- If $(M_y - N_x)/N$ is a function of x alone, then an integrating factor for equation (11) is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}. \tag{13}$$

- If $(N_x - M_y)/M$ is a function of y alone, then an integrating factor for equation (11) is

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}. \tag{14}$$

EXAMPLE 4 A Nonexact DE Made Exact

The nonlinear first-order differential equation $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$ is not exact. With the identifications $M = xy$, $N = 2x^2 + 3y^2 - 20$ we find the partial derivatives $M_y = x$ and $N_x = 4x$. The first quotient from (13) gets us nowhere since

$$\frac{M_y - N_x}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20}$$

depends on x and y. However (14) yields a quotient that depends only on y:

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y}.$$

The integrating factor is then $e^{\int 3dy/y} = e^{3\ln y} = e^{\ln y^3} = y^3$. After multiplying the given DE by $\mu(y) = y^3$ the resulting equation is

$$xy^4\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0.$$

You should verify that the last equation is now exact as well as show, using the method of this section, that a family of solutions is $\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$. ≡

Remarks

(*i*) When testing an equation for exactness, make sure it is of the precise form $M(x, y)\,dx + N(x, y)\,dy = 0$. Sometimes a differential equation is written $G(x, y)\,dx = H(x, y)\,dy$. In this case, first rewrite it as $G(x, y)\,dx - H(x, y)\,dy = 0$, and then identify $M(x, y) = G(x, y)$ and $N(x, y) = -H(x, y)$ before using (4).

(*ii*) In some texts on differential equations the study of exact equations precedes that of linear DEs. If this were so, the method for finding integrating factors just discussed can be used to derive an integrating factor for $y' + P(x)y = f(x)$. By rewriting the last equation in the differential form $(P(x)y - f(x))\,dx + dy = 0$ we see that

$$\frac{M_y - N_x}{N} = P(x).$$

From (13) we arrive at the already familiar integrating factor $e^{\int P(x)\,dx}$ used in Section 2.3.

2.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-2.

In Problems 1–20, determine whether the given differential equation is exact. If it is exact, solve it.

1. $(2x - 1)\,dx + (3y + 7)\,dy = 0$
2. $(2x + y)\,dx - (x + 6y)\,dy = 0$
3. $(5x + 4y)\,dx + (4x - 8y^3)\,dy = 0$
4. $(\sin y - y \sin x)\,dx + (\cos x + x \cos y - y)\,dy = 0$
5. $(2xy^2 - 3)\,dx + (2x^2y + 4)\,dy = 0$
6. $\left(2y - \dfrac{1}{x} + \cos 3x\right)\dfrac{dy}{dx} + \dfrac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$
7. $(x^2 - y^2)\,dx + (x^2 - 2xy)\,dy = 0$

8. $\left(1 + \ln x + \dfrac{y}{x}\right) dx = (1 - \ln x)\, dy$

9. $(x - y^3 + y^2 \sin x)\, dx = (3xy^2 + 2y \cos x)\, dy$

10. $(x^3 + y^3)\, dx + 3xy^2\, dy = 0$

11. $(y \ln y - e^{-xy})\, dx + \left(\dfrac{1}{y} + x \ln y\right) dy = 0$

12. $(3x^2y + e^y)\, dx + (x^3 + xe^y - 2y)\, dy = 0$

13. $x\dfrac{dy}{dx} = 2xe^x - y + 6x^2$

14. $\left(1 - \dfrac{3}{y} + x\right)\dfrac{dy}{dx} + y = \dfrac{3}{x} - 1$

15. $\left(x^2y^3 - \dfrac{1}{1 + 9x^2}\right)\dfrac{dx}{dy} + x^3y^2 = 0$

16. $(5y - 2x)y' - 2y = 0$

17. $(\tan x - \sin x \sin y)\, dx + \cos x \cos y\, dy = 0$

18. $(2y \sin x \cos x - y + 2y^2e^{xy^2})\, dx = (x - \sin^2 x - 4xye^{xy^2})\, dy$

19. $(4t^3y - 15t^2 - y)\, dt + (t^4 + 3y^2 - t)\, dy = 0$

20. $\left(\dfrac{1}{t} + \dfrac{1}{t^2} - \dfrac{y}{t^2 + y^2}\right) dt + \left(ye^y + \dfrac{1}{t^2 + y^2}\right) dy = 0$

In Problems 21–26, solve the given initial-value problem.

21. $(x + y)^2\, dx + (2xy + x^2 - 1)\, dy = 0, \quad y(1) = 1$

22. $(e^x + y)\, dx + (2 + x + ye^y)\, dy = 0, \quad y(0) = 1$

23. $(4y + 2t - 5)\, dt + (6y + 4t - 1)\, dy = 0, \quad y(-1) = 2$

24. $\left(\dfrac{3y^2 - t^2}{y^5}\right)\dfrac{dy}{dt} + \dfrac{t}{2y^4} = 0, \quad y(1) = 1$

25. $(y^2 \cos x - 3x^2y - 2x)\, dx + (2y \sin x - x^3 + \ln y)\, dy = 0,$ $y(0) = e$

26. $\left(\dfrac{1}{1 + y^2} + \cos x - 2xy\right)\dfrac{dy}{dx} = y(y + \sin x), \quad y(0) = 1$

In Problems 27 and 28, find the value of k so that the given differential equation is exact.

27. $(y^3 + kxy^4 - 2x)\, dx + (3xy^2 + 20x^2y^3)\, dy = 0$

28. $(6xy^3 + \cos y)\, dx + (2kx^2y^2 - x \sin y)\, dy = 0$

In Problems 29 and 30, verify that the given differential equation is not exact. Multiply the given differential equation by the indicated integrating factor $\mu(x, y)$ and verify that the new equation is exact. Solve.

29. $(-xy \sin x + 2y \cos x)\, dx + 2x \cos x\, dy = 0; \quad \mu(x, y) = xy$

30. $(x^2 + 2xy - y^2)\, dx + (y^2 + 2xy - x^2)\, dy = 0; \quad \mu(x, y) = (x + y)^{-2}$

In Problems 31–36, solve the given differential equation by finding, as in Example 4, an appropriate integrating factor.

31. $(2y^2 + 3x)\, dx + 2xy\, dy = 0$

32. $y(x + y + 1)\, dx + (x + 2y)\, dy = 0$

33. $6xy\, dx + (4y + 9x^2)\, dy = 0$

34. $\cos x\, dx + \left(1 + \dfrac{2}{y}\right)\sin x\, dy = 0$

35. $(10 - 6y + e^{-3x})\, dx - 2\, dy = 0$

36. $(y^2 + xy^3)\, dx + (5y^2 - xy + y^3 \sin y)\, dy = 0$

In Problems 37 and 38, solve the given initial-value problem by finding, as in Example 4, an appropriate integrating factor.

37. $x\, dx + (x^2y + 4y)\, dy = 0, \quad y(4) = 0$

38. $(x^2 + y^2 - 5)\, dx = (y + xy)\, dy, \quad y(0) = 1$

39. **(a)** Show that a one-parameter family of solutions of the equation

$$(4xy + 3x^2)\, dx + (2y + 2x^2)\, dy = 0$$

is $x^3 + 2x^2y + y^2 = c$.

(b) Show that the initial conditions $y(0) = -2$ and $y(1) = 1$ determine the same implicit solution.

(c) Find explicit solutions $y_1(x)$ and $y_2(x)$ of the differential equation in part (a) such that $y_1(0) = -2$ and $y_2(1) = 1$. Use a graphing utility to graph $y_1(x)$ and $y_2(x)$.

≡ Discussion Problems

40. Consider the concept of an integrating factor used in Problems 29–38. Are the two equations $M\, dx + N\, dy = 0$ and $\mu M\, dx + \mu N\, dy = 0$ necessarily equivalent in the sense that a solution of one is also a solution of the other? Discuss.

41. Reread Example 3 and then discuss why we can conclude that the interval of definition of the explicit solution of the IVP (the blue curve in Figure 2.4.1) is $(-1, 1)$.

42. Discuss how the functions $M(x, y)$ and $N(x, y)$ can be found so that each differential equation is exact. Carry out your ideas.

(a) $M(x, y)\, dx + \left(xe^{xy} + 2xy + \dfrac{1}{x}\right) dy = 0$

(b) $\left(x^{-1/2}y^{1/2} + \dfrac{x}{x^2 + y}\right) dx + N(x, y)\, dy = 0$

43. Differential equations are sometimes solved by having a clever idea. Here is a little exercise in cleverness: Although the differential equation

$$(x - \sqrt{x^2 + y^2})\, dx + y\, dy = 0$$

is not exact, show how the rearrangement

$$\frac{x\, dx + y\, dy}{\sqrt{x^2 + y^2}} = dx$$

and the observation $\frac{1}{2}d(x^2 + y^2) = x\, dx + y\, dy$ can lead to a solution.

44. True or False: Every separable first-order equation $dy/dx = g(x)h(y)$ is exact.

Computer Lab Assignment

45. (a) The solution of the differential equation

$$\frac{2xy}{(x^2+y^2)^2}\,dx + \left[1 + \frac{y^2 - x^2}{(x^2+y^2)^2}\right]dy = 0$$

is a family of curves that can be interpreted as **streamlines** of a fluid flow around a circular object whose boundary is described by the equation $x^2 + y^2 = 1$. Solve this DE and note the solution $f(x, y) = c$ for $c = 0$.

(b) Use a CAS to plot the streamlines for $c = 0, \pm 0.2, \pm 0.4, \pm 0.6$, and ± 0.8 in three different ways. First, use the **contourplot** of a CAS. Second, solve for x in terms of the variable y. Plot the resulting two functions of y for the given values of c, and then combine the graphs. Third, use the CAS to solve a cubic equation for y in terms of x.

2.5 Solutions by Substitutions

Introduction

We usually solve a differential equation by recognizing it as a certain kind of equation (say, separable) and then carrying out a procedure, consisting of equation-specific mathematical steps, that yields a function that satisfies the equation. Often the first step in solving a given differential equation consists of transforming it into another differential equation by means of a **substitution**. For example, suppose we wish to transform the first-order equation $dy/dx = f(x, y)$ by the substitution $y = g(x, u)$, where u is regarded as a function of the variable x.

If g possesses first-partial derivatives, then the Chain Rule gives

See (10) on page 484.

$$\frac{dy}{dx} = g_x(x, u) + g_u(x, u)\frac{du}{dx}.$$

By replacing dy/dx by $f(x, y)$ and y by $g(x, u)$ in the foregoing derivative, we get the new first-order differential equation

$$f(x, g(x, u)) = g_x(x, u) + g_u(x, u)\frac{du}{dx},$$

which, after solving for du/dx, has the form $du/dx = F(x, u)$. If we can determine a solution $u = \phi(x)$ of this second equation, then a solution of the original differential equation is $y = g(x, \phi(x))$.

Homogeneous Equations

If a function f possesses the property $f(tx, ty) = t^{\alpha} f(x, y)$ for some real number α, then f is said to be a **homogeneous function** of degree α. For example, $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3 since

$$f(tx, ty) = (tx)^3 + (ty)^3 = t^3(x^3 + y^3) = t^3 f(x, y),$$

whereas $f(x, y) = x^3 + y^3 + 1$ is seen not to be homogeneous. A first-order DE in differential form

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

is said to be **homogeneous** if both coefficients M and N are homogeneous functions of the *same* degree. In other words, (1) is homogeneous if

$$M(tx, ty) = t^{\alpha}M(x, y) \quad \text{and} \quad N(tx, ty) = t^{\alpha}N(x, y).$$

A linear first-order DE $a_1y' + a_0y = g(x)$ is homogeneous when $g(x) = 0$.

The word *homogeneous* as used here does not mean the same as it does when applied to linear differential equations. See Sections 2.3 and 3.1.

If M and N are homogeneous functions of degree α, we can also write

$$M(x, y) = x^{\alpha}M(1, u) \quad \text{and} \quad N(x, y) = x^{\alpha}N(1, u) \quad \text{where } u = y/x, \tag{2}$$

and

$$M(x, y) = y^{\alpha}M(v, 1) \quad \text{and} \quad N(x, y) = y^{\alpha}N(v, 1) \quad \text{where } v = x/y. \tag{3}$$