Matrices

CHAPTER OUTLINE

1.1 INTRODUCTION

A matrix is a rectangular table of elements which may be numbers or any abstract quantities that can be added and multiplied. Matrices are used to describe linear equations, keep track of the coefficients of linear transformation, and record data that depend on multiple parameters. There are many applications of matrices in mathematics, viz., graph theory, probability theory, statistics, computer graphics, geometrical optics, etc.

1.2 MATRIX

A set of mn elements (real or complex) arranged in a rectangular array of m rows and n columns is called a matrix of order m by n, written as $m \times n$.

An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The matrix can also be expressed in the form $A=[a_{ij}]_{m\times n}$, where a_{ij} is the element in the i^{th} row and j^{th} column, written as the $(i,j)^{th}$ element of the matrix.

Matrices with one row or one column are also known as *vectors*.

1.3 SOME DEFINITIONS ASSOCIATED WITH MATRICES

1. *Row Matrix* A matrix having only one row and any number of columns is called a row matrix or row vector, e.g.,

$$[-1 \quad 2],\ [2 \quad 5 \quad -3 \quad 4]$$

2. *Column Matrix* A matrix having only one column and any number of rows is called a column matrix or column vector, e.g.,

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

3. *Zero Matrix or Null Matrix* A matrix whose all the elements are zero is called a zero matrix or null matrix and is denoted by **0**, e.g.,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. *Square Matrix* A matrix in which the number of rows is equal to the number of columns is called a square matrix, e.g.,

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & -5 \\ 2 & 6 & 8 \end{bmatrix}.$$

5. *Diagonal Matrix* A square matrix, all of whose nondiagonal elements are zero and at least one diagonal element is nonzero, is called a diagonal matrix, e.g.,

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

6. *Scalar Matrix* A diagonal matrix, all of whose diagonal elements are equal, is called a scalar matrix, e.g.,

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

7. *Unit or Identity Matrix* A diagonal matrix, all of whose diagonal elements are unity, is called a unit matrix or identity matrix and is denoted by I, e.g.,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. *Upper Triangular Matrix* A square matrix in which all the elements below the diagonal are zero is called an upper triangular matrix, e.g.,

$$\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & -5 \\ 0 & 0 & 8 \end{bmatrix}$$

9. *Lower Triangular Matrix* A square matrix in which all the elements above the diagonal are zero is called a lower triangular matrix, e.g.,

$$\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 6 & 8 \end{bmatrix}$$

10. *Trace of a Matrix* The sum of all the diagonal elements of a square matrix is called the trace of a matrix.

If $A = \begin{bmatrix} 2 & -1 & 0 \\ 4 & 6 & -2 \\ -1 & 0 & 3 \end{bmatrix}$ then trace of $A = 2 + 6 + 3 = 11$

11. *Transpose of a Matrix* A matrix obtained by interchanging rows and columns of a matrix is called the transpose of a matrix and is denoted by A' or A^T.

If $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -4 & 1 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 & -4 \\ -1 & 2 & 1 \end{bmatrix}$

Notes

(i) $(A^T)^T = A$

(ii) $(kA)^T = kA^T$, where k is a scalar

(iii) $(A + B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$

■ **Property 1** Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof Let A be a square matrix.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q$$

where $$P = \frac{1}{2}(A + A^T) \quad \text{and} \quad Q = \frac{1}{2}(A - A^T)$$

$$P^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}\left[A^T + (A^T)^T\right] = \frac{1}{2}(A^T + A) = P$$

Hence, P is a symmetric matrix.

Also, $$Q^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}\left[A^T - (A^T)^T\right] = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -Q$$

Hence, Q is a skew-symmetric matrix.

Thus, every matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Uniqueness Let $A = R + S$, where R is a symmetric and S is a skew-symmetric matrix.

$$A^T = (R + S)^T = R^T + S^T = R - S$$

Now, $$\frac{1}{2}(A + A^T) = \frac{1}{2}[(R + S) + (R - S)] = R = P$$

and $$\frac{1}{2}(A - A^T) = \frac{1}{2}[(R + S) - (R - S)] = S = Q$$

Hence, representation $A = P + Q$ is unique.

EXAMPLE 1.1 *Express the matrix* $A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}$ *as the sum of a symmetric and a skew-symmetric matrix.*

Solution:

$$A = \begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & -1 & 8 \\ 5 & -2 & 2 \\ 7 & -4 & 13 \end{bmatrix}$$

Let $$P = \frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{bmatrix} 1 & 5 & 7 \\ -1 & -2 & -4 \\ 8 & 2 & 13 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 8 \\ 5 & -2 & 2 \\ 7 & -4 & 13 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 2 & 4 & 15 \\ 4 & -4 & -2 \\ 15 & -2 & 26 \end{bmatrix}$$

$$Q=\frac{1}{2}\left(A-A^T\right)=\frac{1}{2}\left\{\begin{bmatrix}1 & 5 & 7\\ -1 & -2 & -4\\ 8 & 2 & 13\end{bmatrix}-\begin{bmatrix}1 & -1 & 8\\ 5 & -2 & 2\\ 7 & -4 & 13\end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix}0 & 6 & -1\\ -6 & 0 & -6\\ 1 & 6 & 0\end{bmatrix}$$

P is a symmetric and Q is a skew-symmetric matrix.

$$A=P+Q=\frac{1}{2}\begin{bmatrix}2 & 4 & 15\\ 4 & -4 & -2\\ 15 & -2 & 26\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & 6 & -1\\ -6 & 0 & -6\\ 1 & 6 & 0\end{bmatrix}$$

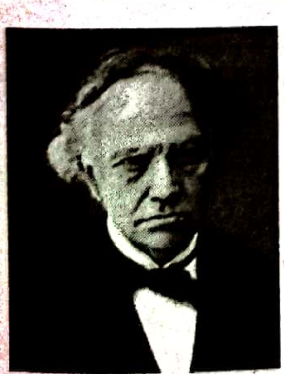
3. *Hermitian Matrix* A square matrix $A=[a_{ij}]$ is called Hermitian if $a_{ij}=\overline{a_{ji}}$ for all i and j, i.e., $A=A^\theta$. Hence, the diagonal elements of a Hermitian matrix are real, e.g.,

$$\begin{bmatrix}2 & 1+i\\ 1-i & 3\end{bmatrix},\ \begin{bmatrix}1 & 2+3i & 3-4i\\ 2-3i & 0 & 2-7i\\ 3+4i & 2+7i & 2\end{bmatrix}$$

4. *Skew-Hermitian Matrix* A square matrix $A=[a_{ij}]$ is called skew-Hermitian if $a_{ij}=-\overline{a_{ji}}$ for all i and j, i.e., $A=-A^\theta$. Hence, the diagonal elements of a skew-Hermitian matrix are either purely imaginary or zero, e.g.,

$$\begin{bmatrix}i & 2+3i\\ -2+3i & 0\end{bmatrix},\ \begin{bmatrix}i & 2 & 3+i\\ -2 & 3i & -1-i\\ -3+i & 1-i & 0\end{bmatrix}$$

HISTORICAL DATA



Charles Hermite (1822–1901) was a French mathematician who did research on number theory, quadratic forms, invariant theory, orthogonal polynomials, elliptic functions, and algebra.

Hermite polynomials, Hermite interpolation, Hermite normal form, Hermitian operators, Hermitian matrices, and cubic Hermite splines are named in his honour. He was the first to prove that e, the base of natural logarithms, is a transcendental number. His methods were later used by Ferdinand von Lindemann to prove that π is transcendental.

In a letter to Thomas Stieltjes in 1893, Hermite famously remarked: "I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives."

■ **Property 1** Every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Proof Let A be a square matrix.

$$A = \frac{1}{2}\left(A + A^\theta\right) + \frac{1}{2}\left(A - A^\theta\right) = P + Q$$

where $$P = \frac{1}{2}\left(A + A^\theta\right) \quad \text{and} \quad Q = \frac{1}{2}\left(A - A^\theta\right)$$

Now, $$P^\theta = \left[\frac{1}{2}\left(A + A^\theta\right)\right]^\theta = \frac{1}{2}\left[A^\theta + \left(A^\theta\right)^\theta\right] = \frac{1}{2}\left(A^\theta + A\right) = P$$

Hence, P is a Hermitian matrix.

Also, $$Q^\theta = \left[\frac{1}{2}\left(A - A^\theta\right)\right]^\theta = \frac{1}{2}\left[A^\theta - \left(A^\theta\right)^\theta\right] = \frac{1}{2}\left(A^\theta - A\right) = -Q$$

Hence, Q is a skew-Hermitian matrix

Thus, every square matrix can be expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

Uniqueness Let $A = R + S$, where R is a Hermitian and S is skew-Hermitian matrix.

$$A^\theta = (R+S)^\theta = R^\theta + S^\theta = R - S$$

Now, $$\frac{1}{2}\left(A + A^\theta\right) = \frac{1}{2}[(R+S) + (R-S)] = R = P$$

and $$\frac{1}{2}\left(A - A^\theta\right) = \frac{1}{2}[(R+S) - (R-S)] = S = Q$$

Hence, representation $A = P + Q$ is unique.

EXAMPLE 1.2 *Express the matrix* $A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ *as the sum of a Hermitian and a skew-Hermitian matrix.*

Solution: $$A = \begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}, \quad A^\theta = (\bar{A})^T = \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}$$

Let $$P = \frac{1}{2}\left(A + A^\theta\right) = \frac{1}{2}\left\{\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix} + \begin{bmatrix} 2-3i & 5 & 1+i \\ 0 & -i & -3-i \\ -4i & 8 & 6 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 4 & 5 & 1+5i \\ 5 & 0 & 5-i \\ 1-5i & 5+i & 12 \end{bmatrix}$$

$$Q=\frac{1}{2}\left(A-A^{\theta}\right)=\frac{1}{2}\left\{\begin{bmatrix}2+3i & 0 & 4i\\ 5 & i & 8\\ 1-i & -3+i & 6\end{bmatrix}-\begin{bmatrix}2-3i & 5 & 1+i\\ 0 & -i & -3-i\\ -4i & 8 & 6\end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix}6i & -5 & -1+3i\\ 5 & 2i & 11+i\\ 1+3i & -11+i & 0\end{bmatrix}$$

P is a Hermitian and Q is a skew-Hermitian matrix.

$$A=P+Q=\frac{1}{2}\begin{bmatrix}4 & 5 & 1+5i\\ 5 & 0 & 5-i\\ 1-5i & 5+i & 12\end{bmatrix}+\frac{1}{2}\begin{bmatrix}6i & -5 & -1+3i\\ 5 & 2i & 11+i\\ 1+3i & -11+i & 0\end{bmatrix}$$

■ **Property 2** Every square matrix can be uniquely expressed as $P + iQ$, where P and Q are Hermitian matrices.

Proof Let A be a square matrix.

$$A=\frac{1}{2}\left(A+A^{\theta}\right)+i\frac{1}{2i}\left(A-A^{\theta}\right)=P+iQ$$

where $$P=\frac{1}{2}\left(A+A^{\theta}\right) \quad\text{and}\quad Q=\frac{1}{2i}\left(A-A^{\theta}\right)$$

Now, $$P^{\theta}=\left[\frac{1}{2}\left(A+A^{\theta}\right)\right]^{\theta}=\frac{1}{2}\left[A^{\theta}+\left(A^{\theta}\right)^{\theta}\right]=\frac{1}{2}\left(A^{\theta}+A\right)=P$$

Hence, P is a Hermitian matrix.

Also, $$Q^{\theta}=\left[\frac{1}{2i}\left(A-A^{\theta}\right)\right]^{\theta}=-\frac{1}{2i}\left[A^{\theta}-\left(A^{\theta}\right)^{\theta}\right]=-\frac{1}{2i}\left(A^{\theta}-A\right)=\frac{1}{2i}\left(A-A^{\theta}\right)=Q$$

Hence, Q is a Hermitian matrix.

Thus, every square matrix can be expressed as $P+iQ$, where P and Q are Hermitian matrices.

Uniqueness Let $A=R+iS$, where R and S are Hermitian matrices.

$$A^{\theta}=(R+iS)^{\theta}=R^{\theta}+(iS)^{\theta}=R-iS$$

Now, $$\frac{1}{2}\left(A+A^{\theta}\right)=\frac{1}{2}[(R+iS)+(R-iS)]=R=P$$

and $$\frac{1}{2}\left(A-A^{\theta}\right)=\frac{1}{2}[(R+iS)-(R-iS)]=iS=iQ$$

Hence, representation $A=P+iQ$ is unique.

EXAMPLE 1.3 *Express the matrix* $A=\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ *as* $P+iQ$, *where* P *and* Q *are both Hermitian.*

Solution:

$$A=\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix},\quad A^\theta=\begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}$$

Let $P=\frac{1}{2}\left(A+A^\theta\right)=\frac{1}{2}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}+\begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix}$

$$Q=\frac{1}{2i}\left(A-A^\theta\right)=\frac{1}{2i}\left\{\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}-\begin{bmatrix} -2i & 0 & 3i \\ -3 & 2-3i & 3-2i \\ 1+i & 1-i & 2+5i \end{bmatrix}\right\}=\frac{1}{2i}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

P and Q are Hermitian matrices.

$$A=P+iQ=\frac{1}{2}\begin{bmatrix} 0 & -3 & 1+2i \\ -3 & 4 & 4-i \\ 1-2i & 4+i & 4 \end{bmatrix}+\frac{1}{2}\begin{bmatrix} 4i & -3 & 1-4i \\ 3 & 6i & -2+3i \\ -1-4i & 2+3i & -10i \end{bmatrix}$$

■ **Property 3** Every Hermitian matrix can be written as $P+iQ$, where P is a real symmetric and Q is a real skew-symmetric matrix.

Proof Let A be a Hermitian matrix.

$$A^\theta = A$$

$$A=\frac{1}{2}\left(A+\bar{A}\right)+i\frac{1}{2i}\left(A-\bar{A}\right)=P+iQ$$

where $P=\frac{1}{2}(A+\bar{A})$ and $Q=\frac{1}{2i}\left(A-\bar{A}\right)$ are real matrices.

Now,

$$P^T=\left[\frac{1}{2}\left(A+\bar{A}\right)\right]^T=\frac{1}{2}\left[A^\theta+\bar{A}\right]^T=\frac{1}{2}\left[\left(\bar{A}\right)^T+\bar{A}\right]^T=\frac{1}{2}\left[\left\{\left(\bar{A}\right)^T\right\}^T+\left(\bar{A}\right)^T\right]=\frac{1}{2}\left(\bar{A}+A^\theta\right)=\frac{1}{2}\left(\bar{A}+A\right)=P$$

Hence, P is a real symmetric matrix.

Also, $Q^T = \left[\frac{1}{2i}\left(A-\bar{A}\right)\right]^T = \frac{1}{2i}\left[A^\theta - \bar{A}\right]^T = \frac{1}{2i}\left[\left(\bar{A}\right)^T - \bar{A}\right]^T = \frac{1}{2i}\left[\left\{\left(\bar{A}\right)^T\right\}^T - \left(\bar{A}\right)^T\right] = \frac{1}{2i}\left(\bar{A} - A^\theta\right)$

$$= \frac{1}{2i}\left(\bar{A}-A\right) = -\frac{1}{2i}\left(A-\bar{A}\right) = -Q$$

Hence, Q is a real skew-symmetric matrix.

Thus, every Hermitian matrix can be written as $P + iQ$, where P is a real symmetric matrix and Q is a real skew-symmetric matrix.

EXAMPLE 1.4 *Express the Hermitian matrix* $A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$ *as* $P + iQ$, *where* P *is a real symmetric matrix and* Q *is a real skew-symmetric matrix.*

Solution:

$$A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}$$

Let $P = \frac{1}{2}\left(A+\bar{A}\right) = \frac{1}{2}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} + \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$

$$Q = \frac{1}{2i}\left(A-\bar{A}\right) = \frac{1}{2i}\left\{\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix} - \begin{bmatrix} 1 & i & 1-i \\ -i & 0 & 2+3i \\ 1+i & 2-3i & 2 \end{bmatrix}\right\}$$

$$= \frac{1}{2i}\begin{bmatrix} 0 & -2i & 2i \\ 2i & 0 & -6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

P is a real symmetric matrix and Q is a real skew-symmetric matrix.

$$A = P + iQ = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -i & i \\ i & 0 & -3i \\ -i & 3i & 0 \end{bmatrix}$$

■ **Property 4** Every skew-Hermitian matrix can be written as $P + iQ$, where P is a real skew-symmetric matrix and Q is a real symmetric matrix.

Proof Let A be a skew-Hermitian matrix.

$$A^\theta = -A$$

$$A = \frac{1}{2}(A+\bar{A}) + i\frac{1}{2i}(A-\bar{A}) = P + iQ$$

where $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$ are real matrices.

Now,
$$P^T = \left[\frac{1}{2}(A+\bar{A})\right]^T = \frac{1}{2}\left[-A^\theta + \bar{A}\right]^T = \frac{1}{2}\left[-(\bar{A})^T + \bar{A}\right]^T = \frac{1}{2}\left[-\left\{(\bar{A})^T\right\}^T + (\bar{A})^T\right]$$

$$= \frac{1}{2}(-\bar{A} + A^\theta) = \frac{1}{2}(-\bar{A} - A) = -\frac{1}{2}(A+\bar{A}) = -P$$

Hence, P is a real skew-symmetric matrix.

$$Q^T = \left[\frac{1}{2i}(A-\bar{A})\right]^T = \frac{1}{2i}\left[-A^\theta - \bar{A}\right]^T = \frac{1}{2i}\left[-(\bar{A})^T - \bar{A}\right]^T = \frac{1}{2i}\left[-\left\{(\bar{A})^T\right\}^T - (\bar{A})^T\right]$$

$$= \frac{1}{2i}(-\bar{A} - A^\theta) = \frac{1}{2i}(-\bar{A} + A) = \frac{1}{2i}(A-\bar{A}) = Q$$

Hence, Q is a real symmetric matrix.

Thus, every skew-Hermitian matrix can be written as $P + iQ$, where P is a real skew-symmetric matrix and Q is a real symmetric matrix.

EXAMPLE 1.5 *Express the skew-Hermitian matrix* $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ *as* $P + iQ$, *where P is a real skew-symmetric matrix and Q is a real symmetric matrix.*

Solution:

$$A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}$$

Let
$$P = \frac{1}{2}(A+\bar{A}) = \frac{1}{2}\left\{\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix} + \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$Q=\frac{1}{2i}\left(A-\bar{A}\right)=\frac{1}{2i}\left\{\begin{bmatrix}2i & 2+i & 1-i\\ -2+i & -i & 3i\\ -1-i & 3i & 0\end{bmatrix}-\begin{bmatrix}-2i & 2-i & 1+i\\ -2-i & i & -3i\\ -1+i & -3i & 0\end{bmatrix}\right\}$$

$$=\frac{1}{2i}\begin{bmatrix}4i & 2i & -2i\\ 2i & -2i & 6i\\ -2i & 6i & 0\end{bmatrix}=\begin{bmatrix}2 & 1 & -1\\ 1 & -1 & 3\\ -1 & 3 & 0\end{bmatrix}$$

P is a real skew-symmetric matrix and Q is a real symmetric matrix.

$$A=P+iQ=\begin{bmatrix}0 & 2 & 1\\ -2 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}+\begin{bmatrix}2i & i & -i\\ i & -i & 3i\\ -i & 3i & 0\end{bmatrix}$$

5. *Orthogonal Matrix* A square matrix A is called orthogonal if $AA^T = A^TA = I$.
Hence, $A^{-1} = A^T$ for an orthogonal matrix.

EXAMPLE 1.6 *Verify if the matrix* $A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & 1\end{bmatrix}$ *is orthogonal and, hence, find its inverse.*

Solution:

$$A=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix},\quad A^T=\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}$$

$$AA^T=\frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ -2 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}=\frac{1}{9}\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

Hence, A is an orthogonal matrix.

For an orthogonal matrix,

$$A^{-1}=A^T=\frac{1}{3}\begin{bmatrix}1 & 2 & -2\\ 2 & 1 & 2\\ 2 & -2 & -1\end{bmatrix}\qquad [\because AA^T = I]$$

EXAMPLE 1.7 *Find l, m, n, and A^{-1} if* $A=\begin{bmatrix}0 & 2m & n\\ l & m & -n\\ l & -m & n\end{bmatrix}$ *is orthogonal.*

Solution: Since the matrix A is orthogonal,

$$AA^T = I$$

$$\begin{bmatrix}0 & 2m & n\\ l & m & -n\\ l & -m & n\end{bmatrix}\begin{bmatrix}0 & l & l\\ 2m & m & -m\\ n & -n & n\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}4m^2+n^2 & 2m^2-n^2 & -2m^2+n^2\\ 2m^2-n^2 & l^2+m^2+n^2 & l^2-m^2-n^2\\ -2m^2+n^2 & l^2-m^2-n^2 & l^2+m^2+n^2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Equating corresponding elements,

$$4m^2+n^2=1 \qquad \text{... (1)}$$

$$2m^2-n^2=0 \qquad \text{... (2)}$$

$$l^2+m^2+n^2=1 \qquad \text{... (3)}$$

Solving Eqs (1), (2), and (3),

$$l^2=\frac{1}{2}, \quad l=\pm\frac{1}{\sqrt{2}}$$

$$m^2=\frac{1}{6}, \quad m=\pm\frac{1}{\sqrt{6}}$$

$$n^2=\frac{1}{3}, \quad n=\pm\frac{1}{\sqrt{3}}$$

For an orthogonal matrix,

$$A^{-1}=A^T=\begin{bmatrix}0 & \pm\frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}}\\ \pm\frac{2}{\sqrt{6}} & \pm\frac{1}{\sqrt{6}} & \mp\frac{1}{\sqrt{6}}\\ \pm\frac{1}{\sqrt{3}} & \mp\frac{1}{\sqrt{3}} & \pm\frac{1}{\sqrt{3}}\end{bmatrix}$$

6. *Unitary Matrix* A square matrix A is called unitary if $AA^\theta = A^\theta A = I$.
Hence, $A^{-1} = A^\theta$ for a unitary matrix.

EXAMPLE 1.8 *If* $A=\begin{bmatrix}\frac{1+i}{2} & \frac{-1+i}{2}\\ \frac{1+i}{2} & \frac{1-i}{2}\end{bmatrix}$, *prove that the matrix A is unitary and, hence, find A^{-1}.*

Solution: $A=\begin{bmatrix}\frac{1+i}{2} & \frac{-1+i}{2}\\ \frac{1+i}{2} & \frac{1-i}{2}\end{bmatrix}$, $A^T=\begin{bmatrix}\frac{1+i}{2} & \frac{1+i}{2}\\ \frac{-1+i}{2} & \frac{1-i}{2}\end{bmatrix}$, $A^\theta=\begin{bmatrix}\frac{1-i}{2} & \frac{1-i}{2}\\ \frac{-1-i}{2} & \frac{1+i}{2}\end{bmatrix}$

$$AA^\theta=\begin{bmatrix}\frac{1+i}{2} & \frac{-1+i}{2}\\ \frac{1+i}{2} & \frac{1-i}{2}\end{bmatrix}\begin{bmatrix}\frac{1-i}{2} & \frac{1-i}{2}\\ \frac{-1-i}{2} & \frac{1+i}{2}\end{bmatrix}=\frac{1}{4}\begin{bmatrix}1-i^2-i^2+1 & 1-i^2+i^2-1\\ 1-i^2-1+i^2 & 1-i^2+1-i^2\end{bmatrix}=\frac{1}{4}\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Hence, A is a unitary matrix.

For a unitary matrix,

$$A^{-1}=A^\theta=\begin{bmatrix}\frac{1-i}{2} & \frac{1-i}{2}\\ \frac{-1-i}{2} & \frac{1+i}{2}\end{bmatrix} \qquad [\because AA^\theta=I]$$

EXAMPLE 1.9 *If* $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$, *prove that the matrix A is unitary and, hence, find A^{-1}.*

Solution: $A=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$, $A^T=\frac{1}{2}\begin{bmatrix}\sqrt{2} & i\sqrt{2} & 0\\ -i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$, $A^\theta=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$

$$AA^\theta=\frac{1}{4}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}=\frac{1}{4}\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$$

Hence, A is a unitary matrix.

For a unitary matrix,

$$A^{-1}=A^\theta=\frac{1}{2}\begin{bmatrix}\sqrt{2} & -i\sqrt{2} & 0\\ i\sqrt{2} & -\sqrt{2} & 0\\ 0 & 0 & 2\end{bmatrix}$$

EXERCISE 1.1

1. Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

(i) $\begin{bmatrix} 0 & 5 & -3 \\ 1 & 1 & 1 \\ 4 & 5 & 9 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

2. Express the following matrices as the sum of a Hermitian matrix and a skew-Hermitian matrix:

(i) $\begin{bmatrix} 2 & 2+i & 3 \\ -2+i & 0 & 4 \\ -i & 3-i & 1-i \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$

3. Express the following matrices as $P + iQ$, where P and Q are both Hermitian:

(i) $\begin{bmatrix} 2 & 3-i & 1+2i \\ i & 0 & 1 \\ 1+2i & 1 & 3i \end{bmatrix}$

(ii) $\begin{bmatrix} 1+2i & 2 & 3-i \\ 2+3i & 2i & 1-2i \\ 1+i & 0 & 3+2i \end{bmatrix}$

4. Express the following Hermitian matrices as $P + iQ$, where P is a real symmetric matrix and Q is a real skew-symmetric matrix:

(i) $\begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$

5. Express the following skew-Hermitian matrices as $P + iQ$, where P is a real and skew-symmetric matrix and Q is a real and symmetric matrix:

(i) $\begin{bmatrix} 0 & 2-3i & 1+i \\ -2-3i & 2i & 2-i \\ -1+i & -2-i & -i \end{bmatrix}$

(ii) $\begin{bmatrix} i & 2i & -1+3i \\ 2i & 2i & 2-i \\ 1+3i & -2-i & 3i \end{bmatrix}$

6. Show that the following matrices are orthogonal and, hence, find their inverses:

(i) $\frac{1}{9}\begin{bmatrix} 8 & -4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

7. Find $l, m, n,$ and A^{-1} if

$$A = \begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$$ is orthogonal.

8. Find a, b, c if $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ is orthogonal.

[Ans. : $a = 1, b = -8, c = 4$]

9. If (a_r, b_r, c_r), where $r = 1, 2, 3$ be the direction cosines of the three mutually perpendicular lines referred to an orthogonal coordinate system then prove that $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is orthogonal.

10. Show that the following matrices are unitary:

(i) $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{i}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{i}{2} \end{bmatrix}$

1.5 ELEMENTARY TRANSFORMATIONS

Elementary transformation is any one of the following operations on a matrix:

(i) The interchange of any two rows (or columns)
(ii) The multiplication of the elements of any row (or column) by any nonzero number
(iii) The addition or subtraction of k times the elements of a row (or column) to the corresponding elements of another row (or column), where $k \neq 0$

Symbols to be used for elementary transformation:

(i) R_{ij} : Interchange of i^{th} and j^{th} rows
(ii) kR_i : Multiplication of i^{th} row by a nonzero number k
(iii) $R_i + kR_j$: Addition of k times the j^{th} row to the i^{th} row

The corresponding column transformations are denoted by C_{ij}, kC_i, and $C_i + kC_j$ respectively.

1.5.1 Elementary Matrices

A matrix obtained from a unit matrix by subjecting it to any row or column transformation is called an elementary matrix.

1.5.2 Equivalence of Matrices

If B is an $m \times n$ matrix obtained from an $m \times n$ matrix by elementary transformation of A then A is called the equivalent to B. Symbolically, it is written as $A \sim B$.

1.5.3 Echelon Form of a Matrix

A matrix A is said to be in echelon form if it satisfies the following properties:

(i) Every zero row of the matrix A occurs below a nonzero row.
(ii) The first nonzero number from the left of a nonzero row is 1. This is called a leading 1.
(iii) For each nonzero row, the leading 1 appears to the right and below any leading 1 in the preceding rows.

The following matrices are in echelon form:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 1.10 *Determine whether the following matrices are in echelon form or not:*

(i) $\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 7 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Solution:

(i) The given matrix is in echelon form since it satisfies properties (i), (ii), and (iii).
(ii) The given matrix is not in echelon form since it does not satisfy the property (iii).
(iii) The given matrix is not in echelon form since it does not satisfy the property (i).

EXAMPLE 1.11 *Find the echelon form of the matrix* $\begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$.

Solution: Let

$$A = \begin{bmatrix} 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\overset{R_{13}}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 2 & 3 & 4 & 5 \\ 0 & -1 & 2 & 3 \\ 3 & 2 & 4 & 1 \end{bmatrix} \overset{R_2-2R_1,\ R_4-3R_1}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -3 & 6 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & -7 & 7 & -5 \end{bmatrix} \overset{R_{23}}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix} \overset{(-1)R_2}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & 6 & 1 \\ 0 & -7 & 7 & -5 \end{bmatrix}$$

$$\overset{R_3+3R_2,\ R_4+7R_2}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & -8 \\ 0 & 0 & -7 & -26 \end{bmatrix} \overset{R_{34}}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -26 \\ 0 & 0 & 0 & -8 \end{bmatrix} \overset{\left(-\frac{1}{7}\right)R_3}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{26}{7} \\ 0 & 0 & 0 & -8 \end{bmatrix} \overset{\left(-\frac{1}{8}\right)R_4}{\sim} \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & \frac{26}{7} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.6 INVERSE OF A MATRIX

If A be any n-rowed square matrix then a matrix B, if it exists such that

$$AB = BA = I_n$$

is called the inverse of A,

i.e.,
$$B = A^{-1}$$

- **Property 1** Every invertible matrix possesses a unique inverse.

- **Property 2** The necessary and sufficient condition for a square matrix A to possess an inverse is that $|A| \neq 0$, i.e., A is nonsingular.

- **Property 3** If A, B are two n-rowed nonsingular matrices then AB is also nonsingular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

i.e., the inverse of a product is the product of the inverses taken in the reverse order.

- **Property 4** If A is an $n \times n$ nonsingular matrix then $(A^{-1})^T = (A^T)^{-1}$.

Inverse of a Matrix by Elementary Transformations

Let A be any nonsingular matrix.

$$A = IA$$

Applying suitable elementary row transformations to A on the LHS and to I on the RHS, A reduces to I and I reduces to any matrix B.

Hence, $I = BA$

$B = A^{-1}$

EXAMPLE 1.12 *Find the inverse of the matrix* $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ *by elementary transformations.*

Solution: Let

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$A = I_3 A$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Reducing the matrix A to echelon form,

R_{13}

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

$R_2 - 4R_1,\ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix} A$$

$\left(-\frac{1}{5}\right) R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & 0 & -2 \end{bmatrix} A$$

$R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ 1 & -\frac{1}{5} & -\frac{6}{5} \end{bmatrix} A$$

$(-1) R_3$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{4}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$R_2 - 3R_3,\ R_1 - 4R_3$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -\frac{4}{5} & -\frac{19}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} A$$

$$I_3 = A^{-1}A$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & \frac{4}{5} & \frac{9}{5} \\ 3 & -\frac{4}{5} & -\frac{14}{5} \\ -1 & \frac{1}{5} & \frac{6}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$$

EXAMPLE 1.13 *Find the inverse of the matrix* $\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$ *by elementary transformations.*

Solution: Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix}$$

$$A = I_4 A$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 2 & 1 & 2 & 1 \\ 3 & -2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

Reducing the matrix A to echelon form,

$R_3 - 2R_1,\ R_4 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 2 & -3 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 - 3R_2,\ R_4 - R_2$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$(-1)R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_2 + R_4,\ R_1 - 2R_4$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -3 & 0 & 0 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 2 & 0 & -2 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix} A$$

$$I_4 = A^{-1} A$$

$$\therefore A^{-1} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -5 & -3 & 1 & 1 \\ 2 & 3 & -1 & 0 \\ -3 & -1 & 0 & 1 \end{bmatrix}$$

EXERCISE 1.2

1. Using elementary row transformations, find the inverses of the following matrices:

(i) $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & -1 & 5 \\ 0 & 2 & 0 \\ -5 & 3 & -15 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 4 & 3 & 2 \\ 3 & 6 & 5 & 2 \\ 2 & 5 & 2 & -3 \\ 4 & 5 & 14 & 14 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$

Ans.:

(i) $\frac{1}{3}\begin{bmatrix} 1 & -2 & -1 \\ 1 & -5 & 2 \\ -3 & 12 & 0 \end{bmatrix}$ (ii) $\frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} -23 & 29 & -\frac{64}{5} & -\frac{18}{5} \\ 10 & -12 & \frac{26}{5} & \frac{7}{5} \\ 1 & -2 & \frac{6}{5} & \frac{2}{5} \\ 2 & -2 & \frac{3}{5} & \frac{1}{5} \end{bmatrix}$ (iv) $\begin{bmatrix} -3 & 3 & -3 & 2 \\ 3 & -4 & 4 & -2 \\ -3 & 4 & -5 & 3 \\ 2 & -2 & 3 & -2 \end{bmatrix}$

2. Find the matrix A if

$$A^{-1} = \begin{bmatrix} -1 & -3 & -3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Ans.: $\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$

1.7 RANK OF A MATRIX

The positive integer r is said to be the rank of a matrix A if it possesses the following properties:

(i) There is at least one minor of order r which is nonzero.

(ii) Every minor of order greater than r is zero.

Rank of the matrix A is denoted by $\rho(A)$.

Notes

(i) The rank of a matrix remains unchanged by elementary transformations.

(ii) The rank of the transpose of a matrix is same as that of the original matrix.

(iii) The rank of the product of two matrices cannot exceed the rank of either matrix.

$$\rho(AB) \le \rho(A) \quad \text{or} \quad \rho(AB) \le \rho(B)$$

(iv) The rank of a null matrix is zero.

(v) The rank of a nonsingular square matrix is always equal to its order.

1.7.1 Rank of a Matrix by Reducing to Echelon Form

The rank of a matrix in echelon form is equal to the number of nonzero rows of the matrix,

The matrix $A=\begin{bmatrix}1 & 3 & -1\\ 0 & 1 & 4\\ 0 & 0 & 0\end{bmatrix}$ is in echelon form and the number of nonzero rows is two. Hence, ran of the matrix is two, i.e.,

$$\rho(A)=2$$

EXAMPLE 1.14 *Find the rank of the matrix* $\begin{bmatrix}5 & 3 & 14 & 4\\ 0 & 1 & 2 & 1\\ 1 & -1 & 2 & 0\end{bmatrix}$ *by reducing it to echelon form*

Solution: Let

$$A=\begin{bmatrix}5 & 3 & 14 & 4\\ 0 & 1 & 2 & 1\\ 1 & -1 & 2 & 0\end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\overset{R_{13}}{\sim}\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 2 & 1\\ 5 & 3 & 14 & 4\end{bmatrix} \quad \overset{R_3-5R_1}{\sim}\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 2 & 1\\ 0 & 8 & 4 & 4\end{bmatrix} \quad \overset{R_3-8R_2}{\sim}\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 2 & 1\\ 0 & 0 & -12 & -4\end{bmatrix} \quad \overset{\left(-\frac{1}{12}\right)R_3}{\sim}\begin{bmatrix}1 & -1 & 2 & 0\\ 0 & 1 & 2 & 1\\ 0 & 0 & 1 & \frac{1}{3}\end{bmatrix}$$

The equivalent matrix is in echelon form.

$\rho(A)=$ number of nonzero rows $=3$

EXAMPLE 1.15 *Find the rank of the matrix* $\begin{bmatrix}1 & 2 & 3 & -1\\ -2 & -1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix}$ *by reducing it to echelon form.*

Solution: Let
$$A=\begin{bmatrix}1 & 2 & 3 & -1\\ -2 & -1 & -3 & -1\\ 1 & 0 & 1 & 1\\ 0 & 1 & 1 & -1\end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\underset{R_2+2R_1,\ R_3-R_1}{\sim}\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 3 & 3 & -3\\ 0 & -2 & -2 & 2\\ 0 & 1 & 1 & -1\end{bmatrix} \quad \underset{R_{24}}{\sim}\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 1 & 1 & -1\\ 0 & -2 & -2 & 2\\ 0 & 3 & 3 & -3\end{bmatrix} \quad \underset{R_3+2R_2,\ R_4-3R_2}{\sim}\begin{bmatrix}1 & 2 & 3 & -1\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The equivalent matrix is in echelon form.

$\rho(A)$ = number of nonzero rows = 2

1.7.2 Rank of a Matrix by Reducing to Normal Form

Any matrix of order $m\times n$ can be reduced to the form $\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}$ by elementary transformations, where r is the rank of the matrix. This form is known as normal form or first canonical form of a matrix.

$$[I_r],\qquad [I_r\ 0],\qquad \begin{bmatrix}I_r\\ 0\end{bmatrix},\qquad \begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}$$

- **Corollary**

(i) The rank of a matrix A of order $m\times n$ is r if and only if it can be reduced to the normal form $\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}$ by elementary transformations.

(ii) If A be an $m\times n$ matrix of rank r then there exist nonsingular matrices P and Q of orders m and n respectively such that
$$PAQ=\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix}$$

Let $A_{m\times n}=I_{m\times m}A_{m\times n}I_{n\times n}$.

The matrix on the LHS is reduced to normal form by applying elementary row or column transformation. If row transformation is applied on the LHS then this transformation is applied on pre-factor of A on the RHS. If column transformation is applied on the LHS then this transformation is applied on the post-factor of A on the RHS.

Note P and Q are not unique. A different sequence of transformations to obtain the normal form gives different P and Q.

EXAMPLE 1.16 *Find the rank of the matrix* $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$ *by reducing it to normal form.*

Solution: Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

$$\underset{R_2 - 3R_1,\, R_3 + R_1}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix} \underset{R_3 + R_2}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{C_2 - 2C_1,\, C_3 + C_1,\, C_4 - 3C_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\left(-\frac{1}{2}\right)C_2,\, \left(\frac{1}{3}\right)C_3,\, \left(-\frac{1}{10}\right)C_4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{C_3 - C_2,\, C_4 - C_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

EXAMPLE 1.17 *Find nonsingular matrices P and Q such that PAQ is in normal form. Hence, find the rank of A, where* $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$.

Solution:

$$A = I_3 A I_4$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2+2R_1,\ R_3-R_1$

$$\begin{bmatrix}1&2&1&0\\0&8&5&0\\0&-2&1&-8\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_2-2C_1,\ C_3-C_1$

$$\begin{bmatrix}1&0&0&0\\0&8&5&0\\0&-2&1&-8\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&-2&-1&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

C_{23}

$$\begin{bmatrix}1&0&0&0\\0&5&8&0\\0&1&-2&-8\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&-1&-2&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

R_{23}

$$\begin{bmatrix}1&0&0&0\\0&1&-2&-8\\0&5&8&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&0&1\\2&1&0\end{bmatrix}A\begin{bmatrix}1&-1&-2&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$R_3-5R_2,$

$$\begin{bmatrix}1&0&0&0\\0&1&-2&-8\\0&0&18&40\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&0&1\\7&1&-5\end{bmatrix}A\begin{bmatrix}1&-1&-2&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

$C_3+2C_2,\ C_4+8C_2$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&18&40\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&0&1\\7&1&-5\end{bmatrix}A\begin{bmatrix}1&-1&-4&-8\\0&0&1&0\\0&1&0&8\\0&0&0&1\end{bmatrix}$$

$\left(\frac{1}{18}\right)C_3,\ \left(\frac{1}{40}\right)C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -\frac{4}{18} & -\frac{8}{40} \\ 0 & 0 & \frac{1}{18} & 0 \\ 0 & 1 & \frac{2}{18} & \frac{8}{40} \\ 0 & 0 & 0 & \frac{1}{40} \end{bmatrix}$$

$C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -\frac{4}{18} & \frac{1}{45} \\ 0 & 0 & \frac{1}{18} & -\frac{1}{18} \\ 0 & 1 & \frac{2}{18} & \frac{4}{45} \\ 0 & 0 & 0 & \frac{1}{40} \end{bmatrix}$$

$$[I_3 \; 0] = PAQ$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 7 & 1 & -5 \end{bmatrix}, Q = \begin{bmatrix} 1 & -1 & -\frac{4}{18} & \frac{1}{45} \\ 0 & 0 & \frac{1}{18} & -\frac{1}{18} \\ 0 & 1 & \frac{2}{18} & \frac{4}{45} \\ 0 & 0 & 0 & \frac{1}{40} \end{bmatrix}$$

$$\rho(A) = 3$$

EXAMPLE 1.18 *Find nonsingular matrices P and Q such that PAQ is in normal form. Hence, find the rank of A, where* $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$.

Solution:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A = I_3\, A\, I_3$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_{13}

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1;\ C_3 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(-1)R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 + 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 7 & -6 & 4 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\left(\frac{1}{5}\right)R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ \frac{7}{5} & -\frac{6}{5} & \frac{4}{5} \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_3 = PAQ$$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ \frac{7}{5} & -\frac{6}{5} & \frac{4}{5} \end{bmatrix}, Q = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 3$$

EXERCISE 1.3

1. Find the ranks of the following matrices by reducing to echelon forms:

(i) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & 6 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & 7 \end{bmatrix}$

[**Ans.:** (i) 2 (ii) 4 (iii) 4 (iv) 2]

2. Find the ranks of the following matrices by reducing to normal forms:

(i) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & 8 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -2 \end{bmatrix}$

[**Ans.:** (i) 2 (ii) 3 (iii) 3 (iv) 4]

3. Find nonsingular matrices P and Q such that PAQ is in normal form. Also, find their ranks.

(i) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$

Ans.:
(i) rank = 3
(ii) rank = 2
(iii) rank = 3
(iv) rank = 3

1.8 SYSTEM OF NONHOMOGENEOUS LINEAR EQUATIONS

A system of m nonhomogeneous linear equations in n unknowns $x_1, x_2, \ldots x_n$ or simply a linear system, is a set of m linear equations each in n unknowns. A linear system is represented by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \quad \vdots \quad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the coefficient matrix of order $m \times n$,

$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector of order $n \times 1$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is any vector of order $m \times 1$.

The matrix
$$[A : B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$
is called the augmented matrix of the given system of linear equations.

Condition for Consistency

When the system of linear equations has one or more solutions, the system is said to be consist otherwise it is inconsistent.

The system of equations $A\mathbf{x} = B$ is consistent if and only if the coefficient matrix A and the augmen matrix $[A : B]$ are of the same rank,

i.e.,
$$\rho(A) = \rho[A : B]$$

There are two cases:

Case I If $\rho(A) = \rho[A : B] = n$, number of unknowns, the system has a unique solution.

Case II If $\rho(A) = \rho[A : B] < n$, number of unknowns, the system has infinite solutions.

In this case, $n - r$ unknowns, called parameters, can be assigned arbitrary values. The remain unknowns can be expressed in terms of these parameters.

When $\rho(A) \neq \rho[A : B]$, the system is said to be inconsistent and has no solution.

Note The linear system has a unique solution if $\det(A) \neq 0$

EXAMPLE 1.19 *Examine the consistency of the system and if consistent, solve the equatio*

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0$$

Solution: The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

The augmented matrix of the system is

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\underset{R_2 - 2R_1,\ R_3 - 3R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \underset{\left(\frac{1}{2}\right)R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{array}\right] \underset{R_3 - 3R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right] \underset{(-2)R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\rho[A] = \rho[A:B] = 3 \quad \text{(number of unknowns)}$$

Hence, the system is consistent and has a unique solution.

The corresponding system of equations is

$$x + y + 2z = 9$$
$$y - \frac{7}{2}z = -\frac{17}{2}$$
$$z = 3$$

Solving these equations,

$$x = 1, y = 2$$

Hence, $x = 1, y = 2, z = 3$ is the solution of the system.

EXAMPLE 1.20 *Examine the consistency of the following system of equations and if consistent, solve the equations:*

$$2x - y - z = 2$$
$$x + 2y + z = 2$$
$$4x - 7y - 5z = 2$$

Solution: The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\underset{R_{12}}{\sim} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{array}\right] \quad \underset{R_2 - 2R_1,\, R_3 - 4R_1}{\sim} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{array}\right] \quad \underset{\left(-\frac{1}{5}\right)R_2}{\sim} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & -15 & -9 & -6 \end{array}\right] \quad \underset{R_3 + 15R_2}{\sim} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 2 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho(A) = \rho[A:B] = 2 < 3 \quad \text{(number of unknowns)}$$

Hence, the system is consistent and has infinite solutions.

Number of parameters = 3 − 2 = 1

The corresponding system of equations is

$$x + 2y + z = 2$$
$$y + \frac{3}{5}z = \frac{2}{5}$$

Assigning the variable z an arbitrary value t,

$$y = \frac{2}{5} - \frac{3}{5}t = \frac{2-3t}{5}$$

$$x = 2 - t - 2\left(\frac{2-3t}{5}\right) = \frac{6+t}{5}$$

Hence, $x = \frac{6+t}{5}$, $y = \frac{2-3t}{5}$, $z = t$ is the solution of the system, where t is a parameter.

EXAMPLE 1.21 *Solve the following system of equations:*

$$2x - y + z = 9$$
$$3x - y + z = 6$$
$$4x - y + 2z = 7$$
$$-x + y - z = 4$$

Solution: The matrix form of the system is

$$A\mathbf{x} = B$$

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -1 & 1 & 9 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ -1 & 1 & -1 & 4 \end{array}\right]$$

Reducing the augmented matrix to echelon form.

$$\overset{R_{14}}{\sim}\left[\begin{array}{ccc:c} -1 & 1 & -1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right] \quad \overset{(-1)R_1}{\sim}\left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \\ 2 & -1 & 1 & 9 \end{array}\right] \quad \overset{R_2-3R_1,\ R_3-4R_1,\ R_4-2R_1}{\sim}\left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 0 & 2 & -2 & 18 \\ 0 & 3 & -2 & 23 \\ 0 & 1 & -1 & 17 \end{array}\right]$$

$$\overset{R_{24}}{\sim}\left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 3 & -2 & 23 \\ 0 & 2 & -2 & 18 \end{array}\right] \quad \overset{R_3-3R_2,\ R_4-2R_2}{\sim}\left[\begin{array}{ccc:c} 1 & -1 & 1 & -4 \\ 0 & 1 & -1 & 17 \\ 0 & 0 & 1 & -28 \\ 0 & 0 & 0 & -16 \end{array}\right]$$

$$\rho(A)=3$$

$$\rho[A:B]=4$$

$$\rho(A)\neq\rho[A:B]$$

Hence, the system is inconsistent and has no solution.

EXAMPLE 1.22 *Solve the following system for x, y, and z:*

$$-\frac{1}{x}+\frac{3}{y}+\frac{4}{z}=30$$

$$\frac{3}{x}+\frac{2}{y}-\frac{1}{z}=9$$

$$\frac{2}{x}-\frac{1}{y}+\frac{2}{z}=10$$

Solution: The matrix form of the system is

$$A\mathbf{x}=B$$

$$\begin{bmatrix} -1 & 3 & 4 \\ 3 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix}=\begin{bmatrix} 30 \\ 9 \\ 10 \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\overset{(-1)R_1}{\sim}\left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right] \quad \overset{R_2-3R_1,\ R_3-2R_1}{\sim}\left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right] \quad \overset{\left(\frac{1}{11}\right)R_2,\ \left(\frac{1}{5}\right)R_3}{\sim}\left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 2 & 14 \end{array}\right] \quad \overset{R_3-R_2}{\sim}\left[\begin{array}{ccc:c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\rho(A) = \rho[A:B] = 3 \quad \text{(number of unknowns)}$$

Hence, the system is consistent and has a unique solution.
The corresponding system of equations is

$$\frac{1}{x}-\frac{3}{y}-\frac{4}{z}=-30$$

$$\frac{1}{y}+\frac{1}{z}=9$$

$$\frac{1}{z}=5$$

Solving these equations,

$$x=\frac{1}{2},\ y=\frac{1}{4},\ z=\frac{1}{5}$$

Hence, $x=\frac{1}{2}, y=\frac{1}{4}, z=\frac{1}{5}$ is the solution of the system.

EXAMPLE 1.23 *Investigate for what values of λ and μ the equations*

$$x+2y+z=8$$

$$2x+2y+2z=13$$

$$3x+4y+\lambda z=\mu$$

have (i) no solution, (ii) unique solution, and (iii) many solutions.

Solution: The matrix form of the system is

$$Ax = B$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 8 \\ 13 \\ \mu \end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c}1 & 2 & 1 & 8\\ 2 & 2 & 2 & 13\\ 3 & 4 & \lambda & \mu\end{array}\right]$$

Reducing the augmented matrix to echelon form,

$$\underset{R_2-2R_1,\ R_3-3R_1}{\sim}\left[\begin{array}{ccc:c}1 & 2 & 1 & 8\\ 0 & -2 & 0 & -3\\ 0 & -2 & \lambda-3 & \mu-24\end{array}\right]\ \underset{\left(-\frac{1}{2}\right)R_2}{\sim}\left[\begin{array}{ccc:c}1 & 2 & 1 & 8\\ 0 & 1 & 0 & \frac{3}{2}\\ 0 & -2 & \lambda-3 & \mu-24\end{array}\right]\ \underset{R_3+2R_2}{\sim}\left[\begin{array}{ccc:c}1 & 2 & 1 & 8\\ 0 & 1 & 0 & \frac{3}{2}\\ 0 & 0 & \lambda-3 & \mu-21\end{array}\right]$$

(i) If $\lambda=3$ and $\mu\neq 21$, $\rho[A]\neq\rho[A:B]$, the system is inconsistent and has no solution.

(ii) If $\lambda\neq 3$ and μ has any value, $\rho[A]=\rho[A:B]=3$ (number of unknowns), the system is consistent and has a unique solution.

(iii) If $\lambda=3$ and $\mu=21$, $\rho[A]=\rho[A:B]=2<3$ (number of unknowns), the system is consistent and has infinite (many) solutions.

EXAMPLE 1.24 *Solve the following system of nonlinear equations for the unknown angles α, β, and γ, where $0\le\alpha\le 2\pi$, $0\le\beta\le 2\pi$, and $0\le\gamma<\pi$.*

$$2\sin\alpha-\cos\beta+3\tan\gamma=3$$
$$4\sin\alpha+2\cos\beta-2\tan\gamma=2$$
$$6\sin\alpha-3\cos\beta+\tan\gamma=9$$

Solution: The matrix form of the system is

$$A\mathbf{x}=B$$

$$\begin{bmatrix}2 & -1 & 3\\ 4 & 2 & -2\\ 6 & -3 & 1\end{bmatrix}\begin{bmatrix}\sin\alpha\\ \cos\beta\\ \tan\gamma\end{bmatrix}=\begin{bmatrix}3\\ 2\\ 9\end{bmatrix}$$

The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc:c}2 & -1 & 3 & 3\\ 4 & 2 & -2 & 2\\ 6 & -3 & 1 & 9\end{array}\right]$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}$$

Writing these equations in matrix form,

$$A\mathbf{x} = \mathbf{0}$$

where A is any matrix of order $m \times n$, $\mathbf{x}$ is a vector of order $n \times 1$, and $\mathbf{0}$ is a null vector of order $m \times 1$. The matrix A is called the coefficient matrix of the system of equations.

Solutions of a System of Linear Equations

The number of linearly independent solutions of the equation $A\mathbf{x} = \mathbf{0}$ is $n - r$, where n is the number of unknowns and r is the rank of the coefficient matrix A.

There are two cases:

Case I If the rank of the matrix A is equal to the number of unknowns, i.e., $r = n$ then the equation $A\mathbf{x} = \mathbf{0}$ will have nonlinearly independent solutions. In this case, $\mathbf{x} = \mathbf{0}$, i.e., $x_1 = x_2 = \ldots = x_n = 0$ is the only solution and is known as the *trivial solution*.

Case II If the rank of the matrix A is less than the number of unknowns, i.e., $r < n$ then the equation $A\mathbf{x} = \mathbf{0}$ will have $n - r$ linearly independent solutions, known as *nontrivial solutions*. The equation $A\mathbf{x} = \mathbf{0}$ will have infinite solutions. In this case, $n - r$ unknowns, called parameters, can be assigned arbitrary values. The remaining unknowns can be expressed in terms of these parameters.

Note The system of equations has a nontrivial solution if $\det(A) = 0$.

EXAMPLE 1.25 *Solve the following system of equations:*

$$\begin{aligned}
2x_1 + x_2 + 3x_3 &= 0 \\
x_1 + 2x_2 &= 0 \\
x_2 + x_3 &= 0
\end{aligned}$$

Solution: The matrix form of the system is

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\underset{R_{12}}{\sim}\begin{bmatrix}1 & 2 & 0\\2 & 1 & 3\\0 & 1 & 1\end{bmatrix} \underset{R_2-2R_1}{\sim}\begin{bmatrix}1 & 2 & 0\\0 & -3 & 3\\0 & 1 & 1\end{bmatrix} \underset{\left(-\frac{1}{3}\right)R_2}{\sim}\begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\\0 & 1 & 1\end{bmatrix} \underset{R_3-R_2}{\sim}\begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\\0 & 0 & 2\end{bmatrix} \underset{\left(\frac{1}{2}\right)R_3}{\sim}\begin{bmatrix}1 & 2 & 0\\0 & 1 & -1\\0 & 0 & 1\end{bmatrix}$$

$$\rho(A) = 3 \quad \text{(number of unknowns)}$$

Hence, the system has a trivial solution, i.e., $x = 0, y = 0, z = 0$.

EXAMPLE 1.26 *Solve the following system of equations:*

$$3x - y - z = 0$$
$$x + y + 2z = 0$$
$$5x + y + 3z = 0$$

Solution: The matrix form of the system is

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix}3 & -1 & -1\\1 & 1 & 2\\5 & 1 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$A = \begin{bmatrix}3 & -1 & -1\\1 & 1 & 2\\5 & 1 & 3\end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\underset{R_{12}}{\sim}\begin{bmatrix}1 & 1 & 2\\3 & -1 & -1\\5 & 1 & 3\end{bmatrix} \underset{R_2-3R_1,\ R_3-5R_1}{\sim}\begin{bmatrix}1 & 1 & 2\\0 & -4 & -7\\0 & -4 & -7\end{bmatrix} \underset{\left(-\frac{1}{4}\right)R_2,\left(-\frac{1}{4}\right)R_3}{\sim}\begin{bmatrix}1 & 1 & 2\\0 & 1 & \frac{7}{4}\\0 & 1 & \frac{7}{4}\end{bmatrix} \underset{R_3-R_2}{\sim}\begin{bmatrix}1 & 1 & 2\\0 & 1 & \frac{7}{4}\\0 & 0 & 0\end{bmatrix}$$

$$\rho(A) = 2 < 3 \quad \text{(number of unknowns)}$$

Hence, the system has a nontrivial solution.

The number of parameters = 3 − 2 = 1

The corresponding system of equations is

$$x+y+2z=0$$
$$y+\frac{7}{4}z=0$$

Assigning the variable z an arbitrary value t,

$$y=-\frac{7}{4}t$$
$$x=\frac{7}{4}t-2t=-\frac{1}{4}t$$

Hence, $x=-\frac{1}{4}t,\ y=-\frac{7}{4}t$ is the nontrivial solution of the system, where t is a parameter.

EXAMPLE 1.27 *Solve the following system of equations:*

$$x+y-z+w=0$$
$$x-y+2z-w=0$$
$$3x+y+w=0$$

Solution: The matrix form of the system is

$$A\mathbf{x}=\mathbf{0}$$

$$\begin{bmatrix}1 & 1 & -1 & 1\\ 1 & -1 & 2 & -1\\ 3 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 1 & -1 & 1\\ 1 & -1 & 2 & -1\\ 3 & 1 & 0 & 1\end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\xrightarrow{R_2-R_1,\ R_3-3R_1}\ \sim\begin{bmatrix}1 & 1 & -1 & 1\\ 0 & -2 & 3 & -2\\ 0 & -2 & 3 & -2\end{bmatrix}\quad \xrightarrow{\left(-\frac{1}{2}\right)R_2,\ \left(-\frac{1}{2}\right)R_3}\ \sim\begin{bmatrix}1 & 1 & -1 & 1\\ 0 & 1 & -\frac{3}{2} & 1\\ 0 & 1 & -\frac{3}{2} & 1\end{bmatrix}\quad \xrightarrow{R_3-R_2}\ \sim\begin{bmatrix}1 & 1 & -1 & 1\\ 0 & 1 & -\frac{3}{2} & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\rho(A) = 2 < 4 \quad \text{(number of unknowns)}$$

Hence, the system has a nontrivial solution.

Number of parameters $= 4 - 2 = 2$

The corresponding system of equations is

$$x + y - z + w = 0$$

$$y - \frac{3}{2}z + w = 0$$

Assigning the variables z and w the arbitrary values t_1 and t_2 respectively,

$$y = \frac{3}{2}t_1 - t_2$$

$$x = -\frac{3}{2}t_1 + t_2 + t_1 - t_2 = -\frac{1}{2}t_1$$

Hence, $x = -\frac{1}{2}t_1$, $y = \frac{3}{2}t_1 - t_2$, $z = t_1$, $w = t_2$ is the nontrivial solution of the system, where t_1 and t_2 are parameters.

EXAMPLE 1.28 *If the following system has a nontrivial solution then prove that $a + b + c = 0$ or $a = b = c$ and, hence, find the solution in each case.*

$$ax + by + cz = 0$$

$$bx + cy + az = 0$$

$$cx + ay + bz = 0$$

Solution: The matrix form of the system is

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has a nontrivial solution if $\det(A) = 0$,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) = 0$$

$$-(a^3 + b^3 + c^3 - 3abc) = 0$$

$$-(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$a+b+c=0$$

or

$$a^2+b^2+c^2-ab-bc-ca=0$$

$$\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]=0$$

$$a-b=0,\ b-c=0,\ c-a=0$$

$$a=b,\ b=c,\ c=a$$

$$a=b=c$$

Hence, the system has a nontrivial solution if $a+b+c=0$ or $a=b=c$.

$$A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

$$R_3+R_1+R_2$$

$$\sim\begin{bmatrix} a & b & c \\ b & c & a \\ a+b+c & a+b+c & a+b+c \end{bmatrix}$$

The corresponding system of equations is

$$ax+by+cz=0$$

$$bx+cy+az=0$$

$$(a+b+c)\,x+(a+b+c)y+(a+b+c)z=0$$

(i) When $a+b+c=0$, the system of equations is

$$ax+by+cz=0$$

$$bx+cy+az=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} b & c \\ c & a \end{vmatrix}}=-\frac{y}{\begin{vmatrix} a & c \\ b & a \end{vmatrix}}=\frac{z}{\begin{vmatrix} a & b \\ b & c \end{vmatrix}}=t,\text{say}$$

$$\frac{x}{ab-c^2}=-\frac{y}{a^2-bc}=\frac{z}{ac-b^2}=t$$

Hence, $x=(ab-c^2)t$, $y=(bc-a^2)t$, $z=(ac-b^2)t$ is the solution of the system, where t is a parameter.

(ii) When $a = b = c$, the system of equations is

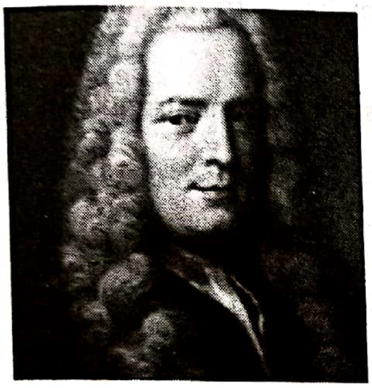
$$x + y + z = 0$$

Let $$y = t_1,\ z = t_2$$

Then, $$x = -t_1 - t_2$$

Hence, $x = -t_1 - t_2$, $y = t_1$, $z = t_2$ is the solution of the system, where t_1 and t_2 are parameters.

HISTORICAL DATA



Gabriel Cramer (1704–1752) was a Swiss mathematician, born in Geneva. Cramer showed promise in mathematics from an early age. At 18, he received his doctorate and at 20, he was co-chair of mathematics. In 1728, he proposed a solution to the St. Petersburg Paradox that came very close to the concept of expected utility theory given ten years later by Daniel Bernoulli.

He published his best-known work in his forties. This was his treatise on algebraic curves (1750). It contains the earliest demonstration that a curve of the n-th degree is determined by $n(n + 3)/2$ points on it, in general position. He edited the works of the two elder Bernoullis; and wrote on the physical cause of the spheroidal shape of the planets and the motion of their apsides (1730), and on Newton's treatment of cubic curves (1746). He did extensive travel throughout Europe in the late 1730s, which greatly influenced his works in mathematics.

EXAMPLE 1.29 *Discuss the solution of the system of equations*

$$2x + 3ky + (3k + 4)z = 0$$
$$x + (k + 4)y + (4k + 2)z = 0$$
$$x + 2(k + 1)y + (3k + 4)z = 0$$

for all values of k.

Solution: The matrix form of the system is

$$A\mathbf{x} = 0$$

$$\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_{12}$$

$$\sim\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2-2R_1,\ R_3-R_1$$

$$\sim\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i) When $k \neq \pm 2$, $\rho[A]=3$ (number of unknowns), the system has a trivial solution, i.e., $x=0, y=0, z=0$.

(ii) When $k=\pm 2$, $\rho[A]=2<3$ (number of unknowns), the system has a nontrivial solution.

Case I When $k=2$,

$$A=\begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\left(-\frac{1}{6}\right)R_2$$

$$\sim\begin{bmatrix} 1 & 6 & 10 \\ 0 & 1 & \dfrac{10}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A)=2<3 \quad \text{(number of unknowns)}$$

Hence, the system has a nontrivial solution.

Number of parameters $=3-2=1$

The corresponding system of equations is

$$x=0$$

$$y+\frac{10}{6}z=0$$

Assigning the variable z any arbitrary value t,

$$y=-\frac{10}{6}t=-\frac{5}{3}t$$

Hence, $x=0$, $y=-\dfrac{5}{3}t$, $z=t$ is the solution of the system, where t is a parameter.

Case II When $k=-2$,

$$A=\begin{bmatrix}1 & 2 & -6\\ 0 & -10 & 10\\ 0 & -4 & 4\end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\left(-\frac{1}{10}\right)R_2,\left(-\frac{1}{4}\right)R_3 \qquad\qquad R_3-R_2$$

$$\sim\begin{bmatrix}1 & 2 & -6\\ 0 & 1 & -1\\ 0 & 1 & -1\end{bmatrix} \qquad \sim\begin{bmatrix}1 & 2 & -6\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

$$\rho(A)=2<3 \quad \text{(number of unknowns)}$$

Hence, the system has a nontrivial solution.

Number of parameters $=3-2=1$

The corresponding system of equations is

$$x-4z=0$$
$$y-z=0$$

Assigning the variable z any arbitrary value t,

$$x=4t$$
$$y=t$$

Hence, $x=4t,\ y=t,\ z=t$ is the solution of the system, where t is a parameter.

EXERCISE 1.5

1. Solve the following equations:

(i)
$$x-y+z=0$$
$$x+2y+z=0$$
$$2x+y+3z=0$$

(ii)
$$x-2y+3z=0$$
$$2x+5y+6z=0$$

(iii)
$$2x-2y+5z+3w=0$$
$$4x-y+z+w=0$$
$$3x-2y+3z+4w=0$$
$$x-3y+7z+6w=0$$

(iv)
$$3x+4y-z-9w=0$$
$$2x+3y+2z-3w=0$$
$$2x+y-14z-12w=0$$
$$x+3y+13z+3w=0$$

[**Ans.:** (i) $x = 0, y = 0, z = 0$
(ii) $x = -3t, y = 0, z = t$
(iii) $x = \frac{211}{9}t, y = 4t, z = \frac{7}{9}t, w = t$
(iv) $x = 11t, y = -8t, z = t, w = 0$]

2. For what value of λ does the following systems of equations possess a nontrivial solution? Obtain the solution for real values of λ.

(i)
$$3x + y - \lambda z = 0$$
$$4x - 2y - 3z = 0$$
$$2\lambda x + 4y - \lambda z = 0$$

(ii)
$$(1-\lambda)x_1 + 2x_2 + 3x_3 = 0$$
$$3x_1 + (1-\lambda)x_2 + 2x_3 = 0$$
$$2x_1 + 3x_2 + (1-\lambda)x_3 = 0$$

[**Ans.:** (i) Nontrivial solution $\lambda = 1, -9$
For $\lambda = 1, x = -t, y = -t, z = -2t$
For $\lambda = -9, x = -3t, y = -9t, z = 2t$
(ii) $\lambda = 6, x = y = z = t$]

3. Show that the system of equations $2x - 2y + z = \lambda x$, $2x - 3y + 2z = \lambda y$, $-x + 2y = \lambda z$ can posses a nontrivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case.

[**Ans.:** For $\lambda = 1, x = 2t_2 - t_1, y = t_2, z = t_1$
For $\lambda = -3, x = -t, y = -2t, z = t$]

1.10 LINEAR DEPENDENCE AND INDEPENDENCE OF VECTORS

1.10.1 Linear Dependence of Vectors

A set of r vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2 \ldots k_r$ not all zero, such that

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r = 0$$

1.10.2 Linear Independence of Vectors

A set of r vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_r$ is said to be linearly independent if there exist r scalars (numbers) $k_1, k_2 \ldots k_r$ such that if

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 \cdots + k_r\mathbf{x}_r = 0$$

then
$$k_1 = k_2 = \cdots = k_r = 0$$

1.10.3 Linear Combination of Vectors

A vector $\mathbf{x}$ which can be expressed in the form

$$\mathbf{x} = k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r$$

is said to be a linear combination of the vectors $\mathbf{x}_1, \mathbf{x}_2 \ldots \mathbf{x}_r$ where $k_1, k_2 \ldots k_r$ are any numbers.

Notes

(i) If a set of vectors is linearly dependent then at least one vector of the set can be expressed as a linear combination of the remaining vectors.

(ii) If a set of vectors is linearly independent then no vector of the set can be expressed as a linear combination of the remaining vectors.

EXAMPLE 1.30 *Examine for linear dependence or independence of vectors:*

$$\mathbf{x}_1 = (1, 1, -1), \mathbf{x}_2 = (2, 3, -5), \mathbf{x}_3 = (2, -1, 4)$$

If dependent, find the relation between them.

Solution: Let

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + k_3\mathbf{x}_3 = \mathbf{0}$$

$$k_1(1, 1, -1) + k_2(2, 3, -5) + k_3(2, -1, 4) = \mathbf{0}$$

$$(k_1 + 2k_2 + 2k_3, k_1 + 3k_2 - k_3, -k_1 - 5k_2 + 4k_3) = (0, 0, 0)$$

Equating corresponding components,

$$k_1 + 2k_2 + 2k_3 = 0$$
$$k_1 + 3k_2 - k_3 = 0$$
$$-k_1 - 5k_2 + 4k_3 = 0$$

The matrix form of the system is

$$A\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & -5 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -1 \\ -1 & -5 & 4 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\underset{R_2 - R_1,\ R_3 + R_1}{\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & -3 & 6 \end{bmatrix}} \qquad \underset{R_3 + 3R_2}{\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix}} \qquad \underset{\left(-\frac{1}{3}\right)R_3}{\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}}$$

Number of nonzero rows = 3

$$\rho(A) = 3 \quad \text{(number of unknowns)}$$

Hence, the system has a trivial solution.

$$k_1 = 0, k_2 = 0, k_3 = 0$$

Since k_1, k_2, k_3 are all zero, the vectors are linearly independent.

EXAMPLE 1.31 *Find the rank of* $A=\begin{bmatrix}1 & 2 & 3 & -2\\ 2 & -2 & 1 & 3\\ 3 & 0 & 4 & 1\end{bmatrix}$ *and the relation of linear dependence among row vectors.*

Solution:

$$A=\begin{bmatrix}1 & 2 & 3 & -2\\ 2 & -2 & 1 & 3\\ 3 & 0 & 4 & 1\end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\underset{R_2-2R_1,\ R_3-3R_1}{\sim}\begin{bmatrix}1 & 2 & 3 & -2\\ 0 & -6 & -5 & 7\\ 0 & -6 & -5 & 7\end{bmatrix}\quad \underset{\left(-\frac{1}{6}\right)R_2}{\sim}\begin{bmatrix}1 & 2 & 3 & -2\\ 0 & 1 & \frac{5}{6} & \frac{7}{6}\\ 0 & -6 & -5 & 7\end{bmatrix}\quad \underset{R_3+6R_2}{\sim}\begin{bmatrix}1 & 2 & 3 & -2\\ 0 & 1 & \frac{5}{6} & \frac{7}{6}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Number of nonzero rows = 2

$$\rho(A)=2$$

Relation of linear dependence among row vectors

$$\mathbf{x}_1=(1, 2, 3, -2),\ \mathbf{x}_2=(2, -2, 1, 3),\ \mathbf{x}_3=(3, 0, 4, 1)$$

Let $\quad k_1\mathbf{x}_1+k_2\mathbf{x}_2+k_3\mathbf{x}_3=\mathbf{0}$

$$k_1(1, 2, 3, -2)+k_2(2, -2, 1, 3)+k_3(3, 0, 4, 1)=\mathbf{0}$$

$$(k_1+2k_2+3k_3,\ 2k_1-2k_2,\ 3k_1+k_2+4k_3,\ -2k_1+3k_2+k_3)=(0, 0, 0)$$

Equating corresponding components,

$$\begin{aligned}k_1+2k_2+3k_3&=0\\ 2k_1-2k_2&=0\\ 3k_1+k_2+4k_3&=0\\ -2k_1+3k_2+k_3&=0\end{aligned}$$

The matrix form of the system is

$$A\mathbf{x}=\mathbf{0}$$

$$\begin{bmatrix}1 & 2 & 3\\ 2 & -2 & 0\\ 3 & 1 & 4\\ -2 & 3 & 1\end{bmatrix}\begin{bmatrix}k_1\\ k_2\\ k_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 0 \\ 3 & 1 & 4 \\ -2 & 3 & 1 \end{bmatrix}$$

Reducing the matrix A to echelon form,

$$\underset{R_2-2R_1,\, R_3-3R_1,\, R_4+2R_1}{\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -6 \\ 0 & -5 & -5 \\ 0 & 7 & 7 \end{bmatrix}} \qquad \underset{\left(-\frac{1}{6}\right)R_2}{\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \\ 0 & 7 & 7 \end{bmatrix}} \qquad \underset{R_3+5R_2,\, R_4-7R_2}{\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

Number of nonzero rows = 2

$$\rho(A) = 2 < 3 \quad \text{(number of unknowns)}$$

Hence, the system has a nontrivial solution.

Number of parameters = 3 − 2 = 1

The corresponding system of equations is

$$k_1 + 2k_2 + 3k_3 = 0$$
$$k_2 + k_3 = 0$$

Assigning the variable k_3 any arbitrary value t,

$$k_2 = -t$$
$$k_1 = -2k_2 - 3k_3 = -2(-t) - 3t = -t$$

Since k_1, k_2, k_3, are not all zero, the vectors are linearly dependent.

Substituting in Eq. (1),

$$-t\mathbf{x}_1 - t\mathbf{x}_2 + t\mathbf{x}_3 = \mathbf{0}$$
$$-\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$$
$$\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_3$$

EXERCISE 1.6

1. Examine whether the following vectors are linearly independent or dependent. If dependent, find the relation between them.

(i) $\mathbf{x}_1 = (3, 1, -4)$, $\mathbf{x}_2 = (2, 2, -3)$, $\mathbf{x}_3 = (0, -4, 1)$

[**Ans. :** Dependent, $2\mathbf{x}_1 = 3\mathbf{x}_2 + \mathbf{x}_3$]

(ii) $\mathbf{x}_1 = (1, 1, 1, 3)$, $\mathbf{x}_2 = (1, 2, 3, 4)$, $\mathbf{x}_3 = (2, 3, 4, 7)$

[**Ans.** : Dependent, $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_3$]

(iii) $\mathbf{x}_1 = (1, -1, 0), \mathbf{x}_2 = (2, 1, 3), \mathbf{x}_3 = (0, 1, 1),$
$\mathbf{x}_4 = (2, 2, 1)$

[**Ans.** : Dependent, $\mathbf{x}_2 = 2\mathbf{x}_1 + 3\mathbf{x}_3$]

(iv) $\mathbf{x}_1 = (1, -1, 2, 2)^T, \mathbf{x}_2 = (2, -3, 4, -1)^T,$
$\mathbf{x}_3 = (-1, 2, -2, 3)^T$

[**Ans.** : Dependent, $\mathbf{x}_1 = \mathbf{x}_2 + \mathbf{x}_3$]

(v) $\mathbf{x}_1 = (2, 2, 1)^T, \mathbf{x}_2 = (1, 3, 1)^T, \mathbf{x}_3 = (1, 2, 2)^T$

[**Ans.** : Independent]

(vi) $\mathbf{x}_1 = (1, 2, -1, 0), \mathbf{x}_2 = (1, 3, 1, 2),$
$\mathbf{x}_3 = (4, 2, 1, 0), \mathbf{x}_4 = (6, 1, 0, 1)$

[**Ans.** : Independent]

1.11 LINEAR TRANSFORMATION

Often it is necessary to transform data from one measurement scale to another, e.g., the conversion of temperature from degree centigrade to degree Farenheit is given by °F = 1.8°C + 32. This is a linear transformation. Hence, linear transformation is a function that converts one type of data into another type of data.

Consider a linear transformation defined by

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \quad & \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{aligned}$$

In matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{y} = A\mathbf{x}$$

The matrix A is called the *standard matrix* of the linear transformation. The linear transformation $\mathbf{y} = A\mathbf{x}$ converts the vector $\mathbf{x}$ to the vector $\mathbf{y}$, i.e., expresses $y_1, y_2, \dots, y_n$ in terms of $x_1, x_2, \dots, x_n$.

If $|A| = 0$ then the transformation matrix A is called *singular* and the linear transformation is also called singular.

If $|A| \neq 0$ then the transformation matrix A is called *nonsingular* and the linear transformation is also called nonsingular or regular. For nonsingular transformation, the inverse transformation $\mathbf{x} = A^{-1}\mathbf{y}$ converts the vector $\mathbf{y}$ back to the vector $\mathbf{x}$.

If two linear transformations are given as

$$\mathbf{y} = A\mathbf{x} \quad \text{and} \quad \mathbf{z} = B\mathbf{y}$$

then

$$\mathbf{z} = B\mathbf{y} = B(A\mathbf{x}) = (BA)\mathbf{x}$$

converts the vector $\mathbf{x}$ to the vector $\mathbf{z}$. This is called *composite transformation.*

EXAMPLE 1.32 *Show that the linear transformation*

$$y_1 = 2x_1 + x_2 + x_3, \quad y_2 = x_1 + x_2 + 2x_3, \quad y_3 = x_1 - 2x_3 \text{ is regular.}$$

Write down the inverse transformation.

Solution:

$$y_1 = 2x_1 + x_2 + x_3$$

$$y_2 = x_1 + x_2 + 2x_3,$$

$$y_3 = x_1 - 2x_3$$

In matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{y} = A\mathbf{x}$$

$$\text{Transformation matrix } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = 2(-2-0) - 1(-2-2) + 1(0-1) = -1 \neq 0$$

Hence, the matrix A is nonsingular and the transformation is regular.
The inverse transformation is given by

$$\mathbf{x} = A^{-1}\mathbf{y}$$

$$A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Hence, $x_1 = 2y_1 - 2y_2 - y_3$, $\quad x_2 = -4y_1 + 5y_2 + 3y_3$, $\quad x_3 = y_1 - y_2 - y_3$ is the inverse transformation.

EXAMPLE 1.33 *If* $y = \begin{bmatrix} 4 & -5 & 1 \\ 3 & 1 & -2 \\ 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *find the coordinates* (x_1, x_2, x_3) *corresponding to* (2, 9, 5) *in y.*

Solution:

$$\mathbf{y} = \begin{bmatrix} 4 & -5 & 1 \\ 3 & 1 & -2 \\ 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{y} = A\mathbf{x}$$

$$A = \begin{bmatrix} 4 & -5 & 1 \\ 3 & 1 & -2 \\ 1 & 4 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{72}\begin{bmatrix} 9 & 9 & 9 \\ -5 & 3 & 11 \\ 11 & -21 & 19 \end{bmatrix}$$

$$\mathbf{x} = A^{-1}\mathbf{y}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{72}\begin{bmatrix} 9 & 9 & 9 \\ -5 & 3 & 11 \\ 11 & -21 & 19 \end{bmatrix}\begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore x_1 = 2, x_2 = 1, x_3 = -1$$

EXAMPLE 1.34 *Express each of the transformations* $x_1 = 3y_1 + 2y_2$, $x_2 = -y_1 + 4y_2$ *and* $y_1 = z_1 + 2z_2$, $y_2 = 3z_1$ *in matrix form and find the composite transformation which expresses* x_1, x_2 *in terms of* z_1, z_2.

Solution:

$$x_1 = 3y_1 + 2y_2$$

$$x_2 = -y_1 + 4y_2$$

In matrix form,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\mathbf{x} = A\mathbf{y}$$

Also,

$$y_1 = z_1 + 2z_2$$

$$y_2 = 3z_1$$

In matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\mathbf{y} = B\mathbf{z}$$

Substituting $\mathbf{y}$ in $\mathbf{x} = A\mathbf{y}$,

$$\mathbf{x} = A\mathbf{y} = A(B\mathbf{z}) = (AB)\mathbf{z}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 11 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\therefore x_1 = 9z_1 + 6z_2,\ x_2 = 11z_1 - 2z_2$$

1.12 ORTHOGONAL TRANSFORMATION

The linear transformation $\mathbf{y} = A\mathbf{x}$ is said to be orthogonal if A is an orthogonal matrix. An orthogonal transformation transforms $x_1^2 + x_2^2 + \cdots + x_n^2$ into $y_1^2 + y_2^2 + \cdots + y_n^2$.

Also,

$$\mathbf{x}^T\mathbf{x} = [x_1 \ x_2 \ldots\ldots x_n] \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \cdots + x_n^2$$

Similarly, $\mathbf{y}^T\mathbf{y} = y_1^2 + y_2^2 + \cdots + y_n^2$

If $\mathbf{y} = A\mathbf{x}$ is an orthogonal transformation,

$$\mathbf{x}^T\mathbf{x} = \mathbf{y}^T\mathbf{y} = (A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x} = \mathbf{x}^T(A^TA)\mathbf{x}$$

which holds only if $A^TA = I$.

But,

$$A^{-1}A = I$$

$$\therefore A^T = A^{-1}$$

Hence, a square matrix A is said to be orthogonal if $AA^T = A^TA = I$.

EXAMPLE 1.35 *Show that* $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *is an orthogonal transformation.*

Solution: Let
$$A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & -\sin\theta\cos\theta + \cos\theta\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, A is an orthogonal matrix and the transformation $\mathbf{y} = A\mathbf{x}$ is an orthogonal transformation.

EXERCISE 1.7

1. Given the transformation
$$\mathbf{y} = \begin{bmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$
find the coordinates (x_1, x_2, x_3) corresponding to $(3, 0, 8)$ in y.

Ans. : $x_1 = \frac{8}{5}, x_2 = 5, x_3 = \frac{9}{5}$

2. Given the transformation
$$\mathbf{y} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$
find the coordinates (x_1, x_2, x_3) corresponding to $(2, 3, 0)$ in y.

Ans. : $x_1 = \frac{21}{19}, x_2 = -\frac{16}{19},$

$x_3 = -\frac{5}{\ldots}$

3. A transformation from the variables x_1, x_2, x_3 to y_1, y_2, y_3 is given by $\mathbf{y} = A\mathbf{x}$ and another transformation from y_1, y_2, y_3 to z_1, z_2, z_3 is given by $\mathbf{z} = B\mathbf{y}$, where
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$
Obtain the transformation from x_1, x_2, x_3 to z_1, z_2, z_3.

[**Ans. :** $z_1 = x_1 + 4x_2 - x_3, z_2 = -x_1 + 9x_2 -$ … $z_3 = 3x_1 + 14x_2 - x_3$]

4. Show that the transformation
$$y_1 = \frac{8}{9}x_1 - \frac{4}{9}x_2 + \frac{1}{9}x_3,$$
$$y_2 = \frac{1}{9}x_1 + \frac{4}{9}x_2 - \frac{8}{9}x_3,$$
$$y_3 = \frac{4}{9}x_1 + \frac{7}{9}x_2 + \frac{4}{9}x_3$$