Translated, this means $y = c_2e^{-x}\cos x + c_1e^{-x}\sin x$ is a solution of $y'' + 2y' + 2y = 0$.

In the classic text *Differential Equations* by Ralph Palmer Agnew* (used by the author as a student), the following statement is made:

> *It is not reasonable to expect students in this course to have computing skills and equipment necessary for efficient solving of equations such as*

$$4.317\frac{d^4y}{dx^4} + 2.179\frac{d^3y}{dx^3} + 1.416\frac{d^2y}{dx^2} + 1.295\frac{dy}{dx} + 3.169y = 0. \tag{15}$$

Although it is debatable whether computing skills have improved in the intervening years, it is a certainty that technology has. If one has access to a computer algebra system, equation (15) could be considered reasonable. After simplification and some relabeling of the output, *Mathematica* yields the (approximate) general solution

$$y = c_1e^{-0.728852x}\cos(0.618605x) + c_2e^{-0.728852x}\sin(0.618605x)$$
$$+ c_3e^{-0.476478x}\cos(0.759081x) + c_4e^{-0.476478x}\sin(0.759081x).$$

We note in passing that the **DSolve** and **dsolve** commands in *Mathematica* and *Maple*, like most aspects of any CAS, have their limitations.

Finally, if we are faced with an initial-value problem consisting of, say, a fourth-order differential equation, then to fit the general solution of the DE to the four initial conditions we must solve a system of four linear equations in four unknowns (the c_1, c_2, c_3, c_4 in the general solution). Using a CAS to solve the system can save lots of time. See Problems 35, 36, 69, and 70 in Exercises 3.3.

*McGraw-Hill, New York, 1960.

Remarks

In case you are wondering, the method of this section also works for homogeneous linear *first-order* differential equations $ay' + by = 0$ with constant coefficients. For example, to solve, say, $2y' + 7y = 0$, we substitute $y = e^{mx}$ into the DE to obtain the auxiliary equation $2m + 7 = 0$. Using $m = -\frac{7}{2}$, the general solution of the DE is then $y = c_1e^{-7x/2}$.

3.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–14, find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$
2. $y'' - 36y = 0$
3. $y'' - y' - 6y = 0$
4. $y'' - 3y' + 2y = 0$
5. $y'' + 8y' + 16y = 0$
6. $y'' - 10y' + 25y = 0$
7. $12y'' - 5y' - 2y = 0$
8. $y'' + 4y' - y = 0$
9. $y'' + 9y = 0$
10. $3y'' + y = 0$
11. $y'' - 4y' + 5y = 0$
12. $2y'' + 2y' + y = 0$
13. $3y'' + 2y' + y = 0$
14. $2y'' - 3y' + 4y = 0$

In Problems 15–28, find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y' = 0$
16. $y''' - y = 0$
17. $y''' - 5y'' + 3y' + 9y = 0$
18. $y''' + 3y'' - 4y' - 12y = 0$
19. $\dfrac{d^3u}{dt^3} + \dfrac{d^2u}{dt^2} - 2u = 0$
20. $\dfrac{d^3x}{dt^3} - \dfrac{d^2x}{dt^2} - 4x = 0$
21. $y''' + 3y'' + 3y' + y = 0$
22. $y''' - 6y'' + 12y' - 8y = 0$
23. $y^{(4)} + y''' + y'' = 0$
24. $y^{(4)} - 2y'' + y = 0$

25. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0$

26. $\dfrac{d^4y}{dx^4} - 7\dfrac{d^2y}{dx^2} - 18y = 0$

27. $\dfrac{d^5u}{dr^5} + 5\dfrac{d^4u}{dr^4} - 2\dfrac{d^3u}{dr^3} - 10\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + 5u = 0$

28. $2\dfrac{d^5x}{ds^5} - 7\dfrac{d^4x}{ds^4} + 12\dfrac{d^3x}{ds^3} + 8\dfrac{d^2x}{ds^2} = 0$

In Problems 29–36, solve the given initial-value problem.

29. $y'' + 16y = 0,\ y(0) = 2, y'(0) = -2$

30. $\dfrac{d^2y}{d\theta^2} + y = 0,\ y(\pi/3) = 0, y'(\pi/3) = 2$

31. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0,\ y(1) = 0, y'(1) = 2$

32. $4y'' - 4y' - 3y = 0,\ y(0) = 1, y'(0) = 5$

33. $y'' + y' + 2y = 0,\ y(0) = y'(0) = 0$

34. $y'' - 2y' + y = 0,\ y(0) = 5, y'(0) = 10$

35. $y''' + 12y'' + 36y' = 0,\ y(0) = 0, y'(0) = 1, y''(0) = -7$

36. $y''' + 2y'' - 5y' - 6y = 0,\ y(0) = y'(0) = 0, y''(0) = 1$

In Problems 37–40, solve the given boundary-value problem.

37. $y'' - 10y' + 25y = 0,\ y(0) = 1, y(1) = 0$

38. $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 0$

39. $y'' + y = 0,\ y'(0) = 0,\ y'(\pi/2) = 0$

40. $y'' - 2y' + 2y = 0,\ y(0) = 1, y(\pi) = 1$

In Problems 41 and 42, solve the given problem first using the form of the general solution given in (10). Solve again, this time using the form given in (11).

41. $y'' - 3y = 0,\ y(0) = 1, y'(0) = 5$

42. $y'' - y = 0,\ y(0) = 1, y'(1) = 0$

In Problems 43–48, each figure represents the graph of a particular solution of one of the following differential equations:

(a) $y'' - 3y' - 4y = 0$ (b) $y'' + 4y = 0$

(c) $y'' + 2y' + y = 0$ (d) $y'' + y = 0$

(e) $y'' + 2y' + 2y = 0$ (f) $y'' - 3y' + 2y = 0$

Match a solution curve with one of the differential equations. Explain your reasoning.

43.

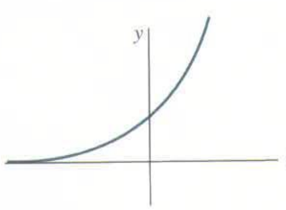

FIGURE 3.3.2 Graph for Problem 43

44.

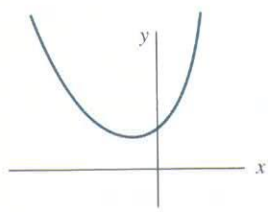

FIGURE 3.3.3 Graph for Problem 44

45.

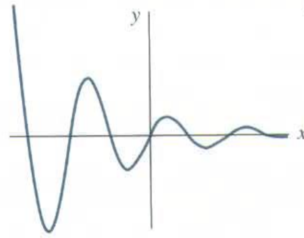

FIGURE 3.3.4 Graph for Problem 45

46.

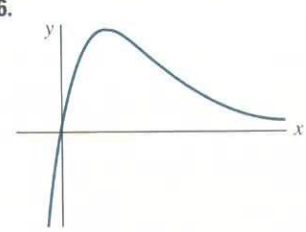

FIGURE 3.3.5 Graph for Problem 46

47.

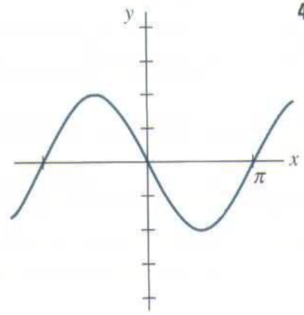

FIGURE 3.3.6 Graph for Problem 47

48.

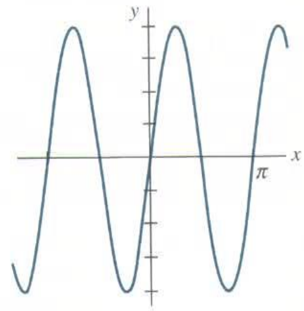

FIGURE 3.3.7 Graph for Problem 48

In Problems 49–58 find a homogeneous linear differential equation with constant coefficients whose general solution is given.

49. $y = c_1e^{x} + c_2e^{6x}$

50. $y = c_1e^{-5x} + c_2e^{-4x}$

51. $y = c_1 + c_2e^{3x}$

52. $y = c_1e^{-10x} + c_2xe^{-10x}$

53. $y = c_1\cos 8x + c_2\sin 8x$

54. $y = c_1\cosh \frac{1}{2}x + c_2\sinh \frac{1}{2}x$

55. $y = c_1e^{x}\cos x + c_2e^{x}\sin x$

56. $y = c_1 + c_2e^{-2x}\cos 5x + c_3e^{-2x}\sin 5x$

57. $y = c_1 + c_2x + c_3e^{7x}$

58. $y = c_1\cos x + c_2\sin x + c_3\cos 3x + c_4\sin 3x$

≡ Discussion Problems

59. Two roots of a cubic auxiliary equation with real coefficients are $m_1 = -\frac{1}{2}$ and $m_2 = 3 + i$. What is the corresponding homogeneous linear differential equation?

60. Find the general solution of $y''' + 6y'' + y' - 34y = 0$ if it is known that $y_1 = e^{-4x}\cos x$ is one solution.

61. To solve $y^{(4)} + y = 0$ we must find the roots of $m^4 + 1 = 0$. This is a trivial problem using a CAS, but it can also be done by hand working with complex numbers. Observe that $m^4 + 1 = (m^2 + 1)^2 - 2m^2$. How does this help? Solve the differential equation.

62. Verify that $y = \sinh x - 2\cos(x + \pi/6)$ is a particular solution of $y^{(4)} - y = 0$. Reconcile this particular solution with the general solution of the DE.

63. Consider the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(\pi/2) = 0$. Discuss: Is it possible to determine values of λ so that the problem possesses (a) trivial solutions? (b) nontrivial solutions?

64. In the study of techniques of integration in calculus, certain indefinite integrals of the form $\int e^{ax} f(x)\, dx$ could be evaluated by applying integration by parts twice, recovering the original integral on the right-hand side, solving for the original integral, and obtaining a constant multiple $k \int e^{ax} f(x)\, dx$ on the left-hand side. Then the value of the integral is found by dividing by k. Discuss: For what kinds of functions f does the described procedure work? Your solution should lead to a differential equation. Carefully analyze this equation and solve for f.

Computer Lab Assignments

In Problems 65–68, use a computer either as an aid in solving the auxiliary equation or as a means of directly obtaining the general solution of the given differential equation. If you use a CAS to obtain the general solution, simplify the output and, if necessary, write the solution in terms of real functions.

65. $y''' - 6y'' + 2y' + y = 0$

66. $6.11y''' + 8.59y'' + 7.93y' + 0.778y = 0$

67. $3.15y^{(4)} - 5.34y'' + 6.33y' - 2.03y = 0$

68. $y^{(4)} + 2y'' - y' + 2y = 0$

In Problems 69 and 70, use a CAS as an aid in solving the auxiliary equation. Form the general solution of the differential equation. Then use a CAS as an aid in solving the system of equations for the coefficients c_i, $i = 1, 2, 3, 4$ that result when the initial conditions are applied to the general solution.

69. $2y^{(4)} + 3y''' - 16y'' + 15y' - 4y = 0$,
$y(0) = -2$, $y'(0) = 6$, $y''(0) = 3$, $y'''(0) = \frac{1}{2}$

70. $y^{(4)} - 3y''' + 3y'' - y' = 0$,
$y(0) = y'(0) = 0$, $y''(0) = y'''(0) = 1$

3.4 Undetermined Coefficients

Introduction To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \tag{1}$$

we must do two things: (*i*) find the complementary function y_c; and (*ii*) find *any* particular solution y_p of the nonhomogeneous equation. Then, as discussed in Section 3.1, the general solution of (1) on an interval I is $y = y_c + y_p$.

The complementary function y_c is the general solution of the associated homogeneous DE of (1), that is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

In the last section we saw how to solve these kinds of equations when the coefficients were constants. Our goal then in the present section is to examine a method for obtaining particular solutions.

Method of Undetermined Coefficients The first of two ways we shall consider for obtaining a particular solution y_p is called the **method of undetermined coefficients**. The underlying idea in this method is a conjecture, an educated guess really, about the form of y_p motivated by the kinds of functions that make up the input function $g(x)$. The general method is limited to nonhomogeneous linear DEs such as (1) where

- the coefficients, a_i, $i = 0, 1, \ldots, n$ are constants, and
- where $g(x)$ is a constant, a polynomial function, exponential function $e^{\alpha x}$, sine or cosine functions $\sin \beta x$ or $\cos \beta x$, or finite sums and products of these functions.

Strictly speaking, $g(x) = k$ (a constant) is a polynomial function. Since a constant function is probably not the first thing that comes to mind when you think of polynomial functions, for emphasis we shall continue to use the redundancy "constant functions, polynomial functions, ..."

A constant k is a polynomial function of degree 0.