4 Integration

4.3 Riemann Sums and Definite Integrals

Objectives

- Understand the definition of a Riemann sum.
- Evaluate a definite integral using limits and geometric formulas.
- Evaluate a definite integral using properties of definite integrals.

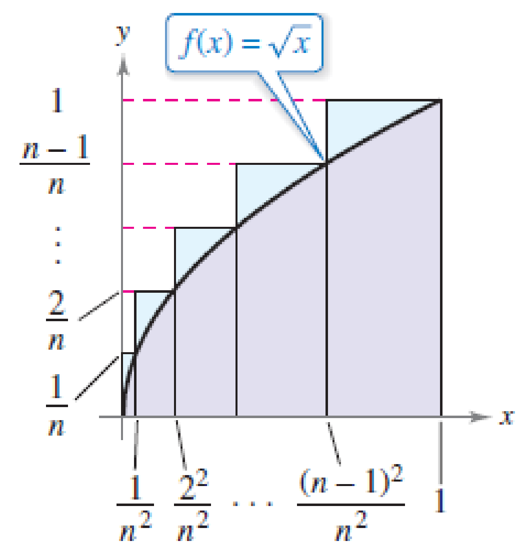
Riemann Sums

Example 1 – *A Partition with Subintervals of Unequal Widths*

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \le x \le 1$, as shown in Figure 4.18. Evaluate the limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

where c_i is the right endpoint of the partition given by $c_i = i^2/n^2$ and Δx_i is the width of the ith interval.



The subintervals do not have equal widths.

Figure 4.18

Example 1 – *Solution*

The width of the *i*th interval is

$$\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}$$

$$= \frac{i^2 - i^2 + 2i - 1}{n^2}$$

$$= \frac{2i - 1}{n^2}.$$

So, the limit is

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}}\left(\frac{2i-1}{n^2}\right)$$

Example 1 – *Solution*

cont'd

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} (2i^2 - i)$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[2\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{n(n+1)}{2}\right]$$

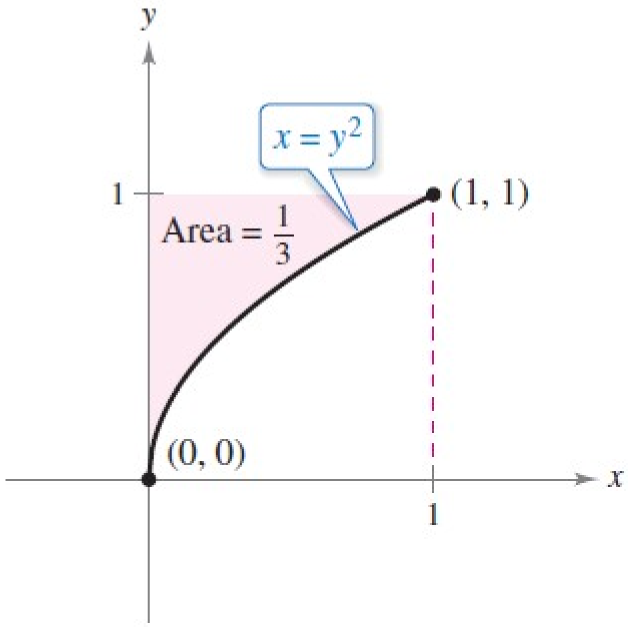
$$= \lim_{n\to\infty} \frac{4n^3 + 3n^2 - n}{6n^3}$$

$$= \lim_{n\to\infty} \left(\frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}\right)$$

$$= \frac{2}{3}.$$

Riemann Sums

We know that the region shown in Figure 4.19 has an area of $\frac{1}{3}$.

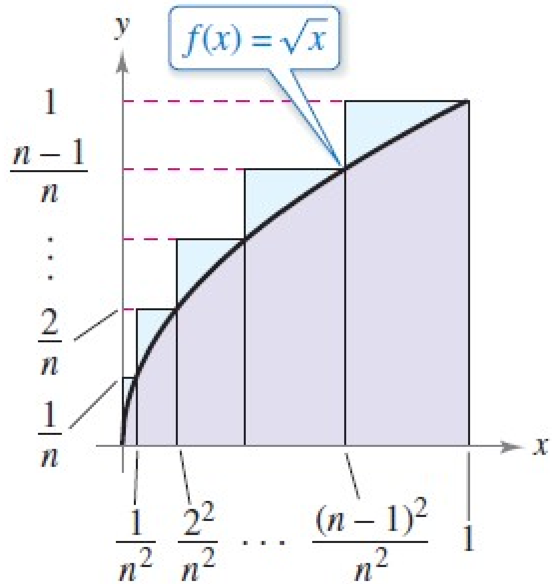


The area of the region bounded by the graph of $x = y^2$ and the y-axis for $0 \le y \le 1$ is $\frac{1}{3}$.

Figure 4.19

Riemann Sums

Because the square bounded by $0 \leq x \leq 1$ and $0 \leq y \leq 1$ has an area of 1, you can conclude that the area of the region shown in Figure 4.18 has an area of $\frac{2}{3}$.



The subintervals do not have equal widths.

Figure 4.18

Riemann Sums

This agrees with the limit found in Example 1, even though that example used a partition having subintervals of unequal widths.

The reason this particular partition gave the proper area is that as *n* increases, the *width of the largest subinterval approaches zero*.

This is a key feature of the development of definite integrals.

Riemann Sums

In the definition of a Riemann sum below, note that the function *f* has no restrictions other than being defined on the interval [*a*, *b*].

Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdot \cdot \cdot < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval

$$[x_{i-1}, x_i].$$

*i*th subinterval

If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ.

Riemann Sums

The width of the largest subinterval of a partition Δ is the **norm** of the partition and is denoted by $||\Delta||$. If every subinterval is of equal width, the partition is **regular** and the norm is denoted by

$$\|\Delta\| = \Delta x = \frac{b-a}{n}. \qquad \text{Regular partition}$$

For a **general partition**, the norm is related to the number of subintervals of [*a*, *b*] in the following way.

$$\frac{b-a}{\|\Delta\|} \leq n \qquad \text{General partition}$$

Riemann Sums

So, the number of subintervals in a partition approaches infinity as the norm of the partition approaches 0.

That is, $||\Delta|| \to 0$ implies that $n \to \infty$.

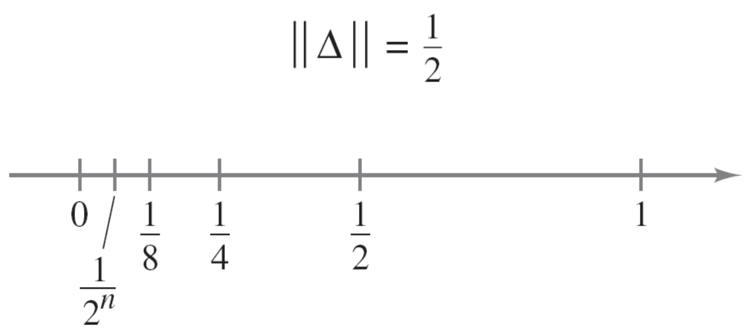
The converse of this statement is not true. For example, let Δ_n be the partition of the interval [0, 1] given by

$$0 < \frac{1}{2^n} < \frac{1}{2^{n-1}} < \cdots < \frac{1}{8} < \frac{1}{4} < \frac{1}{2} < 1.$$

Riemann Sums

As shown in Figure 4.20, for any positive value of *n*, the norm of the partition Δ_n is $\frac{1}{2}$.

So, letting *n* approach infinity does not force $\|\Delta\|$ to approach 0. In a regular partition, however, the statements $\|\Delta\| \to 0$ and $n \to \infty$ are equivalent.



$n \to \infty$ does not imply that $\|\Delta\| \to 0$.

Figure 4.20

Definite Integrals

Definite Integrals

To define the definite integral, consider the limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i = L.$$

To say that this limit exists means there exists a real number L such that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition with $\|\Delta\| < \delta$, it follows that

$$\left| L - \sum_{i=1}^{n} f(c_i)\, \Delta x_i \right| < \varepsilon$$

regardless of the choice of c_i in the ith subinterval of each partition Δ.

Definite Integrals

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\, \Delta x_i = \int_a^b f(x)\, dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Definite Integrals

Though Riemann sums were defined for functions with very few restrictions, a sufficient condition for a function *f* to be integrable on [*a*, *b*] is that it is continuous on [*a*, *b*].

THEOREM 4.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

Example 2 – *Evaluating a Definite Integral as a Limit*

Evaluate the definite integral $\int_{-2}^{1} 2x\, dx.$

Solution:

The function $f(x) = 2x$ is integrable on the interval [–2, 1] because it is continuous on [–2, 1].

Moreover, the definition of integrability implies that any partition whose norm approaches 0 can be used to determine the limit.

Example 2 – *Solution*

cont'd

For computational convenience, define Δ by subdividing [–2, 1] into n subintervals of equal width

$$\Delta x_i = \Delta x = \frac{b - a}{n} = \frac{3}{n}.$$

Choosing c_i as the right endpoint of each subinterval produces

$$c_i = a + i(\Delta x) = -2 + \frac{3i}{n}.$$

Example 2 – *Solution*

cont'd

So, the definite integral is

$$\int_{-2}^{1} 2x\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^{n} \left(-2 + \frac{3i}{n}\right)$$

Example 2 – *Solution*

cont'd

$$= \lim_{n\to\infty} \frac{6}{n}\left(-2\sum_{i=1}^{n} 1 + \frac{3}{n}\sum_{i=1}^{n} i\right)$$

$$= \lim_{n\to\infty} \frac{6}{n}\left\{-2n + \frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\}$$

$$= \lim_{n\to\infty} \left(-12 + 9 + \frac{9}{n}\right)$$

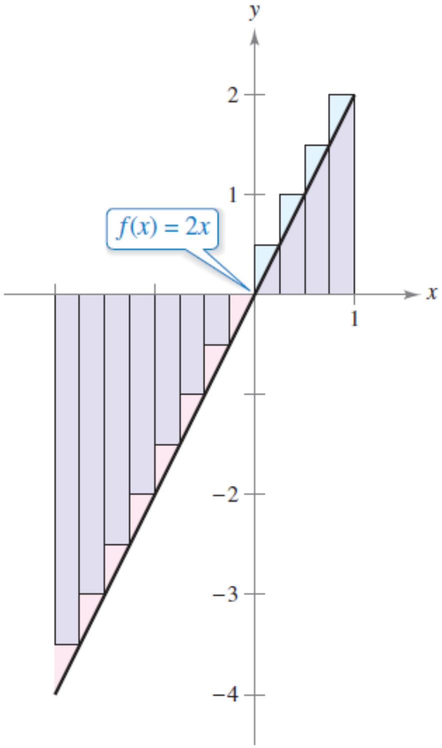
$$= -3.$$

Definite Integrals

Because the definite integral in Example 2 is negative, it *does not* represent the area of the region shown in Figure 4.21.

Definite integrals can be positive, negative, or zero.

For a definite integral to be interpreted as an area, the function *f* must be continuous and nonnegative on [*a*, *b*], as stated in the next theorem.



Because the definite integral is negative, it does not represent the area of the region.

Figure 4.21

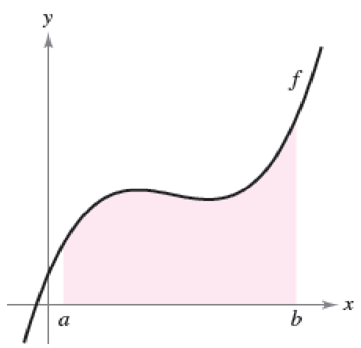
Definite Integrals

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.22.)



You can use a definite integral to find the area of the region bounded by the graph of f, the x-axis, $x = a$, and $x = b$.

Figure 4.22

Definite Integrals

As an example of Theorem 4.5, consider the region bounded by the graph of $f(x) = 4x - x^2$ and the x-axis, as shown in Figure 4.23.

Because f is continuous and nonnegative on the closed interval [0, 4], the area of the region is

$$\text{Area} = \int_0^4 (4x - x^2)\, dx.$$

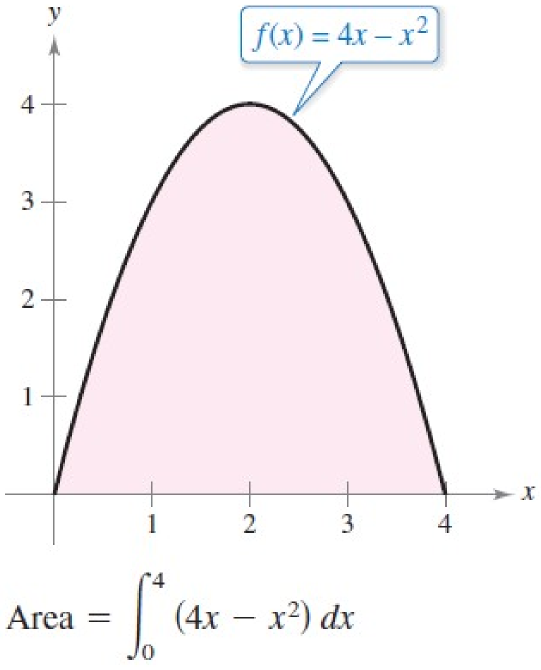


Figure 4.23

Definite Integrals

You can evaluate a definite integral in two ways—you can use the limit definition *or* you can check to see whether the definite integral represents the area of a common geometric region such as a rectangle, triangle, or semicircle.

Example 3 – *Areas of Common Geometric Figures*

Sketch the region corresponding to each definite integral. Then evaluate each integral using a geometric formula.

a. $\displaystyle\int_1^3 4\,dx$

b. $\displaystyle\int_0^3 (x+2)\,dx$

c. $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\,dx$

Example 3(a) – *Solution*

This region is a rectangle of height 4 and width 2.

$$\int_1^3 4\,dx = (\text{Area of rectangle}) = 4(2) = 8$$

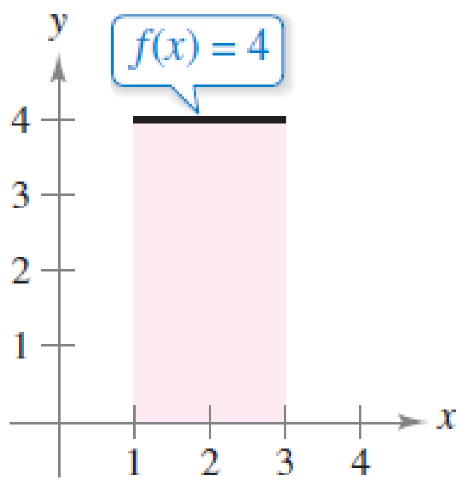


Figure 4.24(a)

Example 3(b) – *Solution*

cont'd

This region is a trapezoid with an altitude of 3 and parallel bases of lengths 2 and 5. The formula for the area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$.

$$\int_0^3 (x + 2)\, dx = (\text{Area of trapezoid}) = \frac{1}{2}(3)(2 + 5) = \frac{21}{2}$$

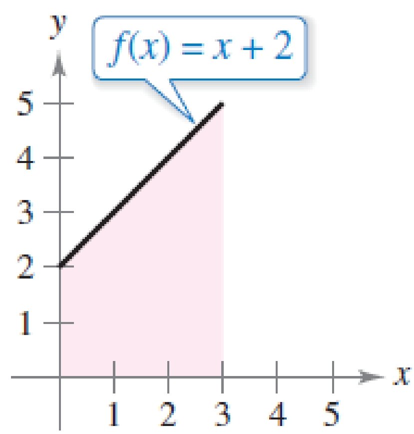


Figure 4.24(b)

Example 3(c) – *Solution*

cont'd

This region is a semicircle of radius 2. The formula for the area of a semicircle is $\frac{1}{2}\pi r^2$.

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = (\text{Area of semicircle}) = \frac{1}{2}\pi(2^2) = 2\pi$$

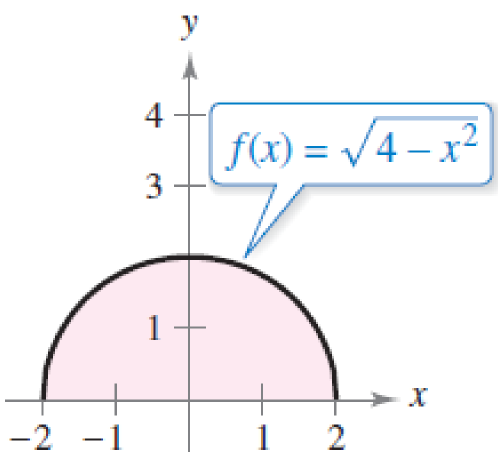


Figure 4.24(c)

Definite Integrals

The variable of integration in a definite integral is sometimes called a *dummy variable* because it can be replaced by any other variable without changing the value of the integral.

For instance, the definite integrals

$$\int_0^3 (x + 2)\, dx \quad \text{and} \quad \int_0^3 (t + 2)\, dt$$

have the same value.

Properties of Definite Integrals

Properties of Definite Integrals

The definition of the definite integral of f on the interval $[a, b]$ specifies that $a < b$.

Now, however, it is convenient to extend the definition to cover cases in which $a = b$ or $a > b$.

Geometrically, the next two definitions seem reasonable.

For instance, it makes sense to define the area of a region of zero width and finite height to be 0.

Properties of Definite Integrals

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Example 4 – *Evaluating Definite Integrals*

Evaluate each definite integral.

a. $\int_{\pi}^{\pi} \sin x \, dx$ **b.** $\int_{3}^{0} (x + 2) \, dx$

Solution:

a. Because the sine function is defined at $x = \pi$, and the upper and lower limits of integration are equal, you can write

$$\int_{\pi}^{\pi} \sin x \, dx = 0.$$

Example 4 – *Solution*

cont'd

b. The integral $\int_3^0 (x + 2)\, dx$ is the same as that given in Example 3(b) except that the upper and lower limits are interchanged.

Because the integral in Example 3(b) has a value of $\frac{21}{2}$, you can write

$$\int_3^0 (x + 2)\, dx = -\int_0^3 (x + 2)\, dx$$

$$= -\frac{21}{2}.$$

Properties of Definite Integrals

In Figure 4.25, the larger region can be divided at $x = c$ into two subregions whose intersection is a line segment.

Because the line segment has zero area, it follows that the area of the larger region is equal to the sum of the areas of the two smaller regions.

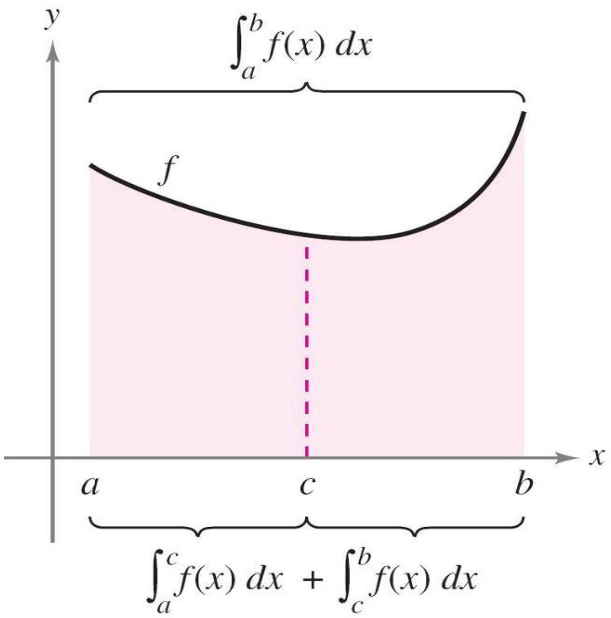


Figure 4.25

THEOREM 4.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

See Figure 4.25.

Example 5 – *Using the Additive Interval Property*

$$\int_{-1}^{1} |x|\, dx = \int_{-1}^{0} -x\, dx + \int_{0}^{1} x\, dx \qquad \text{Theorem 4.6}$$

$$= \frac{1}{2} + \frac{1}{2} \qquad \text{Area of a triangle}$$

$$= 1$$

Properties of Definite Integrals

THEOREM 4.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

2. $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

Note that Property 2 of Theorem 4.7 can be extended to cover any finite number of functions.

Example 6 – *Evaluation of a Definite Integral*

Evaluate $\int_1^3 (-x^2 + 4x - 3)\,dx$ using each of the following values.

$$\int_1^3 x^2\,dx = \frac{26}{3}, \qquad \int_1^3 x\,dx = 4, \qquad \int_1^3 dx = 2$$

Solution:

$$\int_1^3 (-x^2 + 4x - 3)\,dx = \int_1^3 (-x^2)\,dx + \int_1^3 4x\,dx + \int_1^3 (-3)\,dx$$

$$= -\int_1^3 x^2\,dx + 4\int_1^3 x\,dx - 3\int_1^3 dx$$

$$= -\left(\frac{26}{3}\right) + 4(4) - 3(2)$$

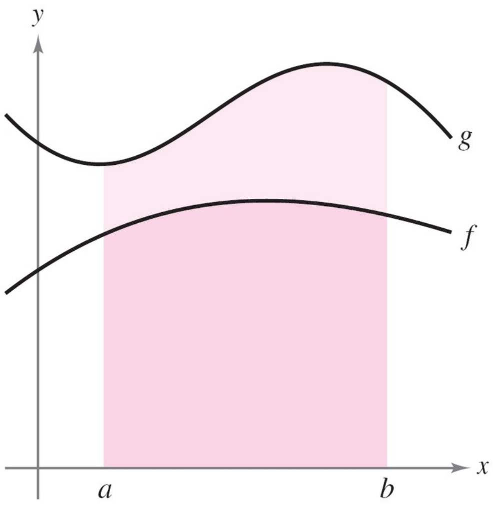
$$= \frac{4}{3}$$

Properties of Definite Integrals

If f and g are continuous on the closed interval $[a, b]$ and $0 \le f(x) \le g(x)$ for $a \le x \le b$, then the following properties are true.

First, the area of the region bounded by the graph of f and the x-axis (between a and b) must be nonnegative.

Second, this area must be less than or equal to the area of the region bounded by the graph of g and the x-axis (between a and b), as shown in Figure 4.26.



$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Figure 4.26

Properties of Definite Integrals

These two properties are generalized in Theorem 4.8.

THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x)\,dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$