Chapter-5

Time Varying Fields and Maxwell's Equations in Different Media

5.1. Introduction

In earlier chapters, we have restricted ourselves to static and time invariant fields. Electrostatic fields are produced due to static (stationary) electric charges whereas magnetic fields are produced due to electric charges moving with uniform velocity (i.e., for steady current) or due to stationary magnetic poles. Electric and magnetic fields are independent of each other in time invariant static fields.

In the present chapter, we deal with time varying electric and magnetic fields. When a charge is moving with an acceleration, it produces a time varying electric field which in turn produces a changing magnetic field. Similarly, a time varying magnetic field produces a changing electric field (i.e., emf) at a point. Thus, we cannot treat time varying electric field and magnetic field separately.

Michael Faraday established the basis of electromagnetic field concept and proposed the fundamental postulate for electromagnetic induction which relates the time varying magnetic field with an electric field.

Subsequently James Clerk Maxwell spanned the subject and gave the postulate that relates the magnetic field with conduction and displacement current.

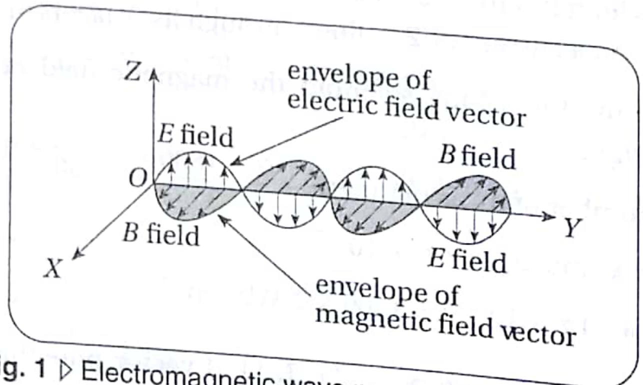

Fig. 1 ▷ Electromagnetic wave propagating along Y-axis

So an oscillating electric field generates an oscillating magnetic field and vice-versa. As a consequence, *the vibrations of the electric and the magnetic fields constitute a wave, known as electromagnetic wave*. Electromagnetic wave transports energy over a very long distance.

Summary

① stationary charges (q) produce electrostatic fields.

② steady current or motion of charges with constant velocity $\left(\frac{dq}{dt} = I\right)$ form magnetostatic fields.

③ time varying magnetic fields or time varying current (or charges with acceleration) i.e., time varying electric fields $\left(\frac{dI}{dt} = \frac{d^2q}{dt^2}\right)$ produce electromagnetic waves or fields.

Now, we will deal with time varying fields and discuss the Faraday's laws of induction and four Maxwell's equations that form the basis of electromagnetic theory.

5.2. Electromagnetic Induction

From Oersted's experiment (in 1820), we know that a steady current carrying conductor can produce a magnetic field.

After Oersted's experiment, Faraday, in the year 1831, wanted to observe whether a magnetic field can produce a current or not. He noticed that as the number of magnetic lines of force associated with a closed coil was changed, an electric current was developed through this coil. This phenomenon is known as **electromagnetic induction.**

5.2.1. Definition of Electromagnetic Induction

The generation of electric current in a closed coil due to the change in the number of magnetic lines of force associated with the coil is known as electromagnetic induction.

The current produced in a closed coil due to electromagnetic induction is called *induced current* and the produced emf in the closed coil is known as *induced emf.*

5.2.2. Laws of Electromagnetic Induction

There are three laws of electromagnetic induction. The first two laws are known as **Faraday's laws** and those are stated below.

1st law **Whenever there is a change in the number of magnetic lines of force passing through a closed coil, an emf is induced in the closed coil. This induced emf sustains as long as the above change occurs.**

2nd law **The magnitude of the induced emf is directly proportional to the rate of change of magnetic flux associated with the coil.**

If ϕ_1 be the magnetic flux passing through the coil at any instant and ϕ_2 be that after a time t, then rate of change of magnetic flux $= \frac{\phi_2 - \phi_1}{t}$.

According to Faraday's second law, the induced emf (e) in the closed coil,

$$e \propto \frac{\phi_2 - \phi_1}{t} \quad \text{or,} \quad e = \frac{k(\phi_2 - \phi_1)}{t}$$, where k is proportionality constant.

$$\text{or,} \quad e = \frac{\phi_2 - \phi_1}{t} \quad [\because \text{ In electromagnetic unit, } k = 1] \qquad \cdots (5.2.2.1)$$

The third law is known as **Lenz's law.**

3rd law The **direction of induced emf is such that it always opposes the cause that produces it**. This is **Lenz's law**.

So the direction of induced emf will be always opposite to the rate of change of magnetic flux.

If $d\phi$ be the change of magnetic flux through a closed coil with n number of turns in an infinitesimal small time dt, the induced emf,

$$e = -n\frac{d\phi}{dt} \quad \cdots (5.2.2.2)$$

The direction of the induced emf may be determined with the help of Fleming's right hand rule.

5.2.3. Integral and Differential Form of Faraday's Law of Electromagnetic Induction

Faraday's law of electromagnetic induction for a closed coil can be expressed as,

$$e = -\frac{d\phi}{dt} \quad \cdots (5.2.3.1)$$

where e is the induced emf and ϕ is the magnetic flux through a closed circuit.

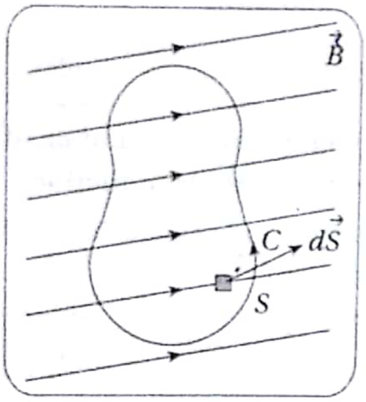

Fig. 2 ▷ Magnetic lines of force through a surface S

The magnetic flux ϕ through a closed surface bounded by a curve C can be written as

$$\phi = \oint_S \vec{B}\cdot d\vec{S} \quad \cdots (5.2.3.2)$$

where $\vec{B}$ is the magnetic flux density and $d\vec{S}$ is an elemental surface on the surface S.

If the electric field induced in space is $\vec{E}$, then the induced emf around the curve C is given by,

$$e = \oint_C \vec{E}\cdot d\vec{l} \quad \cdots (5.2.3.3)$$

where dl is the elemental length on the curve C.

Substituting the values of ϕ and e from equations (5.2.3.2) and (5.2.3.3) into equation (5.2.3.1), we get

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\oint_S \vec{B}\cdot d\vec{S} \quad \cdots (5.2.3.4)$$

or, $$\oint_C \vec{E}\cdot d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} \quad [\because S \text{ is not dependent on time}] \quad \cdots (5.2.3.5)$$

This equation is known as the **integral form of Faraday's law.**

Using Stokes' theorem, equation (5.2.3.5) becomes

$$\oint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\oint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} \quad \cdots (5.2.3.6)$$

or, $$\oint_S \left(\vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S} = 0$$

Since this equation is true for any arbitrary surface S,

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

or, $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \cdots (5.2.3.7)$$

Equation (5.2.3.7) is known as **Faraday's law of electromagnetic induction in differential form**. It relates the spatial rotation of $\vec{E}$ at a particular point to the time rate of change in $\vec{B}$ at that point.

5.3. Displacement Current

If a capacitor is charged, an electric field is developed between its two parallel plates. The magnitude of the electric field is given by

$$E = \frac{\sigma}{\varepsilon_0}$$

where σ is the surface charge density (i.e., charge per unit area) and ε_0 is the permittivity of free space.

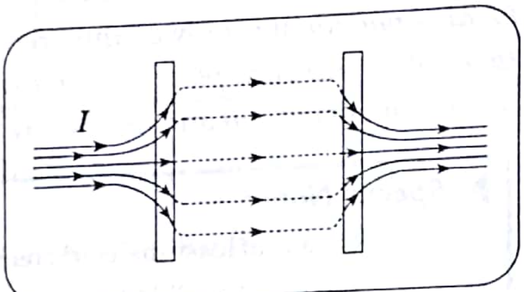

Fig. 3 ▷ Displacement current through a capacitor

Now, if this capacitor is connected with a resistance, this charge will flow between the plates and set up a time varying electric field through the dielectric medium between the two plates. As a result, *this time varying electric field produces a current, known as displacement current. So the continuity of current through the entire circuit is maintained by conduction current through the wire and displacement current through the dielectric medium of the capacitor* [Fig. 3].

In case of ideal capacitor the conduction current through the wire is equal to the displacement current through the dielectric medium of capacitor.

Here the displacement current,

$$I_d = \frac{dq}{dt}, \text{ where } q \text{ is the amount of induced charges on the plates.}$$

or, $$I_d = \frac{d}{dt}(A\varepsilon_0 E) \quad \left[\because E = \frac{\sigma}{\varepsilon_0} = \frac{q}{A\varepsilon_0}, A \text{ is the area of each plate}\right]$$

or, $$I_d = A\varepsilon_0 \frac{\partial E}{\partial t} \quad [\because E = E(x,t)] \qquad \cdots (5.3.1)$$

This is the **relation between displacement current and the magnitude of electric displacement.**

So, displacement current density,

$$\vec{J}_d = \frac{I_d}{A} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_0 \vec{E}) = \frac{\partial \vec{D}}{\partial t}, \text{ where } \vec{D} = \varepsilon_0 \vec{E} = \text{electric displacement vector.}$$

So, $$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} \qquad \cdots (5.3.2)$$

This equation shows the rate of change of electric displacement vector with respect to time is equal the displacement current density.

In Fig. 3 the plates are magnified for clarity. The continuous lines represent the flow of true charge (conduction current) terminating at the inner surface of positive plate and set up a time varying electric field that produces displacement current represented by dotted lines, through the dielectric medium of the capacitor.

So, if a capacitor is connected through a resistance to an alternating emf, then the displacement current is set up due to the time varying electric field between the dielectric medium of the capacitor.

Definition ***The current set up in a dielectric medium due to a time varying electric field through it, is known as displacement current.***

The *displacement current density does not represent a current* which flows directly through *the ideal capacitor.* It is only an *apparent current* representing *the rate at which the flow of charge takes place from one plate to the other* in the external circuit. Hence, this current is renamed as *displacement current* and in this way the term displacement is justified.

Special Note :

In case of lossy dielectric material with conductivity σ and permittivity ε, the capacitor can be represented by an equivalent circuit of a capacitor and a resistance in parallel.

In this case the total current density($\vec{J}$) is equal to the sum of conduction current density ($\vec{J}_c$) and displacement current density ($\vec{J}_d$) [Fig. 4].

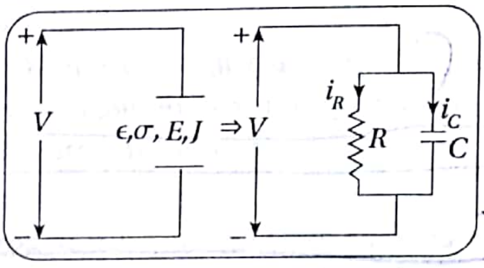

Fig. 4 ▷ An equivalent circuit of lossy capacitor

In empty space, conduction current is zero and the magnetic fields will be set up entirely due to the displacement current. It is to be mentioned that the concept of displacement current was first introduced by *J. C. Maxwell* to account for the production of magnetic field in empty space.

Problem 1

A voltage $60 \cos 1000t$ V is applied to the plates of a parallel plate capacitor with plate area $5\ \text{cm}^2$ and plate separation 3 mm. Calculate the displacement current density between the two plates and conduction current in the wires assuming the dielectric to have permittivity of the three times the permittivity of the free space.

Solution The voltage applied to the parallel plate capacitor

$V = 60 \cos 1000t$

ε = permittivity of the medium = $3\varepsilon_0$

ε_0 = permittivity of the free space = $8.854 \times 10^{-12}\ \text{C}^2 \cdot \text{N}^{-1} \cdot \text{m}^{-2}$

A = area of the plate = $5\ \text{cm}^2 = 0.0005\ \text{m}^2$

d = separation between the two plates = $0.3\ \text{cm} = 0.003\ \text{m}$

∴ The displacement current,

$$I_d = A\varepsilon \cdot \frac{\partial E}{\partial t} = \frac{A \cdot 3\varepsilon_0}{d} \frac{\partial}{\partial t}(Ed) = \frac{3\varepsilon_0 A}{d} \frac{\partial V}{\partial t}$$

$$= \frac{3 \times (8.854 \times 10^{-12}) \times 0.0005}{0.003} \quad [60 \times 1000 \sin 1000t]\ \text{A}$$

$$= 26.56 \times 10^{-8} \sin 1000t \ \text{A (taking only magnitude)}$$

∴ The displacement current density

$$\vec{J_d} = \frac{I_d}{A} = \frac{26.56 \times 10^{-8} \sin 1000t}{0.0005} = 5.31 \times 10^{-4} \sin 1000t \ \text{A} \cdot \text{m}^{-2}$$

In case of ideal capacitor displacement current is equal to conduction current.

Problem 2

In free space electric field intensity is given as $E = \vec{y}\, 20 \cos(\omega t - 50x)$ $\text{V} \cdot \text{m}^{-1}$. Calculate the displacement current density.

Solution Displacement current density can be written as,

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \varepsilon_0 \frac{\partial}{\partial t} [\vec{y}\, 20 \cos(\omega t - 50x)] = -\hat{y}\, \varepsilon_0\, 20\omega \sin(\omega t - 50x)\ \text{A} \cdot \text{m}^{-2}$$

Problem 3

Find the maximum value of displacement current and conduction current through an ideal capacitor of 5 pF. The voltage applied across the capacitor $V = 0.2 \sin 100\pi t$.

Solution Voltage across capacitor $V = 0.2 \sin 100\pi t$

Displacement current,

$$I_d = J_d A = \frac{\partial D}{\partial t} A = \varepsilon_0 \frac{\partial E}{\partial t} A$$

$$= \frac{\partial}{\partial t}(\varepsilon_0 EA) = \frac{dq}{dt}$$

$$= C\frac{dV}{dt} \quad \left[\text{In case of capacitor, electric field } E = \frac{q}{\varepsilon_0 A}\right]$$

$$= 5 \times 10^{-12} \times \frac{d}{dt}(0.2 \sin 100\pi t)$$

$$= 5 \times 10^{-12} \times 0.2 \times 100\pi \cos 100\pi t$$

$$= 3.14 \times 10^{-10} \cos 100\pi t \ \text{A}$$

This is the value of displacement current and its maximum value occurs at $t = 0$

$$\therefore \quad I_d\big|_{\max} = 3.14 \times 10^{-10}\ \text{A}$$

Since, the capacitor is ideal.

$$\therefore \quad I_d = I_c$$

So, maximum value of conduction current or displacement current

$$I_c = 3.14 \times 10^{-10}\ \text{A}$$

Problem 4

An ac voltage source is connected across the two plates of an ideal parallel plate capacitor. If the applied ac voltage $V = V_0 \sin\omega t$, then verify that the displacement current in the ideal capacitor is equal to the conduction current through the wire.

Solution If A is the area of each plate of the capacitor then capacitance,

$$C = \frac{\varepsilon A}{d} \text{ (in SI unit), where } d = \text{separation between the plates.}$$

Now the electric field E in the dielectric medium of the capacitor is

$$E = \frac{V}{d} = \frac{V_0 \sin\omega t}{d}$$

Again, electric displacement vector,

$$D = \varepsilon E = \varepsilon \cdot \frac{V_0 \sin\omega t}{d}$$

Now the displacement current,

$$I_d = \int \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} = \frac{\varepsilon V_0 \omega \cos\omega t}{d} \int dS = \frac{\varepsilon V_0 \omega \cos\omega t}{d} A$$

$$= C V_0 \omega \cos\omega t \quad \left[\because C = \frac{\varepsilon A}{d}\right] \quad \cdots (1)$$

Again, the conduction current through the wire,

$$I_c = C \cdot \frac{dV}{dt} = C V_0 \omega \cos\omega t \quad \cdots (2)$$

So comparing equation (1) and (2), we can write $I_c = I_d$.

Problem 5

Find the conduction and displacement current density in a material having electrical conductivity $\sigma = 10^{-4}$ mho·m^{-1} and relative permittivity $\varepsilon_r = 2.5$. Given the electric field in the material is $E = 6 \times 10^{-5} \sin(8 \times 10^8)t$ V·m^{-1}.

Solution Here, $\sigma = 10^{-4}$ mho·m^{-1}, $\varepsilon_r = 2.5$

and $E = 6 \times 10^{-5} \sin(8 \times 10^8)t$ V·m^{-1}

So the conduction current density,

$$\vec{J} = \sigma \vec{E}$$

$$= 10^{-4} \times 6 \times 10^{-5} \sin(8 \times 10^8)t$$

Hence, amplitude of conduction current density $= 6 \times 10^{-9}$ A·m^{-2}

Again, electric displacement vector,

$$\vec{D} = \varepsilon \vec{E}$$

where $\varepsilon = \varepsilon_0 \varepsilon_r$

Now, displacement current density,

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \varepsilon_0 \varepsilon_r \frac{\partial \vec{E}}{\partial t}$$

$$= 8.854 \times 10^{-12} \times 2.5 \times 6 \times 10^{-5} \times 8 \times 10^8 \cos(8 \times 10^8)t$$

So, amplitude of displacement current density $= 8.854 \times 10^{-12} \times 2.5 \times 6 \times 10^{-5} \times 8 \times 10^8$

$$= 1.06 \times 10^{-6}\ \text{A} \cdot \text{m}^{-2}$$

Problem 6

The parallel plate capacitor has plates each of area $2\,\text{m}^2$ and the plates are separated by a dielectric of thickness 1 mm and dielectric constant 3. The potential difference and conduction current in the connecting wire at certain instant of time is 100 V and 2 mA respectively. Find out the displacement current flowing between the two plates of capacitor.

Solution In case of ideal capacitor, the conduction current (I_c) through the wire is equal to the displacement current (I_d) through the dielectric medium of the capacitor.

$\therefore \quad I_d = I_c \quad$ or, $\quad I_d = 2$ mA

Problem 7

The electric field between two parallel metal plates of area $1\,\text{cm}^2$ changes at the rate of $1.2 \times 10^8\ \text{V} \cdot \text{m}^{-1} \cdot \text{s}^{-1}$. Calculate the displacement current.

Solution The displacement current is $I_d = \varepsilon_0 A \frac{\partial E}{\partial t}$

where ε_0 = dielectric constant

A = area of each parallel plate = $1\ \text{cm}^2 = 10^{-4}\ \text{m}^2$

E = electric field = $1.2 \times 10^8\ \text{V} \cdot \text{m}^{-1}$

$\therefore \quad I_d = (8.85 \times 10^{-12}\ \text{C}^2 \cdot \text{N}^{-1} \cdot \text{m}^{-1}) \cdot (10^{-4}\text{m}^2) \cdot (1.2 \times 10^8\ \text{V} \cdot \text{m}^{-1} \cdot \text{s}^{-1})$

$= 1.062 \times 10^{-7}$ A

5.4 Differences between Conduction Current and Displacement Current

Conduction current	Displacement current
1. It is the actual current that flows through a conducting medium.	1. It is the apparent current which is produced due to the time varying electric field.
2. It obeys Ohm's law, i.e., $I = \frac{V}{R}$	2. It does not obey Ohm's law.
3. The conduction current density is represented by, $\vec{J}_c = \sigma \vec{E}$ where σ is the surface charge density and $\vec{E}$ is the applied electric field.	3. The displacement current density is represented by, $\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \varepsilon \frac{\partial \vec{E}}{\partial t}$ where ε is the permittivity of a medium.

5.5. Maxwell's Equations (for Electric Field $\vec{E}$ and Magnetic Field $\vec{B}$)

All the phenomena in electricity and magnetism (i.e., of electromagnetic theory) can be developed with the help of four differential equations in vector form known as Maxwell's electromagnetic equations (or Maxwell's field equations). *Maxwell's equations are the basic equations of electromagnetism.* They are valid even relativistically.

Differential form of Maxwell's equations (in SI unit)

① $\vec{\nabla} \cdot \vec{D} = \rho$ ··· (5.5.1)

[Divergence of electric displacement vector ($\vec{D}$) is equal to *charge density* (ρ). This is the Gauss' law in electrostatics]

② $\vec{\nabla} \cdot \vec{B} = 0$ ··· (5.5.2)

[Divergence of magnetic flux density or magnetic induction vector ($\vec{B}$) is equal to zero. This is the Gauss' law of magnetostatics]

③ $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ··· (5.5.3)

[The curl of electric field vector is equal to the negative time rate of change of magnetic flux density vector. This is Faraday's law of electromagnetic induction.]

④ $\vec{\nabla} \times \vec{H} = \vec{J}_c + \vec{J}_d = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$ ❶ ··· (5.5.4)

[The curl of magnetic field vector is equal to the sum of conduction current density $\vec{J}_c$ and displacement current density $\vec{J}_d$ $\left(= \frac{\partial \vec{D}}{\partial t}\right)$. This is Maxwell's modified form of Ampere's law.]

In the above equations,

$$\vec{J}_C = \sigma \vec{E},\ \vec{B} = \mu \vec{H},\ \vec{D} = \varepsilon \vec{E}$$

where σ, μ and ε are the conductivity, permeability and permittivity of the medium respectively.

Maxwell's first two equations are steady state equations, as these are time independent whereas **3rd and 4th equations are time varying equations,** as they are time dependent.

Integral form of Maxwell's equations

① $\oint_S \vec{D} \cdot d\vec{S} = \int_V \rho dV$ ··· (5.5.5)

[Surface integral of electric displacement vector is equal to the volume integral of charge density.]

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❶ In CGS unit, Maxwell's equations are,

(I) $\vec{\nabla} \cdot \vec{D} = 4\pi\rho$, (II) $\vec{\nabla} \cdot \vec{B} = 0$, (III) $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ and (IV) $\vec{\nabla} \times \vec{H} = 4\pi \vec{J}_c + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

② $\oint_S \vec{B} \cdot d\vec{S} = 0$ ... (5.5.6)

[Surface integral of magnetic flux density vector is equal to zero.]

③ $\oint_C \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\oint_S \vec{B} \cdot d\vec{S}$ ... (5.5.7)

[The line integral of electric field vector is equal to negative time rate of change of the surface integral of magnetic flux density vector.]

④ $\oint_C \vec{H} \cdot d\vec{l} = \oint_S \vec{J}_c \cdot d\vec{S} + \oint_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$ ... (5.5.8)

[The line integral of magnetic field intensity vector is equal to the sum of surface integral of conduction current density vector and displacement current density vector i.e., it is the sum of conduction current and diplacement current.]

5.5.1. Physical Significance of Maxwell's Equations

Maxwell's 1st equation $\vec{\nabla} \cdot \vec{D} = \rho$

① It states that the total electric flux through any closed surface is $\frac{1}{\varepsilon_0}$ times the total charge included within the surface. This represents **Gauss' law in electrostatics in differential form.**

② If ρ is positive, divergence of electric field is positive and if ρ is negative, divergence is negative. So electric field lines start from positive charges (source) and end at negative charges (sink). Thus, it gives a relation between electric field and charge distribution.

③ It is a steady state equation as the equation does not depend on time.

④ It indicates that the charge density ρ is a scalar quantity.

Maxwell's 2nd equation $\vec{\nabla} \cdot \vec{B} = 0$

① It states that the net magnetic flux through any closed surface is zero. This represents the **Gauss' law in magnetostatics in differential form.**

② As $\vec{\nabla} \cdot \vec{B} = 0$, so, we can conclude that the number of magnetic lines of force entering into any region of a closed surface is equal to the number of magnetic lines of force leaving it. Hence, **an isolated magnetic pole or magnetic monopole cannot exist. It appears only in pairs.**

③ There is no source or sink for magnetic lines of force.

④ It is a steady state equation i.e., independent of time.

Maxwell's 3rd equation $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

① It signifies that an *electric field is produced due to the change of magnetic flux with time.* So the third equation may be written as $e = -\frac{\partial \phi}{\partial t}$, which represents the partial version of Faraday's law of electromagnetic induction. This equation gives the **differential form of Faraday's law of electromagnetic induction.**

② It is a time dependent equation.

Maxwell's 4th equation $\vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$

① It states that *a magnetic field is produced due to the current density* $\vec{J}_c$ (i.e., conduction current density) and a time variation of $\vec{D}$ (or $\vec{E}$), displacement current density (J_d), jointly as well as separately.

② This represents the *modified differential form of Ampere's circuital law for steady current as well as varying current.*

③ This is a *time dependent equation.*

5.5.2. Maxwell's Equations for Free Space (or in Vacuum)

In free space, free charge density $\rho = 0$, conduction current density $\vec{J}_c = \sigma\vec{E} = 0$ as conductivity $\sigma = 0$, electric displacement vector $\vec{D} = \varepsilon_0 \vec{E}$, where ε_0 is the permittivity of free space and magnetic flux density $\vec{B} = \mu_0 \vec{H}$, where μ_0 is the permeability of free space. Hence, Maxwell's equations in differential form for free space become,

① $\vec{\nabla} \cdot \vec{E} = 0$ … (5.5.2.1)

② $\vec{\nabla} \cdot \vec{H} = 0$ … (5.5.2.2)

③ $\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \left(= -\frac{\partial \vec{B}}{\partial t} \right)$ … (5.5.2.3)

④ $\vec{\nabla} \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \left(= \frac{\partial \vec{D}}{\partial t} \right)$ … (5.5.2.4)

5.5.3. Maxwell's Equations for Static Fields (for Time Invariant Fields)

In case of time invariant fields, $\frac{\partial \vec{B}}{\partial t} = 0$, $\frac{\partial \vec{D}}{\partial t} = 0$

So, Maxwell's four equations for static fields are,

① $\vec{\nabla} \cdot \vec{D} = \rho$ … (5.5.3.1)

② $\vec{\nabla} \cdot \vec{B} = 0$ … (5.5.3.2)

③ $\vec{\nabla} \times \vec{E} = 0$ … (5.5.3.3)

④ $\vec{\nabla} \times \vec{H} = \vec{J}_c$ … (5.5.3.4)

and in this case $\vec{\nabla} \cdot \vec{J}_c = 0$.

5.5.4. Maxwell's Equations for Good Conductors

For good conductors, $\rho = 0$ as the amount of negative and positive charges are equal. Again, the displacement current density, $J_d = 0$ as the displacement current cannot flow through a conductor and it flows only through a dielectric medium of the capacitor.

The Maxwell's equations for good conductors are,

① $\vec{\nabla} \cdot \vec{D} = 0$ ··· (5.5.4.1)

② $\vec{\nabla} \cdot \vec{B} = 0$ ··· (5.5.4.2)

③ $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ··· (5.5.4.3)

④ $\vec{\nabla} \times \vec{H} = \vec{J}_c$ ··· (5.5.4.4)

Also, in this case $\vec{\nabla} \cdot \vec{J}_c = 0$.

Now, $\vec{J}_c = \sigma \vec{E}$, $\vec{B} = \mu \vec{H}$, $\vec{D} = \varepsilon \vec{E}$, where σ, μ and ε are the conductivity, permeability and permittivity of the medium respectively.

5.5.5. Maxwell's Equations for Dielectrics

There are no free charges in dielectrics. All the charges are bounded. A dielectric medium has finite magnitudes of permeability μ and permittivity ε and its conductivity $\sigma = 0$.

So, in dielectrics, free charge density, $\rho = 0$ and conductivity, $\sigma = 0$

This implies, $\vec{J}_c = \sigma \vec{E} = 0$

Substituting $\rho = 0$ and $\vec{J}_c = 0$ in equations (5.5.1) and (5.5.4), we get

① $\vec{\nabla} \cdot \vec{D} = 0$ ··· (5.5.5.1)

② $\vec{\nabla} \cdot \vec{B} = 0$ ··· (5.5.5.2)

③ $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ ··· (5.5.5.3)

④ $\vec{\nabla} \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t}$ ··· (5.5.5.4)

and also in this case $\vec{\nabla} \cdot \vec{J}_c = 0$.

5.6. Derivation of Maxwell's Equations

5.6.1. $\vec{\nabla} \cdot \vec{D} = \rho$ (= $4\pi\rho$ in CGS unit)

From Gauss' law in electrostatics, we get

$$\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int_V \rho dV \quad \cdots (5.6.1.1)$$

or, $$\oint_S \vec{D} \cdot d\vec{S} = \int_V \rho dV \quad [\because \vec{D} = \varepsilon_0 \vec{E}] \quad \cdots (5.6.1$$

Now from Gauss' divergence theorem, $\int_V (\vec{\nabla}\cdot\vec{D})\,dV = \oint_S \vec{D}\cdot d\vec{S}$

So, $\int_V \vec{\nabla}\cdot\vec{D}\,dV = \int_V \rho\, dV$ ··· (5.6.1.3)

or, $\int_V (\vec{\nabla}\cdot\vec{D}-\rho)\,dV = 0$

Since the volume element dV is arbitrary,

$\therefore \quad \vec{\nabla}\cdot\vec{D}-\rho = 0$

or, $\vec{\nabla}\cdot\vec{D} = \rho$ ··· (5.6.1.4)

5.6.2. $\vec{\nabla}\cdot\vec{B} = 0$

Since, magnetic monopole does not exist, any closed volume will always contain equal and opposite magnetic poles. Hence, magnetic field of induction across any closed surface is always zero (or the magnetic flux entering into any region is equal to the magnetic flux leaving it).

So, $\oint_S \vec{B}\cdot d\vec{S} = 0$ ··· (5.6.2.1)

or, $\int_V (\vec{\nabla}\cdot\vec{B})\,dV = 0$ [using Gauss' divergence theorem] ··· (5.6.2.2)

or, $\vec{\nabla}\cdot\vec{B} = 0$, since dV is arbitrary. ··· (5.6.2.3)

5.6.3. $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ $\left(= -\frac{1}{c}\frac{\partial B}{\partial t} \text{ in CGS unit}\right)$

From Faraday's law of electromagnetic induction, we know that the induced emf in a closed circuit is equal to the negative time rate of change of magnetic flux through the circuit. So, if e is the induced emf and ϕ is the magnetic flux then,

$e = -\frac{d\phi}{dt}$ ··· (5.6.3.1)

Again, the induced emf e can be expressed in terms of electrostatic field $\vec{E}$ as,

$e = \oint_C \vec{E}\cdot d\vec{l}$ ··· (5.6.3.2)

where C is the closed circuit bounding the surface S.

The magnetic flux ϕ can be expressed in terms of magnetic flux density $\vec{B}$ as,

$\phi = \oint_S \vec{B}\cdot d\vec{S}$ ··· (5.6.3.3)

Substituting the value of ϕ and e in equation (5.6.3.1), we get

$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\oint_S \vec{B}\cdot d\vec{S}$ ··· (5.6.3.4)

Now from Stokes' theorem, $\oint_C \vec{E} \cdot d\vec{l} = \oint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S}$

So, $$\oint_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{S} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \quad \cdots (5.6.3.5)$$

or, $$\oint_S \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{S} = 0$$

Since the surface element $d\vec{S}$ is arbitrary, we may write

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \cdots (5.6.3.6)$$

5.6.4. $\vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t} \left(= 4\pi J + \frac{1}{c}\frac{\partial D}{\partial t} \text{ in CGS unit}\right)$

We know from Ampere's circuital law,

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I \quad \cdots (5.6.4.1)$$

where C is the closed circuit bounding the surface S.

or, $$\oint_C \vec{H} \cdot d\vec{l} = I \quad [\because \vec{B} = \mu_0 \vec{H}] \quad \cdots (5.6.4.2)$$

Now from Stokes' theorem, $\oint_C \vec{H} \cdot d\vec{l} = \oint_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S}$

So, $$\oint_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = I \quad \cdots (5.6.4.3)$$

Now the current I can be written in terms of current density $\vec{J}_c$ as $I = \oint_S \vec{J}_c \cdot d\vec{S}$

So, $$\oint_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \oint_S \vec{J}_c \cdot d\vec{S} \quad \cdots (5.6.4.4)$$

or, $$\oint_S (\vec{\nabla} \times \vec{H} - \vec{J}_c) \cdot d\vec{S} = 0$$

Since $d\vec{S}$ is arbitrary, we can write

$$\vec{\nabla} \times \vec{H} - \vec{J}_c = 0$$

or, $$\vec{\nabla} \times \vec{H} = \vec{J}_c \quad \cdots (5.6.4.5)$$

which is valid only for steady field, not for time varying fields.

To show this, let us take the divergence of equation (5.6.4.5) as,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = \vec{\nabla} \cdot \vec{J}_c$$

or, $$\vec{\nabla} \cdot \vec{J}_c = 0 \quad [\because \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0] \quad \cdots (5.6.4.6)$$

Again from the equation of continuity, we have

$$\vec{\nabla}\cdot\vec{J}_c+\frac{\partial\rho}{\partial t} = 0 \quad \cdots (5.6.4.7)$$

or, $$\vec{\nabla}\cdot\vec{J}_c = -\frac{\partial\rho}{\partial t} \quad \cdots (5.6.4.8)$$

So, if we apply the equation of continuity for steady flow of charge or static charge, we get

$$\frac{\partial\rho}{\partial t} = 0 \quad \cdots (5.6.4.9)$$

Maxwell suggested to include the time varying fields, so that Ampere's law must be modified for static charge as well as for varying currents. In that case, the current density $\vec{J}_c$ should be replaced by the sum of $\vec{J}_c+\vec{J}_d$, where $\vec{J}_d$ is the current density for displacement current.

Hence from equation $(5.6.4.5)$, we have

$$\vec{\nabla}\times\vec{H} = \vec{J}_c+\vec{J}_d \quad \cdots (5.6.4.10)$$

Taking divergence of this equation, we have

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\cdot(\vec{J}_c+\vec{J}_d) \quad \cdots (5.6.4.11)$$

or, $$\vec{\nabla}\cdot\vec{J}_c+\vec{\nabla}\cdot\vec{J}_d = 0 \quad [\because \vec{\nabla}\cdot(\vec{\nabla}\times\vec{H}) = 0]$$

or, $$\vec{\nabla}\cdot\vec{J}_c = -\vec{\nabla}\cdot\vec{J}_d \quad \cdots (5.6.4.12)$$

Now comparing equation $(5.6.4.12)$ with the continuity equation $(5.6.4.8)$, we get

$$\vec{\nabla}\cdot\vec{J}_d = \frac{\partial\rho}{\partial t} \quad \cdots (5.6.4.13)$$

Now from differential form of Gauss' law, $\vec{\nabla}\cdot\vec{D} = \rho$

So, $$\vec{\nabla}\cdot\vec{J}_d = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D}) \quad \cdots (5.6.4.14)$$

or, $$\vec{\nabla}\cdot\vec{J}_d = \vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t}$$

or, $$\vec{J}_d = \frac{\partial\vec{D}}{\partial t} \quad \cdots (5.6.4.15)$$

Hence, substituting the value of $\vec{J}_d$ from the equation $(5.6.4.16)$ we get from equation $(5.6.4.10)$,

$$\vec{\nabla}\times\vec{H} = \vec{J}_c+\frac{\partial\vec{D}}{\partial t} \quad \cdots (5.6.4.16)$$

So, this is the Maxwell's modification of Ampere's law.

Problem 1

Prove that $\vec{E} = \sin(y-t)\,\hat{k}$ and $\vec{B} = \sin(y-t)\,\hat{i}$ constitute a possible electromagnetic field.

[WBUT 2009]

Solution $\vec{E} = \sin(y-t)\,\hat{k}$ and $\vec{B} = \sin(y-t)\,\hat{i}$

So, $$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \sin(y-t) \end{vmatrix} = \left[\frac{\partial}{\partial y}\sin(y-t)\right]\hat{i} + \left[-\frac{\partial}{\partial x}\sin(y-t)\right]\hat{j}$$

$$= \cos(y-t)\,\hat{i} \qquad \cdots(1)$$

Again, $\vec{B} = \sin(y-t)\,\hat{i}$

$$\therefore \quad \frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\sin(y-t)\,\hat{i} = -\cos(y-t)\,\hat{i} \qquad \cdots(2)$$

So, comparing equations (1) and (2), we can write

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

which is *Maxwell's 3rd equation for an electromagnetic field.*

Hence *the given* $\vec{E}$ *and* $\vec{B}$ *constitute a possible electromagnetic field.*

Problem 2

If in some region of space the conduction current density ($\vec{J}_c$) is zero, then calculate the displacement current density and its maximum value for a magnetic field $\vec{H} = H_0\sin(\omega t - \alpha x)\hat{k}$. Also mention the nature of the medium. [Given α is propagation constant]

Solution Here, magnetic field is vibrating along Z-direction and it travels along X-direction as a transverse wave.

$$\therefore \quad H_x = 0,\ H_y = 0 \text{ and } H_z = H_0\sin(\omega t - \alpha x)$$

From Maxwell's 4th equation, we have

$$\vec{\nabla}\times\vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$$

Since $\vec{J}_c = 0$, $\vec{\nabla}\times\vec{H} = \dfrac{\partial \vec{D}}{\partial t} = \vec{J}_d$

Now, $$\vec{\nabla}\times\vec{H} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_0\sin(\omega t-\alpha x) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}H_0\sin(\omega t-\alpha x)\right]\hat{i} + \left[-\frac{\partial}{\partial x}H_0\sin(\omega t-\alpha x)\right]\hat{j} + 0$$

$$= 0 + [H_0\alpha\cos(\omega t-\alpha x)]\hat{j}$$

So the displacement current will be $\vec{J}_d = H_0\alpha\cos(\omega t-\alpha x)\hat{j}$

The amplitude of displacement current is $J_d|_{max} = H_0\alpha$.

The given medium is either dielectric medium or free space as $J_c = 0$ for the medium.

5.7. Maxwell's Wave Equations in Free Space and Its Solution

In free space (charge free medium), volume charge density $\rho = 0$, the current density $\vec{J}_c = 0$, conductance $\sigma = 0$, the permittivity of the medium $= \varepsilon_0$, the magnetic permeability of the medium $= \mu_0$, Electric displacement $\vec{D} = \varepsilon_0 \vec{E}$, magnetic field induction $\vec{B} = \mu_0 \vec{H}$, where $\vec{H}$ is the magnetic field vector, μ_0 is the magnetic permeability of free space and ε_0 is the permittivity or dielectric constant of the medium.

So, Maxwell's equations in free space take the form,

$$\vec{\nabla} \cdot \vec{D} = 0 \quad \cdots (5.7.1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \cdots (5.7.2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \left(= -\mu_0 \frac{\partial \vec{H}}{\partial t} \right) \quad \cdots (5.7.3)$$

and $\vec{\nabla} \times \dfrac{\vec{B}}{\mu_0} = \dfrac{\partial \vec{D}}{\partial t}$

or, $$\vec{\nabla} \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad [\because \vec{D} = \varepsilon_0 \vec{E}] \quad \cdots (5.7.4)$$

Maxwell's wave equation for $\vec{E}$

In order to solve Maxwell's equation for $\vec{E}$, we take the curl of equation (5.7.3) and get,

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) \quad \cdots (5.7.5)$$

or, $$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \quad \cdots (5.7.6)$$

Now from Maxwell's 4th equation, $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

So, $$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left(\frac{\partial \vec{E}}{\partial t} \right) \quad \cdots (5.7.7)$$

Since, for charge free region, $\vec{\nabla} \cdot \vec{D} = 0$ i.e., $\vec{\nabla} \cdot \vec{E} = 0$, the equation (5.7.7) will reduce to

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \cdots (5.7.8)$$

This is the **Maxwell's wave equation for an electric field in free space.**

Maxwell's wave equation for $\vec{B}$

To solve Maxwell's equation for $\vec{B}$, we take the curl of equation (5.7.4) and get,

$$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E})$$

or, $$\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = -\mu_0 \varepsilon_0 \frac{\partial \vec{B}}{\partial t} \quad \left[\because \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \right]$$

But $\vec{\nabla} \cdot \vec{B} = 0$, we can write

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

or,
$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \quad \cdots (5.7.9)$$

The equation represents **Maxwell's wave equations for $\vec{B}$ or $\vec{H}$ in charge free space.**

Now *the general wave equation can be written as,*

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad \cdots (5.7.10)$$

where v is the velocity of the wave.

Comparing the equation (5.7.10) with equations (5.7.8) and (5.7.9) we can say that the *field vectors $\vec{E}$ and $\vec{B}$ or $\vec{H}$* i.e., ***electromagnetic wave propagates in free space*** (i.e., ***in vacuum) with speed***

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.854 \times 10^{-12}}} = 2.998 \times 10^8 \simeq 3 \times 10^8 \text{ m} \cdot \text{s}^{-1}$$

$$[\because \varepsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2 \cdot \text{N}^{-1} \cdot \text{m}^{-2},\ \mu_0 = 4\pi \times 10^{-7}\ \text{Wb} \cdot \text{A}^{-1} \cdot \text{m}^{-1}]$$

This is the speed of light in free space. So, **we can conclude that the electric field intensity $(\vec{E})$ and magnetic field intensity $(\vec{H})$ i.e., electromagnetic wave can propagate in free space with the speed of light. This suggests that light is an electromagnetic wave.**

Solution of Maxwell's wave equations

The plane wave solution for $\vec{E}$ can be written from equation (5.7.8) as

$$\vec{E}(\vec{r}, t) = \vec{E}_0\, e^{i(\vec{K} \cdot \vec{r} - \omega t)} \quad \cdots (5.7.11)$$

and the plane wave solution for $\vec{H}$ is

$$\vec{H}(\vec{r}, t) = \vec{H}_0\, e^{i(\vec{K} \cdot \vec{r} - \omega t)} \quad \cdots (5.7.12)$$

where $\vec{K}$ is the propagation vector and ω is the frequency of the wave. E_0 and H_0 are the peak values of $\vec{E}$ and $\vec{H}$ wave respectively.

Problem 1

A medium is characterized by relative permittivity $\varepsilon_1 = 45$ and relative permeability $\mu_1 = 5$. Calculate the speed of electromagnetic wave in the medium and refractive index of the medium.

Solution Relative permittivity, $\varepsilon_r = \varepsilon_1 = 45$ and relative permeability $\mu_r = \mu_1 = 5$

Now, **Refractive index of the medium** $(\mu) = \dfrac{\text{velocity of e. m. wave in vacuum}}{\text{velocity of e. m. wave in medium}}$

$$= \frac{\frac{1}{\sqrt{\mu_0 \varepsilon_0}}}{\frac{1}{\sqrt{\mu\varepsilon}}} = \sqrt{\frac{\mu}{\mu_0}\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\mu_r \varepsilon_r} \quad \left[\because \mu_r = \frac{\mu}{\mu_0} \text{ and } \varepsilon_r = \frac{\varepsilon}{\varepsilon_0}\right]$$

$$= \sqrt{5 \times 45} = 15$$

So the speed of the e.m. wave in the medium,

$$v = \frac{c}{\mu} = \frac{3 \times 10^8}{15} = 2 \times 10^7 \text{ m} \cdot \text{s}^{-2}$$

Problem 2

Calculate the velocity of an electromagnetic wave propagating through a medium with relative permittivity 3 and relative permeability 1.

Solution The velocity of electromagnetic wave will be,

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0\mu_r\varepsilon_r}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} \quad \left[c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \text{velocity of light in free space}\right]$$

$$= \frac{3 \times 10^8}{\sqrt{3 \times 1}} = \sqrt{3} \times 10^8 = 1.732 \times 10^8 \text{ m} \cdot \text{s}^{-1}$$

5.8. Properties of Electromagnetic Wave

The properties of electromagnetic waves are

1. *Electromagnetic waves are transverse in nature.*
2. *Electric field* $(\vec{E})$, *magnetic field* $(\vec{H})$ *and propagation vector* $(\vec{K})$ *are mutually orthogonal.*
3. *The electric field and magnetic field in electromagnetic wave are in phase.* They become maximum or minimum at the same time.

Now we will prove the above properties.

Transverse nature of electromagnetic waves

The plane wave solution of electric field intensity vector $\vec{E}$ and magnetic field intensity vector $\vec{H}$ can be written from Maxwell wave equation as

$$\vec{E}(\vec{r}, t) = \vec{E_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} = \hat{e}\, E_0\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \cdots (5.8.1)$$

and
$$\vec{H}(\vec{r}, t) = \vec{H_0}\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} = \hat{b}\, H_0\, e^{i(\vec{K}\cdot\vec{r} - \omega t)} \quad \cdots (5.8.2)$$

where $\hat{e}$ and $\hat{b}$ are the unit vectors of $\vec{E}$ and $\vec{H}$.

Here $\hat{K} (= \hat{i}K_x + \hat{j}K_y + \hat{k}K_z)$ is the propagation vector and ω is the frequency of the wave. $\vec{E_0}$ and $\vec{H_0}$ are the peak values of $\vec{E}$ and $\vec{H}$ waves respectively. They are space dependent but time independent vectors.

Now, substituting the value of $\vec{E}$ from equation (5.8.1) in Maxwell's 1st equation $\vec{\nabla}\cdot\vec{E} = 0$, we get

$$\vec{\nabla}\cdot[\hat{e}\,E_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}] = 0 \qquad \cdots(5.8.3)$$

But, we know

$$\vec{\nabla}\cdot(\phi\vec{A}) = \vec{\nabla}\phi\cdot\vec{A} + \phi(\vec{\nabla}\cdot\vec{A}) \qquad \cdots(5.8.4)$$

where $\vec{A}$ is a differentiable vector function and ϕ is a differentiable scalar function.
So, we can write eqn. (5.8.4) as

$$\vec{\nabla}[E_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}]\cdot\hat{e} + E_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}\,(\vec{\nabla}\cdot\hat{e}) = 0$$

Again, we know divergence of constant unit vector is zero, $\vec{\nabla}\cdot\hat{e} = 0$.
Thus the above equation becomes

$$\vec{\nabla}[E_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}]\cdot\hat{e} = 0$$

or, $[(i\vec{K})\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}]\cdot\hat{e} = 0 \quad [\because \vec{K} = \hat{i}K_x + \hat{j}K_y + \hat{k}K_z]$

So, $i(\vec{K}\cdot\hat{e}) = 0 \quad [\because e^{i(\vec{K}\cdot\vec{r}-\omega t)} \neq 0]$

or, $$\boldsymbol{\vec{K}\cdot\hat{e} = 0} \qquad \cdots(5.8.5)$$

This indicates ***the transverse nature of electric field.***

Similarly we can write from Maxwell's second equation $\vec{\nabla}\cdot\vec{H} = 0$ in free space

$$\boldsymbol{\vec{K}\cdot\hat{b} = 0} \qquad \cdots(5.8.6)$$

This indicates ***transverse nature of magnetic field.***

So the equations (5.8.5) and (5.8.6) show that electric field intensity ($\vec{E}$) and magnetic field intensity ($\vec{H}$) are perpendicular to the direction of propagation ($\vec{K}$) of electromagnetic wave. Hence, the **electromagnetic waves are transverse in nature.**

> **Electric field ($\vec{E}$); magnetic field ($\vec{H}$) and propagation vector (K) are mutually orthogonal.**

Maxwell's 3rd equation is

$$\vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t} \qquad \cdots(5.8.7)$$

or, $\vec{\nabla}\times[\hat{e}\,E_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}] = -\mu_0\frac{\partial}{\partial t}[\hat{b}\,H_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}]$

or, $i(\vec{K}\times\hat{e})\,E_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)} = i\mu_0\omega\,\hat{b}\,H_0\,e^{i(\vec{K}\cdot\vec{r}-\omega t)}$

or, $(\vec{K}\times\hat{e})E_0 = \omega B_0 \quad [\because$ Peak value of magnetic induction vector $B_0 = \mu_0 H_0]$

or, $$\boldsymbol{\vec{K}\times\hat{e} = \frac{\omega B_0}{E_0}\hat{b}} \qquad \cdots(5.8.8)$$

This equation indicates $\hat{e}$, $\hat{b}$ and $\hat{K}$ are mutually perpendicular to each other i.e., they form an **orthogonal triad.** Hence, **electric field, magnetic field and the propagation vector of electromagnetic wave are mutually orthogonal.**

5.8.1. Relation between the Amplitudes (Peak Values) of Electric Vector and Magnetic Vector

The equation (5.8.8) can be written as

$$K(\hat{K}\times\hat{e}) = \frac{\omega B_0}{E_0}\hat{b} \qquad \cdots(5.8.1.1)$$

where $\hat{K}$ is the unit vector in the direction of wave propagation and $\vec{K} = K\hat{K}$.

Now equating the magnitudes of both sides of equation (5.8.1.1), we get

$$K = \omega\frac{B_0}{E_0} \quad \text{or,} \quad \frac{\omega}{v} = \omega\frac{B_0}{E_0}$$

$\left[\because \text{The velocity of the wave } i.e. \text{ wave velocity } (= \text{phase velocity})\ v = \frac{\omega}{K}\right]$

$$\text{or,} \quad B_0 = \frac{E_0}{v} \qquad \cdots(5.8.1.2)$$

This is the relation between **magnitude of electric vector and magnetic vector of an e.m. wave.**

In vacuum, the velocity of the electromagnetic wave $v = c$ (velocity of light) $= \frac{1}{\sqrt{\mu_0\varepsilon_0}}$.

So we get from equation (5.8.1.2),

$$B_0 = \frac{E_0}{c} \quad \text{or,} \quad B_0 = E_0\sqrt{\mu_0\varepsilon_0} \quad \text{or,} \quad \frac{E_0}{B_0} = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

$$\text{or,} \quad \frac{E_0}{H_0} = \frac{\mu_0}{\sqrt{\mu_0\varepsilon_0}} \quad [\because B_0 = \mu_0 H_0]$$

$$\text{or,} \quad \frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad \cdots(5.8.1.3)$$

$$\text{or,} \quad \frac{E_0}{H_0} = z_0 \quad \text{(say)} \qquad \cdots(5.8.1.4)$$

where $z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the *impedance of vacuum* (as the dimension of z_0 is the dimension of electrical impedance).

Again we have, $\frac{E_0}{H_0} = \frac{\vec{E}}{\vec{H}}$ and we can write from equation (5.8.1.2)

$$\frac{\vec{E}}{\vec{H}} = \frac{\mu_0}{\sqrt{\mu_0\varepsilon_0}} = \mu_0 v \qquad \cdots(5.8.1.5)$$

This is the **relation between electric vector and magnetic vector of electro magnetic wave.**

Problem 1

The electric vector component of a plane electromagnetic wave vector propagating in a non-magnetic medium is given by $\vec{E} = \hat{y}60\cos(10^8 t + 2z)\ \mathrm{V \cdot m^{-1}}$. Symbols have their usual meanings. Find the relative permittivity of the medium and magnetic vector component of the wave.

Solution The electric vector component of a plane electromagnetic wave is

$$\vec{E} = \hat{y}60\cos(10^8 t + 2z)\ \mathrm{V \cdot m^{-1}}$$

This equation informs us ω = frequency of the wave = 10^8, peak value of $\vec{E} = E_0 = 60$ and the propagation constant $K = 2$.

Now the *velocity of the wave*, $v = \dfrac{\omega}{K} = \dfrac{10^8}{2} = 5 \times 10^7\ \mathrm{m \cdot s^{-1}}$

Refractive index, $n = \dfrac{c(\text{velocity of light in vacuum})}{v(\text{velocity of light in medium})} = \dfrac{\frac{1}{\sqrt{\mu_0\varepsilon_0}}}{\frac{1}{\sqrt{\mu\varepsilon}}} = \sqrt{\dfrac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \sqrt{\mu_r\varepsilon_r}$

or, $\dfrac{c^2}{v^2} = \mu_r\varepsilon_r$

$\therefore$ ***Relative permittivity*** **of the medium,**

$$\varepsilon_r = \frac{c^2}{\mu_r v^2} = \frac{(3 \times 10^8)^2}{1 \times (5 \times 10^7)^2} = 36 \qquad \left[\begin{array}{l}\text{as the medium is} \\ \text{non-magnetic, } \mu_r = 1\end{array}\right]$$

But we know $B_0 = \dfrac{E_0}{v} = \dfrac{60}{5 \times 10^7} = .12 \times 10^{-7}\ \mathrm{T}$

Thus *magnetic vector component of the wave*

$$\vec{B} = \hat{x}(12 \times 10^{-7})\cos(10^8 t + 2z)\ \mathrm{T}$$

So, $\vec{H} = \hat{x}\dfrac{12 \times 10^{-7}}{\mu_0}\cos(10^8 t + 2z)$

$$= \hat{x}\,0.955\cos(10^8 t + 2z)\ \mathrm{A \cdot m^{-1}}$$

Problem 2

Light from laser is propagating in vacuum in $+z$ direction. If the amplitude of the electric field in this light wave is $10.3\ \mathrm{V \cdot m^{-1}}$, calculate the amplitude of the corresponding magnetic field.

Solution Velocity of electromagnetic wave in vacuum $v = c = 3 \times 10^8\ \mathrm{m \cdot s^{-1}}$

Amplitude of electric field $E_0 = 10.3\ \mathrm{V \cdot m^{-1}}$

Now, *amplitude of magnetic field in vacuum*

$$B_0 = \frac{E_0}{c} = \frac{10.3}{3 \times 10^8}\ \mathrm{T} = 3.43 \times 10^{-8}\ \mathrm{T}$$

5.9. Ionised Medium

In a metal or ionised medium (*e.g.* plasma state) free charged particles are present. So these mediums are conducting.

An ionised medium consists of ions and free electrons distributed over a region of space. *In the undisturbed condition, plasma state is neutral.* But, under the influence of electric field and magnetic field of an electromagnetic wave, current arises due to the motion of free electrons and ions.

5.9.1. Electromagnetic Waves in a Charge Free Conducting Medium

Let us consider a uniform[❷] but charge free (i.e., source free) and external current free medium. In uniform medium dielectric constant ε and magnetic permeability μ are constant. The medium is considerd to be charge free as long as it does not contain charges and currents necessary to generate the field. The current existing in the medium is induced only by the propagating electromagnetic wave.

The conduction current density $\vec{J} = \sigma \vec{E}$ as determined by Ohm's law where σ represents conductivity of the medium. Now, the corresponding Maxwell's equations for conducting medium (considering $\rho = 0$) are :

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \cdots (5.9.1.1)$$

$$\vec{\nabla} \cdot \vec{H} = 0 \quad \cdots (5.9.1.2)$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \cdots (5.9.1.3)$$

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \quad \cdots (5.9.1.4)$$

Now we have from equation (5.9.1.3),

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\mu \vec{\nabla} \times \left(\frac{\partial \vec{H}}{\partial t}\right) \quad \text{or,} \quad \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t}$$

Now using equation (5.9.1.1) in the above equation, we get

$$\text{or,} \quad -\nabla^2 \vec{E} = -\mu \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t} \quad [\because \vec{\nabla} \cdot \vec{E} = 0]$$

$$\text{or,} \quad \nabla^2 \vec{E} = \mu \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{H}) \quad \text{or,} \quad \nabla^2 \vec{E} = \mu \frac{\partial}{\partial t} \cdot \left[\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right] \quad \left[\because \vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right]$$

$$\text{or,} \quad \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \cdots (5.9.1.5)$$

$$\text{or,} \quad \nabla^2 \vec{E} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t} = 0$$

---

❷ **Uniform medium :** If a medium is **linear** ($\vec{D}$ is parallel to $\vec{E}$ and $\vec{B}$ is parallel to $\vec{H}$), **homogeneous** (the properties of medium are the same at all points) and **isotropic** (μ and ε are independent of direction, i.e., not depend on position and time i.e., μ and ε are always constant), the medium is called a **uniform medium**.

Similarly, **taking curl of both sides of Maxwell's fourth equation** (i.e., for the $\vec{H}$ field), we have

$$\nabla^2\vec{H} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} - \mu\sigma\frac{\partial\vec{H}}{\partial t} = 0 \quad \cdots(5.9.1.6)$$

These equations (5.9.1.5) and (5.9.1.6) are known as general wave equations for a conducting medium with conductivity σ. These two equations indicate the behaviour of all electromagnetic fields in a uniform but source (charge) free conducting medium.

The presence of *first order term (dissipative term)* $\mu\sigma\frac{\partial E}{\partial t}$ *or* $\mu\sigma\frac{\partial H}{\partial t}$ *in the above wave equations* indicates that *the electromagnetic fields-decay* (i.e., loss of energy) *as they propagate through the medium.* Due to this reason, ***a conducting medium is called a*** **lossy medium.** Thus dissipative term is analogous to the damping term in the equation of a damped oscillator.

But, in a charge free **non-conducting perfectly dielectric medium,** $\rho = 0$ and $\sigma = 0$. So, there is no loss of energy. Thus a **perfectly dielectric medium is called lossless medium.**

5.9.2. Wave Equation in a Charge Free Dielectric or Non-conducting or Lossless Medium

In a charge free, non-conducting medium, $\rho = 0$ and $\sigma = 0$. If the electric permittivity ε and magnetic permeability μ of the medium are independent of time, the corresponding wave equation for $\vec{E}$ and $\vec{H}$ can be obtained in the same way as that of free space.

Thus, by putting $\varepsilon_0 = \varepsilon$ and $\mu_0 = \mu$ in equation (5.7.8) and equation (5.7.9) we get the wave equations as

$$\nabla^2\vec{E} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0 \quad \cdots(5.9.2.1)$$

and $$\nabla^2\vec{H} = \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2} \quad \cdots(5.9.2.2)$$

Comparing these equations with the general wave equation $\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$, we can write **the speed of electromagnetic wave in non-conducting medium** is

$$v = \frac{1}{\sqrt{\varepsilon\mu}} \quad \cdots(5.9.2.3)$$

Since $\varepsilon > \varepsilon_0$ and $\mu > \mu_0$, *the speed of e.m.* wave in non-conducting medium is less than the velocity of light $c\left(=\frac{1}{\sqrt{\varepsilon_0\mu_0}}\right)$ in vacuum.

Thus, the speed of an electromagnetic wave in a non-conducting medium is less than the velocity of light in vacuum.

5.9.3. Electromagnetic Energy Density

The **electric energy density** (i.e., electric energy per unit volume) in the space may be written as

$$\frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}(\varepsilon\vec{E})\cdot\vec{E} = \frac{1}{2}\varepsilon E^2 \quad \cdots(5.9.3.1)$$

The **magnetic energy density** (i.e., magnetic energy per unit volume) in the space may be written as

$$\frac{1}{2}\vec{B}\cdot\vec{H} = \frac{1}{2}(\mu\vec{H})\cdot\vec{H} = \frac{1}{2}\mu H^2 \qquad \cdots(5.9.3.2)$$

Therefore, the **electromagnetic energy density in the space**

$$U_{em}\left(=\frac{1}{2}\vec{D}\cdot\vec{E}+\frac{1}{2}\vec{B}\cdot\vec{H}\right) = \frac{1}{2}\varepsilon E^2+\frac{1}{2}\mu H^2 \qquad \cdots(5.9.3.3)$$

> **Special Note :**
>
> *The electromagnetic energy density* **in vacuum,** $U_{em} = \frac{1}{2}\varepsilon_0 E^2+\frac{1}{2}\mu_0 H^2$.

5.10. Attenuation of Electromagnetic Wave in a Conducting (or Ionized) Medium

The propagation of electromagnetic wave indicates the flow of energy. But, all time varying fields during propagation attenuate very quickly within a conducting medium. For an example, when a wave propagates through a medium, its energy decreases due to continuous ohmic loss of energy. This ohmic loss of energy is seen at any conductor that has high conductivity and large conduction currents.

Now, we will discuss how an electromagnetic wave propagates through a conducting medium.

The *general wave equation* in a charge free and external current free medium is given by

$$\nabla^2\vec{E}-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}-\sigma\mu\frac{\partial\vec{E}}{\partial t} = 0 \quad \textit{(for the electric field vector } \vec{E}\textit{)} \qquad \cdots(5.10.1)$$

and $$\nabla^2\vec{H}-\varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}-\sigma\mu\frac{\partial\vec{H}}{\partial t} = 0 \quad \textit{(for the magnetic field vector } \vec{H}\textit{)} \qquad \cdots(5.10.2)$$

To find a solution to these equations, we assume that the fields ($\vec{E}$ and $\vec{H}$) vary sinusoidally with time. Let the solution be

$$\vec{E}(r,t) = \vec{E}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)} \qquad \cdots(5.10.3)$$

$$\vec{H}(r,t) = \vec{H}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)} \qquad \cdots(5.10.4)$$

Substituting the required value in equation number (5.10.1), we have

$$-K^2\vec{E}+\varepsilon\mu\omega^2\vec{E}+i\sigma\mu\omega\vec{E} = 0 \qquad [\because i^2 = -1]$$

or, $$\vec{E}\,[K^2-\varepsilon\mu\omega^2-i\sigma\mu\omega] = 0 \quad \text{or,} \quad K^2-\varepsilon\mu\omega^2-i\sigma\mu\omega = 0$$

or, $$K^2 = \varepsilon\mu\omega^2\left[1+\frac{i\sigma}{\varepsilon\omega}\right] \qquad \cdots(5.10.5a)$$

$$\therefore \quad K = \omega\sqrt{\varepsilon\mu}\sqrt{1+\frac{i\sigma}{\varepsilon\omega}} \qquad \cdots(5.10.5b)$$

This is the **expression of propagation constant in a conducting or ionised medium.** The term $\frac{\sigma}{\varepsilon\omega}$ **is known as dissipation factor** *as it is the ratio of conduction current density to displacement current density.* This relation is also known as **dispersion relation** as *it relates the propagation constant 'K' and angular frequency 'ω'.*

Since the propagtion vector K in a conducting medium is a complex quantity we can write it as

$$K = \alpha + i\beta \quad \cdots(5.10.6)$$

[Usually, α is a known as **attenuation constant** and β is called **phase constant.** *In a lossless medium, wave does not attenuate. In that case* $\alpha = 0$.]

Now, we can write equation (5.10.5) as

$$(\alpha + i\beta)^2 = \varepsilon\mu\omega^2\left[1 + \frac{i\sigma}{\varepsilon\omega}\right]$$

or, $$(\alpha^2 - \beta^2) + i(2\alpha\beta) = \varepsilon\mu\omega^2\left[1 + \frac{i\sigma}{\varepsilon\omega}\right] \quad \cdots(5.10.7)$$

So, $\alpha^2 - \beta^2 = \varepsilon\mu\omega^2$ and $2\alpha\beta = \sigma\mu\omega$.

On solving we get,

$$\alpha = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\left\{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2} + 1\right]^{1/2} \quad \cdots(5.10.8)$$

and $$\beta = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\left\{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right\}^{1/2} - 1\right]^{1/2} \quad \cdots(5.10.9)$$

Again, since $K = \alpha + i\beta$, the equation for $\vec{E}$ and $\vec{H}$ can be written as

$$\vec{E}(r,t) = \vec{E}_0 e^{-\beta r} e^{i(\alpha r - \omega t)} \quad \text{[from equation 5.10.3]} \quad \cdots(5.10.10)$$

$$\vec{H}(r,t) = \vec{H}_0 e^{-\beta r} e^{i(\alpha r - \omega t)} \quad \text{[from equation 5.10.4]} \quad \cdots(5.10.11)$$

In the above equation, $e^{-\beta r}$ **is called attenuation factor** and $e^{i(\alpha r - \omega t)}$ **is called phase factor.**

The factor $e^{-\beta r}$ *shows an exponential decrease in amplitude with increasing value of 'r'.* Since these **equations suggest that the electromagnetic wave attenuates as it propagates through a conducting medium [Fig. 5].** Here the quantity β **is called the absorption coefficient and is a measure of attenuation.**

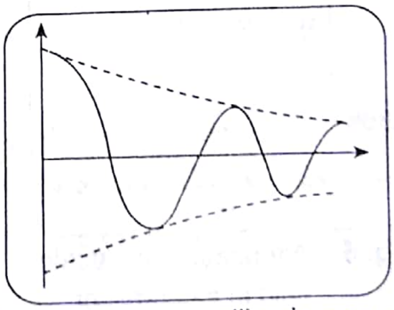

Fig. 5 ▷ Wave travelling in a conducting medium attenuates

5.10.1. Skin (or Penetration) Depth

The general wave equation for electric field vector in a uniform but source free conducting medium is represented by

$$\nabla^2\vec{E} - \mu\sigma\frac{\partial\vec{E}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

The solution of the above equation is $\vec{E}(r,t) = \vec{E}_0 e^{-\beta r} e^{i(\alpha r - \omega t)}$, where β indicates **absorption coefficient**. This equation indicates that ***electromagnetic wave*** attenuates as it ***propagates through a conducting medium.***

The origin of the term, $\mu\varepsilon\frac{\partial^2 \vec{E}}{\partial t^2}$ is due to displacement current while the term $\mu\sigma\frac{\partial \vec{E}}{\partial t}$ is due to conduction current. ***For a good conducting medium*** $\mu\varepsilon\frac{\partial^2 E}{\partial t^2}$ ***is negligible as practically no displacement current exists.*** Therefore, the **general wave equation for a good conducting medium,**

$$\nabla^2\vec{E} - \mu\sigma\frac{\partial \vec{E}}{\partial t} = 0$$

The *solution of the above equation* is

$$\vec{E}(r,t) = \vec{E}_0 e^{-\beta r} e^{i(\alpha r - \omega t)} \quad \cdots(5.10.1.1)$$

Now **for good conductor,** if *the frequency of the electromagnetic wave is not so high,* $\sigma \gg \varepsilon\omega$ i.e., $\frac{\sigma}{\varepsilon\omega} \gg 1$. In that case, we can write from equation (5.10.8) and (5.10.9)

$$\alpha = \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\times\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\mu\sigma\omega}{2}} = \frac{1}{\delta} \text{ (say)} \quad \cdots(5.10.1.2)$$

where $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$ $\cdots(5.10.1.3)$

Hence, the **general solution of electromagnetic wave equation for $\vec{E}$ is obtained** from equation (5.10.1.1) as

$$\vec{E}(r,t) = \vec{E}_0 e^{-\frac{r}{\delta}} e^{i\left(\frac{r}{\delta} - \omega t\right)} \quad \left[\because \alpha = \beta = \frac{1}{\delta}\right] \quad \cdots(5.10.1.4)$$

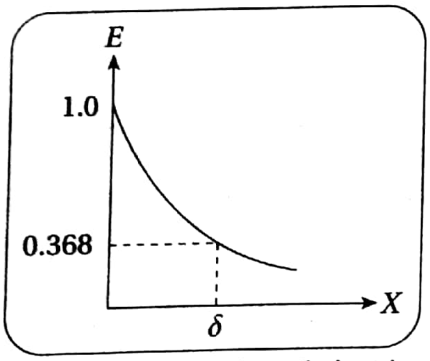

Fig. 6 ▷ Attenuation of electric field in a conductor

Here the amplitude of the electromagnetic wave is $\vec{E}_0 e^{-\frac{r}{\delta}}$. **The distance $r = \delta$, is called the depth of penetration or skin depth, where the amplitude of electric field reduces to $\frac{1}{e}$ times (approximately 37% of original value) of its value at the surface (i.e., $r = 0$) of the conducting medium [Fig. 6].**

*The **skin depth** is given by the expression*

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \quad \cdots(5.10.1.5)$$

The *skin depth decreases with increase in frequency* (ω) *of the electromagnetic wave and the conductivity* (σ) *of the medium.*

5.10.2. Physical Significance of Skin Depth

① The large value of skin depth indicates the less attenuation of the electromagnetic waves in a medium and vice versa.

② *Any high frequency* (ω) *electromagnetic wave* (i.e., *for low value of skin depth*) *can not propagate through conducting media (like metals and ionised gases)*. This fact is used in **electromagnetic shielding** and **reflection of e.m. waves by the ionosphere**.

5.10.3. Relation between Phase Velocity and Propagation Constant of the Electromagnetic Wave in Vacuum

We know from equation (5.10.5), the propagation constant

$$K = \omega\sqrt{\varepsilon\mu}\sqrt{1 + \frac{i\sigma}{\varepsilon\omega}}$$

In vacuum $\sigma = 0$. Now, putting $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$, we can write from the above equation

$$K = \omega\sqrt{\varepsilon_0\mu_0} \quad \cdots(5.10.3.1)$$

But the phase velocity of the wave in vacuum, $v_p = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$. Thus, from equation (5.10.3.1)

$$K = \frac{\omega}{v_p}$$

or, $$v_p = \frac{\omega}{K} \quad \cdots(5.10.3.2)$$

Now, phase velocity = $\frac{\text{velocity of the wave}}{v}$. So, this *is the relation between phase velocity* (v_p), *angular frequency* (ω) *and propagation constant* (K) *of the electromagnetic wave.*

Problem 1

Find the skin depth δ at a frequency 1.6MHz in aluminium where $\sigma = 38.2 \times 10^6 \text{mho} \cdot \text{m}^{-1}$ and $\mu = 4\pi \times 10^{-7} \text{henry} \cdot \text{m}^{-1}$.

Solution Here, $\mu = 4\pi \times 10^{-7}\ \text{H} \cdot \text{m}^{-1}$; $\nu = 1.6 \times 10^6$ Hz,

$\sigma = 38.2 \times 10^6\ \text{mho} \cdot \text{m}^{-1}$

So the skin depth, $\delta = \left(\frac{2}{\mu\omega\sigma}\right)^{\frac{1}{2}} = \left(\frac{2}{\mu 2\pi\nu\sigma}\right)^{\frac{1}{2}} = \left(\frac{1}{\pi\nu\mu\sigma}\right)^{\frac{1}{2}}$

$$= \frac{1}{\sqrt{\pi \times 1.6 \times 10^6 \times 4\pi \times 10^{-7} \times 38.2 \times 10^6}} = 6.43 \times 10^{-5}\ \text{m}$$

5.11. Poynting Vector and Its Physical Significance

The cross product of the electric field vector $(\vec{E})$ and magnetic field vecter $(\vec{H})$ is known as Poynting vector $(\vec{P})$ (after the name of J.S. Poynting who first investigated this properties.)

$$\therefore \quad \vec{P} = \vec{E} \times \vec{H}\left(= \frac{\vec{E} \times \vec{B}}{\mu}\right) \quad \cdots(5.11.1)$$

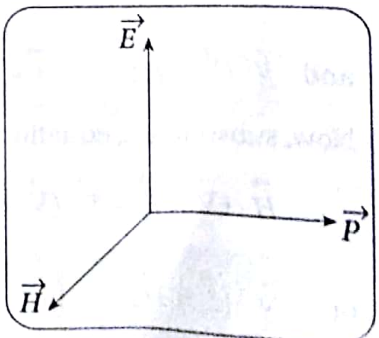

Fig. 7 ▷ Direction of Poynting vector

The direction of Poynting vector is *perpen-dicular to both the electric field* $(\vec{E})$ *and magnetic field* $(\vec{H})$ (i.e., in the *direction of* $\vec{E} \times \vec{H}$) [Fig. 7].

Unit and physical significance :

$$\vec{P} = \vec{E} \times \vec{H} = \frac{\text{voltage}}{\text{length}} \times \frac{\text{current}}{\text{length}} = \frac{\text{power}}{\text{area}} = (\text{V} \cdot \text{m}^{-1})\ (\text{A} \cdot \text{m}^{-1}) = \text{V} \cdot \text{A} \cdot \text{m}^{-2}$$

$$= \text{J} \cdot \text{s}^{-1} \cdot \text{m}^{-2} = \text{W} \cdot \text{m}^{-2} \ [\text{in SI}]$$

Thus, a Poynting vector measures the rate of flow (or radiation) of electromagnetic energy per unit area normal to the direction of flow of energy. Due to this reason, *the Poynting vector is also sometimes called* ***radiation vector.*** This can be explained with the help of the following example.

If we consider a plane polarized electromagnetic wave travelling along Z-axis and the electric field vector $\vec{E}$ along X-axis, E_x, then magnetic field vector will be along Y-axis i.e., H_y.

$\therefore$ Poynting vector,

$$\vec{P} = \vec{E} \times \vec{H} = \hat{i}\, \vec{E}_x \times \hat{j}\, \vec{H}_y = \hat{k}\, \vec{E}_x\, \vec{H}_y$$

So the magnitude of Poynting vector, $P = E_x H_y$.

Thus, **the Poynting vector measures the flow of electric energy per unit area** (**held perpendicular to the direction of propagation of the electromagnetic wave) per unit time.** So, *it can also be called as* ***power flux*** *or the* ***flux vector****. The integration of Poynting vector over a closed surface gives the total power flowing out of that surface.* .

> ***Special Note :***
>
> Poynting vector is a time varying quantity. So, **it cannot be used for *static field*.**

5.11.1. Electromagnetic Energy (or Power) Flow and Poynting Vector

The electromagnetic waves carry energy when they propagate. So, energy can be transported from a transmitter to a receiver by means of electromagnetic waves. The *rate of such energy transportation can be obtained from* **Maxwell's 3rd** and **4th equations** :

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \cdots (5.11.1.1)$$

and $$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \cdots (5.11.1.2)$$

Taking scalar product of equation (5.11.1.1) with $\vec{H}$ and equation (5.11.1.2) with $\vec{E}$, we get

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\left(\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}\right) \qquad \cdots (5.11.1.3)$$

and $$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \qquad \cdots (5.11.1.4)$$

Now, substracting equation (5.11.1.4) from equation (5.11.1.3), we get

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{H}) = -\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - (\vec{E} \cdot \vec{J}) = -\left(\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) - \vec{E} \cdot \vec{J}$$

or, $$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = -\left(\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) - \vec{E} \cdot \vec{J} \qquad \cdots (5.11.1.5)$$

[$\because$ For any vector fields $\vec{A}$ and $\vec{B}$, $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$]

Now, for *linear isotropic medium*

$$\vec{B} = \mu\vec{H} \text{ and } \vec{D} = \varepsilon\vec{E}$$

Substituting the value of $\vec{B}$ and $\vec{D}$ in equation (5.11.1.5), we get

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\vec{H}\cdot\frac{\partial}{\partial t}(\mu\vec{H}) - \vec{E}\cdot\frac{\partial}{\partial t}(\varepsilon\vec{E}) - \vec{E}\cdot\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}(\mu H^2) - \frac{1}{2}\varepsilon\frac{\partial}{\partial t}(\varepsilon E^2) - \vec{E}\cdot\vec{J}$$

or,

$$\vec{\nabla}\cdot(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\left[\frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2\right] - \vec{E}\cdot\vec{J} \qquad \cdots(5.11.1.6)$$

Now using the mathematical expression of Poynting vector $\vec{P}(=\vec{E}\times\vec{H})$ and electromanetic energy density $U_{em} = \frac{1}{2}\mu H^2 + \frac{1}{2}\varepsilon E^2$ in equation (5.11.1.6), we get

$$\vec{\nabla}\cdot\vec{P} = -\frac{\partial}{\partial t}(U_{em}) - \vec{E}\cdot\vec{J}$$

Integrating the above equation over the volume bounded by a closed surface, we get

$$\oint_V (\vec{\nabla}\cdot\vec{P})dV = -\oint_V \frac{\partial}{\partial t}(U_{em})dV - \oint_V (\vec{E}\cdot\vec{J})dV \qquad \cdots(5.11.1.7)$$

Applying Gauss's divergence theorem on the L.H.S. of the equation, we have

$$\underbrace{\oint_S \vec{P}\cdot\vec{dS}}_{\text{Total power leaving the volume}} = -\underbrace{\oint_V \frac{\partial U_{em}}{\partial t}dV}_{\text{Rate of decrease of stored electromagnetic energy}} - \underbrace{\oint_V (\vec{E}\cdot\vec{J})dV}_{\text{Ohmic power dissipated due to motion of charge}} \qquad \cdots(5.11.1.8)$$

This equation (5.11.1.8) is mathematical representation of **Poynting theorem**. Practically, **Poynting theorem is a statement of conservation of energy in electromagnetic field.**

Statement **Poynting theorem states that the net power flowing out of a given volume V is equal to the time rate of decrease of stored electromagnetic energy in that volume minus the conduction losses. This theorem is illustrated in Fig. 8.**

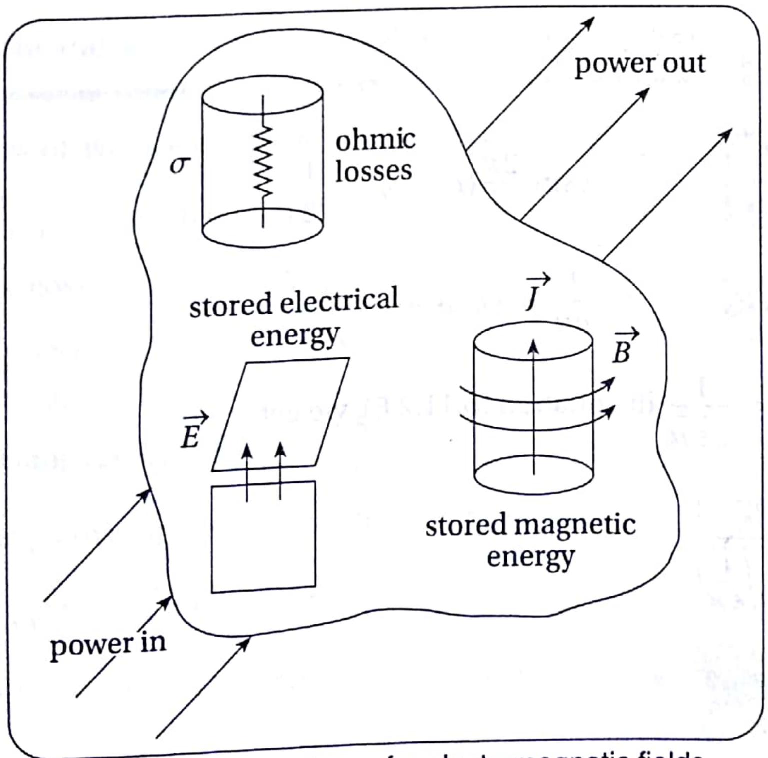

Fig. 8 ▷ Power balance for electromagnetic fields

5.11.2. Average Value of Poynting's Vector for a Plane Electromagnetic Wave

Let us consider a plane electromagnetic wave travelling along Z-axis. The corresponding electric vector (E_x) and magnetic vector (H_y) can be given by

$$E_x = E_0 \sin\frac{2\pi}{\lambda}(vt - z) \qquad \cdots(5.11.2.1)$$

and $$H_y = H_0 \sin\frac{2\pi}{\lambda}(vt - z) \qquad \cdots(5.11.2.2)$$

where v = speed of electromagnetic wave.

For such plane electromagnetic wave, we can write from equation (5.8.1.5) $\left(\text{i.e., } \frac{E_0}{H_0} = \mu v\right)$

$$\frac{E_x}{H_y} = \frac{E_0}{H_0} = \mu v \qquad \cdots(5.11.2.3)$$

In e.m. wave, the electric and magnetic vectors are in the same phase. *So the ratio of magnitudes of the two vectors at any time can be written as*

$$\frac{|\vec{E}|}{|\vec{H}|} = \mu v \qquad \cdots(5.11.2.4)$$

Since, *the electric field and magnetic field for plane polarized e.m. wave are mutually perpendicular,* ***the magnitude of Poynting vector*** *can be written as*

$$\mathbf{P = EH = \frac{E^2}{\mu v}} \qquad \left[\because\ H_y = \frac{E_x}{\mu v}\right] \qquad \cdots(5.11.2.5)$$

Now, **the time average value of Poynting vector** is

$$\langle P\rangle = \frac{E_0^2}{\mu v}\left\langle \sin^2\frac{2\pi}{\lambda}(ct - z)\right\rangle$$

or, $$\langle \mathbf{P}\rangle = \frac{1}{2}\left(\frac{E_0^2}{\mu v}\right) \qquad \left[\because\ \left\langle \sin^2\frac{2\pi}{\lambda}(ct - z)\right\rangle = \frac{1}{2}\right] \qquad \cdots(5.11.2.6)$$

or, $$\langle \mathbf{P}\rangle = \frac{1}{2}\varepsilon v E_0^2 \qquad \left[\because\ \frac{1}{v\mu} = v\varepsilon \text{ as } v^2 = \frac{1}{\mu\varepsilon}\right] \qquad \cdots(5.11.2.7)$$

Again, putting $v = \frac{1}{\sqrt{\varepsilon\mu}}$ in equation (5.11.2.6), we get

$$\langle P\rangle = \frac{1}{2}\left(\frac{E_0^2}{\mu\sqrt{\frac{1}{\varepsilon\mu}}}\right)$$

or, $$\langle \mathbf{P}\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,E_0^2 \qquad \cdots(5.11.2.8)$$

This is the expression of **average Poynting vector in terms of E_0**.

The *average value of Poynting vector is the intensity of the electromagnetic wave.*

$\therefore$ **Intensity of the electromagnetic wave** $= \langle P \rangle = \dfrac{1}{2}\sqrt{\dfrac{\varepsilon}{\mu}}E_0^2$ ...(5.11.2.9)

Average Poynting vector in terms of H_0

We know, $\langle P \rangle = \dfrac{1}{2}\sqrt{\dfrac{\varepsilon}{\mu}}E_0^2$; but $E_0^2 = H_0^2\mu^2 v^2$

$$\therefore \quad \langle P \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\mu^2 v^2 H_0^2 = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\mu^2 \frac{1}{\varepsilon\mu}H_0^2 \qquad \left[\because v^2 = \frac{1}{\varepsilon\mu}\right]$$

$$\text{or,} \quad \langle P \rangle = \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}H_0^2 \qquad \cdots(5.11.2.10)$$

Average Poynting vector in free space

In free space, $v = c =$ velocity of light and $\varepsilon = \varepsilon_0$

$\therefore$ We can write from equation (5.11.2.7), **the average value of Poynting vector for a plane electromagnetic wave is**

$$\langle P \rangle = \frac{1}{2}\varepsilon_0 c E_0^2 \qquad \cdots(5.11.2.11)$$

$$\text{or,} \quad \langle P \rangle = \varepsilon_0 c E_{r.m.s}^2 \qquad \cdots(5.11.2.12)$$

where $E_{r.m.s} = \dfrac{E_0}{\sqrt{2}}$

Problem 1

The electromagnetic wave is propagating in free space with electric vector $\vec{E}(z, t_0) = 150\cos(\omega t - Kz)\hat{x}$. How much average energy is passing through a rectangular hole of length 3 cm and width 1.5 cm on YZ or XZ plane in one minute time.

Solution The flow of average electrical energy per unit area per second in free space is

$\langle P \rangle = c\varepsilon_0 E_{r.m.s}^2$ [from equation 5.11.2.12]

Here, $E(z, t_0) = 150\cos(\omega t - kz)\hat{x}$ with $E_0 = 150$

$$\therefore \quad E_{r.m.s} = \frac{E_0}{\sqrt{2}} = \frac{150}{\sqrt{2}}$$

Now, area of the rectangular hole $= \left(\dfrac{3}{10} \times \dfrac{1.5}{100}\right) \text{m}^2$

$\therefore$ The average energy crossing the rectangular hole in 60 s

$$= c\varepsilon_0 E_{r.m.s}^2 \times \left(\frac{3}{100} \times \frac{1.5}{100}\right)\text{m}^2 \times 60 \text{ s}$$

$$= (3 \times 10^8) \times (8.85 \times 10^{-12}) \times \left(\frac{150}{\sqrt{2}}\right)^2 \times (4.5 \times 10^{-4}) \times 60 \text{ J} \cdot \text{s}^{-1}$$

$$= 0.806 \text{ J} \cdot \text{s}^{-1}$$

Problem 2

The amount of electromagnetic energy received by earth in the form of light from sun is 1300 $W \cdot m^{-1}$. Calculate the root mean square value of the electric vector and magnetic vector of the light wave on the earth surface.

Solution Here, intensity of electromagnetic wave on the earth surface

$$I = 1300 \text{ W} \cdot \text{m}^{-1}$$

The flow of average electrical energy per unit area per second = intensity of the electromagnetic wave

$$\therefore \quad I = c\varepsilon_0 E_{r.m.s}^2 \quad \text{or,} \quad I = c\varepsilon_0 \frac{E_0^2}{\sqrt{2}}$$

$\therefore$ Electric vector of the light wave on the earth surface

$$E_0 = \sqrt{\frac{2I}{c\varepsilon_0}} = \sqrt{\frac{2 \times 1300}{3 \times 10^8 \times 8.85 \times 10^{-12}}} = 989.588 \text{ V} \cdot \text{m}^{-1}$$

$\therefore$ Magnetic vector of the light wave on the earth surface

$$B_0 = \frac{E_0}{c} = \frac{989.588}{3 \times 10^8} = 3.29 \ \mu\text{T}$$

Problem 3

i **Calculate the value of Poynting vector at the surface of the sun if the power radiated by sun is 3.8×10^{26} watt. The radius of the sun is 7×10^8m.**

ii **If the average distance between the sun and the earth is 1.5×10^{11} metre, show that the average solar energy incident on the earth (called solar constant) is $2 \text{cal} \cdot \text{cm}^{-2} \cdot \text{min}^{-1}$.**

Solution

i Here, P' = power radiated by the Sun = 3.8×10^{26} watt

r_S = radius of the sun = 7×10^8 m

If $\langle \vec{S}_S \rangle$ is the average Poynting vector at the surface of the sun, then by its definition

$$\langle \vec{S}_S \rangle (4\pi r_S^2) = P'$$

$$\therefore \quad \langle \vec{S}_S \rangle = \frac{\vec{P}}{4\pi r_S^2} = \frac{3.8 \times 10^{26}}{4\pi \times (7 \times 10^8)^2} = 6.175 \times 10^7 \text{watt} \cdot \text{m}^{-2}$$

ii Let r_E be the radius of the earth.

r_{ES} = the average distance between the sun and earth = 1.5×10^{11} metre.

If $\langle \vec{S}_E \rangle$ is the average value of Poynting vector at the surface of the earth, then

$$\langle \vec{S}_E \rangle 4\pi r_E^2 = \langle \vec{S}_S \rangle 4\pi r_S^2 = P$$

i.e., $$\langle \vec{S}_E \rangle = \frac{r_S^2}{r_E^2}\langle \vec{S}_S \rangle \approx \frac{r_S^2}{r_{ES}^2}\langle S_S \rangle = \frac{r_S^2}{r_{ES}^2}\langle \vec{S}_S \rangle \quad [\because r_{ES} > r_S]$$

$$= \frac{(7 \times 10^8)^2}{(1.5 \times 10^{11})^2} \times 6.175 \times 10^7 = 1.34 \times 10^3 \text{ watt} \cdot \text{m}^{-2}$$

$$= \frac{1.34 \times 10^3 \times 60}{4.2 \times 10^4} \text{ cal} \cdot \text{cm}^{-2} \cdot \text{min}^{-1}$$

$$= 1.91 \text{ cal} \cdot \text{cm}^{-2} \cdot \text{min}^{-1}$$

$$\cong 2 \text{ cal} \cdot \text{cm}^{-2} \cdot \text{min}^{-1}$$

Problem 4

The maximum value of electric field in an electromagnetic wave is 800 $V \cdot m^{-1}$. Find the maximum value of magnetic intensity and the average value of Poynting vector.

Solution The maximum value of electric field $E_0 = 800 \text{ V} \cdot \text{m}^{-1}$

If maximum value of magnetic field is H_0, we can write from equation (5.8.1.3)

$$\frac{E_0}{H_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

or, $$H_0 = E_0\sqrt{\frac{\varepsilon_0}{\mu_0}} = 800 \times \sqrt{\frac{8.85 \times 10^{-12}}{4\pi \times 10^{-7}}} = 2.12 \text{ A} \cdot \text{m}^{-1}$$

The average value of Poynting vector is

$$\langle P \rangle = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}E_0^2 = \frac{1}{2}\frac{H_0}{E_0}E_0^2 \quad \left[\because \frac{E_0}{H_0} = \sqrt{\frac{\mu}{\varepsilon}}\right]$$

$$= \frac{E_0 H_0}{2} = \frac{800 \times 2.12}{2} = 848 \text{ W} \cdot \text{m}^{-2}$$

Problem 5

Assuming that in a plane e.m. wave $\vec{H}$ and $\vec{E}$ are perpendicular to each other and related by the vector relation $\vec{H_0} = \varepsilon_0 c\vec{E_0}$. Prove that the Poynting vector is 'ε_0' times the energy density of the field.

Solution As the energy density of a plane e.m. wave is given by

$$u_{av} = \frac{1}{2}\varepsilon_0 E_{av}^2 + \frac{1}{2}\mu_0 H_{av}^2$$

or, $$u_{av} = \frac{1}{2}\varepsilon_0 E_{av}^2 + \frac{1}{2}\varepsilon_0 E_{av}^2 \quad \left[\text{As } H_{av} = \frac{E_{av}}{\mu_0 c} \text{ and } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right]$$

or, $$u_{av} = \varepsilon_0 E_{av}^2$$

Problem 6

If the electric field associated with a plane wave is given by $E_x = 20\sin(1.5\pi \times 10^7 t - 0.1\pi z)\ \mathrm{V \cdot m^{-1}}$, find the velocity of electromagnetic wave propagation and the corresponding magnetic field for a perfectly dielectric medium.

Solution Here, $E_x = 20\sin(1.5\pi \times 10^7 t - 0.1\pi z)\ \mathrm{V \cdot m^{-1}}$

Comparing this equation with $E = E_0 \sin(\omega t - kz)$, we get

$$\omega = 1.5\pi \times 10^7 \text{ and } k = 0.1\pi$$

So the required velocity,

$$v = \frac{\omega}{k} = \frac{1.5\pi \times 10^7}{0.1\pi} = 15 \times 10^7\ \mathrm{m \cdot s^{-1}}$$

Again, $$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{\mu_0}{\sqrt{\mu_0\varepsilon_0}}\sqrt{\frac{\mu_r}{\varepsilon_r}} = \mu_0 c\sqrt{\frac{\mu_r}{\varepsilon_r}} \quad \left[\because c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right]$$

So, $$\frac{E}{H} = (4\pi \times 10^{-7}) \times 3 \times 10^8 \times \sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}}$$

Now, for a perfect dielectric medium $\mu_r = 1$, $\varepsilon_r = 1$.

$\therefore$ $$\frac{E}{H} = 120\pi$$

or, $$H = \frac{E}{120\pi} = \frac{20}{120\pi}\sin(1.5\pi \times 10^7 t - 0.1\pi z) = 0.05\sin(1.5\pi \times 10^7 t - 0.1\pi z)$$

Exercise

Multiple Choice Questions

1. Any stationary charge can produce—

Ⓐ electrostatic field
Ⓑ magnetic field
Ⓒ electromagnetic field

Ans. Ⓐ

2. An accelerated charge can produce—

Ⓐ electrostatic field
Ⓑ magnetic field
Ⓒ electromagnetic field

Ans. Ⓒ

3. The Faraday's laws of electromagnetic induction can be expressed as—

Ⓐ $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Ⓑ $\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$ Ⓒ $\vec{\nabla} \times \vec{E} = \vec{B}$

Ans. Ⓐ

4. Displacement current arises due to—

Ⓐ posititive charge only
Ⓑ negative charge only
Ⓒ time varying electric field

Ans. Ⓒ
[WBUT 2004]

5. The displacement current density $\vec{J}_d$ is defined as—

Ⓐ $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$ Ⓑ $\vec{J}_d = \epsilon_0 \frac{\partial \vec{D}}{\partial t}$ Ⓒ $\vec{J}_d = \epsilon \frac{\partial \vec{E}}{\partial t}$ **Ans.** Ⓐ

6. In an empty space, magnetic field is produced entirely due to—
Ⓐ displacement current
Ⓑ conduction current
Ⓒ both **Ans.** Ⓐ

7. The displacement current through an ideal capacitor—
Ⓐ is greater than conduction current
Ⓑ equal to conduction current
Ⓒ less than conduction current **Ans.** Ⓑ

8. Any kind of steady current forms—
Ⓐ electrostatic field Ⓑ magnetostatic field Ⓒ both **Ans.** Ⓑ

9. $\vec{\nabla} \cdot \vec{D} = \rho$ ($\vec{D}$ = electric displacement vector, ρ = volume charge density) represents—
Ⓐ Gauss' law in electrostatics
Ⓑ Gauss' law in magnetostatics
Ⓒ Ampere's law **Ans.** Ⓐ

10. The significance of $\vec{\nabla} \cdot \vec{B} = 0$ ($\vec{B}$ is the magnetic field of induction) is that—
Ⓐ magnetic monopole can exist
Ⓑ magnetic monopole cannot exist
Ⓒ none **Ans.** Ⓑ [WBUT 2013]

11. The significance of $\vec{\nabla} \cdot \vec{D} = \rho$ is that—
Ⓐ electric lines start from positive charges and end at negative charges
Ⓑ magnetic monopole cannot exist
Ⓒ none **Ans.** Ⓐ

12. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, ($\vec{E}$ is the electric field intensity and $\vec{B}$ is the magnetic flux density) represents—
Ⓐ Ampere's law Ⓑ Faraday's law Ⓒ Gauss' law **Ans.** Ⓑ

13. The physical significance of $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ is that—
Ⓐ the time rate of change of magnetic flux density produces electric field
Ⓑ the time rate of change of electric flux produces magnetic field
Ⓒ none **Ans.** Ⓐ

14. $\vec{\nabla} \times \vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$, where the symbols have usual meaning represents—
Ⓐ Gauss' law
Ⓑ Maxwell's modification of Ampere's law
Ⓒ Faraday's law **Ans.** Ⓑ [WBUT 2013]

15. The physical significance of $\vec{\nabla} \times \vec{H} = \vec{J}_c + \dfrac{\partial \vec{D}}{\partial t}$ is—

Ⓐ magnetic field is produced due to a conduction current density only

Ⓑ magnetic field is produced due to a conduction current density along with the variation of displacement current

Ⓒ none

Ans. Ⓑ

16. Faraday's law of electromagnetic induction is—

Ⓐ $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ Ⓑ $\vec{\nabla} \times \vec{E} = 0$ Ⓒ $\vec{\nabla} \times \vec{E} = \dfrac{\partial \vec{B}}{\partial t}$ **Ans.** Ⓐ

17. In empty space Ampere's law becomes—

Ⓐ $\vec{\nabla} \times \vec{H} = \sigma \vec{E}$ Ⓑ $\vec{\nabla} \times \vec{H} = \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ Ⓒ $\vec{\nabla} \times \vec{H} = 0$ **Ans.** Ⓑ

18. In case of static field, Faraday's law refuces to—

Ⓐ $\vec{\nabla} \times \vec{E} = 0$ Ⓑ $\vec{\nabla} \times \vec{E} = \dfrac{\partial \vec{B}}{\partial t}$ Ⓒ $\vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$ **Ans.** Ⓐ

19. For a good conductor, Gauss' law in electrostatics $\vec{\nabla} \cdot \vec{D} = \rho$ will be reduced to—

Ⓐ $\vec{\nabla} \cdot \vec{D} = 0$ Ⓑ $\vec{\nabla} \cdot \vec{D} = \varepsilon_0 \rho$ Ⓒ $\vec{\nabla} \cdot \vec{D} = -\rho$ **Ans.** Ⓐ

20. Maxwell's electromagnetic wave equation in terms of electric field vector $\vec{E}$ in free space is—

Ⓐ $\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$ Ⓑ $\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ Ⓒ $\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{E}}{\partial t^2}$ **Ans.** Ⓐ

21. If $\vec{E}$ and $\vec{B}$ are the electric field and magnetic field respectively of an electron in electromagnetic wave travelling in vacuum with propagation vector $\vec{A}$ then—

Ⓐ $\vec{K} \times \vec{E} = -\omega \vec{B}$ Ⓑ $\vec{K} \times \vec{E} = 0$ Ⓒ $\vec{K} \times \vec{E} = \omega \vec{B}$ **Ans.** Ⓒ

22. The velocity of electromagnetic wave in free space is—

Ⓐ equal to velocity of light

Ⓑ greater than the velocity of light

Ⓒ less than the velocity of light

Ans. Ⓐ

[WBUT 2013]

23. Maxwell's electromagnetic wave equation in terms of magnetic field vector $\vec{H}$ in free space is—

Ⓐ $\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{H}}{\partial t^2}$ Ⓑ $\nabla^2 \vec{H} = -\mu_0 \varepsilon_0 \dfrac{\partial \vec{H}}{\partial t}$ Ⓒ $\nabla^2 \vec{H} = -\mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{H}}{\partial t^2}$ **Ans.** Ⓐ

24. Displacement current arises due to—

Ⓐ negative charges only

Ⓑ time varying electric field

Ⓒ positive charges only

Ans. Ⓑ

[W.B.U.T 2007]

25. Waves originating from a point source and travelling in an isotropic medium is described as —

Ⓐ $\phi = \phi_0 \exp i(kr - \omega t)$

Ⓑ $\phi = \phi_0 \exp i\dfrac{(kr - \omega t)}{r}$

Ⓒ $\phi = \phi_0 \exp i\dfrac{(kr - \omega t)}{r^2}$

Ans. Ⓑ

[WBUT 2007]

26. Electromagnetic wave is propagated through a region of vacuum, which does not contain any charge or current. If the electric vector is given by—

Ⓐ in x direction Ⓑ in y direction Ⓒ in z direction **Ans.** Ⓒ

[WBUT 2007]

27. If electric field vector $\vec{E}$ and magnetic field vector $\vec{H}$ are right angle to each other, the wave propagates in the direction of—

Ⓐ $\vec{E} \times \vec{H}$ Ⓑ $\vec{E} \cdot \vec{H}$ Ⓒ none of them **Ans.** Ⓐ

28. The direction poynting vector $P(= \vec{E} \times \vec{H})$ is—

Ⓐ perpendicular to $\vec{E}$ and $\vec{H}$

Ⓑ parallel to vector $\vec{E}$ and $\vec{H}$

Ⓒ none of them

Ans. Ⓐ

29. Poynting vector is expressed as—

Ⓐ $\vec{H} \times \vec{E}$ Ⓑ $\vec{E} \times \vec{H}$ Ⓒ $\vec{E} \times \vec{H} \cdot d\vec{s}$ **Ans.** Ⓑ

where $\vec{E}$ and $\vec{H}$ are electric field vector and magnetic field vector respectively.

30. The energy associated with an electric field $\vec{E}$ is—

Ⓐ $\frac{1}{2}E^2$ Ⓑ $\frac{1}{2}\varepsilon_0 E^2$ Ⓒ none of them **Ans.** Ⓑ

where ε_0 = permittivity of vacuum.

31. The energy associated with a magnetic field $\vec{H}$ is—

Ⓐ $\frac{1}{2}H^2$ Ⓑ $\mu_0 H^2$ Ⓒ $\frac{1}{2}\mu_0 H^2$ **Ans.** Ⓒ

32. Expression of propagation constant for a conducting medium is—

Ⓐ same as per non-conducting medium

Ⓑ less than non-conducting medium

Ⓒ a complex quantity

Ans. Ⓒ

33. In a homogeneous, isotropic conducting medium of permittivity ε, permeability μ and conductivity σ, the skin depth δ is—

Ⓐ $\sqrt{\dfrac{1}{\omega\sigma\mu}}$ Ⓑ $\sqrt{\dfrac{2}{\omega\sigma\mu}}$ Ⓒ none of them **Ans.** Ⓑ

34. An ionized medium consists of—

Ⓐ electrons and ions Ⓑ electrons Ⓒ ions **Ans.** Ⓐ

35. The speed (v) of an electromagnetic wave in a non-conducting medium respectively with permittivity ε and permeability μ is—

Ⓐ $v = \dfrac{1}{\varepsilon\mu}$ Ⓑ $v = \dfrac{1}{\sqrt{\mu\varepsilon}}$ Ⓒ $v = 0$ **Ans.** Ⓑ

36. If μ and σ are the magnetic permeability and conductivity of a medium, the skin depth δ is given by—

Ⓐ $\delta = \dfrac{2}{\mu\sigma\omega}$ Ⓑ $\dfrac{1}{\mu\sigma\omega}$ Ⓒ $\sqrt{\dfrac{2}{\mu\sigma\omega}}$

where ω = frequency of the electromagnetic wave **Ans.** Ⓒ

37. Skin depth for a conductor in reference to electromagnetic wave varies—

Ⓐ inversely as frequency
Ⓑ directly as frequency
Ⓒ inversely as square root of frequency
Ⓓ directly as square root of frequency **Ans.** Ⓒ

[WBUT 2012]

38. The energy associated with a magnetic field $\vec{H}$ is—

Ⓐ $\frac{1}{2}H^2$ Ⓑ $\mu_0 H^2$

Ⓒ $\frac{1}{2}\mu_0 H^2$ Ⓓ $\frac{1}{2\mu_0}H^2$ **Ans.** Ⓒ

[WBUT 2012]

39. The velocity of electromagnetic wave in free space is—

Ⓐ equal to velocity of light
Ⓑ greater than the velocity of light
Ⓒ less than the velocity of light
Ⓓ zero **Ans.** Ⓐ

[WBUT 2013]

40. In an electromagnetic wave in free space, the electric and magnetic fields are—

Ⓐ parallel to each other
Ⓑ perpendicular to each other
Ⓒ inclined at an angle
Ⓓ inclined at an obtuse angle **Ans.** Ⓑ

[WBUT 2013]

41. In a homogeneous, isotropic conducting medium of permittivity ε, permeability μ and conductivity σ, the skin depth is—

Ⓐ $\sqrt{\dfrac{1}{\omega\sigma\mu}}$ Ⓑ $\sqrt{\dfrac{2}{\omega\sigma\mu}}$

Ⓒ $\sqrt{\dfrac{\mu}{\omega\sigma}}$ Ⓓ $\sqrt{\dfrac{2\mu}{\omega\sigma}}$ **Ans.** Ⓑ

[WBUT 2013]

42. In a plane e.m. wave—

Ⓐ $\vec{E}\times\vec{B}=0$ Ⓑ $\vec{E}\parallel\vec{B}$

Ⓒ $\vec{E}\perp\vec{B}$ Ⓓ $\vec{K}\times\vec{B}=0$

Ans. Ⓒ

[WBUT 2013]

43. The magnetic flux linked with a coil at any instant t is given by $\phi_t = 10t^2 - 100t + 50$, the e.m.f. induced in the coil at $t = 4$ seconds is—

Ⓐ 20 V Ⓑ 10 V

Ⓒ 100 V Ⓓ 200 V

Ans. Ⓐ

[WBUT 2014]

44. Depth of penetration of a wave in a lossy dielectric increases with increase in—

Ⓐ conductivity Ⓑ wavelength

Ⓒ permeability Ⓓ permittivity

Ans. Ⓑ

[WBUT 2014]

Short Answer Type Questions

1. [a] Give the statement of Faraday's law of electromagnetic induction and then express it mathematically. [See Article 5.2.2] [See Article 5.2.3]

[b] Derive its differential form. [See Article 5.3]

2. [a] What is displacement current ?

[b] Prove mathematically, that the rate of change of electric displacement vector with respect to time is equal to the displacement current density. [See Article 5.3]

[c] Derive the relation between displacement current and the magnitude of electric displacement. [See Article 5.3]

3. [a] Write down Maxwell's field equation. [See Article 5.5]

[b] State the significance of Gauss' law and Maxwell's modification of Ampere's law. [See Article 5.5.1]

4. Write down Maxwell's equations for free space and static field. [See Article 5.5.2 and 5.5.3]

5. What will be the form of Maxwell's equations in case of good conductors and dielectric medium ? [See Article 5.5.4 and 5.5.5]

6. Derive the Maxwell's modification of Ampere's law. [See Article 5.6.4]

7. Derive the electromagnetic wave equation in terms of electric vector when the wave is passing through vacuum. [See Article 5.7]

8. Prove that $\vec{E} = \vec{E}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}$ and $\vec{H} = \vec{H}_0 e^{i(\vec{K}\cdot\vec{r}-\omega t)}$ are possible solutions of Maxwell's electromagnetic wave equation, where symbols have their usual significance. [See Article 5.7]

9. Derive the differential form of Faraday's law of electromagnetic induction. [See Article 5.2.3]

10. Write down the Maxwell's wave equation for electric field and magnetic field for a conducting ionised medium, specifying each term. [See Article 5.9.1]

11. Write down the Maxwell's wave equations for a charge free non-conducting medium. Hence, prove that the speed of electromagnetic wave in a non-conducting medium is less than the velocity of light in vacuum. [See Article 5.9.2]

12. Define skin depth. The skin depth decreases with increase in frequency (ω) of electromagnetic wave and the conductivity (σ) of the medium. Explain it. [See Article 5.10.1]

13. What is the physical significance of skin depth? [See Article 5.10.2]

14. Establish the relation between phase velocity and propagation constant of an electromagnetic wave in vacuum. [See Article 5.10.3]

15. Prove that electromagnetic wave attenuates as it propagates through a conducting medium. [See Article 5.10]

16. What is Poynting vector? Give its importance. [See Article 5.11]

17. Write the Maxwell electromagnetic equation which follows from the non-existence of isolated magnetic pole.

Long Answer Type Questions

1. **[a]** What is electromagnetic induction ? [See Article 5.2.1]

[b] State Faraday's law and find its differential form. [See Article 5.2.2 and 5.2.3]

[c] What is the significance of Faraday's law ? [See Article 5.5.1]

2. **[a]** Define electromagnetic wave. Can accelerated charged particles produce electromagnetic waves ? Give reasons. [See Article 5.1]

[b] Define displacement current. [See Article 5.3]

[c] An ac voltage is connected across an ideal parallel plate capacitor. Prove that the displacement current at the capacitor is equal to the conduction current through the wire.

[d] Give the differences between displacement and conduction current. [See Article 5.4]

3. **[a]** Write down Maxwell's field equations. [See Article 5.5]

[b] From these equations identify Gauss' law, Ampere's law and Faraday's law. [See Article 5.5]

[c] Prove that $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$, where the symbols have their usual meanings. [See Article 5.6.3] **[WBUT 2013]**

[d] Prove that $\vec{E} = \cos(y-t)\,\hat{k}$ and $\vec{B} = \cos(y-t)\,\hat{i}$ constitute a possible electromagnetic field. **[WBUT 2014]**

4. **[a]** Write down the Maxwell's wave equation for electric field and magnetic field. Find its solutions and prove that the electric field and the magnetic field vector are perpendicular to the direction of propagation of the electromagnetic wave.

or,

Prove that the electromagnetic waves are transverse in nature. [See Article 5.8]

[b] Prove that in an electromagnetic wave magnetic field is always orthogonal to the electric field. [See Article 5.8]

[c] Derive a relation between magnitudes of electric vector and magnetic vector. [See Article 5.8.1]

5. [a] Starting from Maxwell's wave equation, obtain the wave equation for electric field $\vec{E}$ and magnetic field $\vec{H}$ in conducting medium. Identify the dissipative term in the equation. [See Article 5.9.1]

[b] Prove that the speed of an electromagnetic wave in a non-conducting medium is less than the velocity of light in vacuum. [See Article 5.9.2]

6. [a] What is skin depth? Deduce the expression of skin depth and indicate the factors on which it depends. [See Article 5.10.1]

[b] What are the physical significances of skin depth? [See Article 5.10.2]

7. [a] State and prove Poynting's theorem. How does it describe the conservation of energy in electromagnetic field? [See Article 5.11] **[WBUT 2013]**

[b] Show the average value of Poynting's vector for a plane electromagnetic wave is $\frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}H_0^2$. [See Article 5.11.2] **[WBUT 2013]**

8. Write down the Maxwell's equations of an electromagnetic field. Hence, obtain the wave equation for electric field in free space.

[See Article 5.5 and 5.7] **[WBUT 2012]**

Numerical Problems

1. An electromagnetic wave is propagating through a medium in such a manner that electric vector of the wave satisfies the differential wave equation $\nabla^2 E = \mu\varepsilon\frac{\partial^2 E}{\partial t^2}$. How much energy will be absorbed by the medium in 20 seconds?

Hint: The equation represents electromagnetic wave equation for charge free non-conducting medium.
Hence the energy absorbed by the medium is zero.