# 3 Applications of Differentiation

# 3.7 Optimization Problems

# Objective

- Solve applied minimum and maximum problems.

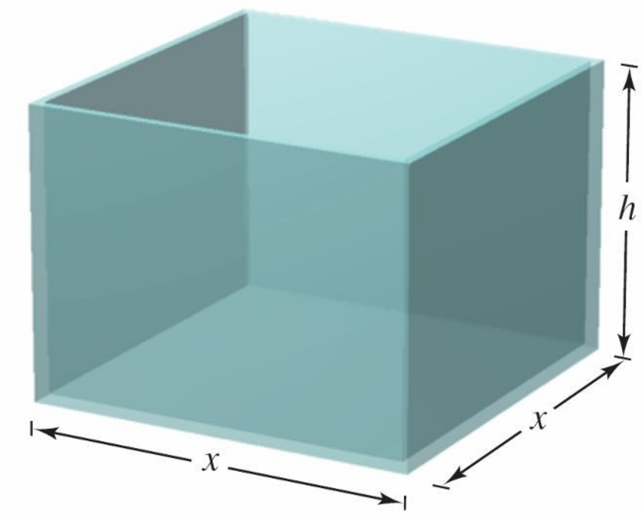
# Applied Minimum and Maximum Problems

# Applied Minimum and Maximum Problems

One of the most common applications of calculus involves the determination of minimum and maximum values.

# Example 1 – *Finding Maximum Volume*

A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, as shown in Figure 3.53. What dimensions will produce a box with maximum volume?




Open box with square base:
$S = x^2 + 4xh = 108$

**Figure 3.53**

# Example 1 – *Solution*

Because the box has a square base, its volume is

$V = x^2h.$ Primary equation

This equation is called the **primary equation** because it gives a formula for the quantity to be optimized.

The surface area of the box is

$S$ = (area of base) + (area of four sides)

$108 = x^2 + 4xh.$ Secondary equation

# Example 1 – *Solution*

cont'd

Because $V$ is to be maximized, you want to write $V$ as a function of just one variable.

To do this, you can solve the equation $x^2 + 4xh = 108$ for $h$ in terms of $x$ to obtain $h = (108 - x^2)/(4x)$.

Substituting into the primary equation produces

$$V = x^2 h \qquad \text{Function of two variables}$$

$$= x^2\left(\frac{108 - x^2}{4x}\right) \qquad \text{Substitute for } h.$$

$$= 27x - \frac{x^3}{4}. \qquad \text{Function of one variable}$$

# Example 1 – *Solution*

cont'd

Before finding which $x$-value will yield a maximum value of $V$, you should determine the *feasible domain*.

That is, what values of $x$ make sense in this problem?

You know that $V \geq 0$. You also know that $x$ must be nonnegative and that the area of the base ($A = x^2$) is at most 108.

So, the feasible domain is

$$0 \leq x \leq \sqrt{108}. \qquad \text{Feasible domain}$$

# Example 1 – *Solution*

cont'd

To maximize $V$, find its critical numbers on the interval $\left(0, \sqrt{108}\right)$.

$$\frac{dV}{dx} = 27 - \frac{3x^2}{4} \qquad \text{Differentiate with respect to } x.$$

$$27 - \frac{3x^2}{4} = 0 \qquad \text{Set derivative equal to 0.}$$

$$3x^2 = 108 \qquad \text{Simplify.}$$

$$x = \pm 6 \qquad \text{Critical numbers}$$

So, the critical numbers are $x = \pm 6$.

You do not need to consider $x = -6$ because it is outside the domain.

# Example 1 – *Solution*

cont'd

Evaluating $V$ at the critical number $x = 6$ and at the endpoints of the domain produces $V(0) = 0$, $V(6) = 108$, and $V(\sqrt{108}) = 0.$

So, $V$ is maximum when $x = 6$, and the dimensions of the box are 6 inches by 6 inches by 3 inches.

# Applied Minimum and Maximum Problems

**GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS**

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented on the formula card inside the back cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques.

# Example 2 – *Finding Minimum Distance*

Which points on the graph of $y = 4 - x^2$ are closest to the point (0, 2)?

Solution:

Figure 3.55 shows that there are two points at a minimum distance from the point (0, 2).

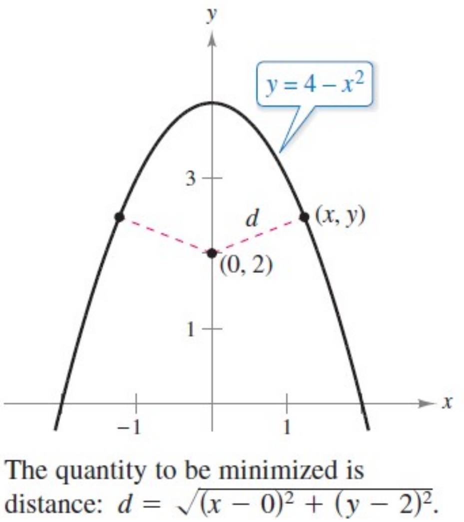



**Figure 3.55**

The distance between the point (0, 2) and a point $(x, y)$ on the graph of $y = 4 - x^2$ is

$$d = \sqrt{(x - 0)^2 + (y - 2)^2}. \qquad \text{Primary equation}$$

# Example 2 – *Solution*

cont'd

Using the secondary equation $y = 4 - x^2$, you can rewrite the primary equation as

$$d = \sqrt{x^2 + (4 - x^2 - 2)^2}$$

$$= \sqrt{x^4 - 3x^2 + 4}.$$

Because $d$ is smallest when the expression inside the radical is smallest, you need only find the critical numbers of $f(x) = x^4 - 3x^2 + 4$.

Note that the domain of $f$ is the entire real number line. So, there are no endpoints of the domain to consider.

# Example 2 – *Solution*

cont'd

Moreover, the derivative of *f*

$$f'(x) = 4x^3 - 6x$$
$$= 2x(2x^2 - 3)$$

is zero when

$$x = 0, \sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}.$$

Testing these critical numbers using the First Derivative Test verifies that *x* = 0 yields a relative maximum, whereas both $x = \sqrt{3/2}$ and $x = -\sqrt{3/2}$ yield a minimum distance.

So, the closest points are $\left(\sqrt{3/2}, 5/2\right)$ and $\left(-\sqrt{3/2}, 5/2\right)$.