We solve the last equation for w, use $w = u'$, and integrate again:

$$u = c_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + c_2.$$

By choosing $c_1 = 1$ and $c_2 = 0$, we find from $y = u(x)y_1(x)$ that a second solution of equation (3) is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)} dx. \tag{5}$$

It makes a good review of differentiation to verify that the function $y_2(x)$ defined in (5) satisfies equation (3) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero.

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution on the interval $(0, \infty)$.

SOLUTION From the standard form of the equation

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0,$$

we find from (5)

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx \quad \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

The general solution on the interval $(0, \infty)$ is given by $y = c_1y_1 + c_2y_2$; that is, $y = c_1x^2 + c_2x^2 \ln x$. ≡

Remarks

We have derived and illustrated how to use (5) because this formula appears again in the next section and in Section 5.2. We use (5) simply to save time in obtaining a desired result. Your instructor will tell you whether you should memorize (5) or whether you should know the first principles of reduction of order.

3.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-4.

In Problems 1–16, the indicated function $y_1(x)$ is a solution of the given equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0$; $\quad y_1 = e^{2x}$
2. $y'' + 2y' + y = 0$; $\quad y_1 = xe^{-x}$
3. $y'' + 16y = 0$; $\quad y_1 = \cos 4x$
4. $y'' + 9y = 0$; $\quad y_1 = \sin 3x$
5. $y'' - y = 0$; $\quad y_1 = \cosh x$
6. $y'' - 25y = 0$; $\quad y_1 = e^{5x}$
7. $9y'' - 12y' + 4y = 0$; $\quad y_1 = e^{2x/3}$
8. $6y'' + y' - y = 0$; $\quad y_1 = e^{x/3}$
9. $x^2y'' - 7xy' + 16y = 0$; $\quad y_1 = x^4$
10. $x^2y'' + 2xy' - 6y = 0$; $\quad y_1 = x^2$
11. $xy'' + y' = 0$; $\quad y_1 = \ln x$
12. $4x^2y'' + y = 0$; $\quad y_1 = x^{1/2} \ln x$
13. $x^2y'' - xy' + 2y = 0$; $\quad y_1 = x \sin(\ln x)$
14. $x^2y'' - 3xy' + 5y = 0$; $\quad y_1 = x^2 \cos(\ln x)$
15. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0$; $\quad y_1 = x + 1$
16. $(1 - x^2)y'' + 2xy' = 0$; $\quad y_1 = 1$

In Problems 17–20, the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution of the given nonhomogeneous equation.

17. $y'' - 4y = 2$; $\quad y_1 = e^{-2x}$
18. $y'' + y' = 1$; $\quad y_1 = 1$
19. $y'' - 3y' + 2y = 5e^{3x}$; $\quad y_1 = e^x$
20. $y'' - 4y' + 3y = x$; $\quad y_1 = e^x$

≡ Discussion Problems

21. (a) Give a convincing demonstration that the second-order equation $ay'' + by' + cy = 0$, a, b, and c constants, always possesses at least one solution of the form $y_1 = e^{m_1 x}$, m_1 a constant.
(b) Explain why the differential equation in part (a) must then have a second solution, either of the form $y_2 = e^{m_2 x}$, or of the form $y_2 = xe^{m_1 x}$, m_1 and m_2 constants.
(c) Reexamine Problems 1–8. Can you explain why the statements in parts (a) and (b) above are not contradicted by the answers to Problems 3–5?

22. Verify that $y_1(x) = x$ is a solution of $xy'' - xy' + y = 0$. Use reduction of order to find a second solution $y_2(x)$ in the form of an infinite series. Conjecture an interval of definition for $y_2(x)$.

≡ Computer Lab Assignment

23. (a) Verify that $y_1(x) = e^x$ is a solution of

$$xy'' - (x + 10)y' + 10y = 0.$$

(b) Use (5) to find a second solution $y_2(x)$. Use a CAS to carry out the required integration.
(c) Explain, using Corollary (a) of Theorem 3.1.2, why the second solution can be written compactly as

$$y_2(x) = \sum_{n=0}^{10} \frac{1}{n!} x^n.$$

3.3 Homogeneous Linear Equations with Constant Coefficients

≡ Introduction We have seen that the linear first-order DE $y' + ay = 0$, where a is a constant, possesses the exponential solution $y = c_1 e^{-ax}$ on the interval $(-\infty, \infty)$. Therefore, it is natural to ask whether exponential solutions exist for homogeneous linear higher-order DEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{1}$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants and $a_n \neq 0$. The surprising fact is that *all* solutions of these higher-order equations are either exponential functions or are constructed out of exponential functions.

□ Auxiliary Equation We begin by considering the special case of a second-order equation

$$ay'' + by' + cy = 0. \tag{2}$$

If we try a solution of the form $y = e^{mx}$, then after substituting $y' = me^{mx}$ and $y'' = m^2 e^{mx}$ equation (2) becomes

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

Since e^{mx} is never zero for real values of x, it is apparent that the only way that this exponential function can satisfy the differential equation (2) is to choose m as a root of the quadratic equation

$$am^2 + bm + c = 0. \tag{3}$$

This last equation is called the **auxiliary equation** of the differential equation (2). Since the two roots of (3) are $m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ and $m_2 = (-b - \sqrt{b^2 - 4ac})/2a$, there will be three forms of the general solution of (1) corresponding to the three cases:

- m_1 and m_2 are real and distinct $(b^2 - 4ac > 0)$,
- m_1 and m_2 are real and equal $(b^2 - 4ac = 0)$, and
- m_1 and m_2 are conjugate complex numbers $(b^2 - 4ac < 0)$.

We discuss each of these cases in turn.