$$w(x) = w_0\left(1 - \frac{2}{L}x\right) - w_0\left(1 - \frac{2}{L}x\right)\mathcal{U}\left(x - \frac{L}{2}\right)$$

$$= \frac{2w_0}{L}\left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right)\mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

Transforming (19) with respect to the variable x gives

$$EI(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) = \frac{2w_0}{L}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right]$$

or

$$s^4Y(s) - sy''(0) - y'''(0) = \frac{2w_0}{EIL}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right].$$

If we let $c_1 = y''(0)$ and $c_2 = y'''(0)$, then

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL}\left[\frac{L/2}{s^5} - \frac{1}{s^6} + \frac{1}{s^6}e^{-Ls/2}\right],$$

and consequently

$$y(x) = \frac{c_1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} + \frac{c_2}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$+ \frac{2w_0}{EIL}\left[\frac{L/2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} - \frac{1}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} + \frac{1}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{s^6}e^{-Ls/2}\right\}\right]$$

$$= \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5\mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

Applying the conditions $y(L) = 0$ and $y'(L) = 0$ to the last result yields a system of equations for c_1 and c_2:

$$c_1\frac{L^2}{2} + c_2\frac{L^3}{6} + \frac{49w_0L^4}{1920EI} = 0$$

$$c_1L + c_2\frac{L^2}{2} + \frac{85w_0L^3}{960EI} = 0.$$

Solving, we find $c_1 = 23w_0L^2/960EI$ and $c_2 = -9w_0L/40EI$. Thus the deflection is

$$y(x) = \frac{23w_0L^2}{1920EI}x^2 - \frac{3w_0L}{80EI}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5\mathcal{U}\left(x - \frac{L}{2}\right)\right]. \quad \equiv$$

4.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-9.

4.3.1 Translation on the s-axis

In Problems 1–20, find either $F(s)$ or $f(t)$, as indicated.

1. $\mathcal{L}\{te^{10t}\}$
2. $\mathcal{L}\{te^{-6t}\}$
3. $\mathcal{L}\{t^3e^{-2t}\}$
4. $\mathcal{L}\{t^{10}e^{-7t}\}$
5. $\mathcal{L}\{t(e^t + e^{2t})^2\}$
6. $\mathcal{L}\{e^{2t}(t - 1)^2\}$
7. $\mathcal{L}\{e^t \sin 3t\}$
8. $\mathcal{L}\{e^{-2t}\cos 4t\}$
9. $\mathcal{L}\{(1 - e^t + 3e^{-4t})\cos 5t\}$
10. $\mathcal{L}\left\{e^{3t}\left(9 - 4t + 10\sin\frac{t}{2}\right)\right\}$
11. $\mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^3}\right\}$
12. $\mathcal{L}^{-1}\left\{\frac{1}{(s - 1)^4}\right\}$
13. $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 6s + 10}\right\}$
14. $\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 2s + 5}\right\}$
15. $\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4s + 5}\right\}$
16. $\mathcal{L}^{-1}\left\{\frac{2s + 5}{s^2 + 6s + 34}\right\}$

17. $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^2}\right\}$

18. $\mathcal{L}^{-1}\left\{\dfrac{5s}{(s-2)^2}\right\}$

19. $\mathcal{L}^{-1}\left\{\dfrac{2s-1}{s^2(s+1)^3}\right\}$

20. $\mathcal{L}^{-1}\left\{\dfrac{(s+1)^2}{(s+2)^4}\right\}$

In Problems 21–30, use the Laplace transform to solve the given initial-value problem.

21. $y' + 4y = e^{-4t}, \quad y(0) = 2$
22. $y' - y = 1 + te^t, \quad y(0) = 0$
23. $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$
24. $y'' - 4y' + 4y = t^3e^{2t}, \quad y(0) = 0, \quad y'(0) = 0$
25. $y'' - 6y' + 9y = t, \quad y(0) = 0, \quad y'(0) = 1$
26. $y'' - 4y' + 4y = t^3, \quad y(0) = 1, \quad y'(0) = 0$
27. $y'' - 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = -3$
28. $2y'' + 20y' + 51y = 0, \quad y(0) = 2, \quad y'(0) = 0$
29. $y'' - y' = e^t \cos t, \quad y(0) = 0, \quad y'(0) = 0$
30. $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, \quad y'(0) = 4$

In Problems 31 and 32, use the Laplace transform and the procedure outlined in Example 10 to solve the given boundary-value problem.

31. $y'' + 2y' + y = 0, \quad y'(0) = 2, \quad y(1) = 2$
32. $y'' + 8y' + 20y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$
33. A 4-lb weight stretches a spring 2 ft. The weight is released from rest 18 in above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.
34. Recall that the differential equation for the instantaneous charge $q(t)$ on the capacitor in an LRC-series circuit is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{20}$$

See Section 3.8. Use the Laplace transform to find $q(t)$ when $L = 1$ h, $R = 20\ \Omega$, $C = 0.005$ f, $E(t) = 150$ V, $t > 0$, $q(0) = 0$, and $i(0) = 0$. What is the current $i(t)$?

35. Consider the battery of constant voltage E_0 that charges the capacitor shown in **FIGURE 4.3.9**. Divide equation (20) by L and define $2\lambda = R/L$ and $\omega^2 = 1/LC$. Use the Laplace transform to show that the solution $q(t)$ of $q'' + 2\lambda q' + \omega^2 q = E_0/L$, subject to $q(0) = 0$, $i(0) = 0$, is

$$q(t) = \begin{cases} E_0C\left[1 - e^{-\lambda t}\left(\cosh\sqrt{\lambda^2-\omega^2}\,t + \dfrac{\lambda}{\sqrt{\lambda^2-\omega^2}}\sinh\sqrt{\lambda^2-\omega^2}\,t\right)\right], & \lambda > \omega \\ E_0C\left[1 - e^{-\lambda t}(1+\lambda t)\right], & \lambda = \omega \\ E_0C\left[1 - e^{-\lambda t}\left(\cos\sqrt{\omega^2-\lambda^2}\,t + \dfrac{\lambda}{\sqrt{\omega^2-\lambda^2}}\sin\sqrt{\omega^2-\lambda^2}\,t\right)\right], & \lambda < \omega. \end{cases}$$

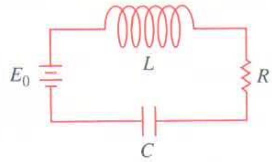


FIGURE 4.3.9 Circuit in Problem 35

36. Use the Laplace transform to find the charge $q(t)$ in an RC-series when $q(0) = 0$ and $E(t) = E_0e^{-kt}$, $k > 0$. Consider two cases: $k \neq 1/RC$ and $k = 1/RC$.

4.3.2 Translation on the t-axis

In Problems 37–48, find either $F(s)$ or $f(t)$, as indicated.

37. $\mathcal{L}\{(t-1)\mathcal{U}(t-1)\}$
38. $\mathcal{L}\{e^{2-t}\mathcal{U}(t-2)\}$
39. $\mathcal{L}\{t\,\mathcal{U}(t-2)\}$
40. $\mathcal{L}\{(3t+1)\mathcal{U}(t-1)\}$
41. $\mathcal{L}\{\cos 2t\,\mathcal{U}(t-\pi)\}$
42. $\mathcal{L}\{\sin t\,\mathcal{U}(t-\pi/2)\}$
43. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\}$
44. $\mathcal{L}^{-1}\left\{\dfrac{(1+e^{-2s})^2}{s+2}\right\}$
45. $\mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$
46. $\mathcal{L}^{-1}\left\{\dfrac{se^{-\pi s/2}}{s^2+4}\right\}$
47. $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$
48. $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\}$

In Problems 49–54, match the given graph with one of the given functions in (a)–(f). The graph of $f(t)$ is given in **FIGURE 4.3.10**.

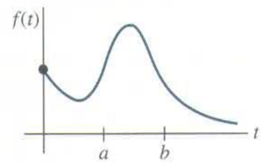


FIGURE 4.3.10 Graph for Problems 49–54

(a) $f(t) - f(t)\,\mathcal{U}(t-a)$
(b) $f(t-b)\,\mathcal{U}(t-b)$
(c) $f(t)\,\mathcal{U}(t-a)$
(d) $f(t) - f(t)\,\mathcal{U}(t-b)$
(e) $f(t)\,\mathcal{U}(t-a) - f(t)\,\mathcal{U}(t-b)$
(f) $f(t-a)\,\mathcal{U}(t-a) - f(t-a)\,\mathcal{U}(t-b)$

49.

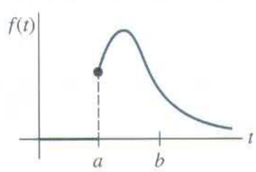


FIGURE 4.3.11 Graph for Problem 49

50.

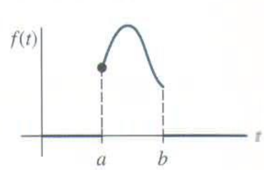


FIGURE 4.3.12 Graph for Problem 50

51.

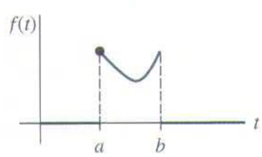


FIGURE 4.3.13 Graph for Problem 51

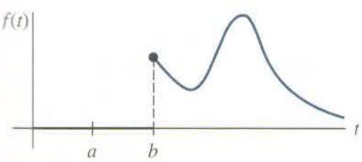
52.


FIGURE 4.3.14 Graph for Problem 52

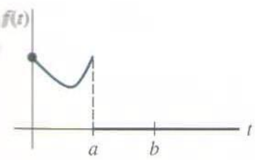
53.


FIGURE 4.3.15 Graph for Problem 53

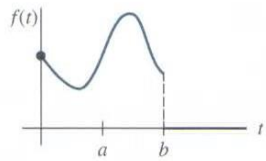
54.


FIGURE 4.3.16 Graph for Problem 54

In Problems 55–62, write each function in terms of unit step functions. Find the Laplace transform of the given function.

55. $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & t \ge 3 \end{cases}$

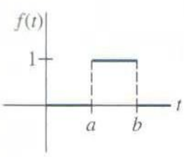
56. $f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & 4 \le t < 5 \\ 1, & t \ge 5 \end{cases}$

57. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & t \ge 1 \end{cases}$

58. $f(t) = \begin{cases} 0, & 0 \le t < 3\pi/2 \\ \sin t, & t \ge 3\pi/2 \end{cases}$

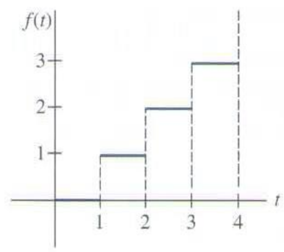
59. $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

60. $f(t) = \begin{cases} \sin t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

61.


rectangular pulse

FIGURE 4.3.17 Graph for Problem 61

62.


staircase function

FIGURE 4.3.18 Graph for Problem 62

In Problems 63–70, use the Laplace transform to solve the given initial-value problem.

63. $y' + y = f(t), \quad y(0) = 0$, where $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 5, & t \ge 1 \end{cases}$

64. $y' + y = f(t), \quad y(0) = 0$, where $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1 \end{cases}$

65. $y' + 2y = f(t), \quad y(0) = 0$, where $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

66. $y'' + 4y = f(t), \quad y(0) = 0, \; y'(0) = -1$, where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

67. $y'' + 4y = \sin t\, \mathcal{U}(t - 2\pi), \quad y(0) = 1, \; y'(0) = 0$

68. $y'' - 5y' + 6y = \mathcal{U}(t - 1), \quad y(0) = 0, \; y'(0) = 1$

69. $y'' + y = f(t), \quad y(0) = 0, \; y'(0) = 1$, where

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

70. $y'' + 4y' + 3y = 1 - \mathcal{U}(t - 2) - \mathcal{U}(t - 4) + \mathcal{U}(t - 6)$, $y(0) = 0, \; y'(0) = 0$

71. Suppose a mass weighing 32 lb stretches a spring 2 ft. If the weight is released from rest at the equilibrium position, find the equation of motion $x(t)$ if an impressed force $f(t) = 20t$ acts on the system for $0 \le t < 5$ and is then removed (see Example 5). Ignore any damping forces. Use a graphing utility to obtain the graph $x(t)$ on the interval $[0, 10]$.

72. Solve Problem 71 if the impressed force $f(t) = \sin t$ acts on the system for $0 \le t < 2\pi$ and is then removed.

In Problems 73 and 74, use the Laplace transform to find the charge $q(t)$ on the capacitor in an RC-series circuit subject to the given conditions.

73. $q(0) = 0, R = 2.5\ \Omega$, $C = 0.08$ f, $E(t)$ given in **FIGURE 4.3.19**

74. $q(0) = q_0, R = 10\ \Omega, C = 0.1$ f, $E(t)$ given in **FIGURE 4.3.20**

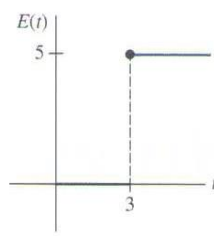


FIGURE 4.3.19 $E(t)$ in Problem 73

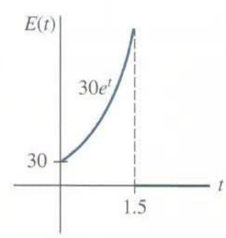


FIGURE 4.3.20 $E(t)$ in Problem 74

75. **(a)** Use the Laplace transform to find the current $i(t)$ in a single-loop LR-series circuit when $i(0) = 0$, $L = 1$ h, $R = 10\ \Omega$, and $E(t)$ is as given in **FIGURE 4.3.21**.

(b) Use a computer graphing program to graph $i(t)$ for $0 \le t \le 6$. Use the graph to estimate i_{max} and i_{min}, the maximum and minimum values of the current, respectively.

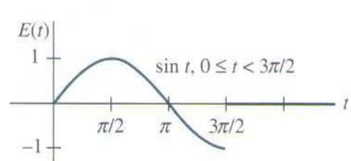


FIGURE 4.3.21 $E(t)$ in Problem 75

76. (a) Use the Laplace transform to find the charge $q(t)$ on the capacitor in an RC-series circuit when $q(0) = 0$, $R = 50\ \Omega$, $C = 0.01$ f, and $E(t)$ is as given in **FIGURE 4.3.22**.
(b) Assume $E_0 = 100$ V. Use a computer graphing program to graph $q(t)$ for $0 \le t \le 6$. Use the graph to estimate q_{max}, the maximum value of the charge.

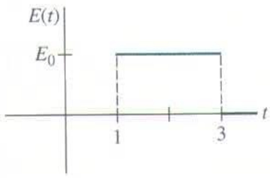


FIGURE 4.3.22 $E(t)$ in Problem 76

77. A cantilever beam is embedded at its left end and free at its right end. Use the Laplace transform to find the deflection $y(x)$ when the load is given by

$$w(x) = \begin{cases} w_0, & 0 < x < L/2 \\ 0, & L/2 \le x < L. \end{cases}$$

78. Solve Problem 77 when the load is given by

$$w(x) = \begin{cases} 0, & 0 < x < L/3 \\ w_0, & L/3 \le x < 2L/3 \\ 0, & 2L/3 \le x < L. \end{cases}$$

79. Find the deflection $y(x)$ of a cantilever beam embedded at its left end and free at its right end when the load is as given in Example 10.

80. A beam is embedded at its left end and simply supported at its right end. Find the deflection $y(x)$ when the load is as given in Problem 77.

81. **Cake Inside an Oven** Reread Example 4 in Section 2.7 on the cooling of a cake that is taken out of an oven.
(a) Devise a mathematical model for the temperature of a cake while it is *inside* the oven based on the following assumptions: At $t = 0$ the cake mixture is at the room temperature of 70°; the oven is not preheated so that at $t = 0$, when the cake mixture is placed into the oven, the temperature inside the oven is also 70°; the temperature of the oven increases linearly until $t = 4$ minutes, when the desired temperature of 300° is attained; the oven temperature is a constant 300° for $t \ge 4$.
(b) Use the Laplace transform to solve the initial-value problem in part (a).

≡ Discussion Problems

82. Discuss how you would fix up each of the following functions so that Theorem 4.3.2 could be used directly to find the given Laplace transform. Check your answers using (16) of this section.
(a) $\mathscr{L}\{(2t + 1)\mathscr{U}(t - 1)\}$
(b) $\mathscr{L}\{e^t\mathscr{U}(t - 5)\}$
(c) $\mathscr{L}\{\cos t\,\mathscr{U}(t - \pi)\}$
(d) $\mathscr{L}\{(t^2 - 3t)\mathscr{U}(t - 2)\}$

83. (a) Assume that Theorem 4.3.1 holds when the symbol a is replaced by ki, where k is a real number and $i^2 = -1$. Show that $\mathscr{L}\{te^{kti}\}$ can be used to deduce

$$\mathscr{L}\{t\cos kt\} = \frac{s^2 - k^2}{(s^2 + k^2)^2}$$

and

$$\mathscr{L}\{t\sin kt\} = \frac{2ks}{(s^2 + k^2)^2}.$$

(b) Now use the Laplace transform to solve the initial-value problem

$$x'' + \omega^2 x = \cos \omega t, \quad x(0) = 0,\ x'(0) = 0.$$

4.4 Additional Operational Properties

≡ Introduction In this section we develop several more operational properties of the Laplace transform. Specifically, we shall see how to find the transform of a function $f(t)$ that is multiplied by a monomial t^n, the transform of a special type of integral, and the transform of a periodic function. The last two transform properties allow us to solve some equations that we have not encountered up to this point: Volterra integral equations, integrodifferential equations, and ordinary differential equations in which the input function is a periodic piecewise-defined function.

4.4.1 Derivatives of Transforms

☐ Multiplying a Function by t^n The Laplace transform of the product of a function $f(t)$ with t can be found by differentiating the Laplace transform of $f(t)$. If $F(s) = \mathscr{L}\{f(t)\}$ and we assume that interchanging of differentiation and integration is possible, then

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\,dt = -\int_0^\infty e^{-st}tf(t)\,dt = -\mathscr{L}\{tf(t)\};$$

that is,

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}.$$