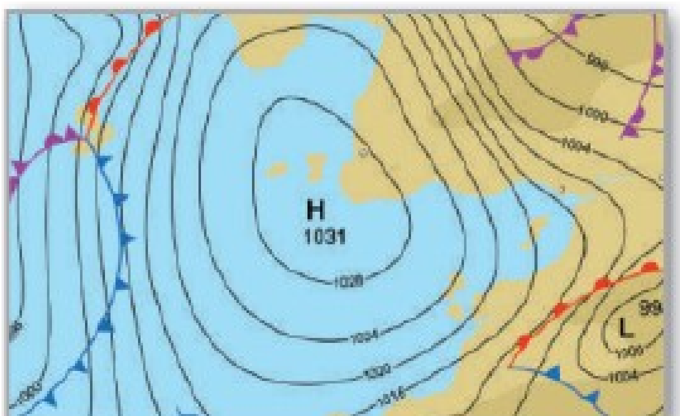
5 Logarithmic, Exponential, and Other Transcendental Functions



5.7 Inverse Trigonometric Functions: Differentiation

Objectives

- Develop properties of the six inverse trigonometric functions.
- Differentiate an inverse trigonometric function.
- Review the basic differentiation rules for elementary functions.

Inverse Trigonometric Functions

Inverse Trigonometric Functions

None of the six basic trigonometric functions has an inverse function.

This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one.

In this section, you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains*.

Inverse Trigonometric Functions

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the next definition.

Definitions of Inverse Trigonometric Functions

Function	Domain	Range
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \ne 0$

Inverse Trigonometric Functions

The graphs of the six inverse trigonometric functions are shown in Figure 5.26.

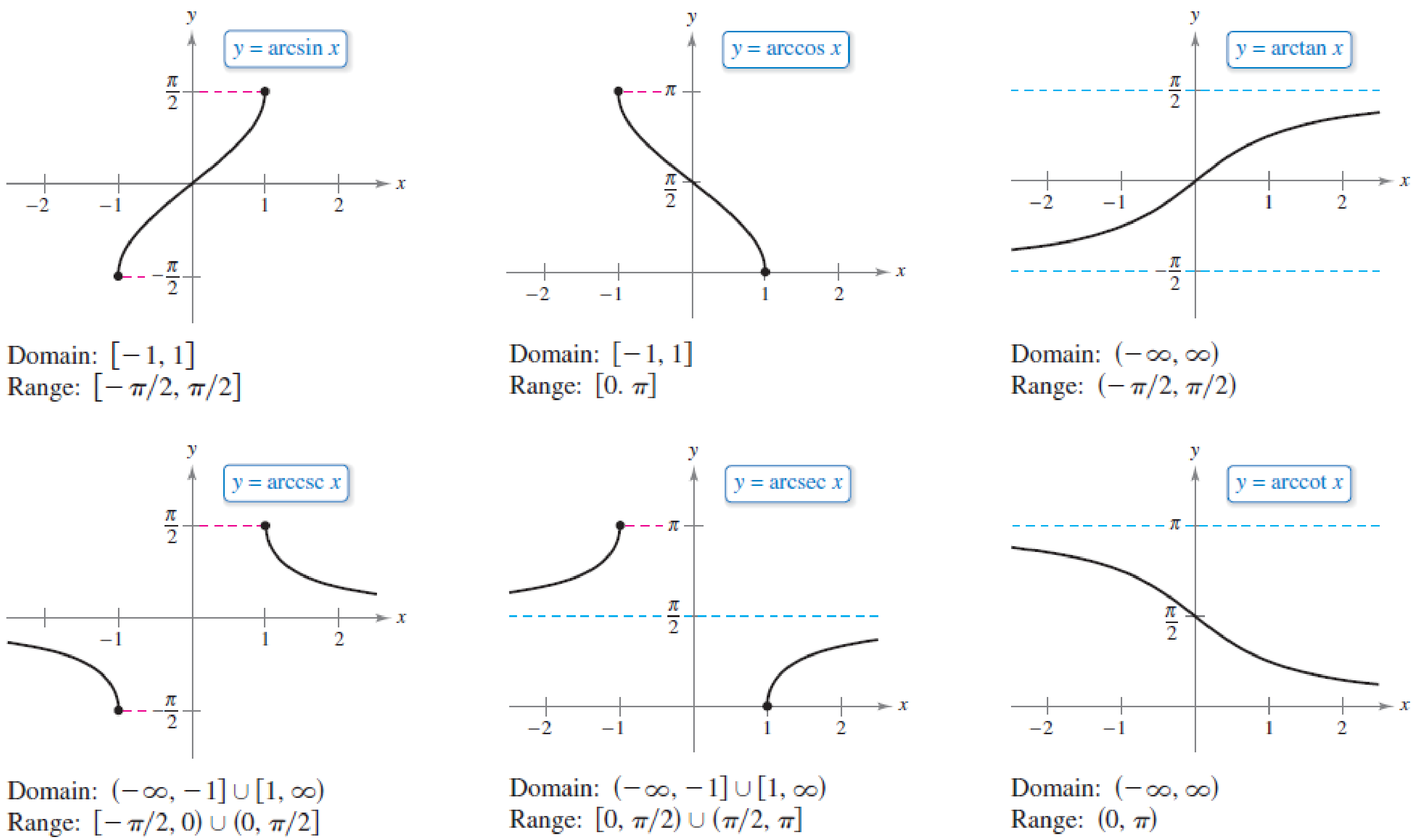


Figure 5.26

Example 1 – *Evaluating Inverse Trigonometric Functions*

Evaluate each function.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

Solution:

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval [–π/2, π/2], the correct value of *y* is –π/6.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

Example 1 – *Solution*

cont'd

b. By definition, y = arccos 0 implies that cos y = 0. In the interval [0,π], you have y = π/2.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval [–π/2, π/2], you have y = π/3.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian* mode produces

$$\arcsin(0.3) \approx 0.305.$$

Inverse Trigonometric Functions

Inverse functions have the properties

$$f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains.

For x-values outside these domains, these two properties do not hold.

For example, arcsin(sin π) is equal to 0, not π.

Inverse Trigonometric Functions

Properties of Inverse Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

Example 2 – *Solving an Equation*

$$\arctan(2x - 3) = \frac{\pi}{4}$$ Original equation

$$\tan[\arctan(2x - 3)] = \tan\frac{\pi}{4}$$ Take tangent of each side.

$$2x - 3 = 1$$ $\tan(\arctan x) = x$

$$x = 2$$ Solve for x.

Derivatives of Inverse Trigonometric Functions

Derivatives of Inverse Trigonometric Functions

The derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$.

You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

Derivatives of Inverse Trigonometric Functions

The next theorem lists the derivatives of the six inverse trigonometric functions. Note that the derivatives of arccos *u,* arccot *u*, and arccsc *u* are the *negatives* of the derivatives of arcsin *u*, arctan *u*, and arcsec *u,* respectively.

THEOREM 5.18 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Example 4 – *Differentiating Inverse Trigonometric Functions*

a. $$\frac{d}{dx}[\arcsin(2x)] = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$$

b. $$\frac{d}{dx}[\arctan(3x)] = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$$

c. $$\frac{d}{dx}\left[\arcsin\sqrt{x}\right] = \frac{(1/2)\,x^{-1/2}}{\sqrt{1-x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1}{2\sqrt{x-x^2}}$$

d. $$\frac{d}{dx}[\operatorname{arcsec} e^{2x}] = \frac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \frac{2}{\sqrt{e^{4x}-1}}$$

The absolute value sign is not necessary because $e^{2x} > 0$.

Review of Basic Differentiation Rules

Review of Basic Differentiation Rules

An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	**Transcendental Functions**
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

Review of Basic Differentiation Rules

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function.

For convenience, these differentiation rules are summarized below.

BASIC DIFFERENTIATION RULES FOR ELEMENTARY FUNCTIONS

1. $\dfrac{d}{dx}[cu] = cu'$
2. $\dfrac{d}{dx}[u \pm v] = u' \pm v'$
3. $\dfrac{d}{dx}[uv] = uv' + vu'$
4. $\dfrac{d}{dx}\left[\dfrac{u}{v}\right] = \dfrac{vu' - uv'}{v^2}$
5. $\dfrac{d}{dx}[c] = 0$
6. $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\dfrac{d}{dx}[x] = 1$
8. $\dfrac{d}{dx}[|u|] = \dfrac{u}{|u|}(u'), \ u \neq 0$
9. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$
10. $\dfrac{d}{dx}[e^u] = e^u u'$

Review of Basic Differentiation Rules cont'd

11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$

12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$

13. $\frac{d}{dx}[\sin u] = (\cos u)u'$

14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$

15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$

16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$

17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$

18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$

19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$

20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$

21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$

22. $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1 + u^2}$

23. $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$

24. $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$