is a polynomial, continued differentiation of $P(x)e^{\alpha x}\sin\beta x$ will generate an independent set containing only a *finite* number of functions—all of the same type, namely, polynomials times $e^{\alpha x}\sin\beta x$ or $e^{\alpha x}\cos\beta x$. On the other hand, repeated differentiations of input functions such as $g(x) = \ln x$ or $g(x) = \tan^{-1}x$ generate an independent set containing an *infinite* number of functions:

$$\textit{derivatives of } \ln x: \quad \frac{1}{x}, \frac{-1}{x^2}, \frac{2}{x^3}, \ldots,$$

$$\textit{derivatives of } \tan^{-1}x: \quad \frac{1}{1+x^2}, \frac{-2x}{(1+x^2)^2}, \frac{-2+6x^2}{(1+x^2)^3}, \ldots.$$

3.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–26, solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$
2. $4y'' + 9y = 15$
3. $y'' - 10y' + 25y = 30x + 3$
4. $y'' + y' - 6y = 2x$
5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$
11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$
12. $y'' - 16y = 2e^{4x}$
13. $y'' + 4y = 3\sin 2x$
14. $y'' - 4y = (x^2 - 3)\sin 2x$
15. $y'' + y = 2x\sin x$
16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$
17. $y'' - 2y' + 5y = e^x\cos 2x$
18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$
19. $y'' + 2y' + y = \sin x + 3\cos 2x$
20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$
21. $y''' - 6y'' = 3 - \cos x$
22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$
23. $y''' - 3y'' + 3y' - y = x - 4e^x$
24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$
25. $y^{(4)} + 2y'' + y = (x - 1)^2$
26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

In Problems 27–36, solve the given initial-value problem.

27. $y'' + 4y = -2,\ y(\pi/8) = \frac{1}{2},\ y'(\pi/8) = 2$
28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11,$ $y(0) = 0,\ y'(0) = 0$
29. $5y'' + y' = -6x,\ y(0) = 0,\ y'(0) = -10$
30. $y'' + 4y' + 4y = (3 + x)e^{-2x},\ y(0) = 2,\ y'(0) = 5$
31. $y'' + 4y' + 5y = 35e^{-4x},\ y(0) = -3,\ y'(0) = 1$
32. $y'' - y = \cosh x,\ y(0) = 2,\ y'(0) = 12$
33. $\dfrac{d^2x}{dt^2} + \omega^2x = F_0\sin\omega t,\ x(0) = 0,\ x'(0) = 0$
34. $\dfrac{d^2x}{dt^2} + \omega^2x = F_0\cos\gamma t,\ x(0) = 0,\ x'(0) = 0$
35. $y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x},\ y(0) = \frac{1}{2},\ y'(0) = \frac{5}{2},$ $y''(0) = -\frac{9}{2}$
36. $y''' + 8y = 2x - 5 + 8e^{-2x},\ y(0) = -5,\ y'(0) = 3,\ y''(0) = -4$

In Problems 37–40, solve the given boundary-value problem.

37. $y'' + y = x^2 + 1,\ y(0) = 5,\ y(1) = 0$
38. $y'' - 2y' + 2y = 2x - 2,\ y(0) = 0,\ y(\pi) = \pi$
39. $y'' + 3y = 6x,\ y(0) = 0,\ y(1) + y'(1) = 0$
40. $y'' + 3y = 6x,\ y(0) + y'(0) = 0,\ y(1) = 0$

In Problems 41 and 42, solve the given initial-value problem in which the input function $g(x)$ is discontinuous. [*Hint*: Solve each problem on two intervals, and then find a solution so that y and y' are continuous at $x = \pi/2$ (Problem 41) and at $x = \pi$ (Problem 42).]

41. $y'' + 4y = g(x),\ y(0) = 1,\ y'(0) = 2$, where

$$g(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & x > \pi/2 \end{cases}$$

42. $y'' - 2y' + 10y = g(x)$, $y(0) = 0$, $y'(0) = 0$, where

$$g(x) = \begin{cases} 20, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

≡ Discussion Problems

43. Consider the differential equation $ay'' + by' + cy = e^{kx}$, where a, b, c, and k are constants. The auxiliary equation of the associated homogeneous equation is

$$am^2 + bm + c = 0.$$

(a) If k is not a root of the auxiliary equation, show that we can find a particular solution of the form $y_p = Ae^{kx}$, where $A = 1/(ak^2 + bk + c)$.

(b) If k is a root of the auxiliary equation of multiplicity one, show that we can find a particular solution of the form $y_p = Axe^{kx}$, where $A = 1/(2ak + b)$. Explain how we know that $k \neq -b/(2a)$.

(c) If k is a root of the auxiliary equation of multiplicity two, show that we can find a particular solution of the form $y = Ax^2e^{kx}$, where $A = 1/(2a)$.

44. Discuss how the method of this section can be used to find a particular solution of $y'' + y = \sin x \cos 2x$. Carry out your idea.

45. Without solving, match a solution curve of $y'' + y = f(x)$ shown in the figure with one of the following functions:

(i) $f(x) = 1$, **(ii)** $f(x) = e^{-x}$,
(iii) $f(x) = e^x$, **(iv)** $f(x) = \sin 2x$,
(v) $f(x) = e^x \sin x$, **(vi)** $f(x) = \sin x$.

Briefly discuss your reasoning.

(a)

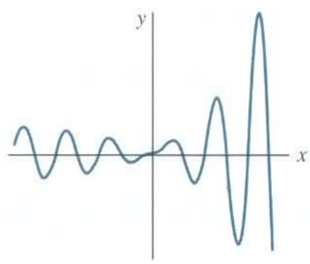

FIGURE 3.4.1 Solution curve

(b)

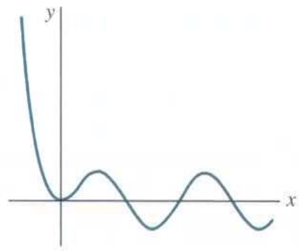

FIGURE 3.4.2 Solution curve

(c)

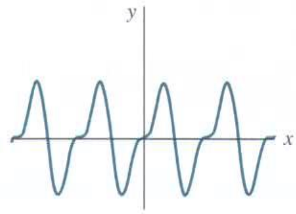

FIGURE 3.4.3 Solution curve

(d)

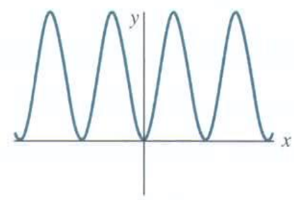

FIGURE 3.4.4 Solution curve

≡ Computer Lab Assignments

In Problems 46 and 47, find a particular solution of the given differential equation. Use a CAS as an aid in carrying out differentiations, simplifications, and algebra.

46. $y'' - 4y' + 8y = (2x^2 - 3x)e^{2x} \cos 2x + (10x^2 - x - 1)e^{2x} \sin 2x$

47. $y^{(4)} + 2y'' + y = 2 \cos x - 3x \sin x$

3.5 Variation of Parameters

≡ Introduction The **method of variation of parameters** used in Section 2.3 to find a particular solution of a linear first-order differential equation is applicable to linear higher-order equations as well. Variation of parameters has a distinct advantage over the method of the preceding section in that it *always* yields a particular solution y_p provided the associated homogeneous equation can be solved. In addition, the method presented in this section, unlike undetermined coefficients, is *not* limited to cases where the input function is a combination of the four types of functions listed on page 126, nor is it limited to differential equations with constant coefficients.

□ Some Assumptions To adapt the method of variation of parameters to a linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \tag{1}$$