5 Logarithmic, Exponential, and Other Transcendental Functions

5.5 Bases Other Than *e* and Applications

Objectives

- Define exponential functions that have bases other than *e*.
- Differentiate and integrate exponential functions that have bases other than *e*.
- Use exponential functions to model compound interest and exponential growth.

Bases Other than *e*

Bases Other than e

The **base** of the natural exponential function is *e*. This "natural" base can be used to assign a meaning to a general base *a*.

Definition of Exponential Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base *a*** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

Bases Other than *e*

Exponential functions obey the usual laws of exponents. For instance, here are some familiar properties.

1. $a^0 = 1$

2. $a^x a^y = a^{x+y}$

3. $\dfrac{a^x}{a^y} = a^{x-y}$

4. $(a^x)^y = a^{xy}$

When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model. (*Half-life* is the number of years required for half of the atoms in a sample of radioactive material to decay.)

Example 1 – *Radioactive Half-Life Model*

The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbon-14. How much will be present in 10,000 years?

Solution:

Let $t = 0$ represent the present time and let y represent the amount (in grams) of carbon-14 in the sample.

Using a base of $\frac{1}{2}$, you can model y by the equation

$$y = \left(\frac{1}{2}\right)^{t/5715}.$$

Notice that when $t = 5715$, the amount is reduced to half of the original amount

$$y = \left(\frac{1}{2}\right)^{5715/5715} = \frac{1}{2} \text{ gram}$$

Example 1 – *Solution*

cont'd

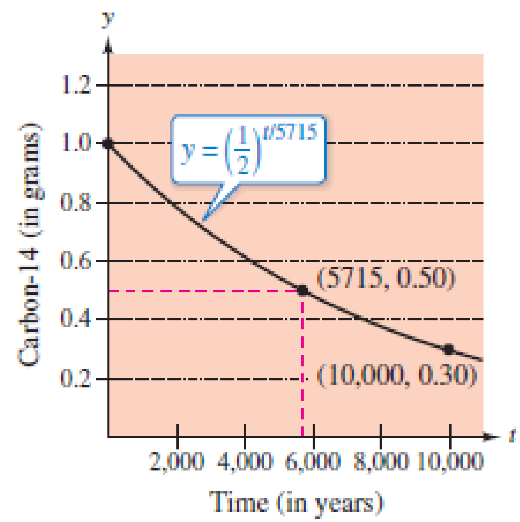
When t = 11,430, the amount is reduced to a quarter of the original amount, and so on.

To find the amount of carbon-14 after 10,000 years, substitute 10,000 for t.

$$y = \left(\frac{1}{2}\right)^{10{,}000/5715}$$

$\approx$ 0.30 gram

The graph of y is shown at the right.



The half-life of carbon-14 is about 5715 years.

Bases Other than *e*

Logarithmic functions to bases other than *e* can be defined in much the same way as exponential functions to other bases are defined.

Definition of Logarithmic Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base *a*** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Bases Other than e

Logarithmic functions to the base a have properties similar to those of the natural logarithmic function.

1. $\log_a 1 = 0$ — Log of 1

2. $\log_a xy = \log_a x + \log_a y$ — Log of a product

3. $\log_a x^n = n \log_a x$ — Log of a power

4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ — Log of a quotient

From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Bases Other than e

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$
2. $a^{\log_a x} = x$, for $x > 0$
3. $\log_a a^x = x$, for all x

The logarithmic function to the base 10 is called the **common logarithmic function**. So, for common logarithms,

$$y = 10^x \quad \text{if and only if} \quad x = \log_{10} y. \qquad \text{Property of Inverse Functions}$$

Example 2 – *Bases Other Than e*

Solve for x in each equation.

a. $3^x = \frac{1}{81}$ **b.** $\log_2 x = -4$

Solution:

a. To solve this equation, you can apply the logarithmic function to the base 3 to each side of the equation.

$$3^x = \frac{1}{81}$$

$$\log_3 3^x = \log_3 \frac{1}{81}$$

$$x = \log_3 3^{-4}$$

$$x = -4$$

Example 2 – *Solution*

cont'd

b. To solve this equation, you can apply the exponential function to the base 2 to each side of the equation.

$$\log_2 x = -4$$

$$2^{\log_2 x} = 2^{-4}$$

$$x = \frac{1}{2^4}$$

$$x = \frac{1}{16}$$

Differentiation and Integration

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options:

(1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions,

(2) use logarithmic differentiation, or

(3) use the differentiation rules for bases other than e given in the next theorem.

Differentiation and Integration

THEOREM 5.13 Derivatives for Bases Other than e

Let a be a positive real number $(a \neq 1)$, and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$

2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u \dfrac{du}{dx}$

3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$

4. $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u} \dfrac{du}{dx}$

Example 3 – *Differentiating Functions to Other Bases*

Find the derivative of each function.

a. $y = 2^x$

b. $y = 2^{3x}$

c. $y = \log_{10} \cos x$

d. $y = \log_3 \dfrac{\sqrt{x}}{x + 5}$

Example 3 – *Solution*

a. $y' = \dfrac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \dfrac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3 \ln 2)2^{3x}$

c. $y' = \dfrac{d}{dx}[\log_{10} \cos x] = \dfrac{-\sin x}{(\ln 10)\cos x} = -\dfrac{1}{\ln 10} \tan x$

d. Before differentiating, rewrite the function using logarithmic properties.

$$y = \log_3 \frac{\sqrt{x}}{x + 5} = \frac{1}{2}\log_3 x - \log_3(x + 5)$$

Example 3 – *Solution*

cont'd

Next, apply Theorem 5.13 to differentiate the function.

$$y' = \frac{d}{dx}\left[\frac{1}{2}\log_3 x - \log_3(x+5)\right]$$

$$= \frac{1}{2(\ln 3)x} - \frac{1}{(\ln 3)(x+5)}$$

$$= \frac{5-x}{2(\ln 3)x(x+5)}$$

Differentiation and Integration

Occasionally, an integrand involves an exponential function to a base other than *e*. When this occurs, there are two options:

(1) convert to base *e* using the formula $a^x = e^{(\ln a)x}$ and then integrate, or

(2) integrate directly, using the integration formula

$$\int a^x \, dx = \left(\frac{1}{\ln a}\right) a^x + C$$

which follows from Theorem 5.13.

Example 4 – *Integrating an Exponential Function to Another Base*

Find $\int 2^x \, dx$.

Solution:

$$\int 2^x \, dx = \frac{1}{\ln 2} 2^x + C$$

Differentiation and Integration

THEOREM 5.14 The Power Rule for Real Exponents

Let n be any real number, and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[x^n] = nx^{n-1}$

2. $\dfrac{d}{dx}[u^n] = nu^{n-1}\dfrac{du}{dx}$

Example 5 – *Comparing Variables and Constants*

a. $\frac{d}{dx}[e^e] = 0$ Constant Rule

b. $\frac{d}{dx}[e^x] = e^x$ Exponential Rule

c. $\frac{d}{dx}[x^e] = ex^{e-1}$ Power Rule

Example 5 – *Comparing Variables and Constants* cont'd

d. $y = x^x$ Use logarithmic differentiation.

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1)$$

$$\frac{y'}{y} = 1 + \ln x$$

$$y' = y(1 + \ln x)$$

$$y' = x^x(1 + \ln x)$$

Applications of Exponential Functions

Applications of Exponential Functions

An amount of P dollars is deposited in an account at an annual interest rate r (in decimal form). What is the balance in the account at the end of 1 year? The answer depends on the number of times n the interest is compounded according to the formula

$$A = P\left(1 + \frac{r}{n}\right)^{n}.$$

Applications of Exponential Functions

For instance, the result for a deposit of $1000 at 8% interest compounded *n* times a year is shown in the table.

n	A
1	$1080.00
2	$1081.60
4	$1082.43
12	$1083.00
365	$1083.28

Applications of Exponential Functions

As *n* increases, the balance *A* approaches a limit. To develop this limit, use the following theorem.

THEOREM 5.15 A Limit Involving *e*

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to \infty} \left(\frac{x+1}{x}\right)^x = e$$

Applications of Exponential Functions

To test the reasonableness of this theorem, try evaluating

$$\left(\frac{x+1}{x}\right)^x$$

for several values of *x*, as shown in the table at the right.

x	$\left(\frac{x+1}{x}\right)^x$
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

Applications of Exponential Functions

Given Theorem 5.15, take another look at the formula for the balance *A* in an account in which the interest is compounded *n* times per year. By taking the limit as *n* approaches infinity, you obtain

$$
\begin{aligned}
A &= \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{n} && \text{Take limit as } n \to \infty. \\
&= P \lim_{n\to\infty} \left[\left(1 + \frac{1}{n/r}\right)^{n/r}\right]^{r} && \text{Rewrite.} \\
&= P\left[\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x}\right]^{r} && \text{Let } x = n/r. \text{ Then } x \to \infty \text{ as } n \to \infty. \\
&= Pe^{r}. && \text{Apply Theorem 5.15.}
\end{aligned}
$$

Applications of Exponential Functions

This limit produces the balance after 1 year of **continuous compounding.** So, for a deposit of \$1000 at 8% interest compounded continuously, the balance at the end of 1 year would be

$A = 1000e^{0.08}$

$\approx \$1083.29.$

Applications of Exponential Functions

SUMMARY OF COMPOUND INTEREST FORMULAS

Let P = amount of deposit, t = number of years, A = balance after t years, r = annual interest rate (in decimal form), and n = number of compoundings per year.

1. Compounded n times per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. Compounded continuously: $A = Pe^{rt}$

Example 6 – *Continuous, Quarterly, and Monthly Compounding*

A deposit of \$2500 is made in an account that pays an annual interest rate of 5%. Find the balance in the account at the end of 5 years if the interest is compounded

(a) quarterly, (b) monthly, and (c) continuously.

Solution:

a. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compounded quarterly

$= 2500\left(1 + \frac{0.05}{4}\right)^{4(5)}$

$= 2500(1.0125)^{20}$

$\approx \$3205.09$

Example 6 – *Solution*

cont'd

b. $A = P\left(1 + \frac{r}{n}\right)^{nt}$ Compounded monthly

$$= 2500\left(1 + \frac{0.05}{12}\right)^{12(5)}$$

$$\approx 2500(1.0041667)^{60}$$

$$\approx \$3208.40$$

c. $A = Pe^{rt}$ Compounded continuously

$$= 2500[e^{0.05(5)}]$$

$$= 2500e^{0.25}$$

$$\approx \$3210.06$$