5 Logarithmic, Exponential, and Other Transcendental Functions

5.6 Indeterminate Forms and L'Hôpital's Rule

Objectives

- Recognize limits that produce indeterminate forms.
- Apply L'Hôpital's Rule to evaluate a limit.

Indeterminate Forms

Indeterminate Forms

We know that the forms 0/0 and ∞/∞ are called *indeterminate* because they do not guarantee that a limit exists, nor do they indicate what the limit is, if one does exist. When you encountered one of these indeterminate forms, you attempted to rewrite the expression by using various algebraic techniques.

Indeterminate Form	Limit	Algebraic Technique
$\frac{0}{0}$	$\lim_{x\to -1} \frac{2x^2-2}{x+1} = \lim_{x\to -1} 2(x-1) = -4$	Divide numerator and denominator by $(x+1)$.
$\frac{\infty}{\infty}$	$\lim_{x\to\infty} \frac{3x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{3-(1/x^2)}{2+(1/x^2)} = \frac{3}{2}$	Divide numerator and denominator by x^2.

Indeterminate Forms

Occasionally, you can extend these algebraic techniques to find limits of transcendental functions. For instance, the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1}$$

produces the indeterminate form 0/0. Factoring and then dividing produces

$$\lim_{x \to 0} \frac{e^{2x} - 1}{e^x - 1} = \lim_{x \to 0} \frac{(e^x + 1)(e^x - 1)}{e^x - 1}$$

$$= \lim_{x \to 0} (e^x + 1)$$

$$= 2.$$

Indeterminate Forms

Not all indeterminate forms, however, can be evaluated by algebraic manipulation. This is often true when *both* algebraic and transcendental functions are involved. For instance, the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

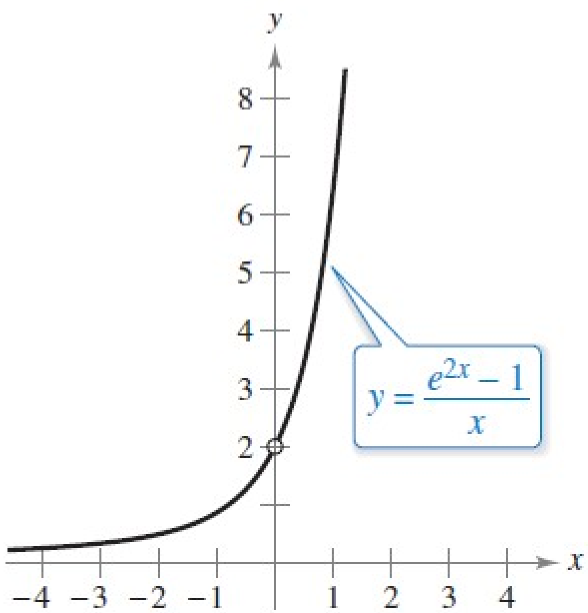
produces the indeterminate form 0/0. Rewriting the expression to obtain

$$\lim_{x \to 0} \left(\frac{e^{2x}}{x} - \frac{1}{x} \right)$$

merely produces another indeterminate form, $\infty - \infty$.

Indeterminate Forms

Of course, you could use technology to estimate the limit, as shown in the table and in Figure 5.23.



The limit as x approaches 0 appears to be 2.

Figure 5.23

Indeterminate Forms

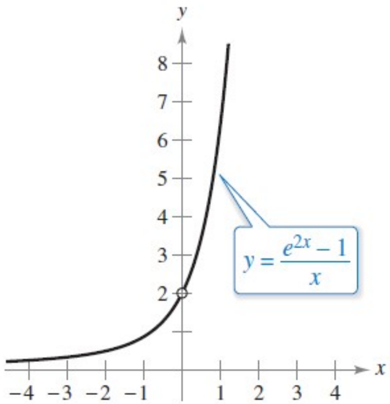
From the table and the graph, the limit appears to be 2.

x	-1	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	1
$\dfrac{e^{2x}-1}{x}$	0.865	1.813	1.980	1.998	?	2.002	2.020	2.214	6.389

L’Hôpital’s Rule

L’Hôpital’s Rule

To find the limit illustrated in Figure 5.23, you can use a theorem called **L’Hôpital’s Rule.**



The limit as x approaches 0 appears to be 2.

Figure 5.23

This theorem states that under certain conditions, the limit of the quotient $f(x)/g(x)$ is determined by the limit of the quotient of the derivatives

$$\frac{f'(x)}{g'(x)}.$$

L'Hôpital's Rule

To prove this theorem, you can use a more general result called the **Extended Mean Value Theorem.**

THEOREM 5.16 The Extended Mean Value Theorem

If f and g are differentiable on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

L'Hôpital's Rule

THEOREM 5.17 L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$.

L’Hôpital’s Rule

L’Hôpital’s Rule can also be applied to one-sided limits. For instance, if the limit of $f(x)/g(x)$ as x approaches c *from the right* produces the indeterminate form $0/0$, then

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)}$$

provided the limit exists (or is infinite).

Example 1 – *Indeterminate Form 0/0*

Evaluate

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}.$$

Solution:

Because direct substitution results in the indeterminate form 0/0

$$\lim_{x \to 0} (e^{2x} - 1) = 0$$

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

$$\lim_{x \to 0} x = 0$$

Example 1 – *Solution*

cont'd

You can apply L'Hôpital's Rule, as shown below.

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}[e^{2x} - 1]}{\frac{d}{dx}[x]}$$ Apply L'Hôpital's Rule.

$$= \lim_{x \to 0} \frac{2e^{2x}}{1}$$ Differentiate numerator and denominator.

$$= 2$$ Evaluate the limit.

L'Hôpital's Rule

Another form of L'Hôpital's Rule states that if the limit of $f(x)/g(x)$ as x approaches ∞ (or $-\infty$) produces the indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

Example 2 – *Indeterminate Form ∞ / ∞*

Evaluate

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Solution:

Because direct substitution results in the indeterminate form ∞/∞, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]} \qquad \text{Apply L'Hôpital's Rule.}$$

$$= \lim_{x\to\infty} \frac{1}{x} \qquad \text{Differentiate numerator and denominator.}$$

$$= 0. \qquad \text{Evaluate the limit.}$$

L'Hôpital's Rule

In addition to the forms $0/0$ and ∞/∞, there are other indeterminate forms such as $0 \cdot \infty$, 1^{∞}, ∞^{0}, 0^{0}, and $\infty - \infty$.

For example, consider the following four limits that lead to the indeterminate form $0 \cdot \infty$.

$$\underbrace{\lim_{x\to 0}\left(\frac{1}{x}\right)(x)}_{\text{Limit is 1.}}, \quad \underbrace{\lim_{x\to 0}\left(\frac{2}{x}\right)(x)}_{\text{Limit is 2.}}, \quad \underbrace{\lim_{x\to \infty}\left(\frac{1}{e^{x}}\right)(x)}_{\text{Limit is 0.}}, \quad \underbrace{\lim_{x\to \infty}\left(\frac{1}{x}\right)(e^{x})}_{\text{Limit is } \infty.}$$

L’Hôpital’s Rule

Because each limit is different, it is clear that the form $0 \cdot \infty$ is indeterminate in the sense that it does not determine the value (or even the existence) of the limit.

Basically, you attempt to convert each of these forms to $0/0$ or ∞/∞ so that L’Hôpital’s Rule can be applied.

Example 4 – *Indeterminate Form 0 ▯ ∞*

Evaluate

$$\lim_{x\to\infty} e^{-x}\sqrt{x}.$$

Solution:

Because direct substitution produces the indeterminate form $0 \cdot \infty,$ you should try to rewrite the limit to fit the form $0/0$ or $\infty/\infty.$

In this case, you can rewrite the limit to fit the second form.

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x}}$$

Example 4 – *Solution*

cont'd

Now, by L'Hôpital's Rule, you have

$$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{e^x}$$ Differentiate numerator and denominator.

$$= \lim_{x\to\infty} \frac{1}{2\sqrt{x}e^x}$$ Simplify.

$$= 0.$$ Evaluate the limit.

L'Hôpital's Rule

The indeterminate forms 1^∞, ∞^0, and 0^0 and 0^0 arise from limits of functions that have variable bases and variable exponents.

Example 5 – *Indeterminate Form* 1^∞

Evaluate

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Solution:

Because direct substitution yields the indeterminate form 1^∞, you can proceed as follows. To begin, assume that the limit exists and is equal to *y*.

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Taking the natural logarithm of each side produces

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right].$$

Example 5 – *Solution*

cont'd

Because the natural logarithmic function is continuous, you can write

$$\ln y = \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] \qquad \text{Indeterminate form } \infty \cdot 0$$

$$= \lim_{x\to\infty}\left(\frac{\ln[1+(1/x)]}{1/x}\right) \qquad \text{Indeterminate form } 0/0$$

$$= \lim_{x\to\infty}\left(\frac{(-1/x^2)\{1/[1+(1/x)]\}}{-1/x^2}\right) \qquad \text{L'Hôpital's Rule}$$

$$= \lim_{x\to\infty}\frac{1}{1+(1/x)}$$

$$= 1.$$

Example 5 – *Solution*

cont'd

Now, because you have shown that

$$\ln y = 1$$

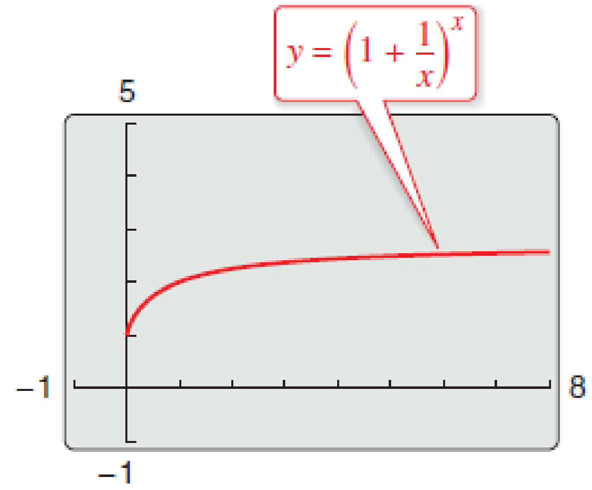
you can conclude that

$$y = e$$

and obtain

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

You can use a graphing utility to confirm this result, as shown in Figure 5.24.



The limit of $[1 + (1/x)]^x$ as x approaches infinity is e.

Figure 5.24

L’Hôpital’s Rule

L’Hôpital’s Rule can also be applied to one-sided limits.

Example 6 – *Indeterminate Form 0^0*

Evaluate

$$\lim_{x \to 0^+} (\sin x)^x.$$

Solution:

Because direct substitution produces the indeterminate form 0^0, you can proceed as shown below. To begin, assume that the limit exists and is equal to *y*.

$$y = \lim_{x \to 0^+} (\sin x)^x \qquad \text{Indeterminate form } 0^0$$

$$\ln y = \ln\left[\lim_{x \to 0^+} (\sin x)^x\right] \qquad \text{Take natural log of each side.}$$

$$= \lim_{x \to 0^+} \left[\ln(\sin x)^x\right] \qquad \text{Continuity}$$

Example 6 – *Solution*

cont'd

$$= \lim_{x \to 0^+} [x \ln(\sin x)] \qquad \text{Indeterminate form } 0 \cdot (-\infty)$$

$$= \lim_{x \to 0^+} \frac{\ln(\sin x)}{1/x} \qquad \text{Indeterminate form } -\infty/\infty$$

$$= \lim_{x \to 0^+} \frac{\cot x}{-1/x^2} \qquad \text{L'Hôpital's Rule}$$

$$= \lim_{x \to 0^+} \frac{-x^2}{\tan x} \qquad \text{Indeterminate form } 0/0$$

$$= \lim_{x \to 0^+} \frac{-2x}{\sec^2 x} \qquad \text{L'Hôpital's Rule}$$

$$= 0$$

Now, because ln *y* = 0, you can conclude that $y = e^0 = 1,$

and it follows that $\lim_{x \to 0^+} (\sin x)^x = 1.$

Example 7 – *Indeterminate Form ∞ - ∞*

Evaluate

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right).$$

Solution:

Because direct substitution yields the indeterminate form $\infty - \infty$, you should try to rewrite the expression to produce a form to which you can apply L'Hôpital's Rule. In this case, you can combine the two fractions to obtain

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}.$$

Example 7 – *Solution*

cont'd

Now, because direct substitution produces the indeterminate form 0/0, you can apply L'Hôpital's Rule to obtain

$$\lim_{x\to 1^+} \frac{x - 1 - \ln x}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{\frac{d}{dx}[x - 1 - \ln x]}{\frac{d}{dx}[(x-1)\ln x]}$$

$$= \lim_{x\to 1^+} \frac{1 - (1/x)}{(x-1)(1/x) + \ln x}$$

$$= \lim_{x\to 1^+} \frac{x-1}{x - 1 + x\ln x}.$$

Example 7 – *Solution*

cont'd

This limit also yields the indeterminate form 0/0, so you can apply L'Hôpital's Rule again to obtain

$$\lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x} = \lim_{x\to 1^+} \frac{1}{1+x(1/x)+\ln x} = \frac{1}{2}.$$

You can check the reasonableness of this solution using a table, as shown at the right.

x	$\frac{1}{\ln x} - \frac{1}{x-1}$
2	0.44270
1.5	0.46630
1.1	0.49206
1.01	0.49917
1.001	0.49992
1.0001	0.49999
1.00001	0.50000

L'Hôpital's Rule

The forms $0/0$, ∞/∞, $\infty - \infty$, $0 \cdot \infty$, 0^0, 1^∞, and ∞^0 have been identified as *indeterminate*.

There are similar forms that you should recognize as "determinate."

$\infty + \infty \rightarrow \infty$	Limit is positive infinity.
$-\infty - \infty \rightarrow -\infty$	Limit is negative infinity.
$0^\infty \rightarrow 0$	Limit is zero.
$0^{-\infty} \rightarrow \infty$	Limit is positive infinity.

L'Hôpital's Rule

As a final comment, remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms 0/0 and ∞/∞. For instance, the application of L'Hôpital's Rule shown below is *incorrect*.

$$\lim_{x \to 0} \frac{e^x}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

Incorrect use of L'Hôpital's Rule

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1, which means that the hypotheses of L'Hôpital's Rule have not been satisfied.