Using the initial condition $y(2) = 6$ we obtain the solution

$$y(x) = 6 + \int_2^x e^{-t^2}\,dt. \qquad \equiv$$

The procedure illustrated in Example 5 works equally well on separable equations $dy/dx = g(x)f(y)$ where, say, $f(y)$ possesses an elementary antiderivative but $g(x)$ does not possess an elementary antiderivative. See Problems 29 and 30 in Exercises 2.2.

Remarks

(i) As we have just seen in Example 5, some functions do not possess an antiderivative that is an elementary function. Integrals of these kinds of functions are called **nonelementary**. For example, $\int_2^x e^{-t^2}dt$ and $\int \sin x^2 dx$ are nonelementary integrals. We will run into this concept again in Section 2.3.

(ii) In some of the preceding examples we saw that the constant in the one-parameter family of solutions for a first-order differential equation can be relabeled when convenient. Also, it can easily happen that two individuals solving the same equation correctly arrive at dissimilar expressions for their answers. For example, by separation of variables, we can show that one-parameter families of solutions for the DE $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$ are

$$\arctan x + \arctan y = c \quad \text{or} \quad \frac{x + y}{1 - xy} = c.$$

As you work your way through the next several sections, keep in mind that families of solutions may be equivalent in the sense that one family may be obtained from another by either relabeling the constant or applying algebra and trigonometry. See Problems 27 and 28 in Exercises 2.2.

2.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–22, solve the given differential equation by separation of variables.

1. $\dfrac{dy}{dx} = \sin 5x$
2. $\dfrac{dy}{dx} = (x + 1)^2$
3. $dx + e^{3x}dy = 0$
4. $dy - (y - 1)^2 dx = 0$
5. $x\dfrac{dy}{dx} = 4y$
6. $\dfrac{dy}{dx} + 2xy^2 = 0$
7. $\dfrac{dy}{dx} = e^{3x+2y}$
8. $e^x y\dfrac{dy}{dx} = e^{-y} + e^{-2x-y}$
9. $y\ln x\dfrac{dx}{dy} = \left(\dfrac{y + 1}{x}\right)^2$
10. $\dfrac{dy}{dx} = \left(\dfrac{2y + 3}{4x + 5}\right)^2$
11. $\csc y\,dx + \sec^2 x\,dy = 0$
12. $\sin 3x\,dx + 2y\cos^3 3x\,dy = 0$
13. $(e^y + 1)^2 e^{-y}dx + (e^x + 1)^3 e^{-x}dy = 0$
14. $x(1 + y^2)^{1/2}dx = y(1 + x^2)^{1/2}dy$
15. $\dfrac{dS}{dr} = kS$
16. $\dfrac{dQ}{dt} = k(Q - 70)$
17. $\dfrac{dP}{dt} = P - P^2$
18. $\dfrac{dN}{dt} + N = Nte^{t+2}$
19. $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$
20. $\dfrac{dy}{dx} = \dfrac{xy + 2y - x - 2}{xy - 3y + x - 3}$
21. $\dfrac{dy}{dx} = x\sqrt{1 - y^2}$
22. $(e^x + e^{-x})\dfrac{dy}{dx} = y^2$

In Problems 23–28, find an implicit and an explicit solution of the given initial-value problem.

23. $\dfrac{dx}{dt} = 4(x^2 + 1), \quad x(\pi/4) = 1$
24. $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 - 1}, \quad y(2) = 2$
25. $x^2\dfrac{dy}{dx} = y - xy, \quad y(-1) = -1$
26. $\dfrac{dy}{dt} + 2y = 1, \quad y(0) = \frac{5}{2}$
27. $\sqrt{1 - y^2}\,dx - \sqrt{1 - x^2}\,dy = 0, \quad y(0) = \sqrt{3}/2$
28. $(1 + x^4)\,dy + x(1 + 4y^2)\,dx = 0, \quad y(1) = 0$

In Problems 29 and 30, proceed as in Example 5 and find an explicit solution of the given initial-value problem.

29. $\dfrac{dy}{dx} = ye^{-x^2}$, $y(4) = 1$

30. $\dfrac{dy}{dx} = y^2 \sin x^2$, $y(-2) = \frac{1}{3}$

31. **(a)** Find a solution of the initial-value problem consisting of the differential equation in Example 3 and the initial conditions $y(0) = 2$, $y(0) = -2$, $y(\frac{1}{4}) = 1$.

(b) Find the solution of the differential equation in Example 4 when $\ln c_1$ is used as the constant of integration on the *left-hand* side in the solution and $4 \ln c_1$ is replaced by $\ln c$. Then solve the same initial-value problems in part (a).

32. Find a solution of $x\dfrac{dy}{dx} = y^2 - y$ that passes through the indicated points.

(a) $(0, 1)$ **(b)** $(0, 0)$ **(c)** $(\frac{1}{2}, \frac{1}{2})$ **(d)** $(2, \frac{1}{4})$

33. Find a singular solution of Problem 21. Of Problem 22.

34. Show that an implicit solution of

$$2x \sin^2 y\, dx - (x^2 + 10) \cos y\, dy = 0$$

is given by $\ln(x^2 + 10) \csc y = c$. Find the constant solutions, if any, that were lost in the solution of the differential equation.

Often a radical change in the form of the solution of a differential equation corresponds to a very small change in either the initial condition or the equation itself. In Problems 35–38, find an explicit solution of the given initial-value problem. Use a graphing utility to plot the graph of each solution. Compare each solution curve in a neighborhood of $(0, 1)$.

35. $\dfrac{dy}{dx} = (y - 1)^2, \quad y(0) = 1$

36. $\dfrac{dy}{dx} = (y - 1)^2, \quad y(0) = 1.01$

37. $\dfrac{dy}{dx} = (y - 1)^2 + 0.01, \quad y(0) = 1$

38. $\dfrac{dy}{dx} = (y - 1)^2 - 0.01, \quad y(0) = 1$

39. Every autonomous first-order equation $dy/dx = f(y)$ is separable. Find explicit solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ of the differential equation $dy/dx = y - y^3$ that satisfy, in turn, the initial conditions $y_1(0) = 2$, $y_2(0) = \frac{1}{2}$, $y_3(0) = -\frac{1}{2}$, and $y_4(0) = -2$. Use a graphing utility to plot the graphs of each solution. Compare these graphs with those predicted in Problem 19 of Exercises 2.1. Give the exact interval of definition for each solution.

40. **(a)** The autonomous first-order differential equation $dy/dx = 1/(y - 3)$ has no critical points. Nevertheless, place 3 on a phase line and obtain a phase portrait of the equation. Compute d^2y/dx^2 to determine where solution curves are concave up and where they are concave down (see Problems 35 and 36 in Exercises 2.1). Use the phase portrait and concavity to sketch, by hand, some typical solution curves.

(b) Find explicit solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ of the differential equation in part (a) that satisfy, in turn, the initial conditions $y_1(0) = 4$, $y_2(0) = 2$, $y_3(1) = 2$, and $y_4(-1) = 4$. Graph each solution and compare with your sketches in part (a). Give the exact interval of definition for each solution.

41. **(a)** Find an explicit solution of the initial-value problem

$$\frac{dy}{dx} = \frac{2x + 1}{2y}, \quad y(-2) = -1.$$

(b) Use a graphing utility to plot the graph of the solution in part (a). Use the graph to estimate the interval I of definition of the solution.

(c) Determine the exact interval I of definition by analytical methods.

42. Repeat parts (a)–(c) of Problem 41 for the IVP consisting of the differential equation in Problem 7 and the condition $y(0) = 0$.

≡ Discussion Problems

43. **(a)** Explain why the interval of definition of the explicit solution $y = \phi_2(x)$ of the initial-value problem in Example 2 is the *open* interval $(-5, 5)$.

(b) Can any solution of the differential equation cross the x-axis? Do you think that $x^2 + y^2 = 1$ is an implicit solution of the initial-value problem $dy/dx = -x/y$, $y(1) = 0$?

44. **(a)** If $a > 0$, discuss the differences, if any, between the solutions of the initial-value problems consisting of the differential equation $dy/dx = x/y$ and each of the initial conditions $y(a) = a$, $y(a) = -a$, $y(-a) = a$, and $y(-a) = -a$.

(b) Does the initial-value problem $dy/dx = x/y$, $y(0) = 0$ have a solution?

(c) Solve $dy/dx = x/y$, $y(1) = 2$, and give the exact interval I of definition of its solution.

45. In Problems 39 and 40 we saw that every autonomous first-order differential equation $dy/dx = f(y)$ is separable. Does this fact help in the solution of the initial-value problem $\dfrac{dy}{dx} = \sqrt{1 + y^2 \sin^2 y}$, $y(0) = \frac{1}{2}$? Discuss. Sketch, by hand, a plausible solution curve of the problem.

46. Without the use of technology, how would you solve

$$(\sqrt{x} + x)\frac{dy}{dx} = \sqrt{y} + y?$$

Carry out your ideas.

47. Find a function whose square plus the square of its derivative is 1.

48. **(a)** The differential equation in Problem 27 is equivalent to the normal form

$$\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$$

in the square region in the xy-plane defined by $|x| < 1$, $|y| < 1$. But the quantity under the radical is nonnegative also in the regions defined by $|x| > 1$, $|y| > 1$. Sketch all regions in the xy-plane for which this differential equation possesses real solutions.

(b) Solve the DE in part (a) in the regions defined by $|x| > 1$, $|y| > 1$. Then find an implicit and an explicit solution of the differential equation subject to $y(2) = 2$.

Mathematical Model

49. Suspension Bridge In (16) of Section 1.3 we saw that a mathematical model for the shape of a flexible cable strung between two vertical supports is

$$\frac{dy}{dx} = \frac{W}{T_1}, \tag{10}$$

where W denotes the portion of the total vertical load between the points P_1 and P_2 shown in Figure 1.3.9. The DE (10) is separable under the following conditions that describe a suspension bridge.

Let us assume that the x- and y-axes are as shown in **FIGURE 2.2.5**—that is, the x-axis runs along the horizontal roadbed, and the y-axis passes through $(0, a)$, which is the lowest point on one cable over the span of the bridge, coinciding with the interval $[-L/2, L/2]$. In the case of a suspension bridge, the usual assumption is that the vertical load in (10) is only a uniform roadbed distributed along the horizontal axis. In other words, it is assumed that the weight of all cables is negligible in comparison to the weight of the roadbed and that the weight per unit length of the roadbed (say, pounds per horizontal foot) is a constant ρ. Use this information to set up and solve an appropriate initial-value problem from which the shape (a curve with equation $y = \phi(x)$) of each of the two cables in a suspension bridge is determined. Express your solution of the IVP in terms of the sag h and span L shown in Figure 2.2.5.

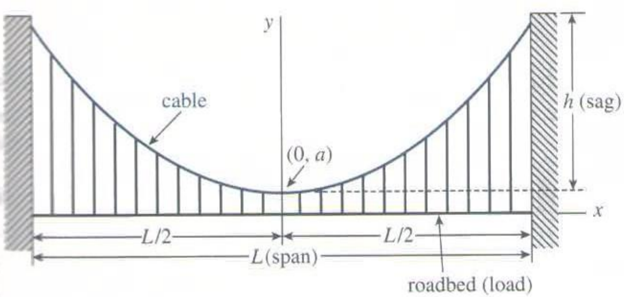

FIGURE 2.2.5 Shape of a cable in Problem 49

Computer Lab Assignments

50. (a) Use a CAS and the concept of level curves to plot representative graphs of members of the family of solutions of the differential equation $\dfrac{dy}{dx} = -\dfrac{8x + 5}{3y^2 + 1}$. Experiment with different numbers of level curves as well as various rectangular regions defined by $a \le x \le b$, $c \le y \le d$.

(b) On separate coordinate axes plot the graphs of the particular solutions corresponding to the initial conditions: $y(0) = -1$; $y(0) = 2$; $y(-1) = 4$; $y(-1) = -3$.

51. (a) Find an implicit solution of the IVP

$$(2y + 2)\,dy - (4x^3 + 6x)\,dx = 0, \quad y(0) = -3.$$

(b) Use part (a) to find an explicit solution $y = \phi(x)$ of the IVP.

(c) Consider your answer to part (b) as a *function* only. Use a graphing utility or a CAS to graph this function, and then use the graph to estimate its domain.

(d) With the aid of a root-finding application of a CAS, determine the approximate largest interval I of definition of the *solution* $y = \phi(x)$ in part (b). Use a graphing utility or a CAS to graph the solution curve for the IVP on this interval.

52. (a) Use a CAS and the concept of level curves to plot representative graphs of members of the family of solutions of the differential equation $\dfrac{dy}{dx} = \dfrac{x(1 - x)}{y(-2 + y)}$. Experiment with different numbers of level curves as well as various rectangular regions in the xy-plane until your result resembles **FIGURE 2.2.6**.

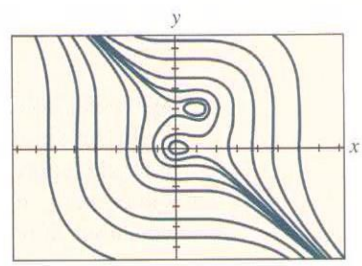

FIGURE 2.2.6 Level curves in Problem 52

(b) On separate coordinate axes, plot the graph of the implicit solution corresponding to the initial condition $y(0) = \frac{3}{2}$. Use a colored pencil to mark off that segment of the graph that corresponds to the solution curve of a solution ϕ that satisfies the initial condition. With the aid of a root-finding application of a CAS, determine the approximate largest interval I of definition of the solution ϕ. [*Hint*: First find the points on the curve in part (a) where the tangent is vertical.]

(c) Repeat part (b) for the initial condition $y(0) = -2$.