5 Logarithmic, Exponential, and Other Transcendental Functions

5.1 The Natural Logarithmic Function: Differentiation

Objectives

- Develop and use properties of the natural logarithmic function.
- Understand the definition of the number *e*.
- Find derivatives of functions involving the natural logarithmic function.

The Natural Logarithmic Function

The Natural Logarithmic Function

The General Power Rule

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \text{General Power Rule}$$

has an important disclaimer—it does not apply when $n = -1$. Consequently, you have not yet found an antiderivative for the function $f(x) = 1/x$.

In this section, you will use the Second Fundamental Theorem of Calculus to *define* such a function.

The Natural Logarithmic Function

It is neither algebraic nor trigonometric, but falls into a new class of functions called *logarithmic functions*. This particular function is the **natural logarithmic function.**

Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

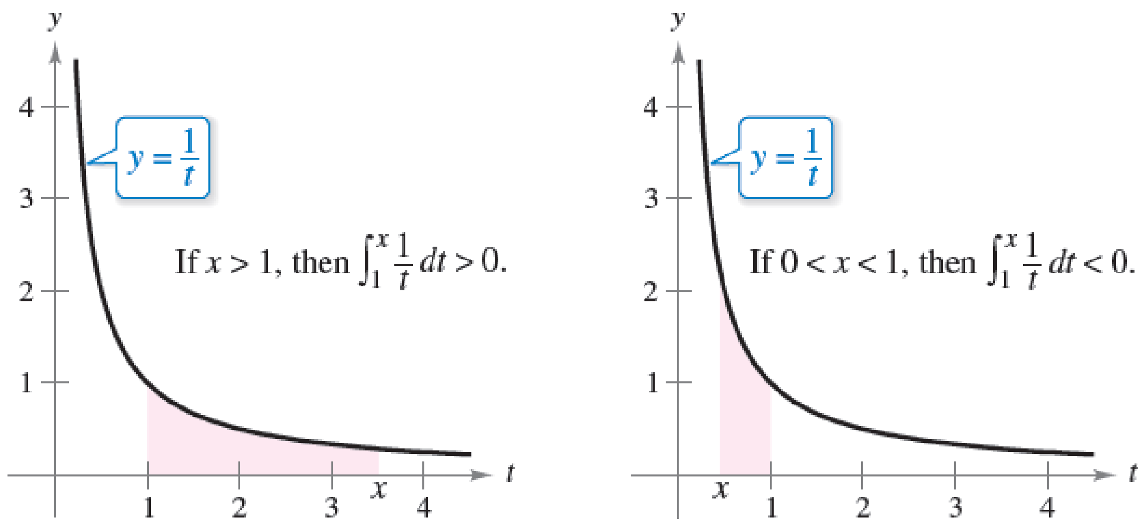
$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

The domain of the natural logarithmic function is the set of all positive real numbers.

The Natural Logarithmic Function

From this definition, you can see that ln *x* is positive for *x* > 1 and negative for 0 < *x* < 1, as shown in Figure 5.1.

Moreover, ln 1 = 0, because the upper and lower limits of integration are equal when *x* = 1.



If $x > 1$, then $\ln x > 0$.

If $0 < x < 1$, then $\ln x < 0$.

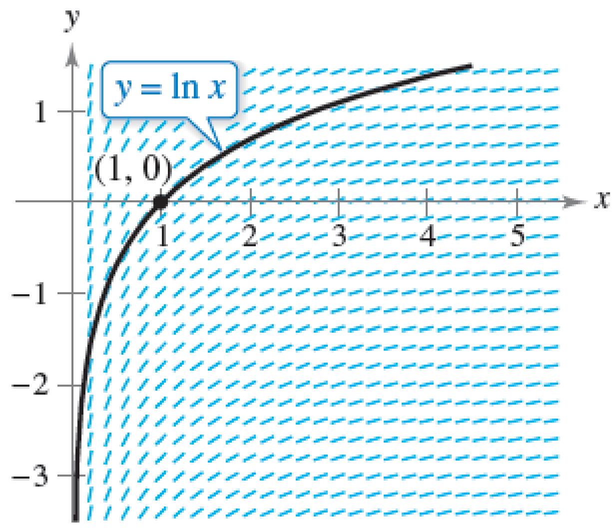
Figure 5.1

The Natural Logarithmic Function

To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an *antiderivative* given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}.$$

Figure 5.2 is a computer-generated graph, called a *slope field (or direction field)*, showing small line segments of slope $1/x$.



Each small line segment has a slope of $\frac{1}{x}$.

Figure 5.2

The graph of $y = \ln x$ is the solution that passes through the point (1, 0).

The Natural Logarithmic Function

The next theorem lists some basic properties of the natural logarithmic function.

THEOREM 5.1 Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

The Natural Logarithmic Function

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties listed below are characteristic of all logarithms.

THEOREM 5.2 Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln 1 = 0$

2. $\ln(ab) = \ln a + \ln b$

3. $\ln(a^n) = n \ln a$

4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

Example 1 – *Expanding Logarithmic Expressions*

a. $\ln \dfrac{10}{9} = \ln 10 - \ln 9$ Property 4

b. $\ln \sqrt{3x + 2} = \ln(3x + 2)^{1/2}$ Rewrite with rational exponent.

$= \dfrac{1}{2} \ln(3x + 2)$ Property 3

c. $\ln \dfrac{6x}{5} = \ln(6x) - \ln 5$ Property 4

$= \ln 6 + \ln x - \ln 5$ Property 2

Example 1 – *Expanding Logarithmic Expressions* cont'd

d. $$\ln \frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} = \ln(x^2+3)^2 - \ln\left(x\sqrt[3]{x^2+1}\right)$$

$$= 2\ln(x^2+3) - [\ln x + \ln(x^2+1)^{1/3}]$$

$$= 2\ln(x^2+3) - \ln x - \ln(x^2+1)^{1/3}$$

$$= 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)$$

The Natural Logarithmic Function

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original.

For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2 \ln x$ is all positive real numbers. (See Figure 5.4.)

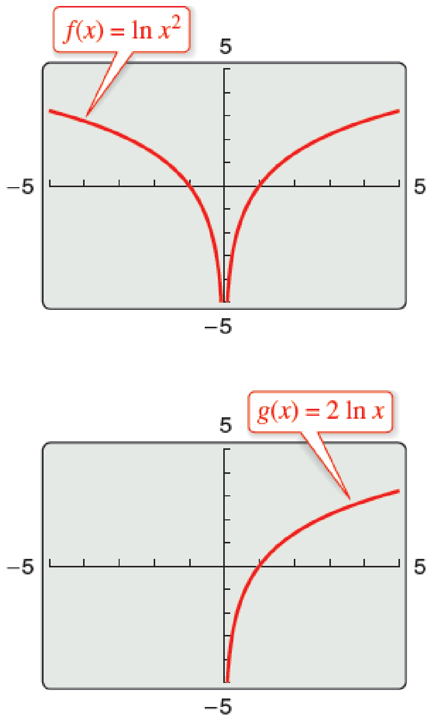


Figure 5.4

The Number *e*

The Number *e*

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number.

For example, common logarithms have a base of 10 and therefore $\log_{10}10 = 1$.

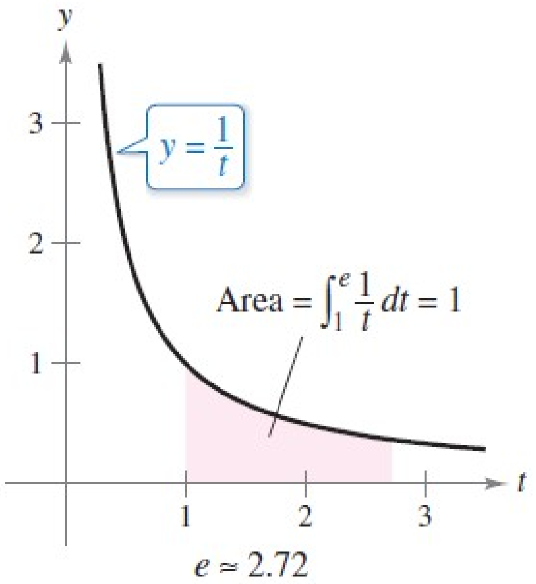
The **base for the natural logarithm** is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$.

The Number e

So, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.5.

This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.

$$e \approx 2.71828182846$$



e is the base for the natural logarithm because $\ln e = 1$.

Figure 5.5

The Number *e*

Definition of *e*

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$

Once you know that ln e = 1, you can use logarithmic properties to evaluate the natural logarithms of several other numbers.

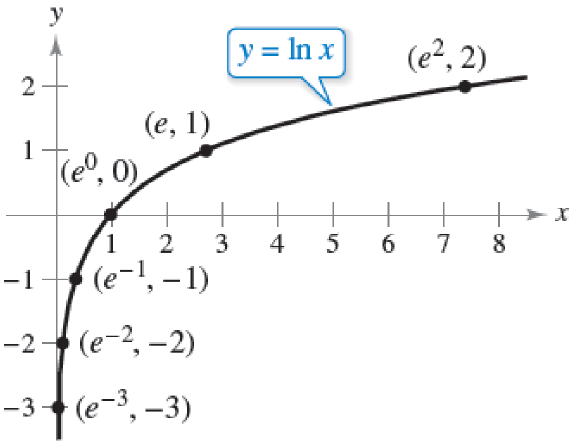
The Number *e*

For example, by using the property

$$\begin{aligned}\ln(e^n) &= n \ln e \\ &= n(1) \\ &= n\end{aligned}$$

you can evaluate $\ln(e^n)$ for various values of n, as shown in the table and in Figure 5.6.

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2



If $x = e^n$, then $\ln x = n$.

Figure 5.6

The Number *e*

The logarithms shown in the table are convenient because the x–values are integer powers of e. Most logarithmic expressions are, however, best evaluated with a calculator.

Example 2 – *Evaluating Natural Logarithmic Expressions*

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$

The Derivative of the Natural Logarithmic Function

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3.

The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$

2. $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

Example 3 – *Differentiation of Logarithmic Functions*

a. $\frac{d}{dx}[\ln(2x)] = \frac{u'}{u} = \frac{2}{2x} = \frac{1}{x}$ $\qquad u = 2x$

b. $\frac{d}{dx}[\ln(x^2 + 1)] = \frac{u'}{u} = \frac{2x}{x^2 + 1}$ $\qquad u = x^2 + 1$

c. $\frac{d}{dx}[x \ln x] = x\left(\frac{d}{dx}[\ln x]\right) + (\ln x)\left(\frac{d}{dx}[x]\right)$ $\qquad$ Product Rule

$$= x\left(\frac{1}{x}\right) + (\ln x)(1)$$

$$= 1 + \ln x$$

Example 3 – *Differentiation of Logarithmic Functions*

cont'd

d. $\frac{d}{dx}[(\ln x)^3] = 3(\ln x)^2 \frac{d}{dx}[\ln x]$ Chain Rule

$$= 3(\ln x)^2 \frac{1}{x}$$

The Derivative of the Natural Logarithmic Function

On occasion, it is convenient to use logarithms as aids in differentiating *nonlogarithmic* functions.

This procedure is called **logarithmic differentiation.**

Example 6 – *Logarithmic Differentiation*

Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$$

Solution:

Note that $y > 0$ for all $x \neq 2$. So, ln y is defined.

Begin by taking the natural logarithm of each side of the equation.

Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

Example 6 – *Solution*

cont'd

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2$$

Write original equation.

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}}$$

Take natural log of each side.

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1)$$

Logarithmic properties

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right)$$

Differentiate.

$$\frac{y'}{y} = \frac{x^2+2x+2}{(x-2)(x^2+1)}$$

Simplify.

Example 6 – *Solution*

cont'd

$$y' = y\left[\frac{x^2 + 2x + 2}{(x - 2)(x^2 + 1)}\right]$$

Solve for y'.

$$y' = \frac{(x - 2)^2}{\sqrt{x^2 + 1}}\left[\frac{x^2 + 2x + 2}{(x - 2)(x^2 + 1)}\right]$$

Substitute for y.

$$y' = \frac{(x - 2)(x^2 + 2x + 2)}{(x^2 + 1)^{3/2}}$$

Simplify.

The Derivative of the Natural Logarithmic Function

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. The next theorem states that you can differentiate functions of the form $y = \ln|u|$ as though the absolute value notation was not present.

THEOREM 5.4 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Example 7 – *Derivative Involving Absolute Value*

Find the derivative of

$$f(x) = \ln|\cos x|.$$

Solution:

Using Theorem 5.4, let $u = \cos x$ and write

$$\frac{d}{dx}[\ln|\cos x|] = \frac{u'}{u} \qquad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

$$= \frac{-\sin x}{\cos x} \qquad u = \cos x$$

$$= -\tan x. \qquad \text{Simplify.}$$