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Lucrare scrisă la analiză matematică sesiune iarnă 2021

Turturică Dorin
grupa 217

1,5p ① Stud. conv. şi abs conv seriei:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \sin \frac{\pi}{\sqrt{n}}$$

1,5p ② Calc integrala improprie

$$\int_1^{\infty} \frac{1}{x^3+x}\, dx$$

2p ③ Det punctele critice şi pct de extrem local (specificând tipul acestora pt funcţia)

$$f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = (x+xy+y^2)\sqrt{e^x}$$

2p ④ Fie $g : (0,\infty)^2 \to \mathbb{R}$ fct de cls C^1, transformaţi relaţia

$$-u \cdot \frac{\partial g}{\partial u}(u,v) + \frac{v}{1+2u} \cdot \frac{\partial g}{\partial v}(u,v) = 1,\ \forall (u,v) \in (0,\infty)^2$$

în var $(x,y) \in (0,\infty)^2$, $u = \frac{y}{x}$, $v = x+2y$ cu transformările

Det apoi o fct g cu prop de mai sus. Verificare

of 1p

① $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \sin \frac{\pi}{\sqrt{n}}$

fie $a_n = \sin \frac{\pi}{\sqrt{n}}$

$0 \le \frac{\pi}{\sqrt{n}} \le \pi \Rightarrow 0 \le \sin \frac{\pi}{\sqrt{n}} \le 1 \Rightarrow a_n$ stp (şir cu termeni pozitivi)

$\lim_{n\to\infty} \sin \frac{\pi}{\sqrt{n}} = \sin 0_+ = 0$

pt $n \ge 2$: a_n strict descrescător (se află în cadranul I şi scade)

$\overset{Leibniz}{\Rightarrow} \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \sin \frac{\pi}{\sqrt{n}}$ convergentă

absolut convergentă: $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \cdot \sin \frac{\pi}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \sin \frac{\pi}{\sqrt{n}} \ge 0$ stp

alegem şirul $y_n = \frac{1}{\sqrt{n}}$ $\quad x_n = \sin \frac{\pi}{\sqrt{n}}$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\sin \frac{\pi}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \underbrace{\frac{\sin \frac{\pi}{\sqrt{n}}}{\frac{\pi}{\sqrt{n}}}}_{\to 1} \pi = \pi \in (0, \infty) \Rightarrow$

$\Rightarrow \sum y_n \sim \sum x_n \Rightarrow \sum_{n=1}^{\infty} \sin \frac{\pi}{\sqrt{n}}$ divergentă $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \sin \frac{\pi}{\sqrt{n}}$ nu este absolut convergentă

② $\int_1^{\infty - 0} \frac{1}{x^3+x} dx = \int_1^{\infty-0} \frac{1}{x(x^2+1)} = \lim_{v\to\infty} \int_1^v \left(\frac{1}{x} - \frac{x}{(x^2+1)}\right) dx =$

$\cancel{= \lim_{v\to\infty} \left(\int_1^v \frac{1}{x} dx - \int_1^v x \cdot \frac{1}{x^2+1} dx\right) = \lim_{v\to\infty} \ln x \Big|_1^v \int_1^v x (\operatorname{arctg} x)' dx} =$

$\cancel{= \lim_{v\to\infty} \ln v - \ln 1 - x \operatorname{arctg} x \Big|_1^v + \int_1^v}$

$= \lim_{v\to\infty} \ln x \Big|_1^v - \frac{1}{2} \int_1^v \frac{(x^2+1)'}{x^2+1} dx = \lim_{v\to\infty} \ln v - 0 - \frac{1}{2} \ln(x^2+1) \Big|_1^v =$

$= \lim_{v\to\infty} \ln v - \frac{1}{2} \ln(v^2+1) + \frac{1}{2} \ln 2 = \lim_{v\to\infty} \ln \frac{v}{(v^2+1)^{\frac{1}{2}}} + \frac{1}{2} \ln 2 =$

$= \lim_{v\to\infty} \ln \frac{v}{\sqrt{v^2+1}} + \ln \sqrt{2} = \lim_{v\to\infty} \ln \frac{\cancel{v}}{\cancel{v}\sqrt{1+\frac{1}{v^2}}} + \ln \sqrt{2} = \ln 1 + \ln \sqrt{2} = \ln \sqrt{2}$

③ $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (x + xy + y^2)\sqrt{e^x} = (x + xy + y^2) e^{\frac{x}{2}}$

$\nabla f(x^0) = 0 \Rightarrow$ pct critic

$\frac{\partial f}{\partial x}(x,y) = (1+y)\sqrt{e^x} + (x + xy + y^2) \cdot e^{\frac{x}{2}} \cdot \frac{1}{2} =$

$= e^{\frac{x}{2}}\left(1 + y + \frac{1}{2}x + \frac{xy}{2} + \frac{y^2}{2}\right) = \frac{1}{2} e^{\frac{x}{2}} (2 + 2y + x + xy + y^2)$

$$\frac{\partial f}{\partial y}(x,y) = \sqrt{e^x}\,(x+2y) = e^{\frac{x}{2}}(x+2y)$$

$$\nabla f(x,y) = \left(\frac{1}{2}e^{\frac{x}{2}}(2+2y+x+xy+y^2),\ e^{\frac{x}{2}}(x+2y)\right)$$

$$\begin{cases} \frac{1}{2}e^{\frac{x}{2}}(2+2y+x+xy+y^2) = 0 \\ e^{\frac{x}{2}}(x+2y) = 0 \iff e^{\frac{x}{2}} = 0 \text{ „imposibil" sau } x+2y=0 \Rightarrow x=-2y \\ \qquad \Rightarrow y = -\frac{x}{2} \end{cases}$$

$$\frac{1}{2}e^{\frac{x}{2}}\left(2\cancel{+4} - \cancel{x} + \cancel{x} - \frac{x^2}{2} + \frac{x^2}{4}\right) = 0 \Rightarrow 2 - \frac{x^2}{4} = 0 \Rightarrow \frac{x^2}{4} = 2 \Rightarrow$$

$$\Rightarrow x^2 = 8 \Rightarrow x = \pm\sqrt{8} \Rightarrow x = \pm 2\sqrt{2}$$

$y = \frac{-x}{2}$ ~~ly2=~~

I $x = 2\sqrt{2} \Rightarrow y = \frac{-2\sqrt{2}}{2} = -\sqrt{2}$

II $x = -2\sqrt{2} \Rightarrow y = \sqrt{2}$

Pct critice: $(2\sqrt{2}, -\sqrt{2})$ și $(-2\sqrt{2}, \sqrt{2})$

$$\frac{\partial^2 f}{\partial x^2} = \left[\frac{1}{2}e^{\frac{x}{2}}(2+2y+x+xy+y^2)\right]'_x = \frac{1}{2}\left[e^{\frac{x}{2}}\cdot\frac{1}{2}(2+2y+x+xy+y^2) + \right.$$

$$\left. + e^{\frac{x}{2}}(1+y)\right] = \frac{1}{4}e^{\frac{x}{2}}(2+2y+x+xy+y^2+2+2y) = \frac{1}{4}e^{\frac{x}{2}}(4+4y+x+xy+y^2)$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \left[\frac{1}{2}e^{\frac{x}{2}}(2+2y+x+xy+y^2)\right]'_y = \frac{1}{2}e^{\frac{x}{2}}(2+x+2y)$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \left[e^{\frac{x}{2}}(x+2y)\right]'_x = e^{\frac{x}{2}}\cdot\frac{1}{2}(x+2y) + e^{\frac{x}{2}} = \frac{1}{2}e^{\frac{x}{2}}(2+x+2y)$$

$$\frac{\partial^2 f}{\partial y^2} = \left[e^{\frac{x}{2}}(x+2y)\right]'_y = 2e^{\frac{x}{2}}$$

$$H(f)(2\sqrt{2},-\sqrt{2}) = \begin{pmatrix} \frac{1}{4}e^{\sqrt{2}}(4-4\sqrt{2}+2\sqrt{2}-4+2) & \frac{1}{2}e^{\sqrt{2}}(2+2\sqrt{2}-2\sqrt{2}) \\ \frac{1}{2}e^{\sqrt{2}}(2+2\sqrt{2}-2\sqrt{2}) & 2e^{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}e^{\sqrt{2}}(2-2\sqrt{2}) & e^{\sqrt{2}} \\ e^{\sqrt{2}} & 2e^{\sqrt{2}} \end{pmatrix} \qquad \begin{array}{l} \Delta_1 > 0 \ (e^{\sqrt{2}} > 0) \\ \Delta_2 = \begin{vmatrix} \frac{1}{4}e^{\sqrt{2}}(2-2\sqrt{2}) & e^{\sqrt{2}} \\ e^{\sqrt{2}} & 2e^{\sqrt{2}} \end{vmatrix} = \end{array}$$

$$= (e^{\sqrt{2}})^2 \begin{vmatrix} \frac{1}{4}(2-2\sqrt{2}) & 1 \\ 1 & 2 \end{vmatrix} = e^{2\sqrt{2}}\left(\frac{1}{2}(2-2\sqrt{2}) - 1\right) = e^{2\sqrt{2}}(\cancel{1} - \sqrt{2} - \cancel{1}) =$$

$$= -\sqrt{2}\,e^{2\sqrt{2}} < 0$$

$\left.\begin{array}{l}\Delta_1 > 0 \\ \Delta_2 < 0\end{array}\right\} \Rightarrow \phi(u_1, u_2)$ indefinit definită $\Rightarrow (2\sqrt{2}, -\sqrt{2})$ pct șa

$$H(f)(-2\sqrt{2}, \sqrt{2}) = \begin{pmatrix} \frac{1}{4} e^{-\sqrt{2}}(\cancel{4} + 4\sqrt{2}\, u - 2\sqrt{2} - \cancel{4} + 2) & \frac{1}{2} e^{-\sqrt{2}}(2 - \cancel{2\sqrt{2}} + \cancel{2\sqrt{2}}) \\ \frac{1}{2} e^{-\sqrt{2}}(2 - \cancel{2\sqrt{2}} + \cancel{2\sqrt{2}}) & 2e^{-\sqrt{2}} \end{pmatrix}$$

$$H(f)(-2\sqrt{2}, \sqrt{2}) = \begin{pmatrix} \frac{1}{4} e^{-\sqrt{2}}(2 + 2\sqrt{2}) & e^{-\sqrt{2}} \\ e^{-\sqrt{2}} & 2e^{-\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} e^{-\sqrt{2}}(1 + \sqrt{2}) & e^{-\sqrt{2}} \\ e^{-\sqrt{2}} & 2e^{-\sqrt{2}} \end{pmatrix}$$

$\Delta_1 > 0 \quad (e^{-\sqrt{2}} > 0)$

$$\Delta_2 = (e^{-\sqrt{2}})^2 \cdot \begin{vmatrix} \frac{1}{2}(1+\sqrt{2}) & 1 \\ 1 & 2 \end{vmatrix} = e^{-2\sqrt{2}} \cdot (\cancel{1} + \sqrt{2} - \cancel{1}) = \sqrt{2} \cdot e^{-2\sqrt{2}} > 0$$

$\Rightarrow \phi(u_1, u_2)$ pozitiv definită $\Rightarrow (-2\sqrt{2}, \sqrt{2})$ pct de minim local

4) $g : (0, \infty)^2 \to \mathbb{R}$ fct de cls C^1.

$$-u \cdot \frac{\partial g}{\partial u}(u, v) + \frac{v}{1+2u} \cdot \frac{\partial g}{\partial v}(u, v) = 1, \ \forall (u, v) \in (0, \infty)^2$$

În var $(x, y) \in (0, \infty)^2$, $u = \frac{y}{x}$, $v = x + 2y$

$$\left.\begin{array}{l} u = \frac{y}{x} \\ v = x + 2y \end{array}\right\} \Rightarrow -\frac{y}{x} \cdot \frac{\partial g}{\partial u}\left(\frac{x}{y}, x+2y\right) + \frac{x+2y}{1 + \frac{2y}{x}} \cdot \frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right) = 1$$

$$\Leftrightarrow -\frac{y}{x} \cdot \frac{\partial g}{\partial u}\left(\cancel{\frac{x}{y}}, x+2y\right) + \frac{x\cancel{(x+2y)}}{\cancel{x+2y}} \cdot \frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right) = 1$$

$$\Leftrightarrow -\frac{y}{x} \cdot \frac{\partial g}{\partial u}\left(\cancel{\frac{x}{y}}, x+2y\right) + x \cdot \frac{\partial g}{\partial v}\left(\frac{y}{x}, x+2y\right) = 1$$

$f : (0, \infty)^2 \to \mathbb{R}^2$, $f(x, y) = \left(\frac{y}{x}, x+2y\right)$

$g \circ f : (0, \infty)^2 \to \mathbb{R}$

$\nabla(g \circ f)(x, y) = \nabla g(f(x, y)) \cdot J(f)(x, y)$.

$$\nabla(g\circ f)(x,y) = \left(\frac{\partial}{\partial x}(g\circ f)(x,y), \frac{\partial}{\partial y}(g\circ f)(x,y)\right) =$$

$$= \left(\frac{\partial g}{\partial u} f(x,y), \frac{\partial g}{\partial v} f(x,y)\right)\cdot \begin{pmatrix} \left(\frac{y}{x}\right)'_x & \left(\frac{y}{x}\right)'_y \\ (x+2y)'_x & (x+2y)'_y \end{pmatrix} =$$

$$= \left(\frac{\partial g}{\partial u} f(x,y), \frac{\partial g}{\partial v} f(x,y)\right)\begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 2 \end{pmatrix}$$

$$\frac{\partial}{\partial x}(g\circ f)(x,y) = \frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right)\cdot\frac{-y}{x^2} + \frac{\partial g}{\partial v}\left(\frac{x}{y}, x+2y\right)\cdot 1 =$$

$$= \frac{1}{x}\left(\frac{\partial g}{\partial u}\left(\frac{y}{x}, x+2y\right)\cdot\frac{-y}{x} + x\,\frac{\partial g}{\partial v}\left(\frac{x}{y}, x+2y\right)\right) =$$

$$= \frac{1}{x}\cdot 1 = \frac{1}{x}, \quad \forall x \in (0,\infty)^2$$

$\frac{\partial h}{\partial x}(x,y) = \frac{1}{x}$ $\qquad$ $\int \frac{1}{x}dx = \ln x + C$

alegem $h(x,y) = \ln x$

$g(f(x,y)) = \ln x \Rightarrow g\left(\frac{y}{x}, x+2y\right) = \ln x \Rightarrow g(u,v)$

$\ln x$ trebuie scris în funcție de ~~x, y~~ u și v

$\ln x = \ln(v-2y) = \ln\left(v - \frac{2y}{x}\cdot x\right) = \ln(v - 2u\cdot x)$