EXAMPLE 4 A Special Case of System (1)

Solve

$$\begin{aligned} x_1'' + 10x_1 \quad\;\; - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned} \tag{13}$$

subject to $x_1(0) = 0, x_1'(0) = 1, x_2(0) = 0, x_2'(0) = -1$.

SOLUTION Using elimination on the equivalent form of the system

$$\begin{aligned} (D^2 + 10)x_1 \quad\;\; - 4x_2 &= 0 \\ -4x_1 + (D^2 + 4)x_2 &= 0 \end{aligned}$$

we find that x_1 and x_2 satisfy, respectively,

$$(D^2 + 2)(D^2 + 12)x_1 = 0 \quad \text{and} \quad (D^2 + 2)(D^2 + 12)x_2 = 0.$$

Thus we find

$$\begin{aligned} x_1(t) &= c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + c_3 \cos 2\sqrt{3}t + c_4 \sin 2\sqrt{3}t \\ x_2(t) &= c_5 \cos \sqrt{2}t + c_6 \sin \sqrt{2}t + c_7 \cos 2\sqrt{3}t + c_8 \sin 2\sqrt{3}t. \end{aligned}$$

Substituting both expressions into the first equation of (13) and simplifying eventually yields $c_5 = 2c_1, c_6 = 2c_2, c_7 = -\frac{1}{2}c_3, c_8 = -\frac{1}{2}c_4$. Thus, a solution of (13) is

$$\begin{aligned} x_1(t) &= c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + c_3 \cos 2\sqrt{3}t + c_4 \sin 2\sqrt{3}t \\ x_2(t) &= 2c_1 \cos \sqrt{2}t + 2c_2 \sin \sqrt{2}t - \tfrac{1}{2}c_3 \cos 2\sqrt{3}t - \tfrac{1}{2}c_4 \sin 2\sqrt{3}t. \end{aligned}$$

The stipulated initial conditions then imply $c_1 = 0, c_2 = -\sqrt{2}/10, c_3 = 0, c_4 = \sqrt{3}/5$. And so the solution of the initial-value problem is

$$\begin{aligned} x_1(t) &= -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t \\ x_2(t) &= -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t. \end{aligned} \tag{14}$$

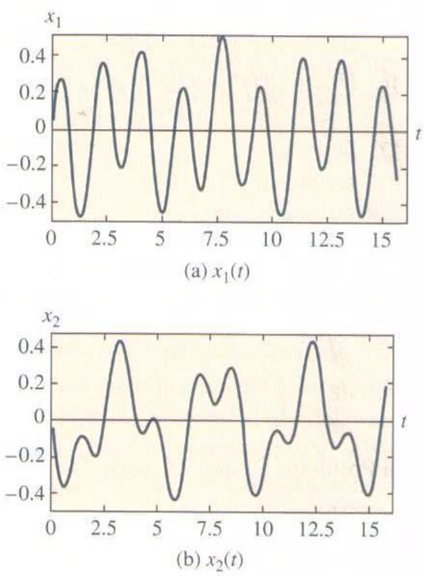

FIGURE 3.12.3 Displacements of the two masses in Example 4

The graphs of x_1 and x_2 in **FIGURE 3.12.3** reveal the complicated oscillatory motion of each mass. ≡

We will revisit Example 4 in Section 4.6, where we will solve the system in (13) by means of the Laplace transform.

3.12 Exercises

Answers to selected odd-numbered problems begin on page ANS-8.

In Problems 1–20, solve the given system of differential equations by systematic elimination.

1. $\dfrac{dx}{dt} = 2x - y$
   $\dfrac{dy}{dt} = x$

2. $\dfrac{dx}{dt} = 4x + 7y$
   $\dfrac{dy}{dt} = x - 2y$

3. $\dfrac{dx}{dt} = -y + t$
   $\dfrac{dy}{dt} = x - t$

4. $\dfrac{dx}{dt} - 4y = 1$
   $\dfrac{dy}{dt} + x = 2$

5. $(D^2 + 5)x - \quad 2y = 0$
   $-2x + (D^2 + 2)y = 0$

6. $(D + 1)x + (D - 1)y = 2$
$3x + (D + 2)y = -1$

7. $\dfrac{d^2x}{dt^2} = 4y + e^t$
$\dfrac{d^2y}{dt^2} = 4x - e^t$

8. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x$
$\dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$

9. $Dx + D^2y = e^{3t}$
$(D + 1)x + (D - 1)y = 4e^{3t}$

10. $D^2x - Dy = t$
$(D + 3)x + (D + 3)y = 2$

11. $(D^2 - 1)x - y = 0$
$(D - 1)x + Dy = 0$

12. $(2D^2 - D - 1)x - (2D + 1)y = 1$
$(D - 1)x + \qquad Dy = -1$

13. $2\dfrac{dx}{dt} - 5x + \dfrac{dy}{dt} = e^t$
$\dfrac{dx}{dt} - x + \dfrac{dy}{dt} = 5e^t$

14. $\dfrac{dx}{dt} + \dfrac{dy}{dt} = e^t$
$-\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + y = 0$

15. $(D - 1)x + (D^2 + 1)y = 1$
$(D^2 - 1)x + (D + 1)y = 2$

16. $D^2x - 2(D^2 + D)y = \sin t$
$x + \qquad Dy = 0$

17. $Dx = y$
$Dy = z$
$Dz = x$

18. $Dx + \qquad z = e^t$
$(D - 1)x + Dy + Dz = 0$
$x + 2y + Dz = e^t$

19. $\dfrac{dx}{dt} = 6y$
$\dfrac{dy}{dt} = x + z$
$\dfrac{dz}{dt} = x + y$

20. $\dfrac{dx}{dt} = -x + z$
$\dfrac{dy}{dt} = -y + z$
$\dfrac{dz}{dt} = -x + y$

In Problems 21 and 22, solve the given initial-value problem.

21. $\dfrac{dx}{dt} = -5x - y$
$\dfrac{dy}{dt} = 4x - y$
$x(1) = 0, y(1) = 1$

22. $\dfrac{dx}{dt} = y - 1$
$\dfrac{dy}{dt} = -3x + 2y$
$x(0) = 0, y(0) = 0$

≡ Mathematical Models

23. **Projectile Motion** A projectile shot from a gun has weight $w = mg$ and velocity $\mathbf{v}$ tangent to its path of motion or trajectory. Ignoring air resistance and all other forces acting on the projectile except its weight, determine a system of differential equations that describes its path of motion. See **FIGURE 3.12.4**. Solve the system. [*Hint*: Use Newton's second law of motion in the x and y directions.]

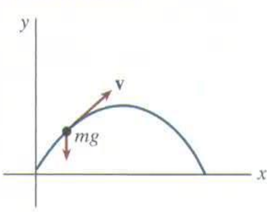

FIGURE 3.12.4 Path of projectile in Problem 23

24. **Projectile Motion with Air Resistance** Determine a system of differential equations that describes the path of motion in Problem 23 if linear air resistance is a retarding force $\mathbf{k}$ (of magnitude k) acting tangent to the path of the projectile but opposite to its motion. See **FIGURE 3.12.5**. Solve the system. [*Hint*: $\mathbf{k}$ is a multiple of velocity, say $\beta\mathbf{v}$.]

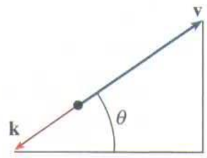

FIGURE 3.12.5 Forces in Problem 24

≡ Computer Lab Assignments

25. Consider the solution $x_1(t)$ and $x_2(t)$ of the initial-value problem given at the end of Example 3. Use a CAS to graph $x_1(t)$ and $x_2(t)$ in the same coordinate plane on the interval $[0, 100]$. In Example 3, $x_1(t)$ denotes the number of pounds of salt in tank A at time t, and $x_2(t)$ the number of pounds of salt in tank B at time t. See Figure 2.9.1. Use a root-finding application to determine when tank B contains more salt than tank A.

26. **(a)** Reread Problem 10 of Exercises 2.9. In that problem you were asked to show that the system of differential equations

$$\frac{dx_1}{dt} = -\frac{1}{50}x_1$$
$$\frac{dx_2}{dt} = \frac{1}{50}x_1 - \frac{2}{75}x_2$$
$$\frac{dx_3}{dt} = \frac{2}{75}x_2 - \frac{1}{25}x_3$$

is a model for the amounts of salt in the connected mixing tanks A, B, and C shown in Figure 2.9.7. Solve the system subject to $x_1(0) = 15, x_2(t) = 10, x_3(t) = 5$.

(b) Use a CAS to graph $x_1(t)$, $x_2(t)$, and $x_3(t)$ in the same coordinate plane on the interval $[0, 200]$.

(c) Since only pure water is pumped into tank A, it stands to reason that the salt will eventually be flushed out of all three tanks. Use a root-finding application of a CAS to determine the time when the amount of salt in each tank is less than or equal to 0.5 pounds. When will the amounts of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ be simultaneously less than or equal to 0.5 pounds?

27. **(a)** Use systematic elimination to solve the system (1) for the coupled spring/mass system when $k_1 = 4$, $k_2 =$ [illegible], $m_1 = 2$, and $m_2 = 1$ and with initial conditions $x_1(0) =$ [illegible], $x_1'(0) = 1$, $x_2(0) = -1$, $x_2'(0) = 1$.

(b) Use a CAS to plot the graphs of $x_1(t)$ and $x_2(t)$ in the tx-plane. What is the fundamental difference in the motions of the masses m_1 and m_2 in this problem and that of the masses illustrated in Figure 3.12.3?

(c) As parametric equations, plot $x_1(t)$ and $x_2(t)$ in the x_1x_2-plane. The curve defined by these parametric equations is called a **Lissajous curve**.