Five

Infinite Series

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5.1 INTRODUCTION

Infinite series are sums that involve infinitely many terms. They play an important role in both mathematics and science. They are used to approximate trigonometric functions and logarithms to solve differential equations, to evaluate definite integrals, to create new functions and to construct mathematical models of physical laws. This chapter covers convergence and divergence of sequences and series. There are various methods to test the convergence and divergence of an infinite series, such as comparison test, D'Alembert's ratio test, Raabe's test, logarithmic test, Cauchy's root test and Cauchy's integral test. The chapter also covers alternating series, absolute and uniform convergence of a series, and power series.

5.2 SEQUENCE

An ordered set of real numbers as $u_1, u_2, u_3, \ldots\ldots.u_n, \ldots\ldots$ is called a sequence and is denoted by $\{u_n\}$. If the number of terms in a sequence is infinite, it is said to be an infinite sequence, otherwise it is a finite sequence and u_n is called the n^{th} term of the sequence.

5.2.1 Limit of a Sequence

A sequence $\{u_n\}$ tends to a finite number l as $n \to \infty$ if for every $\in > 0$ there exists an integer m such that $|u_n - l| < \in$ for all $n > m$, i.e., $\lim\limits_{n\to\infty} u_n = l$.

5.2.2 Convergence, Divergence, and Oscillation of a Sequence

(i) If the sequence $\{u_n\}$ has a finite limit, i.e., $\lim_{n\to\infty} u_n$ is finite, the sequence is said to be convergent, e.g.,

$$\{u_n\} = \left\{\frac{1}{1+\frac{1}{n}}\right\}$$

$$\lim_{n\to\infty} u_n = 1$$

Since limit is finite, the sequence is convergent.

(ii) If the sequence $\{u_n\}$ has infinite limit, i.e., $\lim_{n\to\infty} u_n$ is infinite, the sequence is said to be divergent, e.g.,

$$\{u_n\} = \{2n+1\}$$

$$\lim_{n\to\infty} u_n = \infty$$

Since limit is infinite, the sequence is divergent.

(iii) If the limit of the sequence $\{u_n\}$ is not unique, the sequence is said to be oscillatory, e.g.,

$$\{u_n\} = (-1)^n + \frac{1}{2^n}$$

$$\lim_{n\to\infty} u_n = 1, \text{ if } n \text{ is even}$$

$$= -1, \text{ if } n \text{ is odd}$$

Since limit is not unique, the sequence is oscillatory.

5.2.3 Monotonic Sequence

A sequence is said to be monotonically increasing if $u_{n+1} \geq u_n$ for each value of n and is monotonically decreasing if $u_{n+1} \leq u_n$ for each value of n. The sequence is called alternating sequence if the terms are alternately positive and negative.

(i) $1, 2, 3, 4, \ldots$ is a monotonically increasing sequence.

(ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is a monotonically decreasing sequence.

(iii) $1, -2, 3, -4, \ldots$ is an alternating sequence.

5.2.4 Bounded Sequence

A sequence $\{u_n\}$ is said to be a bounded sequence if there exists numbers m and M such that $m < u_n < M$ for all n.

Notes

(i) Every convergent sequence is bounded but the converse is not true.

(ii) A monotonic increasing sequence converges if it is bounded above and diverges to ∞ if it is not bounded above.

(iii) A monotonic decreasing sequence converges if it is bounded below and diverges to $-\infty$ if it is not bounded below.

(iv) If sequence $\{u_n\}$ and $\{v_n\}$ converges to l_1 and l_2 respectively then

- Sequence $\{u_n + v_n\}$ converges to $l_1 + l_2$
- Sequence $\{u_n \cdot v_n\}$ converges to $l_1 \cdot l_2$
- Sequence $\left\{\dfrac{u_n}{v_n}\right\}$ converges to $\dfrac{l_1}{l_2}$ provided $l_2 \neq 0$

5.2.5 Standard Limits

(i) $\lim\limits_{n\to\infty} \dfrac{\log n}{n} = 0$

(ii) $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e$

(iii) $\lim\limits_{n\to\infty} (n)^{\frac{1}{n}} = 1$

(iv) $\lim\limits_{n\to\infty} (n!)^{\frac{1}{n}} = \infty$

(v) $\lim\limits_{n\to\infty} \left(\dfrac{n!}{n}\right)^{\frac{1}{n}} = \dfrac{1}{e}$

(vi) $\lim\limits_{n\to\infty} x^n = 0$ if $x < 1$

(vii) $\lim\limits_{n\to\infty} x^n = \infty$ if $x > 1$

(viii) $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ for all x

(ix) $\lim\limits_{n\to 0} \left(\dfrac{a^n - 1}{n}\right) = \log a$

(x) $\lim\limits_{n\to\infty} \dfrac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} = \log a$

EXAMPLE 5.1 *Test the convergence of the sequence* $\left\{\dfrac{n^2+n}{2n^2-n}\right\}$.

Solution: Let $u_n = \dfrac{n^2+n}{2n^2-n}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2+n}{2n^2-n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2-\frac{1}{n}} = \frac{1}{2}$$

Hence, $\{u_n\}$ is convergent.

EXAMPLE 5.2 *Show that the sequence $\{u_n\}$ whose n^{th} term is $u_n = \frac{1}{1!} + \frac{1}{2!} + \ldots,$ $n \in N$, is monotonic increasing and bounded. Is it convergent?*

Solution:

$$u_n = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$$

$$u_{n+1} = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} + \frac{1}{(n+1)!}$$

$$u_{n+1} - u_n = \frac{1}{(n+1)!} > 0$$

$$u_{n+1} > u_n$$

Hence, $\{u_n\}$ is a monotonic increasing sequence.

Also,
$$u_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$$

$$< \frac{1\left(1 - \frac{1}{2^{n+1}}\right)}{1 - \frac{1}{2}} \qquad \text{[Using sum of GP]}$$

$$< 2\left(1 - \frac{1}{2^{n+1}}\right) < 2$$

$\{u_n\}$ is bounded above by 2.
Since $\{u_n\}$ is monotonic increasing and bounded above, it is convergent.

EXAMPLE 5.3 *Show that the sequence $\left\{\frac{n}{n^2+1}\right\}$ is monotonic decreasing and bounded. Is it convergent?*

Solution: Let $u_n = \frac{n}{n^2+1}$

$$u_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$u_{n+1} - u_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{(n+1)(n^2+1) - n(n^2+2n+2)}{(n^2+2n+2)(n^2+1)} = \frac{-n^2-n+1}{(n^2+2n+2)(n^2+1)} < 0$$

Hence, $\{u_n\}$ is a monotonic decreasing sequence.

Also,
$$u_n = \frac{n}{n^2+1} > 0$$

$\{u_n\}$ is bounded below by 0.
Since $\{u_n\}$ is monotonic decreasing and bounded below, it is convergent.

5.2.6 Sandwich Theorem for Sequences

Let $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ be three sequences such that $u_n \le v_n \le w_n$ for all n. If $\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = l$ then $\lim_{n\to\infty} v_n = l$

HISTORICAL DATA

In calculus, the **sandwich theorem** (known also as the pinching theorem, the squeeze theorem, the sandwich rule and sometimes the squeeze lemma) is a theorem regarding the limit of a function.

The sandwich theorem is a technical result that is very important in proofs in calculus and mathematical analysis. It is typically used to confirm the limit of a function via comparison with two other functions whose limits are known or easily computed. It was first used geometrically by the mathematicians Archimedes and Eudoxus in an effort to compute π, and was formulated in modern terms by Gauss.

In Italy, China, Chile, Russia, Poland and France, the squeeze theorem is also known as the two *carabinieri* theorem, *two militsioner* theorem, *two gendarmes* theorem, *double-sided* theorem or *two-policemen-and-a-drunk* theorem. The story is that if two policemen are escorting a drunk prisoner between them, and both officers go to a cell then (regardless of the path taken, and the fact that the prisoner may be wobbling about between the policemen) the prisoner must also end up in the cell!

EXAMPLE 5.4 *Show that the sequence $\{u_n\}$, where $u_n = \frac{\sin n}{n}$ converges to zero.*

Solution:

$$-1 \le \sin n \le 1$$

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

$$\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

By Sandwich theorem,

$$\lim_{n\to\infty}\frac{\sin n}{n} = 0$$

Hence, $\{u_n\}$ converges to zero.

EXERCISE 5.1

1. Test the convergence of the following sequences:

(i) $\frac{2n+1}{1-3n}$ (ii) $2+(0.1)^n$

(iii) $1+(-1)^n$ (iv) e^n

(v) $\frac{n^2}{2n-1}\sin\left(\frac{1}{n}\right)$ (vi) $\tan^{-1} n$

[**Ans.:** (i) convergent (ii) convergent (iii) divergent (iv) divergent (v) convergent (vi) convergent]

2. Determine whether the following sequences are monotonically increasing/decreasing, bounded or convergent/divergent.

(i) $1+\frac{1}{n}$ (ii) $\frac{2n-7}{3n+2}$

[**Ans.:** (i) decreasing, bounded, convergent
(ii) increasing, bounded, convergent]

3. Show that the sequence $\{u_n\}$ is convergent, where $u_n = \frac{1}{n+2}+\cdots+\frac{1}{n+n}$.

4. Show that the sequence $\{u_n\}$, wh... $u_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n-1)!}; n \geq 2$, is convergent.

5. Does the sequence $\{u_n\}$ convergent, wh... $u_n = \left(\frac{n+1}{n-1}\right)^n$?

[Ans.: Ye...

5.3 INFINITE SERIES

If $u_1, u_2, u_3, \ldots, u_n, \ldots$ is an infinite sequence of real numbers then the sum of the terms of the sequence $u_1+u_2+u_3+\ldots+u_n+\ldots\infty$ is called an infinite series.

The infinite series $u_1+u_2+u_3+\ldots+u_n+\ldots\infty$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or Σu_n.

The sum of its first n terms is denoted by S_n and is also known as n^{th} partial sum of Σu_n.

5.3.1 Convergence, Divergence, and Oscillation of Infinite Series

Consider the infinite series $\Sigma u_n = u_1+u_2+u_3+\ldots u_n+\ldots\infty$ and let the sum of the first n terms be $S_n = u_1+u_2+u_3+\ldots+u_n$. As $n\to\infty$, three possibilities arise for S_n:

(i) If S_n tends to a finite limit as $n\to\infty$, the series Σu_n is said to be convergent.

(ii) If S_n tends to $\pm\infty$ as $n\to\infty$, the series Σu_n is said to be divergent.

(iii) If S_n does not tend to a unique limit as $n\to\infty$, i.e., limit does not exist, the series Σu_n is said to be oscillatory.

5.3.2 Properties of Infinite Series

(i) The convergence or divergence of an infinite series remains unaffected:

(a) by addition or removal of a finite number of terms

(b) by multiplication of each term with a finite number

(ii) If two series Σu_n and Σv_n are convergent then $\Sigma(u_n+v_n)$ is also convergent.

(iii) If two series Σu_n and Σv_n are divergent then $\Sigma(u_n+v_n)$ may be convergent or divergent.

(iv) If each term of a series Σu_n of positive terms does not exceed the corresponding term of a convergent series Σv_n of positive terms then Σu_n is convergent.

(v) If each term of a series Σu_n of positive terms exceeds the corresponding term of a divergent series Σv_n of positive terms then Σu_n is divergent.

5.3.3 Necessary Condition for Convergence of Infinite Series

If positive term series Σu_n is convergent then $\lim\limits_{n\to\infty} u_n = 0$.

The converse of this result is not true, i.e., if $\lim\limits_{n\to\infty} u_n = 0$, it is not necessary that the series will be convergent, e.g.,

$$\sum u_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} + \ldots \infty$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Now,

$$S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 1 + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots + \frac{1}{\sqrt{n}}$$

$$S_n > \frac{n}{\sqrt{n}}$$

$$S_n > \sqrt{n}$$

and

$$\lim_{n\to\infty} \sqrt{n} = \infty$$

Thus, the series is divergent.

Hence, $\lim\limits_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for convergence of Σu_n.

The above result leads to a test for divergence. If $\lim\limits_{n\to\infty} u_n \neq 0$ or $\lim\limits_{n\to\infty} u_n$ does not exist then Σu_n is divergent.

5.4 GEOMETRIC SERIES

Consider the geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ ... (5.1)

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r}, \qquad \text{if } r < 1$$

$$= \frac{a(r^n - 1)}{r-1}, \qquad \text{if } r > 1$$

(i) When $|r| < 1$,

$$\lim_{n\to\infty} r^n = 0$$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} \text{ is finite.}$$

Hence, the series is convergent.

(ii) When $r > 1$,

$$\lim_{n\to\infty} r^n \to \infty$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} \to \infty$$

Hence, the series is divergent.

(iii) When $r = 1$, $S_n = a + a + a + \ldots = na$

$$\lim_{n\to\infty} S_n \to \infty$$

Hence, the series is divergent.

(iv) When $r = -1$, $S_n = a - a + a - \ldots (-1)^{n-1} a$

$= 0$, if n is even

$= a$, if n is odd

Hence, the series is oscillatory.

(v) When $r < -1$, let $r = -k$, where $k > 0$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a[1-(-k)^n]}{1+k} = \lim_{n\to\infty} \frac{a[1-(-1)^n k^n]}{1+k}$$

$= -\infty$, if n is even

$= \infty$, if n is odd

Hence, the series is oscillatory.

From all the above cases, it can be concluded that the geometric series (5.1) is

(i) convergent if $|r| < 1$

(ii) divergent if $r \geq 1$

(iii) oscillatory if $r \leq -1$

Note The p series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ is

(i) convergent if $p > 1$

(ii) divergent if $p \leq 1$

5.5 COMPARISON TEST

▪ **Statement** If Σu_n and Σv_n are series of positive terms such that $\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (finite and nonzero) then both series converge or diverge together.

Proof $\lim_{n\to\infty} \frac{u_n}{v_n} = l$

By definition of limit, for a positive number $\in$, however small, there exists an integer m such that

$$\left|\frac{u_n}{v_n} - l\right| < \in \quad \text{for all } n > m$$

$$-\in < \frac{u_n}{v_n} - l < \in \quad \text{for all } n > m$$

$$l-\in<\frac{u_n}{v_n}<l+\in \qquad \text{for all } n>m$$

Neglecting the first m terms of Σu_n and Σv_n,

$$l-\in<\frac{u_n}{v_n}<l+\in \qquad \text{for all } n \qquad \dots (5.2)$$

Case I If Σv_n is convergent then $\lim\limits_{n\to\infty}(v_1+v_2+v_3+\ldots+v_n)=\text{finite}=k$, say

From Eq. (5.2),

$$\frac{u_n}{v_n}<l+\in$$

$$u_n<(l+\in)v_n \qquad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\ldots+u_n)<(l+\in)\lim_{n\to\infty}(v_1+v_2+v_3+\ldots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\cdots+u_n)<(l+\in)k \text{ (finite)}$$

Hence, Σu_n is also convergent.

Case II If Σv_n is divergent then

$$\lim_{n\to\infty}(v_1+v_2+v_3+\ldots+v_n)\to\infty \qquad \dots (5.3)$$

From Eq. (5.2),

$$l-\in<\frac{u_n}{v_n}$$

$$u_n>(l-\in)v_n \qquad \text{for all } n$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\ldots+u_n)>(l-\in)\lim_{n\to\infty}(v_1+v_2+v_3+\ldots+v_n)$$

$$\lim_{n\to\infty}(u_1+u_2+u_3+\ldots+u_n)\to\infty \qquad \text{[From Eq. (5.3)]}$$

Hence, Σu_n is also divergent.

EXAMPLE 5.5 *Test the convergence of the series* $\sum\limits_{n=1}^{\infty}\frac{2n-1}{n(n+1)(n+2)}$.

Solution: Let

$$u_n=\frac{2n-1}{n(n+1)(n+2)}=\frac{2-\frac{1}{n}}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

Let

$$v_n=\frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}=2 \qquad \text{[Finite and nonzero]}$$

and the series $\Sigma v_n=\sum\frac{1}{n^2}$ is convergent as $p=2>1$.

Hence, by comparison test, Σu_n is also convergent.

EXAMPLE 5.6 *Test the convergence of the series* $\sum_{n=1}^{\infty}\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}$.

Solution: Let

$$u_n=\frac{(2n^2-1)^{\frac{1}{3}}}{(3n^3+2n+5)^{\frac{1}{4}}}=\frac{n^{\frac{2}{3}}\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{3}{4}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}=\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{n^{\frac{1}{12}}\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}$$

Let

$$v_n=\frac{1}{n^{\frac{1}{12}}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(2-\frac{1}{n^2}\right)^{\frac{1}{3}}}{\left(3+\frac{2}{n^2}+\frac{5}{n^3}\right)^{\frac{1}{4}}}=\frac{(2)^{\frac{1}{3}}}{(3)^{\frac{1}{4}}} \qquad \text{[Finite and nonzero]}$$

and $\Sigma v_n=\sum\frac{1}{n^{\frac{1}{12}}}$ is divergent as $p=\frac{1}{12}<1$.

Hence, by comparison test, Σu_n is also divergent.

EXAMPLE 5.7 *Test the convergence of the series* $\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\cdots$.

Solution: Let

$$u_n=\frac{1}{(2n+1)^p}=\frac{1}{n^p\left(2+\frac{1}{n}\right)^p}$$

Let
$$v_n = \frac{1}{n^p}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\left(2+\frac{1}{n}\right)^p} = \frac{1}{2^p} \qquad \text{[Finite and nonzero]}$$

and $\Sigma v_n = \sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

Hence, by comparison test, Σu_n is also convergent if $p > 1$ and divergent if $p \le 1$.

EXAMPLE 5.8 *Test the convergence of the series* $\frac{2\cdot 1^3+5}{4\cdot 1^5+1}+\frac{2\cdot 2^3+5}{4\cdot 2^5+1}+\dots+\frac{2\cdot n^3+5}{4\cdot n^5+1}+\dots$

Solution: Let
$$u_n = \frac{2n^3+5}{4n^5+1} = \frac{2+\frac{5}{n^3}}{n^2\left(4+\frac{1}{n^5}\right)}$$

Let
$$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\left(2+\frac{5}{n^3}\right)}{\left(4+\frac{1}{n^5}\right)} = \frac{2}{4} = \frac{1}{2} \qquad \text{[Finite and nonzero]}$$

and $\Sigma v_n = \sum \frac{1}{n^2}$ is convergent as $p = 2 > 1$.

Hence, by comparison test, Σu_n is also convergent.

EXERCISE 5.2

1. Test the convergence of the following series:

(i) $\sum \frac{1}{n^2+1}$

(ii) $\sum\left(\sqrt{n+1}-\sqrt{n}\right)$

(iii) $\sum\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

(iv) $\sum\left(\frac{n^p}{\sqrt{n+1}+\sqrt{n}}\right)$

(v) $\sum \frac{n^p}{(n+1)^q}$

(vi) $\sum \frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right)$

(vii) $\sum \tan^{-1}\left(\frac{1}{n}\right)$

(viii) $\sum \frac{1}{n^{\left(a+\frac{b}{n}\right)}}$

[Ans. :
(i) convergent (ii) divergent (iii) convergent
(iv) convergent if $p<-\frac{1}{2}$ divergent if $p\ge -\frac{1}{2}$
(v) convergent if $p-q+1<0$,
divergent if $p-q+1\ge 0$
(vi) convergent (vii) divergent
(viii) convergent if $a>1$, divergent if $a\le 1$]

2. Test the convergence of the series

$$\frac{(1+a)(1+b)}{1\cdot 2\cdot 3}+\frac{(2+a)(2+b)}{2\cdot 3\cdot 4}+\frac{(3+a)(3+b)}{3\cdot 4\cdot 5}+\ldots$$

[Ans. : divergent]

3. Test the convergence of the series

$$\frac{(1+a)(1+b)}{1\cdot 2\cdot 3}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\frac{1}{9^p}+\ldots\ldots$$

[Ans. : convergent if $p>$
divergent if $p\le$

5.6 D'ALEMBERT'S RATIO TEST

- **Statement** If Σu_n is a positive-term series and $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=l$ then
  (i) Σu_n is convergent if $l<1$
  (ii) Σu_n is divergent if $l>1$

Proof

Case I If $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=l<1$

Consider a number $l<r<1$ such that $\frac{u_{n+1}}{u_n}<r$ for all $n>m$ ...(5.

Neglecting the first m terms,

$$\sum_{n=m+1}^{\infty} u_n = u_{m+1}+u_{m+2}+u_{m+3}+\ldots\infty$$

$$=u_{m+1}\left(1+\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+3}}{u_{m+1}}+\frac{u_{m+4}}{u_{m+1}}+\ldots\right)$$

$$=u_{m+1}\left(1+\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+3}}{u_{m+2}}\cdot\frac{u_{m+2}}{u_{m+1}}+\frac{u_{m+4}}{u_{m+3}}\cdot\frac{u_{m+3}}{u_{m+2}}\cdot\frac{u_{m+2}}{u_{m+1}}+\ldots\right)$$

$$<u_{m+1}(1+r+r\cdot r+r\cdot r\cdot r+\ldots)$$ [Using Eq. (5.4)]

$$= u_{m+1}(1 + r + r^2 + r^3 + \dots)$$

$$= u_{m+1} \cdot \frac{1}{1-r} \qquad (r < 1)$$

$$\therefore \quad \sum_{n=m+1}^{\infty} u_n < \frac{u_{m+1}}{1-r} \quad \text{(Finite)}$$

Thus the series $\sum_{n=m+1}^{\infty} u_n$ is convergent.

The nature of a series remains unchanged if a finite number of terms are neglected in the beginning.

Hence, the series $\sum_{n=1}^{\infty} u_n$ is convergent.

Case II If $\quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l > 1$

$$\frac{u_{n+1}}{u_n} > 1 \qquad \text{for all } n > m \qquad \dots (5.5)$$

Neglecting the first m terms,

$$\sum_{n=m+1}^{\infty} u_n = u_{m+1} + u_{m+2} + u_{m+3} + u_{m+4} + \dots\infty = u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+1}} + \dots\right)$$

$$= u_{m+1}\left(1 + \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \frac{u_{m+4}}{u_{m+3}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+1}} + \dots\right) > u_{m+1}(1+1+1+1+\dots)$$

$$\therefore \quad (u_{m+1} + u_{m+2} + \dots \text{to } n \text{ terms}) > u_{m+1}(1+1+1\dots \text{ to } n \text{ terms})$$

$$S_n > u_{m+1} n$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} n u_{m+1} \to \infty \qquad [\because u_{m+1} \text{ is positive}]$$

Thus, the series $\sum_{n=m+1}^{\infty} u_n$ is divergent.

The nature of a series remains unchanged if a finite number of terms are neglected in the beginning.

Hence, the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Notes

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, D'Alembert's ratio test fails, i.e., no conclusion can be drawn about the convergence or divergence of the series.

HISTORICAL DATA

Jean-Baptiste le Rond d'Alembert (1717–1783) was a French mathematician, mechanician, physicist, philosopher and music theorist. Until 1759 he was also co-editor with Denis Diderot of the *Encyclopédie*. D'Alembert's formula for obtaining solutions to the wave equations is named after him. The wave equation is sometimes referred to as D'Alembert's equation.

In July 1739, he made his first contribution to the field of mathematics, pointing out the errors he had detected in *L'analyse démontrée* (published 1708 by Charles René Reynaud) in a communication addressed to the Académie des Sciences. At the time, *L'analyse démontrée* was a standard work, which D'Alembert himself had used to study the foundations of mathematics.

EXAMPLE 5.9 *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n!}$.

Solution: Let

$$u_n = \frac{(n+1)^n}{n!}$$

$$u_{n+1} = \frac{(n+2)^{n+1}}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(n+1)^n} = \frac{[(n+1)+1]^{n+1}}{(n+1)(n!)} \cdot \frac{n!}{(n+1)^n} = \left(1+\frac{1}{n+1}\right)^{n}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1} = e > 1$$

Hence, by D'Alembert's ratio test, the series is divergent.

EXAMPLE 5.10 *Test the convergence of the series* $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\cdots$

Solution: Let

$$u_n = \frac{n^2}{n!}$$

$$u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2} = \frac{n!(n+1)^2}{n^2(n+1)(n!)} = \frac{n+1}{n^2} = \frac{1}{n}+\frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{1}{n}+\frac{1}{n^2}\right) = 0 < 1$$

Hence, by D'Alembert's ratio test, the series is convergent.

[EXA]MPLE 5.11 *Test the convergence of the series* $\frac{2}{1}+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\frac{2\cdot 5\cdot 8\cdot 11}{1\cdot 5\cdot 9\cdot 13}+\cdots$.

[Soluti]on: Let

$$u_n=\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)} \qquad \text{[Using AP]}$$

$$u_{n+1}=\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)(3n+2)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)(4n+1)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\dfrac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)(3n+2)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)(4n+1)}}{\dfrac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)}}=\lim_{n\to\infty}\frac{3n+2}{4n+1}=\lim_{n\to\infty}\frac{3+\frac{2}{n}}{4+\frac{1}{n}}=\frac{3}{4}<1$$

[Hen]ce, by D'Alembert's ratio test, the series is convergent.

[E]XAMPLE 5.12 *Test the convergence of the series* $\sum\frac{\sqrt{n}}{\sqrt{n^2+1}}x^n$.

[So]lution: Let

$$u_n=\sqrt{\frac{n}{n^2+1}}\,x^n$$

$$u_{n+1}=\sqrt{\frac{n+1}{(n+1)^2+1}}\,x^{n+1}$$

$$\frac{u_{n+1}}{u_n}=\sqrt{\frac{n+1}{(n+1)^2+1}}\,x^{n+1}\cdot\sqrt{\frac{n^2+1}{n}}\,\frac{1}{x^n}=\sqrt{\frac{(n+1)(n^2+1)}{n(n^2+2n+2)}}\,x$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\sqrt{\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right)}{1+\frac{2}{n}+\frac{2}{n^2}}}\,x=x$$

By D'Alembert's ratio test, the series is

(i) convergent if $x<1$

(ii) divergent if $x>1$

The test fails for $x=1$.

For $x=1$,

$$u_n=\sqrt{\frac{n}{n^2+1}}=\frac{n^{\frac{1}{2}}}{n\sqrt{1+\frac{1}{n^2}}}=\frac{1}{n^{\frac{1}{2}}\sqrt{1+\frac{1}{n^2}}}$$

Let $$v_n = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$$ [Finite and nonzero]

and $$\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}} \text{ is divergent for } p = \frac{1}{2} < 1.$$

By comparison test, Σu_n is also divergent for $x = 1$.
Hence, the series is convergent for $x < 1$ and is divergent for $x \geq 1$.

EXAMPLE 5.13 *Test the convergence of the series* $1+\frac{3}{2}x+\frac{5}{9}x^2+\frac{7}{28}x^3+\frac{9}{65}x^4+\ldots$

Solution: Let $$u_n = \frac{2n+1}{n^3+1}x^n$$

$$u_{n+1} = \frac{2n+3}{(n+1)^3+1}x^{n+1}$$ [Neglecting the first term]

$$\frac{u_{n+1}}{u_n} = \frac{2n+3}{(n+1)^3+1}x^{n+1} \cdot \frac{n^3+1}{2n+1}\frac{1}{x^n} = \frac{\left(2+\frac{3}{n}\right)\left(1+\frac{1}{n^3}\right)x}{\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]\left(2+\frac{1}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\left(2+\frac{3}{n}\right)\left(1+\frac{1}{n^3}\right)x}{\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]\left(2+\frac{1}{n}\right)} = x$$

By D'Alembert's ratio test, the series is

(i) convergent if $x < 1$

(ii) divergent if $x > 1$

The test fails if $x = 1$.

For $x = 1$, $$u_n = \frac{2n+1}{n^3+1} = \frac{2+\frac{1}{n}}{n^2\left(1+\frac{1}{n^3}\right)}$$

Let $$v_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2+\frac{1}{n}}{\left(1+\frac{1}{n^3}\right)}=2 \qquad \text{[Finite and nonzero]}$$

and $\sum v_n=\sum\frac{1}{n^2}$ is convergent as $p=2>1$.

By comparison test, $\sum u_n$ is also convergent if $x=1$.

Hence, the series is convergent for $x\le 1$ and is divergent for $x>1$.

EXERCISE 5.3

Test the convergence of the following series:

1. $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots\infty$

   [Ans. : Convergent]

2. $\sum_{n=1}^{\infty}\frac{4\cdot7\cdot10\ldots(3n+1)}{1\cdot2\cdot3\cdot4\ldots n}$

   [Ans. : Convergent]

3. $\frac{1}{1+5}+\frac{2}{1+5^2}+\frac{3}{1+5^3}+\ldots\infty$

   [Ans. : Convergent]

4. $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\ldots\infty$

   [Ans. : Convergent]

5. $\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots$

   [Ans. : Convergent]

6. $1+\frac{3}{2!}+\frac{3^2}{3!}+\frac{3^3}{4!}+\frac{3^4}{5!}+\ldots$

   [Ans. : Convergent]

7. $\sum_{n=1}^{\infty}\frac{n^2}{3^n}$

   [Ans. : Convergent]

8. $\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n+1}$

   [Ans. : Convergent]

9. $\sum_{n=1}^{\infty}\frac{1}{n!}$

   [Ans. : Convergent]

10. $\sum_{n=1}^{\infty}\frac{n^2(n+1)^2}{n!}$

    [Ans. : Convergent]

11. $\sum_{n=1}^{\infty}\frac{3^n+4^n}{4^n+5^n}$

    [Ans. : Divergent]

12. $\sum_{n=1}^{\infty}\frac{x^n}{3^n\cdot n^2},\ x>0$

    [Ans. : Convergent for $x<3$, divergent for $x>3$]

13. $\sum_{n=1}^{\infty}\frac{3^n-2}{3^n+1}\cdot x^{n-1},\ x>0$

    [Ans. : Convergent for $x<1$, divergent for $x>3$]

14. $\sum_{n=1}^{\infty} \frac{x^n}{(2^n)!}$

[Ans. : Convergent]

15. $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^3+1}} \cdot x^n, x>0$

[Ans. : Convergent for $x<1$, divergent for $x>1$]

16. $x+2x^2+3x^3+4x^4+\ldots\infty$

[Ans. : Convergent for $x<1$, divergent for $x>1$]

17. $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots\infty$

[Ans. : Convergent for $x<1$, divergent for $x>1$]

18. $\frac{x}{1\cdot 3}+\frac{x^2}{3\cdot 5}+\frac{x^3}{5\cdot 7}+\ldots\infty$

[Ans. : Convergent for $x<1$, divergent for $x>1$]

19. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\ldots+\frac{n^2-1}{n^2+1}x^n+\ldots$

[Ans. : Convergent for $x<1$, divergent for $x>1$]

20. $\frac{x}{2\sqrt{3}}+\frac{x^2}{3\sqrt{4}}+\frac{x^3}{4\sqrt{5}}+\ldots\infty$

[Ans. : Convergent for $x<1$, divergent for $x>1$]

5.7 RAABE'S TEST (HIGHER RATIO TEST)

- **Statement** If Σu_n is a positive term series and $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l$ then

(i) Σu_n is convergent if $l>1$

(ii) Σu_n is divergent if $l<1$

(iii) test fails if $l=1$

Proof

(i) Consider a number p such that $p>1$. The series $\Sigma v_n=\sum\frac{1}{n^p}$ is convergent if $p>1$. By comparison test, Σu_n will be convergent if from and after some term

$$\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}=\frac{(n+1)^p}{n^p}=\left(1+\frac{1}{n}\right)^p$$

$$\frac{u_n}{u_{n+1}}>\left(1+\frac{1}{n}\right)^p=1+\frac{p}{n}+\frac{p(p-1)}{2!n^2}+\ldots$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)>n\left[\frac{p}{n}+\frac{p(p-1)}{2n^2}+\ldots\right]$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)>\lim_{n\to\infty}\left[p+\frac{p(p-1)}{2n}+\ldots\right]$$

$$l>p>1$$

Hence, Σu_n is convergent if $l>1$.

(ii) Consider a number p such that $p < 1$. The series $\Sigma v_n = \sum \frac{1}{n^p}$ is divergent if $p < 1$.

By comparison test, Σu_n will be divergent if from and after some term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

Proceeding as above in the case (i)

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < \lim_{n\to\infty}\left[p + \frac{p(p-1)}{2n} + \ldots\right]$$

$$l < p < 1$$

Hence, Σu_n is divergent if $l < 1$.

(iii) Raabe's test fails if $l = 1$ and other tests are required to check the nature of the series.

HISTORICAL DATA

Joseph Ludwig Raabe (1801–1859) was a Swiss mathematician.

As his parents were quite poor, Raabe was forced to earn his living from a very early age by giving private lessons. He began to study mathematics in 1820 at the Polytechnicum in Vienna, Austria. In autumn 1831, he moved to Zürich, where he became professor of mathematics in 1833. In 1855, he became professor at the newly founded Swiss Polytechnicum.

He is best known for Raabe's ratio test, an extension of D'Alembert's ratio test, which serves to determine the convergence or divergence of an infinite series. He is also known for the Raabe integral of the gamma function

$$\int_a^{a+1} \log \Gamma(t)\, dt = \frac{1}{2}\log 2\pi + a\log a - a, \; a \geq 0.$$

EXAMPLE 5.14 *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{1^2 \cdot 5^2 \cdot 9^2 \ldots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \ldots (4n)^2}$.

Solution: Let

$$u_n = \frac{1^2 \cdot 5^2 \cdot 9^2 \ldots (4n-3)^2}{4^2 \cdot 8^2 \cdot 12^2 \ldots (4n)^2}$$

$$u_{n+1} = \frac{1^2 \cdot 5^2 \cdot 9^2 \ldots (4n-3)^2 (4n+1)^2}{4^2 \cdot 8^2 \cdot 12^2 \ldots (4n)^2 (4n+4)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{1^2 \cdot 5^2 \cdot 9^2 \ldots (4n-3)^2 (4n+1)^2}{4^2 \cdot 8^2 \cdot 12^2 \ldots (4n)^2 (4n+4)^2} \cdot \frac{4^2 \cdot 8^2 \cdot 12^2 \ldots (4n)^2}{1^2 \cdot 5^2 \cdot 9^2 \ldots (4n-3)^2} = \frac{(4n+1)^2}{(4n+4)^2} = \frac{\left(4 + \frac{1}{n}\right)^2}{\left(4 + \frac{4}{n}\right)^2}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\left(4+\frac{1}{n}\right)^2}{\left(4+\frac{4}{n}\right)^2}=1$$

Thus, D'Alembert's ratio test fails.
Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left[\frac{(4n+4)^2}{(4n+1)^2}-1\right]=\lim_{n\to\infty}\frac{n(24n+15)}{(4n+1)^2}=\lim_{n\to\infty}\frac{24+\frac{15}{n}}{\left(4+\frac{1}{n}\right)^2}=\frac{24}{16}=\frac{3}{2}>1$$

Hence, by Raabe's test, the series is convergent.

EXAMPLE 5.15 *Test the convergence of the series* $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\cdots$.

Solution: Let

$$u_n=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots 2n}\cdot\frac{x^{2n+1}}{2n+1}$$ [Neglecting first term]

$$u_{n+1}=\frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots(2n)(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}$$

$$\frac{u_{n+1}}{u_n}=\frac{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots(2n)(2n+2)}\cdot\frac{x^{2n+3}}{2n+3}\cdot\frac{2\cdot 4\cdot 6\ldots 2n}{1\cdot 3\cdot 5\ldots(2n-1)}\cdot\frac{2n+1}{x^{2n+1}}$$

$$=\frac{(2n+1)^2x^2}{(2n+2)(2n+3)}=\frac{\left(2+\frac{1}{n}\right)^2x^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)^2x^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}=x^2$$

By D'Alembert's ratio test, the series is

(i) convergent if $x^2<1$

(ii) divergent if $x^2>1$

The test fails if $x^2 = 1$.

For $x^2 = 1$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(2n+3)}{(2n+1)^2}-1\right] = \lim_{n\to\infty}\frac{n(6n+5)}{(2n+1)^2} = \lim_{n\to\infty}\frac{\left(6+\frac{5}{n}\right)}{\left(2+\frac{1}{n}\right)^2} = \frac{6}{4} > 1$$

By Raabe's test, the series is convergent if $x^2 = 1$.

Hence, the series is convergent for $x^2 \le 1$ and is divergent for $x^2 > 1$.

EXERCISE 5.4

Test the convergence of the following series:

1. $1+\frac{1}{2}+\frac{1}{2\cdot 4}+\frac{1}{2\cdot 4\cdot 6}+\dots$

   [Ans. : Divergent]

2. $1+\frac{(1!)^2}{2!}x+\frac{(2!)^2}{4!}x^2+\frac{(3!)^2}{6!}x^3+\dots$

   [Ans. : Convergent for $x < 4$ and divergent for $x \ge 4$]

3. $1+\frac{2^2}{3^2}+\frac{2^2\cdot 4^2}{3^2\cdot 5^2}+\frac{2^2\cdot 4^2\cdot 6^2}{3^2\cdot 5^2\cdot 7^2}+\dots$

   [Ans. : Divergent]

4. $\frac{a(a+1)}{2!}+\frac{(a+1)(a+2)}{3!}+\frac{(a+2)(a+3)}{4!}+\dots$

   [Ans. : Convergent for $a \le 0$]

5. $\sum \frac{(n!)^2}{(2n)!}x^{2n}$

   [Ans. : Convergent for $x < 4$ and divergent for $x^2 \ge 4$]

5.8 LOGARITHMIC TEST

- **Statement** If Σu_n is a positive term series and if $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = l$ then
  (i) Σu_n is convergent if $l > 1$
  (ii) Σu_n is divergent if $l < 1$

Proof Let $\Sigma v_n = \Sigma\frac{1}{n^p}$ which converges if $p > 1$ and diverges if $p \le 1$.

(i) Let Σv_n be convergent. Σu_n will also be convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} = \frac{(n+1)^p}{n^p}$$

$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)^p$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) > \log\left(1+\frac{1}{n}\right)^p$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) > p\log\left(1+\frac{1}{n}\right)$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) > p\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right)$$

$$n\log\left(\frac{u_n}{u_{n+1}}\right) > p\left(1-\frac{1}{2n}+\frac{1}{3n^2}-\cdots\right)$$

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) > p$$

$$l > p > 1$$

Hence, Σu_n is convergent if $l > 1$. $[\because \Sigma v_n$ is convergent if $p > 1]$

(ii) Let Σv_n be divergent. Σu_n will also be divergent if $\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$.
Proceeding as above,

$$\lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) < p$$

$$l < p \le 1$$ $[\because \Sigma v_n$ is divergent if $p \le 1]$

Hence, Σu_n is divergent if $l < 1$.

EXAMPLE 5.16 *Test the convergence of the series* $\frac{1^2}{4^2}+\frac{5^2}{8^2}+\frac{9^2}{12^2}+\frac{13^2}{16^2}+\cdots$

Solution: Let $u_n = \frac{(4n-3)^2}{(4n)^2}$

$$u_{n+1} = \frac{(4n+1)^2}{(4n+4)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{(4n-3)^2}{(4n)^2}\cdot\frac{(4n+4)^2}{(4n+1)^2} = \left[\frac{\left(1-\frac{3}{4n}\right)\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{4n}\right)}\right]^2$$

$$\log\frac{u_n}{u_{n+1}} = 2\left[\log\left(1-\frac{3}{4n}\right)+\log\left(1+\frac{1}{n}\right)-\log\left(1+\frac{1}{4n}\right)\right]$$

$$= 2\left[\left(-\frac{3}{4n}-\frac{1}{2}\cdot\frac{3^2}{16n^2}-\cdots\right)+\left(\frac{1}{n}-\frac{1}{2}\cdot\frac{1}{n^2}+\cdots\right)-\left(\frac{1}{4n}-\frac{1}{2}\cdot\frac{1}{16n^2}+\cdots\right)\right]$$

$$\left[\because \log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right]$$

$$n\log\frac{u_n}{u_{n+1}} = 2\left[\left(-\frac{3}{4}-\frac{9}{32n}-\cdots\right)+\left(1-\frac{1}{2n}+\cdots\right)-\left(\frac{1}{4}-\frac{1}{32n}+\cdots\right)\right]$$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = 0 < 1$$

Hence, by logarithmic test, the series is divergent.

EXAMPLE 5.17 *Test the convergence of the series* $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\frac{4!}{5^4}x^4+\cdots$

Solution: Let $u_n = \frac{n!}{(n+1)^n}x^n$ [Neglecting first term]

$$u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}\cdot\frac{(n+1)^n}{n!}\frac{1}{x^n} = \frac{n^n\left(1+\frac{1}{n}\right)^n(n+1)n!}{n!\,n^{n+1}\left(1+\frac{2}{n}\right)^{n+1}}\cdot x = \frac{\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}\cdot x$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{e}{e^2}\cdot x = \frac{1}{e}\cdot x \quad \left[\because \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{a}} = e\right]$$

By D'Alembert's ratio test, the series is

(i) convergent if $\frac{x}{e} < 1$ or $x < e$

(ii) divergent if $\frac{x}{e} > 1$ or $x > e$

The test fails if $\frac{e}{x} = 1$ or $x = e$.

For $x = e$,

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{2}{n}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}}\cdot\frac{1}{e}$$

Applying logarithmic test,

$$\log\frac{u_n}{u_{n+1}}=(n+1)\log\left(1+\frac{2}{n}\right)-(n+1)\log\left(1+\frac{1}{n}\right)-\log e$$

$$=(n+1)\left[\left(\frac{2}{n}-\frac{1}{2}\cdot\frac{2^2}{n^2}+\frac{1}{3}\cdot\frac{2^3}{n^3}-\cdots\right)-\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right)\right]-1$$

$$=(n+1)\left(\frac{1}{n}-\frac{3}{2n^2}+\frac{7}{3n^3}-\cdots\right)-1$$

$$=\left(1-\frac{3}{2n}+\frac{7}{3n^2}+\frac{1}{n}-\frac{3}{2n^2}+\frac{7}{3n^3}-\cdots\right)-1=-\frac{1}{2n}+\frac{5}{6n^2}+\frac{7}{3n^3}-\cdots$$

$$\lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\left(-\frac{1}{2}+\frac{5}{6n}+\frac{7}{3n^2}-\cdots\right)=-\frac{1}{2}<1$$

By logarithmic test, the series is divergent if $x=e$.

Hence, the series is convergent if $x<e$ and is divergent if $x\geq e$.

EXERCISE 5.5

Test the convergence of the following series:

1. $x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\frac{5^5x^5}{5!}+\cdots.$

   [**Ans.:** Convergent if $x<\frac{1}{e}$ and divergent if $x\geq\frac{1}{e}$]

2. $1+\frac{2}{2!}x+\frac{3^2}{3!}x^2+\frac{4^3}{4!}x^3+\frac{5^4}{5!}x^4+\cdots\cdots.$

   [**Ans.:** Convergent if $xe\leq 1$ and divergent if $xe>1$]

3. $\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\frac{5^5}{6^6}+\cdots.$

   [**Ans.:** Convergent]

4. $(a+1)\frac{x}{1!}+(a+2)^2\frac{x^2}{2!}+(a+3)^2\frac{x^3}{3!}+\cdots.$

   [**Ans.:** Convergent if $xe<1$ and divergent if $xe\geq 1$]

5.9 *CAUCHY'S ROOT TEST

- **Statement** If Σu_n is a positive term series and if $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}}=l$ then

  (i) Σu_n is convergent if $l<1$

  (ii) Σu_n is divergent if $l>1$

* Refer Chapter 3 for Historical Data of Baron Augustin-Louis Cauchy.

Proof

Case I If $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = l < 1$

Consider a number $l < r < 1$ such that $(u_n)^{\frac{1}{n}} < r$ for all $n > m$

$$u_n < r^n \text{ for all } n > m \qquad \dots (5.6)$$

The geometric series $\quad \Sigma r^n = r + r^2 + r^3 + \dots \infty$

$$S_n = r + r^2 + r^3 + \dots + r^n = \frac{r(1-r^n)}{1-r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{r(1-r^n)}{1-r} = \frac{r}{1-r}, \text{ which is finite} \qquad \left[\begin{array}{l} \because r<1 \\ \therefore \lim\limits_{n\to\infty} r^n = 0 \end{array}\right]$$

Hence, the series Σr^n is convergent.

From Eq. (5.6),
$$u_n < r^n \text{ for all } n > m$$
$$\Sigma u_n < \Sigma r^n$$

Since Σr^n is convergent, Σu_n is also convergent.

Case II If $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = l > 1$

$$(u_n)^{\frac{1}{n}} > 1 \text{ for all } n > m \qquad \dots (5.7)$$

Neglecting the first m terms,

$$\Sigma u_n = (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \dots \infty$$
$$> 1 + 1 + 1 \dots \infty \qquad \text{[Using Eq. (5.7)]}$$

$$S_n = (u_{m+1})^{\frac{1}{m+1}} + (u_{m+2})^{\frac{1}{m+2}} + (u_{m+3})^{\frac{1}{m+3}} + \dots + (u_{m+n})^{\frac{1}{m+n}}$$
$$> 1 + 1 + 1 \dots n \text{ terms} = n$$

$$\lim_{n\to\infty} S_n > \lim_{n\to\infty} n \to \infty$$

$$\lim_{n\to\infty} S_n \to \infty$$

The series $\sum\limits_{n=m+1}^{\infty} u_n$ is divergent.

The nature of a series remains unchanged if a finite number of terms are neglected in the beginning.

Hence, the series $\sum\limits_{n=1}^{\infty} u_n$ is divergent.

Note If $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = 1$, the root test fails, i.e., no conclusion can be drawn about the convergence or divergence of the series.

* Refer Chapter 3 for Historical Data of Cauchy.

EXAMPLE 5.18 *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Solution: Let

$$u_n = \frac{1}{(\log n)^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{1}{\log n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{\log n} = 0 < 1 \qquad [\because \log\infty \to \infty]$$

Hence, by Cauchy's root test, the series is convergent.

EXAMPLE 5.19 *Test the convergence of the series* $\sum \frac{(n-\log n)^n}{2^n \cdot n^n}$.

Solution: Let

$$u_n = \frac{(n-\log n)^n}{2^n \cdot n^n}$$

$$(u_n)^{\frac{1}{n}} = \frac{(n-\log n)}{2n}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{(n-\log n)}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{\log n}{2n}\right) = \frac{1}{2}-\lim_{n\to\infty}\frac{\frac{1}{n}}{2} = \frac{1}{2} < 1 \quad \text{[Using L'Hospital's rule]}$$

Hence, by Cauchy's root test, the series is convergent.

EXAMPLE 5.20 *Test the convergence of the series* $\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$

Solution: Let

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right]^{-n}$$

$$(u_n)^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^{n+1}-\left(\frac{n+1}{n}\right)\right]^{-1} = \left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)-\left(1+\frac{1}{n}\right)\right]^{-1} = (e-1)^{-1} = \frac{1}{e-1} < 1$$

Hence, by Cauchy's root test, the series is convergent.

EXAMPLE 5.21 *Test the convergence of the series* $\sum_{n=1}^{\infty}\frac{n^n x^n}{(n+1)^n}, x > 0.$

Solution: Let

$$u_n = \frac{n^n x^n}{(n+1)^n}$$

$$(u_n)^{\frac{1}{n}} = \left[\frac{n^n x^n}{(n+1)^n}\right]^{\frac{1}{n}} = \frac{nx}{n+1} = \frac{x}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty}(u_n)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{x}{1+\frac{1}{n}} = x$$

By Cauchy's root test, the series is

(i) convergent if $x < 1$

(ii) divergent if $x > 1$

The test fails if $x = 1$.

For $x = 1$,

$$u_n = \frac{n^n}{(n+1)^n}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} \neq 0$$

The series is divergent for $x = 1$.

Hence, the series is convergent if $x < 1$ and is divergent if $x \geq 1$.

EXERCISE 5.6

Test the convergence of the following series:

1. $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots\infty$

   [Ans. : Convergent]

2. $\sum\left(\frac{n+1}{3n}\right)^n$

   [Ans. : Convergent]

3. $\sum\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{\frac{3}{2}}}$

   [Ans. : Convergent]

4. $1+\frac{x}{2}+\frac{x^2}{3^2}+\frac{x^3}{4^3}+\ldots(x > 0)$

   [Ans. : Convergent]

5. $\sum\left(1+\frac{1}{n}\right)^{n^2}$

[**Ans.** : Divergent]

6. $\sum\frac{(1+nx)^n}{n^n}$

[**Ans.** : Convergent if $x<1$ and divergent if $x>1$]

5.10 CAUCHY'S INTEGRAL TEST

■ **Statement** If $\sum_{n=1}^{\infty}u_n=\sum_{n=1}^{\infty}f(n)$ is a positive term series, where $f(n)$ decreases as n increases, and let $\int_1^{\infty}f(x)\mathrm{d}x=I$ then

(i) Σu_n is convergent if I is finite

(ii) Σu_n is divergent if I is infinite

Proof Consider the area under the curve $y=f(x)$ from $x=1$ to $x=n+1$ represented as $\int_1^{n+1}f(x)\mathrm{d}x$ (Fig. 5.1). Plot the terms $f(1)$, $f(2)$, $f(3)$……$f(n)$, $f(n+1)$.

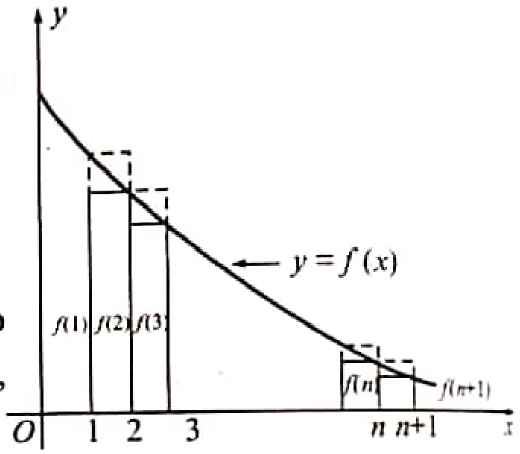

Fig. 5.1 Area under the curve

The area $\int_1^{n+1}f(x)\mathrm{d}x$ lies between the sum of the areas of smaller rectangles and sum of the areas of larger rectangles.

$$f(2)+f(3)+\cdots+f(n+1)\le\int_1^{n+1}f(x)\mathrm{d}x\le f(1)+f(2)+f(3)+\cdots+f(n)$$

$$S_{n+1}-f(1)\le\int_1^{n+1}f(x)\mathrm{d}x\le S_n$$

As $n\to\infty$ first inequality reduces to

$$\lim_{n\to\infty}S_{n+1}\le\int_1^{\infty}f(x)\mathrm{d}x+f(1)$$

This shows that if $\int_1^{\infty}f(x)\mathrm{d}x$ is finite, $\Sigma f(n)=\Sigma u_n$ is convergent. As $n\to\infty$, the second inequality reduces to

$$\int_1^{\infty}f(x)\mathrm{d}x\le\lim_{n\to\infty}S_n$$

or

$$\lim_{n\to\infty}S_n\ge\int_1^{\infty}f(x)\mathrm{d}x$$

This shows that if $\int_1^{\infty}f(x)\mathrm{d}x$ is infinite, $\Sigma f(n)=\Sigma u_n$ is divergent.

EXAMPLE 5.22 *Test the convergence of the series* $\sum_{n=2}^{\infty}\frac{1}{n\log n}$.

Solution: Let $u_n = \frac{1}{n\log n} = f(n)$

$$f(x) = \frac{1}{x\log x}$$

$$\int_2^{\infty} f(x)\mathrm{d}x = \int_2^{\infty}\frac{1}{x\log x}\mathrm{d}x = \lim_{m\to\infty}\int_2^{m}\frac{1}{x\log x}\mathrm{d}x = \lim_{m\to\infty}\left|\log\log x\right|_2^m \qquad \left[\because \int\frac{f'(x)}{f(x)}\mathrm{d}x = \log f(x)\right]$$

$$= \lim_{m\to\infty}(\log\log m - \log\log 2) \to \infty$$

Hence, by Cauchy's integral test, the series is divergent.

EXAMPLE 5.23 *Test the convergence of the series* $\sum_{n=1}^{\infty} n^2 e^{-n^3}$.

Solution: Let $u_n = n^2 e^{-n^3} = f(n)$

$$f(x) = x^2 e^{-x^3}$$

$$\int_1^{\infty} f(x)\mathrm{d}x = \int_1^{\infty} x^2 e^{-x^3}\mathrm{d}x = \lim_{m\to\infty}\left[-\frac{1}{3}\int_1^m e^{-x^3}(-3x^2)\mathrm{d}x\right]$$

$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left|e^{-x^3}\right|_1^m\right] \qquad \left[\because \int e^{f(x)} f'(x)\mathrm{d}x = e^{f(x)}\right]$$

$$= \lim_{m\to\infty}\left[-\frac{1}{3}\left(e^{-m^3} - e^{-1}\right)\right] = -\frac{1}{3}(e^{-\infty} - e^{-1}) = -\frac{1}{3}\left(0 - \frac{1}{e}\right) = \frac{1}{3e} \qquad \text{[Finite]}$$

Hence, by Cauchy's integral test, the series is convergent.

EXAMPLE 5.24 *Show that the harmonic series of order p,*

$$\sum_{n=1}^{\infty}\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\infty \quad \textit{is convergent if } p > 1 \textit{ and is divergent if } p \le 1.$$

Solution: Let $u_n = \frac{1}{n^p} = f(n)$

$$f(x) = \frac{1}{x^p}$$

$$\int_1^{\infty} f(x)\mathrm{d}x = \int_1^{\infty}\frac{1}{x^p}\mathrm{d}x = \lim_{m\to\infty}\left|\frac{x^{-p+1}}{-p+1}\right|_1^m = \lim_{m\to\infty}\left(\frac{m^{1-p}}{1-p} - \frac{1}{1-p}\right) = -\frac{1}{1-p}, \qquad p > 1$$

$$= \infty, \qquad p < 1$$

If $p = 1$,

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x}dx = \lim_{m\to\infty}\int_1^m \frac{1}{x}dx = \lim_{m\to\infty}|\log x|_1^m = \lim_{m\to\infty}(\log m - \log 1) = \log\infty \to \infty$$

The integral $\int_1^\infty f(x)dx$ is finite if $p > 1$ and is infinite if $p \le 1$.
Hence, by Cauchy's integral test, the series is convergent if $p > 1$ and is divergent if $p \le 1$.

EXERCISE 5.7

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

[**Ans.** : Divergent]

2. $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

[**Ans.** : Convergent]

3. $\sum_{n=1}^{\infty} ne^{-n^2}$

[**Ans.** : Convergent]

4. $\sum_{n=1}^{\infty}\frac{1}{n(\log n)^2}$

[**Ans.** : Convergent]

5.11 ALTERNATING SERIES

An infinite series with alternate positive and negative terms is called an alternating series.

Leibnitz's Test for Alternating Series

An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}v_n = \sum_{n=1}^{\infty}u_n$ is convergent if

(i) each term is numerically less than its preceding term, i.e., $|u_{n+1}| < |u_n|$ or $|u_n| > |u_{n+1}|$

(ii) $\lim_{n\to\infty}|u_n| = 0$

EXAMPLE 5.25 *Test the convergence of the series* $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$

Solution: Let $u_n = \frac{(-1)^{n-1}}{n\sqrt{n}}$

$$|u_n| = \frac{1}{n\sqrt{n}}$$

* Refer Chapter 3 for Historical Data of Leibnitz.

The given series is an alternating series.

(i) $$|u_n|-|u_{n+1}|=\frac{1}{n\sqrt{n}}-\frac{1}{(n+1)\sqrt{n+1}}=\frac{(n+1)\sqrt{n+1}-n\sqrt{n}}{(n\sqrt{n})\left[(n+1)\sqrt{(n+1)}\right]}>0 \quad \text{for all } n\in N$$

$$\therefore \quad |u_n|>|u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{1}{n\sqrt{n}}=0$$

Hence, by Leibnitz's test, the series is convergent.

EXAMPLE 5.26 *Test the convergence of the series* $\frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\ldots$.

Solution: Let

$$u_n=(-1)^n\frac{1}{n^p}$$

$$|u_n|=\frac{1}{n^p}$$

The given series is an alternating series.

Case I $p>0$

(i) $$|u_n|-|u_{n+1}|=\frac{1}{n^p}-\frac{1}{(n+1)^p}=\frac{(n+1)^p-n^p}{n^p(n+1)^p}>0 \qquad [\because p>0]$$

$$\therefore \quad |u_n|>|u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{1}{n^p}=0 \qquad [\because p>0]$$

Hence, by Leibnitz's test, the series is convergent if $p>0$.

Case II $p<0$

In this case the conditions (i) and (ii) of the Leibnitz's test are not satisfied.
Hence, the given series is not convergent if $p<0$.

EXAMPLE 5.27 *Test the convergence of the series* $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ for $0<x<1$.

Solution:

Let $$u_n=(-1)^{n-1}\frac{x^n}{n}$$

$$|u_n|=\frac{x^n}{n}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{x^n}{n} - \frac{x^{n+1}}{n+1} = \frac{x^n[(n+1)-nx]}{n(n+1)} = \frac{x^n[1+(1-x)n]}{n(n+1)} > 0 \quad [\because n \geq 1 \text{ and } 0 < x < 1]$$

$$\therefore |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \frac{x^n}{n} = 0 \qquad \left[\because \lim_{n\to\infty} x^n = 0 \text{ if } x < 1\right]$$

Hence, by Leibnitz's test, the series is convergent.

EXAMPLE 5.28 *Test the convergence of the series* $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$

Solution: Let

$$u_n = (-1)^{n-1} \cdot \frac{n}{n+1}$$

$$|u_n| = \frac{n}{n+1}$$

The given series is an alternating series.

(i) $$|u_n| - |u_{n+1}| = \frac{n}{n+1} - \frac{n+1}{n+2} = \frac{n^2+2n-n^2-2n-1}{(n+1)(n+2)} = -\frac{1}{(n+1)(n+2)} < 0$$

Since each term of the series is not numerically less than the preceding term, Leibnitz's test cannot be applied.
The series can be written as

$$\sum_{n=1}^{\infty} u_n = \left(1-\frac{1}{2}\right) - \left(1-\frac{1}{3}\right) + \left(1-\frac{1}{4}\right) - \left(1-\frac{1}{5}\right) + \cdots$$

$$= (1-1+1-1+\cdots) + \left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right) = \sum_{n=1}^{\infty}(-1)^{n-1} + (\log 2 - 1)$$

As $n \to \infty$, the sum of the above series tends to $(-1 + \log 2 - 1)$ or $(1 + \log 2 - 1)$ according as n is even or odd.
Hence, the given series is an oscillatory series.

EXERCISE 5.8

Test the convergence of the following series:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\cdots$

[Ans.: Convergent]

2. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{2n-1}$

[Ans.: Oscillatory]

3. $\frac{1}{2^2}-\frac{1}{3^3}(1+2)+\frac{1}{4^3}(1+2+3)-\frac{1}{5^3}(1+2+3+4)+\ldots$

[**Ans.:** Convergent]

4. $1-2x+3x^2-4x^3+\ldots(x<1)$

[**Ans.:** Convergent]

5. $\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\frac{x^4}{1+x^4}+\ldots(0<x<1)$.

[**Ans.:** Convergent]

5.12 ABSOLUTE AND CONDITIONAL CONVERGENCE OF A SERIES

The series $\sum_{n=1}^{\infty} u_n$ with both positive and negative terms (not necessarily alternative) is called absolutely convergent if the corresponding series $\sum_{n=1}^{\infty} |u_n|$ with all positive terms is convergent.

If the series $\sum_{n=1}^{\infty} u_n$ is convergent and $\sum_{n=1}^{\infty} |u_n|$ is divergent then the series $\sum_{n=1}^{\infty} u_n$ is called conditionally convergent.

Notes

(i) Every absolutely convergent series is a convergent series but converse is not true.
(ii) Any convergent series of positive terms is also absolutely convergent.

EXAMPLE 5.29 *Test the absolute convergence of the series*

$$1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\cdots$$

Solution: Let

$$u_n=(-1)^{n-1}\frac{1}{n\sqrt{n}}$$

$$|u_n|=\frac{1}{n\sqrt{n}}=\frac{1}{n^{\frac{3}{2}}}$$

By comparison test, $\sum_{n=1}^{\infty}|u_n|$ is convergent as $p=\frac{3}{2}>1$.

Hence, the series is absolutely convergent.

EXAMPLE 5.30 *Determine absolute or conditional convergence of the series* $\sum_{n=1}^{\infty}(-1)^n\cdot\frac{n^2}{n^3+1}$.

Solution: Let

$$u_n=(-1)^n\cdot\frac{n^2}{n^3+1}$$

$$|u_n| = \frac{n^2}{n^3+1} = \frac{1}{n\left(1+\frac{1}{n^3}\right)}$$

Let
$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{|u_n|}{v_n} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n^3}} = 1$$ [Finite and nonzero]

and $\sum v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, $\sum |u_n|$ is also divergent.

Hence, $\sum u_n$ is not absolutely convergent.

To check the conditional convergence, applying Leibnitz's test,

(i) $$|u_n| - |u_{n+1}| = \frac{n^2}{n^3+1} - \frac{(n+1)^2}{(n+1)^3+1}$$

$$= \frac{n^2(n^3+3n^2+3n+2)-(n^3+1)(n^2+2n+1)}{(n^3+1)[(n+1)^3+1]} = \frac{n^4+2n^3+n^2-2n-1}{(n^3+1)[(n+1)^3+1]}$$

$$= \frac{n^4+n^2(2n+1)-1(2n+1)}{(n^3+1)[(n+1)^3+1]} = \frac{n^4+(2n+1)(n^2-1)}{(n^3+1)[(n+1)^3+1]} > 0 \quad \text{for all } n \in N$$

$$\therefore |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{n^2}{n^3+1} = \lim_{n\to\infty}\frac{1}{n\left(1+\frac{1}{n^3}\right)} = 0$$

By Leibnitz's test, Σu_n is convergent.

The series Σu_n is convergent and $\sum |u_n|$ is divergent.

Hence, the series is conditionally convergent.

EXAMPLE 5.31 *Test the series for absolute or conditional convergence*

$$\frac{2}{3} - \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} - \frac{5}{6}\cdot\frac{1}{4} + \cdots.$$

Solution: Let
$$u_n = (-1)^{n-1}\left(\frac{n+1}{n+2}\cdot\frac{1}{n}\right)$$

$$\sum_{n=1}^{\infty}|u_n| = \frac{2}{3} + \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} + \frac{5}{6}\cdot\frac{1}{4} + \cdots.$$

$$|u_n| = \frac{n+1}{n+2}\cdot\frac{1}{n}$$

Let

$$v_n = \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{|u_n|}{v_n} = \lim_{n\to\infty}\frac{n+1}{n+2} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 1 \qquad \text{[Finite and nonzero]}$$

and $\sum v_n = \sum \frac{1}{n}$ is divergent as $p = 1$.

By comparison test, $\sum |u_n|$ is also divergent.

Hence, the series is not absolutely convergent.

To check the conditional convergence, applying Leibnitz's test,

(i) $$|u_n| - |u_{n+1}| = \frac{n+1}{n(n+2)} - \frac{n+2}{(n+1)(n+3)} = \frac{n^2+3n+3}{n(n+1)(n+2)(n+3)} > 0 \qquad \text{for all } n \in N$$

$$\therefore |u_n| > |u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{n+1}{n(n+2)} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{n\left(1+\frac{2}{n}\right)} = 0$$

By Leibnitz's test, $\sum u_n$ is convergent.

The series $\sum u_n$ is convergent and $\sum |u_n|$ is divergent.

Hence, the series is conditionally convergent.

EXAMPLE 5.32 *Test the convergence of the series* $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, x > 0.$

Solution: Let

$$u_n = (-1)^{n-1}\frac{x^{2n-1}}{2n-1}$$

$$|u_n| = \frac{x^{2n-1}}{2n-1}$$

$$|u_{n+1}| = \frac{x^{2n+1}}{2n+1}$$

$$\frac{|u_{n+1}|}{|u_n|} = \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} = \left(\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right) \cdot x^2$$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \left(\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right) \cdot x^2 = x^2$$

By D'Alembert's ratio test, $\sum |u_n|$ is convergent if $x^2 < 1$ or $x < 1$ $[\because x > 0]$

Thus, the given series is absolutely convergent and hence, is convergent for $x < 1$.

If $x^2 = 1$ or $x = 1$ $[\because x > 0]$,

$$u_n = \frac{(-1)^{n-1}}{2n-1}$$

$$|u_n| = \frac{1}{2n-1}$$

The given series is an alternating series.

(i) $|u_n| - |u_{n+1}| = \dfrac{1}{2n-1} - \dfrac{1}{2n+1} = \dfrac{2}{4n^2-1} > 0$ for all $n \in N$

(ii) $\lim\limits_{n\to\infty} |u_n| = \lim\limits_{n\to\infty} \dfrac{1}{2n-1} = 0$

By Leibnitz's test, the series is convergent for $x = 1$.

Hence, the series is convergent for $x \le 1$.

EXERCISE 5.9

Test the following series for absolute or conditional convergence:

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

[Ans.: Conditionally convergent]

2. $1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \dots$

[Ans.: Absolutely convergent]

3. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

[Ans.: Conditionally convergent]

4. $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \dots$

[Ans.: Absolutely convergent]

5.13 UNIFORM CONVERGENCE OF A SERIES

The series $\sum_{n=1}^{\infty} u_n(x)$ of real valued functions defined in the interval (a, b) is said to converge uniformly to a function $S(x)$ if for a given $\in > 0$, there exists a number m independent of x such that for every $x \in (a, b)$,

$$|S_n(x) - S(x)| < \in \text{ for all } n > m$$

where

$$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

Weierstrass's M-Test

The series $\sum_{n=1}^{\infty} u_n(x)$ is said to converge uniformly in an interval (a, b), if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that

$$|u_n(x)| \le M_n \text{ for all } x \in (a, b)$$

Proof Let $\sum_{n=1}^{\infty} M_n$ be convergent then for a given $\in > 0$, there exists a number m such that $|C - C_n| < \in$ for all $n > m$,

where $C = M_1 + M_2 + M_3 + \ldots \infty$ and $C_n = M_1 + M_2 + \ldots + M_n$

then $\quad |M_{n+1} + M_{n+2} + \ldots| < \in$ for all $n > m$

$(M_{n+1} + M_{n+2} + \ldots) < \in$ for all $n > m$ $\quad$ [$\because M_n$ is positive constant]

Now, $\quad |u_n(x)| \le M_n$ for all $x \in (a, b)$

$$|u_{n+1}(x) + u_{n+2}(x) + \ldots| \le |u_{n+1}(x)| + |u_{n+2}(x)| + \ldots$$
$$\le M_{n+1} + M_{n+2} + \ldots$$
$$< \in \quad \text{for all } n > m$$

$$\therefore |S(x) - S_n(x)| < \in \text{ for all } n > m$$

where

$$S_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

Since m does not depend on x, the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly in the interval (a, b).

HISTORICAL DATA

Karl Theodor Wilhelm Weierstrass (1815–1897) was a German mathematician who is often cited as the "father of modern analysis".

Delta-epsilon proofs are first found in the works of Cauchy in the 1820s. Cauchy did not clearly distinguish between continuity and uniform continuity on an interval. Notably, in his 1821 *Cours d'analyse*, Cauchy argued that the (pointwise) limit of (pointwise) continuous functions was itself (pointwise) continuous, a statement interpreted as being incorrect by many scholars. The correct statement is rather that the uniform limit of continuous functions is continuous (also, the uniform limit of uniformly continuous functions is uniformly continuous). This required the concept of uniform convergence, which was first observed by Weierstrass's advisor, Christoph Gudermann, in an 1838 paper, where Gudermann noted the phenomenon but did not define it or elaborate on it. Weierstrass saw the importance of the concept, and both formalized it and applied it widely throughout the foundations of calculus.

EXAMPLE 5.33 *Test the series* $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^3x^2}$ *for uniform convergence.*

Solution: Let

$$u_n(x) = \frac{1}{n^4 + n^3x^2}$$

$$|u_n(x)| = \left|\frac{1}{n^4 + n^3x^2}\right| < \frac{1}{n^4} \quad \text{for all } x \in R$$

$$M_n = \frac{1}{n^4}$$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent since $p = 4 > 1$.

Hence, by M-test, the series is uniformly convergent for all real values of x.

EXAMPLE 5.34 *Test the series* $\sum_{n=1}^{\infty} \frac{\sin(x^2 + n^2x)}{n(n^2+2)}$ *for uniform convergence.*

Solution: Let

$$u_n(x) = \frac{\sin(x^2 + n^2x)}{n(n^2+2)}$$

$$|u_n(x)| = \left|\frac{\sin(x^2 + n^2x)}{n(n^2+2)}\right| = \frac{|\sin(x^2 + n^2x)|}{n(n^2+2)}$$

$$\le \frac{1}{n^3 + 2n} \quad \text{for all } x \in R \qquad \left[\begin{array}{r} \because -1 \le \sin\theta \le 1 \\ |\sin\theta| \le 1 \end{array}\right]$$

$$< \frac{1}{n^3} \quad \text{for all } n \in N$$

$$M_n = \frac{1}{n^3}$$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent since $p = 3 > 1$.

Hence, by M-test, the series is uniformly convergent for all real values of x.

EXAMPLE 5.35 *Test the series* $\sin x - \frac{\sin 2x}{2\sqrt{2}} + \frac{\sin 3x}{3\sqrt{3}} - \frac{\sin 4x}{4\sqrt{4}} + \ldots$ *for uniform convergence.*

Solution: Let $u_n(x) = (-1)^{n-1} \frac{\sin nx}{n\sqrt{n}}$

$$|u_n(x)| = \left|\frac{\sin nx}{n\sqrt{n}}\right| \le \frac{1}{n^{\frac{3}{2}}} \quad \text{for all } x \in R \quad \left[\begin{array}{r} \because -1 \le \sin\theta \le 1 \\ |\sin\theta| \le 1 \end{array}\right]$$

$$M_n = \frac{1}{n^{\frac{3}{2}}}$$

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is convergent since $p = \frac{3}{2} > 1$.

Hence, by M-test, the series is uniformly convergent for all real values of x.

EXAMPLE 5.36 *Show that if $0 < r < 1$, the series* $\sum_{n=1}^{\infty} r^n \cos n^2 x$ *is uniformly convergent.*

Solution: Let $u_n(x) = r^n \cos n^2 x$

$$|u_n(x)| = \left| r^n \cos n^2 x \right| \le |r^n| \quad \text{for all } x \in R \quad \left[\begin{array}{l} \because -1 \le \cos\theta \le 1 \\ |\cos\theta| \le 1 \end{array}\right]$$

$$= r^n, \; 0 < r < 1$$

$$M_n = r^n$$

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \ldots$$

which is convergent being a geometric series with $0 < r < 1$.
Hence, by M-test, the series is uniformly convergent for all real values of x.

EXERCISE 5.10

1. Test the following series for uniform convergence:

   (i) $\sum_{n=1}^{\infty} \frac{\sin(x^2 + nx)}{n(n+2)}$; for all real x.

   [Ans. : uniformly convergent]

   (ii) $\frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \dots$

   [Ans. : uniformly convergent]

2. Show that if $0 < r < 1$ then the series $\sum_{n=1}^{\infty} r^n \sin a^n x$ is uniformly convergent for all real values of x.

3. Show that

   $$\frac{1}{1+x^2} - \frac{1}{2+x^2} + \frac{1}{3+x^2} - \frac{1}{4+x^2} + \dots$$

   converges uniformly in the interval $x \geq 0$ but not absolutely.

5.14 POWER SERIES

A power series is an infinite series of the form $\sum_{n=1}^{\infty} a_n x^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$, where a_n represents the coefficient of the n^{th} term, c is a constant and x varies around c. When $c = 0$, the series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

5.14.1 Interval and Radius of Convergence

A power series will converge only for certain values of x. An interval $(-R, R)$ in which a power series converges is called the interval of convergence. The number R is called the radius of convergence. e.g., if a power series converges for all the values of x then interval of convergence will be $(-\infty, \infty)$ and the radius of convergence will be ∞.

5.14.2 Test for Convergence

Since a power series may be positive, alternating or mixed series, the concept of absolute convergence is used to test the convergence of a power series. Applying D'Alembert's ratio test,

$$u_n = a_n x^n$$

$$u_{n+1} = a_{n+1} x^{n+1}$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{l}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|\frac{1}{l} = \left|\frac{x}{l}\right|$$

By D'Alembert's ratio test, the series is absolutely convergent, and hence, is convergent

if $\left|\frac{x}{l}\right| < 1$, i.e., $|x| < l$, $-l < x < l$.

The interval of convergence of the series is $(-l, l)$ and the radius of convergence is l.

EXAMPLE 5.37 *Obtain the range of convergence of* $\sum_{n=1}^{\infty}\frac{x^n}{a+\sqrt{n}}$, $x > 0$, $a > 0$.

Solution: Let
$$u_n = \frac{x^n}{a+\sqrt{n}}$$

$$u_{n+1} = \frac{x^{n+1}}{a+\sqrt{n+1}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{n+1}}{a+\sqrt{n+1}}\cdot\frac{a+\sqrt{n}}{x^n} = \lim_{n\to\infty}\frac{\frac{a}{\sqrt{n}}+1}{\frac{a}{\sqrt{n}}+\sqrt{1+\frac{1}{n}}}\cdot x = x \quad [\because x > 0]$$

By D'Alembert's ratio test, the series is

(i) convergent if $x < 1$

(ii) divergent if $x > 1$ $\quad [\because x > 0]$

The test fails if $x = 1$.

For $x = 1$,
$$u_n = \frac{1}{a+\sqrt{n}} = \frac{1}{\sqrt{n}}\cdot\left(\frac{1}{\frac{a}{\sqrt{n}}+1}\right)$$

Let
$$v_n = \frac{1}{\sqrt{n}}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\frac{a}{\sqrt{n}}+1} = 1 \qquad \text{[Finite and nonzero]}$$

and $\Sigma v_n = \sum \frac{1}{n^{\frac{1}{2}}}$ is divergent as $p = \frac{1}{2} < 1$.

By comparison test, Σu_n is also divergent for $x = 1$.

Hence, the series is convergent for $0 < x < 1$ and the range of convergence is $0 < x < 1$.

EXAMPLE 5.38 *Obtain the range of convergence of*

$$1 + \frac{3}{7}x + \frac{3\cdot 6}{7\cdot 10}x^2 + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3 + \cdots, x > 0.$$

Solution: Let $u_n = \frac{3\cdot 6\cdot 9 \ldots 3n}{7\cdot 10\cdot 13 \ldots (3n+4)}x^n$ $\left[\text{Considering first term as } \frac{3}{7}x\right]$

$$u_{n+1} = \frac{3\cdot 6\cdot 9 \ldots 3n\,(3n+3)x^{n+1}}{7\cdot 10\cdot 13 \ldots (3n+4)(3n+7)}$$

Since $x > 0$, the given series is positive term series. The range of convergence can be obtained by applying D'Alembert's ratio test directly to the given series.

$$\frac{u_{n+1}}{u_n} = \frac{3\cdot 6\cdot 9 \ldots 3n(3n+3)x^{n+1}}{7\cdot 10\cdot 13 \ldots (3n+4)(3n+7)} \cdot \frac{7\cdot 10\cdot 13 \ldots (3n+4)}{3\cdot 6\cdot 9 \ldots 3n} \frac{1}{x^n} = \left(\frac{3n+3}{3n+7}\right)x = \left(\frac{3+\frac{3}{n}}{3+\frac{7}{n}}\right)x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{3+\frac{3}{n}}{3+\frac{7}{n}}\right)x = x$$

By D'Alembert's ratio test, the series is convergent if $0 < x < 1$.

At $x = 1$, the test fails.

Applying Raabe's test at $x = 1$,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{3n+7}{3n+3} - 1\right) = \lim_{n\to\infty} \frac{4n}{3n+3} = \lim_{n\to\infty} \frac{4}{3+\frac{3}{n}} = \frac{4}{3} > 1$$

The series is convergent at $x = 1$.

Hence, the series is convergent for $0 < x \le 1$ and the range of convergence is $0 < x \le 1$.

EXAMPLE 5.39 *Obtain the range of convergence of the series* $\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3\ldots n}{4\cdot 7\cdot 10\ldots(3n+1)}x^n$.

Solution: Let
$$u_n = \frac{1\cdot 2\cdot 3\ldots n}{4\cdot 7\cdot 10\ldots(3n+1)}x^n$$

$$u_{n+1} = \frac{1\cdot 2\cdot 3\ldots n(n+1)}{4\cdot 7\cdot 10\ldots(3n+1)(3n+4)}x^{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{1\cdot 2\cdot 3\ldots n(n+1)}{4\cdot 7\cdot 10\ldots(3n+1)(3n+4)}x^{n+1}\cdot\frac{4\cdot 7\cdot 10\ldots(3n+1)}{1\cdot 2\cdot 3\ldots n}\frac{1}{x^n} = \left(\frac{n+1}{3n+4}\right)x = \left(\frac{1+\frac{1}{n}}{3+\frac{4}{n}}\right)x$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\left(\frac{1+\frac{1}{n}}{3+\frac{4}{n}}\right)x\right| = \left|\frac{x}{3}\right|$$

By D'Alembert's ratio test, the series is convergent if $\left|\frac{x}{3}\right| < 1$ i.e., $|x| < 3$, i.e., $-3 < x < 3$.

At $x = 3$,
$$\frac{u_n}{u_{n+1}} = \frac{3n+4}{n+1}\cdot\frac{1}{3}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{3n+4}{3n+3}-1\right) = \lim_{n\to\infty}\frac{n}{3n+3} = \lim_{n\to\infty}\frac{1}{3+\frac{3}{n}} = \frac{1}{3} < 1$$

By Raabe's test, the series is divergent for $x = 3$.

At $x = -3$,
$$u_n = \frac{1\cdot 2\cdot 3\ldots n}{4\cdot 7\cdot 10\ldots(3n+1)}(-3)^n$$
$$|u_n| = \frac{1\cdot 2\cdot 3\ldots n}{4\cdot 7\cdot 10\ldots(3n+1)}3^n$$

The given series is an alternating series.

(i) $$|u_n|-|u_{n+1}|=\frac{(1\cdot2\cdot3\ldots n)3^n}{4\cdot7\cdot10\ldots(3n+1)}-\frac{[1\cdot2\cdot3\ldots n(n+1)]3^{n+1}}{4\cdot7\cdot10\ldots(3n+1)(3n+4)}$$

$$=\frac{(1\cdot2\cdot3\ldots n)3^n}{4\cdot7\cdot10\ldots(3n+1)}\left[1-\frac{(n+1)\cdot3}{3n+4}\right]$$

$$=\frac{(1\cdot2\cdot3\ldots n)3^n}{4\cdot7\cdot10\ldots(3n+1)}\cdot\frac{1}{(3n+4)}>0 \qquad \text{for all } n\in N$$

$$\therefore |u_n|>|u_{n+1}|$$

(ii) $$\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{1\cdot2\cdot3\ldots n}{4\cdot7\cdot10\ldots(3n+1)}\cdot3^n\neq0$$

By Leibnitz's test, the series is divergent for $x=-3$.
Hence, the series is convergent for $-3<x<3$ and the range of convergence is $-3<x<3$.

EXAMPLE 5.40 *Obtain the range of convergence of* $\sum_{n=1}^{\infty}\frac{n+1}{2n+1}\cdot\frac{(x-3)^n}{2^n}$.

Solution: Let $$u_n=\frac{n+1}{2n+1}\cdot\frac{(x-3)^n}{2^n}$$

$$u_{n+1}=\frac{n+2}{2n+3}\cdot\frac{(x-3)^{n+1}}{2^{n+1}}$$

$$\frac{u_{n+1}}{u_n}=\frac{n+2}{2n+3}\cdot\frac{(x-3)^{n+1}}{2^{n+1}}\cdot\frac{2n+1}{n+1}\cdot\frac{2n}{(x-3)^n}=\frac{\left(2+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{2\left(1+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)}\cdot(x-3)$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{\left(2+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{2\left(1+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)}\cdot(x-3)\right|=\lim_{n\to\infty}\left|\frac{x-3}{2}\right|$$

By D'Alembert's ratio test, the series is convergent if

$$\left|\frac{x-3}{2}\right|<1, \text{i.e.,} |x-3|<2, \text{ i.e.,} -2<x-3<2, \text{i.e.,} 1<x<5$$

At $x=1$,

$$u_n=\frac{n+1}{2n+1}\cdot\frac{(-2)^n}{2^n}=(-1)^n\left(\frac{n+1}{2n+1}\right)$$

series $\sum u_n$ is an alternating series.

$$|u_n| = \frac{n+1}{2n+1}$$

$$\lim_{n\to\infty}|u_n| = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2+\frac{1}{n}} = \frac{1}{2} \neq 0$$

By Leibnitz's test, the series is not convergent at $x = 1$.

$x = 5$,

$$u_n = \frac{n+1}{2n+1}$$

$$u_{n+1} = \frac{n+2}{2n+3}$$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)}{(2n+1)}\cdot\frac{(2n+3)}{(n+2)}$$

Applying Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(n+1)(2n+3)}{(2n+1)(n+2)}-1\right] = \lim_{n\to\infty}\frac{n}{(2n+1)(n+2)} = \lim_{n\to\infty}\frac{1}{\left(2+\frac{1}{n}\right)(n+2)} = 0 < 1$$

The series is divergent at $x = 5$.

Hence, the series is convergent for $1 < x < 5$ and the range of convergence is $1 < x < 5$.

EXERCISE 5.11

Obtain the range of convergence of the following series:

1. $1+x+2x^2+3x^3+\cdots+nx^n+\cdots$ **[Ans. : $-1<x<1$]**

2. $\frac{1}{2}+\frac{x}{3}+\frac{x^2}{4}+\frac{x^3}{5}+\cdots+\frac{x^n}{n+2}+\cdots$ **[Ans. : $-1<x<1$]**

3. $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{(n+1)}$ **[Ans. : $|x|\le 1$]**

4. $\sum_{n=0}^{\infty}\frac{(x+2)}{\sqrt{n+1}}$ **[Ans. : $-3\le x\le -1$]**

5. $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{\log(n+1)}$ **[Ans. : $|x|<1$]**

6. $\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n$ **$\left[\text{Ans. : } \frac{1}{2}<x<\frac{3}{2}\right]$**

7. $\sum_{n=1}^{\infty}n!(x-1)^n$ **[Ans. : $x=1$]**

8. $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}x^n$ **[Ans. : $|x|<4$]**

9. $\sum_{n=1}^{\infty}\frac{(-2)^n(2x+1)^n}{n^2}$ **$\left[\text{Ans. : } -\frac{3}{4}\le x\le -\frac{1}{4}\right]$**

10. $\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)^{\frac{3}{2}}}$ **[Ans. : $-1\le x\le 1$]**