Solving the last equation for u and then resubstituting gives the solution

$$u = \frac{3(1 + ce^{6x})}{1 - ce^{6x}} \quad \text{or} \quad y = 2x + \frac{3(1 + ce^{6x})}{1 - ce^{6x}}. \tag{6}$$

Finally, applying the initial condition $y(0) = 0$ to the last equation in (6) gives $c = -1$. With the aid of a graphing utility we have shown in **FIGURE 2.5.1** the graph of the particular solution

$$y = 2x + \frac{3(1 - e^{6x})}{1 + e^{6x}}$$

in blue along with the graphs of some other members of the family solutions (6). ≡

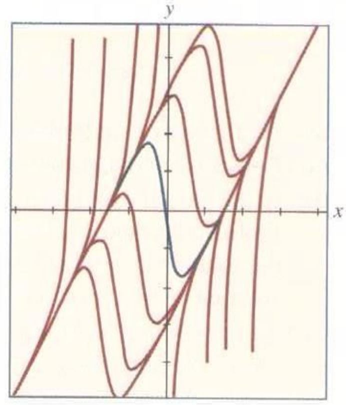

FIGURE 2.5.1 Some solutions of the DE in Example 3

2.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-3.

Each DE in Problems 1–14 is homogeneous.

In Problems 1–10, solve the given differential equation by using an appropriate substitution.

1. $(x - y)\,dx + x\,dy = 0$
2. $(x + y)\,dx + x\,dy = 0$
3. $x\,dx + (y - 2x)\,dy = 0$
4. $y\,dx = 2(x + y)\,dy$
5. $(y^2 + yx)\,dx - x^2\,dy = 0$
6. $(y^2 + yx)\,dx + x^2\,dy = 0$
7. $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$
8. $\dfrac{dy}{dx} = \dfrac{x + 3y}{3x + y}$
9. $-y\,dx + (x + \sqrt{xy})\,dy = 0$
10. $x\dfrac{dy}{dx} = y + \sqrt{x^2 - y^2}, x > 0$

In Problems 11–14, solve the given initial-value problem.

11. $xy^2\dfrac{dy}{dx} = y^3 - x^3, y(1) = 2$
12. $(x^2 + 2y^2)\dfrac{dx}{dy} = xy, y(-1) = 1$
13. $(x + ye^{y/x})\,dx - xe^{y/x}\,dy = 0, \quad y(1) = 0$
14. $y\,dx + x(\ln x - \ln y - 1)\,dy = 0, \quad y(1) = e$

Each DE in Problems 15–22 is a Bernoulli equation.

In Problems 15–20, solve the given differential equation by using an appropriate substitution.

15. $x\dfrac{dy}{dx} + y = \dfrac{1}{y^2}$
16. $\dfrac{dy}{dx} - y = e^x y^2$
17. $\dfrac{dy}{dx} = y(xy^3 - 1)$
18. $x\dfrac{dy}{dx} - (1 + x)y = xy^2$
19. $t^2\dfrac{dy}{dt} + y^2 = ty$
20. $3(1 + t^2)\dfrac{dy}{dt} = 2ty(y^3 - 1)$

In Problems 21 and 22, solve the given initial-value problem.

21. $x^2\dfrac{dy}{dx} - 2xy = 3y^4, \quad y(1) = \dfrac{1}{2}$
22. $y^{1/2}\dfrac{dy}{dx} + y^{3/2} = 1, \quad y(0) = 4$

Each DE in Problems 23–30 is of the form given in (5).

In Problems 23–28, solve the given differential equation by using an appropriate substitution.

23. $\dfrac{dy}{dx} = (x + y + 1)^2$
24. $\dfrac{dy}{dx} = \dfrac{1 - x - y}{x + y}$
25. $\dfrac{dy}{dx} = \tan^2(x + y)$
26. $\dfrac{dy}{dx} = \sin(x + y)$
27. $\dfrac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$
28. $\dfrac{dy}{dx} = 1 + e^{y - x + 5}$

In Problems 29 and 30, solve the given initial-value problem.

29. $\dfrac{dy}{dx} = \cos(x + y), \quad y(0) = \pi/4$
30. $\dfrac{dy}{dx} = \dfrac{3x + 2y}{3x + 2y + 2}, \quad y(-1) = -1$

≡ Discussion Problems

31. Explain why it is always possible to express any homogeneous differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

You might start by proving that

$$M(x, y) = x^\alpha M(1, y/x) \quad \text{and} \quad N(x, y) = x^\alpha N(1, y/x).$$

32. Put the homogeneous differential equation

$$(5x^2 - 2y^2)\,dx - xy\,dy = 0$$

into the form given in Problem 31.

33. (a) Determine two singular solutions of the DE in Problem 10.
(b) If the initial condition $y(5) = 0$ is as prescribed in Problem 10, then what is the largest interval I over which the solution is defined? Use a graphing utility to plot the solution curve for the IVP.

34. In Example 3, the solution $y(x)$ becomes unbounded as $x \to \pm\infty$. Nevertheless $y(x)$ is asymptotic to a curve as $x \to -\infty$ and to a different curve as $x \to \infty$. Find the equations of these curves.

35. The differential equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

is known as **Riccati's equation**.
(a) A Riccati equation can be solved by a succession of two substitutions *provided* we know a particular solution y_1 of the equation. Show that the substitution $y = y_1 + u$ reduces Riccati's equation to a Bernoulli equation (4) with $n = 2$. The Bernoulli equation can then be reduced to a linear equation by the substitution $w = u^{-1}$.
(b) Find a one-parameter family of solutions for the differential equation

$$\frac{dy}{dx} = -\frac{4}{x^2} - \frac{1}{x}y + y^2,$$

where $y_1 = 2/x$ is a known solution of the equation.

36. Devise an appropriate substitution to solve

$$xy' = y\ln(xy).$$

≡ Mathematical Model

37. **Population Growth** In the study of population dynamics one of the most famous models for a growing but bounded population is the **logistic equation**

$$\frac{dP}{dt} = P(a - bP),$$

where a and b are positive constants. Although we will come back to this equation and solve it by an alternative method in Section 2.8, solve the DE this first time using the fact that it is a Bernoulli equation.

2.6 A Numerical Method

≡ Introduction In Section 2.1 we saw that we could glean *qualitative* information from a first-order DE about its solutions even before we attempted to solve the equation. In Sections 2.2–2.5 we examined first-order DEs *analytically*; that is, we developed procedures for actually obtaining explicit and implicit solutions. But many differential equations possess solutions and yet these solutions cannot be obtained analytically. In this case we "solve" the differential equation *numerically*; this means that the DE is used as the cornerstone of an algorithm for *approximating* the unknown solution. It is common practice to refer to the algorithm as a *numerical method*, the approximate solution as a *numerical solution*, and the graph of a numerical solution as a *numerical solution curve*.

In this section we are going to consider only the simplest of numerical methods. A more extensive treatment of this subject is found in Chapter 6.

□ Using the Tangent Line Let us assume that the first-order initial-value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

possesses a solution. One of the simplest techniques for approximating this solution is to use tangent lines. For example, let $y(x)$ denote the unknown solution of the first-order initial-value problem $y' = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$. The nonlinear differential equation cannot be solved directly by the methods considered in Sections 2.2, 2.4, and 2.5; nevertheless we can still find approximate numerical values of the unknown $y(x)$. Specifically, suppose we wish to know the