3 Applications of Differentiation

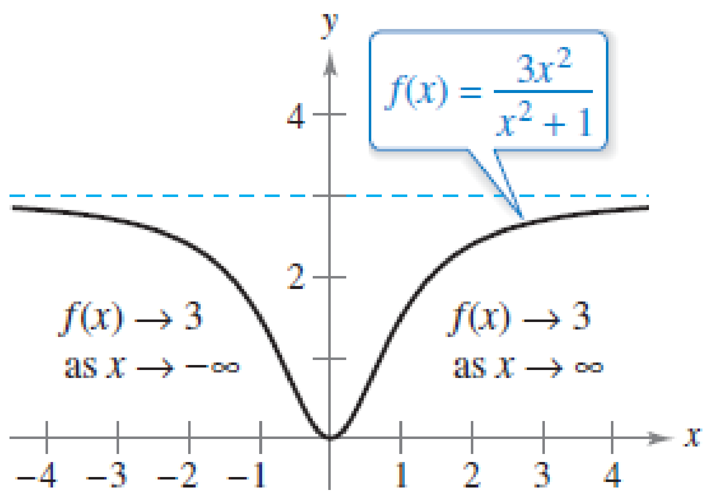
3.5 Limits at Infinity

Objectives

- Determine (finite) limits at infinity.
- Determine the horizontal asymptotes, if any, of the graph of a function.
- Determine infinite limits at infinity.

Limits at Infinity

This section discusses the "end behavior" of a function on an *infinite* interval. Consider the graph of $f(x) = \dfrac{3x^2}{x^2 + 1}$ as shown in Figure 3.32.



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.

Figure 3.32

Limits at Infinity

Graphically, you can see that $f(x)$ appears to approach 3 as x increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.

x decreases without bound. $\quad$ x increases without bound.

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\rightarrow \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.9703	1.5	0	1.5	2.9703	2.9997	$\rightarrow 3$

$f(x)$ approaches 3. $\quad$ $f(x)$ approaches 3.

Limits at Infinity

The table suggests that *f*(*x*) approaches 3 as *x* increases without bound $(x \to \infty)$. Similarly, *f*(*x*) approaches 3 as *x* decreases without bound $(x \to -\infty)$.

These **limits at infinity** are denoted by

$$\lim_{x \to -\infty} f(x) = 3$$ Limit at negative infinity

and

$$\lim_{x \to \infty} f(x) = 3.$$ Limit at positive infinity

Limits at Infinity

To say that a statement is true as x increases *without bound* means that for some (large) real number M, the statement is true for *all* x in the interval $\{x\colon\ x > M\}$.

The next definition uses this concept.

Definition of Limits at Infinity

Let L be a real number.

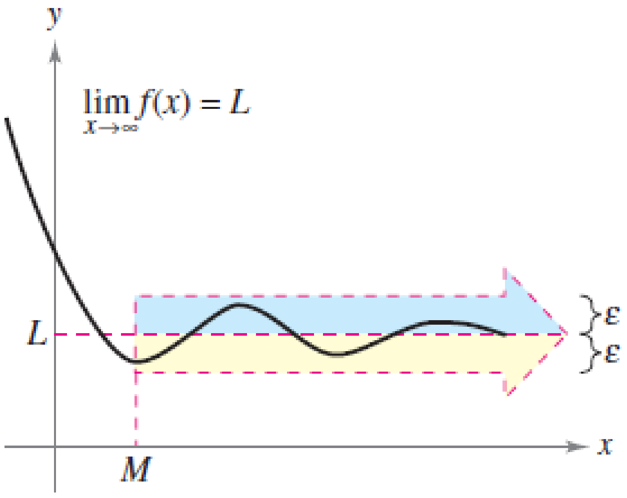
1. The statement $\lim_{x \to \infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.

2. The statement $\lim_{x \to -\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Limits at Infinity

The definition of a limit at infinity is shown in Figure 3.33. In this figure, note that for a given positive number ε, there exists a positive number M such that, for $x > M$, the graph of f will lie between the horizontal lines

$y = L + \varepsilon$ and $y = L - \varepsilon$.



$f(x)$ is within ε units of L as $x \to \infty$.

Figure 3.33

Horizontal Asymptotes

Horizontal Asymptotes

In Figure 3.33, the graph of *f* approaches the line $y = L$ as x increases without bound. The line $y = L$ is called a **horizontal asymptote** of the graph of *f*.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L.$$

Note that from this definition, it follows that the graph of a *function* of x can have at most two horizontal asymptotes—one to the right and one to the left.

Horizontal Asymptotes

Limits at infinity have many of the same properties of limits discussed earlier. For example, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then

$$\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

and

$$\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right].$$

Similar properties hold for limits at $-\infty$.

Horizontal Asymptotes

When evaluating limits at infinity, the next theorem is helpful.

THEOREM 3.10 Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

Example 1 – *Finding a Limit at Infinity*

Find the limit: $\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right)$.

Solution:

Using Theorem 3.10, you can write

$$\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right) = \lim_{x \to \infty} 5 - \lim_{x \to \infty} \frac{2}{x^2} \qquad \text{Property of limits}$$

$$= 5 - 0$$

$$= 5.$$

Example 1 – *Solution*

cont'd

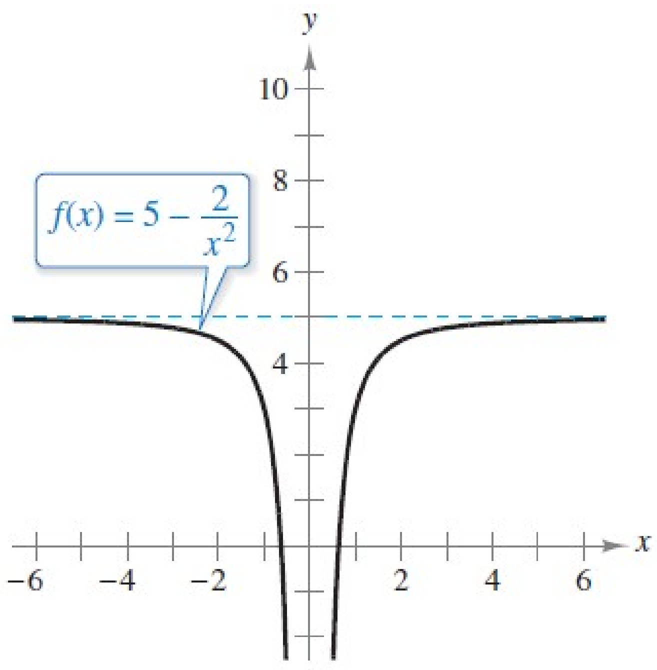
So, the line $y = 5$ is a horizontal asymptote to the right. By finding the limit

$$\lim_{x \to -\infty} \left(5 - \frac{2}{x^2}\right)$$

Limit as $x \to -\infty$

you can see that $y = 5$ is also a horizontal asymptote to the left.

The graph of the function $f(x) = 5 - (2/x^2)$ is shown in Figure 3.34.



$y = 5$ is a horizontal asymptote.

Figure 3.34

Example 2 – *Finding a Limit at Infinity*

Find the limit: $\lim_{x \to \infty} \frac{2x - 1}{x + 1}$.

Solution:

Note that both the numerator and the denominator approach infinity as *x* approaches infinity.

$$\lim_{x \to \infty} (2x - 1) \to \infty$$

$$\lim_{x \to \infty} \frac{2x - 1}{x + 1}$$

$$\lim_{x \to \infty} (x + 1) \to \infty$$

Example 2 – *Solution*

cont'd

This results in ∞/∞, an **indeterminate form.** To resolve this problem, you can divide both the numerator and the denominator by *x.* After dividing, the limit may be evaluated as shown.

$$\lim_{x\to\infty} \frac{2x-1}{x+1} = \lim_{x\to\infty} \frac{\dfrac{2x-1}{x}}{\dfrac{x+1}{x}}$$ Divide numerator and denominator by x.

$$= \lim_{x\to\infty} \frac{2-\dfrac{1}{x}}{1+\dfrac{1}{x}}$$ Simplify.

$$= \frac{\displaystyle\lim_{x\to\infty} 2 - \lim_{x\to\infty} \frac{1}{x}}{\displaystyle\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x}}$$ Take limits of numerator and denominator.

Example 2 – *Solution*

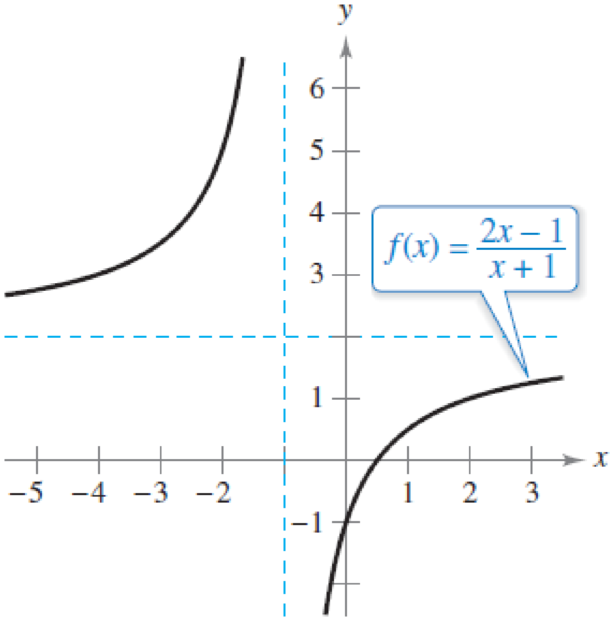
cont'd

$$= \frac{2 - 0}{1 + 0}$$ Apply Theorem 3.10.

$$= 2$$

So, the line $y = 2$ is a horizontal asymptote to the right.
By taking the limit as $x \to -\infty$, you can see that $y = 2$ is also a horizontal asymptote to the left.

The graph of the function is shown in Figure 3.35.



$y = 2$ is a horizontal asymptote.

Figure 3.35

Horizontal Asymptotes

GUIDELINES FOR FINDING LIMITS AT $\pm\infty$ OF RATIONAL FUNCTIONS

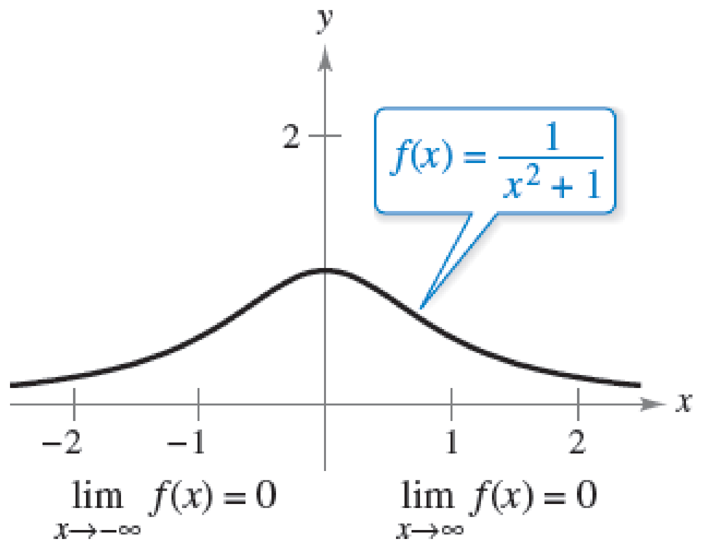
1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Horizontal Asymptotes

The guidelines for finding limits at infinity of rational functions seem reasonable when you consider that for large values of *x*, the highest-power term of the rational function is the most "influential" in determining the limit. For instance,

$$\lim_{x \to \infty} \frac{1}{x^2 + 1}$$

is 0 because the denominator overpowers the numerator as *x* increases or decreases without bound, as shown in Figure 3.37.



f has a horizontal asymptote at $y = 0$.

Figure 3.37

Horizontal Asymptotes

The function shown in Figure 3.37 is a special case of a type of curve studied by the Italian mathematician Maria Gaetana Agnesi.

The general form of this function is

$$f(x) = \frac{8a^3}{x^2 + 4a^2} \qquad \text{Witch of Agnesi}$$

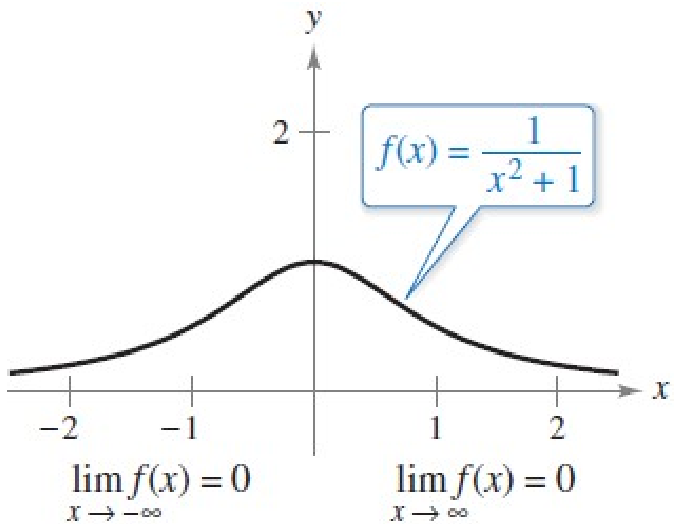
The curve has come to be known as the Witch of Agnesi.

Horizontal Asymptotes

In Figure 3.37, you can see that the function

$$f(x) = \frac{1}{x^2 + 1}$$

approaches the same horizontal asymptote to the right and to the left.



f has a horizontal asymptote at $y = 0$.

Figure 3.37

This is always true of rational functions. Functions that are not rational, however, may approach different horizontal asymptotes to the right and to the left.

Example 4 – *A Function with Two Horizontal Asymptotes*

Find each limit.

a. $\lim\limits_{x \to \infty} \dfrac{3x - 2}{\sqrt{2x^2 + 1}}$

b. $\lim\limits_{x \to -\infty} \dfrac{3x - 2}{\sqrt{2x^2 + 1}}$

Example 4(a) – *Solution*

For $x > 0$, you can write $x = \sqrt{x^2}$.

So, dividing both the numerator and the denominator by x produces

$$\frac{3x - 2}{\sqrt{2x^2 + 1}} = \frac{\dfrac{3x - 2}{x}}{\dfrac{\sqrt{2x^2 + 1}}{\sqrt{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{\dfrac{2x^2 + 1}{x^2}}} = \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}}$$

and you can take the limit as follows.

$$\lim_{x\to\infty} \frac{3x - 2}{\sqrt{2x^2 + 1}} = \lim_{x\to\infty} \frac{3 - \dfrac{2}{x}}{\sqrt{2 + \dfrac{1}{x^2}}} = \frac{3 - 0}{\sqrt{2 + 0}} = \frac{3}{\sqrt{2}}$$

Example 4(b) – *Solution*

cont'd

For $x < 0$, you can write $x = -\sqrt{x^2}$.

So, dividing both the numerator and the denominator by x produces

$$\frac{3x-2}{\sqrt{2x^2+1}} = \frac{\dfrac{3x-2}{x}}{\dfrac{\sqrt{2x^2+1}}{-\sqrt{x^2}}} = \frac{3-\dfrac{2}{x}}{-\sqrt{\dfrac{2x^2+1}{x^2}}} = \frac{3-\dfrac{2}{x}}{-\sqrt{2+\dfrac{1}{x^2}}}$$

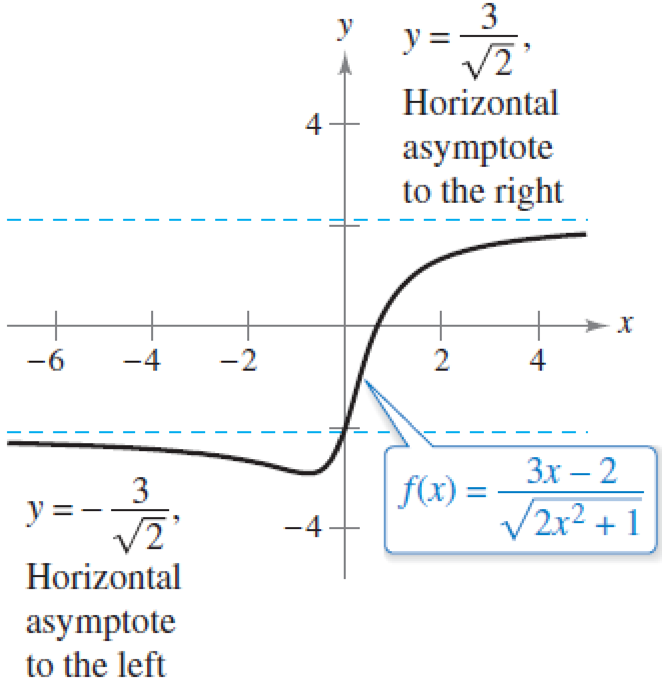
and you can take the limit as follows.

$$\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}} = \lim_{x\to-\infty}\frac{3-\dfrac{2}{x}}{-\sqrt{2+\dfrac{1}{x^2}}} = \frac{3-0}{-\sqrt{2+0}} = -\frac{3}{\sqrt{2}}$$

Example 4(b) – *Solution*

cont'd

The graph of $f(x) = (3x - 2)/\sqrt{2x^2 + 1}$ is shown in Figure 3.38.



Functions that are not rational may have different right and left horizontal asymptotes.

Figure 3.38

Infinite Limits at Infinity

Infinite Limits at Infinity

Many functions do not approach a finite limit as *x* increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity. The next definition is used to describe the behavior of polynomial and other functions at infinity.

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim_{x \to \infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x \to \infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Similar definitions can be given for the statements

$$\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} f(x) = -\infty.$$

Example 7 – *Finding Infinite Limits at Infinity*

Find each limit.

a. $\lim_{x \to \infty} x^3$ **b.** $\lim_{x \to -\infty} x^3$

Solution:

a. As *x* increases without bound, x^3 also increases without bound. So, you can write

$$\lim_{x \to \infty} x^3 = \infty.$$

b. As *x* decreases without bound, x^3 also decreases without bound. So, you can w $\lim_{x \to -\infty} x^3 = -\infty.$

Example 7 – *Solution*

cont'd

The graph of $f(x) = x^3$ in Figure 3.42 illustrates these two results. These results agree with the Leading Coefficient Test for polynomial functions.

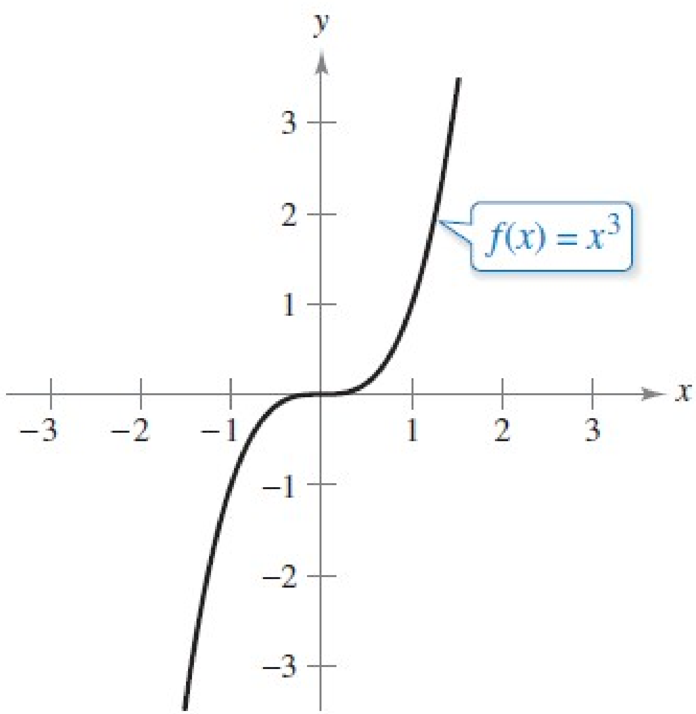


Figure 3.42