Remarks

The similarity between the forms of solutions of Cauchy–Euler equations and solutions of linear equations with constant coefficients is not just a coincidence. For example, when the roots of the auxiliary equations for $ay'' + by' + cy = 0$ and $ax^2y'' + bxy' + cy = 0$ are distinct and real, the respective general solutions are

$$y = c_1e^{m_1x} + c_2e^{m_2x} \quad \text{and} \quad y = c_1x^{m_1} + c_2x^{m_2}, \quad x > 0. \tag{7}$$

In view of the identity $e^{\ln x} = x, x > 0$, the second solution given in (7) can be expressed in the same form as the first solution:

$$y = c_1e^{m_1\ln x} + c_2e^{m_2\ln x} = c_1e^{m_1t} + c_2e^{m_2t},$$

where $t = \ln x$. This last result illustrates another fact of mathematical life: Any Cauchy–Euler equation can *always* be rewritten as a linear differential equation with constant coefficients by means of the substitution $x = e^t$. The idea is to solve the new differential equation in terms of the variable t, using the methods of the previous sections, and once the general solution is obtained, resubstitute $t = \ln x$. Since this procedure provides a good review of the Chain Rule of differentiation, you are urged to work Problems 43–48 in Exercises 3.6.

3.6 Exercises

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–18, solve the given differential equation.

1. $x^2y'' - 2y = 0$
2. $4x^2y'' + y = 0$
3. $xy'' + y' = 0$
4. $xy'' - 3y' = 0$
5. $x^2y'' + xy' + 4y = 0$
6. $x^2y'' + 5xy' + 3y = 0$
7. $x^2y'' - 3xy' - 2y = 0$
8. $x^2y'' + 3xy' - 4y = 0$
9. $25x^2y'' + 25xy' + y = 0$
10. $4x^2y'' + 4xy' - y = 0$
11. $x^2y'' + 5xy' + 4y = 0$
12. $x^2y'' + 8xy' + 6y = 0$
13. $3x^2y'' + 6xy' + y = 0$
14. $x^2y'' - 7xy' + 41y = 0$
15. $x^3y''' - 6y = 0$
16. $x^3y''' + xy' - y = 0$
17. $xy^{(4)} + 6y''' = 0$
18. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$

In Problems 19–24, solve the given differential equation by variation of parameters.

19. $xy'' - 4y' = x^4$
20. $2x^2y'' + 5xy' + y = x^2 - x$
21. $x^2y'' - xy' + y = 2x$
22. $x^2y'' - 2xy' + 2y = x^4e^x$
23. $x^2y'' + xy' - y = \ln x$
24. $x^2y'' + xy' - y = \dfrac{1}{x+1}$

In Problems 25–30, solve the given initial-value problem. Use a graphing utility to graph the solution curve.

25. $x^2y'' + 3xy' = 0,\ y(1) = 0, y'(1) = 4$
26. $x^2y'' - 5xy' + 8y = 0,\ y(2) = 32, y'(2) = 0$
27. $x^2y'' + xy' + y = 0,\ y(1) = 1, y'(1) = 2$
28. $x^2y'' - 3xy' + 4y = 0,\ y(1) = 5, y'(1) = 3$
29. $xy'' + y' = x,\ y(1) = 1, y'(1) = -\frac{1}{2}$
30. $x^2y'' - 5xy' + 8y = 8x^6,\ y(\frac{1}{2}) = 0, y'(\frac{1}{2}) = 0$

In Problems 31 and 32, solve the given boundary-value problem.

31. $xy'' - 7xy' + 12y = 0,\ y(0) = 0, y(1) = 0$
32. $x^2y'' - 3xy' + 5y = 0,\ y(1) = 0, y(e) = 1$

In Problems 33–38, find a homogeneous Cauchy–Euler differential equation whose general solution is given.

33. $y = c_1x^4 + c_2x^{-2}$
34. $y = c_1 + c_2x^5$
35. $y = c_1x^{-3} + c_2x^{-3}\ln x$
36. $y = c_1 + c_2x + c_3x\ln x$
37. $y = c_1\cos(\ln x) + c_2\sin(\ln x)$
38. $y = c_1x^{1/2}\cos(\frac{1}{2}\ln x) + c_2x^{1/2}\sin(\frac{1}{2}\ln x)$

In Problems 39–42, use the substitution $y = (x - x_0)^m$ to solve the given equation.

39. $(x+3)^2y'' - 8(x+3)y' + 14y = 0$
40. $(x-1)^2y'' - (x-1)y' + 5y = 0$
41. $(x+2)^2y'' + (x+2)y' + y = 0$
42. $(x-4)^2y'' - 5(x-4)y' + 9y = 0$

In Problems 43–48, use the substitution $x = e^t$ to transform the given Cauchy–Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using the procedures in Sections 3.3–3.5.

43. $x^2y'' + 9xy' - 20y = 0$
44. $x^2y'' - 9xy' + 25y = 0$
45. $x^2y'' + 10xy' + 8y = x^2$
46. $x^2y'' - 4xy' + 6y = \ln x^2$
47. $x^2y'' - 3xy' + 13y = 4 + 3x$
48. $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

In Problems 49 and 50, use the substitution $t = -x$ to solve the given initial-value problem on the interval $(-\infty, 0)$.

49. $4x^2y'' + y = 0,\ y(-1) = 2, y'(-1) = 4$

50. $x^2y'' - 4xy' + 6y = 0,\ y(-2) = 8, y'(-2) = 0$

Contributed Problem

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51. **Temperature of a Fluid** A very long cylindrical shell is formed by two concentric circular cylinders of different radii. A chemically reactive fluid fills the space between the concentric cylinders as shown in green in **FIGURE 3.6.2**. The inner cylinder has a radius of 1 and is thermally insulated, while the outer cylinder has a radius of 2 and is maintained at a constant temperature T_0. The rate of heat generation in the fluid due to the chemical reactions is proportional to T/r^2, where $T(r)$ is the temperature of the fluid within the space bounded between the cylinders defined by $1 < r < 2$. Under these conditions the temperature of the fluid is defined by the following boundary-value problem:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = \frac{T}{r^2},\quad 1 < r < 2,$$

$$\left.\frac{dT}{dr}\right|_{r=1} = 0,\quad T(2) = T_0.$$

(a) Find the temperature distribution $T(r)$ within the fluid.

(b) Find the minimum and maximum values of $T(r)$ on the interval defined by $1 \le r \le 2$. Why do these values make intuitive sense?

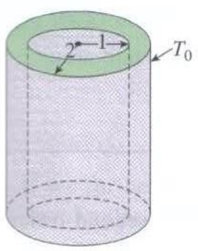

FIGURE 3.6.2 Cylindrical shell in Problem 51

≡ Discussion Problems

52. Find a Cauchy–Euler differential equation of lowest order with real coefficients if it is known that 2 and $1 - i$ are two roots of its auxiliary equation.

53. The initial conditions $y(0) = y_0$, $y'(0) = y_1$, apply to each of the following differential equations:

$$x^2y'' = 0,$$
$$x^2y'' - 2xy' + 2y = 0,$$
$$x^2y'' - 4xy' + 6y = 0.$$

For what values of y_0 and y_1 does each initial-value problem have a solution?

54. What are the x-intercepts of the solution curve shown in Figure 3.6.1? How many x-intercepts are there in the interval defined by $0 < x < \frac{1}{2}$?

≡ Computer Lab Assignments

In Problems 55–58, solve the given differential equation by using a CAS to find the (approximate) roots of the auxiliary equation.

55. $2x^3y''' - 10.98x^2y'' + 8.5xy' + 1.3y = 0$

56. $x^3y''' + 4x^2y'' + 5xy' - 9y = 0$

57. $x^4y^{(4)} + 6x^3y''' + 3x^2y'' - 3xy' + 4y = 0$

58. $x^4y^{(4)} - 6x^3y''' + 33x^2y'' - 105xy' + 169y = 0$

In Problems 59 and 60, use a CAS as an aid in computing roots of the auxiliary equation, the determinants given in (10) of Section 3.5, and integrations.

59. $x^3y''' - x^2y'' - 2xy' + 6y = x^2$

60. $x^3y''' - 2x^2y'' - 8xy' + 12y = x^{-4}$

3.7 Nonlinear Equations

≡ Introduction The difficulties that surround higher-order *nonlinear* DEs and the few methods that yield analytic solutions are examined next.

Some Differences There are several significant differences between linear and nonlinear differential equations. We saw in Section 3.1 that homogeneous linear equations of order two or