3.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-4.

3.1.1 Initial-Value and Boundary-Value Problems

In Problems 1–4, the given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

1. $y = c_1e^x + c_2e^{-x}$, $(-\infty, \infty)$; $y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$
2. $y = c_1e^{4x} + c_2e^{-x}$, $(-\infty, \infty)$; $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$
3. $y = c_1x + c_2x \ln x$, $(0, \infty)$; $x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = -1$
4. $y = c_1 + c_2 \cos x + c_3 \sin x$, $(-\infty, \infty)$; $y''' + y' = 0$, $y(\pi) = 0$, $y'(\pi) = 2$, $y''(\pi) = -1$
5. Given that $y = c_1 + c_2x^2$ is a two-parameter family of solutions of $xy'' - y' = 0$ on the interval $(-\infty, \infty)$, show that constants c_1 and c_2 cannot be found so that a member of the family satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$. Explain why this does not violate Theorem 3.1.1.
6. Find two members of the family of solutions in Problem 5 that satisfy the initial conditions $y(0) = 0$, $y'(0) = 0$.
7. Given that $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ is the general solution of $x'' + \omega^2x = 0$ on the interval $(-\infty, \infty)$, show that a solution satisfying the initial conditions $x(0) = x_0$, $x'(0) = x_1$, is given by
$$x(t) = x_0 \cos \omega t + \frac{x_1}{\omega} \sin \omega t.$$
8. Use the general solution of $x'' + \omega^2x = 0$ given in Problem 7 to show that a solution satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$, is the solution given in Problem 7 shifted by an amount t_0:
$$x(t) = x_0 \cos \omega (t - t_0) + \frac{x_1}{\omega} \sin \omega (t - t_0).$$

In Problems 9 and 10, find an interval centered about $x = 0$ for which the given initial-value problem has a unique solution.

9. $(x - 2)y'' + 3y = x$, $y(0) = 0$, $y'(0) = 1$
10. $y'' + (\tan x)y = e^x$, $y(0) = 1$, $y'(0) = 0$
11. **(a)** Use the family in Problem 1 to find a solution of $y'' - y = 0$ that satisfies the boundary conditions $y(0) = 0$, $y(1) = 1$.
    (b) The DE in part (a) has the alternative general solution $y = c_3 \cosh x + c_4 \sinh x$ on $(-\infty, \infty)$. Use this family to find a solution that satisfies the boundary conditions in part (a).
    (c) Show that the solutions in parts (a) and (b) are equivalent.
12. Use the family in Problem 5 to find a solution of $xy'' - y' = 0$ that satisfies the boundary conditions $y(0) = 1$, $y'(1) = 6$.

In Problems 13 and 14, the given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions.

13. $y = c_1e^x \cos x + c_2e^x \sin x$; $y'' - 2y' + 2y = 0$
    (a) $y(0) = 1$, $y'(\pi) = 0$ **(b)** $y(0) = 1$, $y(\pi) = -1$
    (c) $y(0) = 1$, $y(\pi/2) = 1$ **(d)** $y(0) = 0$, $y(\pi) = 0$
14. $y = c_1x^2 + c_2x^4 + 3$; $x^2y'' - 5xy' + 8y = 24$
    (a) $y(-1) = 0$, $y(1) = 4$ **(b)** $y(0) = 1$, $y(1) = 2$
    (c) $y(0) = 3$, $y(1) = 0$ **(d)** $y(1) = 3$, $y(2) = 15$

3.1.2 Homogeneous Equations

In Problems 15–22, determine whether the given set of functions is linearly dependent or linearly independent on the interval $(-\infty, \infty)$.

15. $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 4x - 3x^2$
16. $f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = e^x$
17. $f_1(x) = 5$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$
18. $f_1(x) = \cos 2x$, $f_2(x) = 1$, $f_3(x) = \cos^2 x$
19. $f_1(x) = x$, $f_2(x) = x - 1$, $f_3(x) = x + 3$
20. $f_1(x) = 2 + x$, $f_2(x) = 2 + |x|$
21. $f_1(x) = 1 + x$, $f_2(x) = x$, $f_3(x) = x^2$
22. $f_1(x) = e^x$, $f_2(x) = e^{-x}$, $f_3(x) = \sinh x$

In Problems 23–30, verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

23. $y'' - y' - 12y = 0$; e^{-3x}, e^{4x}, $(-\infty, \infty)$
24. $y'' - 4y = 0$; $\cosh 2x, \sinh 2x$, $(-\infty, \infty)$
25. $y'' - 2y' + 5y = 0$; $e^x \cos 2x, e^x \sin 2x$, $(-\infty, \infty)$
26. $4y'' - 4y' + y = 0$; $e^{x/2}, xe^{x/2}$, $(-\infty, \infty)$
27. $x^2y'' - 6xy' + 12y = 0$; x^3, x^4, $(0, \infty)$
28. $x^2y'' + xy' + y = 0$; $\cos(\ln x), \sin(\ln x)$, $(0, \infty)$
29. $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$; $x, x^{-2}, x^{-2} \ln x$, $(0, \infty)$
30. $y^{(4)} + y'' = 0$; $1, x, \cos x, \sin x$, $(-\infty, \infty)$

3.1.3 Nonhomogeneous Equations

In Problems 31–34, verify that the given two-parameter family of functions is the general solution of the nonhomogeneous differential equation on the indicated interval.

31. $y'' - 7y' + 10y = 24e^x$;
    $y = c_1e^{2x} + c_2e^{5x} + 6e^x$, $(-\infty, \infty)$
32. $y'' + y = \sec x$;
    $y = c_1 \cos x + c_2 \sin x + x \sin x + (\cos x) \ln(\cos x)$, $(-\pi/2, \pi/2)$
33. $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$;
    $y = c_1e^{2x} + c_2xe^{2x} + x^2e^{2x} + x - 2$, $(-\infty, \infty)$

34. $2x^2y'' + 5xy' + y = x^2 - x$;
$y = c_1x^{-1/2} + c_2x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x$, $(0, \infty)$

35. (a) Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are, respectively, particular solutions of

$$y'' - 6y' + 5y = -9e^{2x}$$

and $$y'' - 6y' + 5y = 5x^2 + 3x - 16.$$

(b) Use part (a) to find particular solutions of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$$

and $$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}.$$

36. (a) By inspection, find a particular solution of

$$y'' + 2y = 10.$$

(b) By inspection, find a particular solution of

$$y'' + 2y = -4x.$$

(c) Find a particular solution of $y'' + 2y = -4x + 10$.
(d) Find a particular solution of $y'' + 2y = 8x + 5$.

Discussion Problems

37. Let $n = 1, 2, 3, \ldots$. Discuss how the observations $D^nx^{n-1} = 0$ and $D^nx^n = n!$ can be used to find the general solutions of the given differential equations.
(a) $y'' = 0$
(b) $y''' = 0$
(c) $y^{(4)} = 0$
(d) $y'' = 2$
(e) $y''' = 6$
(f) $y^{(4)} = 24$

38. Suppose that $y_1 = e^x$ and $y_2 = e^{-x}$ are two solutions of a homogeneous linear differential equation. Explain why $y_3 = \cosh x$ and $y_4 = \sinh x$ are also solutions of the equation.

39. (a) Verify that $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent solutions of the differential equation $x^2y'' - 4xy' + 6y = 0$ on the interval $(-\infty, \infty)$.
(b) Show that $W(y_1, y_2) = 0$ for every real number x. Does this result violate Theorem 3.1.3? Explain.
(c) Verify that $Y_1 = x^3$ and $Y_2 = x^2$ are also linearly independent solutions of the differential equation in part (a) on the interval $(-\infty, \infty)$.
(d) Find a solution of the differential equation satisfying $y(0) = 0$, $y'(0) = 0$.
(e) By the superposition principle, Theorem 3.1.2, both linear combinations $y = c_1y_1 + c_2y_2$ and $Y = c_1Y_1 + c_2Y_2$ are solutions of the differential equation. Discuss whether one, both, or neither of the linear combinations is a general solution of the differential equation on the interval $(-\infty, \infty)$.

40. Is the set of functions $f_1(x) = e^{x+2}$, $f_2(x) = e^{x-3}$ linearly dependent or linearly independent on the interval $(-\infty, \infty)$? Discuss.

41. Suppose $y_1, y_2, \ldots, y_k$ are k linearly independent solutions on $(-\infty, \infty)$ of a homogeneous linear nth-order differential equation with constant coefficients. By Theorem 3.1.2 it follows that $y_{k+1} = 0$ is also a solution of the differential equation. Is the set of solutions $y_1, y_2, \ldots, y_k, y_{k+1}$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

42. Suppose that $y_1, y_2, \ldots, y_k$ are k nontrivial solutions of a homogeneous linear nth-order differential equation with constant coefficients and that $k = n + 1$. Is the set of solutions $y_1, y_2, \ldots, y_k$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

3.2 Reduction of Order

Introduction In Section 3.1 we saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

was a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. Beginning in the next section we examine a method for determining these solutions when the coefficients of the DE in (1) are constants. This method, which is a straightforward exercise in algebra, breaks down in a few cases and yields only a single solution y_1 of the DE. It turns out that we can construct a second solution y_2 of a homogeneous equation (1) (even when the coefficients in (1) are variable) provided that we know one nontrivial solution y_1 of the DE. The basic idea described in this section is that the linear *second-order* equation (1) can be reduced to a linear *first-order* DE by means of a substitution involving the known solution y_1. A second solution, y_2 of (1), is apparent after this first-order DE is solved.

Reduction of Order Suppose $y(x)$ denotes a known solution of equation (1). We seek a second solution $y_2(x)$ of (1) so that y_1 and y_2 are linearly independent on some interval I. Recall