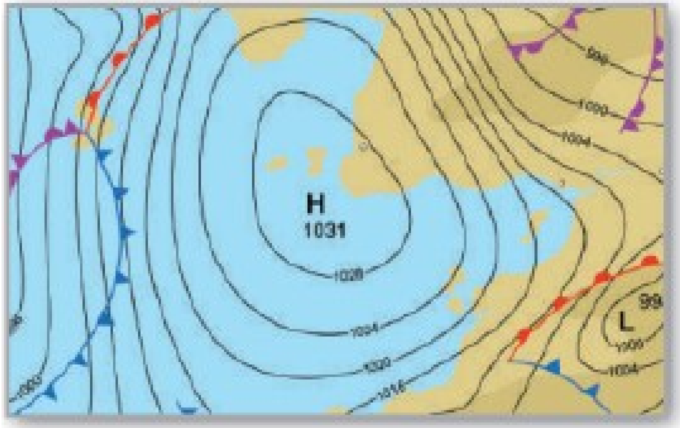
5 Logarithmic, Exponential, and Other Transcendental Functions



5.4 Exponential Functions: Differentiation and Integration

Objectives

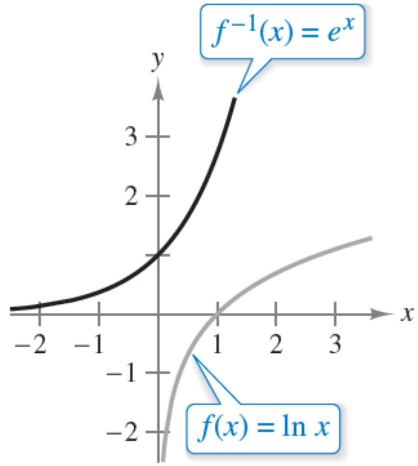
- Develop properties of the natural exponential function.
- Differentiate natural exponential functions.
- Integrate natural exponential functions.

The Natural Exponential Function

The Natural Exponential Function

The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}.

The domain of f^{-1} is the set of all real numbers, and the range is the set of positive real numbers, as shown in Figure 5.18.



The inverse function of the natural logarithmic function is the natural exponential function.

Figure 5.18

The Natural Exponential Function

So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$

If x is rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x.

The Natural Exponential Function

The next definition extends the meaning of e^x to include *all* real values of x.

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

Example 1 – *Solving an Exponential Equation*

Solve $7 = e^{x+1}$.

Solution:

You can convert from exponential form to logarithmic form by *taking the natural logarithm of each side* of the equation.

$7 = e^{x+1}$	Write original equation.
$\ln 7 = \ln(e^{x+1})$	Take natural logarithm of each side.
$\ln 7 = x + 1$	Apply inverse property.
$-1 + \ln 7 = x$	Solve for x.

So, the solution is $-1 + \ln 7 \approx 0.946$.

Example 1 – *Solution*

cont'd

You can check the solution as shown.

$7 = e^{x+1}$	Write original equation.
$7 \stackrel{?}{=} e^{(-1+\ln 7)+1}$	Substitute $-1 + \ln 7$ for x in original equation.
$7 \stackrel{?}{=} e^{\ln 7}$	Simplify.
$7 = 7$ ✓	Solution checks.

The Natural Exponential Function

The familiar rules for operating with rational exponents can be extended to the natural exponential function, as shown in the next theorem.

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$

2. $\dfrac{e^a}{e^b} = e^{a-b}$

The Natural Exponential Function

An inverse function f^{-1} shares many properties with f. So, the natural exponential function inherits the properties listed below from the natural logarithmic function.

Properties of the Natural Exponential Function

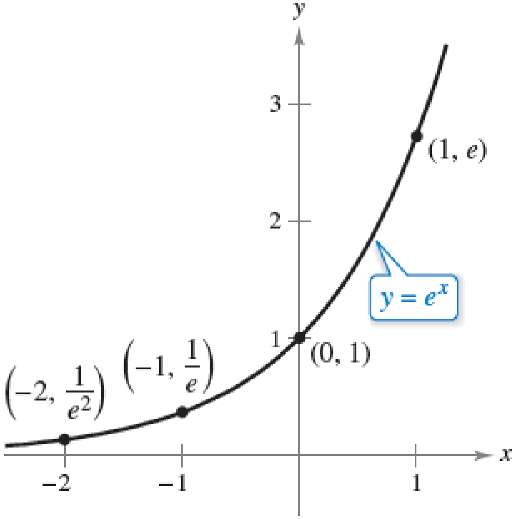
1. The domain of $f(x) = e^x$ is

   $$(-\infty, \infty)$$

   and the range is

   $$(0, \infty).$$

2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x \to -\infty} e^x = 0$
5. $\lim_{x \to \infty} e^x = \infty$



The natural exponential function is increasing, and its graph is concave upward.

Derivatives of Exponential Functions

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*.

In other words, it is a solution to the differential equation $y' = y$. This result is stated in the next theorem.

THEOREM 5.11 Derivatives of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[e^x] = e^x$

2. $\dfrac{d}{dx}[e^u] = e^u \dfrac{du}{dx}$

Example 3 – *Differentiating Exponential Functions*

a. $\dfrac{d}{dx}[e^{2x-1}] = e^u \dfrac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\dfrac{d}{dx}[e^{-3/x}] = e^u \dfrac{du}{dx} = \left(\dfrac{3}{x^2}\right)e^{-3/x} = \dfrac{3e^{-3/x}}{x^2}$ $\qquad u = -\dfrac{3}{x}$

c. $\dfrac{d}{dx}[x^2e^x] = x^2(e^x) + e^x(2x) = xe^x(x + 2)$ $\qquad$ Product Rule and Theorem 5.11

d. $\dfrac{d}{dx}\left[\dfrac{e^{3x}}{e^x + 1}\right] = \dfrac{(e^x + 1)(3e^{3x}) - e^{3x}(e^x)}{(e^x + 1)^2} = \dfrac{3e^{4x} + 3e^{3x} - e^{4x}}{(e^x + 1)^2} = \dfrac{e^{3x}(2e^x + 3)}{(e^x + 1)^2}$

Integrals of Exponential Functions

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\int e^x \, dx = e^x + C$

2. $\int e^u \, du = e^u + C$

Example 7 – *Integrating Exponential Functions*

Find the indefinite integral. $\int e^{3x+1}\,dx$

Solution:

If you let $u = 3x + 1$, then $du = 3\,dx$.

$$\int e^{3x+1}\,dx = \frac{1}{3}\int e^{3x+1}(3)\,dx \qquad \text{Multiply and divide by 3.}$$

$$= \frac{1}{3}\int e^{u}\,du \qquad \text{Substitute: } u = 3x + 1.$$

$$= \frac{1}{3}e^{u} + C \qquad \text{Apply Exponential Rule.}$$

$$= \frac{e^{3x+1}}{3} + C \qquad \text{Back-substitute.}$$