is recognized as linear in the variable x. You should verify that the integrating factor $e^{\int(-1)dy} = e^{-y}$ and integration by parts yield an implicit solution of the first equation: $x = -y^2 - 2y - 2 + ce^y$.

(*ii*) Because mathematicians thought they were appropriately descriptive, certain words were "adopted" from engineering and made their own. The word *transient*, used earlier, is one of these terms. In future discussions the words *input* and *output* will occasionally pop up. The function f in (2) is called the **input** or **driving function**; a solution of the differential equation for a given input is called the **output** or **response**.

2.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–24, find the general solution of the given differential equation. Give the largest interval over which the general solution is defined. Determine whether there are any transient terms in the general solution.

1. $\dfrac{dy}{dx} = 5y$
2. $\dfrac{dy}{dx} + 2y = 0$
3. $\dfrac{dy}{dx} + y = e^{3x}$
4. $3\dfrac{dy}{dx} + 12y = 4$
5. $y' + 3x^2y = x^2$
6. $y' + 2xy = x^3$
7. $x^2y' + xy = 1$
8. $y' = 2y + x^2 + 5$
9. $x\dfrac{dy}{dx} - y = x^2\sin x$
10. $x\dfrac{dy}{dx} + 2y = 3$
11. $x\dfrac{dy}{dx} + 4y = x^3 - x$
12. $(1 + x)\dfrac{dy}{dx} - xy = x + x^2$
13. $x^2y' + x(x + 2)y = e^x$
14. $xy' + (1 + x)y = e^{-x}\sin 2x$
15. $y\,dx - 4(x + y^6)\,dy = 0$
16. $y\,dx = (ye^y - 2x)\,dy$
17. $\cos x\dfrac{dy}{dx} + (\sin x)y = 1$
18. $\cos^2x \sin x\dfrac{dy}{dx} + (\cos^3x)y = 1$
19. $(x + 1)\dfrac{dy}{dx} + (x + 2)y = 2xe^{-x}$
20. $(x + 2)^2\dfrac{dy}{dx} = 5 - 8y - 4xy$
21. $\dfrac{dr}{d\theta} + r\sec\theta = \cos\theta$
22. $\dfrac{dP}{dt} + 2tP = P + 4t - 2$
23. $x\dfrac{dy}{dx} + (3x + 1)y = e^{-3x}$
24. $(x^2 - 1)\dfrac{dy}{dx} + 2y = (x + 1)^2$

In Problems 25–32, solve the given initial-value problem. Give the largest interval I over which the solution is defined.

25. $xy' + y = e^x, \quad y(1) = 2$
26. $y\dfrac{dx}{dy} - x = 2y^2, \quad y(1) = 5$
27. $L\dfrac{di}{dt} + Ri = E; \quad i(0) = i_0$, L, R, E, and i_0 constants
28. $\dfrac{dT}{dt} = k(T - T_m); \quad T(0) = T_0$, K, T_m, and T_0 constants
29. $(x + 1)\dfrac{dy}{dx} + y = \ln x, \quad y(1) = 10$
30. $y' + (\tan x)y = \cos^2x, \quad y(0) = -1$
31. $\left(\dfrac{e^{-2\sqrt{x}} - y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1, \quad y(1) = 1$
32. $(1 + t^2)\dfrac{dx}{dt} + x = \tan^{-1}t, \quad x(0) = 4$

    [*Hint*: In your solution let $u = \tan^{-1}t$.]

In Problems 33–36, proceed as in Example 5 to solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

33. $\dfrac{dy}{dx} + 2y = f(x)$, $y(0) = 0$, where

$$f(x) = \begin{cases} 1, & 0 \le x \le 3 \\ 0, & x > 3 \end{cases}$$

34. $\dfrac{dy}{dx} + y = f(x)$, $y(0) = 1$, where

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

35. $\dfrac{dy}{dx} + 2xy = f(x)$, $y(0) = 2$, where

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

36. $(1 + x^2)\dfrac{dy}{dx} + 2xy = f(x)$, $y(0) = 0$, where

$$f(x) = \begin{cases} x, & -0 \le x < 1 \\ -x, & x \ge 1 \end{cases}$$

37. Proceed in a manner analogous to Example 5 to solve the initial-value problem $y' + P(x)y = 4x$, $y(0) = 3$, where

$$P(x) = \begin{cases} 2, & 0 \le x \le 1 \\ -2/x, & x > 1. \end{cases}$$

Use a graphing utility to graph the continuous function $y(x)$.

38. Consider the initial-value problem $y' + e^x y = f(x)$, $y(0) = 1$. Express the solution of the IVP for $x > 0$ as a nonelementary integral when $f(x) = 1$. What is the solution when $f(x) = 0$? When $f(x) = e^x$?

39. Express the solution of the initial-value problem $y' - 2xy = 1$, $y(1) = 1$, in terms of $\operatorname{erf}(x)$.

≡ Discussion Problems

40. Reread the discussion following Example 1. Construct a linear first-order differential equation for which all nonconstant solutions approach the horizontal asymptote $y = 4$ as $x \to \infty$.

41. Reread Example 2 and then discuss, with reference to Theorem 1.2.1, the existence and uniqueness of a solution of the initial-value problem consisting of $xy' - 4y = x^6 e^x$ and the given initial condition.
    (a) $y(0) = 0$
    (b) $y(0) = y_0, \quad y_0 > 0$
    (c) $y(x_0) = y_0, \quad x_0 > 0, \quad y_0 > 0$

42. Reread Example 3 and then find the general solution of the differential equation on the interval $(-3, 3)$.

43. Reread the discussion following Example 4. Construct a linear first-order differential equation for which all solutions are asymptotic to the line $y = 3x - 5$ as $x \to \infty$.

44. Reread Example 5 and then discuss why it is technically incorrect to say that the function in (13) is a *solution* of the IVP on the interval $[0, \infty)$.

45. **(a)** Construct a linear first-order differential equation of the form $xy' + a_0(x)y = g(x)$ for which $y_c = c/x^3$ and $y_p = x^3$. Give an interval on which $y = x^3 + c/x^3$ is the general solution of the DE.
    (b) Give an initial condition $y(x_0) = y_0$ for the DE found in part (a) so that the solution of the IVP is $y = x^3 - 1/x^3$. Repeat if the solution is $y = x^3 + 2/x^3$. Give an interval I of definition of each of these solutions. Graph the solution curves. Is there an initial-value problem whose solution is defined on the interval $(-\infty, \infty)$?
    (c) Is each IVP found in part (b) unique? That is, can there be more than one IVP for which, say, $y = x^3 - 1/x^3$, x in some interval I, is the solution?

46. In determining the integrating factor (5), we did not use a constant of integration in the evaluation of $\int P(x)dx$. Explain why using $\int P(x)dx + c$ has no effect on the solution of (2).

47. Suppose $P(x)$ is continuous on some interval I and a is a number in I. What can be said about the solution of the initial-value problem $y' + P(x)y = 0$, $y(a) = 0$?

≡ Mathematical Models

48. **Radioactive Decay Series** The following system of differential equations is encountered in the study of the decay of a special type of radioactive series of elements:

$$\frac{dx}{dt} = -\lambda_1 x,$$
$$\frac{dy}{dt} = \lambda_1 x - \lambda_2 y,$$

where λ_1 and λ_2 are constants. Discuss how to solve this system subject to $x(0) = x_0$, $y(0) = y_0$. Carry out your ideas.

49. **Heart Pacemaker** A heart pacemaker consists of a switch, a battery of constant voltage E_0, a capacitor with constant capacitance C, and the heart as a resistor with constant resistance R. When the switch is closed, the capacitor charges; when the switch is open, the capacitor discharges, sending an electrical stimulus to the heart. During the time the heart is being stimulated, the voltage E across the heart satisfies the linear differential equation

$$\frac{dE}{dt} = -\frac{1}{RC}E.$$

Solve the DE subject to $E(4) = E_0$.

≡ Computer Lab Assignments

50. **(a)** Express the solution of the initial-value problem $y' - 2xy = -1$, $y(0) = \sqrt{\pi}/2$, in terms of $\operatorname{erfc}(x)$.
    (b) Use tables or a CAS to find the value of $y(2)$. Use a CAS to graph the solution curve for the IVP on the interval $(-\infty, \infty)$.

51. **(a)** The **sine integral function** is defined by $\operatorname{Si}(x) = \int_0^x (\sin t/t)\,dt$, where the integrand is defined to be 1 at $t = 0$. Express the solution $y(x)$ of the initial-value problem $x^3y' + 2x^2y = 10 \sin x$, $y(1) = 0$, in terms of $\operatorname{Si}(x)$.
    (b) Use a CAS to graph the solution curve for the IVP for $x > 0$.
    (c) Use a CAS to find the value of the absolute maximum of the solution $y(x)$ for $x > 0$.

52. **(a)** The **Fresnel sine integral** is defined by $S(x) = \int_0^x \sin(\pi t^2/2)\,dt$. Express the solution $y(x)$ of the initial-value problem $y' - (\sin x^2)y = 0$, $y(0) = 5$, in terms of $S(x)$.
    (b) Use a CAS to graph the solution curve for the IVP on $(-\infty, \infty)$.
    (c) It is known that $S(x) \to \frac{1}{2}$ as $x \to \infty$ and $S(x) \to -\frac{1}{2}$ as $x \to -\infty$. What does the solution $y(x)$ approach as $x \to \infty$? As $x \to -\infty$?
    (d) Use a CAS to find the values of the absolute maximum and the absolute minimum of the solution $y(x)$.