The first $n - 1$ equations in this system, like $y_1u_1' + y_2u_2' = 0$ in (4), are assumptions made to simplify the resulting equation after $y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ is substituted in (8). In this case, Cramer's rule gives

$$u_k' = \frac{W_k}{W}, \quad k = 1, 2, \ldots, n,$$

where W is the Wronskian of $y_1, y_2, \ldots, y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column consisting of the right-hand side of (9), that is, the column $(0, 0, \ldots, f(x))$. When $n = 2$ we get (5). When $n = 3$, the particular solution is $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where y_1, y_2, and y_3 constitute a linearly independent set of solutions of the associated homogeneous DE, and u_1, u_2, u_3 are determined from

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W}, \tag{10}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}, \text{ and } W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

See Problems 25 and 26 in Exercises 3.5.

Remarks

In the problems that follow do not hesitate to simplify the form of y_p. Depending on how the antiderivatives of u_1' and u_2' are found, you may not obtain the same y_p as given in the answer section. For example, in Problem 3 in Exercises 3.5, both $y_p = \frac{1}{2}\sin x - \frac{1}{2}x\cos x$ and $y_p = \frac{1}{4}\sin x - \frac{1}{2}x\cos x$ are valid answers. In either case the general solution $y = y_c + y_p$ simplifies to $y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$. Why?

3.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–18, solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$

2. $y'' + y = \tan x$

3. $y'' + y = \sin x$

4. $y'' + y = \sec\theta\tan\theta$

5. $y'' + y = \cos^2 x$

6. $y'' + y = \sec^2 x$

7. $y'' - y = \cosh x$

8. $y'' - y = \sinh 2x$

9. $y'' - 4y = \dfrac{e^{2x}}{x}$

10. $y'' - 9y = \dfrac{9x}{e^{3x}}$

11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$

12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$

13. $y'' + 3y' + 2y = \sin e^x$

14. $y'' - 2y' + y = e^t \arctan t$

15. $y'' + 2y' + y = e^{-t}\ln t$

16. $2y'' + 2y' + y = 4\sqrt{x}$

17. $3y'' - 6y' + 6y = e^x \sec x$

18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

In Problems 19–22, solve each differential equation by variation of parameters subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

19. $4y'' - y = xe^{x/2}$

20. $2y'' + y' - y = x + 1$

21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$

22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

In Problems 23 and 24, the indicated functions are known linearly independent solutions of the associated homogeneous differential equation on the interval $(0, \infty)$. Find the general solution of the given nonhomogeneous equation.

23. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2}$; $y_1 = x^{-1/2}\cos x$, $y_2 = x^{-1/2}\sin x$

24. $x^2y'' + xy' + y = \sec(\ln x)$; $y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$

In Problems 25 and 26, solve the given third-order differential equation by variation of parameters.

25. $y''' + y' = \tan x$ **26.** $y''' + 4y' = \sec 2x$

Discussion Problems

In Problems 27 and 28, discuss how the methods of undetermined coefficients and variation of parameters can be combined to solve the given differential equation. Carry out your ideas.

27. $3y'' - 6y' + 30y = 15 \sin x + e^x \tan 3x$

28. $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$

29. What are the intervals of definition of the general solutions in Problems 1, 7, 9, and 18? Discuss why the interval of definition of the general solution in Problem 24 is *not* $(0, \infty)$.

30. Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

Computer Lab Assignments

In Problems 31 and 32, the indefinite integrals of the equations in (5) are nonelementary. Use a CAS to find the first four nonzero terms of a Maclaurin series of each integrand and then integrate the result. Find a particular solution of the given differential equation.

31. $y'' + y = \sqrt{1 + x^2}$

32. $4y'' - y = e^{x^2}$

3.6 Cauchy–Euler Equations

Introduction The relative ease with which we were able to find explicit solutions of linear higher-order differential equations with *constant coefficients* in the preceding sections does not, in general, carry over to linear equations with *variable coefficients*. We shall see in Chapter 5 that when a linear differential equation has variable coefficients, the best that we can usually expect is to find a solution in the form of an infinite series. However, the type of differential equation considered in this section is an exception to this rule; it is an equation with variable coefficients whose general solution can always be expressed in terms of powers of x, sines, cosines, logarithmic, and exponential functions. Moreover, its method of solution is quite similar to that for constant equations.

Cauchy–Euler Equation Any linear differential equation of the form

$$a_nx^n\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1x\frac{dy}{dx} + a_0y = g(x),$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known diversely as a **Cauchy–Euler equation**, an **Euler–Cauchy equation**, an **Euler equation**, or an **equidimensional equation**. The differential equation is named in honor of two of the most prolific mathematicians of all time, **Augustin-Louis Cauchy** (French, 1789–1857) and **Leonhard Euler** (Swiss, 1707–1783). The observable characteristic of this type of equation is that the degree $k = n, n-1, \ldots, 1, 0$ of the monomial coefficients x^k matches the order k of differentiation d^ky/dx^k:

$$\overset{\text{same}}{a_n\underset{\downarrow}{x^n}\frac{d^{\underset{\downarrow}{n}}y}{dx^n}} + \overset{\text{same}}{a_{n-1}\underset{\downarrow}{x^{n-1}}\frac{d^{\underset{\downarrow}{n-1}}y}{dx^{n-1}}} + \cdots.$$

As in Section 3.3, we start the discussion with a detailed examination of the forms of the general solutions of the homogeneous second-order equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0. \tag{1}$$

The solution of higher-order equations follows analogously. Also, we can solve the nonhomogeneous equation $ax^2y'' + bxy' + cy = g(x)$ by variation of parameters, once we have determined the complementary function $y_c(x)$ of (1).