3 Applications of Differentiation

3.2 Rolle's Theorem and the Mean Value Theorem

Objectives

- Understand and use Rolle's Theorem.
- Understand and use the Mean Value Theorem.

Rolle’s Theorem

Rolle's Theorem

The Extreme Value Theorem states that a continuous function on a closed interval [*a*, *b*] must have both a minimum and a maximum on the interval. Both of these values, however, can occur at the endpoints.

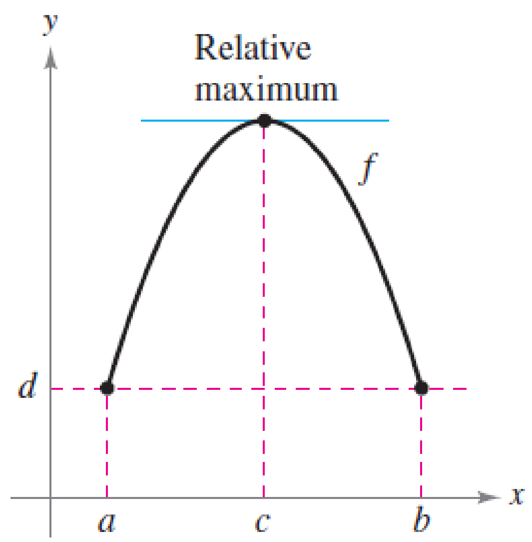
Rolle's Theorem, named after the French mathematician Michel Rolle (1652–1719), gives conditions that guarantee the existence of an extreme value in the *interior* of a closed interval.

THEOREM 3.3 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

Rolle's Theorem

From Rolle's Theorem, you can see that if a function *f* is continuous on [*a*, *b*] and differentiable on (*a*, *b*), and if *f*(*a*) = *f*(*b*), there must be at least one *x*-value between *a* and *b* at which the graph of *f* has a horizontal tangent, as shown in Figure 3.8(a).

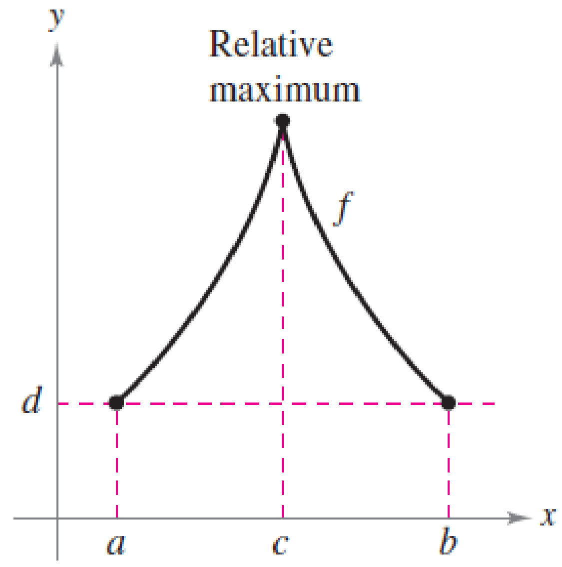


(a) *f* is continuous on [*a*, *b*] and differentiable on (*a*, *b*).

Figure 3.8

Rolle's Theorem

When the differentiability requirement is dropped from Rolle's Theorem, *f* will still have a critical number in (a, b), but it may not yield a horizontal tangent. Such a case is shown in Figure 3.8(b).



(b) *f* is continuous on $[a, b]$.

Figure 3.8

Example 1 – *Illustrating Rolle's Theorem*

Find the two x-intercepts of

$$f(x) = x^2 - 3x + 2$$

and show that $f'(x) = 0$ at some point between the two x-intercepts.

Solution:

Note that f is differentiable on the entire real line. Setting $f(x)$ equal to 0 produces

$$x^2 - 3x + 2 = 0 \qquad \text{Set } f(x) \text{ equal to 0.}$$

$$(x - 1)(x - 2) = 0. \qquad \text{Factor.}$$

$$x = 1, 2. \qquad \text{Solve for } x.$$

Example 1 – *Solution*

cont'd

So, $f(1) = f(2) = 0$, and from Rolle's Theorem you know that there *exists* at least one c in the interval $(1, 2)$ such that $f'(c) = 0$.

To *find* such a c, differentiate f to obtain

$$f'(x) = 2x - 3 \qquad \text{Differentiate.}$$

and then determine that $f'(x) = 0$ when $x = \frac{3}{2}$.

Example 1 – *Solution*

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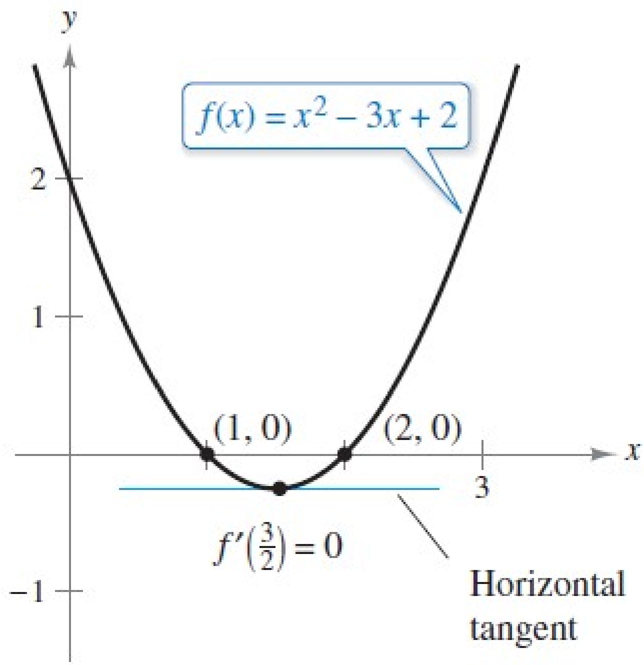
Note that this *x*-value lies in the open interval (1, 2), as shown in Figure 3.9.



The x-value for which $f'(x) = 0$ is between the two x-intercepts.

Figure 3.9

The Mean Value Theorem

The Mean Value Theorem

Rolle's Theorem can be used to prove another theorem —the **Mean Value Theorem.**

THEOREM 3.4 The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 3 – *Finding a Tangent Line*

For $f(x) = 5 – (4/x)$, find all values of c in the open interval (1, 4) such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Solution:

The slope of the secant line through (1, $f(1)$) and (4, $f(4)$) is

$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{4 - 1} = 1. \qquad \text{Slope of secant line}$$

Note that the function satisfies the conditions of the Mean Value Theorem.

Example 3 – *Solution*

cont'd

That is, f is continuous on the interval [1, 4] and differentiable on the interval (1, 4).

So, there exists at least one number c in (1, 4) such that $f'(c) = 1$.

Solving the equation $f'(x) = 1$ yields

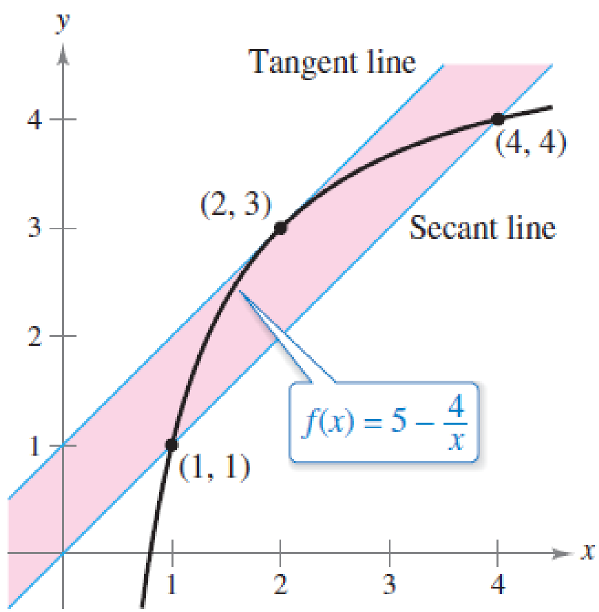
$$\frac{4}{x^2} = 1 \qquad \text{Set } f'(x) \text{ equal to 1.}$$

which implies that $x = \pm 2$.

Example 3 – *Solution*

cont'd

So, in the interval (1, 4), you can conclude that $c = 2$, as shown in Figure 3.13.



The tangent line at (2, 3) is parallel to the secant line through (1, 1) and (4, 4).

Figure 3.13

The Mean Value Theorem

A useful alternative form of the Mean Value Theorem is:
If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f(b) = f(a) + (b - a)f'(c).$$

Alternative form of Mean Value Theorem

Keep in mind that polynomial functions, rational functions, and trigonometric functions are differentiable at all points in their domains.