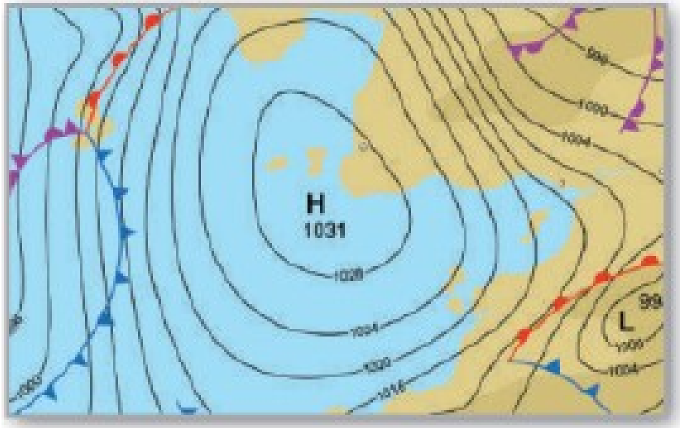
5 Logarithmic, Exponential, and Other Transcendental Functions



5.8 Inverse Trigonometric Functions: Integration

Objectives

- Integrate functions whose antiderivatives involve inverse trigonometric functions.
- Use the method of completing the square to integrate a function.
- Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

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The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other.

For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1 - x^2}}.$$

Integrals Involving Inverse Trigonometric Functions

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair.

It is conventional to use arcsin *x* as the antiderivative of $1/\sqrt{1 - x^2}$, rather than –arccos *x*.

Integrals Involving Inverse Trigonometric Functions

THEOREM 5.19 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

Example 1 – *Integration with Inverse Trigonometric Functions*

a. $$\int \frac{dx}{\sqrt{4 - x^2}} = \arcsin \frac{x}{2} + C$$ $u = x, a = 2$

b. $$\int \frac{dx}{2 + 9x^2} = \frac{1}{3} \int \frac{3\, dx}{\left(\sqrt{2}\right)^2 + (3x)^2}$$ $u = 3x, \ a = \sqrt{2}$

$$= \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$$

c. $$\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2\, dx}{2x\sqrt{(2x)^2 - 3^2}}$$ $u = 2x, \ a = 3$

$$= \frac{1}{3} \operatorname{arcsec} \frac{|2x|}{3} + C$$

Completing the Square

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand.

For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Example 4 – *Completing the Square*

Find $\displaystyle\int \frac{dx}{x^2 - 4x + 7}$.

Solution:

You can write the denominator as the sum of two squares, as follows.

$$x^2 - 4x + 7 = (x^2 - 4x + 4) - 4 + 7$$

$$= (x - 2)^2 + 3$$

$$= u^2 + a^2$$

Example 4 – *Solution*

cont'd

Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x-2)^2 + 3}$$

$$= \frac{1}{\sqrt{3}} \arctan \frac{x-2}{\sqrt{3}} + C$$

Review of Basic Integration Rules

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules**. To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$

Review of Basic Integration Rules cont'd

BASIC INTEGRATION RULES ($a > 0$)

9. $\int \cos u \, du = \sin u + C$

10. $\int \tan u \, du = -\ln|\cos u| + C$

11. $\int \cot u \, du = \ln|\sin u| + C$

12. $\int \sec u \, du = \ln|\sec u + \tan u| + C$

13. $\int \csc u \, du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u \, du = \tan u + C$

15. $\int \csc^2 u \, du = -\cot u + C$

16. $\int \sec u \tan u \, du = \sec u + C$

17. $\int \csc u \cot u \, du = -\csc u + C$

18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$

19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

Example 6 – *Comparing Integration Problems*

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$

b. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 - 1}}$

c. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 1}}$

Example 6 – *Solution*

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\int \frac{x\,dx}{\sqrt{x^2 - 1}} = \frac{1}{2}\int (x^2 - 1)^{-1/2}(2x)\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2 - 1)^{1/2}}{1/2}\right] + C$$

$$= \sqrt{x^2 - 1} + C$$

c. You *cannot* find this integral using the techniques you have studied so far.