3 Applications of Differentiation

3.8 Newton's Method

Objective

- Approximate a zero of a function using Newton’s Method.

Newton's Method

The technique for approximating the real zeros of a function is called **Newton's Method,** and it uses tangent lines to approximate the graph of the function near its x-intercepts.

To see how Newton's Method works, consider a function f that is continuous on the interval $[a, b]$ and differentiable on the interval (a, b).

If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem, f must have at least one zero in the interval (a, b).

Newton's Method

To estimate this zero, you choose

$$x = x_1 \qquad \text{First estimate}$$

as shown in Figure 3.60(a).

Newton's Method is based on the assumption that the graph of f and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at *about* the same point.

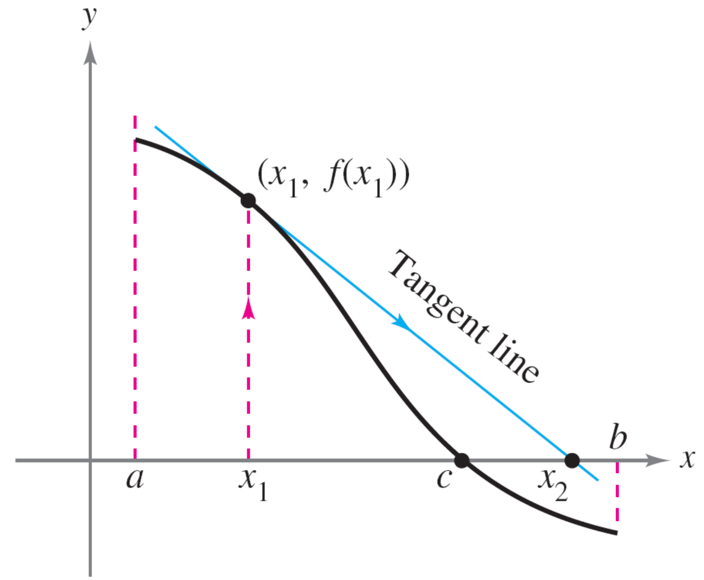


Figure 3.60(a)

Newton's Method

Because you can easily calculate the x-intercept for this tangent line, you can use it as a second (and, usually, better) estimate of the zero of f.

The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$.

In point-slope form, the equation of the tangent line is

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y &= f'(x_1)(x - x_1) + f(x_1). \end{aligned}$$

Newton's Method

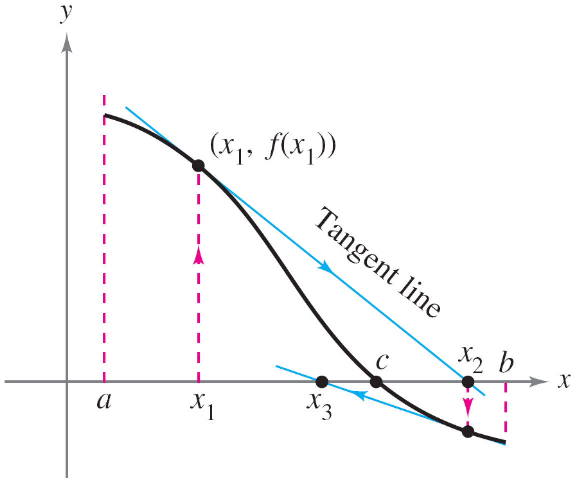
Letting $y = 0$ and solving for x produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

So, from the initial estimate x_1, you

obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$ Second estimate [see Figure 3.60(b)]



The x-intercept of the tangent line approximates the zero of f.

Figure 3.60(b)

Newton's Method

You can improve on x_2 and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}. \qquad \text{Third estimate}$$

Repeated application of this process is called Newton's Method.

Newton's Method

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use these steps.

1. Make an initial estimate x_1 that is close to c. (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. When $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration.**

Example 1 – *Using Newton's Method*

Calculate three iterations of Newton's Method to approximate a zero of $f(x) = x^2 - 2$. Use $x_1 = 1$ as the initial guess.

Solution:

Because $f(x) = x^2 - 2$, you have $f'(x) = 2x$, and the iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}.$$

Example 1 – *Solution*

cont'd

The calculations for three iterations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.000000	−1.000000	2.000000	−0.500000	1.500000
2	1.500000	0.250000	3.000000	0.083333	1.416667
3	1.416667	0.006945	2.833334	0.002451	1.414216
4	1.414216				

Example 1 – *Solution*

cont'd

Of course, in this case you know that the two zeros of the function are $\pm\sqrt{2}$.

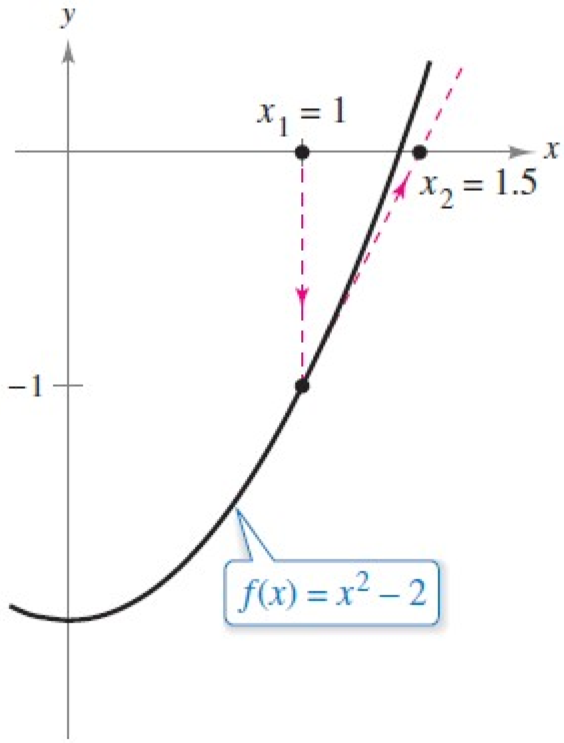
To six decimal places, $\sqrt{2} = 1.414214$.

So, after only three iterations of Newton's Method, you have obtained an approximation that is within 0.000002 of an actual root.

Example 1 – *Solution*

cont'd

The first iteration of this process is shown in Figure 3.61.



The first iteration of Newton's Method

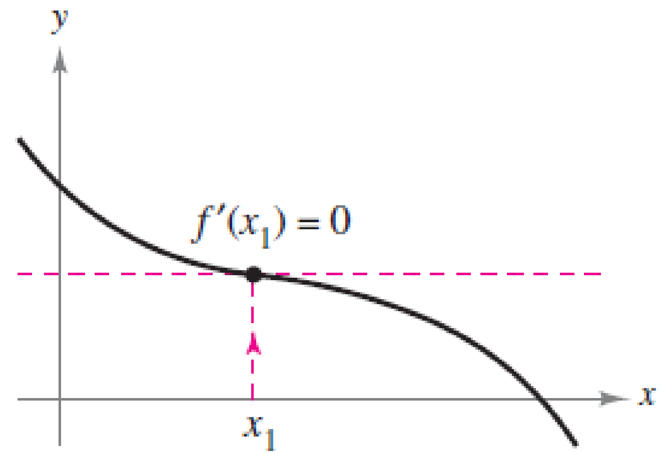
Figure 3.61

Newton's Method

When, as in Example 1, the approximations approach a limit, the sequence of approximations x_1, x_2, x_3, . . . , x_n, . . . is said to **converge.** Moreover, when the limit is *c*, it can be shown that *c* must be a zero of *f*.

Newton's Method does not always yield a convergent sequence.

One way it can fail to do so is shown in Figure 3.63.



Newton's Method fails to converge when $f'(x_n) = 0$.

Figure 3.63

Newton's Method

Because Newton's Method involves division by $f'(x_n)$, it is clear that the method will fail when the derivative is zero for any x_n in the sequence.

When you encounter this problem, you can usually overcome it by choosing a different value for x_1.

Another way Newton's Method can fail is shown in the next example.

Example 3 – *An Example in Which Newton's Method Fails*

The function $f(x) = x^{1/3}$ is not differentiable at $x = 0$. Show that Newton's Method fails to converge using $x_1 = 0.1$.

Solution:

Because $f'(x) = \frac{1}{3}x^{-2/3}$, the iterative formula is

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\
&= x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} \\
&= x_n - 3x_n \\
&= -2x_n.
\end{aligned}
$$

Example 3 – *Solution*

cont'd

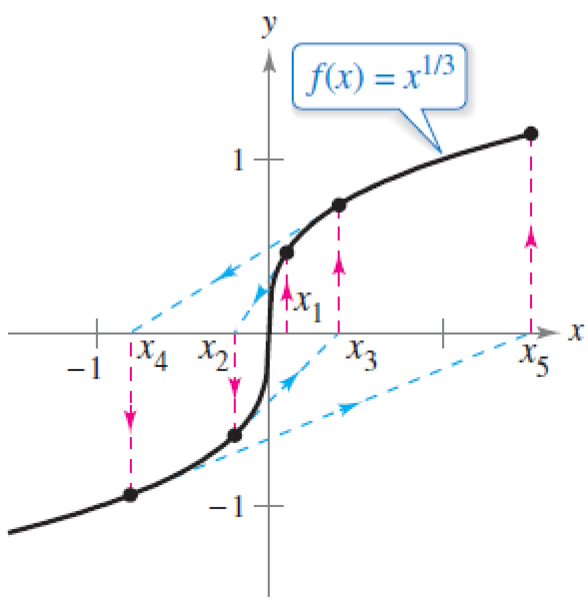
The calculations are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.10000	0.46416	1.54720	0.30000	−0.20000
2	−0.20000	−0.58480	0.97467	−0.60000	0.40000
3	0.40000	0.73681	0.61401	1.20000	−0.80000
4	−0.80000	−0.92832	0.38680	−2.40000	1.60000

Example 3 – *Solution*

cont'd

This table and Figure 3.64 indicate that x_n continues to increase in magnitude as $n \to \infty$, and so the limit of the sequence does not exist.



Newton's Method fails to converge for every x-value other than the actual zero of f.

Figure 3.64

Newton's Method

It can be shown that a condition sufficient to produce convergence of Newton's Method to a zero of *f* is that

$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1 \qquad \text{Condition for convergence}$$

on an open interval containing the zero.

Newton's Method

You have learned several techniques for finding the zeros of functions. The zeros of some functions, such as

$$f(x) = x^3 - 2x^2 - x + 2$$

can be found by simple algebraic techniques, such as factoring.

The zeros of other functions, such as

$$f(x) = x^3 - x + 1$$

cannot be found by *elementary* algebraic methods.

Newton's Method

This particular function has only one real zero, and by using more advanced algebraic techniques, you can determine the zero to be

$$x = -\sqrt[3]{\frac{3 - \sqrt{23/3}}{6}} - \sqrt[3]{\frac{3 + \sqrt{23/3}}{6}}.$$

Because the *exact* solution is written in terms of square roots and cube roots, it is called a **solution by radicals.**