4 Integration

4.4 The Fundamental Theorem of Calculus

Objectives

- Evaluate a definite integral using the Fundamental Theorem of Calculus.
- Understand and use the Mean Value Theorem for Integrals.
- Find the average value of a function over a closed interval.
- Understand and use the Second Fundamental Theorem of Calculus.
- Understand and use the Net Change Theorem.

The Fundamental Theorem of Calculus

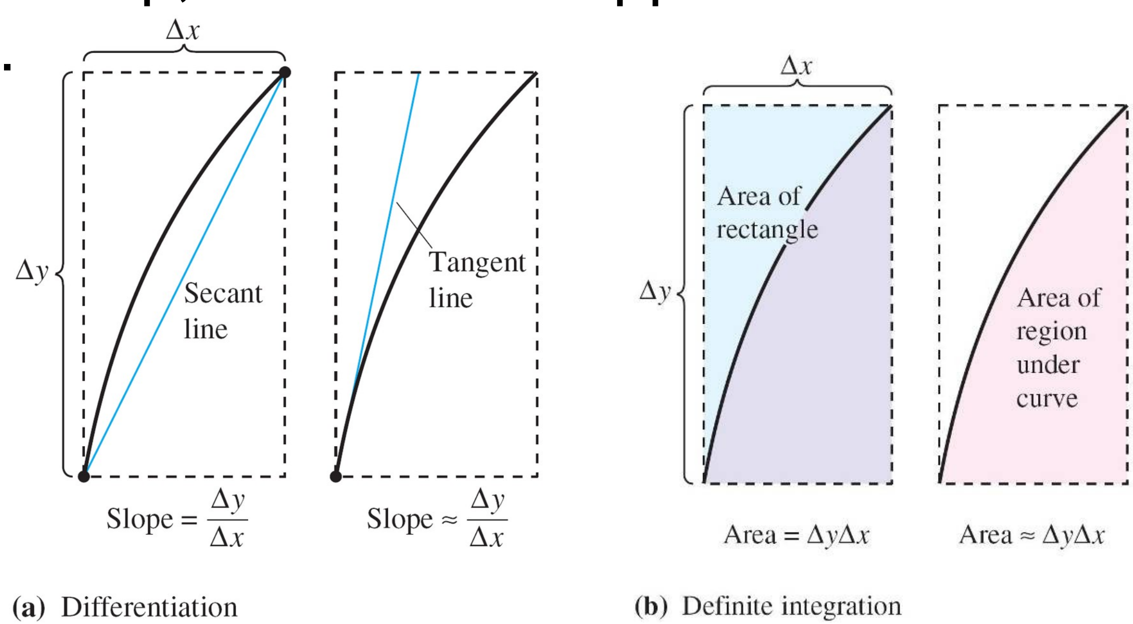
The Fundamental Theorem of Calculus

You have now been introduced to the two major branches of calculus: differential calculus and integral calculus. At this point, these two problems might seem unrelated—but there is a very close connection.

The connection was discovered independently by Isaac Newton and Gottfried Leibniz and is stated in the **Fundamental Theorem of Calculus.**

The Fundamental Theorem of Calculus

Informally, the theorem states that differentiation and (definite) integration are inverse operations, in the same sense that division and multiplication are inverse operations. To see how Newton and Leibniz might have anticipated this relationship, consider the approximations shown in Figure 4.27.



Differentiation and definite integration have an "inverse" relationship.

Figure 4.27

The Fundamental Theorem of Calculus

The slope of the tangent line was defined using the *quotient* $\Delta y/\Delta x$ (the slope of the secant line).

Similarly, the area of a region under a curve was defined using the *product* $\Delta y \Delta x$ (the area of a rectangle).

So, at least in the primitive approximation stage, the operations of differentiation and definite integration appear to have an inverse relationship in the same sense that division and multiplication are inverse operations.

The Fundamental Theorem of Calculus states that the limit processes (used to define the derivative and definite integral) preserve this inverse relationship.

The Fundamental Theorem of Calculus

THEOREM 4.9 The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus

The following guidelines can help you understand the use of the Fundamental Theorem of Calculus.

GUIDELINES FOR USING THE FUNDAMENTAL THEOREM OF CALCULUS

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the notation shown below is convenient.

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative.

$$\int_a^b f(x)\,dx = \Big[F(x) + C\Big]_a^b = [F(b) + C] - [F(a) + C] = F(b) - F(a)$$

Example 1 – *Evaluating a Definite Integral*

Evaluate each definite integral.

a. $\displaystyle\int_1^2 (x^2 - 3)\, dx$

b. $\displaystyle\int_1^4 3\sqrt{x}\, dx$

c. $\displaystyle\int_0^{\pi/4} \sec^2 x\, dx$

Example 1 – *Solution*

a. $$\int_1^2 (x^2 - 3)\,dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$$

b. $$\int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{1/2}\,dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2} - 2(1)^{3/2} = 14$$

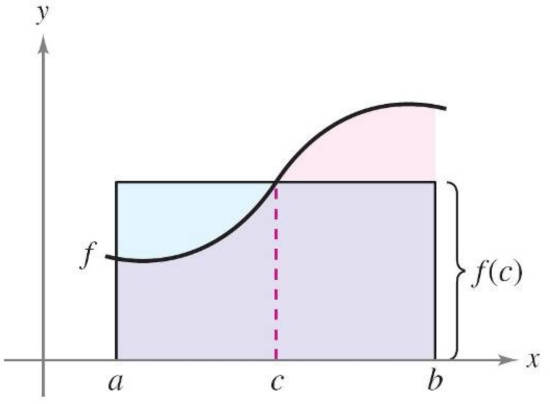
c. $$\int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1 - 0 = 1$$

The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals

You saw that the area of a region under a curve is greater than the area of an inscribed rectangle and less than the area of a circumscribed rectangle.

The Mean Value Theorem for Integrals states that somewhere "between" the inscribed and circumscribed rectangles, there is a rectangle whose area is precisely equal to the area of the region under the curve, as shown in Figure 4.30.



Mean value rectangle:

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

Figure 4.30

The Mean Value Theorem for Integrals

THEOREM 4.10 Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Average Value of a Function

Average Value of a Function

The value of $f(c)$ given in the Mean Value Theorem for Integrals is called the **average value** of f on the interval $[a, b]$.

Definition of the Average Value of a Function on an Interval

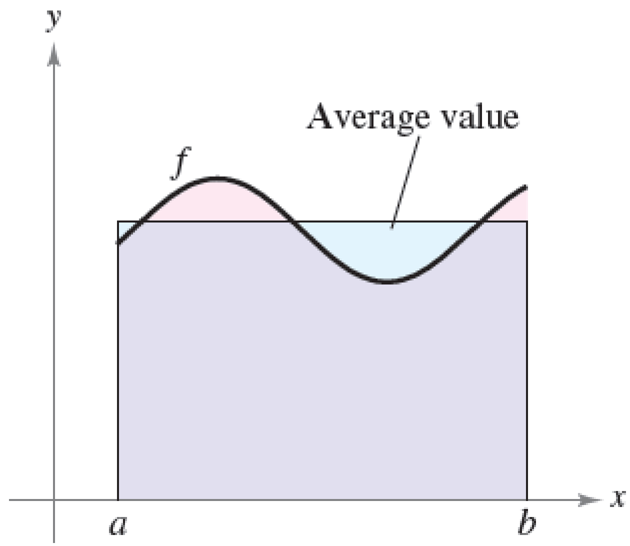
If f is integrable on the closed interval $[a, b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

See Figure 4.32.

Average Value of a Function

In Figure 4.32, the area of the region under the graph of *f* is equal to the area of the rectangle whose height is the average value.



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Figure 4.32

Average Value of a Function

To see why the average value of f is defined in this way, partition $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{b - a}{n}.$$

If c_i is any point in the ith subinterval, then the arithmetic average (or mean) of the function values at the c_i's is

$$a_n = \frac{1}{n}[f(c_1) + f(c_2) + \cdots + f(c_n)]. \qquad \text{Average of } f(c_1), \ldots, f(c_n)$$

Average Value of a Function

By writing the sum using summation notation and then multiplying and dividing by (b – a), you can write the average as

$$a_n = \frac{1}{n}\sum_{i=1}^{n} f(c_i) \qquad \text{Rewrite using summation notation.}$$

$$= \frac{1}{n}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{b-a}\right) \qquad \text{Multiply and divide by } (b-a).$$

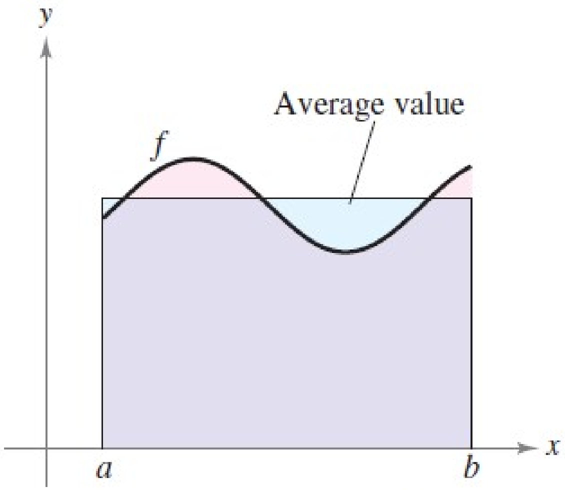
$$= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\left(\frac{b-a}{n}\right) \qquad \text{Rewrite.}$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(c_i)\,\Delta x. \qquad \Delta x = \frac{b-a}{n}$$

Average Value of a Function

Finally, taking the limit as $n \to \infty$ produces the average value of *f* on the interval [*a*, *b*], as given in the definition above.

In Figure 4.32, notice that the area of the region under the graph of *f* is equal to the area of the rectangle whose height is the average value.



$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Figure 4.32

Example 4 – *Finding the Average Value of a Function*

Find the average value of $f(x) = 3x^2 - 2x$ on the interval [1, 4].

Solution:

The average value is

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 (3x^2 - 2x)\,dx$$

$$= \frac{1}{3}\Big[x^3 - x^2\Big]_1^4$$

Example 4 – *Solution* cont'd

$$= \frac{1}{3}[64 - 16 - (1 - 1)]$$

$$= \frac{48}{3}$$

$$= 16.$$ (See Figure 4.33.)

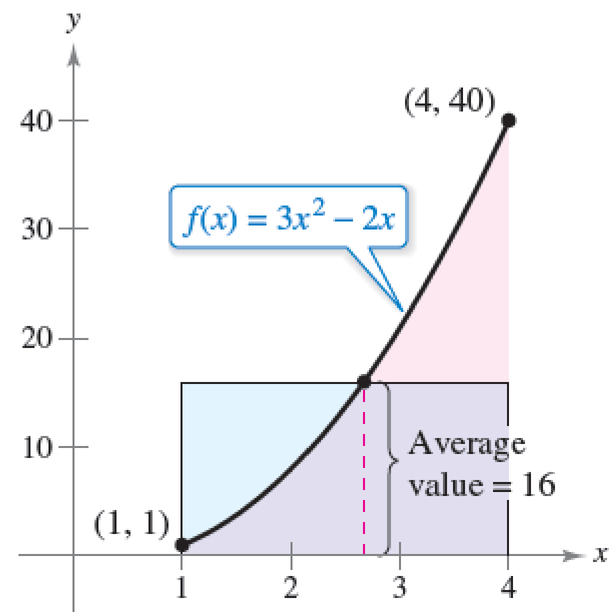


Figure 4.33

The Second Fundamental Theorem of Calculus

The Second Fundamental Theorem of Calculus

Earlier you saw that the definite integral of f on the interval $[a, b]$ was defined using the constant b as the upper limit of integration and x as the variable of integration.

However, a slightly different situation may arise in which the variable x is used in the upper limit of integration.

To avoid the confusion of using x in two different ways, t is temporarily used as the variable of integration. (Remember that the definite integral is *not* a function of its variable of integration.)

The Second Fundamental Theorem of Calculus

The Definite Integral as a Number

Constant

$$\int_a^b f(x)\,dx$$

Constant

f is a function of *x*.

The Definite Integral as a Function of *x*

F is a function of *x*.

$$F(x) = \int_a^x f(t)\,dt$$

Constant

f is a function of *t*.

Example 6 – *The Definite Integral as a Function*

Evaluate the function

$$F(x) = \int_0^x \cos t \, dt$$

at $x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$.

Solution:

You could evaluate five different definite integrals, one for each of the given upper limits.

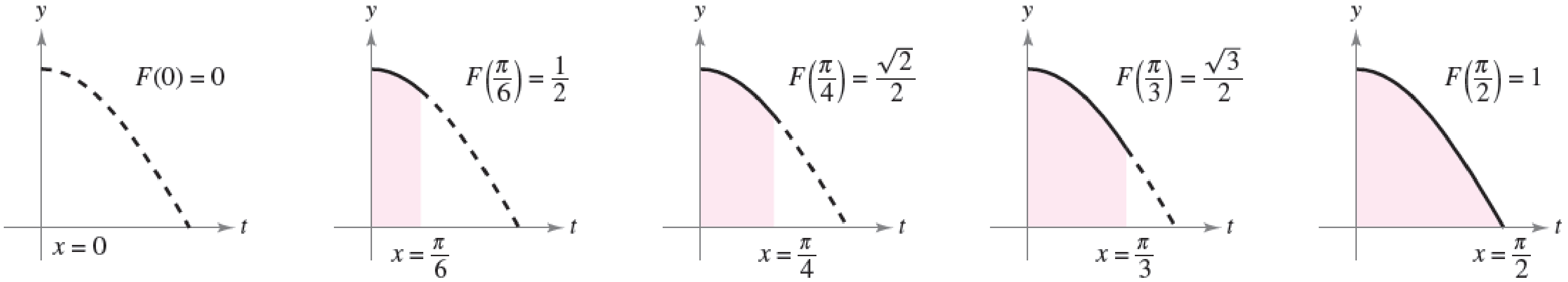
However, it is much simpler to fix *x* (as a constant) temporarily to obtain

$$\int_0^x \cos t \, dt = \sin t \Big]_0^x = \sin x - \sin 0 = \sin x.$$

Example 6 – *Solution*

cont'd

Now, using $F(x) = \sin x$, you can obtain the results shown in Figure 4.35.



$F(x) = \int_0^x \cos t \, dt$ is the area under the curve $f(t) = \cos t$ from 0 to x.

Figure 4.35

The Second Fundamental Theorem of Calculus

You can think of the function $F(x)$ as *accumulating* the area under the curve $f(t) = \cos t$ from $t = 0$ to $t = x$.

For $x = 0$, the area is 0 and $F(0) = 0$. For $x = \pi/2$, $F(\pi/2) = 1$ gives the accumulated area under the cosine curve on the entire interval $[0, \pi/2]$.

This interpretation of an integral as an **accumulation function** is used often in applications of integration.

The Second Fundamental Theorem of Calculus

In Example 6, note that the derivative of *F* is the original integrand (with only the variable changed). That is,

$$\frac{d}{dx}[F(x)] = \frac{d}{dx}[\sin x] = \frac{d}{dx}\left[\int_0^x \cos t \, dt\right] = \cos x.$$

This result is generalized in the following theorem, called the **Second Fundamental Theorem of Calculus.**

THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

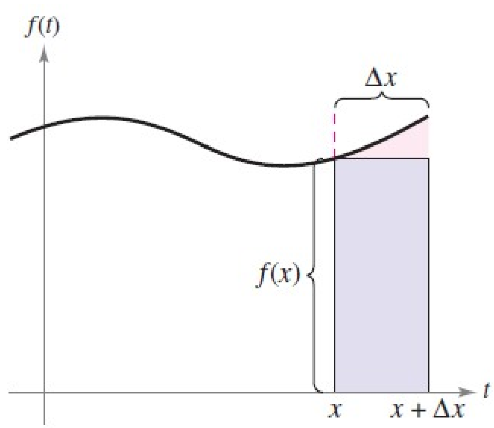
$$\frac{d}{dx}\left[\int_a^x f(t) \, dt\right] = f(x).$$

The Second Fundamental Theorem of Calculus

Using the area model for definite integrals, the approximation

$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\,dt$$

can be viewed as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x + \Delta x]$ as shown in the figure below.



$$f(x)\,\Delta x \approx \int_x^{x+\Delta x} f(t)\,dt$$

Example 7 – *The Second Fundamental Theorem of Calculus*

Evaluate $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right]$.

Solution:

Note that $f(t) = \sqrt{t^2+1}$ is continuous on the entire real number line.

So, using the Second Fundamental Theorem of Calculus, you can write

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}.$$

Net Change Theorem

Net Change Theorem

The Fundamental Theorem of Calculus states that if f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

But because $F'(x) = f(x)$, this statement can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

where the quantity $F(b) - F(a)$ represents the *net change of* $F(x)$ on the interval $[a, b]$.

Net Change Theorem

THEOREM 4.12 The Net Change Theorem

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or **net change**, of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F(x)$$

Example 9 – *Using the Net Change Theorem*

A chemical flows into a storage tank at a rate of $(180 + 3t)$ liters per minute, where t is the time in minutes and $0 \leq t \leq 60$. Find the amount of the chemical that flows into the tank during the first 20 minutes.

Solution:

Let $c(t)$ be the amount of the chemical in the tank at time t.

Then $c'(t)$ represents the rate at which the chemical flows into the tank at time t.

Example 9 – *Solution*

cont'd

During the first 20 minutes, the amount that flows into the tank is

$$\int_0^{20} c'(t)\,dt = \int_0^{20} (180 + 3t)\,dt$$

$$= \left[180t + \frac{3}{2}t^2\right]_0^{20}$$

$$= 3600 + 600 = 4200.$$

So, the amount of the chemical that flows into the tank during the first 20 minutes is 4200 liters.

Net Change Theorem

Another way to illustrate the Net Change Theorem is to examine the velocity of a particle moving along a straight line, where $s(t)$ is the position at time t. Then its velocity is $v(t) = s'(t)$ and

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

This definite integral represents the net change in position, or **displacement,** of the particle.

Net Change Theorem

When calculating the *total* distance traveled by the particle, you must consider the intervals where $v(t) \leq 0$ and the intervals where $v(t) \geq 0$.

When $v(t) \leq 0$, the particle moves to the left, and when $v(t) \geq 0$, the particle moves to the right.

To calculate the total distance traveled, integrate the absolute value of velocity $|v(t)|$.

Net Change Theorem

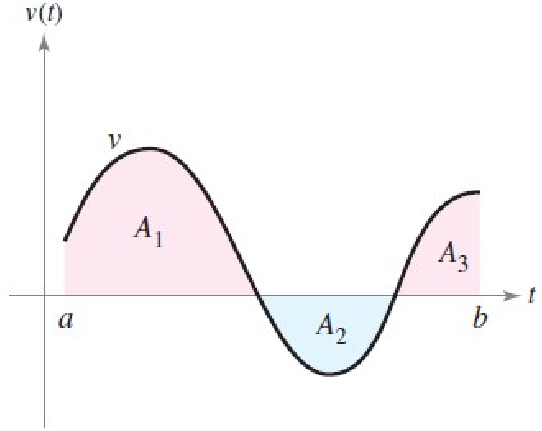
So, the **displacement** of the particle on the interval [*a*, *b*] is

$$\text{Displacement on } [a, b] = \int_a^b v(t)\,dt = A_1 - A_2 + A_3$$

and the **total distance traveled** by the particle on [*a*, *b*] is

$$\text{Total distance traveled on } [a, b] = \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3.$$

(See Figure 4.36).



A_1, A_2, and A_3 are the areas of the shaded regions.

Figure 4.36

Example 10 – *Solving a Particle Motion Problem*

The velocity (in feet per second) of a particle moving along a line is

$$v(t) = t^3 - 10t^2 + 29t - 20$$

where t is the time in seconds.

a. What is the displacement of the particle on the time interval $1 \le t \le 5$?

b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

Example 10(a) – *Solution*

By definition, you know that the displacement is

$$\int_1^5 v(t)\,dt = \int_1^5 (t^3 - 10t^2 + 29t - 20)\,dt$$

$$= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^5$$

$$= \frac{25}{12} - \left(-\frac{103}{12}\right)$$

$$= \frac{128}{12}$$

$$= \frac{32}{3}.$$

So, the particle moves $\frac{32}{3}$ feet to the right.

Example 10(b) – *Solution*

cont'd

To find the total distance traveled, calculate $\int_1^5 |v(t)|\, dt.$

Using Figure 4.37 and the fact that $v(t)$ can be factored as $(t - 1)(t - 4)(t - 5)$, you can determine that $v(t) \geq 0$ on $[1, 4]$ and $v(t) \leq 0$ on $[4, 5]$.

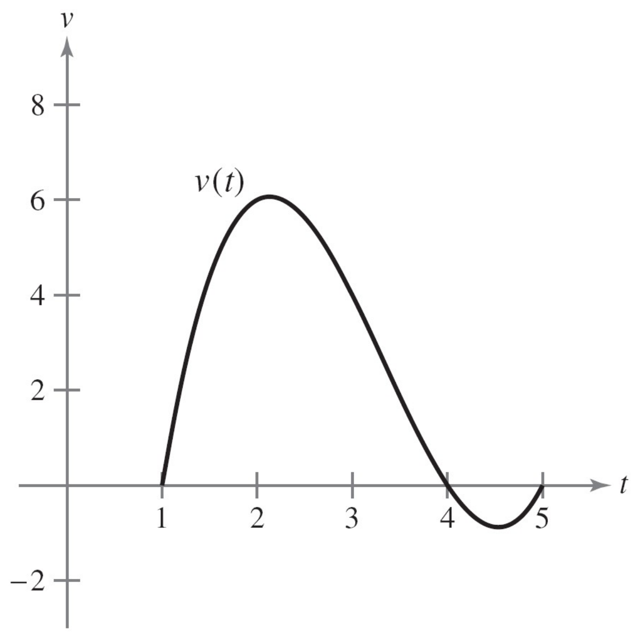


Figure 4.37

Example 10(b) – *Solution*

cont'd

So, the total distance traveled is

$$\int_1^5 |v(t)|\, dt = \int_1^4 v(t)\, dt - \int_4^5 v(t)\, dt$$

$$= \int_1^4 (t^3 - 10t^2 + 29t - 20)\, dt - \int_4^5 (t^3 - 10t^2 + 29t - 20)\, dt$$

$$= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^4 - \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_4^5$$

$$= \frac{45}{4} - \left(-\frac{7}{12}\right)$$

$$= \frac{71}{6} \text{ feet.}$$