5 Logarithmic, Exponential, and Other Transcendental Functions

5.3 Inverse Functions

Objectives

- Verify that one function is the inverse function of another function.
- Determine whether a function has an inverse function.
- Find the derivative of an inverse function.

Inverse Functions

The function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

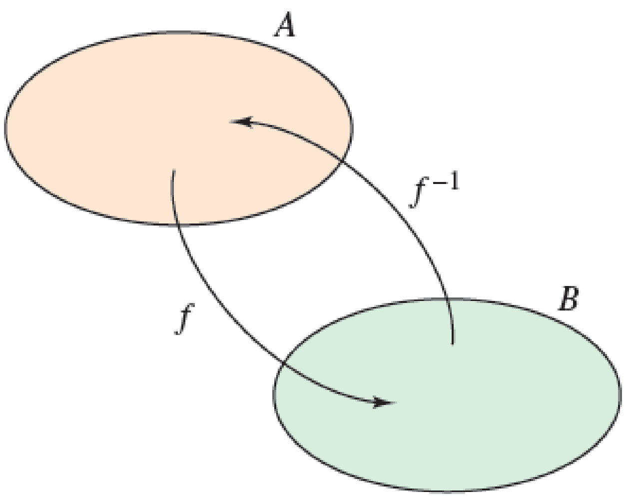
By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Inverse Functions

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 5.9. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f

Figure 5.9

Inverse Functions

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

Inverse Functions

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.

2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.

3. A function need not have an inverse function, but when it does, the inverse function is unique.

Inverse Functions

You can think of f^{-1} as undoing what has been done by f.

For example, subtraction can be used to undo addition, and division can be used to undo multiplication. So,

$$f(x) = x + c \quad \text{and} \quad f^{-1}(x) = x - c$$ Subtraction can be used to undo addition.

are inverse functions of each other and

$$f(x) = cx \quad \text{and} \quad f^{-1}(x) = \frac{x}{c}, \quad c \neq 0$$ Division can be used to undo multiplication.

are inverse functions of each other.

Example 1 – *Verifying Inverse Functions*

Show that the functions are inverse functions of each other.

$f(x) = 2x^3 - 1$ and $g(x) = \sqrt[3]{\dfrac{x+1}{2}}$

Solution:

Because the domains and ranges of both *f* and *g* consist of all real numbers, you can conclude that both composite functions exist for all *x*.

The composition of *f* with *g* is

$$f(g(x)) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1$$

$$= 2\left(\frac{x+1}{2}\right) - 1$$

Example 1 – *Solution*

cont'd

$$= x + 1 - 1$$

$$= x.$$

The composition of *g* with *f* is

$$g(f(x)) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}}$$

$$= \sqrt[3]{\frac{2x^3}{2}}$$

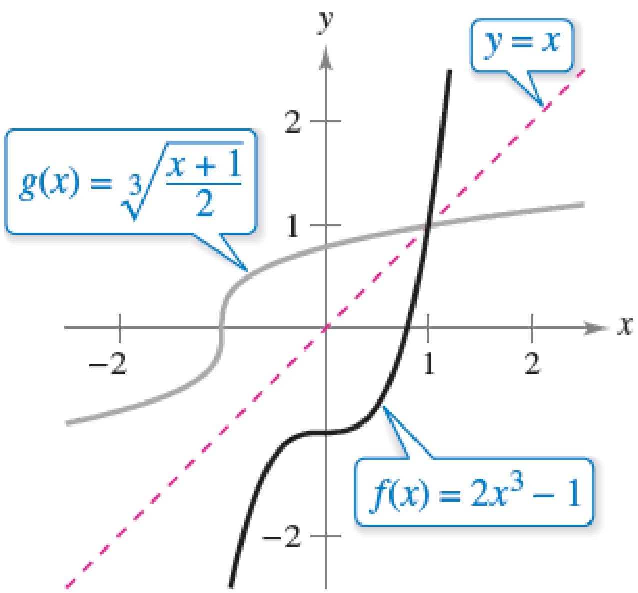
$$= \sqrt[3]{x^3}$$

$$= x.$$

Example 1 – *Solution*

cont'd

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude f and g are inverse functions of each other (see Figure 5.10).



f and *g* are inverse functions of each other.

Figure 5.10

Inverse Functions

In Figure 5.10, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$.

This idea is generalized in the next theorem.

THEOREM 5.6 Reflective Property of Inverse Functions

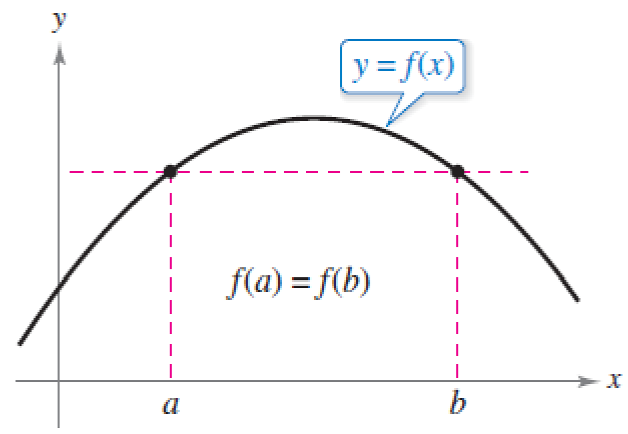
The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

Existence of an Inverse Function

Existence of an Inverse Function

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function.

This test states that a function *f* has an inverse function if and only if every horizontal line intersects the graph of *f* at most once (see Figure 5.12).



If a horizontal line intersects the graph of f twice, then f is not one-to-one.

Figure 5.12

Existence of an Inverse Function

The next theorem formally states why the Horizontal Line Test is valid.

THEOREM 5.7 The Existence of an Inverse Function

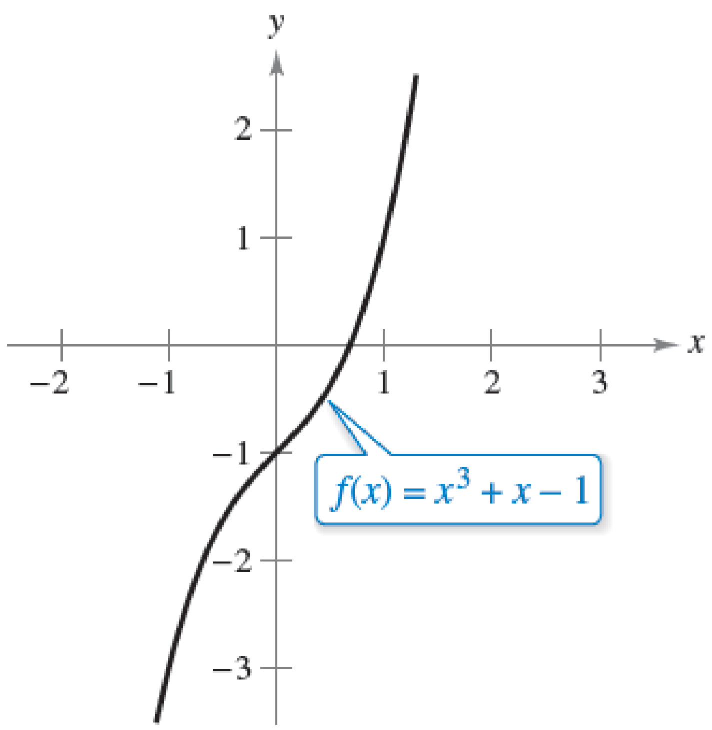
1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Example 2(a) – *The Existence of an Inverse Function*

From the graph of $f(x) = x^3 + x - 1$ shown in Figure 5.13(a), it appears that f is increasing over its entire domain.

To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x.

So, f is strictly monotonic, and it must have an inverse function.



(a) Because f is increasing over its entire domain, it has an inverse function.

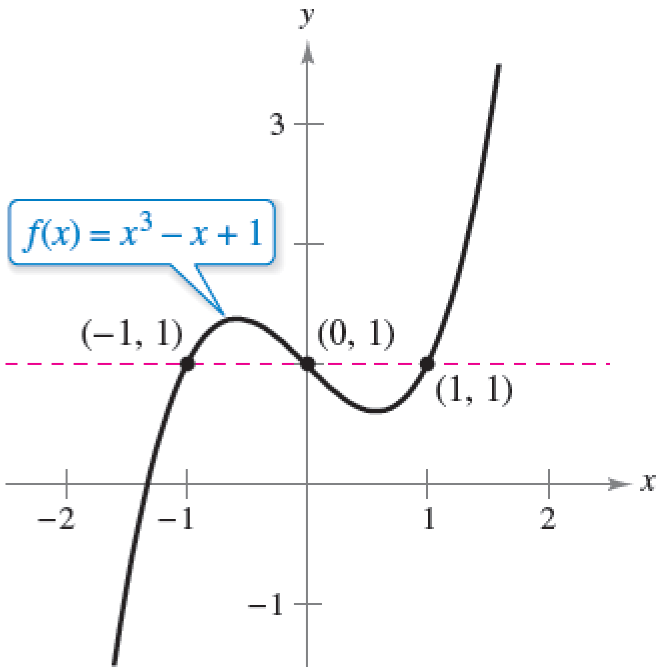
Figure 5.13

Example 2(b) – *The Existence of an Inverse Function*
cont'd

From the graph of $f(x) = x^3 - x + 1$ shown in Figure 5.13(b), you can see that the function does not pass the Horizontal Line Test.

In other words, it is not one-to-one. For instance, f has the same value when $x = -1, 0,$ and 1.

$f(-1) = f(1) = f(0) = 1$ Not one-to-one



(b) Because f is not one-to-one, it does not have an inverse function.

Figure 5.13

So, by Theorem 5.7, f does not have an inverse function.

Existence of an Inverse Function

The following guidelines suggest a procedure for finding an inverse function.

GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Use Theorem 5.7 to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 3 – *Finding an Inverse Function*

Find the inverse function of $f(x) = \sqrt{2x - 3}$.

Solution:

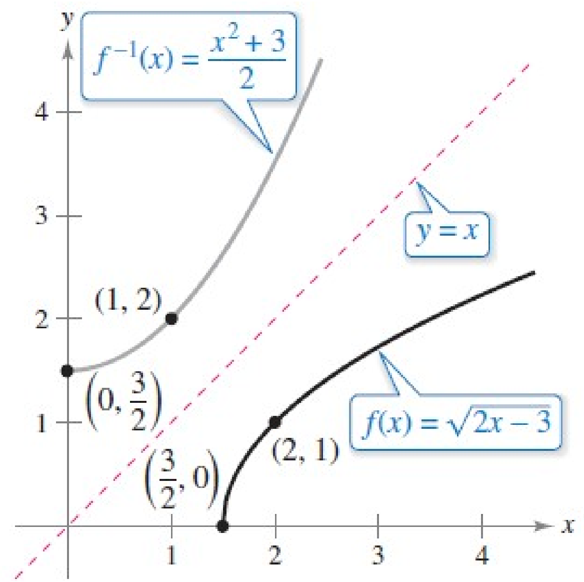
From the graph of *f* in Figure 5.14, it appears that *f* is increasing over its entire domain, $[3/2, \infty)$.

To verify this, note that

$$f'(x) = \frac{1}{\sqrt{2x - 3}}$$

is positive on the domain of *f*.

So, *f* is strictly monotonic and it must have an inverse function.



The domain of f^{-1}, $[0, \infty)$, is the range of f.

Figure 5.14

Example 3 – *Solution*

cont'd

To find an equation for the inverse function, let $y = f(x)$, and solve for x in terms of y.

$$\sqrt{2x - 3} = y \qquad \text{Let } y = f(x).$$

$$2x - 3 = y^2 \qquad \text{Square each side.}$$

$$x = \frac{y^2 + 3}{2} \qquad \text{Solve for } x.$$

$$y = \frac{x^2 + 3}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$f^{-1}(x) = \frac{x^2 + 3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

Example 3 – *Solution*

cont'd

The domain of f^{-1} is the range of f, which is $[0, \infty)$.

You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2 + 3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x - 3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \quad x \geq \frac{3}{2}$$

Existence of an Inverse Function

You are given a function that is *not* one-to-one on its domain.

By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

Example 4 – *Testing Whether a Function Is One-to-One*

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real number line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, on which f is strictly monotonic.

Example 4 – *Solution*

It is clear that f is not one-to-one, because many different x-values yield the same y-value.

For instance,

$$\sin(0) = 0 = \sin(\pi)$$

Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

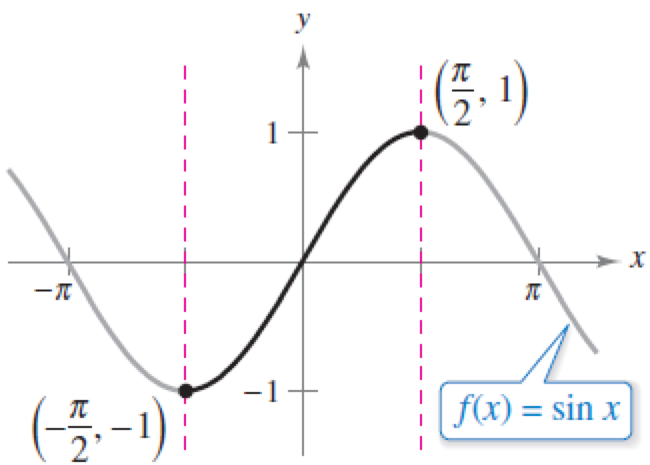
$$f'(x) = \cos x$$

is positive there.

Example 4 – *Solution*

cont'd

Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that *f* is increasing on the closed interval [–π/2, π/2] *and* that on any larger interval the function is not strictly monotonic (see Figure 5.15).



f is one-to-one on the interval $[-\pi/2, \pi/2]$.

Figure 5.15

Derivative of an Inverse Function

Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function.

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

Derivative of an Inverse Function

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

Example 5 – *Evaluating the Derivative of an Inverse Function*

Let $f(x) = \frac{1}{4}x^3 + x - 1.$

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

Solution:

Notice that f is one-to-one and therefore has an inverse function.

a. Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

Example 5 – *Solution*

cont'd

b. Because the function *f* is differentiable and has an inverse function, you can apply Theorem 5.9

(with $g = f^{-1}$) to write

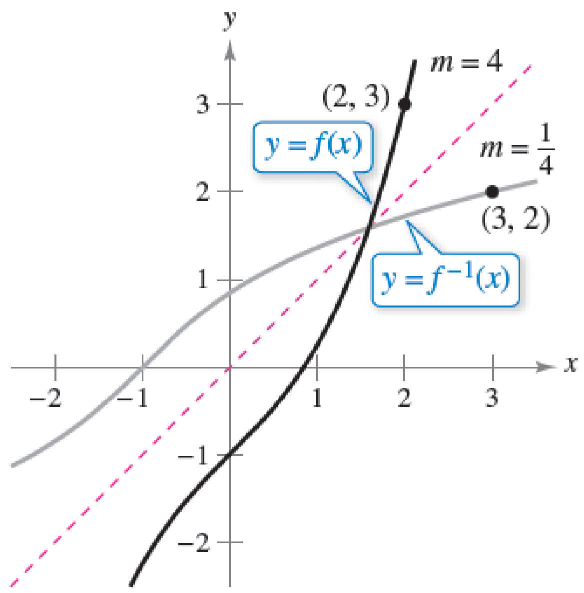
$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1,$ you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$

Derivative of an Inverse Function

In Example 5, note that at the point (2, 3), the slope of the graph of f is $m = 4$, and at the point (3, 2), the slope of the graph of f^{-1} is $m = \frac{1}{4}$ as shown in Figure 5.16.



The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).

Figure 5.16

Derivative of an Inverse Function

In general, if $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \dfrac{dx}{dy}$. It follows from Theorem 5.9 that

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}.$$

This reciprocal relationship is sometimes written as

$$\frac{dy}{dx} = \frac{1}{dx/dy}.$$

Example 6 – *Graphs of Inverse Functions Have Reciprocal Slopes*

Let $f(x) = x^2$ (for $x \geq 0$), and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. (2, 4) and (4, 2) **b.** (3, 9) and (9, 3)

Solution:

The derivative of f and f^{-1} are

$f'(x) = 2x$ and $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x}}$.

a. At (2, 4), the slope of the graph of f is $f'(2) = 2(2) = 4$.

At ($(f^{-1})'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{2(2)} = \dfrac{1}{4}$. of f^{-1} is

Example 6 – *Solution*

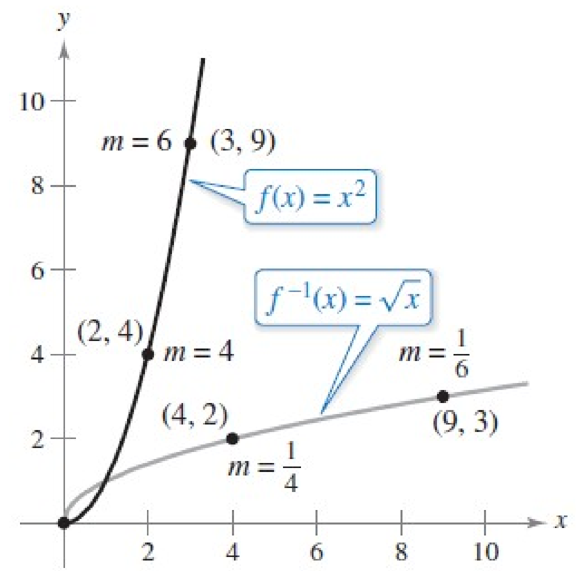
cont'd

b. At (3, 9), the slope of the graph of f is $f'(3) = 2(3) = 6$.

At (9, 3), the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 5.17.



At (0, 0), the derivative of f is 0, and the derivative of f^{-1} does not exist.

Figure 5.17