4 Integration

4.2 Area

Objectives

- Use sigma notation to write and evaluate a sum.
- Understand the concept of area.
- Approximate the area of a plane region.
- Find the area of a plane region using limits.

Sigma Notation

Sigma Notation

This section begins by introducing a concise notation for sums. This notation is called **sigma notation** because it uses the uppercase Greek letter sigma, written as Σ.

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation,** a_i is the ***i*th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

Example 1 – *Examples of Sigma Notation*

a. $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b. $\sum_{i=0}^{5} (i + 1) = 1 + 2 + 3 + 4 + 5 + 6$

c. $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d. $\sum_{j=1}^{5} \frac{1}{\sqrt{j}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}}$

Example 1 – *Examples of Sigma Notation* cont'd

$$\textbf{e.}\ \sum_{k=1}^{n} \frac{1}{n}(k^2 + 1) = \frac{1}{n}(1^2 + 1) + \frac{1}{n}(2^2 + 1) + \cdots + \frac{1}{n}(n^2 + 1)$$

$$\textbf{f.}\ \sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x$$

From parts (a) and (b), notice that the same sum can be represented in different ways using sigma notation.

Sigma Notation

The properties of summation shown below can be derived using the Associative and Commutative Properties of Addition and the Distributive Property of Addition over Multiplication. (In the first property, *k* is a constant.)

1. $$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$$

2. $$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

Sigma Notation

The next theorem lists some useful formulas for sums of powers.

THEOREM 4.2 Summation Formulas

1. $\displaystyle\sum_{i=1}^{n} c = cn$, c is a constant

2. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 2 – *Evaluating a Sum*

Evaluate $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000$, and $10{,}000$.

Solution:

$$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{1}{n^2} \sum_{i=1}^{n} (i+1)$$

Factor the constant $1/n^2$ out of sum.

$$= \frac{1}{n^2} \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right)$$

Write as two sums.

$$= \frac{1}{n^2} \left[\frac{n(n+1)}{2} + n \right]$$

Apply Theorem 4.2.

Example 2 – *Solution*

cont'd

$$= \frac{1}{n^2}\left[\frac{n^2 + 3n}{2}\right] \qquad \text{Simplify.}$$

$$= \frac{n + 3}{2n} \qquad \text{Simplify.}$$

Now you can evaluate the sum by substituting the appropriate values of *n*, as shown in the table below.

n	10	100	1000	10,000
$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{n+3}{2n}$	0.65000	0.51500	0.50150	0.50015

Area

Area

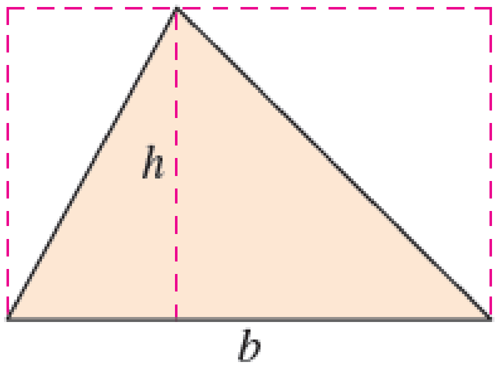
In Euclidean geometry, the simplest type of plane region is a rectangle.

Although people often say that the *formula* for the area of a rectangle is $A = bh$ it is actually more proper to say that this is the *definition* of the **area of a rectangle.**

From this definition, you can develop formulas for the areas of many other plane regions.

Area

For example, to determine the area of a triangle, you can form a rectangle whose area is twice that of the triangle, as shown in Figure 4.5.



Triangle: $A = \frac{1}{2}bh$

Figure 4.5

Area

Once you know how to find the area of a triangle, you can determine the area of any polygon by subdividing the polygon into triangular regions, as shown in Figure 4.6.

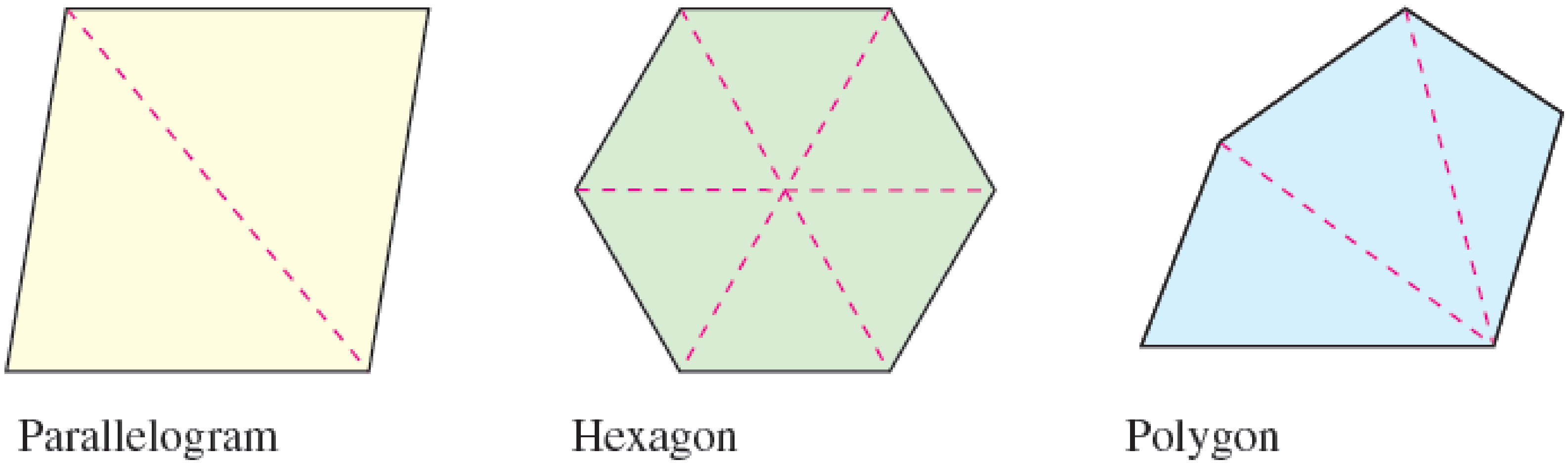


Figure 4.6

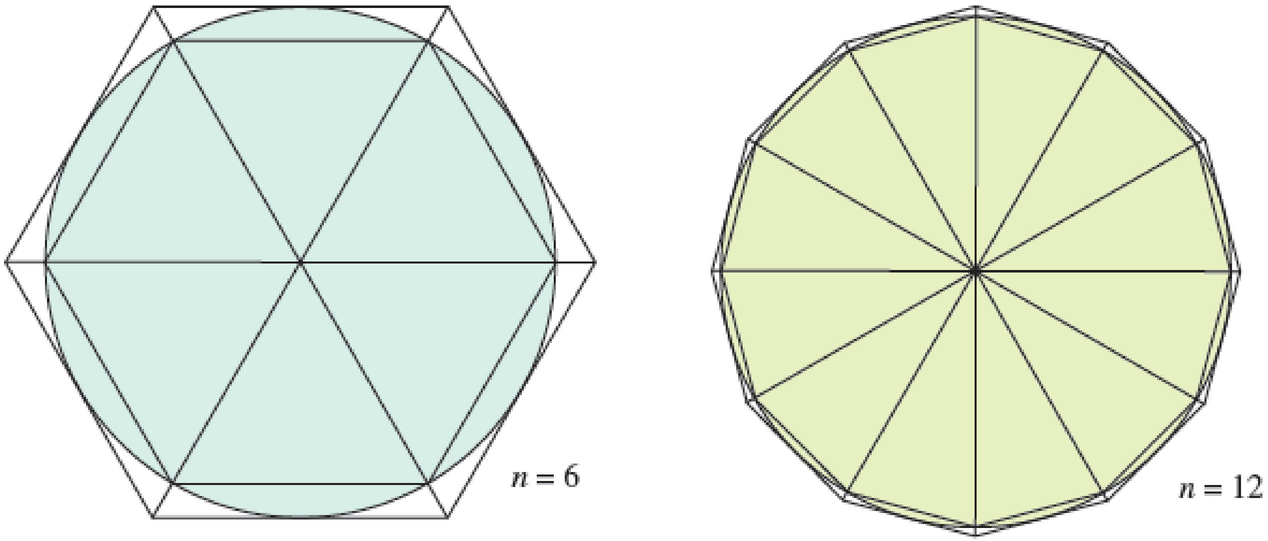
Area

Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions (principally those bounded by conics) by the *exhaustion* method.

The clearest description of this method was given by Archimedes. Essentially, the method is a limiting process in which the area is squeezed between two polygons—one inscribed in the region and one circumscribed about the region.

Area

For instance, in Figure 4.7, the area of a circular region is approximated by an *n*-sided inscribed polygon and an *n*-sided circumscribed polygon.



The exhaustion method for finding the area of a circular region

Figure 4.7

Area

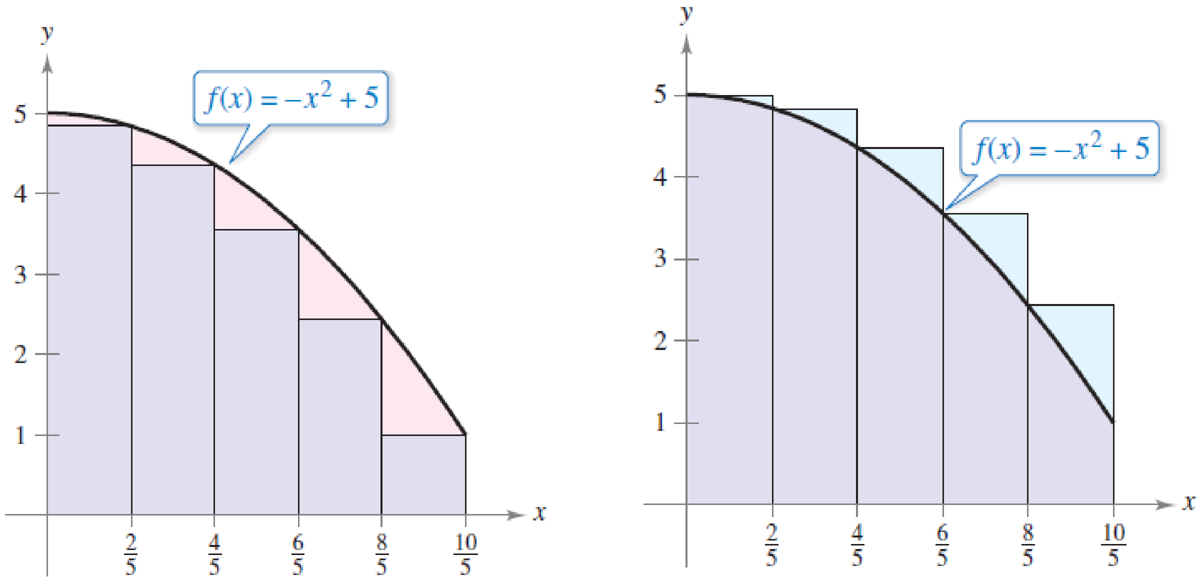
For each value of n, the area of the inscribed polygon is less than the area of the circle, and the area of the circumscribed polygon is greater than the area of the circle.

Moreover, as n increases, the areas of both polygons become better and better approximations of the area of the circle.

The Area of a Plane Region

Example 3 – *Approximating the Area of a Plane Region*

Use the five rectangles in Figure 4.8(a) and (b) to find *two* approximations of the area of the region lying between the graph of $f(x) = -x^2 + 5$ and the x-axis between $x = 0$ and $x = 2$.



(a) The area of the parabolic region is greater than the area of the rectangles.

(b) The area of the parabolic region is less than the area of the rectangles.

Figure 4.8

Example 3(a) – *Solution*

The right endpoints of the five intervals are

$$\frac{2}{5}i \qquad \text{Right endpoints}$$

where $i = 1, 2, 3, 4, 5$.

The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the right endpoint of each interval.

$$\left[0, \frac{2}{5}\right], \left[\frac{2}{5}, \frac{4}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \left[\frac{6}{5}, \frac{8}{5}\right], \left[\frac{8}{5}, \frac{10}{5}\right]$$

Evaluate f at the right endpoints of these intervals.

Example 3(a) – *Solution*

cont'd

The sum of the areas of the five rectangles is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i}{5}\right)}_{\text{Height}}\underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5}\left[-\left(\frac{2i}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{162}{25} = 6.48.$$

Because each of the five rectangles lies inside the parabolic region, you can conclude that the area of the parabolic region is greater than 6.48.

Example 3(b) – *Solution*

cont'd

The left endpoints of the five intervals are

$$\frac{2}{5}(i-1) \qquad \text{Left endpoints}$$

where i = 1, 2, 3, 4, 5.

The width of each rectangle is $\frac{2}{5}$, and the height of each rectangle can be obtained by evaluating f at the left endpoint of each interval. So, the sum is

$$\sum_{i=1}^{5} \underbrace{f\left(\frac{2i-2}{5}\right)}_{\text{Height}} \underbrace{\left(\frac{2}{5}\right)}_{\text{Width}} = \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.08.$$

Example 3(b) – *Solution*

cont'd

Because the parabolic region lies within the union of the five rectangular regions, you can conclude that the area of the parabolic region is less than 8.08.

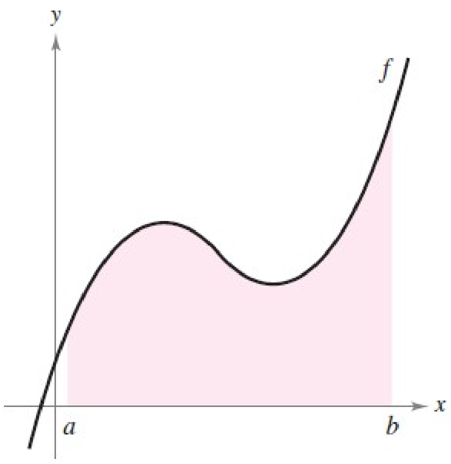
By combining the results in parts (a) and (b), you can conclude that 6.48 < (Area of region) < 8.08.

Finding Area by the Limit Definition

Finding Area by the Limit Definition

Consider a plane region bounded above by the graph of a nonnegative, continuous function $y = f(x)$ as shown in Figure 4.9.

The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.



The region under a curve

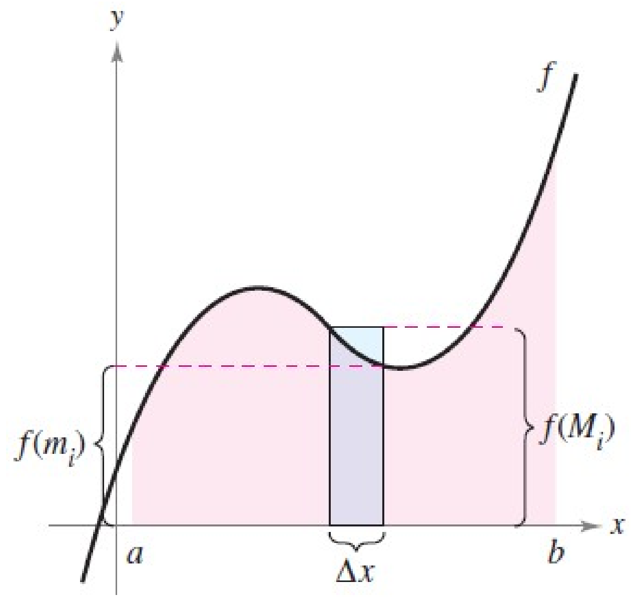
Figure 4.9

Finding Area by the Limit Definition

To approximate the area of the region, begin by subdividing the interval [*a*, *b*] into *n* subintervals, each of width

$$\Delta x = \frac{b - a}{n}$$

as shown in Figure 4.10.



The interval $[a, b]$ is divided into n subintervals of width $\Delta x = \dfrac{b-a}{n}$.

Figure 4.10

Finding Area by the Limit Definition

The endpoints of the intervals are

$$\underbrace{a + 0(\Delta x)}_{a = x_0} < \underbrace{a + 1(\Delta x)}_{x_1} < \underbrace{a + 2(\Delta x)}_{x_2} < \cdots < \underbrace{a + n(\Delta x)}_{x_n = b}.$$

Because *f* is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$f(m_i)$ = Minimum value of $f(x)$ in *i*th subinterval

$f(M_i)$ = Maximum value of $f(x)$ in *i*th subinterval

Finding Area by the Limit Definition

Next, define an **inscribed rectangle** lying *inside* the *i*th subregion and a **circumscribed rectangle** extending *outside* the *i*th subregion. The height of the *i*th inscribed rectangle is $f(m_i)$ and the height of the *i*th circumscribed rectangle is $f(M_i)$.

For *each i*, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix}\text{Area of inscribed}\\ \text{rectangle}\end{pmatrix} = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \begin{pmatrix}\text{Area of circumscribed}\\ \text{rectangle}\end{pmatrix}$$

Finding Area by the Limit Definition

The sum of the areas of the inscribed rectangles is called a **lower sum,** and the sum of the areas of the circumscribed rectangles is called an **upper sum.**

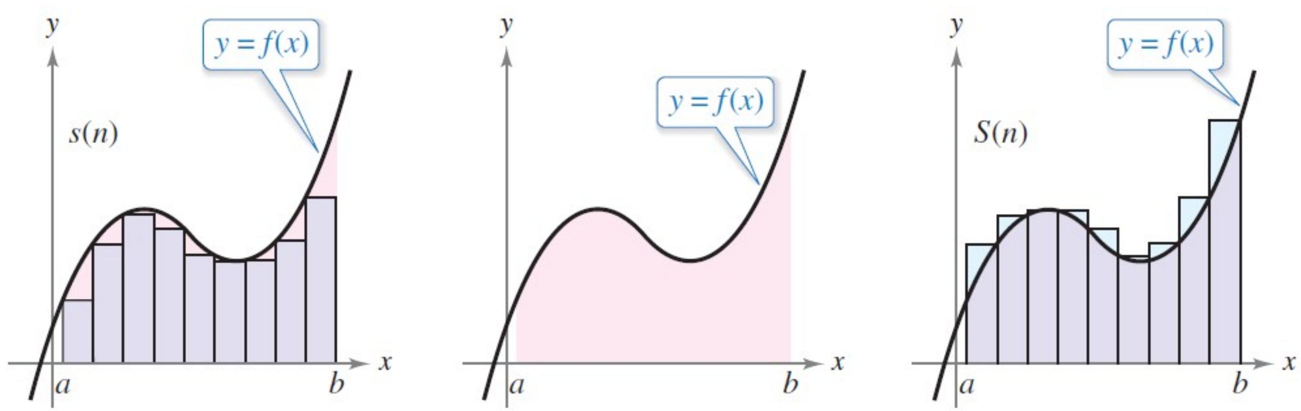
$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x \qquad \text{Area of inscribed rectangles}$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x \qquad \text{Area of circumscribed rectangles}$$

Finding Area by the Limit Definition

From Figure 4.11, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

$$s(n) \leq (\text{Area of region}) \leq S(n)$$



Area of inscribed rectangles is less than area of region.

Area of region

Area of circumscribed rectangles is greater than area of region.

Figure 4.11

Example 4 – *Finding Upper and Lower Sums for a Region*

Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x = 0$ and $x = 2$.

Solution:

To begin, partition the interval [0, 2] into n subintervals, each of width

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}.$$

Example 4 – *Solution*

cont'd

Figure 4.12 shows the endpoints of the subintervals and several inscribed and circumscribed rectangles.

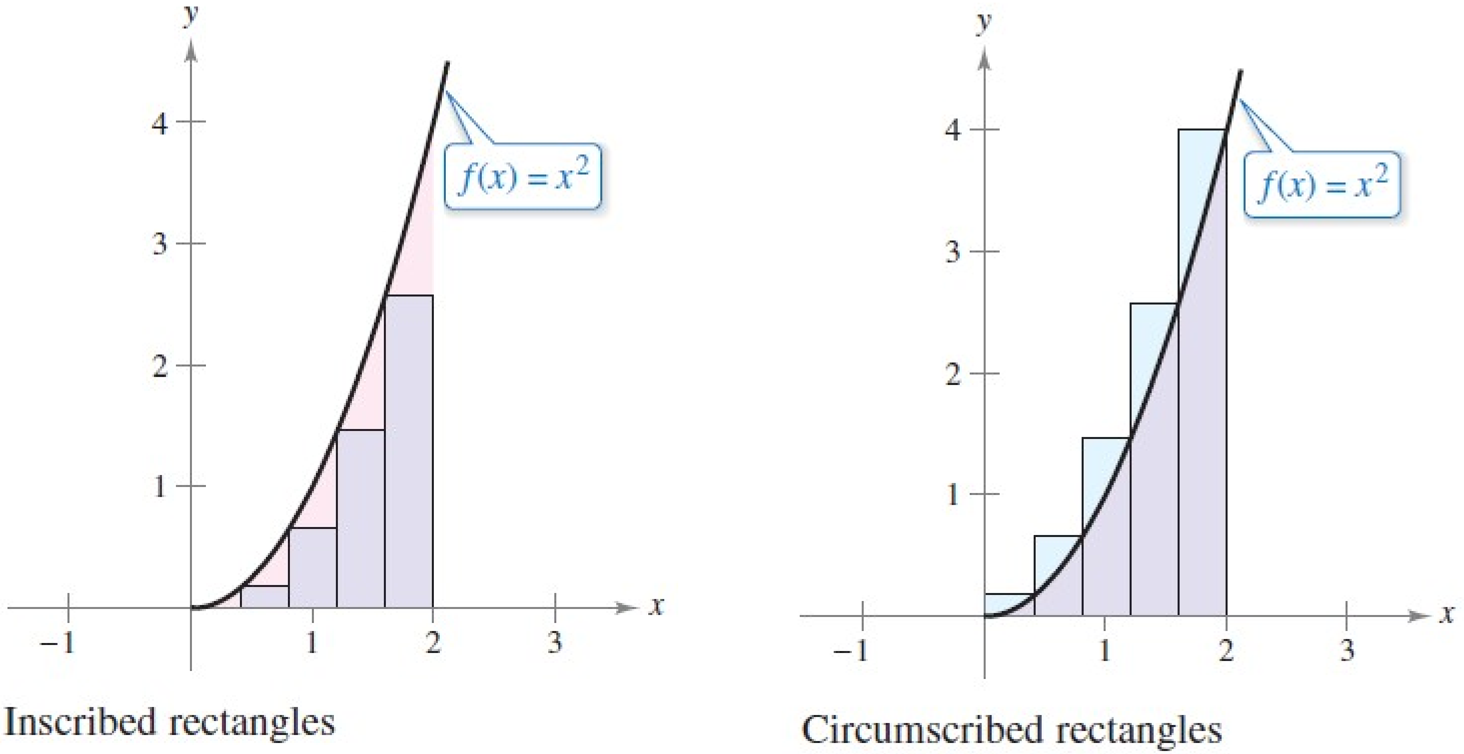


Figure 4.12

Example 4 – *Solution*

cont'd

Because *f* is increasing on the interval [0, 2], the minimum value on each subinterval occurs at the left endpoint, and the maximum value occurs at the right endpoint.

Left Endpoints

$$m_i = 0 + (i - 1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n}$$

Right Endpoints

$$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

Using the left endpoints, the lower sum is

$$s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x = \sum_{i=1}^{n} f\left[\frac{2(i-1)}{n}\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[\frac{2(i-1)}{n}\right]^2\left(\frac{2}{n}\right)$$

Example 4 – *Solution*

cont'd

$$= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right)(i^2 - 2i + 1)$$

$$= \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$$

$$= \frac{8}{n^3}\left\{\frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n\right\}$$

$$= \frac{4}{3n^3}(2n^3 - 3n^2 + n)$$

$$= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.$$

Lower sum

Example 4 – *Solution*

cont'd

Using the right endpoints, the upper sum is

$$S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2 \left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \left(\frac{8}{n^3}\right) i^2$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

Example 4 – *Solution*

cont'd

$$= \frac{4}{3n^3}(2n^3 + 3n^2 + n)$$

$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}. \qquad \text{Upper sum}$$

Finding Area by the Limit Definition

The next theorem shows that the equivalence of the limits (as $n \to \infty$) of the upper and lower sums is not mere coincidence. It is true for all functions that are continuous and nonnegative on the closed interval [*a*, *b*].

THEOREM 4.3 Limits of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\,\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\,\Delta x \\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the ith subinterval.

Finding Area by the Limit Definition

In Theorem 4.3, the same limit is attained for both the minimum value $f(m_i)$ and the maximum value $f(M_i)$.

So, it follows from the Squeeze Theorem (Theorem 1.8) that the choice of x in the ith subinterval does not affect the limit.

THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.

Finding Area by the Limit Definition

This means that you are free to choose an *arbitrary* x-value in the ith subinterval, as shown in the *definition of the area of a region in the plane*.

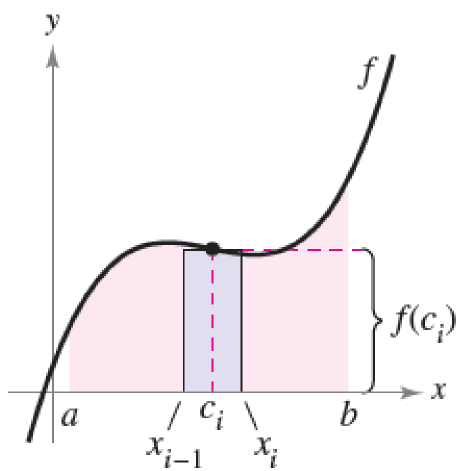
Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. (See Figure 4.13.) The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

where $x_{i-1} \le c_i \le x_i$ and

$$\Delta x = \frac{b-a}{n}.$$

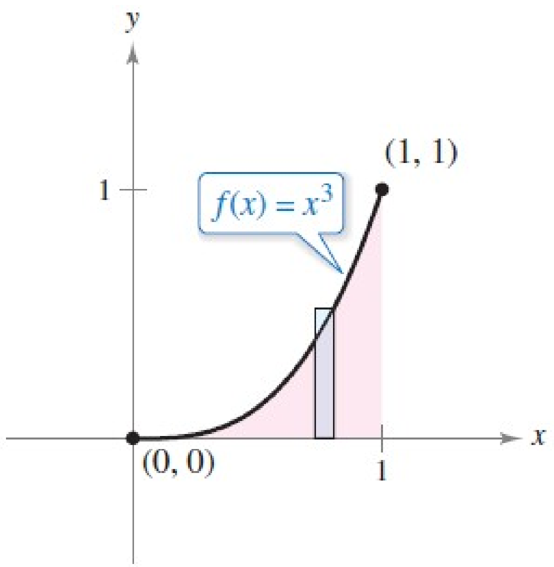


The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.

Figure 4.13

Example 5 – *Finding Area by the Limit Definition*

Find the area of the region bounded by the graph of $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$, as shown in Figure 4.14.



The area of the region bounded by the graph of f, the x-axis, $x = 0$, and $x = 1$ is $\frac{1}{4}$.

Figure 4.14

Example 5 – *Solution*

Begin by noting that f is continuous and nonnegative on the interval [0, 1]. Next, partition the interval [0, 1] into n subintervals, each of width $\Delta x = 1/n$.

According to the definition of area, you can choose any x-value in the ith subinterval.

For this example, the right endpoints $c_i = i/n$ are convenient.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) \qquad \text{Right endpoints: } c_i = \frac{i}{n}$$

Example 5 – *Solution*

cont'd

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \left[\frac{n^2(n+1)^2}{4} \right]$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} \right)$$

$$= \frac{1}{4}$$

The area of the region is $\frac{1}{4}$.

Finding Area by the Limit Definition

In general, a good value to choose is the midpoint of the interval, $c_i = (x_{i-1} + x_i)/2$, and apply the **Midpoint Rule.**

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x. \qquad \text{Midpoint Rule}$$