3 Applications of Differentiation





3.9 Differentials

Objectives

- Understand the concept of a tangent line approximation.
- Compare the value of the differential, dy, with the actual change in y, Δy.
- Estimate a propagated error using a differential.
- Find the differential of a function using differentiation formulas.

Tangent Line Approximations

Tangent Line Approximations

Consider a function *f* that is differentiable at *c*. The equation for the tangent line at the point (*c*, *f*(*c*)) is

$$y - f(c) = f'(c)(x - c)$$

$$y = f(c) + f'(c)(x - c)$$

and is called the **tangent line approximation** (or **linear approximation) of *f* at *c***.

Because *c* is a constant, *y* is a linear function of *x*.

Tangent Line Approximations

Moreover, by restricting the values of x to those sufficiently close to c, the values of y can be used as approximations (to any desired degree of accuracy) of the values of the function f.

In other words, as x approaches c, the limit of y is $f(c)$.

Example 1 – *Using a Tangent Line Approximation*

Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

at the point (0, 1). Then use a table to compare the *y*-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

Solution:

The derivative of f is

$$f'(x) = \cos x. \qquad \text{First derivative}$$

Example 1 – *Solution*

cont'd

So, the equation of the tangent line to the graph of f at the point (0, 1) is

$$y = f(0) + f'(0)(x - 0)$$

$$y = 1 + (1)(x - 0)$$

$$y = 1 + x. \qquad \text{Tangent line approximation}$$

Example 1 – *Solution*

cont'd

The table compares the values of *y* given by this linear approximation with the values of *f*(*x*) near *x* = 0.

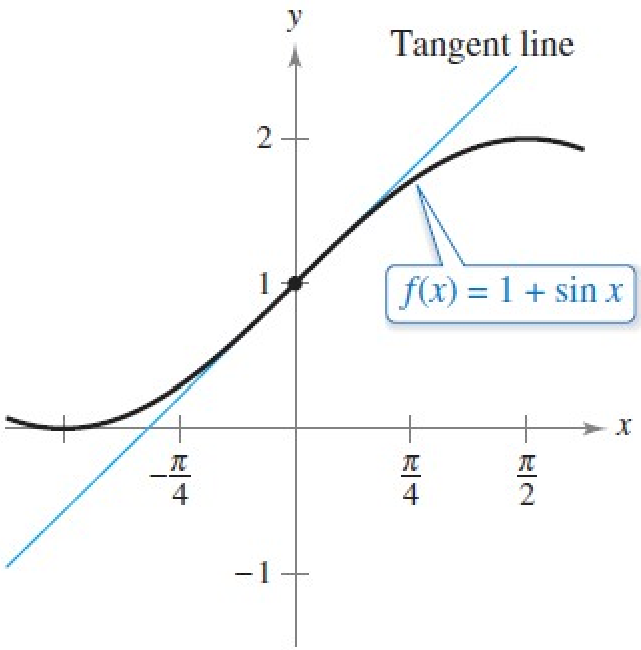
Notice that the closer *x* is to 0, the better the approximation.

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x) = 1 + \sin x$	0.521	0.9002	0.9900002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5

Example 1 – *Solution*

cont'd

This conclusion is reinforced by the graph shown in Figure 3.65.



The tangent line approximation of f at the point $(0, 1)$

Figure 3.65

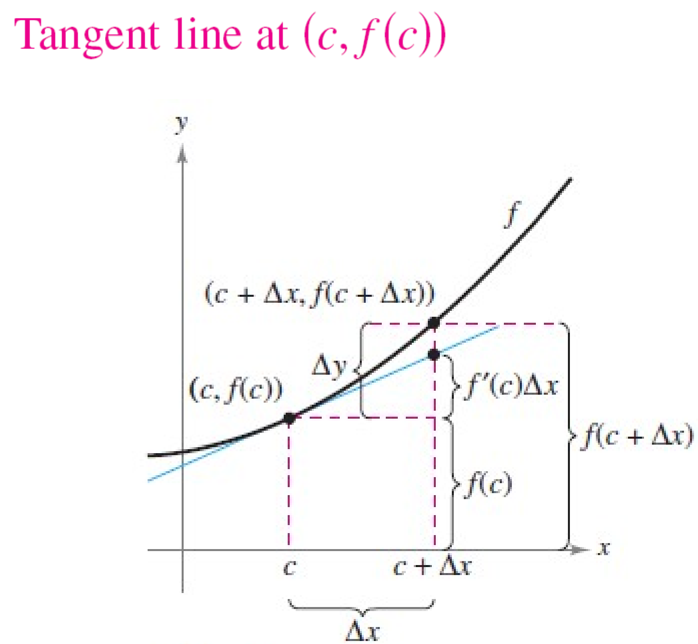
Differentials

Differentials

When the tangent line to the graph of *f* at the point (c, $f(c)$)

$$y = f(c) + f'(c)(x - c)$$

is used as an approximation of the graph of *f*, the quantity $x - c$ is called the change in x, and is denoted by Δx, as shown in Figure 3.66.



When Δx is small, $\Delta y = f(c + \Delta x) - f(c)$ is approximated by $f'(c)\Delta x$.

Figure 3.66

Differentials

When Δx is small, the change in y (denoted by Δy) can be approximated as shown.

$$\Delta y = f(c + \Delta x) - f(c) \qquad \text{Actual change in } y$$

$$\approx f'(c)\Delta x \qquad \text{Approximate change in } y$$

For such an approximation, the quantity Δx is traditionally denoted by dx and is called the **differential of *x***.

The expression $f'(x)dx$ is denoted by dy and is called the **differential of *y***.

Differentials

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential of** x (denoted by dx) is any nonzero real number. The **differential of** y (denoted by dy) is

$$dy = f'(x)\,dx.$$

In many types of applications, the differential of *y* can be used as an approximation of the change in *y*. That is,

$$\Delta y \approx dy \quad \text{or} \quad \Delta y \approx f'(x)\,dx.$$

Example 2 – *Comparing Δy and dy*

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$.
Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

Solution:

Because $y = f(x) = x^2$, you have $f'(x) = 2x$, and the differential dy is

$$dy = f'(x)dx = f'(1)(0.01) = 2(0.01) = 0.02. \quad \text{Differential of } y$$

Now, using $\Delta x = 0.01$, the change in y is

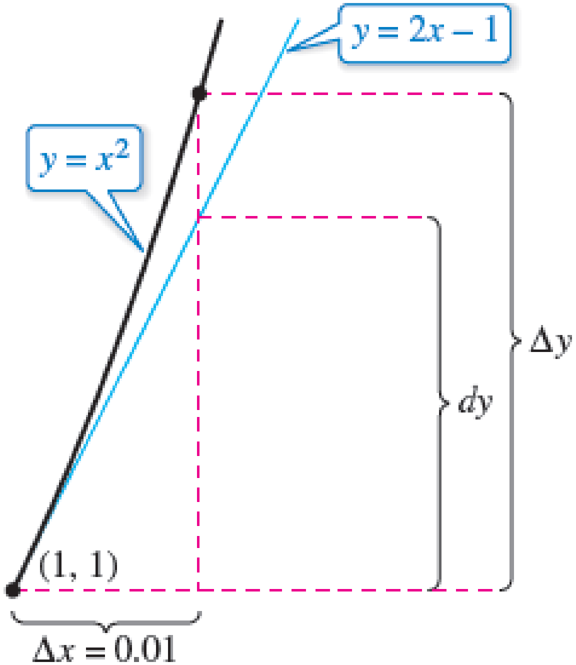
$$\Delta y = f(x + \Delta x) - f(x) = f(1.01) - f(1)$$

$$= (1.01)^2 - 1^2 = 0.0201.$$

Example 2 – *Solution*

cont'd

Figure 3.67 shows the geometric comparison of dy and Δy.



The change in y, Δy, is approximated by the differential of y, dy.

Figure 3.67

Error Propagation

Error Propagation

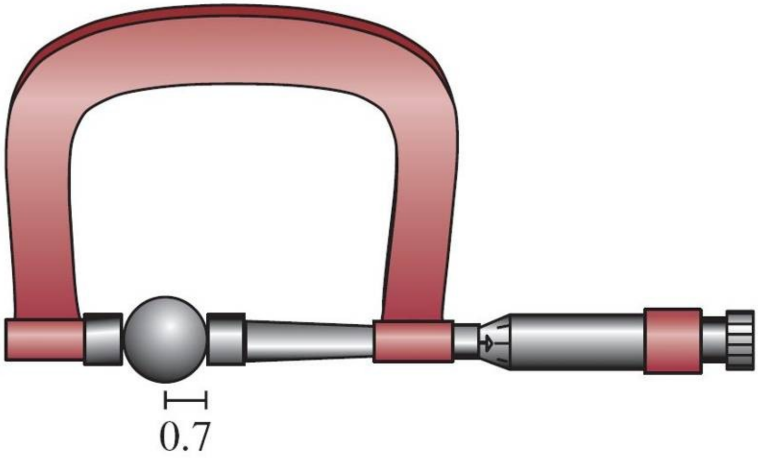
If you let *x* represent the measured value of a variable and let *x* + Δ*x* represent the exact value, then Δ*x* is the *error in measurement*.

Finally, if the measured value *x* is used to compute another value *f*(*x*), the difference between *f*(*x* +Δ*x*) and *f*(*x*) is the **propagated error**.

$$\underbrace{f(x + \overbrace{\Delta x}^{\text{Measurement error}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overbrace{\Delta y}^{\text{Propagated error}}$$

Example 3 – *Estimation of Error*

The measured radius of a ball bearing is 0.7 inch, as shown in the figure below. The measurement is correct to within 0.01 inch. Estimate the propagated error in the volume *V* of the ball bearing.



Ball bearing with measured radius that is correct to within 0.01 inch.

Example 3 – *Solution*

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

where *r* is the radius of the sphere.

So, you can write

$$r = 0.7 \qquad \text{Measured radius}$$

and

$$-0.01 \le \Delta r \le 0.01. \qquad \text{Possible error}$$

To approximate the propagated error in the volume, differentiate *V* to obtain $dV/dr = 4\pi r^2$ and write

$$\Delta V \approx dV \qquad \text{Approximate } \Delta V \text{ by } dV.$$

$$= 4\pi r^2\, dr$$

Example 3 – *Solution*

cont'd

$$= 4\pi(0.7)^2(\pm 0.01) \qquad \text{Substitute for } r \text{ and } dr.$$

$$\approx \pm 0.06158 \text{ cubic inch.}$$

So, the volume has a propagated error of about 0.06 cubic inch.

Error Propagation

The ratio

$$\frac{dV}{V} = \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} \qquad \text{Ratio of } dV \text{ to } V$$

$$= \frac{3\,dr}{r} \qquad \text{Simplify.}$$

$$\approx \frac{3(\pm 0.01)}{0.7} \qquad \text{Substitute for } dr \text{ and } r.$$

$$\approx \pm 0.0429$$

is called the **relative error.** The corresponding **percent error** is approximately 4.29%.

Calculating Differentials

Calculating Differentials

Each of the differentiation rules can be written in **differential form.**

Differential Formulas

Let u and v be differentiable functions of x.

Constant multiple: $d[cu] = c\,du$

Sum or difference: $d[u \pm v] = du \pm dv$

Product: $d[uv] = u\,dv + v\,du$

Quotient: $d\left[\frac{u}{v}\right] = \frac{v\,du - u\,dv}{v^2}$

Example 4 – *Finding Differentials*

Function	Derivative	Differential
a. $y = x^2$	$\dfrac{dy}{dx} = 2x$	$dy = 2x\,dx$
b. $y = \sqrt{x}$	$\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$	$dy = \dfrac{dx}{2\sqrt{x}}$
c. $y = 2\sin x$	$\dfrac{dy}{dx} = 2\cos x$	$dy = 2\cos x\,dx$
d. $y = x\cos x$	$\dfrac{dy}{dx} = -x\sin x + \cos x$	$dy = (-x\sin x + \cos x)\,dx$
e. $y = \dfrac{1}{x}$	$\dfrac{dy}{dx} = -\dfrac{1}{x^2}$	$dy = -\dfrac{dx}{x^2}$

Calculating Differentials

The notation in Example 4 is called the **Leibniz notation** for derivatives and differentials, named after the German mathematician Gottfried Wilhelm Leibniz.

Differentials can be used to approximate function values. To do this for the function given by $y = f(x)$, use the formula

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx$$

which is derived from the approximation

$$\Delta y = f(x + \Delta x) - f(x) \approx dy.$$

The key to using this formula is to choose a value for x that makes the calculations easier.

Example 7 – *Approximating Function Values*

Use differentials to approximate $\sqrt{16.5}$.

Solution:

Using $f(x) = \sqrt{x}$, you can write

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx.$$

Now, choosing *x* = 16 and *dx* = 0.5, you obtain the following approximation.

$$f(x + \Delta x) = \sqrt{16.5} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(0.5) = 4 + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right) = 4.0625$$

So, $\sqrt{16.5} \approx 4.0625.$

Calculating Differentials

The tangent line approximation to $f(x) = \sqrt{x}$ at *x* = 16 is the line $g(x) = \frac{1}{8}x + 2$.

For *x*-values near 16, the graphs of *f* and *g* are close together, as shown in Figure 3.68.

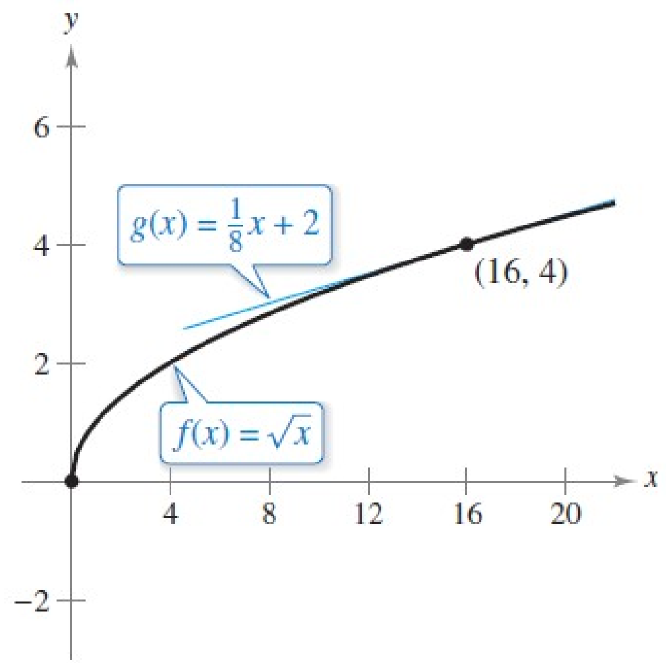


Figure 3.68