3 Applications of Differentiation

3.4 Concavity and the Second Derivative Test

Objectives

- Determine intervals on which a function is concave upward or concave downward.
- Find any points of inflection of the graph of a function.
- Apply the Second Derivative Test to find relative extrema of a function.

Concavity

Concavity

You have seen that locating the intervals on which a function f increases or decreases helps to describe its graph. In this section, you will see how locating the intervals on which f' increases or decreases can be used to determine where the graph of f is *curving upward* or *curving downward*.

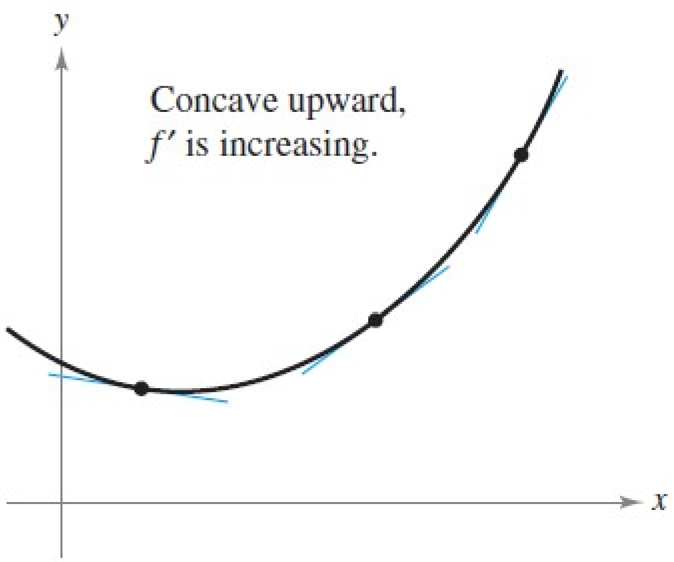
Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.

Concavity

The following graphical interpretation of concavity is useful.

1. Let *f* be differentiable on an open interval *I.* If the graph of *f* is concave *upward* on *I*, then the graph of *f* lies *above* all of its tangent lines on *I*. [See Figure 3.23(a).]

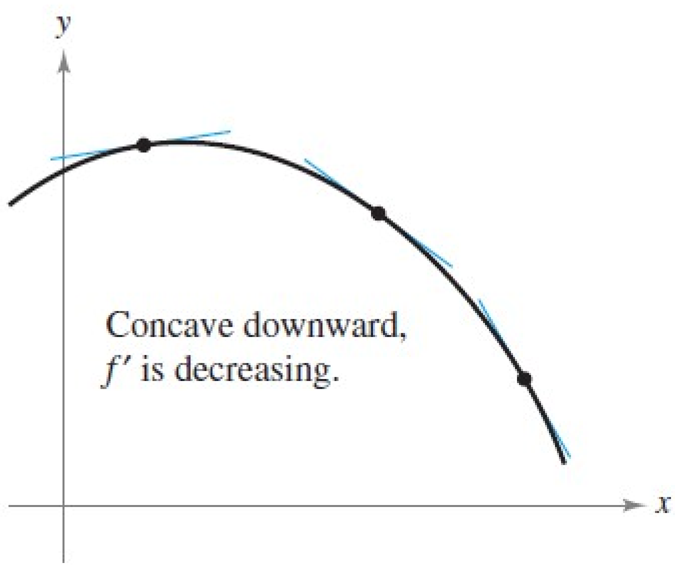


(**a**) The graph of *f* lies above its tangent lines.

Figure 3.23

Concavity

2. Let f be differentiable on an open interval I. If the graph of f is concave *downward* on I, then the graph of f lies *below* all of its tangent lines on I. [See Figure 3.23(b).]



(b) The graph of f lies below its tangent lines.

Figure 3.23

Concavity

To find the open intervals on which the graph of a function f is concave upward or concave downward, you need to find the intervals on which f' is increasing or decreasing.

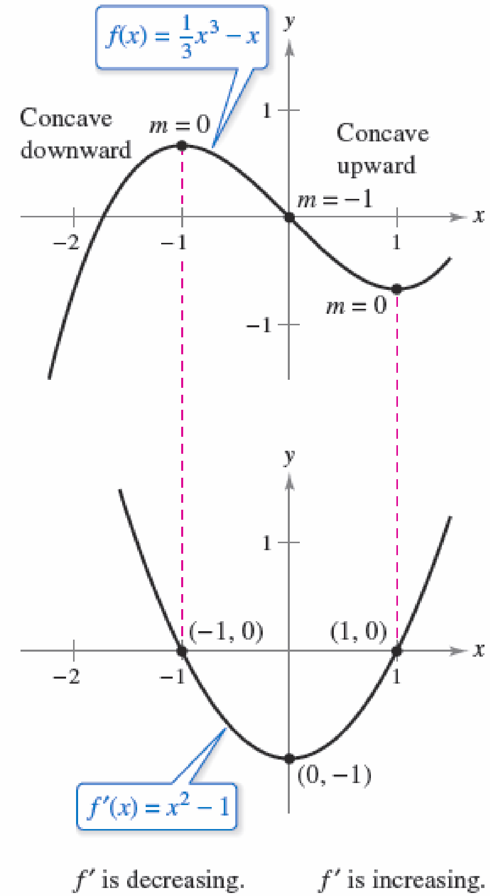
For instance, the graph of

$$f(x) = \frac{1}{3}x^3 - x$$

is concave downward on the open interval $(-\infty, 0)$ because

$$f'(x) = x^2 - 1$$

is decreasing there. (See Figure 3.24)



The concavity of f is related to the slope of the derivative.

Figure 3.24

Concavity

Similarly, the graph of *f* is concave upward on the interval $(0, \infty)$ because *f*′ is increasing on $(0, \infty)$.

The next theorem shows how to use the *second* derivative of a function *f* to determine intervals on which the graph of *f* is concave upward or concave downward.

THEOREM 3.7 Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Concavity

To apply Theorem 3.7, locate the x-values at which $f''(x) = 0$ or $f''(x)$ does not exist. Use these x-values to determine test intervals. Finally, test the sign of $f''(x)$ in each of the test intervals.

Example 1 – *Determining Concavity*

Determine the open intervals on which the graph of

$$f(x) = \frac{6}{x^2 + 3}$$

is concave upward or concave downward.

Solution:

Begin by observing that *f* is continuous on the entire real number line.

Next, find the second derivative of *f*.

$$f(x) = 6(x^2 + 3)^{-1}$$ Rewrite original function.

Example 1 – *Solution*

cont'd

$$f'(x) = (-6)(x^2 + 3)^{-2}(2x) \qquad \text{Differentiate.}$$

$$= \frac{-12x}{(x^2 + 3)^2} \qquad \text{First derivative}$$

$$f''(x) = \frac{(x^2 + 3)^2(-12) - (-12x)(2)(x^2 + 3)(2x)}{(x^2 + 3)^4} \qquad \text{Differentiate.}$$

$$= \frac{36(x^2 - 1)}{(x^2 + 3)^3} \qquad \text{Second derivative}$$

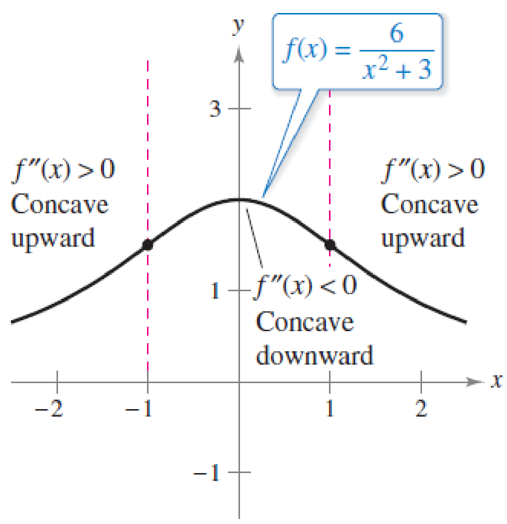
Because $f''(x) = 0$ when $x = \pm 1$ and f'' is defined on the entire real number line, you should test f'' in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

Example 1 – *Solution*

cont'd

The results are shown in the table and in Figure 3.25.

Interval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Test Value	$x = -2$	$x = 0$	$x = 2$
Sign of $f''(x)$	$f''(-2) > 0$	$f''(0) < 0$	$f''(2) > 0$
Conclusion	Concave upward	Concave downward	Concave upward



From the sign of f'', you can determine the concavity of the graph of f.

Figure 3.25

Concavity

cont'd

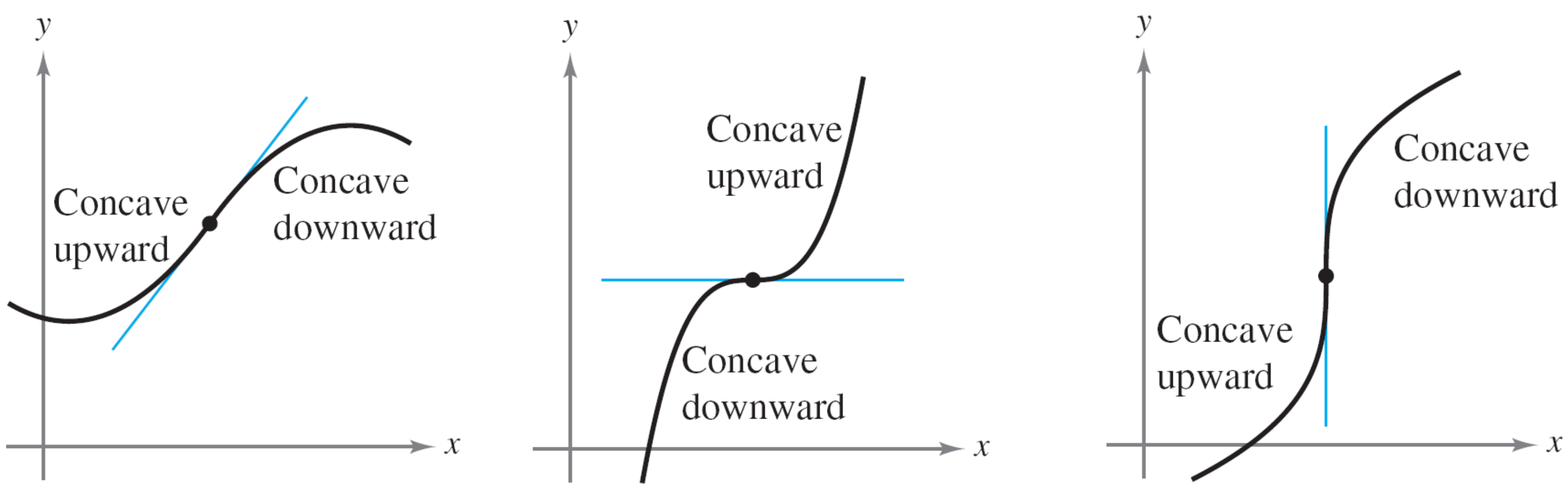
The function given in Example 1 is continuous on the entire real number line.

When there are x-values at which the function is not continuous, these values should be used, along with the points at which $f''(x) = 0$ or $f''(x)$ does not exist, to form the test intervals.

Points of Inflection

Points of Inflection

If the tangent line to the graph exists at such a point where the concavity changes, then that point is a **point of inflection.** Three types of points of inflection are shown in Figure 3.27.



The concavity of *f* changes at a point of inflection. Note that the graph crosses its tangent line at a point of inflection.

Figure 3.27

Points of Inflection

Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph of f has a tangent line at the point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at the point.

To locate *possible* points of inflection, you can determine the values of x for which $f''(x) = 0$ or $f''(x)$ does not exist. This is similar to the procedure for locating relative extrema of f.

THEOREM 3.8 Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Example 3 – *Finding Points of Inflection*

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$.

Solution:

Differentiating twice produces the following.

$$f(x) = x^4 - 4x^3 \qquad \text{Write original function.}$$

$$f'(x) = 4x^3 - 12x^2 \qquad \text{Find first derivative.}$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2) \qquad \text{Find second derivative.}$$

Example 3 – *Solution*

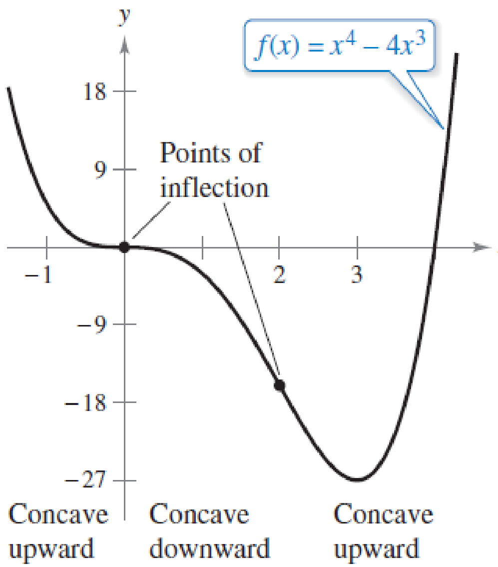
cont'd

Setting $f''(x) = 0$, you can determine that the possible points of inflection occur at $x = 0$ and $x = 2$.

By testing the intervals determined by these x-values, you can conclude that they both yield points of inflection.

A summary of this testing is shown in the table, and the graph of f is shown in Figure 3.28.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Test Value	$x = -1$	$x = 1$	$x = 3$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(1) < 0$	$f''(3) > 0$
Conclusion	Concave upward	Concave downward	Concave upward



Points of inflection can occur where $f''(x) = 0$ or f'' does not exist.

Figure 3.28

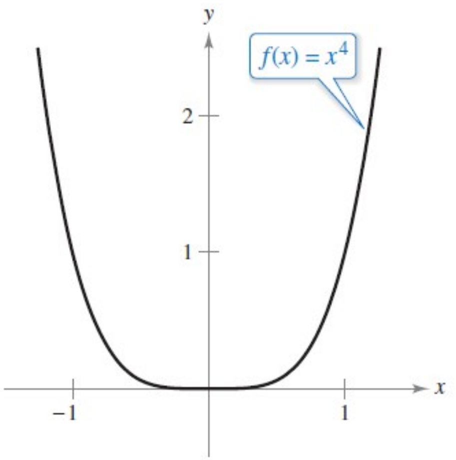
Points of Inflection

The converse of Theorem 3.8 is not generally true. That is, it is possible for the second derivative to be 0 at a point that is *not* a point of inflection.

For instance, the graph of $f(x) = x^4$ is shown in Figure 3.29.

The second derivative is 0 when $x = 0$, but the point (0,0) is not a point of inflection because the graph of f is concave upward on the intervals

$$-\infty < x < 0 \text{ and } 0 < x < \infty.$$



$f''(x) = 0$, but $(0, 0)$ is not a point of inflection.

Figure 3.29

The Second Derivative Test

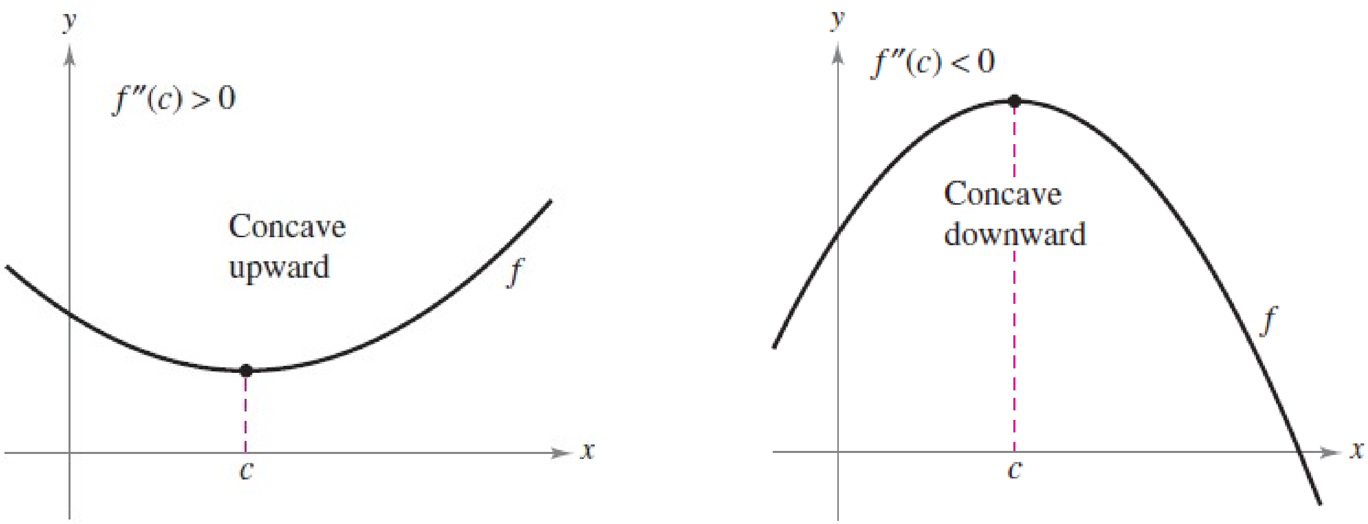
The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima.

The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, then $f(c)$ must be a relative minimum of f.

The Second Derivative Test

Similarly, if the graph of a function *f* is concave downward on an open interval containing *c*, and $f'(c) = 0$, then $f(c)$ must be a relative maximum of *f*. (See Figure 3.30.)



If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a relative minimum.

If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a relative maximum.

Figure 3.30

The Second Derivative Test

THEOREM 3.9 Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$.

If $f''(c) = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Example 4 – *Using the Second Derivative Test*

Find the relative extrema of $f(x) = -3x^5 + 5x^3$.

Solution:

Begin by finding the first derivative of *f*.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2)$$

From this derivative, you can see that $x = -1$, 0, and 1 are the only critical numbers of *f*.

By finding the second derivative

$$f''(x) = -60x^3 + 30x = 30x(1 - 2x^2)$$

you can apply the Second Derivative Test.

Example 4 – *Solution*

cont'd

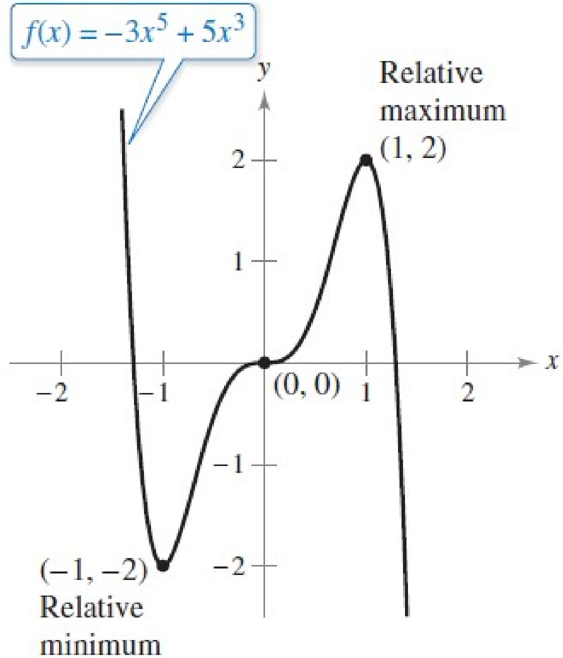
Point	$(-1, -2)$	$(0, 0)$	$(1, 2)$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(0) = 0$	$f''(1) < 0$
Conclusion	Relative minimum	Test fails	Relative maximum

Because the Second Derivative Test fails at (0, 0), you can use the First Derivative Test and observe that *f* increases to the left and right of $x = 0$.

Example 4 – *Solution*

cont'd

So, (0, 0) is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of *f* is shown in Figure 3.31.



(0, 0) is neither a relative minimum nor a relative maximum.

Figure 3.31