi - 1}{4}$$

EXERCISE 7.3

Change the order of integration of the following integrals:

1. $\int_0^6 \int_{2-x}^{2+x} f(x, y)\, dy\, dx$

$\left[\textbf{Ans.:} \int_{-4}^{2} \int_{2-y}^{6} f(x, y)\, dy\, dx + \int_2^8 \int_{y-2}^{6} f(x, y)\, dy\, dx\right]$

2. $\int_0^1 \int_x^{2x} f(x, y)\, dy\, dx$

$\left[\textbf{Ans.:} \int_0^1 \int_{\frac{y}{2}}^{y} f(x, y)\, dx\, dy + \int_1^2 \int_{\frac{y}{2}}^{1} f(x, y)\, dx\, dy\right]$

3. $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx\, dy$

$\left[\textbf{Ans.:} \int_{-1}^1 \int_{x^2}^{1} f(x, y)\, dx\, dy\right]$

4. $\int_{-a}^a \int_0^{\frac{y^2}{a}} f(x, y)\, dx\, dy$.

$\left[\textbf{Ans.:} \int_0^a \int_{-a}^{-\sqrt{ax}} f(x, y)\, dy\, dx + \int_0^a \int_{\sqrt{ax}}^{a} f(x, y)\, dy\, dx\right]$

5. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x, y)\, dy\, dx$

$\left[\textbf{Ans.:} \int_0^a \int_0^{\sqrt{ay}} f(x, y)\, dx\, dy + \int_a^{2a} \int_0^{2a-y} f(x, y)\, dx\, dy\right]$

6. $\int_{-2}^3 \int_{y^2-6}^{y} f(x, y)\, dx\, dy$

$\left[\textbf{Ans.:} \int_{-6}^{-2} \int_{-\sqrt{x+6}}^{\sqrt{x+6}} f(x, y)\, dy\, dx + \int_{-2}^{3} \int_{x}^{\sqrt{x+6}} f(x, y)\, dy\, dx\right]$

7. $\int_0^1 \int_{2y}^{2(1+\sqrt{1-y})} f(x, y)\,dx\,dy$

$$\left[\text{Ans.}: \int_0^2 \int_0^{\frac{x}{2}} f(x, y)\,dy\,dx + \int_2^4 \int_0^{\frac{4x-x^2}{4}} f(x, y)\,dy\,dx\right]$$

8. $\int_0^2 \int_{\frac{x^2+4}{4}}^{\frac{6-x}{2}} f(x, y)\,dy\,dx$

$$\left[\text{Ans.}: \int_1^2 \int_0^{2\sqrt{y-1}} f(x, y)\,dx\,dy + \int_2^3 \int_0^{6-2y} f(x, y)\,dx\,dy\right]$$

9. $\int_0^2 \int_{\sqrt{4-x^2}}^{x+6a} f(x, y)\,dy\,dx$

$$\left[\text{Ans.}: \int_0^2 \int_{\sqrt{4-y^2}}^2 f(x, y)\,dx\,dy + \int_2^{6a} \int_0^2 f(x, y)\,dx\,dy + \int_{6a}^{6a+2} \int_{y-6a}^2 f(x, y)\,dx\,dy\right]$$

10. $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x, y)\,dx\,dy$

$$\left[\text{Ans.}: \int_0^a \int_{\sqrt{a^2-x^2}}^a f(x, y)\,dy\,dx + \int_a^{2a} \int_{x-a}^a f(x, y)\,dy\,dx\right]$$

11. $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)\,dx\,dy$

$$\left[\text{Ans.}: \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)\,dx\,dy + \int_0^a \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x, y)\,dx\,dy + \int_a^{2a} \int_{\frac{y^2}{2a}}^{2a} f(x, y)\,dx\,dy\right]$$

12. $\int_0^a \int_{\sqrt{\frac{a^2-x^2}{4}}}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx$

$$\left[\text{Ans.}: \int_0^{\frac{a}{2}} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy + \int_{\frac{a}{2}}^a \int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy\right]$$

13. $\int_0^a \int_x^{\frac{a^2}{x}} f(x, y)\,dy\,dx$

$$\left[\text{Ans.}: \int_0^a \int_0^y f(x, y)\,dx\,dy + \int_a^\infty \int_0^{\frac{a^2}{y}} f(x, y)\,dx\,dy\right]$$

14. $\int_a^b \int_{\frac{k}{x}}^{mx} f(x, y)\,dy\,dx$

$$\left[\text{Ans.}: \int_{\frac{k}{b}}^{\frac{k}{a}} \int_{\frac{k}{y}}^b f(x, y)\,dx\,dy + \int_{\frac{k}{a}}^{ma} \int_a^b f(x, y)\,dx\,dy + \int_{ma}^{mb} \int_{\frac{y}{m}}^b f(x, y)\,dx\,dy\right]$$

15. $\int_0^1 \int_1^{e^x} f(x, y)\,dy\,dx$

$$\left[\text{Ans.}: \int_1^e \int_{\log y}^1 f(x, y)\,dx\,dy\right]$$

16. $\int_0^2 \int_0^{x^3} f(x, y)\,dy\,dx$

$$\left[\text{Ans.}: \int_0^8 \int_{\frac{1}{y^3}}^2 f(x, y)\,dx\,dy\right]$$

17. $\int_0^{\frac{1}{2}} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dx\,dy$

$$\left[\text{Ans.}: \int_0^1 \int_0^{\frac{\sqrt{1-x^2}}{2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dy\,dx = \frac{2\pi}{3}\right]$$

18. $\int_0^2 \int_0^{\frac{x^2}{2}} \frac{x}{\sqrt{x^2+y^2+1}} \, dy \, dx$

$$\left[\textbf{Ans.:} \int_0^2 \int_{\sqrt{2y}}^2 \frac{x}{\sqrt{x^2+y^2+1}} \, dx \, dy = \frac{1}{4}(5\log 5 - 4)\right]$$

19. $\int_0^a \int_y^a \frac{x}{x^2+y^2} \, dy \, dx$

$$\left[\textbf{Ans.:} \frac{\pi a}{4}\right]$$

7.4 DOUBLE INTEGRALS IN POLAR COORDINATES

The integral $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) dr d\theta$ represents the polar form of double integration. This integral is first integrated w.r.t. r keeping θ constant and then the resulting expression is integrated w.r.t. θ.

Limits of Integration If the limits of integration are not given then these limits are obtained from the equations of the given curves. Let the region be bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.

The region of integration is $PQRS$ (Fig. 7.29). Draw an elementary radius vector AB from the origin which enters in the region from the curve $r = r_1(\theta)$ and leaves at the curve $r = r_2(\theta)$. Hence, limits for r are $r_1(\theta)$ to $r_2(\theta)$.

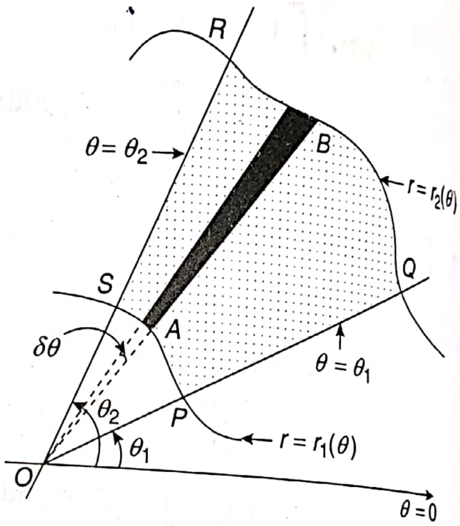

Fig. 7.29 Double integration in polar coordinates

To cover the entire region $PQRS$, rotate the elementary radius vector AB from PQ to RS. Therefore, θ varies from θ_1 to θ_2.

$$\iint f(r, \theta) \, dr \, d\theta = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) \, dr \, d\theta$$

Type I Evaluation of Double Integrals

EXAMPLE 7.18 *Evaluate* $\int_0^{\frac{\pi}{4}} \int_0^1 r \, dr \, d\theta$.

Solution: $\int_0^{\frac{\pi}{4}} \int_0^1 r \, dr \, d\theta = \int_0^{\frac{\pi}{4}} \left[\int_0^1 r \, dr\right] d\theta = \int_0^{\frac{\pi}{4}} \left|\frac{r^2}{2}\right|_0^1 d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2} d\theta = \frac{1}{2}|\theta|_0^{\frac{\pi}{4}} = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}$

Another Method Since both the limits are constant and the integrand (function) is explicit in r and θ, the integral can be written as

$$\int_0^{\frac{\pi}{4}}\int_0^1 r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{4}}\mathrm{d}\theta\cdot\int_0^1 r\,\mathrm{d}r = \left|\theta\right|_0^{\frac{\pi}{4}}\cdot\left|\frac{r^2}{2}\right|_0^1 = \frac{\pi}{4}\cdot\frac{1}{2} = \frac{\pi}{8}$$

EXAMPLE 7.19 *Evaluate* $\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}}\frac{r}{(1+r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta.$

Solution:

$$\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}}\frac{1}{2}\cdot\frac{2r}{(1+r^2)^2}\,\mathrm{d}r\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\int_0^{\sqrt{\cos 2\theta}}(1+r^2)^{-2}\cdot 2r\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left|-(1+r^2)^{-1}\right|_0^{\sqrt{\cos 2\theta}}\mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1},\ n \neq -1\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\cos 2\theta}-1\right)\mathrm{d}\theta = -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2\cos^2\theta}-1\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\sec^2\theta - 1\right)\mathrm{d}\theta = -\frac{1}{2}\left|\frac{1}{2}\tan\theta - \theta\right|_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\tan\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{1}{8}(\pi - 2)$$

Type II Evaluation of Double Integrals Over a Given Region

EXAMPLE 7.20 *Evaluate* $\iint r\sqrt{a^2-r^2}\,\mathrm{d}r\,\mathrm{d}\theta$, *over the upper half of the circle* $r = a\cos\theta$.

Solution:

1. The region of integration is the upper half of the circle $r = a\cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = a\cos\theta$ (Fig. 7.30).

Limits of r: $r = 0$ to $r = a\cos\theta$

Limits of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

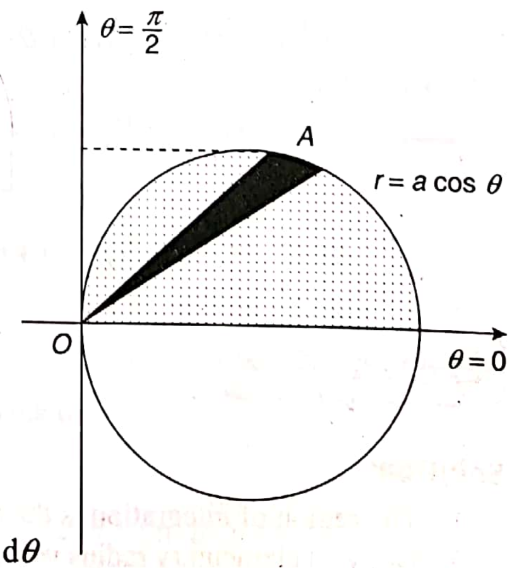

Fig. 7.30

$$I = \iint r\sqrt{a^2-r^2}\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\int_0^{a\cos\theta}\left(-\frac{1}{2}\right)(a^2-r^2)^{\frac{1}{2}}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left|\frac{2(a^2-r^2)^{\frac{3}{2}}}{3}\right|_0^{a\cos\theta} d\theta \qquad \left[\because \int [f(r)]^n f'(r)\,dr = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}}(a^3\sin^3\theta - a^3)\,d\theta = -\frac{a^3}{3}\int_0^{\frac{\pi}{2}}\left(\frac{3\sin\theta-\sin 3\theta}{4}-1\right)d\theta$$

$$= -\frac{a^3}{3}\left|\frac{1}{4}\left(-3\cos\theta+\frac{\cos 3\theta}{3}\right)-\theta\right|_0^{\frac{\pi}{2}} = -\frac{a^3}{3}\left(-\frac{3}{4}\cos\frac{\pi}{2}+\frac{1}{12}\cos\frac{3\pi}{2}-\frac{\pi}{2}+\frac{3}{4}\cos 0-\frac{1}{12}\cos 0\right)$$

$$= -\frac{a^3}{3}\left(-\frac{\pi}{2}+\frac{3}{4}-\frac{1}{12}\right) = -\frac{a^3}{3}\left(\frac{2}{3}-\frac{\pi}{2}\right)$$

EXAMPLE 7.21 *Evaluate* $\iint r^2 \sin\theta \, dr\, d\theta$, *over the cardioid* $r = a(1+\cos\theta)$ *above the initial line.*

Solution:

1. The region of integration is the part of the cardioid $r = a(1+\cos\theta)$ above the initial line ($\theta = 0$).
2. Draw an elementary radius vector OA which starts from the origin and terminates on the cardioid $r = a(1+\cos\theta)$ (Fig. 7.31).
3. Limits of r: $r = 0$ to $r = a(1+\cos\theta)$
   Limits of θ: $\theta = 0$ to $\theta = \pi$

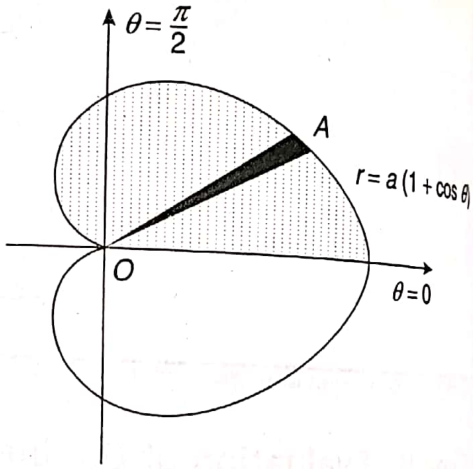

Fig. 7.31

$$I = \iint r^2 \sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi}\left|\frac{r^3}{3}\right|_0^{a(1+\cos\theta)} \sin\theta \, d\theta$$

$$= \frac{1}{3}\int_0^{\pi} a^3(1+\cos\theta)^3 \sin\theta\cdot d\theta = -\frac{a^3}{3}\int_0^{\pi}(1+\cos\theta)^3(-\sin\theta)\,d\theta$$

$$= -\frac{a^3}{3}\left|\frac{(1+\cos\theta)^4}{4}\right|_0^{\pi} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= -\frac{a^3}{12}\left[(1+\cos\pi)^4-(1+\cos 0)^4\right] = -\frac{a^3}{12}(0-16) = \frac{4}{3}a^2$$

EXAMPLE 7.22 *Evaluate* $\iint r^2 dr\, d\theta$, *over the area between the circles* $r = a\sin\theta$ *and* $r = 2a\sin\theta$.

Solution:

1. The region of integration is the area bounded between the circle $r = a\sin\theta$ and $r = 2a\sin\theta$.
2. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = a\sin\theta$ and leaves at the circle $r = 2a\sin\theta$ (Fig. 7.32)

3. Limits of r: $r = a\sin\theta$ to $r = 2a\sin\theta$
Limits of θ: $\theta = 0$ to $\theta = \pi$

$$I = \iint r^2 dr\, d\theta = \int_0^{\pi}\int_{a\sin\theta}^{2a\sin\theta} r^2 dr\, d\theta = \int_0^{\pi}\left|\frac{r^3}{3}\right|_{a\sin\theta}^{2a\sin\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi}(8a^3\sin^3\theta - a^3\sin^3\theta)d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi}\sin^3\theta\, d\theta = \frac{7a^3}{3}\int_0^{\pi}\frac{3\sin\theta - \sin 3\theta}{4}d\theta$$

$$= \frac{7a^3}{12}\left|-3\cos\theta + \frac{\cos 3\theta}{3}\right|_0^{\pi}$$

$$= \frac{7a^3}{12}\left[-3(\cos\pi - \cos 0) + \frac{1}{3}(\cos 3\pi - \cos 0)\right] = \frac{7a^3}{12}\left(\frac{16}{3}\right) = \frac{28}{9}a^3$$

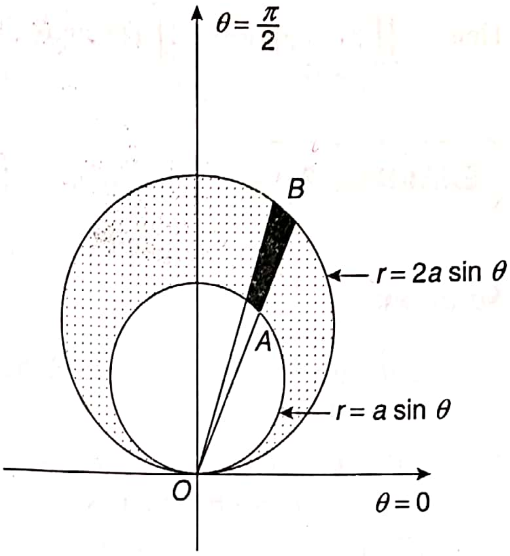

Fig. 7.32

EXERCISE 7.4

Evaluate the following integrals:

1. $\iint re^{\frac{-r^2}{a^2}}\cos\theta\sin\theta\, dr\, d\theta$, over the upper half of the circle $r = 2a\cos\theta$

$$\left[\textbf{Ans.: } \frac{a^2}{16}\left(3 + \frac{1}{e^4}\right)\right]$$

2. $\iint r^3 dr\, d\theta$, over the region between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

$$\left[\textbf{Ans.: } \frac{45\pi}{2}\right]$$

3. $\iint r\sin\theta\, dA$, over the cardioid $r = a(1+\cos\theta)$ above the initial line

$$\left[\textbf{Ans.: } \frac{4}{3}a^3\right]$$

4. $\iint \frac{r}{\sqrt{r^2+4}}dr\, d\theta$, over one loop of the lemniscate $r^2 = 4\cos 2\theta$

$$[\textbf{Ans.: } (4 - \pi)]$$

7.5 CHANGE OF VARIABLES

7.5.1 Change of Variables from Cartesian to Polar Coordinates

The double integral can be changed from Cartesian coordinates (x, y) to polar coordinates (r, θ) by putting $x = r\cos\theta$, $y = r\sin\theta$. Then $\iint f(x, y)dy\,dx = \iint f(r\cos\theta, r\sin\theta)|J|\,dr\,d\theta$, where J is the Jacobian (functional determinant) defined as

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

Hence, $\iint f(x, y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)|r|\,dr\,d\theta = \iint f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$

EXAMPLE 7.23 *Evaluate* $\iint \frac{(x^2+y^2)^2}{x^2y^2}dx\,dy$, *over the region common to the circles* $x^2+y^2=ax$ *and* $x^2+y^2=by$ $(a, b > 0)$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, the polar form of
   (i) the circle $x^2 + y^2 = ax$ is $r^2 = ar\cos\theta$, $r = a\cos\theta$
   (ii) the circle $x^2 + y^2 = by$ is $r^2 = br\sin\theta$, $r = b\sin\theta$
2. The region of integration is the common part of the circles $r = a\cos\theta$ and $r = b\sin\theta$ (Fig. 7.33).
3. The point of intersection of the circle $r = a\cos\theta$ and $r = b\sin\theta$, is obtained as

$$b\sin\theta = a\cos\theta$$

$$\tan\theta = \frac{a}{b}$$

$$\theta = \tan^{-1}\frac{a}{b}$$

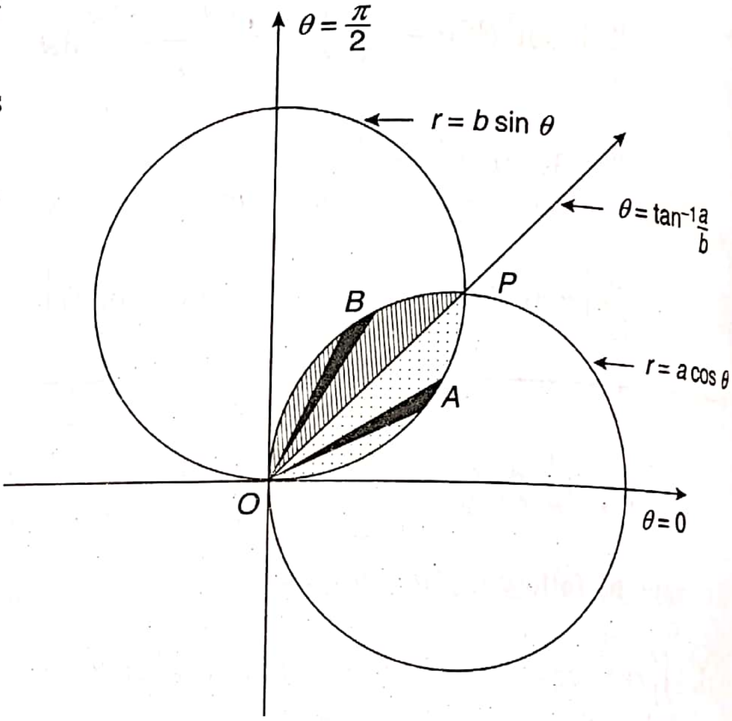

Fig. 7.33

Hence, $\theta = \tan^{-1}\frac{a}{b}$ at P.

4. Divide the region into two subregions OAP and OBP. Draw an elementary radius vector OA and OB in each subregion.
   (i) In the subregion OAP, the elementary radius vector OA starts from the origin and terminates on the circle $r = b\sin\theta$.
   Limits of r: $r = 0$ to $r = b\sin\theta$
   Limits of θ: $\theta = 0$ to $\theta = \tan^{-1}\frac{a}{b}$
   (ii) In the subregion OBP, the elementary radius vector OB starts from the origin and terminates on the circle $r = a\cos\theta$.
   Limits of r: $r = 0$ to $r = a\cos\theta$
   Limits of θ: $\theta = \tan^{-1}\frac{a}{b}$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \iint \frac{(x^2+y^2)^2}{x^2y^2}dx\,dy$$

$$= \int_0^{\tan^{-1}\frac{a}{b}} \int_0^{b\sin\theta} \frac{r^4}{r^4 \sin^2\theta \cos^2\theta} \cdot r\,dr\,d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} \frac{r^4}{r^4 \sin^2\theta\cos^2\theta} \cdot r\,dr\,d\theta$$

$$= \int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta} \left| \frac{r^2}{2} \right|_0^{b\sin\theta} d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta} \left| \frac{r^2}{2} \right|_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta} \cdot b^2 \sin^2\theta\, d\theta + \frac{1}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta} \cdot a^2\cos^2\theta\, d\theta$$

$$= \frac{b^2}{2}\int_0^{\tan^{-1}\frac{a}{b}} \sec^2\theta\, d\theta + \frac{a^2}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \text{cosec}^2\theta\, d\theta$$

$$= \frac{b^2}{2} \left| \tan\theta \right|_0^{\tan^{-1}\frac{a}{b}} + \frac{a^2}{2} \left| -\cot\theta \right|_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}$$

$$= \frac{b^2}{2}\left[\tan\tan^{-1}\left(\frac{a}{b}\right) - \tan 0\right] - \frac{a^2}{2}\left[\cot\frac{\pi}{2} - \cot\left(\tan^{-1}\frac{a}{b}\right)\right]$$

$$= \frac{b^2}{2}\left[\frac{a}{b} - 0\right] - \frac{a^2}{2}\left[0 - \frac{b}{a}\right] = \frac{ab}{2} + \frac{ab}{2} = ab$$

EXAMPLE 7.24 Evaluate $\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xye^{-(x^2+y^2)}}{x^2+y^2}\, dx\, dy.$

Solution:

1. Limits of $y: y = \sqrt{x - x^2}$ to $y = \sqrt{1-x^2}$
   Limits of $x: x = 0$ to $x = 1$
2. The region of integration is the part of the first quadrant bounded by the circles $x^2 + y^2 - x = 0$ and $x^2 + y^2 = 1$ [Fig. 7.34].
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, the polar form of
   (i) the circle $x^2 + y^2 - x = 0$ is
   $r^2 - r\cos\theta = 0$, $r = \cos\theta$
   (ii) the circle $x^2 + y^2 = 1$ is $r^2 = 1$, $r = 1$
4. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = \cos\theta$ and terminates on the circle $r = 1$[Fig. 7.35].
   Limits of r : $r = \cos\theta$ to $r = 1$
   Limits of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

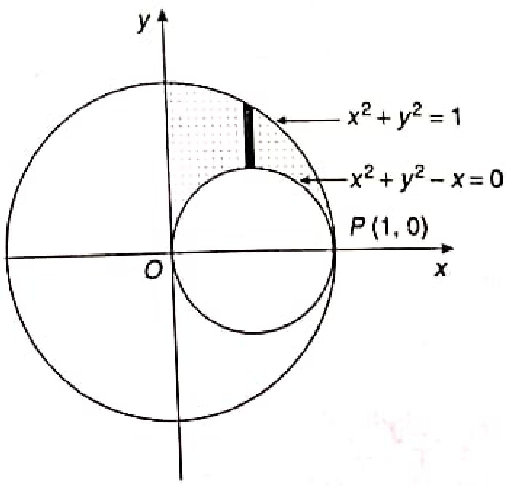

Fig. 7.34

Hence, the polar form of the given integral is

$$I = \int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xye^{-(x^2+y^2)}}{x^2+y^2} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \int_{\cos\theta}^{1} \frac{r^2 \sin\theta \cos\theta e^{-r^2}}{r^2} \cdot r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \int_{\cos\theta}^{1} e^{-r^2}(-2r)\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta \left| e^{-r^2} \right|_{\cos\theta}^{1} \mathrm{d}\theta$$

$$\left[\because \int e^{f(r)} f'(r)\mathrm{d}r = e^{f(r)}\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta (e^{-1} - e^{-\cos^2\theta})\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{2}\left(\frac{1}{e}\sin 2\theta - e^{-\cos^2\theta}\cdot 2\sin\theta\cos\theta\right)\mathrm{d}\theta$$

$$= -\frac{1}{4}\left| \frac{1}{e}\left(-\frac{\cos 2\theta}{2}\right) - e^{-\cos^2\theta} \right|_0^{\frac{\pi}{2}}$$

$$\left[\because \int e^{f(\theta)} f'(\theta)\mathrm{d}\theta = e^{f(\theta)}\right]$$

$$= -\frac{1}{4}\left[-\frac{1}{2e}(\cos\pi - \cos 0) - e^{-\left(\cos^2\frac{\pi}{2}\right)} + e^{-\cos^2 0}\right]$$

$$= -\frac{1}{4}\left[-\frac{1}{2e}(-2) - e^0 + e^{-1}\right] = -\frac{1}{4}\left[\frac{1}{e} - 1 + \frac{1}{e}\right] = \frac{1}{4}\left[1 - \frac{2}{e}\right]$$

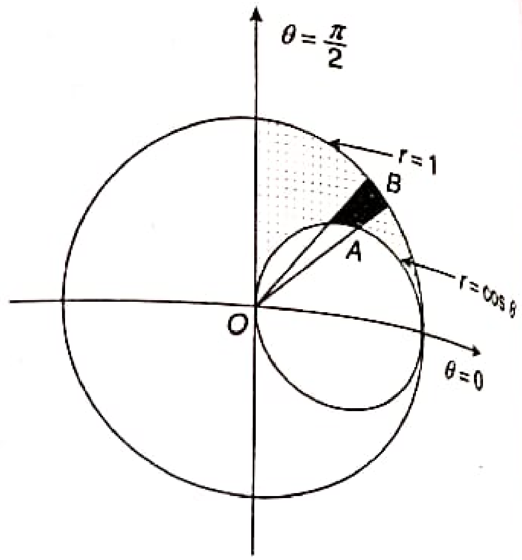

Fig. 7.35

EXAMPLE 7.25 *Evaluate* $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\mathrm{d}x\,\mathrm{d}y.$

Solution:

1. Limits of $x : x = \dfrac{y^2}{4a}$ to $x = y$

   Limits of $y : y = 0$ to $y = 4a$.
2. The region of integration is bounded by the line $y = x$ and the parabola $y^2 = 4ax$ [Fig. 7.36].
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, the polar form of
   (i) the line $y = x$ is $r\sin\theta = r\cos\theta$, $\tan\theta = 1$, $\theta = \dfrac{\pi}{4}$
   (ii) the parabola $y^2 = 4ax$ is
   $r^2\sin^2\theta = 4ar\cos\theta$, $r = 4a\cot\theta\,\mathrm{cosec}\,\theta$

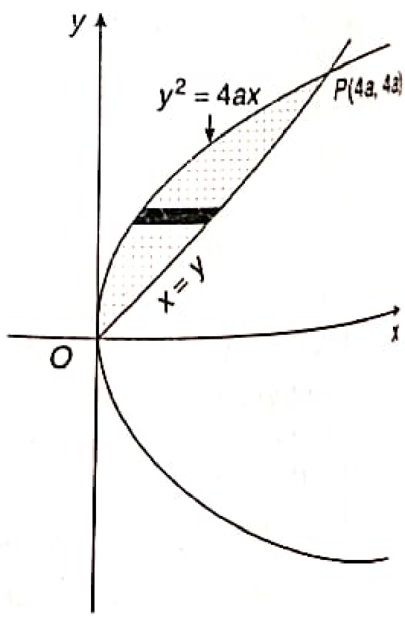

Fig. 7.36

4. Draw an elementary radius vector OA which starts from the origin and terminates on the parabola $r = 4a\cot\theta\,\text{cosec}\,\theta$ [Fig. 7.37].

Limits of r: $r = 0$ to $r = 4a\cot\theta\,\text{cosec}\,\theta$

Limits of θ: $\theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4a\cot\theta\,\text{cosec}\,\theta} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)\left|\frac{r^2}{2}\right|_0^{4a\cot\theta\,\text{cosec}\,\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)(4a)^2\cot^2\theta\,\text{cosec}^2\theta\,d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^2\theta\,\text{cosec}^2\theta - 2\cot^2\theta)\,d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\left\{-(\cot^2\theta)(-\text{cosec}^2\theta)\right\} - 2\,\text{cosec}^2\theta + 2\right]d\theta$$

$$= 8a^2\left| -\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 8a^2\left[-\frac{1}{3}\left(\cot^3\frac{\pi}{2} - \cot^3\frac{\pi}{4}\right) + 2\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= 8a^2\left[-\frac{1}{3}(-1) + 2(-1) + 2\cdot\frac{\pi}{4}\right] = 8a^2\left[-\frac{5}{3} + \frac{\pi}{2}\right] = 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

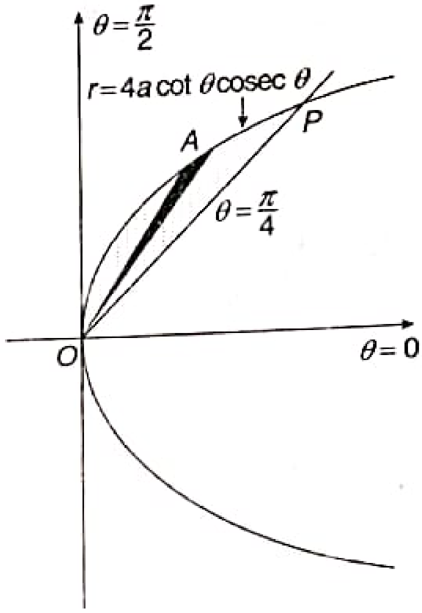

Fig. 7.37

EXERCISE 7.5

Change to polar coordinates and evaluate the following integrals:

1. $\iint \frac{1}{\sqrt{xy}}\,dx\,dy$, over the region bounded by the semicircle $x^2 + y^2 - x = 0, y \geq 0$

$$\left[\textbf{Ans.:}\ \frac{\pi}{\sqrt{2}}\right]$$

2. $\iint y^2\,dx\,dy$, over the area outside the circle $x^2 + y^2 - ax = 0$ and inside the circle $x^2 + y^2 - 2ax = 0$

$$\left[\textbf{Ans.:}\ \frac{15\pi a^4}{64}\right]$$

3. $\iint \sin(x^2+y^2)\,dx\,dy$, over the circle $x^2+y^2=a^2$

$\left[\textbf{Ans.: } \pi(1-\cos a^2)\right]$

4. $\iint xy(x^2+y^2)^{\frac{3}{2}}\,dx\,dy$, over the first quadrant of the circle $x^2+y^2=a^2$

$\left[\textbf{Ans.: } \frac{a^7}{14}\right]$

5. $\int_0^3\int_0^{\sqrt{3x}} \frac{dy\,dx}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans.: } \frac{3}{2}\log 3\right]$

6. $\int_0^a\int_0^x \frac{x^3\,dx\,dy}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans.: } \frac{a^4}{4}\log\left(1+\sqrt{2}\right)\right]$

7. $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sin\left[\frac{\pi}{a^2}(a^2-x^2-y^2)\right]dx\,dy$

$\left[\textbf{Ans.: } \frac{a^2}{2}\right]$

8. $\int_0^a\int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dx\,dy$

$\left[\textbf{Ans.: } \frac{\pi}{4}\left(1-e^{-a^2}\right)\right]$

9. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans.: } \frac{3\pi a^4}{4}\right]$

10. $\int_0^1\int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2}e^{-(x^2+y^2)}\,dx\,dy$

$\left[\textbf{Ans.: } \frac{1}{e}\right]$

11. $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} \log_e(x^2+y^2)\,dx\,dy$

$\left[\textbf{Ans.: } \frac{\pi}{4}a^2\left(\log a-\frac{1}{2}\right)\right]$

12. $\int_0^a\int_y^{a+\sqrt{a^2-y^2}} \frac{dx\,dy}{(4a^2+x^2+y^2)^3}$

$\left[\textbf{Ans.: } \frac{1}{8a^2}\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right)\right]$

13. $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$

$\left[\textbf{Ans.: } a\right]$

14. $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy}{x^2+y^2}e^{-(x^2+y^2)}\,dx\,dy$

$\left[\textbf{Ans.: } \frac{1}{4a^2}\left[1-(1+a^2)e^{-a^2}\right]\right]$

15. $\int_0^1\int_{x^2}^{x} \frac{dx\,dy}{\sqrt{x^2+y^2}}$

$\left[\textbf{Ans.: } \sqrt{2}-1\right]$

7.5.2 Change of Variables from Cartesian to other Coordinates

In some cases, evaluation of double integral becomes easier by changing the variables. Let the variables x, y be replaced by new variables u, v by the transformation $x = f_1(u,v), y = f_2(u,v)$. Then

$$\iint f(x,y)\,dx\,dy = \iint f(f_1,f_2)\,|J|\,du\,dv \qquad \ldots(7.2)$$

where

$$\text{Jacobian, } J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Using Eq. (7.2), the double integral can be transformed to new variables.

EXAMPLE 7.26 *Using the transformation $x-y=u$, $x+y=v$, evaluate $\iint \cos\left(\frac{x-y}{x+y}\right) dx\,dy$ over the region bounded by the lines $x=0$, $y=0$, $x+y=1$.*

Solution:

$$x-y=u,\ x+y=v$$

$$x=\frac{u+v}{2},\ y=\frac{v-u}{2}$$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$dx\,dy=|J|\,du\,dv=\frac{1}{2}du\,dv$$

The region bounded by the lines $x=0$, $y=0$ and $x+y=1$ in xy-plane is a triangle OPQ [Fig. 7.38].

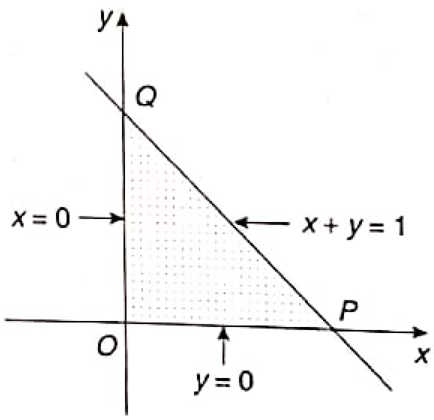

Fig. 7.38

Under the transformation $x=\frac{u+v}{2}$ and $y=\frac{v-u}{2}$,

(i) the line $x=0$ gets transformed to the line $u=-v$
(ii) the line $y=0$ gets transformed to the line $u=v$
(iii) the line $x+y=1$ gets transformed to the line $v=1$

Thus, the triangle OPQ in the xy-plane gets transformed to the triangle $OP'Q'$ in the uv-plane bounded by the lines $u=v$, $u=-v$ and $v=1$ [Fig. 7.39].

In the region, draw a horizontal strip AB parallel to the u-axis which starts from the line $u=-v$ and terminates on the line $u=v$.

Limits of u : $u=-v$ to $u=v$

Limits of v : $v=0$ to $v=1$

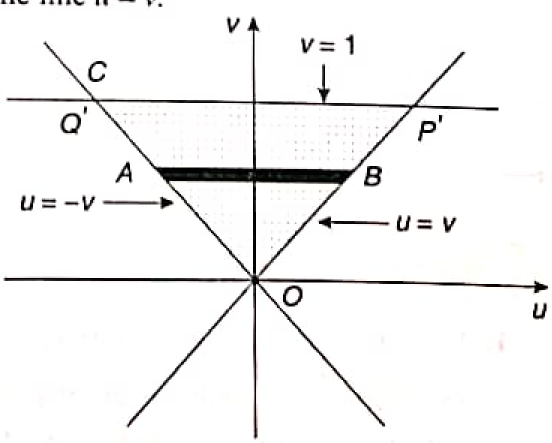

Fig. 7.39

$$I=\iint\cos\left(\frac{x-y}{x+y}\right)dx\,dy=\int_0^1\int_{-v}^{v}\cos\left(\frac{u}{v}\right)\frac{1}{2}du\,dv$$

$$=\frac{1}{2}\int_0^1\left|v\sin\left(\frac{u}{v}\right)\right|_{-v}^{v}dv=\frac{1}{2}\int_0^1 v[\sin 1-\sin(-1)]dv$$

$$=\frac{1}{2}\cdot 2\sin 1\left|\frac{v^2}{2}\right|_0^1=\frac{1}{2}\sin 1$$

EXAMPLE 7.27 *Evaluate* $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dx\,dy$, *over the first quadrant of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution:

Under the transformation $x = ar\cos\theta$, $y = br\sin\theta$, the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the xy-plane [Fig. 7.40] gets transformed to $r^2 = 1$ or $r = 1$, a circle with centre (0, 0) and radius 1 in the $r\theta$-plane [Fig. 7.41].

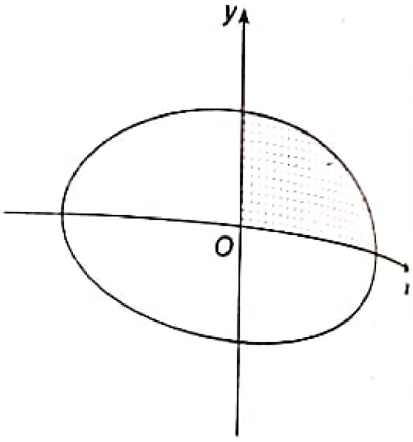

Fig. 7.40

$$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}a\cos\theta & -ar\sin\theta\\ b\sin\theta & br\cos\theta\end{vmatrix}=abr$$

$$dx\,dy=|J|\,dr\,d\theta=ab\,r\,dr\,d\theta$$

The region of integration is the part of the circle $r = 1$ in the first quadrant in the $r\theta$-plane. In the region, draw an elementary radius vector OA from the pole which terminates on the circle $r = 1$.

Limits of r : $r = 0$ to $r = 1$

Limits of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

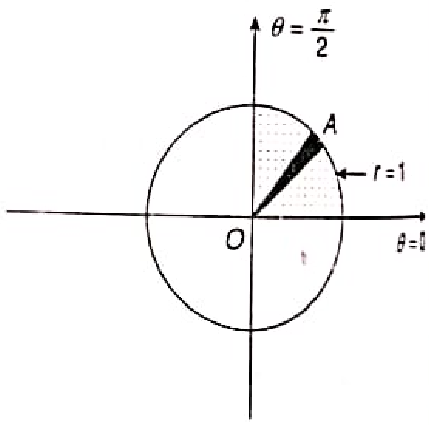

Fig. 7.41

$$I=\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}}dx\,dy$$

$$=\int_0^{\frac{\pi}{2}}\int_0^1 abr^2\cos\theta\sin\theta(r^2)^{\frac{n}{2}}ab\,r\,dr\,d\theta$$

$$=a^2b^2\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\int_0^1(r)^{n+3}dr=\frac{a^2b^2}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\left|\frac{r^{n+4}}{n+4}\right|_0^1$$

$$=\frac{a^2b^2}{4}(-\cos\pi+\cos 0)\cdot\frac{1}{n+4}=\frac{a^2b^2}{2(n+4)}$$

EXERCISE 7.6

1. Using the transformation $x+y=u, x-y=v$, evaluate $\iint e^{\frac{x-y}{x+y}}dx\,dy$ over the region bounded by $x = 0, y = 0$ and $x + y = 1$.

$$\left[\textbf{Ans.: } \frac{1}{4}\left(e-\frac{1}{e}\right)\right]$$

2. Using the transformation $x^2-y^2=u, 2xy=v$, evaluate $\iint(x^2-y^2)\,dx\,dy$ over the region bounded by the hyperbolas $x^2-y^2=1$, $x^2-y^2=9$, $xy=2$ and $xy=4$. [Ans.: 4]

3. Using the transformation $x + y = u$, $y = uv$, evaluate $\int_0^{\infty}\int_0^{\infty} e^{-(x+y)} x^{p-1} y^{q-1}\, dx\, dy$.

$\left[\text{Ans.}: \overline{|p}\,\overline{|q}\right]$

4. Using the transformation $x = u$, $y = uv$, evaluate $\int_0^1\int_0^x \sqrt{x^2+y^2}\, dx\, dy$.

$\left[\text{Ans.}: \frac{1}{3}\left[\frac{\sqrt{2}}{2}+\frac{1}{2}\log\left(1+\sqrt{2}\right)\right]\right]$

5. Evaluate $\iint (x+y)^2\, dx\, dy$ by changing the variables over the region bounded by the parallelogram with sides $x+y=0$, $x+y=2$, $3x-2y=0$ and $3x-2y=3$.

$\left[\text{Ans.}: \frac{8}{5}\right]$

6. Evaluate $\iint (x-y)^4 e^{x+y}\, dx\, dy$, by changing the variables over the region bounded by the square with vertices at $(1, 0)$, $(2, 1)$, $(1, 2)$, $(0, 1)$.

$\left[\text{Ans.}: \frac{e^3-e}{5}\right]$

7. Evaluate $\iint [xy(1-x-y)]^{\frac{1}{2}}\, dx\, dy$, by changing the variables over the region bounded by the triangle with sides $x=0$, $y=0$, $x+y=1$.

$\left[\text{Ans.}: \frac{2\pi}{105}\right]$

7.6 AREA ENCLOSED BY PLANE CURVES USING DOUBLE INTEGRALS

7.6.1 Area in Cartesian Coordinates

(i) The area A bounded by the curves $y = y_1(x)$ and $y = y_2(x)$ intersecting at the points $P(a, b)$ and $Q(c, d)$ (Fig 7.42) is

$$A = \int_a^c \int_{y_1(x)}^{y_2(x)} dy\, dx$$

(ii) If the equation of the curves are represented as $x = x_1(y)$ and $x = x_2(y)$ then

$$A = \int_b^d \int_{x_1(y)}^{x_2(y)} dx\, dy$$

Note Consider the symmetricity of the region while finding the area.

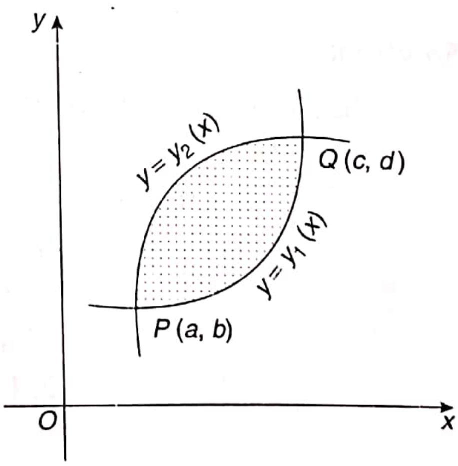

Fig. 7.42 Area bounded by curves

EXAMPLE 7.28 *Find the area enclosed by the parabola $y^2 = 4ax$ and the lines $x + y = 3a$, $y = 0$ in the first quadrant.*

Solution:

1. The points of intersection of the parabola $y^2 = 4ax$ and the line $x + y = 3a$ are obtained as

$$y^2 = 4a(3a - y)$$
$$y^2 + 4ay - 12a^2 = 0$$

$$y = 2a, -6a$$
$$\therefore \quad x = a, 9a$$

The point of intersection is $Q\,(a, 2a)$ which lies in the first quadrant.

2. Area enclosed in the first quadrant is OPQ.
Draw a horizontal strip AB which starts from the parabola $y^2 = 4ax$ and terminates on the line $x + y = 3a$ (Fig. 7.43).

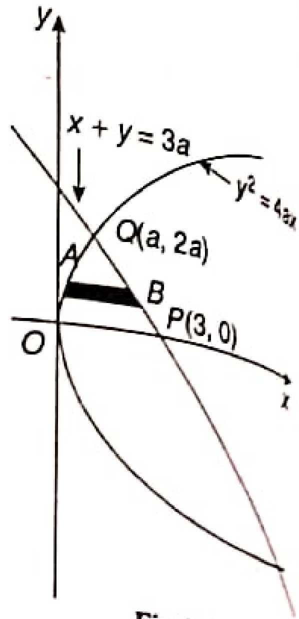

Fig. 7.43

Limits of $x : x = \dfrac{y^2}{4a}$ to $x = 3a - y$

Limits of $y : y = 0$ to $y = 2a$

$$A = \int_0^{2a}\int_{\frac{y^2}{4a}}^{3a-y} dx\,dy = \int_0^{2a} |x|_{\frac{y^2}{4a}}^{3a-y}\,dy$$

$$= \int_0^{2a}\left(3a - y - \frac{y^2}{4a}\right)dy = \left|3ay - \frac{y^2}{2} - \frac{1}{4a}\cdot\frac{y^3}{3}\right|_0^{2a} = 6a^2 - 2a^2 - \frac{1}{4a}\cdot\frac{8a^3}{3} = \frac{10}{3}a^2$$

Note Here, two vertical strips are required to cover the entire region. Hence, one horizontal strip is preferred over the vertical strip.

EXAMPLE 7.29 *Find the area between the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$.*

Solution:

1. The points of intersection of the rectangular hyperbola $3xy = 2$ and the line $12x + y = 6$ are obtained as

$$3x(6 - 12x) = 2$$
$$18x^2 - 9x + 1 = 0$$
$$x = \frac{1}{3}, \frac{1}{6}$$
$$\therefore y = 2, 4$$

The points of intersection are $P\left(\dfrac{1}{3}, 2\right)$ and $Q\left(\dfrac{1}{6}, 4\right)$.

2. Draw a vertical strip AB in the region which starts from the rectangular hyperbola $3xy = 2$ and terminates on the line $12x + y = 6$ Fig (7.44).

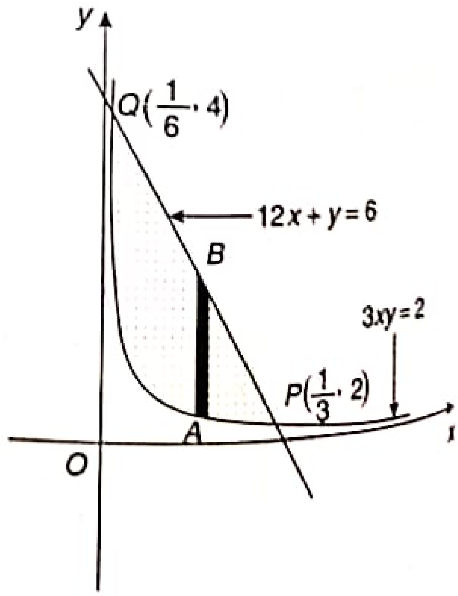

Fig. 7.44

Limits of y : $y = \dfrac{2}{3x}$ to $y = 6 - 12x$

Limits of x : $x = \dfrac{1}{6}$ to $x = \dfrac{1}{3}$

$$A=\int_{\frac{1}{6}}^{\frac{1}{3}}\int_{\frac{2}{3x}}^{6-12x}dy\,dx=\int_{\frac{1}{6}}^{\frac{1}{3}}|y|_{\frac{2}{3x}}^{6-12x}dx$$

$$=\int_{\frac{1}{6}}^{\frac{1}{3}}\left(6-12x-\frac{2}{3x}\right)dx=\left|6x-6x^2-\frac{2}{3}\log x\right|_{\frac{1}{6}}^{\frac{1}{3}}$$

$$=(2-1)-6\left(\frac{1}{9}-\frac{1}{36}\right)-\frac{2}{3}\left(\log\frac{1}{3}-\log\frac{1}{6}\right)=\frac{1}{2}-\frac{2}{3}\log 2$$

EXAMPLE 7.30 *Find the area bounded by the hypocycloid* $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

Solution:

1. The hypocycloid is symmetric in all the quadrants.
   Total area = 4 (area in the first quadrant)
2. Draw a vertical strip AB parallel to the y-axis in the region which lies in the first quadrant. AB starts from the x-axis and terminates on the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$ (Fig. 7.45).

Limits of y : $y=0$ to $y=b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}$

Limits of x : $x=0$ to $x=a$

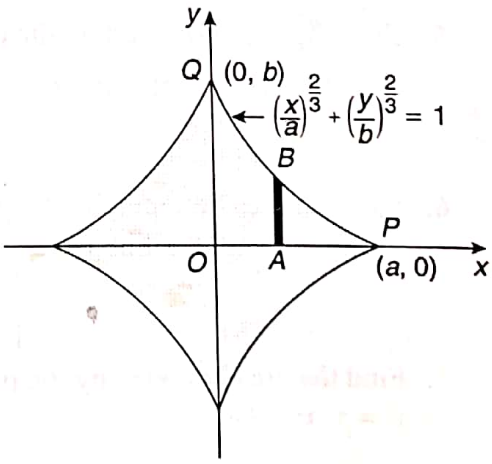

Fig. 7.45

$$A=4\int_0^a\int_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}}dy\,dx=4\int_0^a|y|_0^{b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}}dx$$

$$=4\int_0^a b\left[1-\left(\frac{x}{a}\right)^{\frac{2}{3}}\right]^{\frac{3}{2}}dx$$

Putting $x=a\cos^3 t$, $dx=3a\cos^2 t\,(-\sin t)\,dt$

When $x=0, t=\frac{\pi}{2}$

When $x=a, t=0$

$$A=4\int_{\frac{\pi}{2}}^{0}b(1-\cos^2 t)^{\frac{3}{2}}(-3a\cos^2 t\sin t)dt=12ab\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t\,dt$$

$$= 12ab\frac{1}{2}B\left(\frac{5}{2},\frac{3}{2}\right) = 6ab\frac{\left\lceil\frac{5}{2}\right.\left\lceil\frac{3}{2}\right.}{\lceil 4} = 6ab\frac{\frac{3}{2}\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.\cdot\frac{1}{2}\left\lceil\frac{1}{2}\right.}{3!} = \frac{3}{8}\pi$$

EXERCISE 7.7

1. Find the area bounded by the y-axis, the line $y = 2x$ and the line $y = 4$.
[Ans.: 4]

2. Find the area bounded by the lines $y = 2 + x$, $y = 2 - x$ and $x = 5$.
[Ans.: 25]

3. Find the area bounded by the parabola $y^2 + x = 0$, and the line $y = x + 2$.
$\left[\textbf{Ans.: } \frac{9}{2}\right]$

4. Find the area bounded by the parabola $x = y + y^2$ and the line $x + y = 0$.
$\left[\textbf{Ans.: } \frac{4}{3}\right]$

5. Find the area bounded by the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.
$\left[\textbf{Ans.: } \frac{1}{3}\right]$

6. Find the area bounded by the parabola $y = x^2 - 3x$ and the line $y = 2x$.
$\left[\textbf{Ans.: } \frac{125}{6}\right]$

7. Find the area bounded by the parabolas $y^2 = x$, $x^2 = -8y$.
$\left[\textbf{Ans.: } \frac{8}{3}\right]$

8. Find the area bounded by the parabolas $y = ax^2$ and $y = 1 - \frac{x^2}{a}$, where $a > 0$.
$\left[\textbf{Ans.: } \frac{4}{3}\sqrt{\frac{a}{a^2+1}}\right]$

9. Find the area bounded by the curve $y^2(2a - x) = x^3$ and its asymptote.
[Ans.: $3\pi a^2$]

10. Find the area of the loop of the curve $y^2 = x^2\left(\frac{a+x}{a-x}\right)$
$\left[\textbf{Ans.: } 2a^2\left(\frac{\pi}{4}-1\right)\right]$

11. Find the area of one of the loops of $x^4 + y^4 = 2a^2xy$.
$\left[\textbf{Ans.: } \frac{\pi a^2}{4}\right]$

12. Find the area enclosed by the curve $9xy = 4$ and the line $2x + y = 2$.
$\left[\textbf{Ans.: } \frac{1}{3} - \frac{4}{9}\log 2\right]$

13. Find the area of the smaller region bounded by the circle $x^2 + y^2 = 9$ and a straight line $x = 3 - y$.
$\left[\textbf{Ans.: } 4\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right]$

14. Find the area bounded by the x-axis, circle $x^2 + y^2 = 16$ and the line $y = x$.
[Ans.: 2π]

15. Find the area bounded between the curves $y = 3x^2 - x - 3$ and $y = -2x^2 + 4x + 7$.
$\left[\textbf{Ans.: } \frac{45}{2}\right]$

16. Find the area bounded by the asteroid $(x)^{\frac{2}{3}}+(y)^{\frac{2}{3}}=(a)^{\frac{2}{3}}$. $\left[\text{Ans.}: \frac{3}{8}\pi a^2\right]$

7.6.2 Area in Polar Coordinates

The area A bounded by the curves $r=r_1(\theta)$, $r=r_2(\theta)$ and the lines $\theta=\theta_1$ and $\theta=\theta_2$ (Fig. 7.46) is

$$A=\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta$$

Note Consider the symmetricity of the region while finding the area.

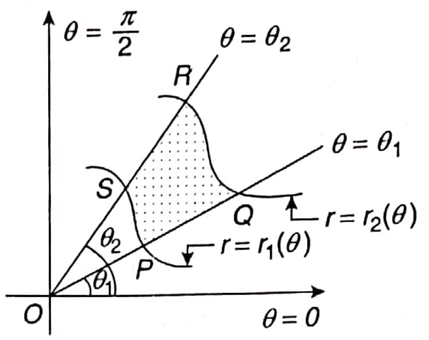

Fig. 7.46 Area bounded by two curves

EXAMPLE 7.31 *Find the area which lies inside the circle $r=3a\cos\theta$ and outside the cardioid $r=a(1+\cos\theta)$.*

Solution:

1. The points of intersection of the circle $r=3a\cos\theta$ and the cardioid $r=a(1+\cos\theta)$ are obtained as

$$3a\cos\theta=a(1+\cos\theta)$$

$$\cos\theta=\frac{1}{2}$$

$$\theta=\pm\frac{\pi}{3}$$

Hence, $\theta=\frac{\pi}{3}$ at R.

2. The region is symmetric about the initial line $\theta=0$. Total area $=2$ (area above the initial line)
3. Draw an elementary radius vector OAB from the origin in the region above the initial line (Fig. 7.47). OAB enters in the region from the cardioid $r=a(1+\cos\theta)$ and terminates on the circle $r=3a\cos\theta$.

Limits of $r: r=a(1+\cos\theta)$ to $r=3a\cos\theta$

Limits of $\theta: \theta=0$ to $\theta=\frac{\pi}{3}$

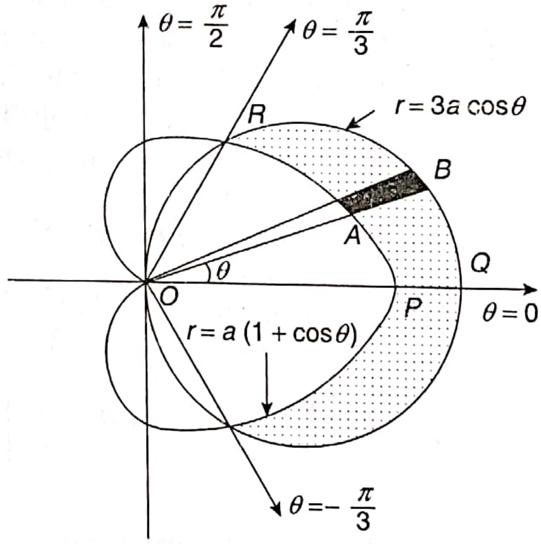

Fig. 7.47

$$A=2\int_0^{\frac{\pi}{3}}\int_{a(1+\cos\theta)}^{3a\cos\theta} r\,\mathrm{d}r\,\mathrm{d}\theta=2\int_0^{\frac{\pi}{3}}\left|\frac{r^2}{2}\right|_{a(1+\cos\theta)}^{3a\cos\theta}\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{3}} \left[9a^2\cos^2\theta - a^2(1+\cos\theta)^2\right] d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left[8a^2\cos^2\theta - a^2 - 2a^2\cos\theta\right] d\theta$$

$$= a^2\int_0^{\frac{\pi}{3}} [4(1+\cos 2\theta) - 1 - 2\cos\theta]\, d\theta$$

$$= a^2\left|3\theta + \frac{4\sin 2\theta}{2} - 2\sin\theta\right|_0^{\frac{\pi}{3}}$$

$$= a^2\left(3\frac{\pi}{3} + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}\right) = \pi a^2$$

EXAMPLE 7.32 *Find the area between the curve $r = a\ (\sec\theta + \cos\theta)$ and its asymptote $r = a \sec\theta$.*

Solution:

1. The region is symmetric about the initial line $\theta = 0$
   Total area = 2(area above the initial line)
2. Draw an elementary radius vector OAB in the region above the initial line (Fig. 7.48).
   OAB enters in the region from the line $r = a\sec\theta$ and terminates on the curve $r = a\,(\sec\theta + \cos\theta)$.
   Limits of r: $r = a\sec\theta$ to $r = a\,(\sec\theta + \cos\theta)$
   Limits of θ: $\theta = 0$ to $\theta = \frac{\pi}{2}$

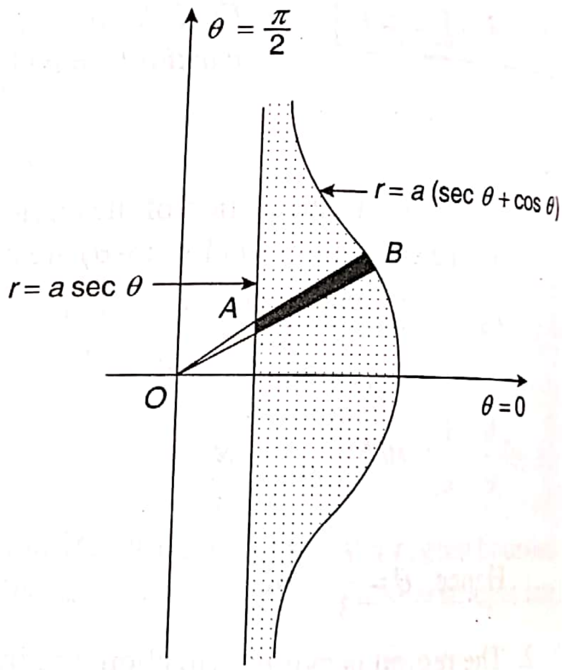

Fig. 7.48

$$A = 2\int_0^{\frac{\pi}{2}}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\, dr\, d\theta = 2\int_0^{\frac{\pi}{2}}\left|\frac{r^2}{2}\right|_{a\sec\theta}^{a(\sec\theta+\cos\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{2}} [a^2(\sec\theta + \cos\theta)^2 - a^2\sec^2\theta]\, d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}} (\cos^2\theta + 2)\, d\theta$$

$$= a^2\left[\frac{1}{2}B\left(\frac{3}{2}, \frac{1}{2}\right) + |2\theta|_0^{\frac{\pi}{2}}\right]$$

$$= a^2\left[\frac{1}{2}\frac{\Gamma\frac{3}{2}\,\Gamma\frac{1}{2}}{\Gamma 2} + \frac{2\pi}{2}\right] = \frac{5\pi}{4}a^2$$

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EXERCISE 7.8

1. Find the area common to the circles $r = a$ and $r = 2a\cos\theta$.

$$\left[\textbf{Ans.: } a^2\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\right]$$

2. Find the area of the crescent bounded by the circles $r=\sqrt{2}$ and $r=2\cos\theta$.

$$[\textbf{Ans.: } 1]$$

3. Find the area which lies inside the circle $r = a \sin\theta$ and outside the cardioid $r=a(1-\cos\theta)$.

$$\left[\textbf{Ans.: } \frac{a^2(4-\pi)}{4}\right]$$

4. Find the area which lies inside the cardioid $r = 2a(1 + \cos\theta)$ and outside the parabola $r=\dfrac{2a}{1+\cos\theta}$.

$$\left[\textbf{Ans.: } 3\pi a^2+\frac{16a^2}{3}\right]$$

5. Find the area bounded between the circles $r=2a\sin\theta, r=2b\sin\theta\, (b>a)$.

$$\left[\textbf{Ans.: } \pi(b^2-a^2)\right]$$

6. Find the area outside the circle $r = a$ and inside the cardioid $r=a(1+\cos\theta)$.

$$\left[\textbf{Ans.: } \frac{a^2}{4}(\pi+8)\right]$$

7.7 AREA OF A CURVED SURFACE USING DOUBLE INTEGRALS

Let R_1 be the projection of the surface $z=f(x, y)$ on the xy-plane. Let δS be the surface area of a small element of the surface.

$\delta x \delta y$ = projection of δS on the xy-plane = $\delta S\cos\gamma$

where γ is the angle between the z-axis and normal at δS (Fig. 7.49).

Let $F(x, y, z)=f(x, y)-z$

Direction cosines normal to the surface $F(x, y, z)=0$ are proportional to

$$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \quad \text{i.e., } -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1$$

Direction cosines of the z-axis are 0, 0, 1.

$$\cos\gamma=\frac{\left(-\frac{\partial z}{\partial x}\right)0+\left(-\frac{\partial z}{\partial y}\right)0+(1)1}{\sqrt{\left(-\frac{\partial z}{\partial x}\right)^2+\left(-\frac{\partial z}{\partial y}\right)^2+(1)^2}\sqrt{(0)^2+(0)+(1)^2}}$$

$$=\frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}}$$

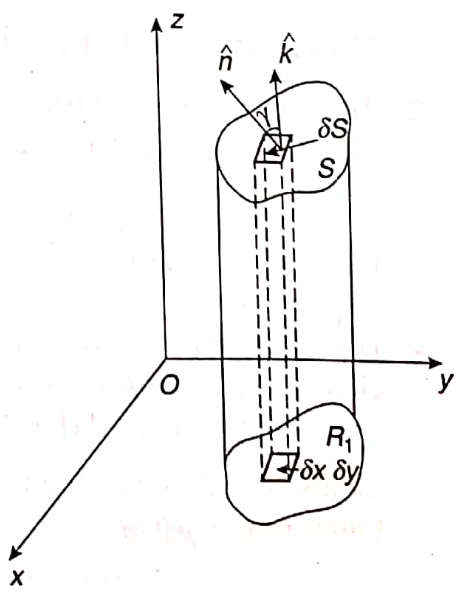

Fig. 7.49

$$\delta S = \frac{\delta x\,\delta y}{\cos\gamma} = \left[\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\right]\delta x\,\delta y$$

Total surface area is

$$S = \lim_{\delta S \to 0} \sum \delta S = \iint_{R_1} \left[\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\right] \mathrm{d}x\mathrm{d}y$$

Note If R_2 and R_3 are projections of S on the yz- and zx-planes respectively then

$$S = \iint_{R_2} \left[\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\right] \mathrm{d}y\mathrm{d}z$$

$$\text{and } S = \iint_{R_3} \left[\sqrt{\left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + 1}\right] \mathrm{d}z\mathrm{d}x$$

EXAMPLE 7.33 *Find the area of the portion of the cylinder $x^2 + z^2 = 4$ lying inside the cylinder $x^2 + y^2 = 4$.*

Solution:

1. For the cylinder $x^2 + z^2 = 4$,

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = 0$$

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2} = \frac{z^2 + x^2}{z^2} = \frac{4}{4 - x^2}$$

The region is symmetric in all octants.

2. Total surface area = 8 (surface area of the upper portion of $x^2 + z^2 = 4$ lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)
3. The projection of the given surface on the xy-plane is the part of the circle $x^2 + y^2 = 4$ in the positive quadrant (Fig. 7.50)
4. Draw a vertical strip AB in the positive quadrant of the circle $x^2 + y^2 = 4$. AB starts from the x-axis and terminates on the circle $x^2 + y^2 = 4$ (Fig. 7.51).

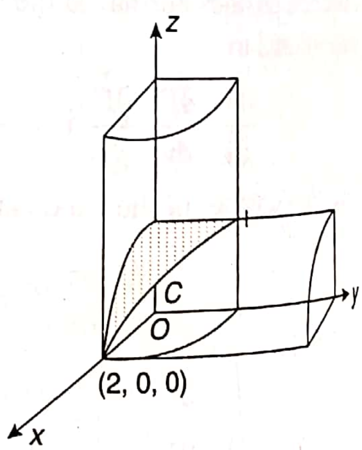

Fig. 7.50

Limits of $y : y = 0$ to $y = \sqrt{4 - x^2}$
Limits of $x : x = 0$ to $x = 2$

$$S = 8\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \mathrm{d}y\mathrm{d}x$$

$$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{\frac{4}{4-x^2}}\, dy dx = 8\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\, dy dx$$

$$= 16\int_0^2 \left| \frac{1}{\sqrt{4-x^2}}\, y \right|_0^{\sqrt{4-x^2}} dx = 16\int_0^2 \frac{1}{\sqrt{4-x^2}} \sqrt{4-x^2}\, dx$$

$$= 16\int_0^2 dx = 16|x|_0^2 = 32$$

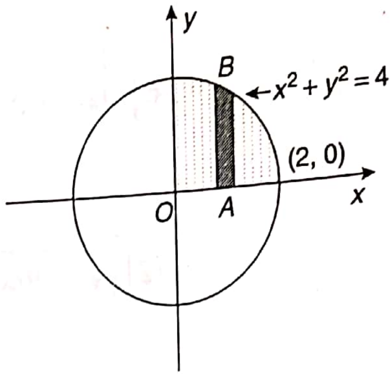

Fig. 7.51

EXAMPLE 7.34 *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution:

1. For the sphere $x^2 + y^2 + z^2 = 9$,

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{z^2 + x^2 + y^2}{z^2} = \frac{9}{9 - x^2 - y^2}$$

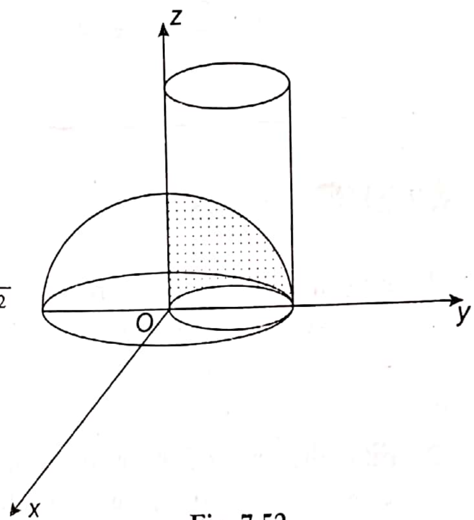

Fig. 7.52

2. The region is symmetric about the y and z-axis but not about the x-axis (Fig. 7.52).
   Total surface area = 4 (surface area of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$)
3. The projection of the given surface on the xy-plane is the circle $x^2 + y^2 = 3y$ (Fig. 7.53)
4. Putting $x = r\cos\theta$, $y = r\sin\theta$, the polar form of the circle $x^2 + y^2 = 3y$ is $r^2 = 3r\sin\theta$, $r = 3\sin\theta$
5. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 3\sin\theta$.
   Limits of r : $r = 0$ to $r = 3\sin\theta$
   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$ (in the positive quadrant)

$$S = 4\iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx dy$$

$$= 4\iint \frac{3}{\sqrt{9 - x^2 - y^2}}\, dx\, dy$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^{3\cos\theta} \frac{3}{\sqrt{9 - r^2}}\, r\, dr\, d\theta$$

$$= 12\int_0^{\frac{\pi}{2}} \left[\int_0^{3\cos\theta} (9 - r^2)^{-\frac{1}{2}} \frac{(-2r)}{(-2)}\, dr \right] d\theta$$

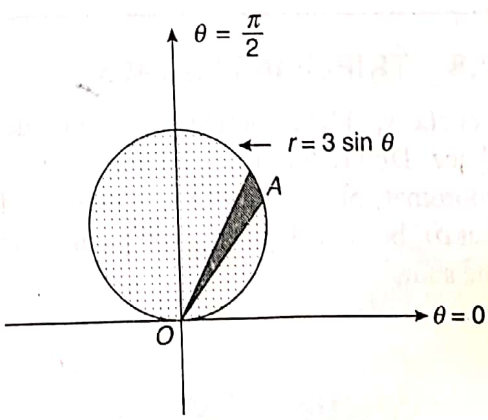

Fig. 7.53

$$= -6\int_0^{\frac{\pi}{2}} \left| 2(9-r^2)^{\frac{1}{2}} \right|_0^{3\cos\theta} d\theta$$

$$= -12\int_0^{\frac{\pi}{2}} \left(\sqrt{9-9\cos^2\theta} - \sqrt{9}\right) d\theta = -36\int_0^{\frac{\pi}{2}} (\sin\theta - 1) d\theta$$

$$= -36\left| -\cos\theta - \theta \right|_0^{\frac{\pi}{2}} = 36\left(\cos\frac{\pi}{2} + \frac{\pi}{2} - \cos 0\right)$$

$$= 36\left(\frac{\pi}{2} - 1\right) = 18(\pi - 2)$$

EXERCISE 7.9

1. Find the area of the portion of the cylinder $x^2 + y^2 = 4y$ lying inside the sphere $x^2 + y^2 + z^2 = 16$.

   [**Ans.:** 64]

2. Find the area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ lying inside the cylinder $x^2 + y^2 = ax$. [**Ans.:** $2(\pi - 2)a^2$]

3. Find the area of the surface of the cone $x^2 + y^2 = z^2$ cut off by the surface of the cylinder $x^2 + y^2 = a^2$ above the xy-plane.

   [**Ans.:** $2\pi a^2$]

4. Find the area of that part of the plane $x + y + z = 2a$ which lies in the first octant and is bounded by the cylinder $x^2 + y^2 = a^2$

   $\left[\textbf{Ans.:} \dfrac{3\pi a^2}{4}\right]$

5. Find the surface area of the cone $x^2 + y^2 = 3z^2$ which lies above the plane xoy and inside the cylinder $x^2 + y^2 = 4y$.

   $\left[\textbf{Ans.:} \dfrac{8\sqrt{3}}{3}\pi\right]$

7.8 TRIPLE INTEGRALS

Let $f(x, y, z)$ be a continuous function defined in a closed and bounded region V in three-dimensional space. Divide the region V into small elementary parallelepipeds by drawing planes parallel to the coordinate planes. Let the total number of complete parallelepipeds which lie inside the region V be n. Let δV_r be the volume of the r^{th} parallelepiped and (x_r, y_r, z_r) be any point in this parallelepiped. Consider the sum

$$S = \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r \qquad ...(7.3)$$

where

$$\delta V_r = \delta x_r \cdot \delta y_r \cdot \delta z_r$$

If the number of elementary parallelepipeds, n is increased then the volume of each parallelepiped decreases. Hence, as $n \to \infty$, $\delta V_r \to 0$.

The limit of the sum given by Eq. (7.3), if it exists is called the triple integral of $f(x, y, z)$ over the region V and is denoted by $\iiint_V f(x, y, z)\mathrm{d}V$

Hence,
$$\iiint_V f(x, y, z)\mathrm{d}V = \lim_{\substack{n\to\infty \\ \delta V_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\delta V_r$$

where
$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

7.8.1 Triple Integrals in Cartesian Coordinates

Triple integral of a continuous function $f(x, y, z)$ over a region V can be evaluated by three successive integrations.

Let the region V be bounded below by a surface $z = z_1(x, y)$ and above by a surface $z = z_2(x, y)$. Let the projection of the region V in the xy-plane be R which is bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and $x = a, x = b$. Then the triple integral is defined as

$$I = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} \left\{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\,\mathrm{d}z\right\} \mathrm{d}y\right] \mathrm{d}x$$

Note The order of variables in dx dy dz indicates the order of integration. In some cases, this order is not maintained. Therefore, it is advisable to identify the order of integration with the help of the limits.

7.8.2 Triple Integrals in Cylindrical Coordinates

Cylindrical coordinates r, θ, z are used to evaluate the integral in the regions which are bounded by cylinders along the z-axis, planes through the z-axis, planes perpendicular to the z-axis [Fig. 7.54].

Relations between Cartesian (rectangular) coordinates (x, y, z) and cylindrical coordinates (r, θ, ϕ) are given as

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Then

$$\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\cos\theta, r\sin\theta, z)|J|\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

where
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

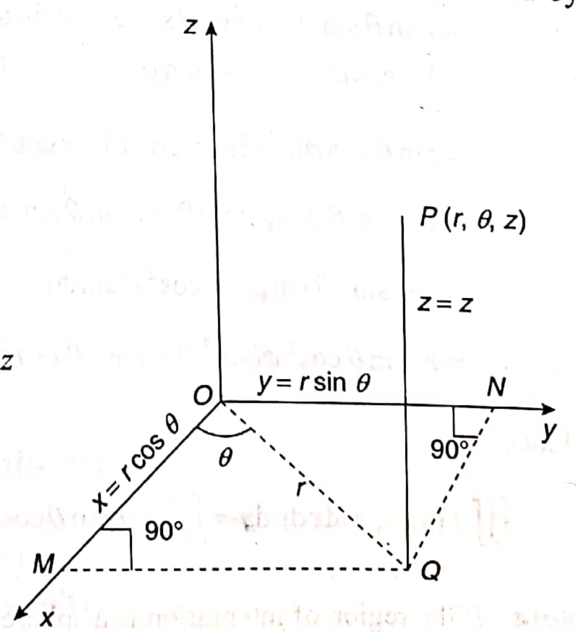

Fig. 7.54 Triple integration in cylindrical coordinates

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta(r\cos\theta) + r\sin\theta(\sin\theta) = r$$

Hence, $\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

7.8.3 Triple Integrals in Spherical Coordinates

Spherical coordinates (r, θ, ϕ) are used to evaluate the integral in the regions which are bounded by the sphere with centre at the origin, [Fig. 7.55].

Relations between Cartesian (rectangular) coordinates (x, y, z) and spherical coordinates (r, θ, ϕ) are given as

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Then $\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)|J|\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$

where $J = \dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi(r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi$$
$$(-r\sin\theta\cos\phi\cos\theta) - r\sin\theta\sin\phi$$
$$(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$
$$= r^2\sin\theta\cos^2\phi(\sin^2\theta + \cos^2\theta) + r^2\sin\theta\sin^2\phi = r^2\sin\theta$$

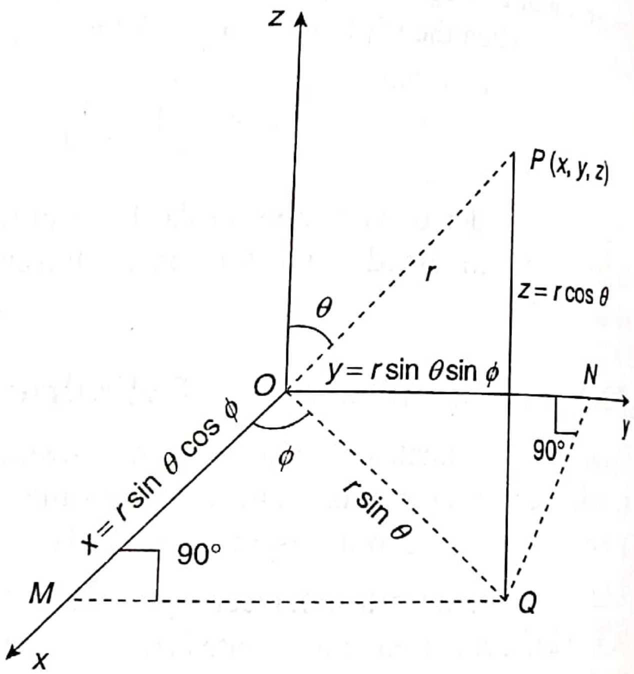

Fig. 7.55 Triple integration in spherical coordinates

Hence,

$$\iiint f(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta) r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Note If the region of integration is a sphere $x^2 + y^2 + z^2 = a^2$ with centre at $(0, 0, 0)$ and radius a then limits of r, θ, ϕ are

(i) For a positive octant of the sphere,

$$r : r = 0 \quad \text{to} \quad r = a$$

$$\theta : \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$$

$$\phi : \phi = 0 \quad \text{to} \quad \phi = \frac{\pi}{2}$$

(ii) For a hemisphere,

$$r : r = 0 \quad \text{to} \quad r = a$$

$$\theta : \theta = 0 \quad \text{to} \quad \theta = \frac{\pi}{2}$$

$$\phi : \phi = 0 \quad \text{to} \quad \phi = 2\pi$$

(iii) For the complete sphere,

$$r : r = 0 \quad \text{to} \quad r = a$$

$$\theta : \theta = 0 \quad \text{to} \quad \theta = \pi$$

$$\phi : \phi = 0 \quad \text{to} \quad \phi = 2\pi$$

7.8.4 Change of Variables

In some cases, evaluation of a triple integral becomes easier by changing the variables. Let the variables x, y, z be replaced by new variables u, v, w by the transformation $x = f_1(u, v, w)$, $y = f_2(u, v, w)$, $z = f_3(u, v, w)$.

Then $$\iiint f(x, y, z)\,dx\,dy\,dz = \iiint f(f_1, f_2, f_3)\,|J|\,du\,dv\,dw$$

where $$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

7.8.5 Working Rule for Evaluation of Triple Integrals

1. Draw all the planes and surfaces and identify the region of integration.
2. Draw an elementary volume parallel to the z (y or x) axis.
3. Find the variation of z (y or x) along the elementary volume.
4. Lower and upper limits of z (y or x) are obtained from the equation of the surface (or plane), where elementary volume starts and terminates respectively.

5. Find the projection of the region on the xy (zx or yz) plane.
6. Draw the region of projection in the xy (zx or yz) plane.
7. Follow the steps of double integration to find the limits of x and y (z and x or y and z).

Notes

(i) If the region is bounded by the cylinders along the z-axis, planes through the z-axis, the planes perpendicular to the z-axis then the cylindrical coordinates are used.

(ii) If the region is bounded by the sphere then the spherical coordinates are used.

Type I Evaluation of Triple Integrals

EXAMPLE 7.35 *Evaluate* $\int_0^1\int_0^2\int_0^e dy\,dx\,dz.$

Solution:

$$\int_0^1\int_0^2\int_0^e dy\,dx\,dz = \int_0^1\int_0^2\left[\int_0^e dy\right]dx\,dz = \int_0^1\int_0^2 \left|y\right|_0^e dx\,dz$$

$$= \int_0^1\left[\int_0^2 e\,dx\right]dz = e\int_0^1 \left|x\right|_0^2 dz = e\int_0^1 2\,dz = 2e\left|z\right|_0^1 = 2e$$

Another Method Since all the limits are constant and integrand (function) is explicit in x, y and z, the integral can be written as

$$\int_0^1\int_0^2\int_0^e dy\,dx\,dz = \int_0^1 dz\cdot\int_0^2 dx\cdot\int_0^e dy = \left|z\right|_0^1\cdot\left|x\right|_0^2\cdot\left|y\right|_0^e$$
$$= 1\cdot 2\cdot e = 2e$$

EXAMPLE 7.36 *Evaluate* $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dx\,dy\,dz.$

Solution: The innermost limits depend on x and y. Hence, integrating first w.r.t. z,

$$I = \int_0^1\int_0^{1-x}\left[\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dz\right]dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left|\frac{1}{-2(x+y+z+1)^2}\right|_0^{1-x-y} dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{\{x+y+(1-x-y)+1\}^2} - \frac{1}{(x+y+1)^2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\left[\int_0^{1-x}\left\{\frac{1}{4} - \frac{1}{(x+y+1)^2}\right\}dy\right]dx$$

$$= -\frac{1}{2}\int_0^1 \left|\frac{y}{4} + \frac{1}{x+y+1}\right|_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{x+(1-x)+1} - \frac{1}{x+1}\right] dx$$

$$= -\frac{1}{2}\int_0^1 \left(\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right) dx$$

$$= -\frac{1}{2}\left|\frac{x}{4} - \frac{x^2}{8} + \frac{x}{2} - \log(x+1)\right|_0^1 = -\frac{1}{2}\left(\frac{5}{8} - \log 2\right)$$

EXAMPLE 7.37 Evaluate $\int_0^\infty \int_0^\infty \int_0^\infty \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$.

Solution:

1. It is difficult to integrate this integral in Cartesian form. Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta \sin\phi$, $z = r\cos\theta$, the integral changes to spherical form.
2. Limits of $x : x = 0$ to $x \to \infty$

   Limits of $y : y = 0$ to $y \to \infty$

   Limits of $z : z = 0$ to $z \to \infty$

   The region of integration is the positive octant of the plane [Fig. 7.56].

   Limits of $r : r = 0$ to $r \to \infty$

   Limits of $\theta = 0$ to $\theta = \frac{\pi}{2}$

   Limits of $\phi : \phi = 0$ to $\phi = \frac{\pi}{2}$

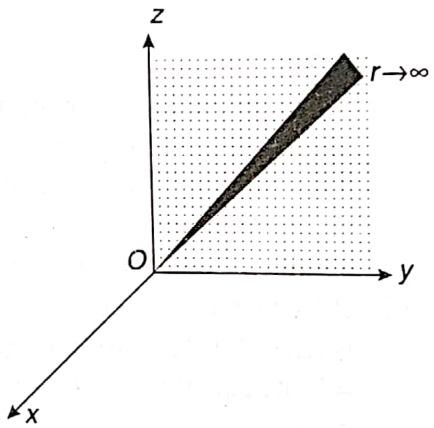

Fig. 7.56

Hence, the spherical form of the given integral is

$$I = \int_0^\infty \int_0^\infty \int_0^\infty \frac{dxdydz}{(1+x^2+y^2+z^2)^2} = \int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{r^2 \sin\theta\, dr\, d\theta\, d\phi}{(1+r^2)^2}$$

$$= \int_0^\infty \frac{r^2 dr}{(1+r^2)^2} \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} d\phi$$

Putting $r = \tan t$, $dr = \sec^2 t\, dt$

When $r = 0, t = 0$

When $r \to \infty, t = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{\sec^4 t} \sec^2 t \, dt \int_0^{\frac{\pi}{2}} d\phi = \left|-\cos\theta\right|_0^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{2}} \sin^2 t \, dt\right) |\phi|_0^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{2} + \cos 0\right) \cdot \left(\int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2} dt\right) \cdot \frac{\pi}{2} = \frac{1}{2}\left| t - \frac{\sin 2t}{2}\right|_0^{\frac{\pi}{2}} \frac{\pi}{2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}(\sin\pi - \sin 0)\right]\frac{\pi}{2} = \frac{\pi^2}{8}$$

Type II Evaluation of Triple Integrals Over the Given Region

EXAMPLE 7.38 *Evaluate* $\iiint x^2 yz \, dx \, dy \, dz$ *over the region bounded by the planes* $x = 0$, $y = 0, z = 0$ *and* $x + y + z = 1$.

Solution:

1. Draw an elementary volume AB parallel to the z-axis in the region [Fig. 7.57]. AB starts from xy-plane and terminates on the plane $x + y + z = 1$.
   Limits of $z : z = 0$ to $z = 1 - x - y$
2. Projection of the plane $x + y + z = 1$ in xy-plane is ΔOPQ [Fig. 7.58]. Putting $z = 0$ in $x + y + z = 1$, the equation of the line PQ is obtained as $x + y = 1$.
3. Draw a vertical strip $A'B'$ in the region OPQ. $A'B'$ starts from the x-axis and terminates on the line $x + y = 1$.
   Limits of $y : y = 0$ to $y = 1 - x$
   Limits of $x : x = 0$ to $x = 1$

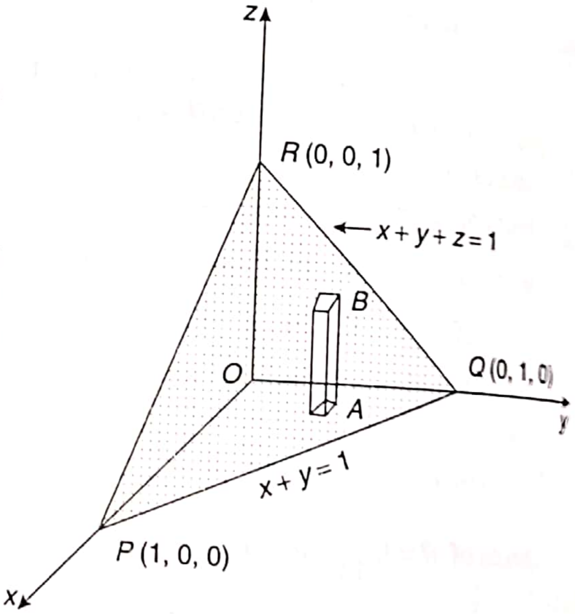

Fig. 7.57

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2 yz \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} x^2 y \left|\frac{z^2}{2}\right|_0^{1-x-y} dy \, dx$$

$$= \frac{1}{2}\int_0^1 x^2 \left[\int_0^{1-x} y\left\{(1-x)^2 + y^2 - 2y(1-x)\right\} dy\right] dx$$

$$= \frac{1}{2}\int_0^1 x^2 \left[\int_0^{1-x} \left\{y(1-x)^2 + y^3 - 2y^2(1-x)\right\} dy\right] dx$$

$$= \frac{1}{2}\int_0^1 x^2 \left|(1-x)^2 \frac{y^2}{2} + \frac{y^4}{4} - 2(1-x)\frac{y^3}{3}\right|_0^{1-x} dx$$

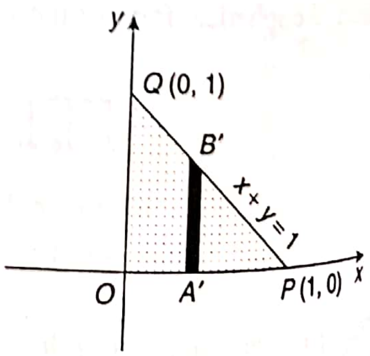

Fig. 7.58

$$= \frac{1}{2}\int_0^1 x^2\left[(1-x)^2 \cdot \frac{(1-x)^2}{2} + \frac{(1-x)^4}{4} - 2(1-x)\cdot\frac{(1-x)^3}{3}\right]dx$$

$$= \frac{1}{2}\int_0^1 \frac{x^2}{12}(1-x)^4 dx = \frac{1}{24}\left|\frac{(1-x)^5}{-5}\cdot x^2 - \frac{(1-x)^6}{30}\cdot 2x + \frac{(1-x)^7}{-210}\cdot 2\right|_0^1$$

$$= \frac{1}{24}\left(0 + \frac{1}{105}\right) = \frac{1}{2520}$$

EXAMPLE 7.39 *Evaluate* $\iiint xyz\,dx\,dy\,dz$ *over the region bounded by the planes* $x = 0$, $y = 0$, $z = 0$, $z = 1$ *and the cylinder* $x^2 + y^2 = 1$.

Solution:

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the equation of the cylinder $x^2 + y^2 = 1$ reduces to $r^2 = 1$, $r = 1$.
2. Draw an elementary volume AB parallel to the z-axis in the region [Fig. 7.59]. This elementary volume AB starts from xy-plane and terminates on the plane $z = 1$.
   Limits of z : $z = 0$ to $z = 1$
3. Projection of the region in $r\theta$-plane is the part of the circle $r = 1$ in the first quadrant [Fig. 7.60]
4. Draw an elementary radius vector OA' in the region in the $r\theta$-plane which starts from the origin and terminates on the circle $r = 1$.

   Limits of $r : r = 0$ to $r = 1$

   Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

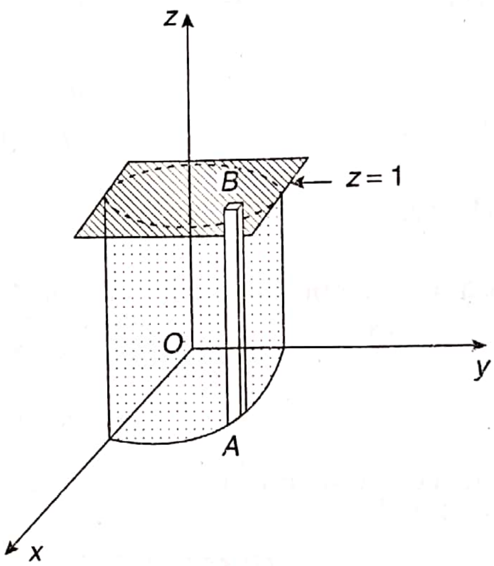

Fig. 7.59

Hence, the cylindrical form of the given integral is

$$I = \iiint xyz\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\cos\theta\sin\theta\cdot zr\,dr\,d\theta\,dz$$

$$= \int_0^1 z\,dz\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}d\theta\int_0^1 r^3 dr$$

$$= \left|\frac{z^2}{2}\right|_0^1 \left|-\frac{\cos 2\theta}{4}\right|_0^{\frac{\pi}{2}} \left|\frac{r^4}{4}\right|_0^1$$

$$= \frac{1}{16}$$

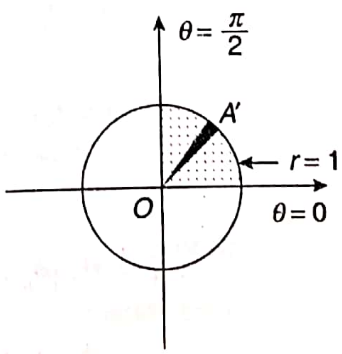

Fig. 7.60

EXAMPLE 7.40 *Evaluate* $\iiint \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz$, *where V is the volume of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: It is difficult to integrate this integral in Cartesian form. Hence, the ellipsoid is transformed into a sphere using the following change of variables.

Putting $x = ar\sin\theta\cos\phi, y = br\sin\theta\sin\phi, z = cr\cos\theta$ the equation of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ reduces to $r^2 = 1$ i.e., $r = 1$, which is a sphere of radius 1 and centre at the origin.

$$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$$

where
$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} a\sin\theta\cos\phi & ar\cos\theta\cos\phi & -ar\sin\theta\sin\phi \\ b\sin\theta\sin\phi & br\cos\theta\sin\phi & br\sin\theta\cos\phi \\ c\cos\theta & -cr\sin\theta & 0 \end{vmatrix} = abc$$

Hence,
$$dx\,dy\,dz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

For the complete sphere, limits of $r : r = 0$ to $r = 1$ (radius of sphere)

limits of $\theta : \theta = 0$ to $\theta = \pi$

limits of $\phi : \phi = 0$ to $\phi = 2\pi$

Hence, the spherical form of the given integral is

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^{1}\sqrt{1-r^2}\,abc\,r^2\sin\theta\,dr\,d\theta\,d\phi = abc\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^1 r^2\sqrt{1-r^2}\,dr$$

Putting $r = \sin t, dr = \cos t\,dt$

When $r = 0, t = 0$

When $r = 1, t = \frac{\pi}{2}$

$$\therefore\ I = abc|\phi|_0^{2\pi}\,|-\cos\theta|_0^{\pi}\int_0^{\frac{\pi}{2}}\sin^2 t\cdot\cos t\cdot\cos t\,dt$$

$$= abc(2\pi)(2)\int_0^{\frac{\pi}{2}}\sin^2 t\cos^2 t\,dt$$

$$= 4\pi abc \cdot \frac{1}{2} B\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= 2\pi abc \frac{\left\lceil\frac{3}{2}\right.\left\lceil\frac{3}{2}\right.}{\left\lceil 3\right.} = 2\pi abc \left(\frac{\frac{1}{2}\left\lceil\frac{1}{2}\right.}{2}\right)^2$$

$$= \frac{abc\pi(\pi)}{4} = \frac{\pi^2 abc}{4}$$

EXERCISE 7.10

(I) Evaluate the following integrals:

1. $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 yz \, dz$ $\quad$ [Ans.: 1]

2. $\int_0^{\log 2} \int_0^x \int_0^{x+y} e^{x+y+z} dz\, dy\, dx$ $\quad$ $\left[\text{Ans.: } \frac{5}{8}\right]$

3. $\int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} \int_0^{\sqrt{a^2-r^2}} r\, dz\, dr\, d\theta$ $\quad$ $\left[\text{Ans.: } \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right)\right]$

4. $\int_0^{\pi} \int_0^{a(1+\cos\theta)} \int_0^{h} 2\left[1 - \frac{r}{a(1+\cos\theta)}\right] r\, dz\, dr\, d\theta$ $\quad$ $\left[\text{Ans.: } \frac{\pi a^2 h}{2}\right]$

5. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy\, dx\, dz$ $\quad$ [Ans.: 8π]

6. $\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dz\, dr\, d\theta$ $\quad$ $\left[\text{Ans.: } \frac{5a^3}{64}\right]$

7. $\int_0^2 \int_0^y \int_{x-y}^{x+y} (x+y+z)\, dz\, dx\, dy$ $\quad$ [Ans.: 16]

8. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\, dz\, dy\, dx.$ $\quad$ $\left[\text{Ans.: } \frac{a^6}{48}\right]$

(II) Evaluate the following integrals over the given region of integration:

1. $\iiint (x+y+z)\, dx\, dy\, dz$, over the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$ $\quad$ $\left[\text{Ans.: } \frac{1}{8}\right]$

2. $\iiint \frac{dx\, dy\, dz}{(1+x+y+z)^3}$, over the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$ $\quad$ $\left[\text{Ans.: } \frac{1}{2}\left(\log 2 - \frac{5}{8}\right)\right]$

3. $\iiint xyz\,dx\,dy\,dz$, over the positive octant of the sphere $x^2+y^2+z^2=a^2$

$$\left[\textbf{Ans.}: \frac{a^6}{48}\right]$$

4. $\iiint xyz(x^2+y^2+z^2)dx\,dy\,dz$, over the positive octant of the sphere $x^2+y^2+z^2=a^2$

$$\left[\textbf{Ans.}: \frac{a^8}{64}\right]$$

5. $\iiint (y^2z^2+z^2x^2+x^2y^2)\,dx\,dy\,dz$, over the sphere of radius a and centre at the origin

$$\left[\textbf{Ans.}: \frac{4\pi a^7}{35}\right]$$

6. $\iiint \frac{z^2}{x^2+y^2+z^2}dx\,dy\,dz$, over the sphere $x^2+y^2+z^2=2$

$$\left[\textbf{Ans.}: \frac{8\pi\sqrt{2}}{9}\right]$$

7. $\iiint \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}}$, over the region bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2,\ a>b>0$

$$\left[\textbf{Ans.}: 4\pi\log\left(\frac{a}{b}\right)\right]$$

8. $\iiint z^2 dx\,dy\,dz$, over the region common to the spheres $x^2+y^2+z^2=a^2$ and cylinder $x^2+y^2=ax$

$$\left[\textbf{Ans.}: \frac{2\pi a^5}{15}\right]$$

9. $\iiint (x^2+y^2)dx\,dy\,dz$, over the region bounded by the paraboloid $x^2+y^2=2z$ and the plane $z=2$

$$\left[\textbf{Ans.}: \frac{16\pi}{3}\right]$$

10. $\iiint x^2yz\,dx\,dy\,dz$, over the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$\left[\textbf{Ans.}: \frac{a^3b^2c^2}{2520}\right]$$

11. $\iiint xyz\,dx\,dy\,dz$, over the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

$$\left[\textbf{Ans.}: \frac{a^2b^2c^2}{48}\right]$$

12. $\iiint \sqrt{\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}}\,dx\,dy\,dz$ over the region bounded by the ellipsoid $\frac{x^2}{1}+\frac{y^2}{4}+\frac{z^2}{9}=1$

$$[\textbf{Ans.}: 8\pi]$$

7.9 VOLUME OF SOLIDS USING TRIPLE INTEGRALS

Volume V of a solid contained in the region V is given as

$$V=\iiint_V dx\,dy\,dz$$

In cylindrical coordinates,

$$V=\iiint_V r\,dr\,d\theta\,dz$$

In spherical coordinates,

$$V = \iiint_V r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Note Consider symmetricity of the region while finding the volume.

EXAMPLE 7.41 *Find the volume cut off from the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$.*

Solution:

1. The region is bounded by the cone and sphere. Putting $x = r\cos\theta, y = r\sin\theta, z = z$,
   (i) the equation of the sphere $x^2+y^2+z^2=a^2$ reduces to $r^2+z^2=a^2$
   (ii) the equation of the cone $x^2+y^2=z^2$ reduces to $r^2=z^2, r=z$

2. Draw an elementary volume AB parallel to the z-axis in the region. AB starts from the cone $r=z$ and terminates on the sphere $r^2+z^2=a^2$ [Fig. 7.61].

   Limits of $z: z=r$ to $z=\sqrt{a^2-r^2}$.

$$V = \iiint_r^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta$$

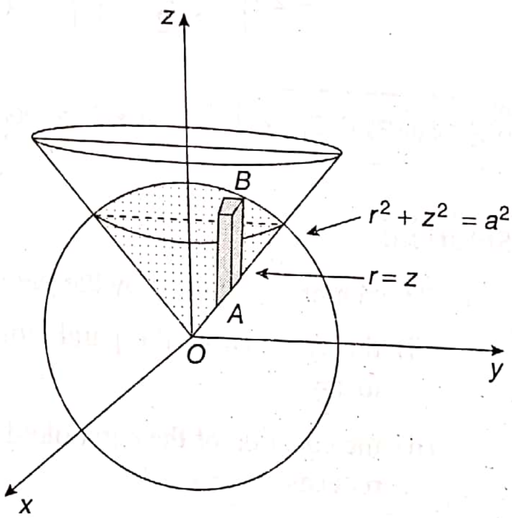

Fig. 7.61

3. Projection of the region in the $r\theta$-plane is the curve obtained by the intersection of the sphere $r^2+z^2=a^2$ and the cone $r=z$ as

$$r^2+r^2=a^2$$

$$r = \frac{a}{\sqrt{2}}$$

   which is a circle with centre at the origin and radius $\frac{a}{\sqrt{2}}$ [Fig. 7.62].

4. The region (circle) is symmetric in all the quadrants. Draw an elementary radius vector OA' in the first quadrant of the region. OA' starts from the origin and terminates on the circle $r=\frac{a}{\sqrt{2}}$.

   Limits of $r: r=0$ to $r=\frac{a}{\sqrt{2}}$

   Limits of $\theta: \theta=0$ to $\theta=\frac{\pi}{2}$ (first quadrant)

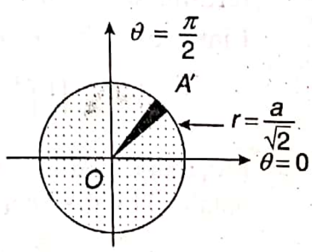

Fig. 7.62

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{a}{\sqrt{2}}}\int_r^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{a}{\sqrt{2}}} r\left|z\right|_r^{\sqrt{a^2-r^2}}\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{a}{\sqrt{2}}}\left[r\left(\sqrt{a^2-r^2}-r\right)\right]dr$$

$$= 4\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{a}{\sqrt{2}}}\left[-\frac{1}{2}\sqrt{a^2-r^2}\,(-2r)-r^2\right]dr$$

$$= 4\left|\theta\right|_0^{\frac{\pi}{2}}\left|-\frac{1}{2}\cdot\frac{2(a^2-r^2)^{\frac{3}{2}}}{3}-\frac{r^3}{3}\right|_0^{\frac{a}{\sqrt{2}}} \qquad \left[\because \int[f(r)]^n f'(r)dr = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= 4\cdot\frac{\pi}{2}\left[-\frac{1}{3}\left\{\left(a^2-\frac{a^2}{2}\right)^{\frac{3}{2}}-a^3\right\}-\frac{a^3}{6\sqrt{2}}\right]$$

$$= 2\pi\left[-\frac{a^3}{3\sqrt{2}}+\frac{a^3}{3}\right] = \frac{\pi a^3}{3}(-\sqrt{2}+2)$$

EXAMPLE 7.42 *Find the volume of the solid bounded by the paraboloids* $z = x^2 + y^2$ *and* $z = 4 - 3(x^2 + y^2)$.

Solution:

1. The region is bounded by the paraboloids. Putting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,
   (i) the equation of the paraboloid $z = x^2 + y^2$ reduces to $z = r^2$
   (ii) the equation of the paraboloid $z = 4 - 3(x^2 + y^2)$ reduces to $z = 4 - 3r^2$
2. Draw an elementary volume AB in the region [Fig. 7.63]. AB starts from the paraboloid $z = r^2$ and terminates on the paraboloid $z = 4 - 3r^2$.
   Limits of $z : z = r^2$ to $z = 4 - 3r^2$.

$$V = \iint\int_{r^2}^{4-3r^2} r\,dz\,dr\,d\theta$$

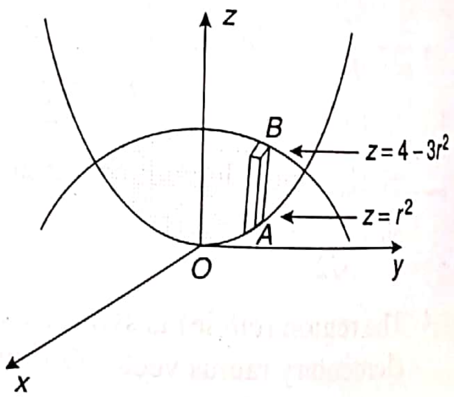

Fig. 7.63

3. Projection of the region in all the quadrants is the curve obtained by the intersection of the paraboloids $z = r^2$ and $z = 4 - 3r^2$ as

$$r^2 = 4 - 3r^2$$
$$r^2 = 1$$
$$r = 1$$

which is a circle with centre at the origin and radius 1 [Fig. 7.64].

4. The region (circle) is symmetric in all the quadrants. Draw an elementary radius vector OA' in the first quadrant of the region. OA' starts from the origin and terminates on the circle $r = 1$.

Limits of $r : r = 0$ to $r = 1$

Limits of $\theta : \theta = 0$ to $\theta = \frac{\pi}{2}$

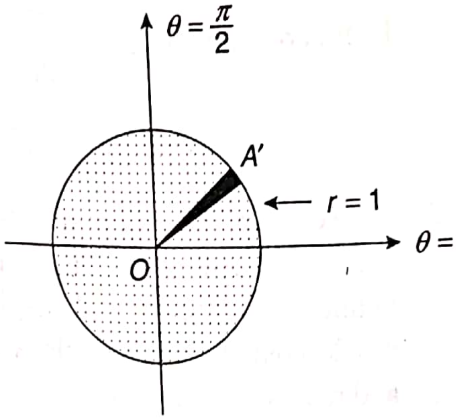

Fig. 7.64

$$V = 4\int_0^{\frac{\pi}{2}} \int_0^1 \int_{r^2}^{4-3r^2} r \, dz \, dr \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^1 r \left| z \right|_{r^2}^{4-3r^2} dr \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^1 r(4 - 3r^2 - r^2) dr \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} d\theta \int_0^1 (4r - 4r^3) dr = 4\left| \theta \right|_0^{\frac{\pi}{2}} \left| 2r^2 - r^4 \right|_0^1$$

$$= 4 \cdot \frac{\pi}{2} = 2\pi$$

EXAMPLE 7.43 *Find the volume of the cylinder $x^2 + y^2 = 2ax$ intercepted between the paraboloid $z = \dfrac{x^2 + y^2}{2a}$ and the xy-plane.*

Solution:

1. The region is bounded by the cylinder, paraboloid and xy-plane. Putting $x = r \cos \theta$, $y = r \sin \theta$, $z = z$
   (i) the equation of the cylinder $x^2 + y^2 = 2ax$ reduces to
   $r^2 = 2ar \cos \theta$, $r = 2a \cos \theta$
   (ii) the equation of the paraboloid $z = \dfrac{x^2 + y^2}{2a}$ reduces to $z = \dfrac{r^2}{2a}$
2. Draw an elementary volume AB parallel to the z-axis in the region [Fig. 7.65]. AB starts from the xy-plane and terminates on the paraboloid $z = \dfrac{r^2}{2a}$.

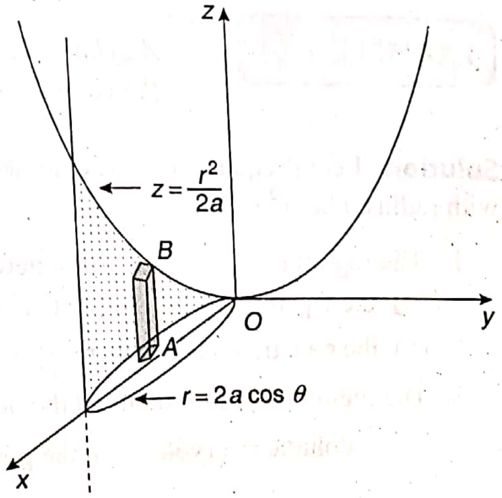

Fig. 7.65

Limits of z : $z = 0$ to $z = \dfrac{r^2}{2a}$.

$$V = \iint \int_0^{\frac{r^2}{2a}} r \cdot dz\, dr\, d\theta$$

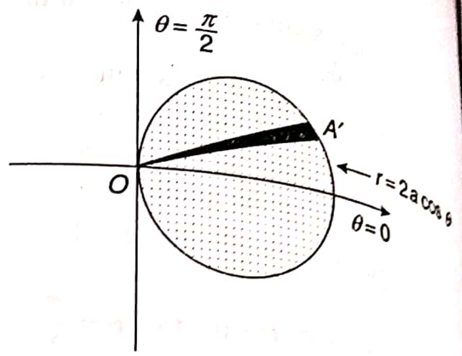

Fig. 7.66

3. Projection of the region in $r\theta$-plane is the curve obtained by the intersection of the cylinder $r = 2a\cos\theta$ and $r\theta$-plane $(z = 0)$ as $r = 2a\cos\theta$, which is circle with centre $(a, 0)$ and radius a [Fig. 7.66].
4. The region (circle) is symmetric about the initial line $\theta = 0$. Draw an elementary radius vector OA' in the region above the initial line $\theta = 0$. OA' starts from the origin and terminates on the circle $r = 2a\cos\theta$.

Limits of r : $r = 0$ to $r = 2a\cos\theta$

Limits of θ: $\theta = 0$ to $\theta = \dfrac{\pi}{2}$ (first quadrant)

$$V = 2\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \int_0^{\frac{r^2}{2a}} r\, dz\, dr\, d\theta = 2\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r \left| z \right|_0^{\frac{r^2}{2a}} dr\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r\left(\frac{r^2}{2a}\right) dr\, d\theta = \frac{1}{a}\int_0^{\frac{\pi}{2}} \left| \frac{r^4}{4} \right|_0^{2a\cos\theta} d\theta$$

$$= 4a^3 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 4a^3 \cdot \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right) = 2a^3 \frac{\sqrt{\frac{5}{2}}\sqrt{\frac{1}{2}}}{\sqrt{3}} = 2a^3 \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}}}{2}$$

$$= a^3 \cdot \frac{3\pi}{4} = \frac{3}{4}\pi a^3$$

EXAMPLE 7.44 *A cylindrical hole of radius b is bored through a sphere of radius a. Find the volume of the remaining solid.*

Solution: Let the equation of the cylinder with radius b be $x^2 + y^2 = b^2$ and the equation of the sphere with radius a be $x^2 + y^2 + z^2 = a^2$.

1. The region is bounded by the sphere and cylinder. Putting $x = r\cos\theta, y = r\sin\theta, z = z$,
   (i) the equation of the cylinder $x^2 + y^2 = b^2$ reduces to $r^2 = b^2, r = b$
   (ii) the equation of the sphere $x^2 + y^2 + z^2 = a^2$ reduces to $r^2 + z^2 = a^2$
2. The region is symmetric in all the octants.

Volume = 8 (volume in the positive octant)

3. Draw an elementary volume AB in the positive octant of the region [Fig. 7.67]. AB starts from xy-plane and terminates on the sphere $r^2+z^2=a^2$.

Limits of $z: z=0$ to $z=\sqrt{a^2-r^2}$

$$V=\iint\int_0^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

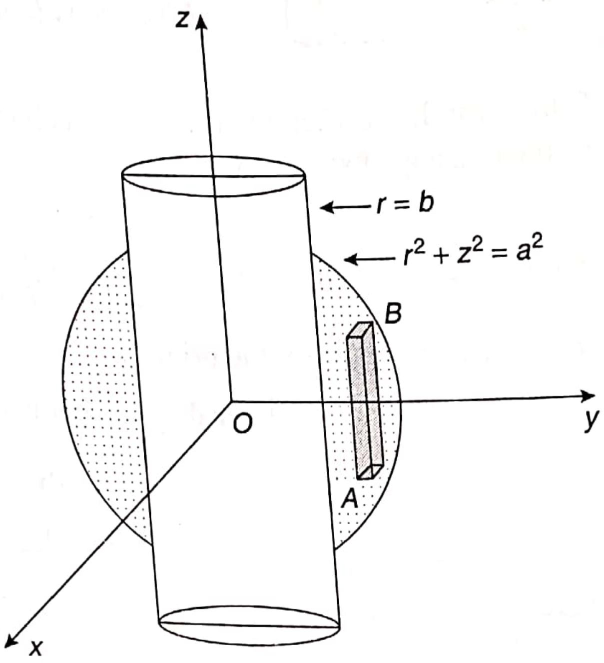

Fig. 7.67

4. Projection of sphere $r^2+z^2=a^2$ in the $r\theta$-plane $(z=0)$ is $r^2=a^2$, $r=a$. Projection of the region in the $r\theta$-plane is the region bounded by the circles $r=b$ and $r=a$ [Fig. 7.68].

5. Draw an elementary radius vector $OA'B'$ in the region in the first quadrant. $OA'B'$ enters in the region from the circle $r=b$ and terminates on the circle $r=a$.

Limits of $r: r=b$ to $r=a$

Limits of $\theta: \theta=0$ to $\theta=\frac{\pi}{2}$ (first quadrant)

$$V=8\int_0^{\frac{\pi}{2}}\int_b^a\int_0^{\sqrt{a^2-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=8\int_0^{\frac{\pi}{2}}\int_b^a r\left|z\right|_0^{\sqrt{a^2-r^2}}\mathrm{d}r\,\mathrm{d}\theta$$

$$=8\int_0^{\frac{\pi}{2}}\int_b^a r\sqrt{a^2-r^2}\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=8\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\left[\int_b^a -\frac{1}{2}\left(\sqrt{a^2-r^2}\right)(-2r)\right]\mathrm{d}r$$

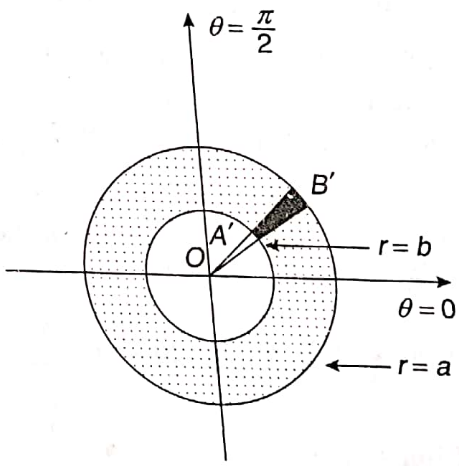

Fig. 7.68

$$=-4\left|\theta\right|_0^{\frac{\pi}{2}}\left|\frac{2(a^2-r^2)^{\frac{3}{2}}}{3}\right|_b^a \qquad \left[\because \int [f(r)]^n f'(r)\,\mathrm{d}r=\frac{[f(r)]^{n+1}}{n+1}\right]$$

$$=-4\cdot\frac{\pi}{2}\cdot\left[0-\frac{2(a^2-b^2)^{\frac{3}{2}}}{3}\right]$$

$$=\frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}.$$

EXAMPLE 7.45 *Find the volume bounded by the solid* $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1.$

Solution: It is difficult to integrate this integral in Cartesian form. Hence, the solid is transformed into a sphere using change of variables.

Putting $x=au^3, y=bv^3, z=cw^3$, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{z}{c}\right)^{\frac{2}{3}}=1$ reduces to $u^2+v^2+w^2=1$, which is a sphere of radius 1 and centre at the origin.

$$dx\,dy\,dz=|J|\,du\,dv\,dw$$

where
$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}3au^2 & 0 & 0\\ 0 & 3bv^2 & 0\\ 0 & 0 & 3cw^2\end{vmatrix}=27abc\,u^2v^2w^2$$

Hence, $dx\,dy\,dz=27abc\,u^2v^2w^2\,du\,dv\,dw$

In the new coordinate system u, v, w, the region is bounded by a sphere.
Putting $u=r\sin\theta\cos\phi$, $v=r\sin\theta\sin\phi$, $w=r\cos\theta$, the sphere $u^2+v^2+w^2=1$ reduces to $r=1$.
Since the region is symmetric in all the octants,

$$V=8\text{ (volume in the positive octant)}$$

Limits in the positive octant of the sphere are
Limits of r: $r=0$ to $r=1$

Limits of θ: $\theta=0$ to $\theta=\frac{\pi}{2}$

Limits of ϕ: $\phi=0$ to $\phi=\frac{\pi}{2}$

$$V=8\iiint dx\,dy\,dz=8\iiint 27abc\,u^2v^2w^2\,du\,dv\,dw$$

$$=216abc\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1(r^2\sin^2\theta\cos^2\phi)(r^2\sin^2\theta\sin^2\phi)(r^2\cos^2\theta)r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$=216abc\int_0^{\frac{\pi}{2}}\cos^2\phi\sin^2\phi\,d\phi\int_0^{\frac{\pi}{2}}\sin^5\theta\cos^2\theta\,d\theta\int_0^1 r^8\,dr$$

$$=216abc\cdot\frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right)\cdot\frac{1}{2}B\left(3,\frac{3}{2}\right)\left|\frac{r^9}{9}\right|_0^1$$

$$= 6abc \frac{\left(\left\lceil\frac{3}{2}\right)^2}{\lceil 3} \cdot \frac{\lceil 3 \left\lceil\frac{3}{2}\right.}{\left\lceil\frac{9}{2}\right.}$$

$$= 6abc \left(\frac{1}{2}\left\lceil\frac{1}{2}\right)^2 \cdot \frac{\left\lceil\frac{3}{2}\right.}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\left\lceil\frac{3}{2}\right.}$$

$$= \frac{4}{35}\pi abc$$

EXERCISE 7.11

1. Find the volume of the sphere $x^2+y^2+z^2=a^2$ cut off by the planes $z=0$ and the cylinder $x^2+y^2=ax$.

$\left[\textbf{Ans.: } \frac{2}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)\right]$

2. Find the volume common to the sphere $x^2+y^2+z^2=a^2$ and the cylinder $x^2+y^2=b^2, (a>b)$.

$\left[\textbf{Ans.: } 2\pi b^2\sqrt{a^2-b^2}\right]$

3. Find the volume bounded by the cylinders $y^2=x, x^2=y$ and the planes $z=0, x+y+z=2$.

$\left[\textbf{Ans.: } \frac{11}{30}\right]$

4. Find the volume of the cylinder $x^2+y^2-4x=0$ cut by the cylinder $z^2=4x$.

$\left[\textbf{Ans.: } \frac{1024}{15}\right]$

5. Find the volume of the paraboloid $x^2+y^2=4z$ cut off by the plane $z=4$.

$[\textbf{Ans.: } 32\pi]$

6. Find the volume of the paraboloid $x^2+\frac{y^2}{9}+z=1$ cut off by the plane $z=0$.

$\left[\textbf{Ans.: } \frac{3\pi}{2}\right]$

7. Find the volume of the solid bounded by the paraboloids $z=4-x^2-\frac{y^2}{4}$ and $z=3x^2+\frac{y^4}{4}$.

$[\textbf{Ans.: } 4\sqrt{2}\pi]$

8. Find the volume of the solid bounded by the plane $z=0$, the paraboloid $3z=x^2+y^2$ and the cylinder $x^2+y^2=9$.

$\left[\textbf{Ans.: } \frac{27\pi}{2}\right]$