Networks An electrical network having more than one loop also gives rise to simultaneous differential equations. As shown in **FIGURE 2.9.3**, the current $i_1(t)$ splits in the directions shown at point B_1, called a *branch point* of the network. By Kirchhoff's first law we can write

$$i_1(t) = i_2(t) + i_3(t). \tag{14}$$

In addition, we can also apply **Kirchhoff's second law** to each loop. For loop $A_1B_1B_2A_2A_1$, summing the voltage drops across each part of the loop gives

$$E(t) = i_1R_1 + L_1\frac{di_2}{dt} + i_2R_2. \tag{15}$$

Similarly, for loop $A_1B_1C_1C_2B_2A_2A_1$ we find

$$E(t) = i_1R_1 + L_2\frac{di_3}{dt}. \tag{16}$$

Using (14) to eliminate i_1 in (15) and (16) yields two linear first-order equations for the currents $i_2(t)$ and $i_3(t)$:

$$\begin{aligned} L_1\frac{di_2}{dt} + (R_1 + R_2)i_2 + R_1i_3 &= E(t) \\ L_2\frac{di_3}{dt} + R_1i_2 + R_1i_3 &= E(t). \end{aligned} \tag{17}$$

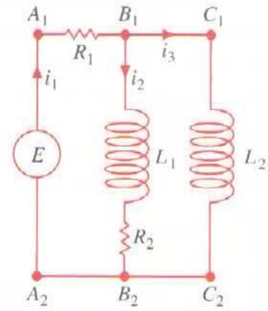

FIGURE 2.9.3 Network whose model is given in (17)

We leave it as an exercise (see Problem 16 in Exercises 2.9) to show that the system of differential equations describing the currents $i_1(t)$ and $i_2(t)$ in the network containing a resistor, an inductor, and a capacitor shown in **FIGURE 2.9.4** is

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 &= E(t) \\ RC\frac{di_2}{dt} + i_2 - i_1 &= 0. \end{aligned} \tag{18}$$

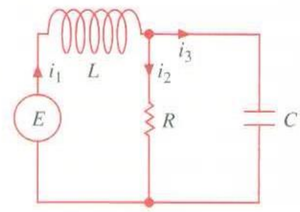

FIGURE 2.9.4 Network whose model is given in (18)

2.9 Exercises

Answers to selected odd-numbered problems begin on page ANS-4.

Radioactive Series

1. We have not discussed methods by which systems of first-order differential equations can be solved. Nevertheless, systems such as (2) can be solved with no knowledge other than how to solve a single linear first-order equation. Find a solution of (2) subject to the initial conditions $x(0) = x_0$, $y(0) = 0$, $z(0) = 0$.

2. In Problem 1, suppose that time is measured in days, that the decay constants are $k_1 = -0.138629$ and $k_2 = -0.004951$, and that $x_0 = 20$. Use a graphing utility to obtain the graphs of the solutions $x(t)$, $y(t)$, and $z(t)$ on the same set of coordinate axes. Use the graphs to approximate the half-lives of substances X and Y.

3. Use the graphs in Problem 2 to approximate the times when the amounts $x(t)$ and $y(t)$ are the same, the times when the amounts $x(t)$ and $z(t)$ are the same, and the times when the amounts $y(t)$ and $z(t)$ are the same. Why does the time determined when the amounts $y(t)$ and $z(t)$ are the same make intuitive sense?

4. Construct a mathematical model for a radioactive series of four elements W, X, Y, and Z, where Z is a stable element.

Contibuted Problems

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5. **Potassium-40 Decay** The mineral potassium, whose chemical symbol is K, is the eighth most abundant element in the Earth's crust, making up about 2% of it by weight, and one of its naturally occurring isotopes, K-40, is radioactive. The radioactive decay of K-40 is more complex than that of carbon-14 because each of its atoms decays through one of two different nuclear decay reactions into one of two different substances: the mineral calcium-40 (Ca-40) or the gas argon-40 (Ar-40). Dating methods have been developed

using both of these decay products. In each case, the age of a sample is calculated using the ratio of two numbers: the amount of the *parent* isotope K-40 in the sample and the amount of the *daughter* isotope (Ca-40 or Ar-40) in the sample that is **radiogenic**, in other words, the substance which originates from the decay of the parent isotope after the formation of the rock.

An igneous rock is solidified magma

The amount of K-40 in a sample is easy to calculate. K-40 comprises 1.17% of naturally occurring potassium, and this small percentage is distributed quite uniformly, so that the mass of K-40 in the sample is just 1.17% of the total mass of potassium in the sample, which can be measured. But for several reasons it is complicated, and sometimes problematic, to determine how much of the Ca-40 in a sample is radiogenic. In contrast, when an igneous rock is formed by volcanic activity, all of the argon (and other) gas previously trapped in the rock is driven away by the intense heat. At the moment when the rock cools and solidifies, the gas trapped inside the rock has the same composition as the atmosphere. There are three stable isotopes of argon, and in the atmosphere they occur in the following relative abundances: 0.063% Ar-38, 0.337% Ar-36, and 99.60% Ar-40. Of these, just one, Ar-36, is not created radiogenically by the decay of any element, so any Ar-40 in excess of 99.60/(0.337) = 295.5 times the amount of Ar-36 must be radiogenic. So the amount of radiogenic Ar-40 in the sample can be determined from the amounts of Ar-38 and Ar-36 in the sample, which can be measured.

Assuming that we have a sample of rock for which the amount of K-40 and the amount of radiogenic Ar-40 have been determined, how can we calculate the age of the rock? Let $P(t)$ be the amount of K-40, $A(t)$ the amount of radiogenic Ar-40, and $C(t)$ the amount of radiogenic Ca-40 in the sample as functions of time t in years since the formation of the rock. Then a mathematical model for the decay of K-40 is the system of linear first-order differential equations

$$\begin{aligned}\frac{dA}{dt} &= \lambda_A P\\ \frac{dC}{dt} &= \lambda_C P\\ \frac{dP}{dt} &= -(\lambda_A + \lambda_C)P,\end{aligned}$$

where $\lambda_A = 0.581 \times 10^{-10}$ and $\lambda_C = 4.962 \times 10^{-10}$.

(a) From the system of differential equations find $P(t)$ i $P(0) = P_0$.

(b) Determine the half-life of K-40.

(c) Use $P(t)$ from part (a) to find $A(t)$ and $C(t)$ if $A(0) =$ and $C(0) = 0$.

(d) Use your solution for $A(t)$ in part (c) to determine the per- centage of an initial amount P_0 of K-40 that decays in Ar-40 over a very long period of time (that is, $t \to \infty$). What percentage of P_0 decays into Ca-40?

6. Potassium–Argon Dating **(a)** Use the solutions in parts (and (c) of Problem 5 to show that

$$\frac{A(t)}{P(t)} = \frac{\lambda_A}{\lambda_A + \lambda_C}[e^{(\lambda_A+\lambda_C)t} - 1].$$

(b) Solve the expression in part (a) for t in terms $A(t)$, $P($ λ_A, and λ_C.

(c) Suppose it is found that each gram of a rock sampl contains 8.6×10^{-7} grams of radiogenic Ar-40 an 5.3×10^{-6} grams of K-40. Use the equation obtaine in part (b) to determine the approximate age of the roc

≡ Mixtures

7. Consider two tanks A and B, with liquid being pumped in an out at the same rates, as described by the system of equatio (3). What is the system of differential equations if, instead o pure water, a brine solution containing 2 pounds of salt pe gallon is pumped into tank A?

8. Use the information given in **FIGURE 2.9.5** to construct a ma ematical model for the number of pounds of salt $x_1(t)$, $x_2($ and $x_3(t)$ at time t in tanks A, B, and C, respectively.

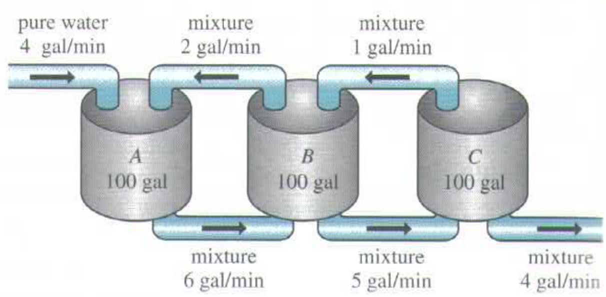

FIGURE 2.9.5 Mixing tanks in Problem 8

9. Two very large tanks A and B are each partially filled w 100 gallons of brine. Initially, 100 pounds of salt is dissol in the solution in tank A and 50 pounds of salt is dissolved the solution in tank B. The system is closed in that the wel stirred liquid is pumped only between the tanks, as shown **FIGURE 2.9.6**.

(a) Use the information given in the figure to construc mathematical model for the number of pounds of s $x_1(t)$ and $x_2(t)$ at time t in tanks A and B, respectively.

(b) Find a relationship between the variables $x_1(t)$ and $x_2(t)$ that holds at time t. Explain why this relationship makes intuitive sense. Use this relationship to help find the amount of salt in tank B at $t = 30$ min.

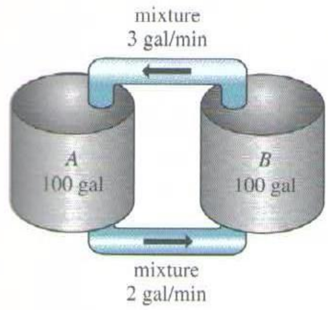

FIGURE 2.9.6 Mixing tanks in Problem 9

10. Three large tanks contain brine, as shown in **FIGURE 2.9.7**. Use the information in the figure to construct a mathematical model for the number of pounds of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ at time t in tanks A, B, and C, respectively. Without solving the system, predict limiting values of $x_1(t)$, $x_2(t)$, and $x_3(t)$ as $t \to \infty$.

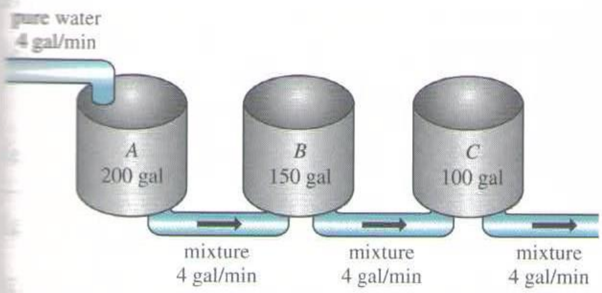

FIGURE 2.9.7 Mixing tanks in Problem 10

≡ Predator–Prey Models

11. Consider the Lotka–Volterra predator–prey model defined by

$$\frac{dx}{dt} = -0.1x + 0.02xy$$
$$\frac{dy}{dt} = 0.2y - 0.025xy,$$

where the populations $x(t)$ (predators) and $y(t)$ (prey) are measured in the thousands. Suppose $x(0) = 6$ and $y(0) = 6$. Use a numerical solver to graph $x(t)$ and $y(t)$. Use the graphs to approximate the time $t > 0$ when the two populations are first equal. Use the graphs to approximate the period of each population.

≡ Competition Models

12. Consider the competition model defined by

$$\frac{dx}{dt} = x(2 - 0.4x - 0.3y)$$
$$\frac{dy}{dt} = y(1 - 0.1y - 0.3x),$$

where the populations $x(t)$ and $y(t)$ are measured in the thousands and t in years. Use a numerical solver to analyze the populations over a long period of time for each of the cases:

(a) $x(0) = 1.5$, $y(0) = 3.5$
(b) $x(0) = 1$, $y(0) = 1$
(c) $x(0) = 2$, $y(0) = 7$
(d) $x(0) = 4.5$, $y(0) = 0.5$

13. Consider the competition model defined by

$$\frac{dx}{dt} = x(1 - 0.1x - 0.05y)$$
$$\frac{dy}{dt} = y(1.7 - 0.1y - 0.15x),$$

where the populations $x(t)$ and $y(t)$ are measured in the thousands and t in years. Use a numerical solver to analyze the populations over a long period of time for each of the cases:

(a) $x(0) = 1$, $y(0) = 1$
(b) $x(0) = 4$, $y(0) = 10$
(c) $x(0) = 9$, $y(0) = 4$
(d) $x(0) = 5.5$, $y(0) = 3.5$

≡ Networks

14. Show that a system of differential equations that describes the currents $i_2(t)$ and $i_3(t)$ in the electrical network shown in **FIGURE 2.9.8** is

$$L\frac{di_2}{dt} + L\frac{di_3}{dt} + R_1 i_2 = E(t)$$
$$-R_1\frac{di_2}{dt} + R_2\frac{di_3}{dt} + \frac{1}{C}i_3 = 0.$$

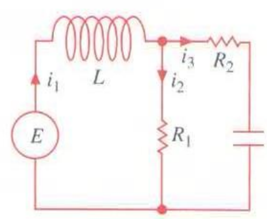

FIGURE 2.9.8 Network in Problem 14

15. Determine a system of first-order differential equations that describe the currents $i_2(t)$ and $i_3(t)$ in the electrical network shown in **FIGURE 2.9.9**.

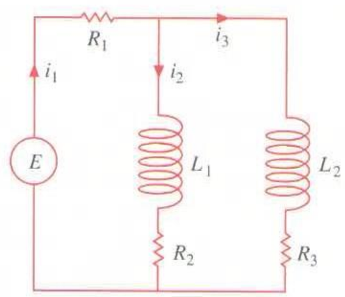

FIGURE 2.9.9 Network in Problem 15

16. Show that the linear system given in (18) describes the currents $i_1(t)$ and $i_2(t)$ in the network shown in Figure 2.9.4. [*Hint*: $dq/dt = i_3$.]

Miscellaneous Mathematical Models

17. **SIR Model** A communicable disease is spread throughout a small community, with a fixed population of n people, by contact between infected individuals and people who are susceptible to the disease. Suppose initially that everyone is susceptible to the disease and that no one leaves the community while the epidemic is spreading. At time t, let $s(t)$, $i(t)$, and $r(t)$ denote, in turn, the number of people in the community (measured in hundreds) who are *susceptible* to the disease but not yet infected with it, the number of people who are *infected* with the disease, and the number of people who have *recovered* from the disease. Explain why the system of differential equations

$$\frac{ds}{dt} = -k_1 si$$
$$\frac{di}{dt} = -k_2 i + k_1 si$$
$$\frac{dr}{dt} = k_2 i,$$

where k_1 (called the *infection rate*) and k_2 (called the *removal rate*) are positive constants, is a reasonable mathematical model, commonly called a **SIR model**, for the spread of the epidemic throughout the community. Give plausible initial conditions associated with this system of equations. Show that the system implies that

$$\frac{d}{dt}(s + i + r) = 0.$$

Why is this consistent with the assumptions?

18. **(a)** In Problem 17 explain why it is sufficient to analyze only

$$\frac{ds}{dt} = -k_1 si$$
$$\frac{di}{dt} = -k_2 i + k_1 si.$$

(b) Suppose $k_1 = 0.2$, $k_2 = 0.7$, and $n = 10$. Choose various values of $i(0) = i_0$, $0 < i_0 < 10$. Use a numerical solver to determine what the model predicts about the epidemic in the two cases $s_0 > k_2/k_1$ and $s_0 \leq k_2/k_1$. In the case of an epidemic, estimate the number of people who are eventually infected.