4.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-8.

[illegible] 1–18, use Definition 4.1.1 to find $\mathscr{L}\{f(t)\}$.

[illegible] $\begin{cases} -1, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

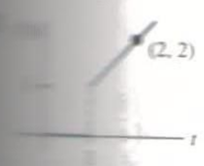
[illegible] $\begin{cases} 4, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

[illegible] $\begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

[illegible] $\begin{cases} 2t + 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

[illegible] $\begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

[illegible] $\begin{cases} \sin t, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases}$



[illegible] Graph for [illegible]

8.

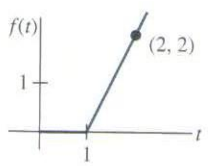


FIGURE 4.1.7 Graph for Problem 8

[illegible] Graph for [illegible]

10.

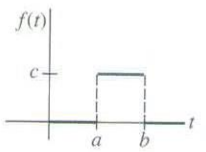


FIGURE 4.1.9 Graph for Problem 10

12. $f(t) = e^{-2t-5}$

14. $f(t) = t^2e^{-2t}$

16. $f(t) = e^t \cos t$

18. $f(t) = t \sin t$

[illegible] use Theorem 4.1.1 to find $\mathscr{L}\{f(t)\}$.

20. $f(t) = t^5$

22. $f(t) = 7t + 3$

24. $f(t) = -4t^2 + 16t + 9$

26. $f(t) = (2t - 1)^3$

28. $f(t) = t^2 - e^{-9t} + 5$

30. $f(t) = (e^t - e^{-t})^2$

32. $f(t) = \cos 5t + \sin 2t$

34. $f(t) = \cosh kt$

36. $f(t) = e^{-t} \cosh t$

[illegible] find $\mathscr{L}\{f(t)\}$ by first using an appropriate [illegible].

38. $f(t) = \cos^2 t$

40. $f(t) = 10 \cos(t - \pi/6)$

41. One definition of the **gamma function** $\Gamma(\alpha)$ is given by the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \ \alpha > 0.$$

Use this definition to show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

42. Use Problem 41 to show that

$$\mathscr{L}\{t^\alpha\} = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}, \ \alpha > -1.$$

This result is a generalization of Theorem 4.1.1(b).

In Problems 43–46, use the results in Problems 41 and 42 and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to find the Laplace transform of the given function.

43. $f(t) = t^{-1/2}$

44. $f(t) = t^{1/2}$

45. $f(t) = t^{3/2}$

46. $f(t) = 6t^{1/2} - 24t^{5/2}$

≡ Discussion Problems

47. Make up a function $F(t)$ that is of exponential order, but $f(t) = F'(t)$ is not of exponential order. Make up a function $f(t)$ that is not of exponential order, but whose Laplace transform exists.

48. Suppose that $\mathscr{L}\{f_1(t)\} = F_1(s)$ for $s > c_1$ and that $\mathscr{L}\{f_2(t)\} = F_2(s)$ for $s > c_2$. When does $\mathscr{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$?

49. Figure 4.1.4 suggests, but does not prove, that the function $f(t) = e^{t^2}$ is not of exponential order. How does the observation that $t^2 > \ln M + ct$, for $M > 0$ and t sufficiently large, show that $e^{t^2} > Me^{ct}$ for any c?

50. Use part (c) of Theorem 4.1.1 to show that

$$\mathscr{L}\{e^{(a+ib)t}\} = \frac{s - a + ib}{(s - a)^2 + b^2},$$

where a and b are real and $i^2 = -1$. Show how Euler's formula (page 119) can then be used to deduce the results

$$\mathscr{L}\{e^{at}\cos bt\} = \frac{s - a}{(s - a)^2 + b^2}$$

and

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s - a)^2 + b^2}.$$

51. Under what conditions is a linear function $f(x) = mx + b$, $m \neq 0$, a linear transform?

52. The proof of part (b) of Theorem 4.1.1 requires the use of mathematical induction. Show that if

$$\mathscr{L}\{t^{n-1}\} = (n - 1)!/s^n$$

is *assumed* to be true, then $\mathscr{L}\{t^n\} = n!/s^{n+1}$ follows.

53. The function $f(t) = 2te^{t^2}\cos e^{t^2}$ is not of exponential order. Nevertheless, show that the Laplace transform $\mathscr{L}\{2te^{t^2}\cos e^{t^2}\}$ exists. [*Hint*: Use integration by parts.]

54. If $\mathscr{L}\{f(t)\} = F(s)$ and $a > 0$ is a constant, show that

$$\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$$

This result is known as the **change of scale theorem**.

In Problems 55–58, use the given Laplace transform and the result in Problem 54 to find the indicated Laplace transform. Assume that a and k are positive constants.

55. $\mathscr{L}\{e^t\} = \dfrac{1}{s-1}$; $\mathscr{L}\{e^{at}\}$

56. $\mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1}$; $\mathscr{L}\{\cos kt\}$

57. $\mathscr{L}\{t - \sin t\} = \dfrac{1}{s^2(s^2+1)}$; $\mathscr{L}\{kt - \sin kt\}$

58. $\mathscr{L}\{\cos t \sinh t\} = \dfrac{s^2-2}{s^4+4}$; $\mathscr{L}\{\cos kt \sinh kt\}$

4.2 The Inverse Transform and Transforms of Derivatives

Introduction In this section we take a few small steps into an investigation of how the Laplace transform can be used to solve certain types of equations. After we discuss the concept of the *inverse* Laplace transform and examine the transforms of derivatives we then use the Laplace transform to solve some simple ordinary differential equations.

4.2.1 Inverse Transforms

The Inverse Problem If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathscr{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $f(t) = \mathscr{L}^{-1}\{F(s)\}$. For example, from Examples 1, 2, and 3 in Section 4.1 we have, respectively,

$$1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}, \quad t = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}, \quad \text{and} \quad e^{-3t} = \mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\}.$$

The analogue of Theorem 4.1.1 for the inverse transform is presented next.

Theorem 4.2.1 Some Inverse Transforms

$$\textbf{(a)}\ 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\textbf{(b)}\ t^n = \mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots \qquad \textbf{(c)}\ e^{at} = \mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\textbf{(d)}\ \sin kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad \textbf{(e)}\ \cos kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$\textbf{(f)}\ \sinh kt = \mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad \textbf{(g)}\ \cosh kt = \mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$

When evaluating inverse transforms, it often happens that a function of s under consideration does not match *exactly* the form of a Laplace transform $F(s)$ given in a table. It may be necessary to "fix up" the function of s by multiplying and dividing by an appropriate constant.

EXAMPLE 1 Applying Theorem 4.2.1

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+7}\right\}$.

SOLUTION **(a)** To match the form given in part (b) of Theorem 4.2.1, we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$