## 1.13 EIGENVALUES AND EIGENVECTORS

Eigenvalues and eigenvectors are important concepts in linear algebra. They are derived from the German word '*eigen*' which means proper or characteristic. Eigenvectors are nonzero vectors that get mapped into scalar multiples of themselves under a linear operator.

Any nonzero vector **x** is said to be a characteristic vector or eigenvector of a square matrix $A$ if there exists a number $\lambda$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

where $A = [a_{ij}]_{n\times n}$ is an $n$-rowed square matrix and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a nonzero column vector.

Also, $\lambda$ is said to be a characteristic root or a characteristic value or an eigenvalue of the matrix $A$.

Depending on the sign and the magnitude of the eigenvalue $\lambda$ corresponding to **x**, the linear operator $A\mathbf{x} = \lambda\mathbf{x}$ compresses or stretches the eigenvector **x** by a factor $\lambda$. If $\lambda$ is negative, the direction of the eigenvector reverses (Fig. 1.1).

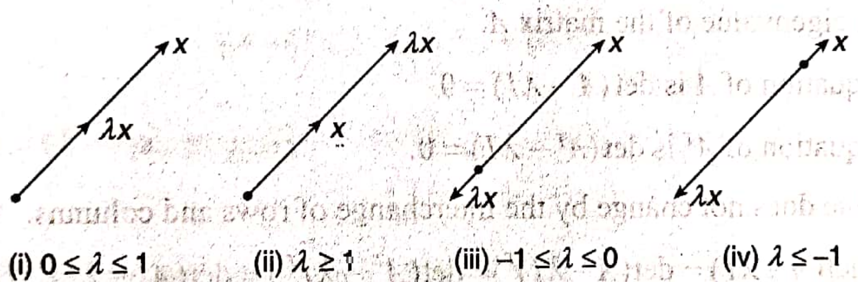


**Fig. 1.1** Geometrical interpretation of eigenvalues and eigenvectors

Now,

$$A\mathbf{x} = \lambda\mathbf{x} = \lambda I\mathbf{x}$$
$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The matrix $A - \lambda I$ is called the *characteristic matrix* of $A$, where $I$ is the unit matrix of order $n$.

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} \qquad \ldots(1.1)$$

which is an ordinary polynomial in $\lambda$ of degree $n$ is called the *characteristic polynomial* of $A$.

The equation $\det(A - \lambda I) = 0$ is called the *characteristic equation* of $A$ and the roots of this equation are called the eigenvalues of the matrix $A$. The set of all eigenvectors is called the *eigenspace* of $A$ corresponding to $\lambda$. The set of all eigenvalues of $A$ is called the *spectrum* of $A$.

**Notes**

(i) The characteristic equation of the matrix $A$ of order 2 can be obtained from

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where $S_1$ = sum of principal diagonal elements

$S_2$ = determinant $A$

(ii) The characteristic equation of the matrix $A$ of order 3 can be obtained from

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of principal diagonal elements

$S_2$ = sum of minors of principal diagonal elements

$S_3$ = determinant $A$

(iii) The sum of the eigenvalues of a matrix is the sum of its principal diagonal elements.

(iv) The product of the eigenvalues of a matrix is the determinant of the matrix.

### 1.13.1 Properties of Eigenvalues

- **Property 1** If $\lambda$ is an eigenvalue of the matrix $A$ then $\lambda$ is also an eigenvalue of $A^T$.

***Proof*** Let $\lambda$ be an eigenvalue of the matrix $A$.

The characteristic equation of $A$ is $\det(A - \lambda I) = 0$.

The characteristic equation of $A^T$ is $\det(A^T - \lambda I) = 0$.

The determinant value does not change by the interchange of rows and columns.

$$\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I^T) = \det(A^T - \lambda I) \quad \text{...(1.2)}$$

The characteristic equations are same for both $A$ and $A^T$.

Hence, $\lambda$ is also an eigenvalue of $A^T$.

- **Property 2** If $\lambda$ is an eigenvalue of the nonsingular matrix $A$ then $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

***Proof*** Let $\lambda$ be an eigenvalue of the nonsingular matrix $A$.

$$A\mathbf{x} = \lambda\mathbf{x} \quad \text{...(1.3)}$$

Premultiplying both sides of Eq. (1.3) by $A^{-1}$,

$$A^{-1}A\mathbf{x} = A^{-1}\lambda\mathbf{x}$$

$$I\mathbf{x} = \lambda A^{-1}\mathbf{x}$$

$$\mathbf{x} = \lambda A^{-1}\mathbf{x}$$

$$A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$$

Hence, $\frac{1}{\lambda}$ is an eigenvalue of $A^{-1}$.

■ **Property 3** If $\lambda$ is an eigenvalue of the matrix $A$ then $\lambda^k$ is an eigenvalue of $A^k$.

***Proof*** Let $\lambda$ be an eigenvalue of the matrix $A$.

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{...(1.4)}$$

Premultiplying both sides of Eq. (1.4) by $A$,

$$AA\mathbf{x} = A\lambda\mathbf{x}$$
$$A^2\mathbf{x} = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$$

Similarly, $A^3\mathbf{x} = \lambda^3\mathbf{x}$. In general, $A^k\mathbf{x} = \lambda^k\mathbf{x}$. Hence, $\lambda^k$ is an eigenvalue of $A^k$.

■ **Property 4** If $\lambda$ is an eigenvalue of the matrix $A$ then $\lambda \pm k$ is an eigenvalue of $A \pm kI$.

***Proof*** Let $\lambda$ be an eigenvalue of the matrix $A$.

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{...(1.5)}$$

Adding $kI\mathbf{x}$ on both sides of Eq. (1.5),

$$A\mathbf{x} + kI\mathbf{x} = \lambda\mathbf{x} + kI\mathbf{x}$$
$$(A + kI)\mathbf{x} = \lambda\mathbf{x} + k\mathbf{x} = (\lambda + k)\mathbf{x}$$

Similarly, $(A - kI)\mathbf{x} = (\lambda - k)\mathbf{x}$. In general, $(A \pm kI)x = (\lambda \pm k)x$. Hence, $\lambda \pm k$ is an eigenvalue of $A \pm kI$.

■ **Property 5** If $\lambda$ is an eigenvalue of the matrix $A$ then $k\lambda$ is an eigenvalue of $kA$, where $k$ is a nonzero scalar.

***Proof*** Let $\lambda$ be an eigenvalue of the matrix $A$.

$$A\mathbf{x} = \lambda\mathbf{x}$$

Multiplying both sides of Eq. (1.5) by the nonzero scalar $k$,

$$kA\mathbf{x} = k\lambda\mathbf{x}$$

Hence, $k\lambda$ is an eigenvalue of $kA$.

■ **Property 6** The eigenvalues of a triangular matrix are the diagonal elements of the matrix.

***Proof*** Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ be a upper triangular matrix of order $n$.

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn}-\lambda \end{bmatrix}=0$$

$$(a_{11}-\lambda)(a_{22}-\lambda)\dots(a_{nn}-\lambda)=0$$

$$\lambda=a_{11}, a_{22}, \dots, a_{nn}$$

Hence, the eigenvalues of $A$ are $a_{11}, a_{22}, \dots, a_{nn}$ which are the diagonal elements of the matrix.

■ **Property 7** The eigenvalues of a real symmetric matrix are real.

***Proof*** Let $\lambda$ be an eigenvalue of the real symmetric matrix.

$$A\mathbf{x}=\lambda\mathbf{x} \qquad \text{...(1.6)}$$

Premultiplying both sides of Eq. (1.6) by $\bar{\mathbf{x}}^T$,

$$\bar{\mathbf{x}}^T A\mathbf{x}=\lambda\bar{\mathbf{x}}^T\mathbf{x} \qquad \text{...(1.7)}$$

Taking complex conjugate on both sides of Eq. (1.7),

$$\mathbf{x}^T \bar{A}\bar{\mathbf{x}}=\bar{\lambda}\mathbf{x}^T\bar{\mathbf{x}}$$

$$\mathbf{x}^T A\bar{\mathbf{x}}=\bar{\lambda}\mathbf{x}^T\bar{\mathbf{x}} \quad (\because \bar{A}=A \text{ for real matrix}) \qquad \text{...(1.8)}$$

Taking transpose on both sides of Eq. (1.8),

$$\bar{\mathbf{x}}^T A^T\mathbf{x}=\bar{\lambda}\bar{\mathbf{x}}^T\mathbf{x}$$

$$\bar{\mathbf{x}}^T A\mathbf{x}=\bar{\lambda}\bar{\mathbf{x}}^T\mathbf{x} \quad (\because A^T=A \text{ for symmetric matrix}) \qquad \text{...(1.9)}$$

From Eqs (1.7) and (1.9),

$$\lambda\bar{\mathbf{x}}^T\mathbf{x}=\bar{\lambda}\bar{\mathbf{x}}^T\mathbf{x}$$

$$(\lambda-\bar{\lambda})\bar{\mathbf{x}}^T\mathbf{x}=0$$

$\bar{\mathbf{x}}^T\mathbf{x}$ is a $1\times 1$ matrix, i.e., a single element which is positive,

$$\lambda-\bar{\lambda}=0$$

i.e., $\lambda$ is real.

Hence, the eigenvalues of a real symmetric matrix are real.

**EXAMPLE 1.36** *Form the matrix whose eigenvalues are $\alpha-5, \beta-5, \gamma-5$, where $\alpha, \beta, \gamma$ are the eigenvalues of*

$$A=\begin{bmatrix}-1 & -2 & -3\\ 4 & 5 & -6\\ 7 & -8 & 9\end{bmatrix}.$$

**Solution:** If $\lambda_1, \lambda_2$, and $\lambda_3$ are eigenvalues of the matrix $A$ then $\lambda_1-k, \lambda_2-k$, and $\lambda_3-k$ are the eigenvalues of $A-kI$.

$$\text{Required matrix } = A-5I=\begin{bmatrix}-1 & -2 & -3\\ 4 & 5 & -6\\ 7 & -8 & 9\end{bmatrix}-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-6 & -2 & -3\\ 4 & 0 & -6\\ 7 & -8 & 4\end{bmatrix}$$

**EXAMPLE 1.37** *If $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$, find the eigenvalues for the following matrices:*

*(i) $A$ (ii) $A^T$ (iii) $A^{-1}$ (iv) $4A^{-1}$ (v) $A^2$ (vi) $A^2-2A+I$ (vii) $A^3+2I$*

**Solution:**

$$A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}3-\lambda & -1 & 1\\ -1 & 5-\lambda & -1\\ 1 & -1 & 3-\lambda\end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where $S_1$ = sum of the principal diagonal elements of $A=3+5+3=11$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$=\begin{vmatrix}5 & -1\\ -1 & 3\end{vmatrix}+\begin{vmatrix}3 & 1\\ 1 & 3\end{vmatrix}+\begin{vmatrix}3 & -1\\ -1 & 5\end{vmatrix}=(15-1)+(9-1)+(15-1)=14+8+14=36$$

$$S_3=\det(A)=\begin{vmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{vmatrix}=3(15-1)+1(-3+1)+1(1-5)=42-2-4=36$$

Hence, the characteristic equation is

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 2, 3, 6$$

| | | | |
|---|---|---|---|
| (i) | Eigenvalues of $A = \lambda$ | : | 2, 3, 6 |
| (ii) | Eigenvalues of $A^T = \lambda$ | : | 2, 3, 6 |
| (iii) | Eigenvalues of $A^{-1} = \lambda^{-1}$ | : | $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ |
| (iv) | Eigenvalues of $4A^{-1} = 4\lambda^{-1}$ | : | $2, \frac{4}{3}, \frac{2}{3}$ |
| (v) | Eigenvalues of $A^2 = \lambda^2$ | : | 4, 9, 36 |
| (vi) | Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$ | : | 1, 4, 25 |
| (vii) | Eigenvalues of $A^3 + 2I = \lambda^3 + 2$ | : | 10, 29, 218 |

## 1.13.2 Properties of Eigenvectors

- **Property 1** If $\mathbf{x}$ is an eigenvector of a matrix $A$ corresponding to the eigenvalue $\lambda$ then $k\mathbf{x}$ is also an eigenvector of $A$ corresponding to the same eigenvalue $\lambda$, where $k$ is a nonzero scalar.

***Proof*** Let $\mathbf{x}$ be an eigenvector of a matrix $A$ corresponding to the eigenvalue $\lambda$.

$$A\mathbf{x} = \lambda\mathbf{x}$$

and

$$\mathbf{x} \neq \mathbf{0}$$

If $k$ is any nonzero scalar then $k\mathbf{x} \neq \mathbf{0}$.

Also, $A(k\mathbf{x}) = k(A\mathbf{x}) = k(\lambda\mathbf{x}) = \lambda(k\mathbf{x})$

Hence, $k\mathbf{x}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. Thus, corresponding to an eigenvalue $\lambda$, there is more than one eigenvector.

- **Property 2** If $\mathbf{x}$ is an eigenvector of a matrix $A$ then $\mathbf{x}$ cannot correspond to more than one eigenvalue of $A$.

***Proof*** Let $\mathbf{x}$ be an eigenvector of a matrix $A$ corresponding to two eigenvalues $\lambda_1$ and $\lambda_2$.

$$A\mathbf{x} = \lambda_1\mathbf{x}$$

and

$$A\mathbf{x} = \lambda_2\mathbf{x}$$

$$\therefore \quad \lambda_1\mathbf{x} = \lambda_2\mathbf{x}$$

$$(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$$

$$\lambda_1 - \lambda_2 = 0 \qquad (\because \mathbf{x} \neq \mathbf{0})$$

$$\lambda_1 = \lambda_2$$

... cannot correspond to more than one eigenvalue of $A$.

■ **Property 3** The eigenvectors corresponding to distinct eigenvalues of a matrix are linearly independent.

***Proof*** Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be the eigenvectors of a matrix $A$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$.

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i, \, i = 1, 2, \ldots, m$$

Let the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ be linearly dependent. $r$ is chosen such that the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ are linearly independent but $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r, \mathbf{x}_{r+1}$ are linearly dependent, where $1 \le r < m$.

$$k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r + k_{r+1}\mathbf{x}_{r+1} = \mathbf{0} \quad \text{...(1.10)}$$

where $k_1, k_2, \ldots, k_r, k_{r+1}$ are scalars not all zero.

$$A(k_1\mathbf{x}_1 + k_2\mathbf{x}_2 + \cdots + k_r\mathbf{x}_r + k_{r+1}\mathbf{x}_{r+1}) = \mathbf{0}$$

$$k_1(A\mathbf{x}_1) + k_2(A\mathbf{x}_2) + \cdots + k_r(A\mathbf{x}_r) + k_{r+1}(A\mathbf{x}_{r+1}) = \mathbf{0}$$

$$k_1(\lambda_1\mathbf{x}_1) + k_2(\lambda_2\mathbf{x}_2) + \cdots + k_r(\lambda_r\mathbf{x}_r) + k_{r+1}(\lambda_{r+1}\mathbf{x}_{r+1}) = \mathbf{0} \quad \text{...(1.11)}$$

Multiplying Eq. (1.10) by $\lambda_{r+1}$ and subtracting from Eq. (1.11),

$$k_1(\lambda_1 - \lambda_{r+1})\mathbf{x}_1 + k_2(\lambda_2 - \lambda_{r+1})\mathbf{x}_2 + \cdots + k_r(\lambda_r - \lambda_{r+1})\mathbf{x}_r = \mathbf{0} \quad \text{...(1.12)}$$

Since $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ are linearly independent,

$$k_1(\lambda_1 - \lambda_{r+1}) = 0, \, k_2(\lambda_2 - \lambda_{r+1}) = 0, \ldots \, k_r(\lambda_r - \lambda_{r+1}) = 0$$

But $\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}$ are distinct.

$$\therefore \, k_1 = 0, \, k_2 = 0, \ldots, \, k_r = 0$$

Putting in Eq. (1.10),

$$k_{r+1}\mathbf{x}_{r+1} = \mathbf{0}$$

$$k_{r+1} = 0 \quad [\because \; \mathbf{x}_{r+1} \neq \mathbf{0}]$$

Thus, for Eq. (1.10),

$$k_1 = 0, \, k_2 = 0, \quad k_r = 0, \, k_{r+1} = 0$$

which contradicts our assumption that $k_1, k_2, \ldots, k_r, k_{r+1}$ are not all zero.

Thus, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{r+1}$ are linearly independent and, hence, $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are linearly independent.

***Note*** If two or more eigenvalues are equal then the corresponding eigenvectors may or may not be linearly independent.

■ **Property 4** The eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.

***Proof*** Let $\lambda_1$ and $\lambda_2$ be the two distinct eigenvalues of a real symmetric matrix $A$, and $\mathbf{x}_1$ and $\mathbf{x}_2$ be the corresponding eigenvectors respectively.

$$A\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \quad \text{...(1.13)}$$

and

$$A\mathbf{x}_2 = \lambda_2\mathbf{x}_2 \quad \text{...(1.14)}$$

Premultiplying both sides of Eq. (1.13) by $\mathbf{x}_2^T$,

$$\mathbf{x}_2^T A\mathbf{x}_1 = \lambda_1\mathbf{x}_2^T\mathbf{x}_1$$

Taking the transpose on both sides,

$$\mathbf{x}_1^T A\mathbf{x}_2 = \lambda_1\mathbf{x}_1^T\mathbf{x}_2 \quad (\because A^T = A) \quad \text{...(1.15)}$$

Premultiplying both sides of Eq. (1.14) by $\mathbf{x}_1^T$,

$$\mathbf{x}_1^T A\mathbf{x}_2 = \lambda_2\mathbf{x}_1^T\mathbf{x}_2 \quad \text{...(1.16)}$$

From Eqs (1.15) and (1.16),

$$\lambda_1\mathbf{x}_1^T\mathbf{x}_2 = \lambda_2\mathbf{x}_1^T\mathbf{x}_2$$

$$(\lambda_1 - \lambda_2)\mathbf{x}_1^T\mathbf{x}_2 = 0$$

$$\mathbf{x}_1^T\mathbf{x}_2 = 0 \quad (\because \lambda_1 \neq \lambda_2)$$

Hence, the eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal.

### 1.13.3 Working Rule for Finding Eigenvalues and Eigenvectors

(i) Write the characteristic equation det $(A - \lambda I) = 0$ for the given square matrix.

(ii) Find the eigenvalues of the matrix by solving the characteristic equation.

(iii) Find the eigenvectors corresponding to each eigenvalue from the equation $[A - \lambda I]\,\mathbf{x} = \mathbf{0}$.

**EXAMPLE 1.38** *Find the eigenvalues and eigenvectors of the matrix* $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$.

**Solution:** Let

$$A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = 4+3-3=4$
$S_2$ = sum of the minors of principal diagonal elements of $A$

$$= \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix} = (-9+8)+(-12+6)+(12-6) = -1-6+6=-1$$

$$S_3 = \det(A) = \begin{vmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{vmatrix} = 4(-9+8)-6(-3+2)+6(-4+3) = -4+6-6=-4$$

Hence, the characteristic equation is

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\lambda = -1, 1, 4$$

(i) For $\lambda=-1$,

$$[A-\lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x+6y+6z=0$$

$$x+4y+2z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ 4 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix}} = t, \text{ say}$$

$$\frac{x}{-12} = \frac{y}{-4} = \frac{z}{14} = t$$

$$\frac{x}{-6} = \frac{y}{-2} = \frac{z}{7} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6t \\ -2t \\ 7t \end{bmatrix} = t\begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix} = t\mathbf{x}_1, \text{ where } \mathbf{x}_1 \text{ is an eigenvector corresponding to } \lambda=-1.$$

(ii) For $\lambda=1$,

$$[A-\lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x+2y+2z=0$$

$$-x-4y-4z=0$$

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By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 2 \\ -4 & -4 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix}} = t, \text{ say}$$

$$\frac{x}{0} = \frac{y}{2} = \frac{z}{-2} = t$$

$$\frac{x}{0} = \frac{y}{1} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = t\mathbf{x}_2, \text{ where } \mathbf{x}_2 \text{ is an eigenvector corresponding to } \lambda = 1.$$

(iii) For $\lambda = 4$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0x + 6y + 6z = 0$$

$$x - y + 2z = 0$$

$$x - 4y - 7z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 6 & 6 \\ -1 & 2 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & 6 \\ 1 & -1 \end{vmatrix}} = t, \text{ say}$$

$$\frac{x}{18} = \frac{y}{6} = \frac{z}{-6} = t$$

$$\frac{x}{3} = \frac{y}{1} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t \\ t \\ -t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = t\mathbf{x}_3, \text{ where } \mathbf{x}_3 \text{ is an eigenvector corresponding to } \lambda = 4.$$

**EXAMPLE 1.39** *Find the eigenvalues and eigenvectors of the matrix* $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

**Solution:** Let

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where $S_1$ = sum of the principal diagonal elements of $A=-2+1+0=-1$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$=\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix}+\begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix}+\begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}=(0-12)+(0-3)+(-2-4)=-12-3-6=-21$$

$$S_3=\det(A)=\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}=(-2)(0-12)-2(0-6)-3(-4+1)=24+12+9=45$$

Hence, the characteristic equation is

$$\lambda^3+\lambda^2-21\lambda-45=0$$

$$\lambda=5,-3,-3$$

(i) For $\lambda=5$, $\quad [A-\lambda I]\mathbf{x}=0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x+2y-3z=0$$
$$2x-4y-6z=0$$
$$-x-2y-5z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix}}=\frac{z}{\begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix}}=t\text{, say}$$

$$\frac{x}{-24}=\frac{y}{-48}=\frac{z}{24}=t$$

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}=t$$

$$\mathbf{x}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} t \\ 2t \\ -t \end{bmatrix}=t\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}=t\mathbf{x}_1\text{, where } \mathbf{x}_1 \text{ is an eigenvector corresponding to } \lambda=5.$$

(ii) For $\lambda = -3$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y - 3z = 0$$

Let

$$y = t_1 \text{ and } z = t_2$$

$$x = -2t_1 + 3t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 + 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = -3$.

**EXAMPLE 1.40** *Find the eigenvalues and eigenvectors of the matrix* $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$.

**Solution:** Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & -3 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = 0 + 0 + 3 = 3$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$= \begin{vmatrix} 0 & 1 \\ -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = (0+3) + (0) + (0) = 3$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{vmatrix} = 0 - 1(0-1) + 0 = 1$$

Hence, the characteristic equation is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$$

$$\lambda = 1, 1, 1$$

For $\lambda = 1$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x + y + 0z = 0$$

$$0x - y + z = 0$$

$$x - 3y + 2z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}} = t, \text{ say}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = t\mathbf{x}_1, \text{ where } \mathbf{x}_1 \text{ is an eigenvector corresponding to } \lambda = 1.$$

Hence, there is only one eigenvector corresponding to the repeated root $\lambda = 1$.

**EXAMPLE 1.41** *Find the values of $\mu$ which satisfy the equation $A^{100}\mathbf{x} = \mu\mathbf{x}$*

*where* $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$.

**Solution:**

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & -2-\lambda & -2 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = 2 - 2 + 0 = 0$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$= \begin{vmatrix} -2 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} = (0+2)+(0+1)+(-4-0) = -1$$

$$S_3 = \det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 0 & -2 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2(0+2) - 1(0+2) - 1(0+2) = 4 - 2 - 2 = 0$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda = 0$$

$$\lambda = 0, 1, -1$$

If $\lambda$ is an eigenvalue of $A$, it satisfies the equation $A\mathbf{x} = \lambda\mathbf{x}$.

For the equation $A^{100}\mathbf{x} = \mu\mathbf{x}$, $\mu$ represents the eigenvalues of $A^{100}$. Eigenvalues of $A^{100} = \lambda^{100}$, i.e., 0, 1, 1.

Hence, values of $\mu$ are 0, 1, 1.

**EXAMPLE 1.42** *Find the characteristic roots and characteristic vectors of* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *and verify that characteristic roots are of unit modulus.*

**Solution:**

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$(\cos\theta - \lambda)^2 = -\sin^2\theta$$

$$\cos\theta - \lambda = \pm i\sin\theta$$

$$\therefore \lambda = \cos\theta \pm i\sin\theta$$

$$|\lambda| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Hence, the characteristic roots are of unit modules.

(i) For $\lambda = \cos\theta + i\sin\theta$, $\quad [A - \lambda I]\mathbf{x} = 0$

$$\begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-i\sin\theta x - \sin\theta y = 0$$

Let

$$y = t$$
$$x = it$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix} = t\mathbf{x}_1,$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = \cos\theta + i\sin\theta$.

(ii) For $\lambda = \cos\theta - i\sin\theta$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$i\sin\theta x - \sin\theta y = 0$$

Let

$$y = t$$
$$x = -it$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t\begin{bmatrix} -i \\ 1 \end{bmatrix} = t\mathbf{x}_2,$$ where $\mathbf{x}_2$ is an eigenvector corresponding to $\lambda = \cos\theta - i\sin\theta$.

**EXAMPLE 1.43** *Find orthogonal eigenvectors for the matrix* $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

**Solution:** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = 1 + 4 + 9 = 14$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$= \begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (36-36) + (9-9) + (4-4) = 0$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 1(36-36) - 2(18-18) + 3(12-12) = 0$$

Hence, the characteristic equation is

$$\lambda^3 - 14\lambda^2 = 0$$

$$\lambda = 0, 0, 14$$

(i) For $\lambda = 14$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-13x + 2y + 3z = 0$$

$$2x - 10y + 6z = 0$$

$$3x + 6y - 5z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 2 & 3 \\ -10 & 6 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -13 & 3 \\ 2 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -13 & 2 \\ 2 & -10 \end{vmatrix}} = t, \text{ say}$$

$$\frac{x}{42} = \frac{y}{84} = \frac{z}{126} = t$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ 2t \\ 3t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = t\mathbf{x}_1,$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda = 14$.

(ii) For $\lambda = 0$,

$$[A - \lambda I]\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y + 3z = 0$$

Let

$$y = t_1 \text{ and } z = t_2$$

$$x = -2t_1 - 3t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t_1 - 3t_2 \\ t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} -2t_1 \\ t_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t_2 \\ 0 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3$$

where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda = 0$.
Since $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal, $\mathbf{x}_3$ is chosen such that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are orthogonal.

Let $$\mathbf{x}_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

For $\mathbf{x}_1$ and $\mathbf{x}_3$ to be orthogonal, $$\mathbf{x}_1^T \mathbf{x}_3 = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$l + 2m + 3n = 0 \qquad \ldots(1)$$

For $\mathbf{x}_2$ and $\mathbf{x}_3$ to be orthogonal, $$\mathbf{x}_2^T \mathbf{x}_3 = 0$$

$$\begin{bmatrix} -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$-2l + m = 0 \qquad \ldots(2)$$

Solving Eqs (1) and (2) by Cramer's rule,

$$\frac{l}{\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}} = t, \text{ say}$$

$$\frac{l}{-3} = \frac{m}{-6} = \frac{n}{5} = t$$

$$\frac{l}{3} = \frac{m}{6} = \frac{n}{-5} = t$$

$$\mathbf{x} = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 3t \\ 6t \\ -5t \end{bmatrix} = t \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} = t\mathbf{x}_3,$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 0$.

## EXERCISE 1.8

1. Find the eigenvalues and eigenvectors for the following matrices:

(i) $\begin{bmatrix} 9 & -1 & 9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$

(ii) $\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

(vii) $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

(viii) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

**Ans.:** (i) $-1, 0, 2; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}$

(ii) $-1, -2, -3; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

(iii) $1, 1, 7; \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(iv) $5, 1, 1; \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(v) $1, 2, 2; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(vi) $3, 2, 2; \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$

(vii) $1, 1, 1; \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$

(viii) $2, 2, 2; \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

**2.** If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigenvalues of the following matrices:

(i) $A^3 + I$

(ii) $A^{-1}$

(iii) $A^2 - 2A + I$

(iv) $A^3 - 3A^2 + A$

**Ans.:** (i) 2, 2, 126 (ii) $1, 1, \frac{1}{5}$

(iii) 0, 0, 16

(iv) $-1, -1, 55$

**3.** Verify that $\mathbf{x} = [2, 3, -2, -3]^T$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$ of the matrix

$$A = \begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}.$$

4. If $A=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ then check whether the eigenvectors of $A$ are orthogonal.

5. If $A=\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ then verify whether the eigenvectors of $A$ are linearly independent or not.

## 1.14 CAYLEY-HAMILTON THEOREM

- **Statement** Every square matrix satisfies its own characteristic equation.

***Proof*** Let $A$ be an $n$-rowed square matrix. Its characteristic equation is

$$|A-\lambda I|=(-1)^n(\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_n)$$

$$(A-\lambda I)\ \mathrm{adj}\,(A-\lambda I)=|A-\lambda I|I \qquad \ldots(1.17)$$

$$[\because A\ \mathrm{adj}(A)=|A|I]$$

Since adj $(A-\lambda I)$ has elements as cofactors of elements of $|A-\lambda I|$, the elements of adj $(A-\lambda I)$ are polynomials in $\lambda$ of degree $n-1$ or less. Hence, adj $(A-\lambda I)$ can be written as a matrix polynomial in $\lambda$.

$$\mathrm{adj}\,(A-\lambda I)=B_0\lambda^{n-1}+B_1\lambda^{n-2}+\cdots+B_{n-2}\lambda+B_{n-1}$$

where $B_0, B_1, \ldots, B_{n-1}$ are matrices of order $n$.

$$(A-\lambda I)\ \mathrm{adj}(A-\lambda I)=(A-\lambda I)[B_0\lambda^{n-1}+B_1\lambda^{n-2}+\cdots+B_{n-2}\lambda+B_{n-1}]$$

$$|A-\lambda I|I=(A-\lambda I)[B_0\lambda^{n-1}+B_1\lambda^{n-2}+\cdots+B_{n-2}\lambda+B_{n-1}]$$

$$(-1)^n[I\lambda^n+a_1I\lambda^{n-1}+a_2I\lambda^{n-2}+\cdots+a_{n-1}I\lambda+a_nI]$$

$$=(-IB_0)\lambda^n+(AB_0-IB_1)\lambda^{n-1}+(AB_1-IB_2)\lambda^{n-2}+\cdots+(AB_{n-2}-IB_{n-1})\lambda+AB_{n-1}$$

Equating corresponding coefficients,

$$-IB_0=(-1)^nI$$

$$AB_0-IB_1=(-1)^na_1I$$

$$AB_1-IB_2=(-1)^na_2I$$

$$\vdots \qquad \vdots$$

$$AB_{n-2}-IB_{n-1}=(-1)^na_{n-1}I$$

$$AB_{n-1}=(-1)^na_nI$$

Premultiplying the above equations successively by $A^n, A^{n-1}, A^{n-2}, \ldots I$ and adding,

$$(-1)^n[A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI] = 0$$

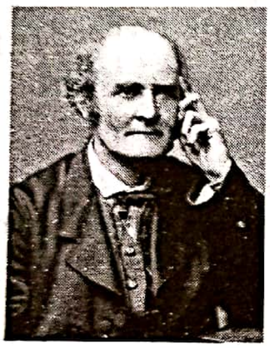
Hence,
$$A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_nI = 0 \qquad \dots(1.18)$$

■ **Corollary** If $A$ is a nonsingular matrix, i.e., det $(A) \neq 0$ then premultiplying Eq. (1.18) by $A^{-1}$,

$$A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \dots + a_nA^{-1} = 0$$

$$A^{-1} = -\frac{1}{a^n}[A^{n-1} + a_1A^{n-2} + \dots + a_{n-1}I]$$

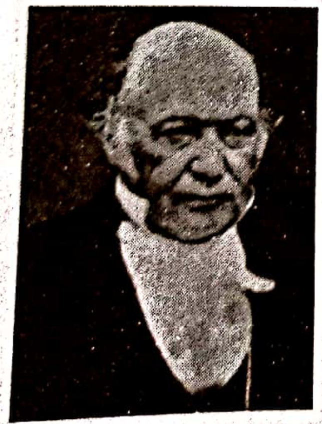
## HISTORICAL DATA



**Arthur Cayley** (1821–1895) was a British mathematician. He helped found the modern British school of pure mathematics. As a child, Cayley enjoyed solving complex mathematics problems for amusement. He entered Trinity College, Cambridge, where he excelled in Greek, French, German, and Italian, as well as mathematics. He worked as a lawyer for 14 years.

He postulated the Cayley–Hamilton theorem—that every square matrix is a root of its own characteristic polynomial, and verified it for matrices of order 2 and 3. He was the first to define the concept of a group in the modern way—as a set with a binary operation satisfying certain laws. Formerly, when mathematicians spoke of "groups", they had meant permutation groups.

## HISTORICAL DATA



**Sir William Rowan Hamilton** (1805–1865) was an Irish physicist, astronomer, and mathematician, who made important contributions to classical mechanics, optics, and algebra. His studies of mechanical and optical systems led him to discover new mathematical concepts and techniques. His greatest contribution is perhaps the reformulation of Newtonian mechanics, now called Hamiltonian mechanics. This work has proven central to the modern study of classical field theories such as electromagnetism, and to the development of quantum mechanics. In mathematics, he is perhaps best known as the inventor of quaternions.

Hamilton is said to have shown immense talent at a very early age. Astronomer Bishop Dr John Brinkley remarked of the 18-year-old Hamilton, "This young man, I do not say will be, but is, the first mathematician of his age."

**EXAMPLE 1.44** *Verify the Cayley–Hamilton theorem for the matrix* $A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$ *and, hence, find* $A^{-1}$ *and* $A^4$.

**Solution:**

$$A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix}2-\lambda&-1&1\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where $S_1$ = sum of the principal diagonal elements of $A=2+2+2=6$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$=\begin{vmatrix}2&-1\\-1&2\end{vmatrix}+\begin{vmatrix}2&1\\1&2\end{vmatrix}+\begin{vmatrix}2&-1\\-1&2\end{vmatrix}=(4-1)+(4-1)+(4-1)=9$$

$$S_3=\det(A)=\begin{vmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{vmatrix}=2(4-1)+1(-2+1)+1(1-2)=6-1-1=4$$

Hence, the characteristic equation is

$$\lambda^3-6\lambda^2+9\lambda-4=0$$

$$A^2=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}=\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}$$

$$A^3=\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}=\begin{bmatrix}22&-21&21\\-21&22&-21\\21&-21&22\end{bmatrix}$$

$A^3-6A^2+9A-4I$

$$=\begin{bmatrix}22&-21&21\\-21&22&-21\\21&-21&22\end{bmatrix}-\begin{bmatrix}36&-30&30\\-30&36&-30\\30&-30&36\end{bmatrix}+\begin{bmatrix}18&-9&9\\-9&18&-9\\9&-9&18\end{bmatrix}-\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=0 \quad \text{...(1)}$$

The matrix $A$ satisfies its own characteristic equation. Hence, the Cayley–Hamilton theorem is verified.

Premultiplying Eq. (1) by $A^{-1}$,

$$A^{-1}(A^3 - 6A^2 + 9A - 4I) = \mathbf{0}$$

$$A^2 - 6A + 9I - 4A^{-1} = \mathbf{0}$$

$$4A^{-1} = (A^2 - 6A + 9I) = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Multiplying Eq. (1) by $A$,

$$A(A^3 - 6A^2 + 9A - 4I) = \mathbf{0}$$

$$A^4 - 6A^3 + 9A^2 - 4A = \mathbf{0}$$

$$A^4 = 6A^3 - 9A^2 + 4A$$

$$= \begin{bmatrix} 132 & -126 & 126 \\ -126 & 132 & -126 \\ 126 & -126 & 132 \end{bmatrix} - \begin{bmatrix} 54 & -45 & 45 \\ -45 & 54 & -45 \\ 45 & -45 & 54 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 4 \\ -4 & 8 & -4 \\ 4 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 86 & -85 & 85 \\ -85 & 86 & -85 \\ 85 & -85 & 86 \end{bmatrix}$$

**EXAMPLE 1.45** *Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify the Cayley–Hamilton theorem for this matrix. Find $A^{-1}$ and also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in $A$.*

**Solution:**

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = 1 + 3 = 4$

$$S_2 = \det(A) = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$$

Hence, the characteristic equation is

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda = -1, 5$$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0} \quad \ldots(1)$$

The matrix $A$ satisfies its own characteristic equation. Hence, the Cayley–Hamilton theorem is verified. Premultiplying Eq. (1) by $A^{-1}$,

$$A^{-1}(A^2 - 4A - 5I) = \mathbf{0}$$

$$A - 4I - 5A^{-1} = \mathbf{0}$$

$$A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A^3(A^2 - 4A - 5I) - 2A(A^2 - 4A - 5I) + 3(A^2 - 4A - 5I) + A + 5I$

$$= (A^2 - 4A - 5I)(A^3 - 2A + 3I) + (A + 5I)$$

$$= \mathbf{0} + (A + 5I) \quad \text{[Using Eq. (1)]}$$

$$= A + 5I$$

which is a linear polynomial in $A$.

## EXERCISE 1.9

1. Verify the Cayley–Hamilton theorem for the matrix $A$ and, hence, find $A^{-1}$ and $A^4$.

(i) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

**Ans.:**

(i) $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}, \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & 13 \end{bmatrix}$

(ii) $\frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}$

2. Verify that the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ satisfies its characteristic equation and, hence, find $A^{-2}$.

$\left[\textbf{Ans.: } A^3 + A^2 - 5A - 5I = 0, \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]$

3. Use the Cayley–Hamilton theorem to find $2A^5 - 3A^4 + A^2 - 4I$, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$.

$\left[\textbf{Ans.: } 138A - 403I = \begin{bmatrix} 11 & 138 \\ -138 & 127 \end{bmatrix}\right]$

4. If $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$, find $A^7 - 9A^2 + I$.

[**Ans.:** $609A + 640I$]

5. Verify the Cayley–Hamilton theorem for (i) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ and, hence, find $A^{-1}$ and $A^3 - 5A^2$.

$\left[\begin{array}{l}\textbf{Ans.: } \text{(i)} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}, 2A \\ \text{(ii) } A^{-1} \text{ does not exist, } A^2 \end{array}\right]$

6. Compute $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

$\left[\textbf{Ans.: } \begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}\right]$

## 1.15 SIMILARITY OF MATRICES

If $A$ and $B$ are two square matrices of order $n$ then $B$ is said to be similar to $A$ if there exists a nonsingular matrix $P$ such that

$$B = P^{-1}AP$$

**Notes**

(i) Similarity of matrices is an equivalence relation.

(ii) Similar matrices have the same determinant.

(iii) Similar matrices have the same characteristic polynomial and, hence, the same eigenvalues. If $\mathbf{x}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda$ then $P^{-1}\mathbf{x}$ is an eigenvector of $B$ corresponding to the eigenvalue $\lambda$, where $B = P^{-1}AP$.

## 1.16 DIAGONALIZATION

A matrix $A$ is said to be *diagonalizable* if it is similar to a diagonal matrix.

A matrix $A$ is diagonalizable if there exists an invertible matrix $P$ such that $P^{-1}AP = D$, where $D$ is a diagonal matrix, also known as the *spectral matrix*. The matrix $P$ is then said to diagonalize $A$ or transform $A$ to a diagonal form. $P$ is known as the *modal matrix*.

**Notes**

(i) An $n \times n$ matrix is diagonalizable if and only if it possesses $n$ linearly independent eigenvectors.

(ii) If the eigenvalues of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

(iii) If $A$ is similar to a diagonal matrix $D$, the diagonal elements of $D$ are the eigenvalues of $A$.

### 1.16.1 Orthogonally Similar Matrices

If $A$ and $B$ are two square matrices of order $n$ then $B$ is said to be orthogonally similar to $A$ if there exists an orthogonal matrix $P$ such that

$$B = P^{-1}AP$$

Since $P$ is orthogonal, $P^{-1} = P^T$.

$$B = P^{-1}AP = P^TAP$$

To find the orthogonal matrix $P$, each element of the eigenvector is divided by its norm (length).

**Notes**

(i) Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

(ii) A real symmetric matrix of order $n$ has $n$ mutually orthogonal real eigenvectors.

(iii) Any two eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.

### 1.16.2 Unitarily Similar Matrices

If $A$ and $B$ are two square matrices of order $n$ then $B$ is said to be unitarily similar to $A$ if there exists a unitary matrix $P$ such that

$$B = P^{-1}AP$$

Since $P$ is unitary, $P^{-1} = P^{\theta}$.

$$B = P^{-1}AP = P^{\theta}AP$$

**Notes**

(i) Every Hermitian matrix is unitarily similar to a diagonal matrix.

(ii) A Hermitian matrix of order $n$ has $n$ mutually orthogonal eigenvectors in the complex vector space.

(iii) Any two eigenvectors corresponding to two distinct eigenvalues of a Hermitian matrix are orthogonal.

### 1.16.3 Working Rule for Diagonalization of a Square Matrix A

(i) Find the eigenvalues of the square matrix $A$.

(ii) Find the eigenvectors corresponding to each eigenvalue.

(iii) Find the modal matrix $P$ having the eigenvectors as its column vectors.

(iv) Find the diagonal matrix $D = P^TAP$. The diagonal matrix $D$ has eigenvalues as its diagonal elements.

**EXAMPLE 1.46** *Show that the matrix* $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ *is not diagonalizable.*

**Solution:** Let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = 1 + 2 + 2 = 5$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$= \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = (4-2)+(2+2)+(2-0) = 2+4+2 = 8$$

$$S_3 = \det(A) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 1(4-2) - 2(0+1) + 2(0+2) = 2 - 2 + 4 = 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

**(i)** For $\lambda = 1$, number of linearly independent eigenvectors = 1

**(ii)** For $\lambda = 2$,

$$[A - \lambda I]\mathbf{x} = 0$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 - R_1$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 + 2R_2$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of matrix = 2

Number of unknowns = 3

Number of linearly independent eigenvectors = 3 − 2 = 1

Since the matrix $A$ has a total of 2 linearly independent eigenvectors which is less than its order 3, the matrix $A$ is not diagonalizable.

**EXAMPLE 1.47** *Show that the matrix* $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ *is similar to its diagonal matrix. Find the diagonal and modal matrices.*

**Solution:** Let

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = -9 + 3 + 7 = 1$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$= \begin{vmatrix} 3 & 4 \\ 8 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -16 & 7 \end{vmatrix} + \begin{vmatrix} -9 & 4 \\ -8 & 3 \end{vmatrix} = (21-32)+(-63+64)+(-27+32) = -11+1+5 = -5$$

$$S_3 = \det(A) = \begin{vmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{vmatrix} = -9(21-32) - 4(-56+64) + 4(-64+48) = 99 - 32 - 64 = 3$$

Hence, the characteristic equation is

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$$\lambda = -1, -1, 3$$

(i) For $\lambda=-1$,

$$[A-\lambda I]\mathbf{x}=\mathbf{0}$$

$$\begin{bmatrix}-8 & 4 & 4\\ -8 & 4 & 4\\ -16 & 8 & 8\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-8x+4y+4z=0$$

Let $y=t_1$ and $z=t_2$

$$x=\frac{1}{2}t_1+\frac{1}{2}t_2$$

$$\mathbf{x}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}\frac{1}{2}t_1+\frac{1}{2}t_2\\ t_1\\ t_2\end{bmatrix}=t_1\begin{bmatrix}\frac{1}{2}\\ 1\\ 0\end{bmatrix}+t_2\begin{bmatrix}\frac{1}{2}\\ 0\\ 1\end{bmatrix}=t_1\mathbf{x}_1+t_2\mathbf{x}_2$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent eigenvectors corresponding to $\lambda=-1$.

(ii) For $\lambda=3$,

$$[A-\lambda I]\mathbf{x}=\mathbf{0}$$

$$\begin{bmatrix}-12 & 4 & 4\\ -8 & 0 & 4\\ -16 & 8 & 4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$-12x+4y+4z=0$$
$$-8x+0y+4z=0$$
$$-16x+8y+4z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}4 & 4\\ 0 & 4\end{vmatrix}}=\frac{-y}{\begin{vmatrix}-12 & 4\\ -8 & 4\end{vmatrix}}=\frac{z}{\begin{vmatrix}-12 & 4\\ -8 & 0\end{vmatrix}}=t, \text{ say}$$

$$\frac{x}{16}=\frac{y}{16}=\frac{z}{32}=t$$

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{2}=t$$

$$\mathbf{x}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}t\\ t\\ 2t\end{bmatrix}=t\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}=t\mathbf{x}_3,$$ where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda=3$.

Since the matrix $A$ has a total of 3 linearly independent eigenvectors which is same as its order, the matrix $A$ is diagonalizable.

The modal matrix $P$ has eigenvectors as its column vectors.

$$P=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The diagonal matrix $D$ has eigenvalues as its diagonal elements.

$$D=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

**EXAMPLE 1.48** *Determine a diagonal matrix orthogonally similar to the real symmetric matrix* $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$. *Also, find the modal matrix.*

**Solution:** Let

$$A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}=0$$

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

where $S_1$ = sum of the principal diagonal elements of $A=6+3+3=12$

$S_2$ = sum of the minors of principal diagonal elements of $A$

$$=\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix}+\begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}+\begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}=(9-1)+(18-4)+(18-4)=8+14+14=36$$

$$S_3=\det(A)=\begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}=6(9-1)+2(-6+2)+2(2-6)=48-8-8=32$$

Hence, the characteristic equation is

$$\lambda^3-12\lambda^2+36\lambda-32=0$$
$$\lambda=2,2,8$$

(i) For $\lambda=8$,

$$[A-\lambda I]\,\mathbf{x}=\mathbf{0}$$
$$\begin{bmatrix}-2&-2&2\\-2&-5&-1\\2&-1&-5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$-2x-2y+2z=0$$
$$-2x-5y-\ z=0$$
$$2x-\ y-5z=0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix}-2&2\\-5&-1\end{vmatrix}}=\frac{-y}{\begin{vmatrix}-2&2\\-2&-1\end{vmatrix}}=\frac{z}{\begin{vmatrix}-2&-2\\-2&-5\end{vmatrix}}=t,\text{ say}$$
$$\frac{x}{12}=\frac{y}{-6}=\frac{z}{6}=t$$
$$\frac{x}{2}=\frac{y}{-1}=\frac{z}{1}=t$$

$$\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2t\\-t\\t\end{bmatrix}=t\begin{bmatrix}2\\-1\\1\end{bmatrix}=t\mathbf{x}_1,$$ where $\mathbf{x}_1$ is an eigenvector corresponding to $\lambda=8$.

(ii) For $\lambda=2$,

$$[A-\lambda I]\,\mathbf{x}=\mathbf{0}$$
$$\begin{bmatrix}4&-2&2\\-2&1&-1\\2&-1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$
$$4x-2y+2z=0$$

Let $y=t_1$ and $z=t_2$

$$x=\frac{1}{2}t_1-\frac{1}{2}t_2$$

$$\mathbf{x}=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{2}t_1-\frac{1}{2}t_2\\t_1\\t_2\end{bmatrix}=\begin{bmatrix}\frac{1}{2}t_1\\t_1\\0\end{bmatrix}+\begin{bmatrix}-\frac{1}{2}t_2\\0\\t_2\end{bmatrix}=t_1\begin{bmatrix}\frac{1}{2}\\1\\0\end{bmatrix}+t_2\begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}=t_1\mathbf{x}_2+t_2\mathbf{x}_3,$$ where $\mathbf{x}_2$ and $\mathbf{x}_3$ are linearly independent eigenvectors corresponding to $\lambda=2$.

The orthogonal matrix $P$ has mutually orthogonal eigenvectors. Since $\mathbf{x}_2$ and $\mathbf{x}_3$ are not orthogonal, $\mathbf{x}_3$ is chosen such that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are orthogonal.

Let
$$\mathbf{x}_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

For orthogonality of eigenvectors,

$$\mathbf{x}_1^T \mathbf{x}_3 = 0 \text{ and } \mathbf{x}_2^T \mathbf{x}_3 = 0$$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} \frac{1}{2} & 1 & 0 \end{bmatrix}\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$$

$$2l - m + n = 0 \quad \text{and} \quad \frac{1}{2}l + m = 0$$

By Cramer's rule,

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix}} = -\frac{m}{\begin{vmatrix} 2 & 1 \\ \frac{1}{2} & 0 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 2 & -1 \\ \frac{1}{2} & 1 \end{vmatrix}} = t, \text{ say}$$

$$\frac{l}{-1} = \frac{m}{\frac{1}{2}} = \frac{n}{\frac{5}{2}} = t$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{5} = t$$

$$\mathbf{x} = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ 5t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = t\mathbf{x}_3$$

where $\mathbf{x}_3$ is an eigenvector corresponding to $\lambda = 2$.

$$\text{Length of eigenvector } \mathbf{x}_1 = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{6}$$

$$\text{Length of eigenvector } \mathbf{x}_2 = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2 + 0^2} = \frac{\sqrt{5}}{2}$$

$$\text{Length of eigenvector } \mathbf{x}_3 = \sqrt{(-2)^2 + (1)^2 + (5)^2} = \sqrt{30}$$

The normalized eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{bmatrix}$$

The modal matrix $P$ has normalized eigenvectors as its column vectors.

$$P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$$

$$P^{-1} = P^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{30}} & \frac{1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \end{bmatrix}$$

The diagonal matrix $D$ has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

## 1.16.4 Powers of a Matrix

If $A$ is an $n \times n$ matrix and $P$ is an invertible matrix then

$$(P^{-1}AP)^k = P^{-1}A^kP$$

If the matrix $A$ is diagonalizable and $D = P^{-1}AP$ is a diagonal matrix then

$$D^k = (P^{-1}AP)^k = P^{-1}A^kP$$

Premultiplying $D^k$ by $P$ and post-multiplying by $P^{-1}$,

$$PD^kP^{-1} = P(P^{-1}A^kP)P^{-1} = (PP^{-1})A^k(PP^{-1}) = IA^kI = A^k$$
$$\therefore A^k = PD^kP^{-1}$$

**EXAMPLE 1.49** *Find a matrix P that diagonalizes* $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. *Hence, find* $A^{13}$.

**Solution:**
$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The characteristic equation is

$$\det(A - \lambda I) = 0$$
$$\begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$
$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1$ = sum of the principal diagonal elements of $A = 0 + 2 + 3 = 5$

$S_2$ = sum of the minors of principal diagonal elements of $A$.

$$= \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} = (6-0) + (0+2) + (0-0) = 8$$

$$S_3 = \det(A) = \begin{vmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} = 0 + 0 - 2(0-2) = 4$$

Hence, the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$
$$\lambda = 1, 2, 2$$

(i) For $\lambda = 1$,
$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$-x + 0y - 2z = 0$$
$$x + y + z = 0$$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}} = -\frac{y}{\begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}} = t, \text{ say}$$

$$\frac{x}{2} = \frac{y}{-1} = \frac{z}{-1} = t$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ -t \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = t\mathbf{x}_1, \text{ where } \mathbf{x}_1 \text{ is an eigenvector corresponding to } \lambda = 1.$$

(ii) For $\lambda = 2$,

$$[A - \lambda I]\,\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 0y + z = 0$$

Let

$$y = t_1 \quad \text{and} \quad z = t_2$$

$$x = -t_2$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = t_1\mathbf{x}_2 + t_2\mathbf{x}_3, \text{ where } \mathbf{x}_2 \text{ and } \mathbf{x}_3 \text{ are linearly independent}$$

eigenvectors corresponding to $\lambda = 2$.

The modal matrix $P$ has eigenvectors as its column vectors.

$$P = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

The diagonal matrix $D$ has eigenvalues as its diagonal elements.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 2^{13} \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$

## EXERCISE 1.10

1. Show that the following matrices are not similar to diagonal matrices:

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

2. Show that the following matrices are similar to diagonal matrices. Find the diagonal and modal matrix in each case.

(i) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

**Ans.:**

(i) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

(ii) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -2 & 1 \\ 3 & 3 & -2 \end{bmatrix}$

3. Determine diagonal matrices orthogonally similar to the following real symmetric matrices. Also, find the modal matrix in each case.

(i) $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & 8 \end{bmatrix}$

**Ans.:**

(i) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}$,

$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

(ii) $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$,

$P = \begin{bmatrix} \frac{4}{\sqrt{18}} & 0 & \frac{1}{3} \\ \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & -\frac{2}{3} \\ -\frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{bmatrix}$