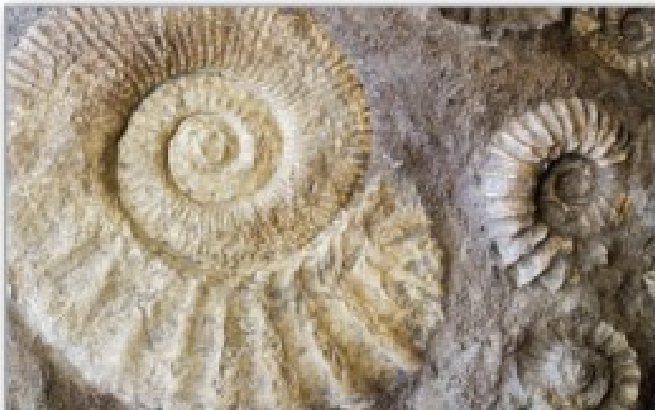
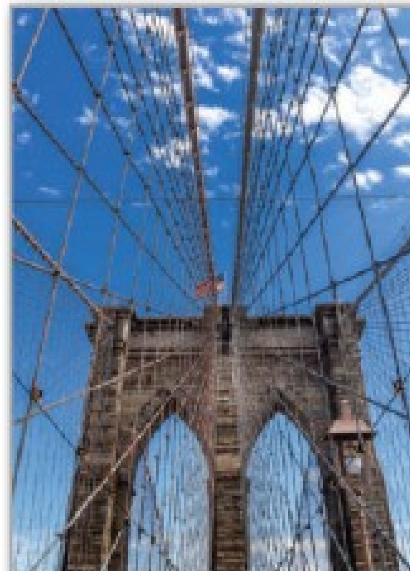
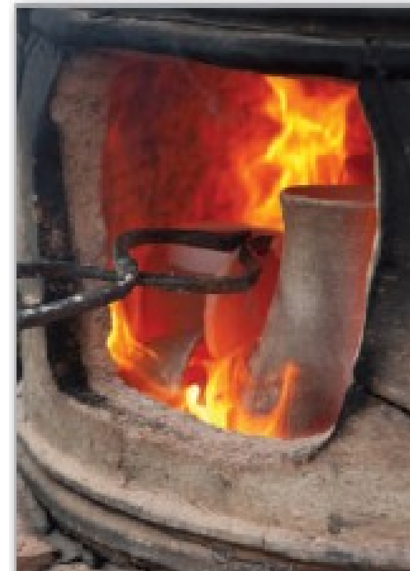
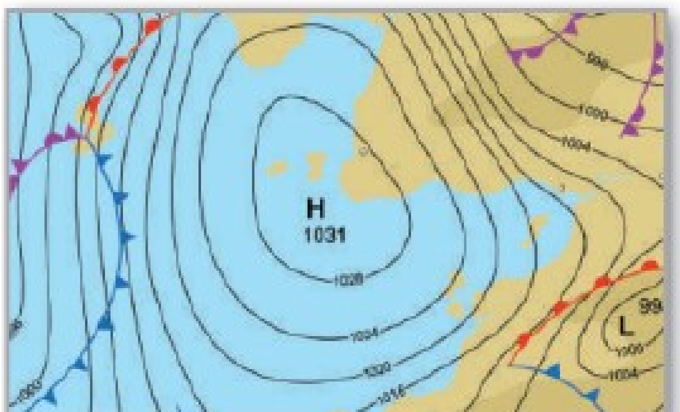
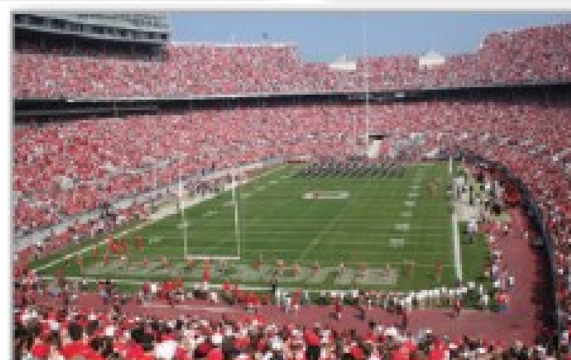
5 Logarithmic, Exponential, and Other Transcendental Functions











5.2

The Natural Logarithmic Function: Integration

Objectives

- Use the Log Rule for Integration to integrate a rational function.
- Integrate trigonometric functions.

Log Rule for Integration

Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

produce the following integration rule.

THEOREM 5.5 Log Rule for Integration

Let u be a differentiable function of x.

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$

Log Rule for Integration

Because $du = u'\,dx,$ the second formula can also be written as

$$\int \frac{u'}{u}\,dx = \ln|u| + C. \qquad \text{Alternative form of Log Rule}$$

Example 1 – *Using the Log Rule for Integration*

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx \qquad \text{Constant Multiple Rule}$$

$$= 2\ln|x| + C \qquad \text{Log Rule for Integration}$$

$$= \ln(x^2) + C \qquad \text{Property of logarithms}$$

Because x^2 cannot be negative, the absolute value notation is unnecessary in the final form of the antiderivative.

Log Rule for Integration

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, when a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule.

This is shown in Example 5.

Example 5 – *Using Long Division Before Integrating*

Find the indefinite integral.

$$\int \frac{x^2 + x + 1}{x^2 + 1}\, dx.$$

Solution:

Begin by using long division to rewrite the integrand.

$$\frac{x^2 + x + 1}{x^2 + 1} \Rightarrow \begin{array}{r} 1 \\ x^2 + 1 \overline{) x^2 + x + 1} \\ \underline{x^2 \qquad + 1} \\ x \end{array} \Rightarrow 1 + \frac{x}{x^2 + 1}$$

Now, you can integrate to obtain

$$\int \frac{x^2 + x + 1}{x^2 + 1}\, dx = \int \left(1 + \frac{x}{x^2 + 1}\right) dx$$

Rewrite using long division.

Example 5 – *Solution*

cont'd

$$= \int dx + \frac{1}{2}\int \frac{2x}{x^2 + 1}\,dx$$ Rewrite as two integrals.

$$= x + \frac{1}{2}\ln(x^2 + 1) + C.$$ Integrate.

Check this result by differentiating to obtain the original integrand.

Log Rule for Integration

The following are guidelines you can use for integration.

GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas.
2. Find an integration formula that resembles all or part of the integrand and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. When you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.
5. Check your result by differentiating to obtain the original integrand.

Example 7 – *u-Substitution and the Log Rule*

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$.

Solution:

The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x}\, dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule.

Example 7 – *Solution*

cont'd

There are three basic choices for u. The choices $u = x$ and $u = x \ln x$ fail to fit the u'/u form of the Log Rule.

However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x}\,dx = \int \frac{1/x}{\ln x}\,dx \qquad \text{Divide numerator and denominator by } x.$$

$$= \int \frac{u'}{u}\,dx \qquad \text{Substitute: } u = \ln x.$$

$$= \ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \ln|\ln x| + C \qquad \text{Back-substitute.}$$

So, the solution is $y = \ln|\ln x| + C$.

Integrals of Trigonometric Functions

Example 8 – *Using a Trigonometric Identity*

Find $\int \tan x \, dx.$

Solution:

This integral does not seem to fit any formulas on our basic list.

However, by using a trigonometric identity, you obtain

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx$$

Apply trigonometric identity and multiply and divide by -1.

Example 8 – *Solution*

cont'd

$$= -\int \frac{u'}{u}\,dx$$ Substitute: $u = \cos x$.

$$= -\ln|u| + C$$ Apply Log Rule.

$$= -\ln|\cos x| + C.$$ Back-substitute.

Integrals of Trigonometric Functions

The integrals of the six basic trigonometric functions are summarized below.

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u\, du = -\cos u + C \qquad \int \cos u\, du = \sin u + C$$

$$\int \tan u\, du = -\ln|\cos u| + C \qquad \int \cot u\, du = \ln|\sin u| + C$$

$$\int \sec u\, du = \ln|\sec u + \tan u| + C \qquad \int \csc u\, du = -\ln|\csc u + \cot u| + C$$