3 Applications of Differentiation

3.3 Increasing and Decreasing Functions and the First Derivative Test

Objectives

- Determine intervals on which a function is increasing or decreasing.

- Apply the First Derivative Test to find relative extrema of a function.

Increasing and Decreasing Functions

Increasing and Decreasing Functions

You will learn how derivatives can be used to *classify* relative extrema as either relative minima or relative maxima. First, it is important to define increasing and decreasing functions.

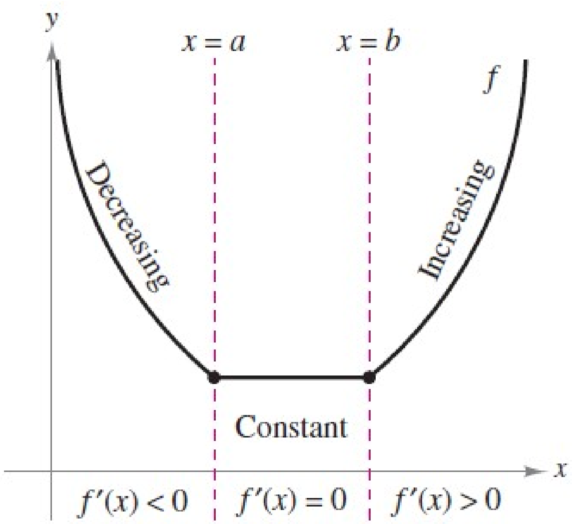
Definitions of Increasing and Decreasing Functions

A function f is **increasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval when, for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

Increasing and Decreasing Functions

A function is increasing when, *as x moves to the right*, its graph moves up, and is decreasing when its graph moves down. For example, the function in Figure 3.15 is decreasing on the interval $(-\infty, a)$, is constant on the interval (a, b), and is increasing on the interval (b, ∞).



The derivative is related to the slope of a function.

Figure 3.15

Increasing and Decreasing Functions

As shown in Theorem 3.5 below, a positive derivative implies that the function is increasing, a negative derivative implies that the function is decreasing, and a zero derivative on an entire interval implies that the function is constant on that interval.

THEOREM 3.5 Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Example 1 – *Intervals on Which f Is Increasing or Decreasing*

Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Solution:

Note that *f* is differentiable on the entire real number line and the derivative of *f* is

$$f(x) = x^3 - \frac{3}{2}x^2 \qquad \text{Write original function.}$$

$$f'(x) = 3x^2 - 3x. \qquad \text{Differentiate.}$$

Example 1 – *Solution*

cont'd

To determine the critical numbers of f, set $f'(x)$ equal to zero.

$3x^2 - 3x = 0$	Set $f'(x)$ equal to 0.
$3(x)(x - 1) = 0$	Factor.
$x = 0, 1$	Critical numbers

Example 1 – *Solution*

cont'd

Because there are no points for which f' does not exist, you can conclude that $x = 0$ and $x = 1$ are the only critical numbers.

The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
Test Value	$x = -1$	$x = \frac{1}{2}$	$x = 2$
Sign of $f'(x)$	$f'(-1) = 6 > 0$	$f'\left(\frac{1}{2}\right) = -\frac{3}{4} < 0$	$f'(2) = 6 > 0$
Conclusion	Increasing	Decreasing	Increasing

Example 1 – *Solution*

cont'd

By Theorem 3.5, *f* is increasing on the intervals $(-\infty, 0)$ and $(1, \infty)$ and decreasing on the interval (0,1), as shown in Figure 3.16.

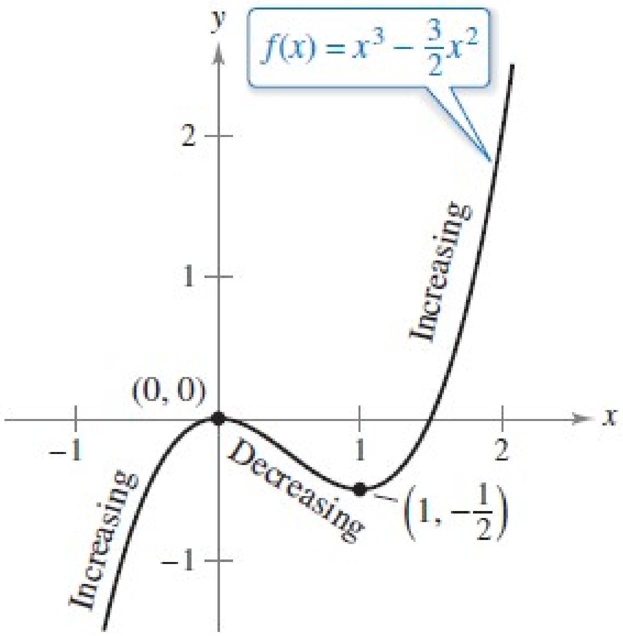


Figure 3.16

Increasing and Decreasing Functions

Example 1 gives you one instance of how to find intervals on which a function is increasing or decreasing. The guidelines below summarize the steps followed in that example.

GUIDELINES FOR FINDING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

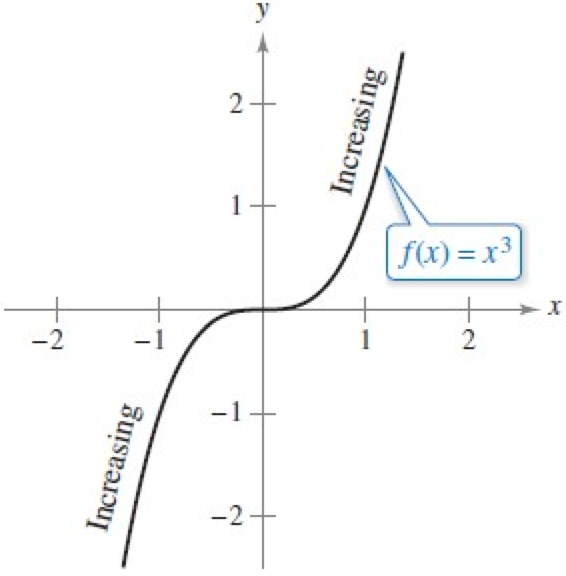
Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

1. Locate the critical numbers of f in (a, b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use Theorem 3.5 to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid when the interval (a, b) is replaced by an interval of the form $(-\infty, b)$, (a, ∞), or $(-\infty, \infty)$.

Increasing and Decreasing Functions

A function is **strictly monotonic** on an interval when it is either increasing on the entire interval or decreasing on the entire interval. For instance, the function $f(x) = x^3$ is strictly monotonic on the entire real number line because it is increasing on the entire real number line, as shown in Figure 3.17(a).

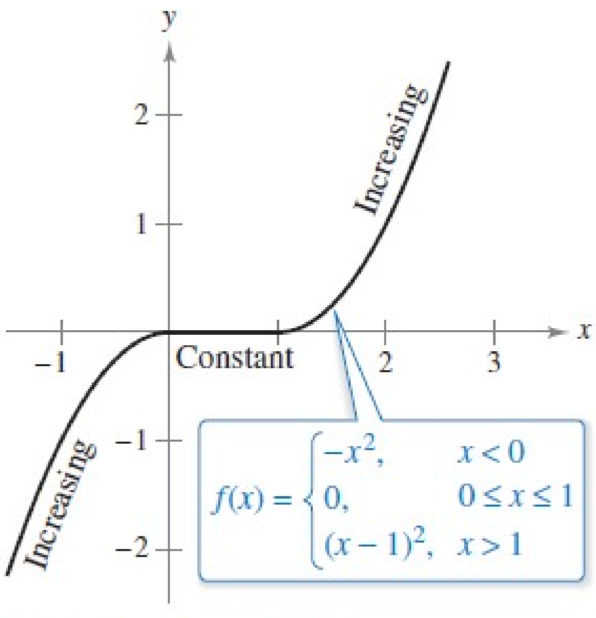


(a) Strictly monotonic function

Figure 3.17

Increasing and Decreasing Functions

The function shown in Figure 3.17(b) is not strictly monotonic on the entire real number line because it is constant on the interval [0, 1].



(b) Not strictly monotonic

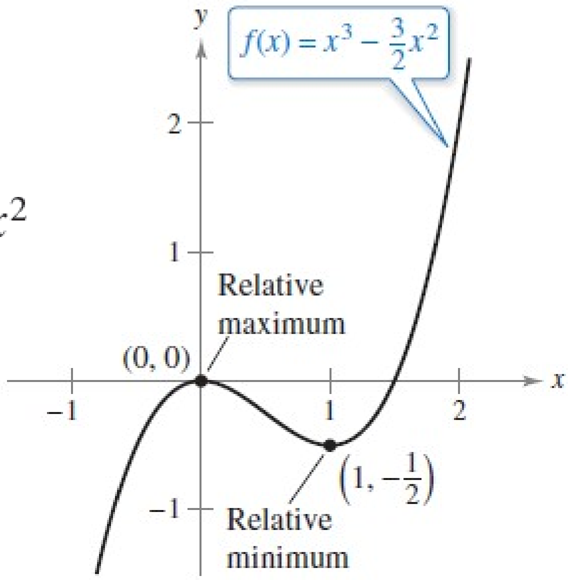
Figure 3.17

The First Derivative Test

The First Derivative Test

After you have determined the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function.

For instance, in Figure 3.18 (from Example 1), the function $f(x) = x^3 - \frac{3}{2}x^2$ has a relative maximum at the point (0, 0) because *f* is increasing immediately to the left of $x = 0$ and decreasing immediately to the right of $x = 0$.



Relative extrema of *f*

Figure 3.18

The First Derivative Test

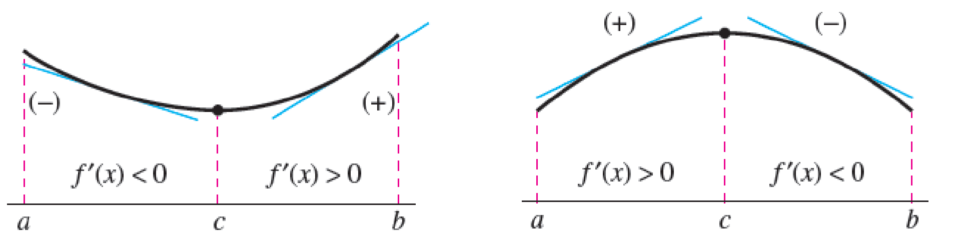
Similarly, *f* has a relative minimum at the point $\left(1, -\frac{1}{2}\right)$ because *f* is decreasing immediately to the left of $x = 1$ and increasing immediately to the right of $x = 1$. The next theorem makes this more explicit.

The First Derivative Test

THEOREM 3.6 The First Derivative Test

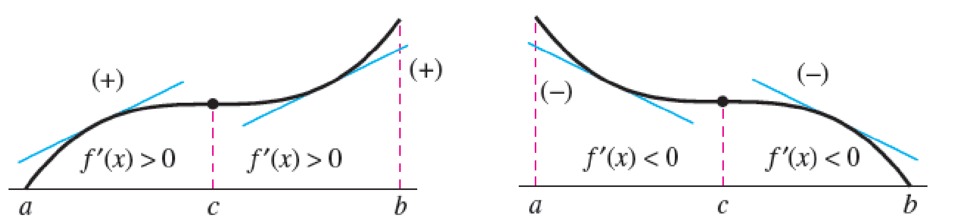
Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum nor a relative maximum.



Relative minimum

Relative maximum



Neither relative minimum nor relative maximum

Example 2 – *Applying the First Derivative Test*

Find the relative extrema of $f(x) = \frac{1}{2}x - \sin x$ in the interval (0, 2*π*).

Solution:

Note that *f* is continuous on the interval (0, 2*π*). The derivative of *f* is

$$f'(x) = \frac{1}{2} - \cos x.$$

To determine the critical numbers of *f* in this interval, set $f'(x)$ equal to 0.

Example 2 – *Solution*

cont'd

$$\frac{1}{2} - \cos x = 0$$ Set $f'(x)$ equal to 0.

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$ Critical numbers

Because there are no points for which f' does not exist, you can conclude that $x = \pi/3$ and $x = 5\pi/3$ are the only critical numbers.

Example 2 – *Solution*

cont'd

The table summarizes the testing of the three intervals determined by these two critical numbers.

Interval	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test Value	$x = \frac{\pi}{4}$	$x = \pi$	$x = \frac{7\pi}{4}$
Sign of $f'(x)$	$f'\left(\frac{\pi}{4}\right) < 0$	$f'(\pi) > 0$	$f'\left(\frac{7\pi}{4}\right) < 0$
Conclusion	Decreasing	Increasing	Decreasing

Example 2 – *Solution*

cont'd

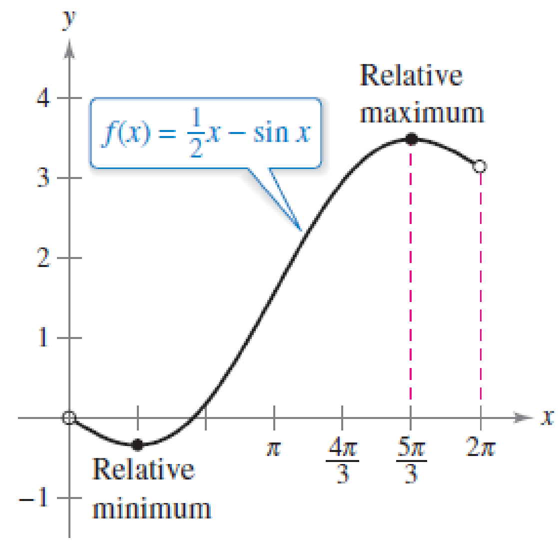
By applying the First Derivative Test, you can conclude that *f* has a relative minimum at the point where

$$x = \frac{\pi}{3}$$

and a relative maximum at the point where

$$x = \frac{5\pi}{3},$$

as shown in Figure 3.19.



A relative minimum occurs where *f* changes from decreasing to increasing, and a relative maximum occurs where *f* changes from increasing to decreasing.

Figure 3.19