4 Integration

4.5 Integration by Substitution

Objectives

- Use pattern recognition to find an indefinite integral.
- Use a change of variables to find an indefinite integral.
- Use the General Power Rule for Integration to find an indefinite integral.
- Use a change of variables to evaluate a definite integral.
- Evaluate a definite integral involving an even or odd function.

Pattern Recognition

Pattern Recognition

In this section, you will study techniques for integrating composite functions.

The discussion is split into two parts—*pattern recognition* and *change of variables*. Both techniques involve a ***u*-substitution.**

With pattern recognition, you perform the substitution mentally, and with change of variables, you write the substitution steps.

The role of substitution in integration is comparable to the role of the Chain Rule in differentiation.

Pattern Recognition

Recall that for the differentiable functions

$y = F(u)$ and $u = g(x)$

the Chain Rule states that

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x).$$

From the definition of an antiderivative, it follows that

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

Pattern Recognition

These results are summarized in the following theorem.

THEOREM 4.13 Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C.$$

Pattern Recognition

Example 1 shows how to apply Theorem 4.13 *directly*, by recognizing the presence of $f(g(x))$ and $g'(x)$.

Note that the composite function in the integrand has an *outside function f* and an *inside function g*. Moreover, the derivative $g'(x)$ is present as a factor of the integrand.

$$\int \overset{\text{Outside function}}{f}(\underset{\text{Inside function}}{g(x)})\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

Example 1 – *Recognizing the f(g(x))g'(x) Pattern*

Find $\int (x^2 + 1)^2(2x)\, dx.$

Solution:

Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern.

Example 1 – *Solution*

cont'd

Using the Power Rule for Integration and Theorem 4.13, you can write

$$\int \underbrace{(x^2+1)^2}_{f(g(x))}\underbrace{(2x)}_{g'(x)}\,dx = \frac{1}{3}(x^2+1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2+1)^3 + C$ is the integrand of the original integral.

Pattern Recognition

The integrand in Example 1 fits the $f(g(x))g'(x)$ pattern exactly—you only had to recognize the pattern.

You can extend this technique considerably with the Constant Multiple Rule

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Many integrands contain the essential part (the variable part) of $g'(x)$ but are missing a constant multiple.

In such cases, you can multiply and divide by the necessary constant multiple, as shown in Example 3.

Example 3 – *Multiplying and Dividing by a Constant*

Find the indefinite integral.

$$\int x(x^2 + 1)^2 \, dx.$$

Solution:

This is similar to the integral given in Example 1, except that the integrand is missing a factor of 2.

Recognizing that $2x$ is the derivative of $x^2 + 1$, you can let

$$g(x) = x^2 + 1$$

and $\int x(x^2 + 1)^2 \, dx = \int (x^2 + 1)^2 \left(\frac{1}{2}\right)(2x) \, dx$ Multiply and divide by 2.

Example 3 – *Solution*

cont'd

$$= \frac{1}{2}\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\ \overbrace{(2x)}^{g'(x)}\,dx \qquad \text{Constant Multiple Rule}$$

$$= \frac{1}{2}\left[\frac{(x^2 + 1)^3}{3}\right] + C \qquad \text{Integrate.}$$

$$= \frac{1}{6}(x^2 + 1)^3 + C \qquad \text{Simplify.}$$

Change of Variables for Indefinite Integrals

Change of Variables for Indefinite Integrals

With a formal **change of variables,** you completely rewrite the integral in terms of *u* and *du* (or any other convenient variable).

Although this procedure can involve more written steps than the pattern recognition illustrated in Examples 1 and 3, it is useful for complicated integrands.

The change of variables technique uses the Leibniz notation for the differential. That is, if $u = g(x)$, then $du = g'(x)dx$, and the integral in Theorem 4.13 takes the form

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du = F(u) + C.$$

Example 4 – *Change of Variables*

Find $\int \sqrt{2x - 1}\, dx.$

Solution:

First, let u be the inner function, $u = 2x - 1$.

Then calculate the differential du to be $du = 2dx$.

Now, using $\sqrt{2x - 1} = \sqrt{u}$ and $dx = du/2,$ substitute to obtain

$$\int \sqrt{2x - 1}\, dx = \int \sqrt{u} \left(\frac{du}{2}\right) \qquad \text{Integral in terms of } u$$

Example 4 – *Solution*

cont'd

$$= \frac{1}{2}\int u^{1/2}\,du$$ Constant Multiple Rule

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2}\right) + C$$ Antiderivative in terms of u

$$= \frac{1}{3}u^{3/2} + C$$ Simplify.

$$= \frac{1}{3}(2x - 1)^{3/2} + C.$$ Antiderivative in terms of x

Change of Variables for Indefinite Integrals

The steps used for integration by substitution are summarized in the following guidelines.

GUIDELINES FOR MAKING A CHANGE OF VARIABLES

1. Choose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

The General Power Rule for Integration

One of the most common *u*-substitutions involves quantities in the integrand that are raised to a power.

Because of the importance of this type of substitution, it is given a special name—the **General Power Rule for Integration.**

THEOREM 4.14 The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)\, dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1.$$

Equivalently, if $u = g(x)$, then

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1.$$

Example 7 – *Substitution and the General Power Rule*

$$\textbf{a.}\ \int 3(3x-1)^4\,dx = \int \overbrace{(3x-1)^4}^{u^4}\overbrace{(3)}^{du}\,dx = \overbrace{\frac{(3x-1)^5}{5}}^{u^5/5} + C$$

$$\textbf{b.}\ \int (2x+1)(x^2+x)\,dx = \int \overbrace{(x^2+x)^1}^{u^1}\overbrace{(2x+1)\,dx}^{du} = \overbrace{\frac{(x^2+x)^2}{2}}^{u^2/2} + C$$

$$\textbf{c.}\ \int 3x^2\sqrt{x^3-2}\,dx = \int \overbrace{(x^3-2)^{1/2}}^{u^{1/2}}\overbrace{(3x^2)\,dx}^{du} = \overbrace{\frac{(x^3-2)^{3/2}}{3/2}}^{u^{3/2}/(3/2)} + C$$

$$= \frac{2}{3}(x^3-2)^{3/2} + C$$

Example 7 – *Substitution and the General Power Rule* cont'd

d. $$\int \frac{-4x}{(1-2x^2)^2}\,dx = \int \overbrace{(1-2x^2)^{-2}}^{u^{-2}}\overbrace{(-4x)\,dx}^{du} = \frac{\overbrace{(1-2x^2)^{-1}}^{u^{-1}/(-1)}}{-1} + C$$

$$= -\frac{1}{1-2x^2} + C$$

e. $$\int \cos^2 x \sin x\,dx = -\int \overbrace{(\cos x)^2}^{u^2}\overbrace{(-\sin x)\,dx}^{du}$$

$$= -\frac{\overbrace{(\cos x)^3}^{u^3/3}}{3} + C$$

Change of Variables for Definite Integrals

Change of Variables for Definite Integrals

When using *u*-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable *u* rather than to convert the antiderivative back to the variable *x* and evaluate at the original limits.

This change of variables is stated explicitly in the next theorem.

THEOREM 4.15 Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example 8 – *Change of Variables*

Evaluate $\int_0^1 x(x^2 + 1)^3 \, dx.$

Solution:

To evaluate this integral, let $u = x^2 + 1$.

Then, you obtain

$$du = 2x \, dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	**Upper Limit**
When $x = 0,\ u = 0^2 + 1 = 1.$	When $x = 1,\ u = 1^2 + 1 = 2.$

Example 8 – *Solution*

cont'd

Now, you can substitute to obtain

$$\int_0^1 x(x^2+1)^3\,dx = \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,dx$$

Integration limits for x

$$= \frac{1}{2}\int_1^2 u^3\,du$$

Integration limits for u

$$= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left(4-\frac{1}{4}\right) = \frac{15}{8}.$$

Example 8 – *Solution*

cont'd

Notice that you obtain the same result when you rewrite the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable *x* and evaluate the definite integral at the original limits of integration, as shown below.

$$\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 = \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(4 - \frac{1}{4}\right)$$

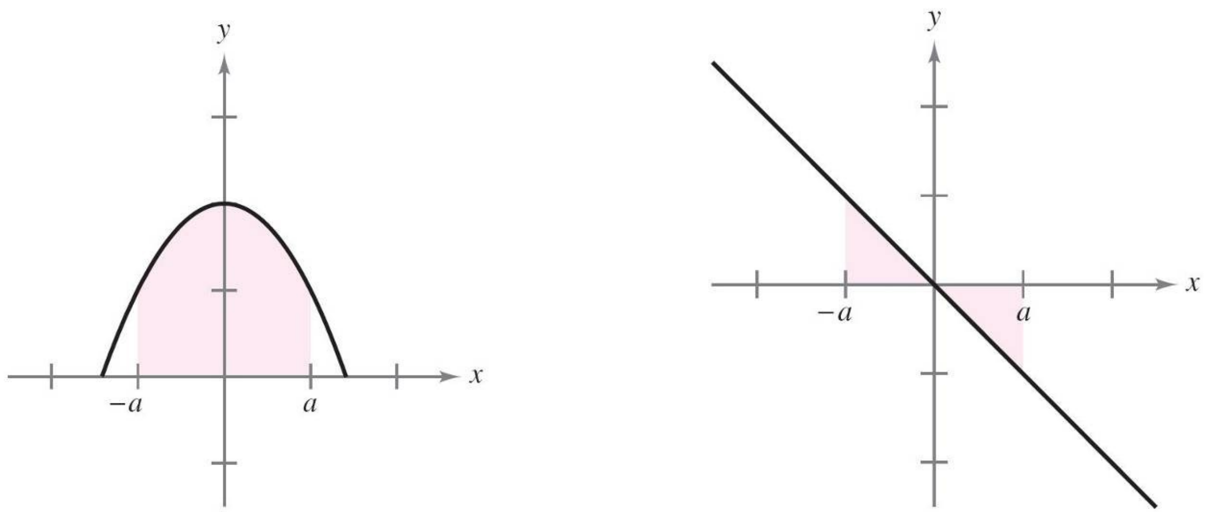
$$= \frac{15}{8}$$

Integration of Even and Odd Functions

Integration of Even and Odd Functions

Even with a change of variables, integration can be difficult.

Occasionally, you can simplify the evaluation of a definite integral over an interval that is symmetric about the *y*-axis or about the origin by recognizing the integrand to be an even or odd function (see Figure 4.40).



Even function   Odd function

Figure 4.40

Integration of Even and Odd Functions

THEOREM 4.16 Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

1. If f is an *even* function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

2. If f is an *odd* function, then $\int_{-a}^{a} f(x)\,dx = 0.$

Example 10 – *Integration of an Odd Function*

Evaluate the definite integral.

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\, dx.$$

Solution:

Letting $f(x) = \sin^3 x \cos x + \sin x \cos x$ produces

$$f(-x) = \sin^3(-x) \cos(-x) + \sin(-x) \cos(-x)$$

$$= -\sin^3 x \cos x - \sin x \cos x$$

$$= -f(x).$$

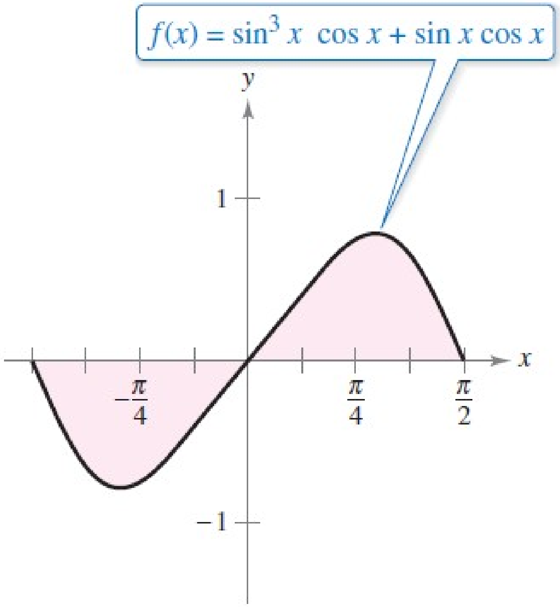
Example 10 – *Solution* cont'd

So, *f* is an odd function, and because *f* is symmetric about the origin over [–π/2, π/2], you can apply Theorem 4.16 to conclude that

$$\int_{-\pi/2}^{\pi/2} (\sin^3 x \cos x + \sin x \cos x)\, dx = 0.$$

From Figure 4.41, you can see that the two regions on either side of the *y*-axis have the same area.

However, because one lies below the *x*-axis and one lies above it, integration produces a cancellation effect.



Because *f* is an odd function,
$\int_{-\pi/2}^{\pi/2} f(x)\, dx = 0.$

Figure 4.41