

1.2 Finding Limits Graphically and Numerically.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \rightarrow c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

then

$$|f(x) - L| < \varepsilon.$$

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \rightarrow c} b = b$
2. $\lim_{x \rightarrow c} x = c$
3. $\lim_{x \rightarrow c} x^n = c^n$

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = K.$$

1. Scalar multiple: $\lim_{x \rightarrow c} [bf(x)] = bL$
2. Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \rightarrow c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$
5. Power: $\lim_{x \rightarrow c} [f(x)]^n = L^n$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \rightarrow c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \rightarrow c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

THEOREM 1.3 Limits of Polynomial and Rational Functions

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THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x \rightarrow c} \sqrt[n]{x} = \sqrt[n]{c}$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \rightarrow c} g(x) = L$ and $\lim_{x \rightarrow L} f(x) = f(L)$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right) = f(L).$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \rightarrow c} \sin x = \sin c$
2. $\lim_{x \rightarrow c} \cos x = \cos c$
3. $\lim_{x \rightarrow c} \tan x = \tan c$
4. $\lim_{x \rightarrow c} \cot x = \cot c$
5. $\lim_{x \rightarrow c} \sec x = \sec c$
6. $\lim_{x \rightarrow c} \csc x = \csc c$

THEOREM 1.7 Functions That Agree at All but One Point

Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c . If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} g(x).$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 1.8 The Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c , except possibly at c itself, and if

$$\lim_{x \rightarrow c} h(x) = L = \lim_{x \rightarrow c} g(x)$$

then $\lim_{x \rightarrow c} f(x)$ exists and is equal to L .

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 1.9 Two Special Trigonometric Limits

1. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$

1.4 Continuity and One-Sided Limits.

Definition of Continuity

Continuity at a Point

A function f is continuous at c when these three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \rightarrow c} f(x)$ exists.
3. $\lim_{x \rightarrow c} f(x) = f(c)$

Continuity on an Open Interval

A function is continuous on an open interval (a, b) when the function is continuous at each point in the interval. A function that is continuous on the entire real number line $(-\infty, \infty)$ is everywhere continuous.

THEOREM 1.10 The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$

Definition of Continuity on a Closed Interval

A function f is continuous on the closed interval $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and

$$\lim_{x \rightarrow b^-} f(x) = f(b).$$

The function f is continuous from the right at a and continuous from the left at b (see Figure 1.31).

THEOREM 1.11 Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c .

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$, $g(c) \neq 0$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 1.12 Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f(g(x))$ is continuous at c .

THEOREM 1.13 Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k.$$

1.5 Infinite Limits.

Definition of Infinite Limits

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x \rightarrow c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$ (see Figure 1.40). Similarly, the statement

$$\lim_{x \rightarrow c} f(x) = -\infty$$

means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever

$$0 < |x - c| < \delta.$$

To define the infinite limit from the left, replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the infinite limit from the right, replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f .

THEOREM 1.14 Vertical Asymptotes

Let f and g be continuous on an open interval containing c . If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 1.15 Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = L.$$

1. Sum or difference: $\lim_{x \rightarrow c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x \rightarrow c} [f(x)g(x)] = \infty, \quad L > 0$
 $\lim_{x \rightarrow c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\lim_{x \rightarrow c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$ [see Example 5(d)].

2.1 The Derivative and the Tangent Line Problem.

Definition of Tangent Line with Slope m

If f is defined on an open interval containing c , and if the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

Definition of the Derivative of a Function

The derivative of f at x is

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x .

THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

2.2 Basic Differentiation Rules and Rates of Change.

THEOREM 2.2 The Constant Rule

The derivative of a constant function is 0. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0. \quad (\text{See Figure 2.14.})$$

THEOREM 2.3 The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing 0.

THEOREM 2.4 The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also

differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.

THEOREM 2.5 The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g .

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \quad \text{Sum Rule}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \quad \text{Difference Rule}$$

THEOREM 2.6 Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin x] = \cos x \quad \frac{d}{dx}[\cos x] = -\sin x$$

2.3 Product and Quotient Rules and Higher-Order Derivatives.

THEOREM 2.7 The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

THEOREM 2.8 The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

THEOREM 2.9 Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

2.4 The Chain Rule

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1} \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'.$$

SUMMARY OF DIFFERENTIATION RULES

General Differentiation Rules Let f , g , and u be differentiable functions of x .

Constant Multiple Rule:

$$\frac{d}{dx}[cf] = cf'$$

Sum or Difference Rule:

$$\frac{d}{dx}[f \pm g] = f' \pm g'$$

Product Rule:

$$\frac{d}{dx}[fg] = fg' + gf'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$

Derivatives of Algebraic Functions

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

(Simple) Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

Chain Rule

Chain Rule:

$$\frac{d}{dx}[f(u)] = f'(u) u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

2.5 Implicit Differentiation.

GUIDELINES FOR IMPLICIT DIFFERENTIATION

1. Differentiate both sides of the equation *with respect to x*.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx .

2.6 Related Rates.

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. After completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

3 Applications of Differentiation.

3.1 Extrema on an Interval.

Definition of Extrema

Let f be defined on an interval I containing c .

1. $f(c)$ is the **minimum of f on I** when $f(c) \leq f(x)$ for all x in I .
2. $f(c)$ is the **maximum of f on I** when $f(c) \geq f(x)$ for all x in I .

The minimum and maximum of a function on an interval are the **extreme values, or extrema** (the singular form of extrema is **extremum**), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval. Extrema can occur at interior points or endpoints of an interval (see Figure 3.1). Extrema that occur at the endpoints are called **endpoint extrema**.

THEOREM 3.1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f , or you can say that f has a **relative maximum at $(c, f(c))$** .
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f , or you can say that f has a **relative minimum at $(c, f(c))$** .

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum**, respectively.

Definition of a Critical Number

Let f be defined at c . If $f'(c) = 0$ or if f is not differentiable at c , then c is a critical number of f .

THEOREM 3.2 Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f .

GUIDELINES FOR FINDING EXTREMA ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a, b]$, use these steps.

1. Find the critical numbers of f in (a, b) .
2. Evaluate f at each critical number in (a, b) .
3. Evaluate f at each endpoint of $[a, b]$.
4. The least of these values is the minimum. The greatest is the maximum.

3.2 Rolle's Theorem and the Mean Value Theorem.

THEOREM 3.3 Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) . If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

THEOREM 3.4 The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3.3 Increasing and Decreasing Functions and the First Derivative Test.

Definitions of Increasing and Decreasing Functions

A function f is **increasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

THEOREM 3.5 Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) .

1. If $f'(x) > 0$ for all x in (a, b) , then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b) , then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b) , then f is constant on $[a, b]$.

GUIDELINES FOR FINDING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

Let f be continuous on the interval (a, b) . To find the open intervals on which f is increasing or decreasing, use the following steps.

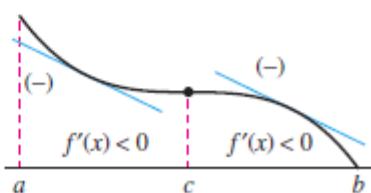
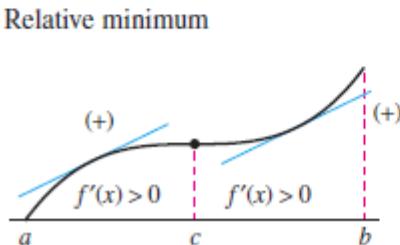
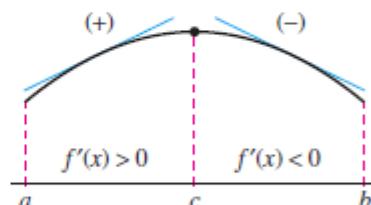
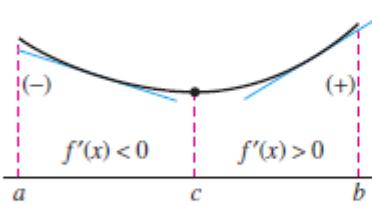
1. Locate the critical numbers of f in (a, b) , and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid when the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞) , or $(-\infty, \infty)$.

THEOREM 3.6 The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c . If f is differentiable on the interval, except possibly at c , then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c , then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c , then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c , then $f(c)$ is neither a relative minimum nor a relative maximum.



3.4 Concavity and the Second Derivative Test.

Definition of Concavity

Let f be differentiable on an open interval I . The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I .

1. If $f''(x) > 0$ for all x in I , then the graph of f is concave upward on I .
2. If $f''(x) < 0$ for all x in I , then the graph of f is concave downward on I .

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at the point.

THEOREM 3.8 Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f , then either $f''(c) = 0$ or f'' does not exist at $x = c$.

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c .

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

3.5 Limits of Infinity.

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x \rightarrow \infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x \rightarrow -\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Definition of a Horizontal Asymptote

The line $y = L$ is a horizontal asymptote of the graph of f when

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow \infty} f(x) = L.$$

THEOREM 3.10 Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x \rightarrow \infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x \rightarrow -\infty} \frac{c}{x^r} = 0.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞) .

1. The statement $\lim_{x \rightarrow \infty} f(x) = \infty$ means that for each positive number M , there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x \rightarrow \infty} f(x) = -\infty$ means that for each negative number M , there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x \rightarrow -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = -\infty.$$

3.7 Optimization Problems.

GUIDELINES FOR SOLVING APPLIED MINIMUM AND MAXIMUM PROBLEMS

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.
2. Write a **primary equation** for the quantity that is to be maximized or minimized. (A review of several useful formulas from geometry is presented inside the back cover.)
3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 through 3.4.

3.8 Newton's Method

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c . Then, to approximate c , use these steps.

1. Make an initial estimate x_1 that is close to c . (A graph is helpful.)
2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

3. When $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration**.

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x . The **differential of x** (denoted by dx) is any nonzero real number. The **differential of y** (denoted by dy) is

$$dy = f'(x) dx.$$

Differential Formulas

Let u and v be differentiable functions of x .

Constant multiple: $d[cu] = c du$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = u dv + v du$

Quotient: $d\left[\frac{u}{v}\right] = \frac{v du - u dv}{v^2}$

4 Integration.

4.1 Antiderivatives and Indefinite Integration.

Definition of Antiderivative

A function F is an antiderivative of f on an interval I when $F'(x) = f(x)$ for all x in I .

THEOREM 4.1 Representation of Antiderivatives

If F is an antiderivative of f on an interval I , then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

Basic Integration Rules

Differentiation Formula

$$\frac{d}{dx}[C] = 0$$

$$\frac{d}{dx}[kx] = k$$

$$\frac{d}{dx}[kf(x)] = kf'(x)$$

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

Integration Formula

$$\int 0 \, dx = C$$

$$\int k \, dx = kx + C$$

$$\int kf(x) \, dx = k \int f(x) \, dx$$

$$\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \quad \text{Power Rule}$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

4.2 Area

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \dots, a_n$ is written as

$$\sum_{i=1}^n a_i = a_1 + a_2 + a_3 + \dots + a_n$$

where i is the index of summation, a_i is the i th term of the sum, and the upper and lower bounds of summation are n and 1.

THEOREM 4.2 Summation Formulas

1. $\sum_{i=1}^n c = cn$, c is a constant
2. $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 4.3 Limits of the Lower and Upper Sums

Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \rightarrow \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n \rightarrow \infty} s(n) &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(m_i) \Delta x \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(M_i) \Delta x \\ &= \lim_{n \rightarrow \infty} S(n)\end{aligned}$$

where $\Delta x = (b - a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

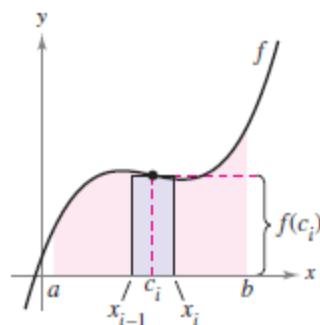
Definition of the Area of a Region in the Plane

Let f be continuous and nonnegative on the interval $[a, b]$. (See Figure 4.13.) The area of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x$$

where $x_{i-1} \leq c_i \leq x_i$ and

$$\Delta x = \frac{b - a}{n}.$$



The width of the i th subinterval is $\Delta x = x_i - x_{i-1}$.

Figure 4.13

4.3 Riemann Sums and Definite Integrals.

Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

where Δx_i is the width of the i th subinterval

$$[x_{i-1}, x_i].$$

ith subinterval

If c_i is any point in the i th subinterval, then the sum

$$\sum_{i=1}^n f(c_i) \Delta x_i, \quad x_{i-1} \leq c_i \leq x_i$$

is called a **Riemann sum** of f for the partition Δ . (The sums in Section 4.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there.)

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(c_i) \Delta x_i = \int_a^b f(x) dx.$$

The limit is called the **definite integral** of f from a to b . The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

THEOREM 4.4 Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x) dx$ exists.

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f , the x -axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x) dx.$$

(See Figure 4.22.)

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x) dx = 0$.
2. If f is integrable on $[a, b]$, then $\int_b^a f(x) dx = - \int_a^b f(x) dx$.

THEOREM 4.6 Additive Interval Property

If f is integrable on the three closed intervals determined by a , b , and c , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

THEOREM 4.7 Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

1. $\int_a^b kf(x) dx = k \int_a^b f(x) dx$
2. $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx.$

THEOREM 4.8 Preservation of Inequality

1. If f is integrable and nonnegative on the closed interval $[a, b]$, then

$$0 \leq \int_a^b f(x) dx.$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

4.4 The Fundamental Theorem of Calculus.

THEOREM 4.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a).$$

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. Provided you can find an antiderivative of f , you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the notation shown below is convenient.

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3 dx$, you can write

$$\int_1^3 x^3 dx = \frac{x^4}{4} \Big|_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative.

$$\int_a^b f(x) dx = \left[F(x) + C \right]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

THEOREM 4.10 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x) dx = f(c)(b - a).$$

Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

$$\frac{1}{b - a} \int_a^b f(x) dx.$$

See Figure 4.32.

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a , then, for every x in the interval,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x).$$

THEOREM 4.12 The Net Change Theorem

The definite integral of the rate of change of quantity $F'(x)$ gives the total change, or net change, in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x) dx = F(b) - F(a) \quad \text{Net change of } F$$

4.5 Integration by Substitution.

THEOREM 4.13 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I , and let f be a function that is continuous on I . If g is differentiable on its domain and F is an antiderivative of f on I , then

$$\int f(g(x))g'(x) dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x) dx$ and

$$\int f(u) du = F(u) + C.$$

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x) dx$.
3. Rewrite the integral in terms of the variable u .
4. Find the resulting integral in terms of u .
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x .
6. Check your answers by differentiating.

THEOREM 4.14 The General Power Rule for Integration

If g is a differentiable function of x , then

$$\int [g(x)]^n g'(x) dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

THEOREM 4.15 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g , then

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du.$$

THEOREM 4.16 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[a, -a]$.

1. If f is an *even* function, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
2. If f is an *odd* function, then $\int_{-a}^a f(x) dx = 0$.

4.6 Numerical Integration.

THEOREM 4.17 The Trapezoidal Rule

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x) dx$ is

$$\int_a^b f(x) dx \approx \frac{b-a}{2n} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

Moreover, as $n \rightarrow \infty$, the right-hand side approaches $\int_a^b f(x) dx$.

THEOREM 4.18 Integral of $p(x) = Ax^2 + Bx + C$

If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x) dx = \left(\frac{b-a}{6} \right) \left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b) \right].$$

THEOREM 4.19 Simpson's Rule

Let f be continuous on $[a, b]$ and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x) dx$ is

$$\begin{aligned} \int_a^b f(x) dx &\approx \frac{b-a}{3n} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots \\ &\quad + 4f(x_{n-1}) + f(x_n)]. \end{aligned}$$

Moreover, as $n \rightarrow \infty$, the right-hand side approaches $\int_a^b f(x) dx$.

THEOREM 4.20 Errors in the Trapezoidal Rule and Simpson's Rule

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x) dx$ by the Trapezoidal Rule is

$$|E| \leq \frac{(b-a)^3}{12n^2} [\max |f''(x)|], \quad a \leq x \leq b. \quad \text{Trapezoidal Rule}$$

Moreover, if f has a continuous fourth derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x) dx$ by Simpson's Rule is

$$|E| \leq \frac{(b-a)^5}{180n^4} [\max |f^{(4)}(x)|], \quad a \leq x \leq b. \quad \text{Simpson's Rule}$$

5 Logarithmic, Exponential, and Other Transcendental Functions.

5.1 The Natural Logarithmic Function: Differentiation.

Definition of the Natural Logarithmic Function

The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

THEOREM 5.1 Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

THEOREM 5.2 Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Definition of e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t} dt = 1.$$

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x .

1. $\frac{d}{dx} [\ln x] = \frac{1}{x}, \quad x > 0$
2. $\frac{d}{dx} [\ln u] = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

THEOREM 5.4 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx} [\ln|u|] = \frac{u'}{u}.$$

5.2 The Natural Logarithmic Function: Integration.

THEOREM 5.5 Log Rule for Integration

Let u be a differentiable function of x .

1. $\int \frac{1}{x} dx = \ln|x| + C$
2. $\int \frac{1}{u} du = \ln|u| + C$

GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and 10 trigonometric rules. By the end of Section 5.7, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. When you cannot find a u -substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C$$

5.3 Inverse Functions.

Definition of Inverse Function

A function g is the inverse function of the function f when

$$f(g(x)) = x \text{ for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \text{ for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1} (read “ f inverse”).

THEOREM 5.6 Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a) .

THEOREM 5.7 The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Use Theorem 5.7 to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y : $x = g(y) = f^{-1}(y)$.
3. Interchange x and y . The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f .
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I . If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I . If f has an inverse function g , then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

5.4 Exponential Functions: Differentiation and Integration.

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \text{ if and only if } x = \ln y.$$

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

$$1. e^a e^b = e^{a+b}$$

$$2. \frac{e^a}{e^b} = e^{a-b}$$

Properties of the Natural Exponential Function

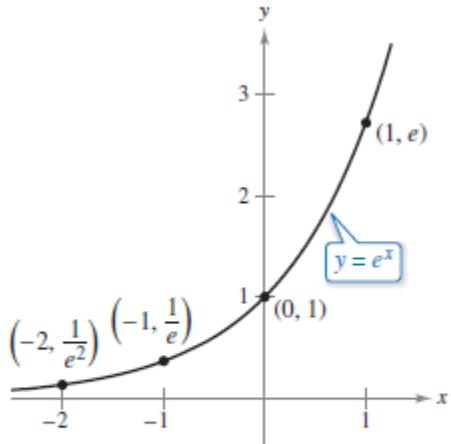
1. The domain of $f(x) = e^x$ is

$$(-\infty, \infty)$$

and the range is

$$(0, \infty).$$

2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x \rightarrow -\infty} e^x = 0$
5. $\lim_{x \rightarrow \infty} e^x = \infty$



The natural exponential function is increasing, and its graph is concave upward.

THEOREM 5.11 Derivatives of the Natural Exponential Function

Let u be a differentiable function of x .

$$1. \frac{d}{dx}[e^x] = e^x$$

$$2. \frac{d}{dx}[e^u] = e^u \frac{du}{dx}$$

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x .

$$1. \int e^x dx = e^x + C$$

$$2. \int e^u du = e^u + C$$

5.5 Bases Other than e and Applications.

Definition of Exponential Function to Base a

If a is a positive real number ($a \neq 1$) and x is any real number, then the exponential function to the base a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Definition of Logarithmic Function to Base a

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the logarithmic function to the base a is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$

2. $a^{\log_a x} = x$, for $x > 0$

3. $\log_a a^x = x$, for all x

THEOREM 5.13 Derivatives for Bases Other than e

Let a be a positive real number ($a \neq 1$), and let u be a differentiable function of x .

$$1. \frac{d}{dx}[a^x] = (\ln a)a^x$$

$$2. \frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$$

$$3. \frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$$

$$4. \frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$$

THEOREM 5.14 The Power Rule for Real Exponents

Let n be any number, and let u be a differentiable function of x .

$$1. \frac{d}{dx}[x^n] = nx^{n-1}$$

$$2. \frac{d}{dx}[u^n] = nu^{n-1} \frac{du}{dx}$$

THEOREM 5.15 A Limit Involving e

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow \infty} \left(\frac{x+1}{x}\right)^x = e$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

SUMMARY OF COMPOUND INTEREST FORMULAS

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (decimal form), and n = number of compoundings per year.

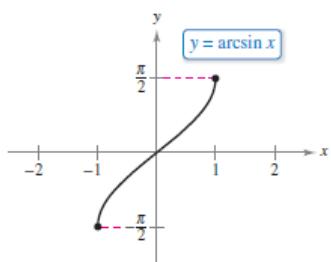
1. Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. Compounded continuously: $A = Pe^{rt}$

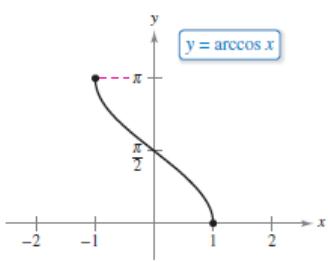
5.6 Inverse Trigonometric Functions: Differentiation

Definitions of Inverse Trigonometric Functions

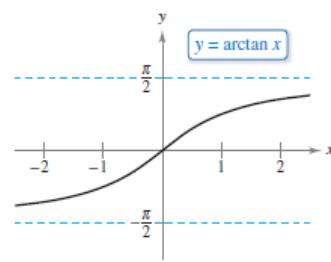
Function	Domain	Range
$y = \arcsin x$ iff $\sin y = x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arcot } x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \text{arcsec } x$ iff $\sec y = x$	$ x \geq 1$	$0 \leq y \leq \pi, \quad y \neq \frac{\pi}{2}$
$y = \text{arccsc } x$ iff $\csc y = x$	$ x \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq 0$



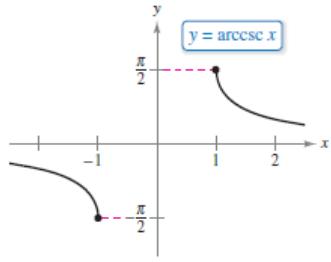
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



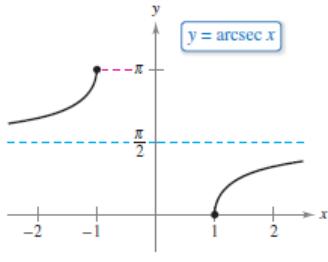
Domain: $[-1, 1]$
Range: $[0, \pi]$



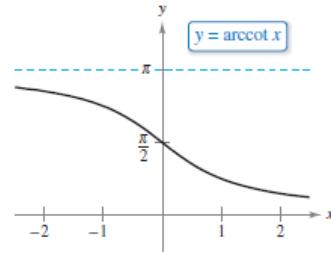
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$



Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Properties of Inverse Trigonometric Functions

If $-1 \leq x \leq 1$ and $-\pi/2 \leq y \leq \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \geq 1$ and $0 \leq y < \pi/2$ or $\pi/2 < y \leq \pi$, then

$$\sec(\arcsec x) = x \quad \text{and} \quad \arcsec(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

THEOREM 5.16 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x .

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\text{arcsec } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Proofs for $\arcsin u$ and $\arccos u$ are given in Appendix A. [The proofs for the other rules are left as an exercise (see Exercise 98).]

See *LarsonCalculus.com* for Bruce Edwards's video of this proof.

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

$$1. \frac{d}{dx}[cu] = cu'$$

$$2. \frac{d}{dx}[u \pm v] = u' \pm v'$$

$$3. \frac{d}{dx}[uv] = uv' + vu'$$

$$4. \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$5. \frac{d}{dx}[c] = 0$$

$$6. \frac{d}{dx}[u^n] = nu^{n-1}u'$$

$$7. \frac{d}{dx}[x] = 1$$

$$8. \frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$$

$$9. \frac{d}{dx}[\ln u] = \frac{u'}{u}$$

$$10. \frac{d}{dx}[e^u] = e^u u'$$

$$11. \frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$$

$$12. \frac{d}{dx}[a^u] = (\ln a)a^u u'$$

$$13. \frac{d}{dx}[\sin u] = (\cos u)u'$$

$$14. \frac{d}{dx}[\cos u] = -(\sin u)u'$$

$$15. \frac{d}{dx}[\tan u] = (\sec^2 u)u'$$

$$16. \frac{d}{dx}[\cot u] = -(\csc^2 u)u'$$

$$17. \frac{d}{dx}[\sec u] = (\sec u \tan u)u'$$

$$18. \frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$$

$$19. \frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$20. \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$21. \frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$22. \frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1+u^2}$$

$$23. \frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$24. \frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

5.7 Inverse Trigonometric Functions: Integration

THEOREM 5.17 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x , and let $a > 0$.

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u) du = k \int f(u) du$
2. $\int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$
3. $\int du = u + C$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u du = e^u + C$
7. $\int a^u du = \left(\frac{1}{\ln a} \right) a^u + C$
8. $\int \sin u du = -\cos u + C$
9. $\int \cos u du = \sin u + C$
10. $\int \tan u du = -\ln|\cos u| + C$
11. $\int \cot u du = \ln|\sin u| + C$
12. $\int \sec u du = \ln|\sec u + \tan u| + C$
13. $\int \csc u du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u du = \tan u + C$
15. $\int \csc^2 u du = -\cot u + C$
16. $\int \sec u \tan u du = \sec u + C$
17. $\int \csc u \cot u du = -\csc u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

5.8 Hyperbolic Functions.

Definitions of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\csc x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

HYPERBOLIC IDENTITIES

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\tanh^2 x + \operatorname{sech}^2 x = 1$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\coth^2 x - \csc^2 x = 1$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh^2 x = \frac{-1 + \cosh 2x}{2}$$

$$\cosh^2 x = \frac{1 + \cosh 2x}{2}$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

THEOREM 5.18 Derivatives and Integrals of Hyperbolic Functions

Let u be a differentiable function of x .

$$\frac{d}{dx} [\sinh u] = (\cosh u)u'$$

$$\int \cosh u \, du = \sinh u + C$$

$$\frac{d}{dx} [\cosh u] = (\sinh u)u'$$

$$\int \sinh u \, du = \cosh u + C$$

$$\frac{d}{dx} [\tanh u] = (\operatorname{sech}^2 u)u'$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\frac{d}{dx} [\coth u] = -(\csc^2 u)u'$$

$$\int \csc^2 u \, du = -\coth u + C$$

$$\frac{d}{dx} [\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u'$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx} [\csc u] = -(\csc u \coth u)u'$$

$$\int \csc u \coth u \, du = -\csc u + C$$

THEOREM 5.19 Inverse Hyperbolic Functions

Function	Domain
$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$	$[1, \infty)$
$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}$	$(-1, 1)$
$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln \left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{ x } \right)$	$(-\infty, 0) \cup (0, \infty)$

THEOREM 5.20 Differentiation and Integration Involving Inverse Hyperbolic Functions

Let u be a differentiable function of x .

$$\begin{aligned}
 \frac{d}{dx}[\sinh^{-1} u] &= \frac{u'}{\sqrt{u^2 + 1}} & \frac{d}{dx}[\cosh^{-1} u] &= \frac{u'}{\sqrt{u^2 - 1}} \\
 \frac{d}{dx}[\tanh^{-1} u] &= \frac{u'}{1 - u^2} & \frac{d}{dx}[\coth^{-1} u] &= \frac{u'}{1 - u^2} \\
 \frac{d}{dx}[\operatorname{sech}^{-1} u] &= \frac{-u'}{u\sqrt{1 - u^2}} & \frac{d}{dx}[\operatorname{csch}^{-1} u] &= \frac{-u'}{|u|\sqrt{1 + u^2}}
 \end{aligned}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln(u + \sqrt{u^2 \pm a^2}) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a} \ln \frac{a + \sqrt{a^2 \pm u^2}}{|u|} + C$$

7 Application of Integration.

7.1 Area of a Region Between Two Curves.

Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx.$$

7.2 Volume: The Disk Method.

THE DISK METHOD

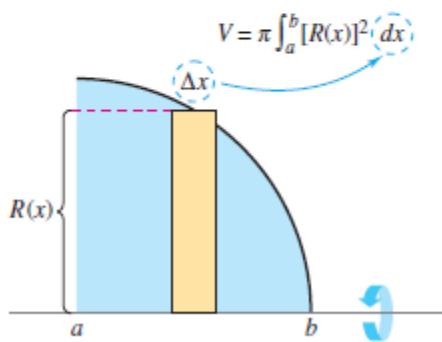
To find the volume of a solid of revolution with the disk method, use one of the formulas below. (See Figure 7.15.)

Horizontal Axis of Revolution

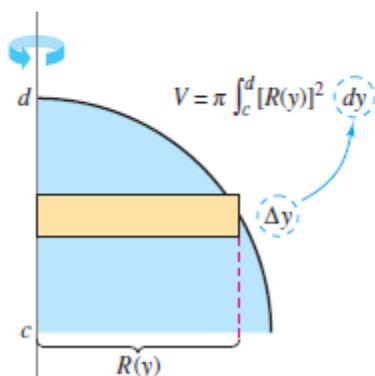
$$\text{Volume} = V = \pi \int_a^b [R(x)]^2 dx$$

Vertical Axis of Revolution

$$\text{Volume} = V = \pi \int_c^d [R(y)]^2 dy$$



Horizontal axis of revolution



Vertical axis of revolution

$$V = \pi \int_a^b ([R(x)]^2 - [r(x)]^2) dx.$$

Washer method

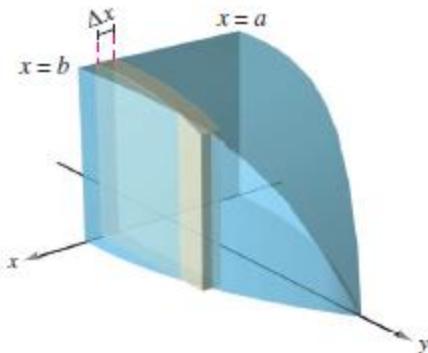
VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

1. For cross sections of area $A(x)$ taken perpendicular to the x -axis,

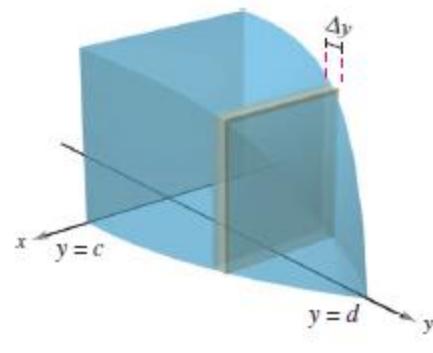
$$\text{Volume} = \int_a^b A(x) dx. \quad \text{See Figure 7.24(a).}$$

2. For cross sections of area $A(y)$ taken perpendicular to the y -axis,

$$\text{Volume} = \int_c^d A(y) dy. \quad \text{See Figure 7.24(b).}$$



(a) Cross sections perpendicular to x -axis



(b) Cross sections perpendicular to y -axis

7.3 Volume: The Shell Method

THE SHELL METHOD

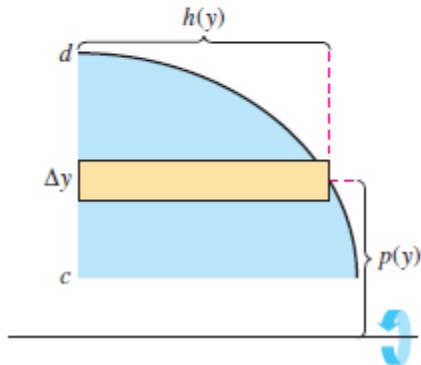
To find the volume of a solid of revolution with the **shell method**, use one of the formulas below. (See Figure 7.29.)

Horizontal Axis of Revolution

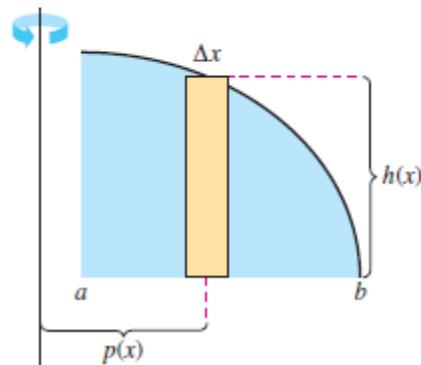
$$\text{Volume} = V = 2\pi \int_c^d p(y)h(y) dy$$

Vertical Axis of Revolution

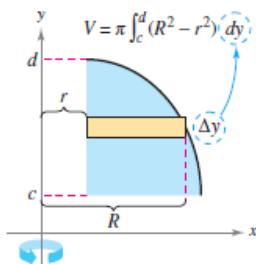
$$\text{Volume} = V = 2\pi \int_a^b p(x)h(x) dx$$



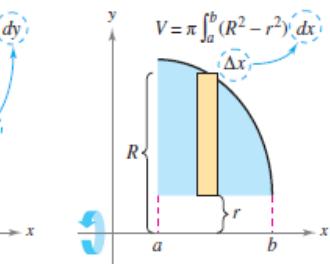
Horizontal axis of revolution



Vertical axis of revolution

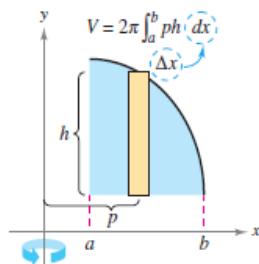


Vertical axis of revolution

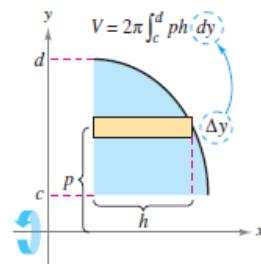


Horizontal axis of revolution

Disk method: Representative rectangle is perpendicular to the axis of revolution.



Vertical axis of revolution



Horizontal axis of revolution

Shell method: Representative rectangle is parallel to the axis of revolution.

7.4 Arc Length and Surfaces of Revolution

Definition of Arc Length

Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

Similarly, for a smooth curve $x = g(y)$, the arc length of g between c and d is

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Definition of Surface of Revolution

When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution**.

Definition of the Area of a Surface of Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2} dx \quad y \text{ is a function of } x.$$

where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y) \sqrt{1 + [g'(y)]^2} dy \quad x \text{ is a function of } y.$$

where $r(y)$ is the distance between the graph of g and the axis of revolution.

7.5 Work

Definition of Work Done by a Constant Force

If an object is moved a distance D in the direction of an applied constant force F , then the **work** W done by the force is defined as $W = FD$.

Definition of Work Done by a Variable Force

If an object is moved along a straight line by a continuously varying force $F(x)$, then the work W done by the force as the object is moved from

$$x = a \quad \text{to} \quad x = b$$

is given by

$$\begin{aligned} W &= \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n \Delta W_i \\ &= \int_a^b F(x) dx. \end{aligned}$$

1. **Hooke's Law:** The force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is,

$$F = kd$$

where the constant of proportionality k (the spring constant) depends on the specific nature of the spring.

2. **Newton's Law of Universal Gravitation:** The force F of attraction between two particles of masses m_1 and m_2 is proportional to the product of the masses and inversely proportional to the square of the distance d between the two particles. That is,

$$F = G \frac{m_1 m_2}{d^2}.$$

When m_1 and m_2 are in kilograms and d in meters, F will be in newtons for a value of $G = 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared, where G is the gravitational constant.

3. **Coulomb's Law:** The force F between two charges q_1 and q_2 in a vacuum is proportional to the product of the charges and inversely proportional to the square of the distance d between the two charges. That is,

$$F = k \frac{q_1 q_2}{d^2}.$$

7.6 Moments, Centers of Mass, and Centroids

System of Measurement	Measure of Mass	Measure of Force
U.S.	Slug	Pound = (slug)(ft/sec ²)
International	Kilogram	Newton = (kilogram)(m/sec ²)
C-G-S	Gram	Dyne = (gram)(cm/sec ²)

Conversions:

1 pound = 4.448 newtons	1 slug = 14.59 kilograms
1 newton = 0.2248 pound	1 kilogram = 0.06852 slug
1 dyne = 0.000002248 pound	1 gram = 0.00006852 slug
1 dyne = 0.00001 newton	1 foot = 0.3048 meter

Moments and Center of Mass: One-Dimensional System

Let the point masses m_1, m_2, \dots, m_n be located at x_1, x_2, \dots, x_n .

1. The moment about the origin is

$$M_0 = m_1x_1 + m_2x_2 + \dots + m_nx_n.$$

2. The center of mass is

$$\bar{x} = \frac{M_0}{m}$$

where $m = m_1 + m_2 + \dots + m_n$ is the total mass of the system.

Moment and Center of Mass: Two-Dimensional System

Let the point masses m_1, m_2, \dots, m_n be located at $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.

1. The moment about the y -axis is

$$M_y = m_1x_1 + m_2x_2 + \dots + m_nx_n.$$

2. The moment about the x -axis is

$$M_x = m_1y_1 + m_2y_2 + \dots + m_ny_n.$$

3. The center of mass (\bar{x}, \bar{y}) (or center of gravity) is

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}$$

where

$$m = m_1 + m_2 + \dots + m_n$$

is the total mass of the system.

Moments and Center of Mass of a Planar Lamina

Let f and g be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x)$, $y = g(x)$, and $a \leq x \leq b$.

1. The moments about the x - and y -axes are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2} \right] [f(x) - g(x)] dx$$
$$M_y = \rho \int_a^b x[f(x) - g(x)] dx.$$

2. The center of mass (\bar{x}, \bar{y}) is given by $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$, where

$m = \rho \int_a^b [f(x) - g(x)] dx$ is the mass of the lamina.

THEOREM 7.1 The Theorem of Pappus

Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R , as shown in Figure 7.63. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is

$$V = 2\pi rA$$

where A is the area of R . (Note that $2\pi r$ is the distance traveled by the centroid as the region is revolved about the line.)

7.7 Fluid Pressure and Fluid Force.

Definition of Fluid Pressure

The pressure on an object at depth h in a liquid is

$$\text{Pressure} = P = wh$$

where w is the weight-density of the liquid per unit of volume.

Definition of Force Exerted by a Fluid

The force F exerted by a fluid of constant weight-density w (per unit of volume) against a submerged vertical plane region from $y = c$ to $y = d$ is

$$\begin{aligned} F &= w \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n h(y_i) L(y_i) \Delta y \\ &= w \int_c^d h(y) L(y) dy \end{aligned}$$

where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y .

8 Integration Techniques, L'Hopital's Rule, and Improper Integrals.

8.1 Basic Integration Rules

REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

$$1. \int kf(u) du = k \int f(u) du$$

$$2. \int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$$

$$3. \int du = u + C$$

$$4. \int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$5. \int \frac{du}{u} = \ln|u| + C$$

$$6. \int e^u du = e^u + C$$

$$7. \int a^u du = \left(\frac{1}{\ln a}\right)a^u + C$$

$$8. \int \sin u du = -\cos u + C$$

$$9. \int \cos u du = \sin u + C$$

$$10. \int \tan u du = -\ln|\cos u| + C$$

$$11. \int \cot u du = \ln|\sin u| + C$$

$$12. \int \sec u du = \ln|\sec u + \tan u| + C$$

$$13. \int \csc u du = -\ln|\csc u + \cot u| + C$$

$$14. \int \sec^2 u du = \tan u + C$$

$$13. \int \csc u du = -\ln|\csc u + \cot u| + C$$

$$14. \int \sec^2 u du = \tan u + C$$

$$15. \int \csc^2 u du = -\cot u + C$$

$$16. \int \sec u \tan u du = \sec u + C$$

$$17. \int \csc u \cot u du = -\csc u + C$$

$$18. \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$$

$$19. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$20. \int \frac{du}{u \sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$$

PROCEDURES FOR FITTING INTEGRANDS TO BASIC INTEGRATION RULES

Technique	Example
Expand (numerator).	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator.	$\frac{1+x}{x^2+1} = \frac{1}{x^2+1} + \frac{x}{x^2+1}$
Complete the square.	$\frac{1}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{1-(x-1)^2}}$
Divide improper rational function.	$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$
Add and subtract terms in numerator.	$\begin{aligned}\frac{2x}{x^2+2x+1} &= \frac{2x+2-2}{x^2+2x+1} \\ &= \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2}\end{aligned}$
Use trigonometric identities.	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate.	$\begin{aligned}\frac{1}{1+\sin x} &= \left(\frac{1}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right) \\ &= \frac{1-\sin x}{1-\sin^2 x} \\ &= \frac{1-\sin x}{\cos^2 x} \\ &= \sec^2 x - \frac{\sin x}{\cos^2 x}\end{aligned}$

8.2 Integration by Parts.

THEOREM 8.1 Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u \, dv = uv - \int v \, du.$$

GUIDELINES FOR INTEGRATION BY PARTS

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
2. Try letting u be the portion of the integrand whose derivative is a function simpler than u . Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

SUMMARY: COMMON INTEGRALS USING INTEGRATION BY PARTS

1. For integrals of the form

$$\int x^n e^{ax} dx, \quad \int x^n \sin ax dx, \quad \text{or} \quad \int x^n \cos ax dx$$

let $u = x^n$ and let $dv = e^{ax} dx, \sin ax dx$, or $\cos ax dx$.

2. For integrals of the form

$$\int x^n \ln x dx, \quad \int x^n \arcsin ax dx, \quad \text{or} \quad \int x^n \arctan ax dx$$

let $u = \ln x, \arcsin ax$, or $\arctan ax$ and let $dv = x^n dx$.

3. For integrals of the form

$$\int e^{ax} \sin bx dx \quad \text{or} \quad \int e^{ax} \cos bx dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax} dx$.

8.3 Trigonometric Integrals.

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SINE AND COSINE

1. When the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \overset{\text{Odd}}{\cancel{\sin^{2k+1}}} x \cos^k x dx = \int (\overset{\text{Convert to cosines}}{\cancel{(\sin^2 x)^k}} \cos^k x \overset{\text{Save for } du}{\cancel{\sin x}}) dx = \int (1 - \cos^2 x)^k \cos^k x \sin x dx$$

2. When the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \overset{\text{Odd}}{\cancel{\sin^m x}} \cos^{2k+1} x dx = \int \overset{\text{Convert to sines}}{\cancel{\sin^m x}} (\overset{\text{Save for } du}{\cancel{(\cos^2 x)^k}} \cos x) dx = \int \sin^m x (1 - \sin^2 x)^k \cos x dx \frac{1}{2}$$

3. When the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in the second guideline.

Wallis's Formulas

1. If n is odd ($n \geq 3$), then

$$\int_0^{\pi/2} \cos^n x dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right) \cdots \left(\frac{n-1}{n}\right).$$

2. If n is even ($n \geq 2$), then

$$\int_0^{\pi/2} \cos^n x dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) \cdots \left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right).$$

GUIDELINES FOR EVALUATING INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT

1. When the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then, expand and integrate.

$$\int \sec^{2k} x \tan^n x dx \xrightarrow{\text{Even}} \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \sec^2 x dx \xrightarrow{\text{Save for } du} \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x dx$$

2. When the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then, expand and integrate.

$$\int \sec^m x \tan^{2k+1} x dx \xrightarrow{\text{Odd}} \int \overbrace{\sec^{m-1} x (\tan^2 x)^k}^{\text{Convert to secants}} \sec x \tan x dx \xrightarrow{\text{Save for } du} \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x dx$$

3. When there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x dx \xrightarrow{\text{Convert to secants}} \int \tan^{n-2} x (\tan^2 x) dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$$

4. When the integral is of the form

$$\int \sec^m x dx$$

where m is odd and positive, use integration by parts, as illustrated in Example 5 in Section 8.2.

5. When none of the first four guidelines applies, try converting to sines and cosines.

$$\sin mx \sin nx = \frac{1}{2} (\cos [(m-n)x] - \cos [(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2} (\sin [(m-n)x] + \sin [(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2} (\cos [(m-n)x] + \cos [(m+n)x])$$

8.4 Trigonometric Substitution.

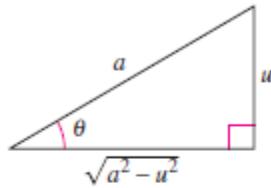
$$\begin{aligned}\cos^2 \theta &= 1 - \sin^2 \theta \\ \sec^2 \theta &= 1 + \tan^2 \theta \\ \tan^2 \theta &= \sec^2 \theta - 1\end{aligned}$$

Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a \sin \theta.$$

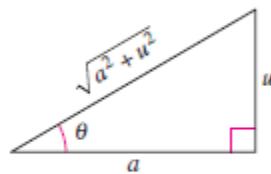
Then $\sqrt{a^2 - u^2} = a \cos \theta$, where
 $-\pi/2 \leq \theta \leq \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let

$$u = a \tan \theta.$$

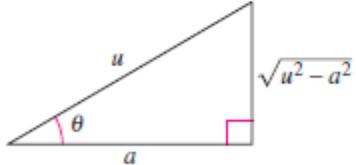
Then $\sqrt{a^2 + u^2} = a \sec \theta$, where
 $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a \sec \theta.$$

Then



$$\sqrt{u^2 - a^2} = \begin{cases} a \tan \theta & \text{for } u > a, \text{ where } 0 \leq \theta \leq \pi/2 \\ -a \tan \theta & \text{for } u < -a, \text{ where } \pi/2 < \theta \leq \pi. \end{cases}$$

THEOREM 8.2 Special Integration Formulas ($a > 0$)

1. $\int \sqrt{a^2 - u^2} du = \frac{1}{2} \left(a^2 \arcsin \frac{u}{a} + u \sqrt{a^2 - u^2} \right) + C$
2. $\int \sqrt{u^2 - a^2} du = \frac{1}{2} \left(u \sqrt{u^2 - a^2} - a^2 \ln |u + \sqrt{u^2 - a^2}| \right) + C, \quad u > a$
3. $\int \sqrt{u^2 + a^2} du = \frac{1}{2} \left(u \sqrt{u^2 + a^2} + a^2 \ln |u + \sqrt{u^2 + a^2}| \right) + C$

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide when improper:** When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

GUIDELINES FOR SOLVING THE BASIC EQUATION

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in the first guideline to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.
2. Collect terms according to powers of x .
3. Equate the coefficients of like powers to obtain a system of linear equations involving A , B , C , and so on.
4. Solve the system of linear equations.

8.6 Integration by Tables and Other Integration Techniques.

Substitution for Rational Functions of Sine and Cosine

For integrals involving rational functions of sine and cosine, the substitution

$$u = \frac{\sin x}{1 + \cos x} = \tan \frac{x}{2}$$

yields

$$\cos x = \frac{1 - u^2}{1 + u^2}, \quad \sin x = \frac{2u}{1 + u^2}, \quad \text{and} \quad dx = \frac{2 du}{1 + u^2}.$$

8.7 Indeterminate Forms and L'Hopital's Rule.

THEOREM 8.3 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b) , then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

A proof of this theorem is given in Appendix A.

See *LarsonCalculus.com* for Bruce Edwards's video of this proof.

THEOREM 8.4 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c , except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b) , except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞ , $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

A proof of this theorem is given in Appendix A.

See *LarsonCalculus.com* for Bruce Edwards's video of this proof.

8.8 Improper Integrals.

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$

where c is any real number (see Exercise 111).

In the first two cases, the improper integral converges when the limit exists—otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a , then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

In the first two cases, the improper integral converges when the limit exists—otherwise, the improper integral diverges. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

THEOREM 8.5 A Special Type of Improper Integral

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \text{diverges,} & p \leq 1 \end{cases}$$

9 Infinite Series

9.1 Sequences.

Definition of the Limit of a Sequence

Let L be a real number. The limit of a sequence $\{a_n\}$ is L , written as

$$\lim_{n \rightarrow \infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence converges to L . If the limit of a sequence does not exist, then the sequence diverges.

THEOREM 9.1 Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x \rightarrow \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n , then

$$\lim_{n \rightarrow \infty} a_n = L.$$

THEOREM 9.2 Properties of Limits of Sequences

Let $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = K$.

1. $\lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm K$
2. $\lim_{n \rightarrow \infty} ca_n = cL$, c is any real number.
3. $\lim_{n \rightarrow \infty} (a_n b_n) = LK$
4. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

THEOREM 9.3 Squeeze Theorem for Sequences

If $\lim_{n \rightarrow \infty} a_n = L = \lim_{n \rightarrow \infty} b_n$ and there exists an integer N such that $a_n \leq c_n \leq b_n$ for all $n > N$, then $\lim_{n \rightarrow \infty} c_n = L$.

THEOREM 9.4 Absolute Value Theorem

For the sequence $\{a_n\}$, if

$$\lim_{n \rightarrow \infty} |a_n| = 0 \quad \text{then} \quad \lim_{n \rightarrow \infty} a_n = 0.$$

Definition of Monotonic Sequence

A sequence $\{a_n\}$ is monotonic when its terms are nondecreasing

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$$

or when its terms are nonincreasing

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots.$$

Definition of Bounded Sequence

1. A sequence $\{a_n\}$ is bounded above when there is a real number M such that $a_n \leq M$ for all n . The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is bounded below when there is a real number N such that $N \leq a_n$ for all n . The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is bounded when it is bounded above and bounded below.

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

9.2 Series and Convergence

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the *n*th partial sum is

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S , then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the **sum of the series**.

$$S = a_1 + a_2 + \cdots + a_n + \cdots \quad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series diverges.

THEOREM 9.6 Convergence of a Geometric Series

A geometric series with ratio r diverges when $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

THEOREM 9.7 Properties of Infinite Series

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A , B , and c be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

THEOREM 9.8 Limit of the *n*th Term of a Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

THEOREM 9.9 *n*th-Term Test for Divergence

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

9.3 The Integral Test and p-Series.

THEOREM 9.10 The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x) dx$$

either both converge or both diverge.

THEOREM 9.11 Convergence of p -Series

The p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

converges for $p > 1$, and diverges for $0 < p \leq 1$.

9.4 Comparisons of Series.

THEOREM 9.12 Direct Comparison Test

Let $0 < a_n \leq b_n$ for all n .

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

THEOREM 9.13 Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

where L is finite and positive, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

9.5 Alternating Series.

THEOREM 9.14 Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge when the two conditions listed below are met.

1. $\lim_{n \rightarrow \infty} a_n = 0$
2. $a_{n+1} \leq a_n$, for all n

THEOREM 9.15 Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 9.16 Absolute Convergence

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.

Definitions of Absolute and Conditional Convergence

1. The series $\sum a_n$ is **absolutely convergent** when $\sum |a_n|$ converges.
2. The series $\sum a_n$ is **conditionally convergent** when $\sum a_n$ converges but $\sum |a_n|$ diverges.

9.6 The Ratio and Root Tests.

THEOREM 9.17 Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. The series $\sum a_n$ converges absolutely when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
2. The series $\sum a_n$ diverges when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
3. The Ratio Test is inconclusive when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

THEOREM 9.18 Root Test

1. The series $\sum a_n$ converges absolutely when $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$.
2. The series $\sum a_n$ diverges when $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive when $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$.

GUIDELINES FOR TESTING A SERIES FOR CONVERGENCE OR DIVERGENCE

1. Does the n th term approach 0? If not, the series diverges.
2. Is the series one of the special types—geometric, p -series, telescoping, or alternating?
3. Can the Integral Test, the Root Test, or the Ratio Test be applied?
4. Can the series be compared favorably to one of the special types?

SUMMARY OF TESTS FOR SERIES

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
n th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$	$0 < r < 1$	$ r \geq 1$	Sum: $S = \frac{a}{1 - r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum: $S = b_1 - L$
p -Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$	
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n \rightarrow \infty} a_n = 0$		Remainder: $ R_N \leq a_{N+1}$

Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n \rightarrow \infty} a_n = 0$		Remainder: $ R_N \leq a_{N+1}$
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x) dx$ converges	$\int_1^{\infty} f(x) dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x) dx$
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } < 1$	$\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } > 1$ or $= \infty$	Test is inconclusive when $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = 1$.
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1$	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1$ or $= \infty$	Test is inconclusive when $\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1$.
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	

9.7 Taylor Polynomials and Approximations.

Definitions of n th Taylor Polynomial and n th Maclaurin Polynomial

If f has n derivatives at c , then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the n th Taylor polynomial for f at c . If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the n th Maclaurin polynomial for f .

THEOREM 9.19 Taylor's Theorem

If a function f is differentiable through order $n + 1$ in an interval I containing c , then, for each x in I , there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x - c)^{n+1}.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

9.8 Power Series.

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series**. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + \cdots + a_n (x - c)^n + \cdots$$

is called a **power series centered at c** , where c is a constant.

THEOREM 9.20 Convergence of a Power Series

For a power series centered at c , precisely one of the following is true.

1. The series converges only at c .
2. There exists a real number $R > 0$ such that the series converges absolutely for

$$|x - c| < R$$

and diverges for

$$|x - c| > R.$$

3. The series converges absolutely for all x .

The number R is the **radius of convergence** of the power series. If the series converges only at c , then the radius of convergence is $R = 0$. If the series converges for all x , then the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THEOREM 9.21 Properties of Functions Defined by Power Series

If the function

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} a_n(x - c)^n \\&= a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \dots\end{aligned}$$

has a radius of convergence $R > 0$, then, on the interval

$$(c - R, c + R)$$

f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x - c)^{n-1}$
 $= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \dots$
2. $\int f(x) dx = C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1}$
 $= C + a_0(x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + \dots$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

9.9 Representation of Functions by Power Series.

Operations with Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$.

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$
2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$
3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

THEOREM 9.22 The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n(x - c)^n$ for all x in an open interval I containing c , then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n = f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c** . Moreover, if $c = 0$, then the series is the **Maclaurin series for f** .

THEOREM 9.23 Convergence of Taylor Series

If $\lim_{n \rightarrow \infty} R_n = 0$ for all x in the interval I , then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

GUIDELINES FOR FINDING A TAYLOR SERIES

1. Differentiate $f(x)$ several times and evaluate each derivative at c .

$$f(c), f'(c), f''(c), f'''(c), \dots, f^{(n)}(c), \dots$$

Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

3. Within this interval of convergence, determine whether the series converges to $f(x)$.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \dots + (-1)^n(x - 1)^n + \dots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots + (-1)^n x^n + \dots$	$-1 < x < 1$
$\ln x = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots + \frac{(-1)^{n-1}(x - 1)^n}{n} + \dots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3 x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \dots + \frac{(2n)! x^{2n+1}}{(2^n n!)^2 (2n+1)} + \dots$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \dots$	$-1 < x < 1^*$

* The convergence at $x = \pm 1$ depends on the value of k .

10 Conics, Parametric Equations, and Polar Coordinates.

10.1 Conics and Calculus.

THEOREM 10.1 Standard Equation of a Parabola

The standard form of the equation of a parabola with vertex (h, k) and directrix $y = k - p$ is

$$(x - h)^2 = 4p(y - k). \quad \text{Vertical axis}$$

For directrix $x = h - p$, the equation is

$$(y - k)^2 = 4p(x - h). \quad \text{Horizontal axis}$$

The focus lies on the axis p units (*directed distance*) from the vertex. The coordinates of the focus are as follows.

$$(h, k + p) \quad \text{Vertical axis}$$

$$(h + p, k) \quad \text{Horizontal axis}$$

THEOREM 10.2 Reflective Property of a Parabola

Let P be a point on a parabola. The tangent line to the parabola at point P makes equal angles with the following two lines.

1. The line passing through P and the focus
2. The line passing through P parallel to the axis of the parabola

THEOREM 10.3 Standard Equation of an Ellipse

The standard form of the equation of an ellipse with center (h, k) and major and minor axes of lengths $2a$ and $2b$, where $a > b$, is

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \quad \text{Major axis is horizontal.}$$

or

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1. \quad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with

$$c^2 = a^2 - b^2.$$

THEOREM 10.4 Reflective Property of an Ellipse

Let P be a point on an ellipse. The tangent line to the ellipse at point P makes equal angles with the lines through P and the foci.

Definition of Eccentricity of an Ellipse

The eccentricity e of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

THEOREM 10.5 Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola with center at (h, k) is

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

or

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center, where $c^2 = a^2 + b^2$.

THEOREM 10.6 Asymptotes of a Hyperbola

For a *horizontal* transverse axis, the equations of the asymptotes are

$$y = k + \frac{b}{a}(x - h) \quad \text{and} \quad y = k - \frac{b}{a}(x - h).$$

For a *vertical* transverse axis, the equations of the asymptotes are

$$y = k + \frac{a}{b}(x - h) \quad \text{and} \quad y = k - \frac{a}{b}(x - h).$$

Definition of Eccentricity of a Hyperbola

The eccentricity e of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

10.2 Plane Curves and Parametric Equations.

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I , then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are **parametric equations** and t is the **parameter**. The set of points (x, y) obtained as t varies over the interval I is the **graph** of the parametric equations. Taken together, the parametric equations and the graph are a **plane curve**, denoted by C .

Definition of a Smooth Curve

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called smooth when f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I . The curve C is called piecewise smooth when it is smooth on each subinterval of some partition of I .

10.3 Parametric Equations.

THEOREM 10.7 Parametric Form of the Derivative

If a smooth curve C is given by the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0.$$

THEOREM 10.8 Arc Length in Parametric Form

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

THEOREM 10.9 Area of a Surface of Revolution

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \leq t \leq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ Revolution about the x -axis: $g(t) \geq 0$
2. $S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ Revolution about the y -axis: $f(t) \geq 0$

10.4 Polar Coordinates and Polar Graphs.

THEOREM 10.10 Coordinate Conversion

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

Polar-to-Rectangular

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Rectangular-to-Polar

$$\tan \theta = \frac{y}{x}$$

$$r^2 = x^2 + y^2$$

THEOREM 10.11 Slope in Polar Form

If f is a differentiable function of θ , then the *slope* of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ) . (See Figure 10.44.)

THEOREM 10.12 Tangent Lines at the Pole

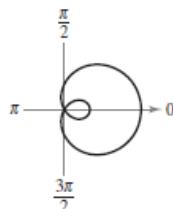
If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Limaçons

$$r = a \pm b \cos \theta$$

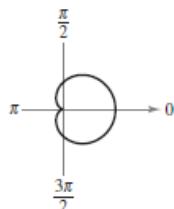
$$r = a \pm b \sin \theta$$

($a > 0, b > 0$)



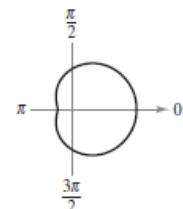
$$\frac{a}{b} < 1$$

Limaçon with
inner loop



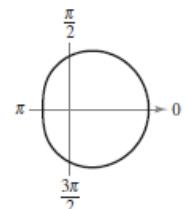
$$\frac{a}{b} = 1$$

Cardioid
(heart-shaped)



$$1 < \frac{a}{b} < 2$$

Dimpled limaçon



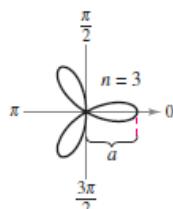
$$\frac{a}{b} \geq 2$$

Convex limaçon

Rose Curves

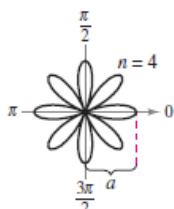
n petals when n is odd

$2n$ petals when n is even ($n \geq 2$)



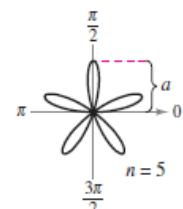
$$r = a \cos n\theta$$

Rose curve



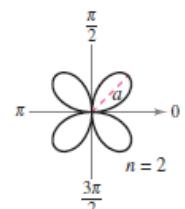
$$r = a \cos n\theta$$

Rose curve



$$r = a \sin n\theta$$

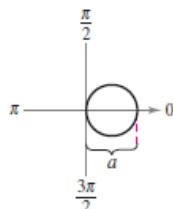
Rose curve



$$r = a \sin n\theta$$

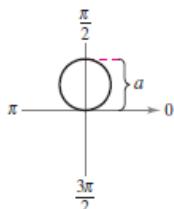
Rose curve

Circles and Lemniscates



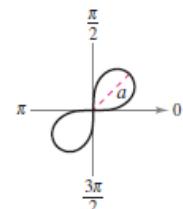
$$r = a \cos \theta$$

Circle



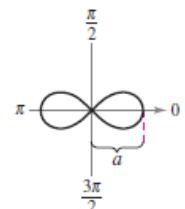
$$r = a \sin \theta$$

Circle



$$r^2 = a^2 \sin 2\theta$$

Lemniscate



$$r^2 = a^2 \cos 2\theta$$

Lemniscate

10.5 Area and Arc Length in Polar Coordinates.

THEOREM 10.13 Area in Polar Coordinates

If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$\begin{aligned} A &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta. \quad 0 < \beta - \alpha \leq 2\pi \end{aligned}$$

THEOREM 10.14 Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

THEOREM 10.15 Area of a Surface of Revolution

Let f be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \sin \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$ About the polar axis
2. $S = 2\pi \int_{\alpha}^{\beta} f(\theta) \cos \theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$ About the line $\theta = \frac{\pi}{2}$

10.6 Polar Equations of Conics and Kepler's Laws.

THEOREM 10.16 Classification of Conics by Eccentricity

Let F be a fixed point (*focus*) and let D be a fixed line (*directrix*) in the plane. Let P be another point in the plane and let e (*eccentricity*) be the ratio of the distance between P and F to the distance between P and D . The collection of all points P with a given eccentricity is a conic.

1. The conic is an ellipse for $0 < e < 1$.
2. The conic is a parabola for $e = 1$.
3. The conic is a hyperbola for $e > 1$.

A proof of this theorem is given in Appendix A.

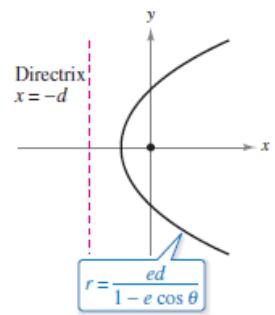
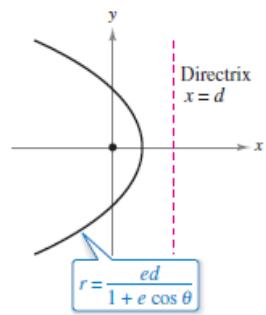
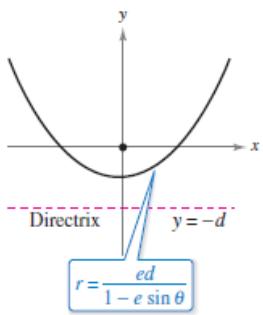
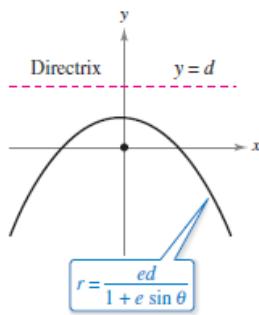
See *LarsonCalculus.com* for Bruce Edwards's video of this proof.

THEOREM 10.17 Polar Equations of Conics

The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix.



The four types of polar equations for a parabola