Seminar 1: Numere reale

Exerciții

1) $x, y \in \mathbb{R}$ arăt. $\max\{x,y\} = \frac{|x-y|+(x+y)}{2}$

$\min\{x,y\}$

$\max\{x,y\} - \min\{x,y\} = |x-y|$

$\max\{x,y\} + \min\{x,y\} = x+y$

$\max\{x,y\} = \frac{|x-y|+(x+y)}{2}$ (+)

$\min\{x,y\} = \frac{(x+y)-|x-y|}{2}$ (−)

2) $x, y \in \mathbb{R}$

Dem. prop:

a) $|x+y| \leq |x|+|y|$

$u \leq v \Leftrightarrow u^2 \leq v^2$, $(\forall)\, u, v \geq 0$ (se întâmplă)

$|x+y|^2 \leq (|x|+|y|)^2 \Leftrightarrow$

$\Leftrightarrow |x+y|^2 \leq |x|^2 + 2|xy| + |y|^2$

$\Leftrightarrow \cancel{x^2} + \cancel{y^2} + 2xy \leq \cancel{x^2} + \cancel{y^2} + 2|xy|$

$\Leftrightarrow xy \leq |xy|$, "A"

b) $|x-y| \geq |x|-|y|$

$\Leftrightarrow |x-y| + |y| \geq |x|$

Folosim a) $|x| = |x-y+y| \leq |x-y|+|y|$

3) $\inf A$, $\sup A$, $\min A$, $\max A$

a) $A = [0,7) \cup [8,+\infty)$

$MIN(A) = (-\infty, 0]$ $\quad \inf A = 0 \quad \min A = 0$

$MAJ(A) = \emptyset \quad \sup A = +\infty \quad \max A =$ cel mai mare el al mulțimii A

$\nexists$

b) $A = [-1,2] \setminus \mathbb{Q}$

$MIN(A) = (-\infty, -1] \quad \inf A = -1 \quad \min A = \nexists$

$MAJ(A) = [2,+\infty) \quad \sup A = 2 \quad \max A = \nexists$

c) $A = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \quad \lim_{n\to\infty} \frac{1}{n} = 0$

$MIN(A) = (-\infty, 0] \quad \inf A = 0 \quad \min A = \nexists$

$MAJ(A) = [1,+\infty) \quad \sup A = 1 \quad \max A = 1$

d) $A = \{ \frac{1}{x-[x]} \mid x \in \mathbb{R}\setminus\mathbb{Z} \} = \{ \frac{1}{\{x\}} \mid x \in \mathbb{R}\setminus\mathbb{Z} \}$

$x - [x] = \{x\} \in (0,1)$

$\frac{1}{x-[x]} \in (1, +\infty)$

$MIN(A) = (-\infty, 1] \quad \inf A = 1 \quad \min A = \nexists$

$MAJ(A) = \emptyset \quad \sup A = +\infty \quad \max A = \nexists$

4) $A \subseteq \mathbb{R}$ Ar. că:

a) Dacă $\exists \min A$ atunci $\min A = \inf A$

b) Dacă $\exists \max A$ atunci $\max A = \sup A$

$m = \inf A \Leftrightarrow \begin{cases} i) \underline{m \in MIN(A)} \\ ii) (\forall) m' \in MIN(A) : m' \leq m \end{cases}$

$m = \min A \Leftrightarrow \begin{cases} i) m \in A \\ ii) \underline{m \in MIN(A)} \end{cases}$

Fie $m = \min A \Rightarrow m \in MIN(A)$

$\left.\begin{array}{l} \forall m' \in MIN(A) \Rightarrow m' \leq a, (\forall) a \in A \\ m \in A \end{array}\right| \Rightarrow m' \leq m \quad \Bigg| \Rightarrow m = \inf A$

5) $A, B \subseteq \mathbb{R}$ nevide și mărginite $A \subseteq B$

a) Ar. că $\inf B \leq \underline{\inf A \leq \sup A} \leq \sup B$ — Temă

b) Dacă $\inf A = \inf B$ și $\sup A = \sup B$ rezultă $A = B$?

a) $\inf A \leq a \leq \sup A$ $(\forall) a \in A$

Arătăm că $\inf B \leq \inf A$

Notăm $m = \inf B \Rightarrow m \in MIN(B) \Rightarrow \left.\begin{array}{l} m \leq b, \forall b \in B \\ A \subseteq B \end{array}\right| \Rightarrow m \leq a, \forall a \in A \Rightarrow$

$\Rightarrow m \in MIN(A) \Rightarrow m \leq \inf A$

b) $A = (0,1)$ $\quad \inf A = \inf B = 0$
$B = [0,1]$ $\quad \sup A = \sup B = 1$, dar $A \neq B$

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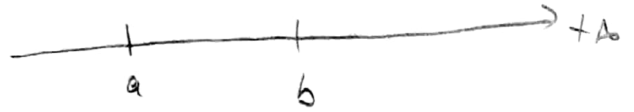
! (6) Dem că între oricare 2 nr. reale distincte există cel puțin un nr. rațional (și cel puțin un nr. irațional)

rat:



$a<b$

$\frac{1}{b-a} > 0 \Rightarrow \exists n \in \mathbb{N}^*: n > \frac{1}{b-a} \Rightarrow \frac{1}{n} < b-a$

$\forall x \in \mathbb{R}, x-1 < [x] \leq x$

Fie $m = [n \cdot a] + 1 \in \mathbb{Z}$

$\Rightarrow n \cdot a - 1 < [n \cdot a] \leq n \cdot a \mid +1$

$\Rightarrow n \cdot a < m \leq n \cdot a + 1 \mid : n$

$\Rightarrow a < \frac{m}{n} \leq a + \frac{1}{n} < a + (b-a) = b \Rightarrow$

$\Rightarrow \frac{m}{n} \in (a,b) \cap \mathbb{Q}$

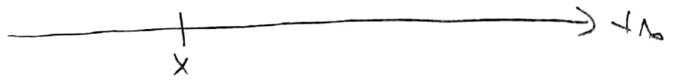
irat:

$a<b \Rightarrow a+\sqrt{2} < b+\sqrt{2} \Rightarrow \exists r \in (a+\sqrt{2}, b+\sqrt{2}) \cap \mathbb{Q} \Rightarrow$

$\sqrt{2} \notin \mathbb{Q}$

$\Rightarrow r - \sqrt{2} \in (a,b) \cap (\mathbb{R} \setminus \mathbb{Q})$

! (7) Dem că ∀ nr. real este limita unui șir de nr. raționale (respectiv iraționale). Dați exemple pt. numerele $x=2$ și $y=\sqrt{2}$



$x \in \mathbb{R}, \left(x - \frac{1}{n}, x + \frac{1}{n}\right), (\forall) n \in \mathbb{N}^* \overset{(6)}{\Longrightarrow} \exists r_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \mathbb{Q}$

$\Rightarrow (r_n)_{n \in \mathbb{N}} \subseteq \mathbb{Q}$; $x - \frac{1}{n} < r_n < x + \frac{1}{n} \mid \lim_{n \to +\infty} \Rightarrow$

x

$$\Rightarrow \lim_{n\to+\infty} r_n = x$$

Analog pt. șir de nr. iraționale ..

$\boxed{x=2}$ $r_n = 2+\frac{1}{n} \to 2$, $r_n \in \mathbb{Q}$

$i_n = 2+\frac{\sqrt{2}}{n} \to 2$, $i_n \in \mathbb{R}\setminus\mathbb{Q}$

$y=\sqrt{2}$ $i_n = \sqrt{2}+\frac{1}{n} \to \sqrt{2}$, $i_n \in \mathbb{R}\setminus\mathbb{Q}$

$r_n = 1 + \underbrace{\frac{1}{2+}\frac{1}{2+}\cdots\frac{1}{2}}_{n \text{ ori}}$, $\forall n\in\mathbb{N}$, $r_n \in \mathbb{Q}$

$r_n \to \sqrt{2}$

Ex2:

$$\begin{cases} r_{n+1} = \frac{r_n}{2} + \frac{1}{r_n}, \ \forall n\in\mathbb{N} \\ r_0 = 1 \end{cases}$$

prin inducție $\Rightarrow r_n \in \mathbb{Q}$

$(r_n) \subseteq \mathbb{Q}$, $r_n \to \sqrt{2}$?

(r_n) mărginit, monoton

Generalizare:

$a>0$

$$\begin{cases} r_{n+1} = \frac{r_n}{2} + \frac{a}{2\cdot r_n}, \ \forall n\in\mathbb{N} \\ r_0 = 1 \end{cases}$$

$$\lim_{n\to\infty} r_n = \sqrt{a}$$

!

Justificati $\sqrt{2} + \sqrt[3]{3} \notin \mathbb{Q}$

$\alpha = \sqrt{2} + \sqrt[3]{3} \Rightarrow \alpha - \sqrt{2} = \sqrt[3]{3} \;\; ()^3 \Rightarrow (\alpha - \sqrt{2})^3 = 3$

$\Rightarrow \alpha^3 - 3\sqrt{2}\alpha(\alpha - \sqrt{2}) - 2\sqrt{2} = 3$

$\Rightarrow \alpha^3 + 6\alpha - 3 = \sqrt{2}(3\alpha^2 + 2) \;\; ()^2$

$\Rightarrow (\alpha^3 + 6\alpha - 3)^2 = 2(3\alpha^2 + 2)^2$

$(\alpha^3 + 6\alpha - 3)^2 = 2(2 + 3\alpha^2)^2 \Rightarrow \quad \alpha^3 + 6\alpha - 3 \quad \alpha^3 - 16\alpha - 3$

$\Rightarrow \alpha^6 + 6\alpha^4 - 3\alpha^3 + 6\alpha^4 + 36\alpha^2 - 18\alpha - 3\alpha^3 - 18\alpha + 9 = 2(4 + 12\alpha^2 + 9\alpha^4)$

$\Rightarrow \alpha^6 + 12\alpha^4 - 6\alpha^3 - 36\alpha + 9 = 8 + 24\alpha^2 + 18\alpha^4$

$\Rightarrow \alpha^6 - 6\alpha^4 - 6\alpha^3 + 12\alpha^2 - 36\alpha + 1 = 0$

PP $\frac{p}{q} \in \mathbb{Q}$ este răd a ec. pol. $x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1 = 0$

$\Rightarrow \begin{matrix} p \mid 1 \\ q \mid 1 \end{matrix} \Rightarrow \frac{p}{q} \in \{\pm 1\}$ - absurd! $\Rightarrow \alpha \notin \mathbb{Q}$

Temă: ex suplimentare

5. a) Temă

$A, B \subseteq \mathbb{R} \quad A \subseteq B$

mac. $\sup A \leq \sup B$

notăm $M = \sup B \Rightarrow M \in MAJ(B) \Rightarrow M \geq b, \forall b \in B$
$A \subseteq B$
$\Big| \Rightarrow M \geq a, \forall a \in A$

$\Rightarrow M \in MAJ(A) \Rightarrow M \geq \sup A \Rightarrow \sup B \geq \sup A$

Ex. suplimentare:

1. a) $A = [-\pi, \pi] \cap \mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3\}$

$MIN(A) = (-\infty, -3]$ $\inf A = -3$ $\min A = -3$

$MAJ(A) = [3, +\infty)$ $\sup A = 3$ $\max A = 3$

b) $A = \left\{ \frac{n}{1-n^2} \mid n \in \mathbb{N}, n \geq 2 \right\} = \left\{ \frac{2}{1-4}, \frac{3}{1-9}, \frac{4}{1-16}, \dots \right\} =$

$= \left\{ -\frac{2}{3}, -\frac{3}{8}, -\frac{4}{15} \dots \right\} \to 0$

$\lim\limits_{n \to \infty} \frac{n}{1-n^2} = 0$

$MIN(A) = (-\infty, -\frac{2}{3}]$ $\inf A = -\frac{2}{3}$ $\min A = -\frac{2}{3}$

$MAJ(A) = [0, +\infty)$ $\sup A = 0$ $\max A = \nexists$

c) $A = \left\{ x + \frac{1}{x} \mid x > 0 \right\}$

$MAJ(A) = \emptyset$ $\sup A = +\infty$ $\max A = \nexists$

d) $A = \{ x \in \mathbb{R} \mid |x^2 - x| \leq 1 \} = \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right]$

$-1 \leq x^2 - x \leq 1$

I $x^2 - x + 1 \geq 0$

$\Delta = -3 < 0 \Rightarrow$

$\Rightarrow x^2 - x + 1 > 0, (\forall) x \in \mathbb{R}$

II $x^2 - x - 1 \leq 0$

$\Delta = 5$

$x_{1/2} = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow x \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right]$

$MIN(A) = (-\infty, \frac{1-\sqrt{5}}{2}]$ $\inf A = \frac{1-\sqrt{5}}{2}$ $\min A = \frac{1-\sqrt{5}}{2}$

$MAJ(A) = [\frac{1+\sqrt{5}}{2}, +\infty)$ $\sup A = \frac{1+\sqrt{5}}{2}$ $\max A = \frac{1+\sqrt{5}}{2}$

2. $A, B \subseteq \mathbb{R}$ mulțimi nevide, mărginite superior

mac. $A \cup B$ mărginită superior și $\sup(A \cup B) = \max\{\sup A, \sup B\}$

A - mărginită superior $\Rightarrow M_1 = \sup A = \min(MAJ(A)) \in \mathbb{R}$

$$M_1 = \sup A \Leftrightarrow \begin{cases} \forall x \in A, x \leq M_1 \\ \forall \varepsilon > 0, \exists y \in A \text{ a.î. } y > M_1 - \varepsilon \end{cases}$$

(1)

B - mărginită superior $\Rightarrow M_2 = \sup B = \min(MAJ(B)) \in \mathbb{R}$

$$M_2 = \sup B \Leftrightarrow \begin{cases} \forall x \in B, x \leq M_2 \\ \forall \varepsilon > 0, \exists y \in B \text{ a.î. } y > M_2 - \varepsilon \end{cases}$$

(2)

$$(1), (2) \Rightarrow \left.\begin{cases} \max A = \sup A \\ \max B = \sup B \end{cases}\right| \Rightarrow \max(A \cup B) = \max\{\max A, \max B\} \in \mathbb{R}$$

$\Rightarrow A \cup B$ mărginită superior

$MAJ(A \cup B) = MAJ(A) \cap MAJ(B) \Rightarrow$

$\Rightarrow \sup(A \cup B) = \max\{\sup A, \sup B\}$

Seminar 2: Șiruri de nr. reale

① $(x_n)_{n\in\mathbb{N}}$

a) $x_n = \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to+\infty} \frac{\sqrt{n}\,(n+1-n)}{\sqrt{n+1}+\sqrt{n}} =$$

$$= \lim_{n\to+\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to+\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{1+\underbrace{\tfrac{1}{n}}_{0}}+1\right)} = \frac{1}{2}$$

b) $x_n = \dfrac{n+\sin(n)}{n+\cos(n)}$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \frac{\cancel{n}\left(1+\frac{\sin n}{n}\right)}{\cancel{n}\left(1+\frac{\cos n}{n}\right)} = \frac{1+0}{1+0} = 1$$

$$-1 \le \sin n \le 1 \;\; |:n \Rightarrow \; -\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n} \;\; \Big| \lim_{n\to+\infty}$$

$$0$$

c) $x_n = \dfrac{(\sqrt{2}+1)^n}{(\sqrt{2})^n+1}$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} \frac{(\sqrt{2}+1)^n}{(\sqrt{2})^n\left(1+\left(\frac{1}{\sqrt{2}}\right)^n\right)} = \lim_{n\to+\infty} \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right)^n \cdot \frac{1}{1+\underbrace{\left(\frac{1}{\sqrt{2}}\right)^n}_{0}} =$$

$$= \infty \cdot \frac{1}{1+0} = +\infty$$

② Justificați cu definiția

a) $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0 \iff \forall \varepsilon>0,\ \underline{\exists n_0\in\mathbb{N}}$ a.î. $\forall n\ge n_0 : \left|\dfrac{1}{\sqrt{n}}-0\right|<\varepsilon$

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$\frac{1}{\sqrt{n}} < \varepsilon \Rightarrow 1 < \varepsilon\sqrt{n} \; |^2 \Rightarrow \varepsilon^2 \cdot n > 1 \Rightarrow n > \frac{1}{\varepsilon^2}$

alegem $n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1$

b) $\lim\limits_{n \to +\infty} \frac{n^2}{n+1} = +\infty$

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 \quad \frac{n^2}{n+1} > \varepsilon$ (definiția pt. limita la $+\infty$)

$\frac{n^2}{n+1} = \frac{n^2 - 1 + 1}{n+1} = \frac{(n-1)(n+1)}{n+1} + \frac{1}{n+1} = \underbrace{(n-1)} + \underbrace{\frac{1}{n+1}}_{\to 0} > n-1 > \varepsilon \Rightarrow$

$n-1 \to$ crește cât mai mult $\to +\infty$

punem

vrem să facem o cantitate mare

$\Rightarrow n > \varepsilon + 1$

alegem $n_0 = [\varepsilon] + 2$

③ $(x_n)_{n \in \mathbb{N}}$ studiați convergența + limita (dacă e posibil)

a) $x_n = a^n, \; a \in \mathbb{R}$

$$\lim_{n \to \infty} a^n = \begin{cases} 0, & a \in (-1,1) \\ +\infty, & a \in (1, +\infty) \\ 1, & a = 1 \\ \nexists, & a \in (-\infty, -1) \end{cases}$$ – se demonstrează cu subșiruri

$a = -1: x_n = (-1)^n$

b) $x_n = \frac{2^n}{n!}$

$\frac{x_n}{x_{n+1}} = \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} = \frac{n+1}{2} > 1, \; \forall n \geq 2 \Rightarrow (x_n)$ descrescător (1)

+ Arătăm că e mărginit inferior

$x_n > 0, \forall n \in \mathbb{N} \Rightarrow (x_n)$ mărginit inferior (2)

(1), (2) $\Rightarrow (x_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow \underbrace{\exists\ L = \lim_{n\to\infty} x_n \in \mathbb{R}}$

$$\frac{x_{n+1}}{x_n} = \frac{2}{n+1} \Rightarrow x_{n+1} = \frac{2}{(n+1)} \cdot x_n \quad \Big|\ \lim_{n\to+\infty}$$

$$\Rightarrow L = 0 \cdot L = 0$$

pt. că $L \in \mathbb{R}$

1\. c) $x_n = \sqrt[n]{n}$ $\qquad \sqrt[n]{n} \geq 1$ (evident)

$$1 \leq \sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}} \quad \Big|\ \lim$$

verificăm $\sqrt[n]{n} < 1 + \sqrt{\frac{2}{n}} \quad \uparrow^n$ + dezvoltare cu combinări

Dem:

Notăm $y_n = \underbrace{\sqrt[n]{n}}_{\geq 1} - 1 \geq 0$

$$(1+y_n)^n = n \Rightarrow 1 + C_n^1 \cdot y_n + \underbrace{C_n^2 y_n^2}_{\text{o alegem pe asta}} + \dots + C_n^n y_n^n = n$$

$$C_n^2 y_n^2 < n, \ \forall n \geq 2$$

$$\frac{\cancel{n}(n-1)}{2} \cdot y_n^2 < \cancel{n} \Rightarrow \frac{n-1}{2} y_n^2 < 1 \Rightarrow$$

$$\Rightarrow 0 \leq y_n < \sqrt{\frac{2}{n-1}}, \ \forall n \geq 2$$

$n \to \infty$

0

$$\Rightarrow \lim_{n\to\infty} y_n = 0 \Rightarrow \lim_{n\to\infty} x_n = 1$$

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d) $x_n = \left(1+\frac{1}{n}\right)^n$ Ar că șirul e convergent

$$x_n = 1 + C_n^1 \cdot \frac{1}{n} + C_n^2 \cdot \frac{1}{n^2} + \dots + C_n^n \cdot \frac{1}{n^n} =$$

$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \dots + \frac{n(n-1)\dots 1}{n!} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{n}\right)}_{<1} + \frac{1}{3!}\underbrace{\left(1-\frac{1}{n}\right)}_{<1}\underbrace{\left(1-\frac{2}{n}\right)}_{<1} + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\dots$$

$$\underbrace{\left(1-\frac{n-1}{n}\right)}_{=\frac{1}{n}}$$

$$x_{n+1} = 1 + 1 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{n+1}\right)}_{\text{mai mare}} + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\dots$$

$$\cdot\left(1-\frac{n-1}{n+1}\right) + \underbrace{\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdot\dots\cdot\left(1-\frac{n}{n+1}\right)}_{>0} \Rightarrow x_n < x_{n+1}, \forall n \geq 1$$

$\Rightarrow (x_n)$ crescător.

În loc de fiecare ( ) de la x_n punem 1

$$x_n < \underbrace{1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!}}_{} < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}\right) = (*)$$

$$\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdot\dots\cdot n} < \frac{1}{1\cdot\underbrace{2\cdot 2\cdot 2\cdot\dots\cdot 2}_{n-1 \text{ ori}}} = \frac{1}{2^{n-1}} \quad \forall n \geq 3$$

$$(*) = 1 + 1\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 1 + 2\left(1 - \underbrace{\frac{1}{2^n}}_{<1}\right) < 3$$

$\Rightarrow x_n < 3, \forall n \geq 1. \Rightarrow (x_n)$ mărginit superior.

$\Rightarrow (x_n)$ convergent ; $\lim\limits_{n\to\infty} x_n \overset{not}{=} e \simeq 2{,}71$

! e) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$ $\quad \lim\limits_{n\to\infty} x_n = e$

convergent

(x_n) crescător

$\Rightarrow x_n < 3, \forall n \in \mathbb{N} \Rightarrow (x_n)$ mărginit superior

d)

$\Rightarrow (x_n)$ convergent. Fie $L = \lim\limits_{n\to\infty} x_n \in \mathbb{R}$

Notăm $y_n = \left(1 + \frac{1}{n}\right)^n$

$y_n < x_n, (\forall) n \geq 1 \Rightarrow \lim\limits_{n\to\infty} y_n \leq \lim\limits_{n\to\infty} x_n \Rightarrow e \leq L$ (1)

$\boxed{n \to \infty}$

Fie $p \in \mathbb{N}$ fixat și $n > p, n \in \mathbb{N}$

$y_n > 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \dots + \frac{1}{p!}\left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{p-1}{n}\right)$

$\underline{n \to \infty}$

$\Rightarrow \lim\limits_{n\to\infty} y_n \geq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{p!} = x_p \Rightarrow e \geq x_p$ (p arbitrar) $\forall p \in \mathbb{N}$ $\Rightarrow$

$p \to +\infty$

$\Rightarrow e \geq \lim\limits_{p\to\infty} x_p = L$ (2)

(1), (2) $\Rightarrow e \leq L \leq e \Rightarrow L = e$

f) $x_n = \frac{\sin(1!)}{1\cdot 2} + \frac{\sin(2!)}{2\cdot 3} + \dots + \frac{\sin(n!)}{n(n+1)}$

- Nu putem stabili monotonia (nu e monoton)

Studiem dacă (x_n) fundamental $(\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0,$ $\forall p \in \mathbb{N} : |x_{n+p} - x_n| < \varepsilon$

$$|x_{n+p} - x_n| = \left| \frac{\sin(1!)}{1\cdot 2} + \frac{\sin(2!)}{2\cdot 3} + \dots + \frac{\sin(n!)}{n(n+1)} + \frac{\sin(n+1)!}{(n+1)(n+2)} + \right.$$

$$\left. \dots + \frac{\sin(n+p)!}{(n+p)(n+p+1)} - \frac{\sin 1!}{1\cdot 2} - \frac{\sin 2!}{2\cdot 3} - \dots - \frac{\sin n!}{n(n+1)} \right| =$$

$$= \left| \frac{\sin(n+1)!}{(n+1)(n+2)} + \dots + \frac{\sin(n+p)!}{(n+p)(n+p+1)} \right| < \frac{|\sin(n+1)!|}{(n+1)(n+2)} + \dots + \frac{|\sin(n+p)!|}{(n+p)(n+p+1)}$$

$0 \leq |\sin| \leq 1$ și de-aia

$$\leq \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+p)(n+p+1)} = \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \dots$$

$$+ \left(\frac{1}{n+p} - \frac{1}{n+p+1}\right) = \frac{1}{n+1} - \frac{1}{n+p+1} < \underbrace{\frac{1}{n+1}}_{\geq 0} < \underbrace{\varepsilon}_{>0} \Rightarrow$$

punem

$\Rightarrow n > \frac{1}{\varepsilon} - 1 \Rightarrow$ alegem $n_0 \in \mathbb{N}; n_0 = \left[\frac{1}{\varepsilon}\right] \Rightarrow (x_n)$ fundamental

$\Rightarrow (x_n)$ convergent $\Rightarrow$

$$\Rightarrow \exists L = \lim_{n\to\infty} x_n \in \mathbb{R}$$

Seminar 3:

(4) de la Seminar 2

Determinați mulțimea punctelor limită, limita inferioară și limita superioară pt. șirurile

a) $x_n = (-1)^n \cdot n \sin \frac{n\pi}{2}$

$x_{2k} = (-1)^{2k} \cdot 2k \cdot \sin \frac{2k\pi}{2} = (-1)^{2k} \cdot 2k \sin k\pi = 2k \cdot 0 = 0$

$\to 0, \quad k \to \infty$

$x_{2k+1} = -(2k+1) \cdot \sin \frac{(2k+1)\pi}{2} = -(2k+1) \cdot \underbrace{\sin\left(k\pi + \frac{\pi}{2}\right)}_{(-1)^k} = (-1)^{k+1}(2k+1)$

$k = 2p$

$x_{4p+1} = -(4p+1) \to -\infty, \quad p \to \infty$

$k = 2p+1$

$x_{4p+3} = 4p+3 \to +\infty, \quad p \to \infty$

$LIM(x_n) = \{0, +\infty, -\infty\} \subseteq \overline{\mathbb{R}}$

$\liminf\limits_{n\to\infty} x_n = -\infty$ (infimumul lui $LIM(x_n)$)

$\limsup\limits_{n\to\infty} x_n = +\infty$ (supremumul lui $LIM(x_n)$)

b) $x_n = \left(1 + \frac{\cos(n\pi)}{n}\right)^n$

$a_n \to 0 \Rightarrow \lim\limits_{n\to\infty} (1+a_n)^{\frac{1}{a_n}} = e$.

<u>Criteriul</u> <u>Stolz</u> - <u>Cesaro</u>: Fie $(a_n)_{n\in\mathbb{N}}$ un șir oarecare de nr. reale și $(b_n)_{n\in\mathbb{N}}$ un șir strict monoton și divergent.
$(\Leftrightarrow b_n \nearrow +\infty)$

Dacă $\exists \lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \ell \in \overline{\mathbb{R}} \Rightarrow \exists \lim\limits_{n\to\infty} \frac{a_n}{b_n} = \ell$

! ① $(x_n)_{n\in\mathbb{N}}$ șir cu termeni strict pozitivi

$$\boxed{\text{Dacă } \exists \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \text{ atunci } \lim_{n\to\infty} \sqrt[n]{x_n} = l}$$

Reciproca e adevărată? → NU.

De reținut!

Alegem $a_n = \ln x_n$

$b_n = n \nearrow +\infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\ln(x_{n+1}) - \ln(x_n)}{(n+1)-n} = \lim_{n\to\infty} \ln \frac{x_{n+1}}{x_n} =$$

continuitatea lui ln (de-aici putem interschimba lim cu f)

$$= \ln\left(\lim_{n\to\infty} \frac{x_{n+1}}{x_n}\right) = \underline{\underline{\ln l}} \overset{S.C.}{\Longrightarrow}$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \ln l.$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n} \cdot \ln(x_n) = \lim_{n\to\infty} \ln x_n^{\frac{1}{n}} = \lim_{n\to\infty} \ln \sqrt[n]{x_n} =$$

$$= \underline{\underline{\ln\left(\lim_{n\to\infty} \sqrt[n]{x_n}\right)}} \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = l.$$

Reciproca:

ex: $x_n = 2^{(-1)^n} \quad \forall n\in\mathbb{N}$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \left[2^{(-1)^n}\right]^{\frac{1}{n}} = \lim_{n\to\infty} 2^{\frac{(-1)^n}{n}} = 2^0 = 1$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{2^{(-1)^{n+1}}}{2^{(-1)^n}} = \lim_{n\to\infty} 2^{(-1)^{n+1} - (-1)^n}$$

$$= \lim_{n\to\infty} 2^{(-1)^{n+1}(1+1)} = \lim_{n\to\infty} 2^{2(-1)^{n+1}} = \nexists$$

②. a) $y_n = \dfrac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}$ – sirul armonic diverge la fel de rapid la infinit ca ln.

$a_n = 1+\frac{1}{2}+\dots+\frac{1}{n}$

$b_n = \ln n \nearrow +\infty$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\frac{a_n}{b_n} \overset{S-C}{=} \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} =$$

$$= \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\ln(n+1)-\ln n} = \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\ln\left(\frac{n+1}{n}\right)} = \lim_{n\to\infty}\frac{1}{\ln\left(1+\frac{1}{n}\right)^{n+1}}$$

$$= \lim_{n\to\infty}\frac{1}{\ln \underbrace{\left(1+\frac{1}{n}\right)^n}_{=e}\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{1}{\ln e} = 1.$$

b) $y_n = \sqrt[n]{n!}$

$x_n = n!$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\sqrt[n]{x_n} \overset{①}{=} \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}(n+1) = +\infty$$

$\exists$

c) $y_n = \dfrac{\sqrt[n]{n!}}{n} = \sqrt[n]{\dfrac{n!}{n^n}}$

$x_n = \dfrac{n!}{n^n}$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\sqrt[n]{x_n} = \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{\cancel{(n+1)!}^{\,n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{\cancel{n!}}$$

$$= \lim_{n\to\infty}\frac{\cancel{(n+1)}}{\cancel{(n+1)}}\cdot\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

③ $a_n = \sum_{k=1}^{n} \frac{1+(-1)^k}{2}$

$\forall n \in \mathbb{N}^*$

$b_n = n$

Calculați $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$ și $\lim_{n\to\infty} \frac{a_n}{b_n}$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\frac{1+(-1)^{n+1}}{2}}{n+1-n} = \lim_{n\to\infty} \frac{1+(-1)^{n+1}}{2} \nexists$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} =$$

$$a_n = \underbrace{0+1+0+1+0+1+\dots}_{n \text{ termeni}} = \begin{cases} \frac{n}{2}, & n - \text{par} \\ \frac{n-1}{2}, & n - \text{impar} \end{cases}$$

$n = 2k+1$

$0 : k+1$

$1 : k$

calculăm $\lim_{k\to\infty} \frac{a_{2k}}{b_{2k}} = \lim_{k\to\infty} \frac{k}{2k} = \frac{1}{2}$

$$\lim_{k\to\infty} \frac{a_{2k+1}}{b_{2k+1}} = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}$$

$$LIM\left(\frac{a_n}{b_n}\right) = \left\{\frac{1}{2}\right\} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$$

Contrazice criteriul Stolz – Cesaro?

Nu contrazice, căci reciproca criteriului nu este în general adevărată.

④ a) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots=\sum_{n=0}^{\infty}\frac{1}{2n+1}=\sum_{n=1}^{\infty}\frac{1}{2n-1}$

b) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots=\sum_{n=0}^{\infty}\frac{1}{2^n}$

c) $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\ldots=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$

⑤ a) $\sum_{n=0}^{\infty}\frac{1}{n!}=\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{k!}=\lim_{n\to\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}\right)=e$

(demonstrat la seminar

b) $\sum_{n=1}^{\infty}\frac{1}{5^n}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{5^k}=\lim_{n\to\infty}\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots+\frac{1}{5^n}=$

$=\lim_{n\to\infty}\frac{1}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^{n-1}}\right)=\lim_{n\to\infty}\frac{1}{5}\cdot\frac{1-\left(\frac{1}{5}\right)^n}{1-\frac{1}{5}}=$

$=\lim_{n\to\infty}\cancel{\frac{1}{5}}\cdot\frac{1-\left(\frac{1}{5}\right)^n}{\cancel{\frac{4}{5}}\,4}=\frac{1}{4}$

$\sum_{n=0}^{\infty}a^n=\frac{1}{1-a},\quad a\in(-1,1)$

$a=\frac{1}{5}:\quad\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n=\sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^n-1=\frac{1}{1-\frac{1}{5}}-1=\frac{5}{4}-1$

$=\frac{1}{4}$

c) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n-1}}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{k}+\sqrt{k-1}}$

$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{k}-\sqrt{k-1}}{k-(k-1)}=\lim_{n\to\infty}\sum_{k=1}^{n}\sqrt{k}-\sqrt{k-1}$

$=\lim_{n\to\infty}\cancel{\sqrt{1}}-\sqrt{0}+\cancel{\sqrt{2}}-\cancel{\sqrt{1}}+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}+\cancel{\sqrt{4}}-\cancel{\sqrt{3}}+\ldots+\sqrt{n}-\cancel{\sqrt{n-1}}$

$=\lim_{n\to\infty}\sqrt{n}=+\infty$

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(2k+1)(2k-1)} =$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2} \frac{(2k+1)-(2k-1)}{(2k+1)(2k-1)} = \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right)$

$= \frac{1}{2} \lim_{n\to\infty} \left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1} \right)$

$= \frac{1}{2} \lim_{n\to\infty} 1 - \frac{1}{2n+1} = \frac{1}{2}$

! e) $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right) =$

$\ln\left(1-\frac{1}{n^2}\right) = \ln\left(1+\frac{1}{n}\right) + \ln\left(1-\frac{1}{n}\right) = \ln\frac{n+1}{n} + \ln\frac{n-1}{n}$

$= \ln\frac{n+1}{n} - \ln\frac{n}{n-1}$

$= \lim_{n\to\infty} \sum_{k=2}^{n} \left(\ln\frac{k+1}{k} - \ln\frac{k}{k-1} \right)$

$= \lim_{n\to\infty} \cancel{\ln\frac{3}{2}} - \ln\frac{2}{1} + \cancel{\ln\frac{4}{3}} - \cancel{\ln\frac{3}{2}} + \cancel{\ln\frac{5}{4}} - \cancel{\ln\frac{4}{3}} + \dots + \cancel{\ln\frac{n}{n-1}} - \cancel{\ln\frac{n-1}{n-2}}$

$+ \ln\frac{n+1}{n} - \cancel{\ln\frac{n}{n-1}} = \lim_{n\to\infty} -\ln 2 + \ln\left(1+\frac{1}{n}\right) = -\ln 2$

! f) $\sum_{n=1}^{\infty} \frac{n\cdot 2^n}{(n+2)!} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\cdot 2^k}{(k+2)!} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(k+2-2)\cdot 2^k}{(k+2)!} =$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\cancel{(k+2)}\cdot 2^k}{\underbrace{\cancel{(k+2)!}}_{(k+1)!}} - \frac{2^{k+1}}{(k+2)!} = \sum_{k=1}^{n} \frac{2^k}{(k+1)!} - \frac{2^{k+1}}{(k+2)!}$

$= \lim_{n\to\infty} \frac{2}{2!} - \frac{2^2}{3!} + \frac{2^2}{3!} - \frac{2^3}{4!} + \frac{2^3}{4!} - \dots + \frac{2^n}{(n+1)!} - \frac{2^{n+1}}{(n+2)!}$

$$= \lim_{n\to\infty} 1 - \frac{2^{n+1}}{(n+2)!} = 1$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+2}}{(n+3)!} \cdot \frac{(n+2)!}{2^{n+1}} = \lim_{n\to\infty} \frac{2}{n+3} = 0 < 1 \Rightarrow \lim_{n\to\infty} a_n = 0$$

Seminar 4: Natura unor serii de numere reale

① Studiați natura următoarelor s.t.p. utilizând criteriile indicate:

i) criteriul comparației

a) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{4n^2-1}}$

$\underline{\sum_{n=1}^{\infty} \frac{1}{n} \text{ divergentă}}$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{4n^2-1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{4n^2-1}} = \lim_{n\to\infty} \frac{n}{n\sqrt{4-\underbrace{\frac{1}{n^2}}_{0}}} = \frac{1}{2} \in (0,+\infty) \Rightarrow$$

seriile au aceeași natură $\Rightarrow$ seria este divergentă

b) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergentă, $\underline{p=2>1}$ (seria armonică generalizată)

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right)^{n^2} = \ln \lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^{n^2} = \ln e = 1 \in (0,+\infty) \Rightarrow$$

$\Rightarrow$ seriile au aceeași natură $\Rightarrow$ seria este convergentă

ii) consecințe ale criteriului lui Kummer

a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$x_n = \frac{2^n}{n!}$$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{n!}} \cdot \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{2^{n+1}}_{\,2}} = \lim_{n\to\infty} \frac{n+1}{2} = +\infty > 1$$

$\Rightarrow$ seria este convergentă

b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}}$

$x_n = \left(\frac{1}{2}\right)^{\sqrt{n}}$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^{\sqrt{n}}}{\left(\frac{1}{2}\right)^{\sqrt{n+1}}} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{\sqrt{n}-\sqrt{n+1}} =$$

$$= \lim_{n\to\infty} 2^{\sqrt{n+1}-\sqrt{n}} = 2^{\lim\limits_{n\to\infty} \sqrt{n+1}-\sqrt{n}} = 2^{\lim\limits_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}} =$$

$= 2^0 = 1$, => criteriul nu decide natura

$$R = \lim_{n\to\infty} n\cdot\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{\left(\frac{1}{2}\right)^{\sqrt{n}}}{\left(\frac{1}{2}\right)^{\sqrt{n+1}}} - 1\right)$$

$$= \lim_{n\to\infty} n\left(2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}} - 1\right) \underset{0\cdot\infty}{=} \lim_{n\to\infty} n\cdot \underbrace{\frac{2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}}-1}{\frac{1}{\sqrt{n+1}+\sqrt{n}}}}_{=\ln 2} \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a,\ a>0}$$

$= \ln 2 \cdot \lim\limits_{n\to\infty} \frac{n}{\sqrt{n+1}+\sqrt{n}} = +\infty > 1$ => seria este convergentă

c) $\sum_{n=1}^{\infty} \left[\frac{(2n)!!}{(2n+1)!!}\right]^2$

<u>Notații</u>: $\boxed{\begin{aligned}(2n)!! &= 2\cdot 4\cdot 6 \cdot\ldots\cdot (2n)\\ (2n+1)!! &= 1\cdot 3\cdot 5\cdot\ldots\cdot(2n+1)\end{aligned}}$

$x_n = \left[\frac{(2n)!!}{(2n+1)!!}\right]^2$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left[\frac{(2n)!!}{\cancel{(2n+1)!!}}\right]^2 \cdot \left[\frac{\overset{(2n+3)^2}{\cancel{(2n+3)!!}}}{\underset{(2n+2)^2}{\cancel{(2n+2)!!}}}\right]^2$$

$= \lim\limits_{n\to\infty} \left(\frac{2n+3}{2n+2}\right)^2 = 1$ => criteriul nu decide natura

$$R=\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty} n\cdot\left(\frac{4n^2+12n+9}{4n^2+8n+4}-1\right)=$$

$$=\lim_{n\to\infty} n\left(\frac{\cancel{4n^2}+12n+9-\cancel{4n^2}-8n-4}{4n^2+8n+4}\right)=\lim_{n\to\infty}\frac{4n^2+5n}{4n^2+8n+4}=1.$$

$\Rightarrow$ criteriul nu decide.

$$B=\lim_{n\to\infty}\ln n\cdot\left[n\left(\frac{x_n}{x_{n+1}}-1\right)-1\right]$$

$$=\lim_{n\to\infty}\ln n\cdot\left[\frac{4n^2+5n}{4n^2+8n+4}-1\right]=\lim_{n\to\infty}(\ln n)\cdot\frac{-3n-4}{4n^2+8n+4}\overset{0\cdot\infty}{=}$$

$$=\lim_{n\to\infty}\underbrace{\frac{\ln n}{n}}_{0}\cdot\underbrace{\lim_{n\to\infty}\frac{-3n^2-4n}{4n^2+8n+4}}_{-\frac{3}{4}}=0\cdot\left(-\frac{3}{4}\right)=0.<1\Rightarrow \text{seria este divergentă}$$

$$\boxed{\lim_{n\to\infty}\sqrt[n]{n}=1}$$

$$\frac{\ln n}{n}=\ln\sqrt[n]{n}\to\ln 1=0$$
!!

iii) criteriul radicalului

$$\sum_{n=1}^{\infty}\frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$x_n=\frac{n^2}{\left(2+\frac{1}{n}\right)^n},$$

$$C=\lim_{n\to\infty}\sqrt[n]{x_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{\left(2+\frac{1}{n}\right)^n}}=\lim_{n\to\infty}\frac{\left(\sqrt[n]{n}\right)^2}{2+\frac{1}{n}}=\frac{1}{2}<1\Rightarrow$$

seria este convergentă

iv) criteriul condensării

$$\sum_{n=2}^{\infty}\frac{1}{n\cdot(\ln n)^p},\ p>0.$$

$x_n=\dfrac{1}{n(\ln n)^p}$ descrescător cu termeni pozitivi

$\ln n\ \nearrow\ p>0$

$$\Rightarrow \sum_{n=2}^{\infty} x_n \underset{\substack{\text{aceeasi} \\ \text{natura}}}{\sim} \sum_{n=1}^{\infty} 2^n \cdot x_{2^n} =$$

$$= \sum_{n=1}^{\infty} 2^n \frac{1}{2^n (\ln 2^n)^p} = \sum_{n=1}^{\infty} \frac{1}{(n \cdot \ln 2)^p} = \frac{1}{(\ln 2)^p} \sum_{n=1}^{\infty} \frac{1}{n^p}$$

serie armonică generalizată

convergentă $\Leftrightarrow p > 1$

caz particular $p = 1$: $\boxed{\sum_{n=2}^{\infty} \frac{1}{n \cdot (\ln n)}}$ divergentă

convergentă

divergentă

$$\frac{1}{n^2} < \frac{1}{n \ln n} < \frac{1}{n}$$

(2) Studiați convergența și absolut convergența seriilor cu termeni oarecare

a) $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{2n+1}{3^n}$

remarcăm că seria este alternată

$$a_n = \frac{2n+1}{3^n}$$

$$\frac{a_n}{a_{n+1}} = \frac{2n+1}{\cancel{3^n}} \cdot \frac{\cancel{3^{n+1}}\,3}{2n+3} = \frac{6n+3}{2n+3} > 1 \Rightarrow (a_n) \text{ descrescător}$$ C.L. $\Rightarrow$

$$\lim_{n \to \infty} a_n = 0$$

$\sum_{n=0}^{\infty} (-1)^n \cdot a_n$ este convergentă

$$\sum_{n=0}^{\infty} \left| (-1)^n \cdot \frac{2n+1}{3^n} \right| = \sum_{n=0}^{\infty} \frac{2n+1}{3^n} \quad \text{s.t.p}$$

$$D = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{6n+3}{2n+3} = 3 > 1 \Rightarrow \text{seria este convergentă} \Rightarrow$$

seria modulelor este conv. $\Rightarrow$ seria din enunț este absolut convergentă

Scanned with CamScanner

b) $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

$\sum_{n=1}^{\infty} \frac{|\sin n|}{2^n}$

$x_n = \frac{1}{2^n}$

$y_n = \frac{|\sin n|}{2^n}$

$|\sin n| \leq 1 \,|:2^n \Rightarrow \quad y_n \leq x_n$

criteriul $\Rightarrow$ comparației $\sum_{n=1}^{\infty} y_n$ convergentă $\Rightarrow$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ este convergentă (serie geometrică) $a = \frac{1}{2} \in (-1,1)$

$\Rightarrow$ seria din enunț este absolut convergentă $\Rightarrow$ este și convergentă

3. $\boxed{\text{Criteriul raportului pentru șiruri}}$

Fie $(x_n)_{n\in\mathbb{N}}$ un șir cu termeni strict pozitivi pentru care

$\exists \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l$. Au loc afirmațiile

i) Dacă $l > 1 \Rightarrow \lim_{n\to\infty} x_n = 0$

ii) Dacă $l < 1 \Rightarrow \lim_{n\to\infty} x_n = +\infty$

-DEM-

Este o observație!! Bag De la teorie

i) Fie seria $\sum_{n=0}^{\infty} x_n$ (s.t.p.), $D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l > 1 \Rightarrow$ seria este convergentă

$\underbrace{\Rightarrow \lim_{n\to\infty} x_n = 0}_{\text{șirul } x_n \text{ descrescător}}$. De ce?

ii) Fie seria $\sum_{n=0}^{\infty} \frac{1}{x_n}$, $D = \lim_{n\to\infty} \frac{\frac{1}{x_n}}{\frac{1}{x_{n+1}}} = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{l} > 1 \Rightarrow$

$\underbrace{\lim_{n\to\infty} \frac{1}{x_n} = 0}$ $\Rightarrow \lim_{n\to\infty} x_n = +\infty$

ex: $x_n = \frac{2n+1}{3^n}$

$\lim\limits_{n\to\infty} \frac{x_n}{x_{n+1}} = 3 > 1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

(5) Fie $\sum\limits_{n=1}^{\infty} x_n$ o stp. Ar că:

$$\sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$$

arătăm că sunt convergente simultan

I) Fie $\sum\limits_{n=1}^{\infty} x_n$ convergentă

$$\frac{x_n}{1+\underbrace{x_n}_{\geq 0}} \leq x_n \quad , \forall n \in \mathbb{N} \xRightarrow[\text{Comparației}]{\text{Criteriul}} \sum_{n=1}^{\infty} \frac{x_n}{1+x_n} \text{ conv.}$$

II) Fie $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ convergentă $\Rightarrow \lim\limits_{n\to\infty} \frac{x_n}{1+x_n} = 0 \Rightarrow \lim\limits_{n\to\infty} \frac{1}{\left(\frac{1}{x_n}+1\right)} = 0 \Rightarrow$

$\lim\limits_{n\to\infty} \frac{\frac{x_n}{1+x_n}}{x_n} = \lim\limits_{n\to\infty} \frac{1}{1+x_n} =$

$= 1 \in (0, +\infty) \Rightarrow$ seriile au aceeași natură

$\Rightarrow \lim\limits_{n\to\infty} \frac{1}{x_n} = +\infty \Rightarrow$

$\lim\limits_{n\to\infty} x_n = 0$

Dacă seria $\sum\limits_{n=1}^{\infty} x_n$ convergentă $\Rightarrow \lim\limits_{n\to\infty} x_n = 0$

Seminar 5:

(1). Justificați afirmațiile:

i) $\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}, \forall n \in \mathbb{N}^*$

$f: (0, +\infty) \to \mathbb{R}$ $f(x) = \ln x$, aplicăm T. de medie a lui Lagrange

pe $[n, n+1]$, $n \in \mathbb{N}^*$ fixat

$\Rightarrow \exists x_0 \in (n, n+1)$: a.î. $f'(x_0) = \frac{f(n+1) - f(n)}{n+1-n} = \ln(n+1) - \ln n \Rightarrow$

$\Rightarrow \frac{1}{x_0} = \ln(n+1) - \ln n$

$x_0 \in (n, n+1) \Rightarrow \frac{1}{x_0} \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ $\Big| \Rightarrow \frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$

$\forall n \in \mathbb{N}^*$

ii) $c_{n+1} - c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n+1} - \ln(n+1) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n\right)$

$= \frac{1}{n+1} - \ln(n+1) + \ln n < 0 \Rightarrow (c_n)$ descrescător.

$c_n \leq c_1 = 1$

$\ln 2 - \ln 1 < 1$

$\ln 3 - \ln 2 < \frac{1}{2}$

$\dots$

$\ln(n+1) - \ln n < \frac{1}{n}$

$\oplus$

$\ln(n+1) - \ln 1 < 1 + \frac{1}{2} + \dots + \frac{1}{n}$ $\Big| - \ln n \Rightarrow$

$\Rightarrow 0 < \ln(n+1) - \ln n < c_n \Rightarrow (c_n)$ mărginit inferior

$\Rightarrow$

$(c_n)_{n \geq 1}$ convergent $\Rightarrow \exists \lim\limits_{n \to \infty} c_n \overset{not}{=} \gamma$ (constanta lui Euler)

$\simeq 0{,}57$

Aplicație:

$\sum\limits_{n=1}^{\infty} \left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \dots + \frac{1}{n}} \sim \sum\limits_{n=1}^{\infty} \frac{1}{n}$ divergentă

Folosim criteriul comparației:

$\lim\limits_{n \to \infty} \frac{\left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \dots + \frac{1}{n}}}{\frac{1}{n}} = \lim\limits_{n \to \infty} \frac{\left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \dots + \frac{1}{n}}}{\left(\frac{1}{e}\right)^{\ln n}} = \lim\limits_{n \to \infty} \left(\frac{1}{e}\right)^{c_n} = \left(\frac{1}{e}\right)^{\gamma} \in (0, +\infty) \Rightarrow$

$\Rightarrow$ seriile au aceeași natură

② Determinați mulțimea punctelor de acumulare A' pentru:

a) $A = \left\{ \frac{1}{2^n} \mid n \in \mathbb{N} \right\}$

$\lim\limits_{n \to \infty} \frac{1}{2^n} = 0 \Rightarrow 0 \in A' \Rightarrow A' = \{0\}$

b) $A = \mathbb{Q}$

$(\forall) r \in \mathbb{R}$, $\exists (x_n) \subseteq \mathbb{Q}$ un şir cu $\lim\limits_{n \to \infty} x_n = r \Rightarrow r \in A'$

$x_n = n \to +\infty$

$x_n = -n \to -\infty$

$\Rightarrow A' = \overline{\mathbb{R}}$

③ Valori extreme. Verif. dacă funcțiile își ating valorile extreme și det. valorile.

a) $f : (-1, 1) \to \mathbb{R}$ $\quad f(x) = \ln \frac{1-x}{1+x}$

valori extreme: $\begin{cases} \inf f(A) \\ \sup \underbrace{f(A)}_{\text{Imaginea}} \end{cases}$

T. lui W nu se aplică pt. că A nu este interval compact - mărginit și închis

$\lim\limits_{x \downarrow -1} \ln \frac{1-x}{1+x} = \ln \frac{+2}{0_+} = +\infty$

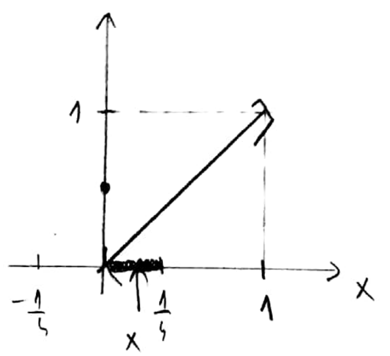
$\lim\limits_{x \uparrow 1} \ln \frac{1-x}{1+x} = \ln \frac{0_+}{2} = -\infty$

$\Rightarrow \inf f(A) = -\infty$

$\sup f(A) = +\infty$, $A = (-1, 1)$

- nu se ating

b) $f : [0, 1] \to \mathbb{R}$ $\quad f(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ x, & x \in (0, 1] \end{cases}$



$A = [0, 1]$

$\sup f(A) = 1 = f(1)$

$\inf f(A) = 0$ - nu se atinge

$\lim\limits_{x \searrow 0} f(x) = 0.$

f nu e continuă în 0 $\quad f(0) = \frac{1}{2} \neq \lim\limits_{x \searrow 0} f(x) = 0 \Rightarrow$ f nu este cont în $x = 0$.

T. lui W nu se aplică pt. că

c) $f:[-1,1]\to\mathbb{R}$ $f(x)=x\sqrt{1-x^2}$

f continuă pe $[-1,1] \overset{T.W.}{\Longrightarrow}$ f mărginită și își atinge valorile extreme

I. cu derivate - fără

II. fără derivate

$A=[-1,1]$, $x\in A$, $x=\sin t$, $t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$f(x)=f(\sin t)=\sin t\cdot\sqrt{1-\sin^2 t}=\sin t\cdot|\cos t|=\sin t\cos t=$

$\frac{1}{2}\underbrace{\sin 2t}_{\in[-1,1]}\in\left[-\frac{1}{2},\frac{1}{2}\right]$

$\Rightarrow \begin{cases}\inf f(A)=-\frac{1}{2}=f\left(-\frac{\sqrt2}{2}\right)\\ \sup f(A)=\frac{1}{2}=f\left(\frac{\sqrt2}{2}\right)\end{cases}$ se ating !!

$\frac{1}{2}\sin 2t=\frac{1}{2}\Rightarrow 2t=\frac{\pi}{2}\Rightarrow t=\frac{\pi}{4}\Rightarrow x=\sin\frac{\pi}{4}=\frac{\sqrt2}{2}$

(5) **Caracterizarea monotoniei cu ajutorul derivatei** - DEMONSTRAȚII -

Fie $f:(a,b)\to\mathbb{R}$ o funcție derivabilă pe (a,b)

$a,b\in\mathbb{R}$, sau $\pm\infty$

a) $f\nearrow$ pe $(a,b) \Leftrightarrow f'(x)\geq 0,\ \forall x\in(a,b)$

b) $f\searrow$ pe $(a,b) \Leftrightarrow f'(x)\leq 0,\ \forall x\in(a,b)$

c) dacă $f'(x)>0,\ \forall x\in(a,b) \Rightarrow f$ este strict crescătoare pe (a,b)

d) dacă $f'(x)<0,\ \forall x\in(a,b) \Rightarrow f$ este strict descrescătoare pe (a,b)

În general reciprocele la c) și d) nu sunt adevărate. Justificați:

a) f cresc. pe $(a,b) \Leftrightarrow \forall x,y\in(a,b)$, $x<y$ avem $f(x)\leq f(y)$

„$\Rightarrow$" Fie $x,y\in(a,b)$, $x<y \Rightarrow \dfrac{f(y)-f(x)}{y-x}\geq 0 \overset{\lim_{y\searrow x}}{\Longrightarrow}$

$\Rightarrow \lim\limits_{y\searrow x}\dfrac{f(y)-f(x)}{y-x}\geq 0 \Rightarrow f'(x)\geq 0$.

dacă f derivabilă $\Rightarrow f'_d(x)=f'(x)$

"$\Leftarrow$" Fie $x, y \in (a, b)$, $x<y$ $\quad x<y$

aplicăm T. de medie pe $[x, y] \Rightarrow \exists\, c \in (x, y)$ a.î.

$$f'(c) = \frac{f(y)-f(x)}{y-x} \geq 0 \Rightarrow$$

$f(y) \geq f(x) \Rightarrow f$ crescătoare

b) Analog!

c), d) similar (doar că avem inegalitate strictă)

Ex: c) $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^3$, strict $\nearrow$

$$x<y \overset{?}{\Rightarrow} x^3 < y^3$$

$$f'(x) = 3x^2 \quad f'(0) = 0$$

5 Det. punctele de extrem local ale funcțiilor de la ex 3.

a) $f: (-1, 1) \to \mathbb{R}$ $f(x) = \ln \frac{1-x}{1+x}$

f derivabilă pe $(-1, 1)$

$$f'(x) = \left(\ln(1-x) - \ln(1+x)\right)' = -\frac{1}{1-x} - \frac{1}{1+x} = \frac{-(1+x)-1+x}{(1-x)(1+x)} = \frac{-2}{\underbrace{1-x^2}_{\geq 0}}$$

$\neq 0, \forall x \in (-1, 1)$

$\Rightarrow f$ nu are puncte de extrem local (T. lui Fermat)

b) $f: [0, 1] \to \mathbb{R}$ $f(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ x, & x \in (0, 1] \end{cases}$

f este derivabilă pe $(0, 1)$ $\quad f'(x) = 1 \neq 0 \Rightarrow f$ nu are puncte de extrem local pe $(0, 1)$

$x = 1$, $f(1) = 1 \geq f(x), \forall x \in [0, 1]$ punct de maxim local

Dem. cu definiția:

$x = 0$ punct de maxim local

$\Leftrightarrow \exists\, \delta > 0$ a.î. $\forall x \in (-\delta, \delta) \cap [0, 1]$ avem $\frac{1}{2} = f(0) \geq f(x)$

(mai trebuie să alegem un δ)

$(\delta \in (0, \frac{1}{2}])$ alegem $\delta = \frac{1}{4}$, $\forall x \in [0, \frac{1}{4})$

$\frac{1}{2} \geq f(x)$

(dacă luăm orice x din $[0, \frac{1}{4})$, $f(x) \leq \frac{1}{2}$)

c) $f : [-1, 1] \to \mathbb{R}$ $f(x) = x \cdot \sqrt{1-x^2}$

f derivabilă pe $(-1, 1)$!!

$$f'(x) = \sqrt{1-x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}} = 0$$

$\Rightarrow 1-2x^2 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x_1 = \frac{1}{\sqrt{2}}$, $x_2 = -\frac{1}{\sqrt{2}}$ ... $x = \pm\frac{\sqrt{2}}{2}$ (puncte critice / candidat)

x	-1		$-\frac{\sqrt{2}}{2}$		$\frac{\sqrt{2}}{2}$		1
		$- -$	0	$+ +$	0	$- -$	
f	0	$\searrow$	$f(-\frac{\sqrt{2}}{2})$	$\nearrow$	$f(\frac{\sqrt{2}}{2})$	$\searrow$	0

$\Rightarrow$ $x_1 = -\frac{\sqrt{2}}{2}$ punct de minim local

$x_2 = \frac{\sqrt{2}}{2}$ punct de maxim local.

$f(-1) = 0$

$f(1) = 0$

din tabel $\Rightarrow$ $x = -1$ pct de maxim local

$x = 1$ pct de minim local

!! Trebuie tratate separat punctele în care funcția nu este derivabilă !!

Lucrare de control $\to$ ex de-astea

(6) Folosind regula lui l'Hopital, calculați limitele

a) $$\lim_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{x} \overset{\frac{0}{0}}{=} \lim_{x \to 0} \frac{-(1+x)^{\frac{1}{x}} \cdot \left(-\frac{1}{x^2} \cdot \ln(1+x) + \frac{1}{x} \cdot \frac{1}{1+x}\right)}{1} = (*)$$

$u = u(x) = 1+x$

$v = v(x) = \frac{1}{x}$

$$(u^v)' = (e^{\ln u^v})' = (e^{v \ln u})' = e^{v \ln u} (v \ln u)'$$

$$= u^v \cdot \left(v' \ln u + v \cdot \frac{1}{u} \cdot u'\right)$$

$$(*) = -e \cdot \lim_{x \to 0} \left(\frac{x - (x+1) \cdot \ln(x+1)}{x^2 (x+1)}\right) \overset{\frac{0}{0}}{=}$$

$$= -e \lim_{x \to 0} \frac{1 - \ln(x+1) - \cancel{(x+1)} \cdot \frac{1}{\cancel{x+1}}}{2x(x+1) - x^2} = -e \lim_{x \to 0} \frac{-\ln(x+1)}{x^2+2x} \overset{\frac{0}{0}}{=}$$

$$= -e \lim_{x \to 0} -\frac{1}{x+1} \cdot \frac{1}{2x+2} = -e \cdot \left(-\frac{1}{2}\right) = \frac{e}{2}$$

5) $\lim\limits_{x \to \infty} \frac{x^\alpha}{e^x}$, $\alpha \in \mathbb{R}$. parametru - ?

Seminar 6: Serii Taylor și serii de puteri

① a) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = +\sin x = f^{(0)}(x)$

$$f^{(n)}(x) = \begin{cases} (-1)^k \cdot \sin x & n = 2k \\ (-1)^k \cos x & n = 2k+1 \end{cases}$$

mulțimea pe care f este indefinit derivabilă : $\mathbb{R}$.

b) $f(x) = \ln(x+1)$ $f : \underbrace{(-1, +\infty)}_{\text{domeniu}} \to \mathbb{R}$

$f'(x) = \frac{1}{x+1} = (x+1)^{-1}$ - indefinit derivabilă pe $(-1, +\infty)$

$f''(x) = -(x+1)^{-2}$

$f'''(x) = 2(x+1)^{-3} = 1 \cdot 2 \cdot (x+1)^{-3}$

$f^{(4)}(x) = -6(x+1)^{-4} = -1 \cdot 2 \cdot 3 \cdot (x+1)^{-4}$

....

$f^{(n)}(x) = (-1)^{n-1} \cdot (n-1)! \cdot (x+1)^{-n} \quad \forall n \geq 1$

- prin inducție se verifică

c) $u = u(x)$ $\quad f(x) = (x^2 - x) \cdot e^x$

$v = v(x)$

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot u^{(k)} \cdot v^{(n-k)} \quad , \forall n \in \mathbb{N}$$

$u(x) = x^2 - x$

$v(x) = e^x$

$u'(x) = 2x - 1$ $\quad v(x) = e^x$

$u''(x) = 2$ $\quad v^{(n)}(x) = e^x, \ \forall n \in \mathbb{N}$

$u'''(x) = 0$

⋮

$u^{(n)}(x) = 0, \ \forall n \geq 3$

$$f^{(n)}(x) = C_n^0 \cdot (x^2-x)\cdot e^x + C_n^1\cdot (2x-1)\cdot e^x + C_n^2 \cdot 2\cdot e^x + 0$$

$$= (x^2-x)\cdot e^x + n(2x-1)\cdot e^x + \frac{n\cdot(n-1)}{\cancel{2}}\cancel{2}\cdot e^x$$

$$= e^x(x^2-x+n(2x-1)+n(n-1))$$

$$= e^x(x^2+(2n-1)x+n^2-2n), \quad \forall n \in \mathbb{N}$$

se verifică și pt. $n=0, n=1$; indefinit derivabilă pe $\mathbb{R}$

$f: (-\infty, 1] \to \mathbb{R}$

$1-x \geq 0$ $1 \geq x$

d) $f(x) = \sqrt{1-x}$

$$f'(x) = \frac{-1}{2\sqrt{1-x}} = -\frac{1}{2}\cdot(\sqrt{1-x})^{-1} = -\frac{1}{2}\cdot(1-x)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}\cdot\left(-\frac{1}{2}\right)(1-x)^{-\frac{1}{2}-1} = -\frac{1}{4}\cdot(1-x)^{-\frac{3}{2}}$$

$$f'''(x) = \left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)\cdot(1-x)^{-\frac{5}{2}} = -\frac{3}{8}(1-x)^{-\frac{5}{2}} = -\frac{1\cdot 3}{2^3}\cdot(1-x)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = -\frac{15}{16}\cdot(1-x)^{-\frac{7}{2}} = -\frac{1\cdot 3\cdot 5}{2^4}\cdot(1-x)^{-\frac{7}{2}}$$

$\dots$

$$f^{(n)}(x) = -\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-3)}{2^n}\cdot(1-x)^{-\frac{(2n-1)}{2}}, \quad \forall n \geq 2.$$

prin inducție dem

f este indefinit derivabilă pe $(-\infty, +1)$

(2) $x_0 = 0$ pt. ex 1.

$n \in \mathbb{N}$

a) Polinomul lui Taylor de grad n asociat funcției f în x_0

b) Mulțimea de convergență a seriei Taylor coresp.

a) $f(x) = \sin x$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot(x-x_0)^k \overset{x_0=0}{=} \sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}\cdot x^k =$$

șirul sumelor parțiale a seriei Taylor

$$f^{(n)}(0) = \begin{cases} 0, & n=2k \\ (-1)^k, & n=2k+1 \end{cases}$$

$$= 0 + \frac{1}{1!}\cdot x + 0 - \frac{1}{3!}\cdot x^3 + 0 + \frac{1}{5!}x^5 + \ldots + \underbrace{\frac{f^{(n)}(0)}{n!}\cdot x^n}$$

$0, \; n = 2k$

$\frac{(-1)^k}{(2k+1)!}\cdot x^{2k+1}, \; n = 2k+1$

Seria Taylor: $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n =$

$\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{(2n+1)!}}_{a_{2n+1}} \cdot x^{2n+1}$

I - mulțimea de convergență

$\left.\begin{matrix} a_{2n+1} \\ a_{2n} = 0 \end{matrix}\right\} \Rightarrow$ nu putem folosi r

serie de puteri: $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ dacă $\exists$ lim

Fie $x \in \mathbb{R}$ fixat, studiem absolut convergența seriei

$\sum_{n=0}^{\infty} \frac{|x|^{2n+1}}{(2n+1)!}$ s.t.p.

$x = 0 \Rightarrow$ seria e conv.

$x \neq 0$, $D = \lim_{n\to\infty} \frac{|x|^{2n+1}}{(2n+1)!} \cdot \frac{(2n+3)!}{|x|^{2n+3}} = \lim_{n\to\infty} \frac{(2n+2)(2n+3)}{\underbrace{|x|^2}_{>0}} = +\infty > 1$, $\forall x \neq 0$.

$\Rightarrow$ serie conv. $\forall x \in \mathbb{R}^*$ $\Rightarrow$ seria inițială este absolut convergentă

$\Rightarrow I = \mathbb{R}$.

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}$, $\forall x \in \mathbb{R}$. (dezvoltarea în serie Taylor a funcției $\sin x$)

5) $f(x) = \ln(x+1)$

$f^{(n)}(0) = (-1)^{n-1} \cdot (n-1)!$, $\forall n \geq 1$

$f(0) = 0$

$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k = 0 + \sum_{k=1}^{n} \frac{(-1)^{k-1} \cdot \cancel{(k-1)!}}{\cancel{k!} \cdot k} \cdot x^k$

$= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot x^k$

Seria Taylor: $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n-1}}{n}}_{a_n} \cdot x^n$

I = mulțimea de convergență

$a_n = \frac{(-1)^{n-1}}{n}$

$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n-1}}}{n} \cdot \frac{n+1}{\cancel{(-1)^{n}}} \right| = 1.$

-1

$\Rightarrow (-1, 1) \subseteq I \subseteq [-1, 1]$

pt. $x=-1$: $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)}{n}}_{!} = -\sum_{n=1}^{\infty} \frac{1}{n}$ divergentă $\Rightarrow -1 \notin I$
seria armonică

pt. $x=1$: $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}}_{!}$ convergentă
seria armonică alternată

$I = (-1,1]$

$$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot x^n, \quad \forall x \in (-1,1]$$

c) – temă ($I=\mathbb{R}$)

d) $f(x) = \sqrt{1-x}$

$$f^{(n)}(x) = -\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-3)}{2^n} \cdot (1-x)^{\frac{-(2n-1)}{2}}, \quad \forall n \geq 2$$

$2^n (2n-3)!!$

$$f^{(n)}(0) = -\frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-3)}{2^n}$$

$f(0) = 1$

$f'(0) = -\frac{1}{2}$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k = 1 - \frac{1}{2}x + \sum_{k=2}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k =$$

$$= 1 - \frac{1}{2}x + \sum_{k=2}^{n} -\frac{(2k-3)!!}{2^k \cdot k!} \cdot x^k$$

Seria Taylor : $1 - \frac{x}{2} - \underbrace{\sum_{n=2}^{\infty} \overbrace{\frac{(2n-3)!!}{2^n \cdot n!}}^{a_n} \cdot x^n}$ ne referim la asta
cealaltă din față nu contează

I = mulțimea de convergență

$$a_n = \frac{(2n-3)!!}{2^n \cdot n!}$$

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(2n-3)!!}}{\cancel{2^n} \cdot \cancel{n!}} \cdot \frac{\overset{2}{\cancel{2^{n+1}}} \cdot \overset{n+1}{\cancel{(n+1)!}}}{\underset{2n-1}{\cancel{(2n-1)!!}}} \right| =$$

$$= \lim_{n\to\infty} \frac{2n+2}{2n-1} = 1$$

$$\Rightarrow (-1,1) \subseteq I \subseteq [-1,1]$$

pt. $x=1$: seria : $\sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!}$ (stp.)

$$D = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{2n+2}{2n-1} = 1$$ nu decide!

$$R = \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n-1} - 1\right) = \lim_{n\to\infty} n \cdot \frac{3}{2n-1} = \frac{3}{2} > 1$$

$\Rightarrow$ serie convergentă $\Rightarrow 1 \in I$

pt. $x = -1$: $\sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot (-1)^n$ este absolut convergentă $\Rightarrow -1 \in I$

$\Rightarrow I = [-1, 1]$

$$\sqrt{1-x} = 1 - \frac{x}{2} - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot x^n, \forall x \in [-1, 1]$$

③ Operații cu serii de puteri

a) $\sum_{n=0}^{\infty} (-1)^n \cdot (n+1) \cdot x^n = \frac{1}{(1+x)^2}, \forall x \in (-1, 1), r = 1$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n, \forall x \in (-1, 1) \quad \Big|'$$

$\Rightarrow \frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot x^{n-1}$ (primul termen e constantă, dacă derivăm devine 0)

re-indexare
$n-1 = m$

$\Rightarrow \frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n \cdot x^{n-1} = \sum_{m=0}^{\infty} (-1)^{m+2} \cdot (m+1) \cdot x^m$ ✓

am justificat egalitatea doar

verificăm intervalul.
r se păstrează după derivare

$\Rightarrow r = 1$ și pt. seria derivată $\Rightarrow (-1, 1) \subseteq I \subseteq [-1, 1]$

pt. $x = -1, x = 1$, în ambele situații seria este divergentă $\Rightarrow$

$$I = (-1, 1)$$

b) prin derivare de la $\sqrt{1-x}$ Temă

Seminar 7:

(5) - de la seminar 6

$$\sum_{n=1}^{\infty} \frac{1}{n^2}(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n \cdot (x-x_0)^n$$

$a_n = \frac{1}{n^2}$

$x_0 = 1$

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = 1.$$

$(x_0 - r, x_0 + r)$

$(0,2) \subseteq I \subseteq [0,2]$

$x = 0$: $\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot (-1)^n$, $a_n = \frac{1}{n^2} \searrow 0 \overset{C.L.}{\Rightarrow}$ serie convergentă $\Rightarrow$ $0 \in I$

$x = 2$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ serie armonică generalizată cu $p = 2 > 1$ $\Rightarrow$ este convergentă $\Rightarrow 2 \in I$

$\Rightarrow I = [0,2]$

(1) a) $\int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\,dx = \ln|e^x + \sqrt{e^{2x}+1}|\Big|_0^1 = \ln|e+\sqrt{e^2+1}| - \ln|1+\sqrt{2}|$

$$\int \frac{u'(x)}{\sqrt{u^2(x)+a^2}}\,dx = \ln|u(x) + \sqrt{u^2(x)+a^2}| + C$$

b) $\int_0^2 \max\{x, x^2\}\,dx = \int_0^1 x\,dx + \int_1^2 x^2\,dx = \frac{x^2}{2}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2$
$= \frac{1}{2} - 0 + \frac{8}{3} - \frac{1}{3}$

$$\max\{x, x^2\} = \begin{cases} x, & x \in [0,1] \\ x^2, & x \in [1,2] \end{cases}$$

c) $\int_1^{\sqrt{3}} \frac{\operatorname{arctg} x}{x^2} dx = \int_1^{\sqrt{3}} \left(-\frac{1}{x}\right)' \cdot \operatorname{arctg} x \, dx$

$= -\frac{1}{x} \cdot \operatorname{arctg} x \Big|_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{1}{x} \cdot (\operatorname{arctg} x)' dx$

$= -\frac{1}{\sqrt{3}} \cdot \frac{\pi}{3} + \frac{\pi}{4} + \int_1^{\sqrt{3}} \frac{1}{x} \cdot \frac{1}{x^2+1} dx.$

$= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$

$= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + \left(\ln x - \frac{1}{2} \cdot \ln(x^2+1)\right) \Big|_1^{\sqrt{3}}$

$= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + \ln\sqrt{3} - \frac{1}{2} \ln 4 + \frac{1}{2} \ln 2$

d) $\int_{-1}^{1} \sqrt{1-x^2} \, dx$

<u>Substituții</u> <u>trigonometrice</u> <u>pentru</u> <u>integrale</u> <u>algebrice</u>:

Fie $R(u,v)$ o <u>funcție rațională</u> și $a>0$,

i) $\int R(x, \sqrt{a^2-x^2}) dx$, $x = a \cdot \sin t$ sau $x = a \cdot \cos t$

ii) $\int R(x, \sqrt{a^2+x^2}) dx$, $x = a \cdot \operatorname{tg} t$ sau $x = a \cdot \operatorname{ctg} t$

iii) $\int R(x, \sqrt{x^2-a^2}) dx$, $x = \frac{a}{\sin t}$ sau $x = \frac{a}{\cos t}$

d) $x = \sin t \Rightarrow dx = +\cos t \, dt.$

$x = 1 \Rightarrow \sin$

$\sqrt{1-x^2} = \sqrt{\cos^2 t} = |\cos t|$

$dx = +\cos t \, dt$

$x = -1 \Rightarrow t = -\frac{\pi}{2}$

$x = 1 \Rightarrow t = \frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos t| \cdot \cos t \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t \, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} dt =$$

$$\cos 2t = 2\cos^2 t - 1 \Rightarrow \cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \frac{t}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{\sin 2t}{4}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

e) $\int_2^4 \frac{\sqrt{x^2-4}}{x} dx =$

$x = \frac{2}{\sin t}$ $\qquad dx = \frac{-2}{\sin^2 t} \cdot \cos t \, dt$

$x=2 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$

$x=4 \Rightarrow \sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$$\frac{\sqrt{x^2-4}}{x} = \frac{\sqrt{\frac{4}{\sin^2 t} - 4}}{\frac{2}{\sin t}} = \frac{\sqrt{\frac{4-4\sin^2 t}{\sin^2 t}}}{\frac{2}{\sin t}}$$

$$= \frac{2 \cdot \left|\frac{\cos t}{\sin t}\right|}{\frac{2}{\sin t}} = \left|\frac{\cos t}{\sin t}\right| \cdot \sin t$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \left|\frac{\cos t}{\sin t}\right| \cdot \sin t \cdot \left(-\frac{2}{\sin^2 t} \cdot \cos t\right) dt$$

$$= 2 \cdot \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t} dt = 2 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1-\sin^2 t}{\sin^2 t} dt = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\sin^2 t} dt - 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 \, dt$$

$$= -2 \operatorname{ctg} t \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} - 2t \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

Seminar 8: Integrale improprii

① a) $\int_0^{\infty} \frac{\operatorname{arctg} x}{1+x^2} dx = \lim_{v \to \infty} \int_0^v (\operatorname{arctg} x)(\operatorname{arctg} x)' dx$

$= \lim_{v \to \infty} \frac{\operatorname{arctg}^2 x}{2} \Big|_0^v = \lim_{v \to \infty} \frac{\operatorname{arctg}^2 v}{2} = \frac{\pi^2}{8}$.

b) $\int_{-1+0}^{1-0} \frac{x+1}{\sqrt{1-x^2}} dx = \lim_{u \searrow -1} \int_u^0 \frac{x+1}{\sqrt{1-x^2}} dx + \lim_{v \nearrow 1} \int_0^v \frac{x+1}{\sqrt{1-x^2}} dx$

$\int \frac{x+1}{\sqrt{1-x^2}} dx = \int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + \arcsin x$

$= \lim_{u \searrow -1} \left(-\sqrt{1-x^2} + \arcsin x\right)\Big|_u^0 + \lim_{v \nearrow 1} \left(-\sqrt{1-x^2} + \arcsin x\right)\Big|_0^v$

$= \lim_{u \searrow -1} \left(-1 + \sqrt{1-u^2} - \arcsin u\right) + \lim_{v \nearrow 1} \left(-\sqrt{1-v^2} + \arcsin v + 1\right)$

$= -1 + \frac{\pi}{2} + \frac{\pi}{2} + 1 = \pi$

c) $\int_0^{\infty} x^n \cdot e^{-x} dx \qquad n \in \mathbb{N}$

$I_n = \int_0^{\infty} x^n \cdot e^{-x} dx = \lim_{v \to \infty} \int_0^v x^n \cdot (-e^{-x})' dx$

$= \lim_{v \to \infty} \left[(-e^{-x} \cdot x^n)\Big|_0^v + \int_0^v n \cdot x^{n-1} \cdot e^{-x} dx \right]$

$= \lim_{v \to \infty} \left[-e^{-v} \cdot v^n + n \int_0^v x^{n-1} \cdot e^{-x} dx \right]$

$\lim_{v \to \infty} \frac{-v^n}{e^v} = \lim_{v \to \infty} \frac{-n \cdot v^{n-1}}{e^v} = \ldots = \lim_{v \to \infty} \frac{-n!}{e^v} = 0.$

$= 0 + n I_{n-1}$

$\Rightarrow I_n = n \cdot I_{n-1}, \ \forall\ n \geq 1.$

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$$I_0 = \lim_{v \to \infty} \int_0^v e^{-x} dx = \lim_{v \to \infty} (-e^{-x}) \Big|_0^v = 1.$$

$$I_1 = 1 \cdot I_0 = 1$$

$$I_2 = 2 \cdot I_1 = 2 \cdot 1 = 2!$$

$$I_3 = 3 \cdot I_2 = 3!$$

$$\dots$$

$$I_n = n!, \ \forall n \in \mathbb{N}$$

d) $$\int_1^{2-0} \frac{1}{\sqrt{x(2-x)}} dx = \lim_{x \nearrow 2} \int_1^v \frac{1}{\sqrt{2x-x^2}} dx = \lim_{x \nearrow 2} \int_1^v \frac{1}{\sqrt{1-(x-1)^2}} dx$$

$$= \lim_{v \nearrow 2} (\arcsin(x-1)) \Big|_1^v = \lim_{v \nearrow 2} \arcsin(v-1) = \frac{\pi}{2}$$

(2) Convergența integralelor improprii

a) $$\int_0^3 \frac{x^3+1}{\sqrt{9-x^2}} dx$$

$f: [0,3) \to [0,+\infty) \quad f(x) = \dfrac{x^3+1}{\sqrt{9-x^2}}$

P1: $$\lambda = \lim_{x \nearrow 3} (3-x)^p \cdot f(x) = \lim_{x \nearrow 3} (3-x)^p \cdot \frac{x^3+1}{\sqrt{(3-x)(3+x)}} =$$

$$= \lim_{x \nearrow 3} (3-x)^{p-\frac{1}{2}} \cdot \frac{x^3+1}{\sqrt{3+x}}$$

i) $p < 1$, $\lambda < +\infty \Rightarrow$ conv

ii) $p \geq 1$, $\lambda > 0 \Rightarrow$ div

alegem $p = \frac{1}{2} \Rightarrow \lambda = \dfrac{28}{\sqrt{6}} < +\infty \Rightarrow$ int. convergentă

b) $$\int_{0+0}^{\infty} \frac{\operatorname{arctg} x}{x} dx = \underbrace{\int_0^1 \frac{\operatorname{arctg} x}{x} dx}_{I_1} + \underbrace{\int_1^{\infty} \frac{\operatorname{arctg} x}{x} dx}_{I_2}$$

P3

pt. I_1: $f(x) = \dfrac{\operatorname{arctg} x}{x} \quad f: (0,1] \to [0,+\infty)$

$$\lambda = \lim_{x\downarrow 0} x^p \cdot f(x) = \lim_{x\downarrow 0} x^p \cdot \frac{\operatorname{arctg} x}{x} = \lim_{x\downarrow 0} x^{p-1} \operatorname{arctg} x$$

$\cancel{p=1,\ \lambda=0}$

$$p=0 \Rightarrow \lambda = \lim_{x\downarrow 0} \frac{\operatorname{arctg} x}{x} \overset{\frac{0}{0}}{=} \lim_{x\downarrow 0} \frac{1}{1+x^2} = 1 < +\infty \quad \Big| \Rightarrow I_1 \text{ conv.}$$

$p = 0 < 1$

$I_2:\ f(x) = \frac{\operatorname{arctg} x}{x} \quad f:[1,+\infty) \to [0,+\infty)$

$\boxed{P_2}$

$$\lambda = \lim_{x\to+\infty} x^p \cdot f(x) = \lim_{x\to+\infty} x^p \cdot \frac{\operatorname{arctg} x}{x}$$

i) $p>1$, $\lambda < +\infty \Rightarrow$ conv

ii) $p \le 1$, $\lambda > 0 \Rightarrow$ div.

$p=1 \Rightarrow \lambda = \frac{\pi}{2} > 0 \Rightarrow I_2$ div.

$$\underset{\substack{\downarrow \\ c}}{I_1} + \underset{\substack{\downarrow \\ +\infty}}{I_2} = \text{divergentă} \qquad c+\infty = +\infty$$

c) $$\int_0^{\pi} x \ln(\sin x)\,dx = -\underbrace{\int_{0+0}^{\frac{\pi}{2}} -x \ln(\sin x)\,dx}_{I_1} + (-1)\underbrace{\int_{\frac{\pi}{2}}^{\pi-0} -x \ln \sin x}_{I_2}$$

$\sin x \in [0,1] \Rightarrow \ln(\sin x) < 0$

pt. I_1: $f : (0, \frac{\pi}{2}] \to [0,+\infty) \quad f(x) = -x \ln(\sin x)$

$\boxed{P_3}$

$$\lambda = \lim_{x\downarrow 0} x^p \cdot (-x \cdot \ln \sin x) = -\lim_{x\downarrow 0} x^{p+1} \cdot \ln \sin x$$

alegem $p = -\frac{1}{2}$, $\lambda = -\lim_{x\downarrow 0} \sqrt{x} \cdot \ln \sin x \overset{0\cdot(-\infty)}{=} 0$ (de la curs)

$$\left.\begin{array}{l} p = -\frac{1}{2} < 1 \\ \lambda < +\infty \end{array}\right| \Rightarrow I_1 \text{ conv.}$$

pt. I_2: $f: [\frac{\pi}{2}, \pi) \to [0,+\infty) \quad f(x) = -x \ln(\sin x)$

$\boxed{P_1}$

$$\lambda = \lim_{x\uparrow\pi} (\pi - x)^p \cdot (-x) \cdot \ln(\sin x)$$

$y = \pi - x \searrow 0$

$$\lambda = \lim_{y \downarrow 0} y^p \cdot (y-\pi) \cdot \ln \sin(\pi - y) = \lim_{y\downarrow 0} y^p (y-\pi) \cdot \ln(\sin y) =$$

$$\left(\sin(\pi - y) = \sin\pi \cos y - \cos\pi \cdot \sin y = \sin y \right)$$

$$= -\pi \cdot \lim_{y \downarrow 0} y^p \cdot \ln(\sin y)$$

alegem $p = \frac{1}{2}$ $\Rightarrow \lambda = -\pi \cdot 0 = 0 < +\infty \Rightarrow I_2$ conv.

$\Rightarrow -I_1 - I_2$ convergentă

③ Studiați convergența

$$I(\alpha) = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha \cdot dx \, , \ \alpha \in \mathbb{R}$$

alegerea lui p depinde de α

+ calculați $I(\frac{1}{2})$

$\rightarrow$ natura integralei $I(\alpha)$

$\underline{\text{cazul I}}$: $\alpha > 0$: $f(x) = \left(\frac{x}{1-x}\right)^\alpha$ $f : [0,1) \to [0,+\infty)$

P1)

$$\lambda = \lim_{x \uparrow 1} (1-x)^p \cdot \frac{x^\alpha}{(1-x)^\alpha} = \lim_{x\uparrow 1} (1-x)^{p-\alpha} \cdot \underbrace{x^\alpha}_{\to 1}$$

$$= \lim_{x \uparrow 1} (1-x)^{p-\alpha}$$

alegem $p = \alpha$, $\lambda = 1$

- dacă $p < 1$ $(\alpha < 1) \to I(\alpha)$ conv.
- dacă $p \geq 1$ $(\alpha \geq 1) \Rightarrow I(\alpha)$ conv.

$\underline{\text{cazul II}}$: $\alpha < 0$ $f(x) = \left(\frac{x}{1-x}\right)^\alpha$ $f : (0,1] \to [0,+\infty)$

P3)

$$f(x) = \left(\frac{1-x}{x}\right)^{-\alpha}.$$

$$\lambda = \lim_{x\downarrow 0} x^p \cdot \frac{x^\alpha}{(1-x)^\alpha} = \lim_{x \downarrow 0} x^{p+\alpha} \cdot \underbrace{\frac{1}{(1-x)^\alpha}}_{\to 1} = \lim_{x\downarrow 0} x^{p+\alpha}$$

alegem $p = -\alpha$, $\lambda = 1$.

- dacă $p < 1$ $(\alpha > -1) \Rightarrow I(\alpha)$ conv.
- dacă $p \geq 1$ $(\alpha \leq -1) \Rightarrow I(\alpha)$ div.

Cazul III : $\alpha = 0$

$$I(0) = \int_0^1 dx = 1 \quad \text{(integrală proprie)} \Rightarrow \text{(conv)}$$

$$\underline{I(\alpha) \text{ convergentă} \Leftrightarrow \alpha \in (-1,1)}$$

$$I\left(\tfrac{1}{2}\right) = \int_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} dx = \underbrace{\int_0^{1-0} \sqrt{\frac{x}{1-x}}\, dx}_{\text{sdv. aici } (*)} = \lim_{v \uparrow 1} \int_0^v \sqrt{\frac{x}{1-x}}\, dx$$

$$\sqrt{\frac{x}{1-x}} = t, \; t \geq 0 \Rightarrow \frac{x}{1-x} = t^2 \Rightarrow x = t^2(1-x)$$

$$\Rightarrow x(1+t^2) = t^2 \Rightarrow$$

$$\Rightarrow x = \frac{t^2}{t^2+1}$$

$x = 0 \Rightarrow t = 0$

$x \uparrow 1 \Rightarrow \lim_{x \uparrow 1} \sqrt{\frac{x}{1-x}} = +\infty \Rightarrow t \to +\infty$

$$dx = \frac{2t(t^2+1) - t^2 \cdot 2t}{(t^2+1)^2} dt$$

$$dx = \frac{2t}{(t^2+1)^2} dt$$

$$(*) I\left(\tfrac{1}{2}\right) = \int_0^\infty t \cdot \frac{2t}{(1+t^2)^2} dt = \int_0^\infty t \cdot \left(-\frac{1}{1+t^2}\right)' dt =$$

$$\left(\frac{1}{1+t^2}\right)' = \frac{-2t}{(1+t^2)^2}$$

$$= -\frac{t}{1+t^2}\Big|_0^\infty + \int_0^\infty \frac{1}{1+t^2} dt$$

$$= \lim_{v \to \infty} \left(-\frac{t}{1+t^2}\Big|_0^v + \operatorname{arctg} t \Big|_0^v\right) = \lim_{v \to \infty} -\frac{v}{1+v^2} + \operatorname{arctg} v = \frac{\pi}{2}$$

) (funcția Γ) Considerăm integrala improprie

$$\boxed{\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx, \quad \alpha \in \mathbb{R}}$$

Demonstrați urm. propr.:

a) $\Gamma(\alpha)$ conv., $\forall \alpha > 0$ (data viitoare)

b) $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$

c) $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$, $\forall \alpha > 0$

d) $\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n} \cdot \Gamma(\frac{1}{2})$, $\forall n \in \mathbb{N}^*$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

[temă - d)]

b) $\Gamma(n+1) = \int_0^\infty x^n \cdot e^{-x}\, dx = n!$, $\forall n \in \mathbb{N}$

c) $\Gamma(\alpha+1) = \int_0^\infty x^\alpha \cdot e^{-x}\, dx = \int_0^\infty x^\alpha \cdot (-e^{-x})'\, dx =$

$= x^\alpha \cdot (-e^{-x}) \Big|_0^\infty - \int_0^\infty \alpha \cdot x^{\alpha-1} \cdot (-e^{-x})\, dx$

$= \lim\limits_{v \to \infty} (-x^\alpha \cdot e^{-x}) \Big|_0^v + \alpha \cdot \Gamma(\alpha)$

$\lim\limits_{v \to \infty} (-v^\alpha \cdot e^{-v}) = -\lim\limits_{v \to \infty} \frac{v^\alpha}{e^v} = 0$, $\forall \alpha \in \mathbb{R}$ (arătat la temă)

$\Rightarrow \Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

$n! = n \cdot (n-1)!$

a) $\Gamma(\alpha) = \int_{0+0} x^{\alpha-1} \cdot e^{-x}\, dx \quad , \alpha \in \mathbb{R}$

$$\Gamma(\alpha) = \underbrace{\int_{0+0}^{1} x^{\alpha-1} e^{-x}\, dx}_{I_1} + \underbrace{\int_{1}^{+\infty} x^{\alpha-1} e^{-x}\, dx}_{I_2}$$

<u>St. I_1</u>:

$x \geq 0 \Rightarrow e^{x} \geq 1 \Rightarrow e^{-x} \leq 1 \,|\cdot x^{\alpha-1}$

$x^{\alpha-1} e^{-x} \leq x^{\alpha-1}, \forall x \in [0,1]$

$\int_0^1 x^{\alpha-1}\, dx = \frac{x^\alpha}{\alpha}\Big|_{\boxed{0}}^{1} = \frac{1^\alpha - 0^\alpha}{\alpha} = \frac{1}{\alpha} \in \mathbb{R}$ (convergentă)

aici am folosit $\alpha > 0$.

$\overset{\text{C.C. (Crit comp)}}{\Longrightarrow}$ I_1 convergentă

<u>St I_2</u>: $f: [1,+\infty) \to [0,+\infty) \quad f(x) = x^{\alpha-1} e^{-x}$

$\lambda = \lim_{x\to\infty} x^p \cdot f(x) = \lim_{x\to\infty} x^{p+\alpha-1} \cdot e^{-x} = 0, \forall p, \alpha \in \mathbb{R}$.

alegem $p=2>1$, $\lambda = 0 < +\infty \Rightarrow I_2$ convergentă

$\Rightarrow \Gamma(\alpha)$ conv., $\forall \alpha > 0$

5) Exprimați cu ajutorul lui Γ valoarea integralelor improprii.

$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx$

a) $\int_0^\infty e^{-x^2} dx$

$x = \sqrt{t} \quad dx = \frac{1}{2\sqrt{t}}\, dt$

$x = 0 \Rightarrow t = 0$

$x \to +\infty \Rightarrow t \to +\infty$

$$\int_0^\infty e^{-t} \cdot \frac{1}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}} \cdot e^{-t}\, dt = \frac{1}{2} \cdot \underbrace{\Gamma\left(\frac{1}{2}\right)}_{\sqrt{\pi}}$$

b) $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx$ — Temă

c) $\int_{0+0}^{1} (\ln x)^{\frac{1}{3}} dx$

$t = -\ln x \Rightarrow x = e^{-t} \quad dx = -e^{-t} dt$

$x \searrow 0 \Rightarrow t \to +\infty$

$x = 1 \Rightarrow t = 0$

$$\int_{+\infty}^{0} (-t)^{\frac{1}{3}} \cdot (-e^{-t})\,dt = -\int_{0}^{\infty} t^{\frac{4}{3}-1} \cdot e^{-t}\,dt = -\Gamma\left(\frac{4}{3}\right) = -\Gamma\left(1+\frac{1}{3}\right)$$

$$= -\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$$

Seminar 9:

① Fie $x=(1,0,-1)$ $y=(3,-1,1)\in\mathbb{R}^3$. Calculați $x+y$, $x\cdot y$, $\|x\|$, $\|-2y\|$ și $\|x-y\|$

$x+y=(1+3,0-1,-1+1)=(4,-1,0)$

$x\cdot y = 1\cdot 3+0\cdot(-1)+(-1)\cdot 1 = 2\in\mathbb{R}$

$\|x\| = \sqrt{x\cdot x} = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}$

$\|-2y\| = |-2|\cdot\|y\| = 2\cdot\sqrt{3^2+(-1)^2+1^2} = 2\sqrt{11}$

$\|x-y\| = \sqrt{(1-3)^2+(0+1)^2+(-1-1)^2} = \sqrt{9} = 3$

② Fie $x,y\in\mathbb{R}^m$ $a=x\cdot y$ $b=\|x\|$ $c=\|y\|$ $a,b,c\in\mathbb{R}$.
Exprimați în fct. de a,b,c.

a) $(x+y)\cdot y = x\cdot y+y\cdot y = a+\|y\|^2 = a+c^2$

b) $x\cdot(2x-y) = 2(x\cdot x)-(x\cdot y) = 2\cdot\|x\|^2-a = 2b^2-a$

c) $\|x-y\| = \sqrt{(x-y)\cdot(x-y)} = \sqrt{x\cdot x-x\cdot y-y\cdot x+y\cdot y} = \sqrt{b^2-2a+c^2}$

③ $x,y\in\mathbb{R}^m$. Demonstrați identitatea paralelogramului.

$$\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$$

suma pătratelor diag = suma pătratelor lat.

$\|x+y\|^2+\|x-y\|^2 =$

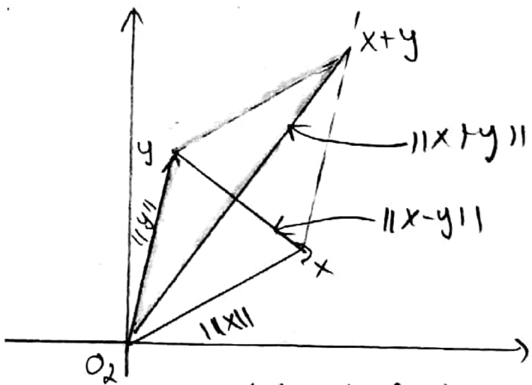
$= \left(\sqrt{(x+y)\cdot(x+y)}\right)^2+\left(\sqrt{(x-y)\cdot(x-y)}\right)^2$

$= x\cdot x+\cancel{2x\cdot y}+y\cdot y+x\cdot x-\cancel{2x\cdot y}+y\cdot y$

$= 2x\cdot x+2\cdot y\cdot y$

$= 2\|x\|^2+2\cdot\|y\|^2 = 2(\|x\|^2+\|y\|^2)$ ✓

$\mathbb{R}^2$



ineg. triunghiului

$\|x+y\| \le \|x\|+\|y\|$

$\forall x,y\in\mathbb{R}: |x+y|\le|x|+|y|$

⑤ Determinați int A, fr A, precum și dacă A este mulțime deschisă, închisă

a) $A = B(O_2,1)\subseteq\mathbb{R}^2$ (bilă de centru O și rază 1)

b) $A = [2,+\infty)\times(2,+\infty)\subseteq\mathbb{R}^2$ ↳ produs cartezian

c) $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$

d) $A = \mathbb{R} \setminus \mathbb{Z} \subseteq \mathbb{R}$

$A \subseteq \mathbb{R}^m$

$\text{int } A = \{x \in \mathbb{R}^m \mid \exists r > 0 : \overset{\text{vecinătatea}}{B(x,r)} \subseteq A\}$

$\text{fr } A = \{x \in \mathbb{R}^m \mid \forall r > 0 : B(x,r) \cap A \neq \emptyset$ și $B(x,r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset\}$

A deschisă $\Leftrightarrow A = \text{int } A$

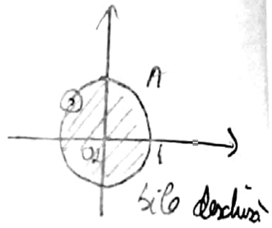
A închisă $\Leftrightarrow \mathbb{R}^m \setminus A$ deschisă

A deschisă $\Leftrightarrow A \cap \text{fr } A = \emptyset$

A închisă $\Leftrightarrow \text{fr } A \subseteq A$

a) $B(0_2, 1) = \{x \in \mathbb{R}^2 \mid \|x - 0_2\| < 1\}$

$x = (x_1, x_2)$ $= \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1)^2 + (x_2)^2 < 1\}$

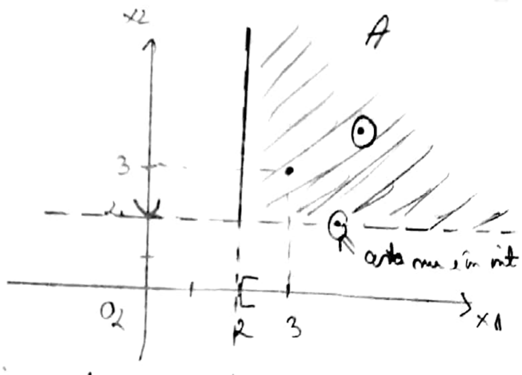


$\text{int } A = A$

$\text{fr } A = \{(x_1, x_2) \in \mathbb{R}^2 \mid (x_1)^2 + (x_2)^2 = 1\}$

A deschisă

b) $A = [2, +\infty) \times (2, +\infty) \subseteq \mathbb{R}^2$



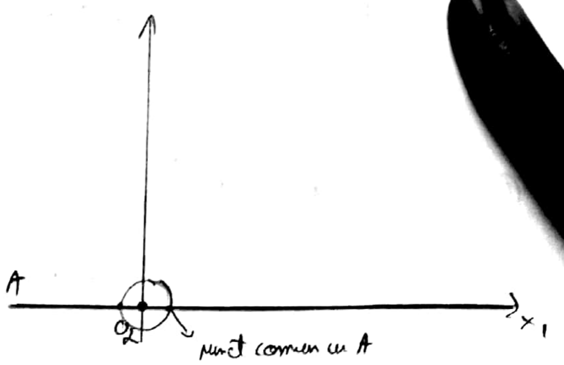
$\text{int } A = (2, +\infty) \times (2, +\infty) = (2, +\infty)^2$

$\text{fr } A = \Big(\underbrace{(2, +\infty) \times \{2\}}_{\text{semidr. orizontală}}\Big) \cup \Big(\underbrace{\{2\} \times [2, +\infty)}_{\text{semidr. verticală}}\Big)$

$A \neq \text{int } A$, $\text{fr } A \not\subseteq A$ frontiera are și semidr. orizontală

A nu este nici deschisă, nici închisă

c) $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$

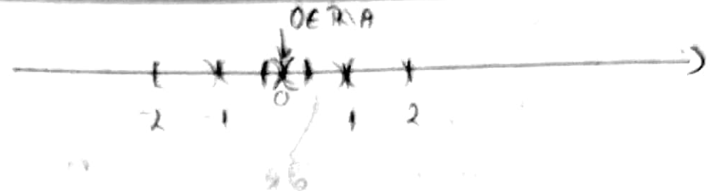


$\text{int } A = \emptyset$

$\text{fr } A = A$

A este închisă

d) $A = \mathbb{R} \setminus \mathbb{Z} \subseteq \mathbb{R}$



$A = \bigcup_{k \in \mathbb{Z}} (k, k+1)$

$\text{int } A = A \Rightarrow A$ deschisă

$\text{fr } A = \mathbb{Z}$

(5) $\forall A \in \mathbb{R}^m$ mulțime nevidă, au loc afirmațiile:

a) $\text{int } A \subseteq A$

b) $\text{int } A \cap \text{fr } A = \emptyset$

c) $A \subseteq \text{int } A \cup \text{fr } A$ (cu egalitate dacă A este închisă)

d) $\text{int } A \cup \text{fr } A \cup \text{int } (\mathbb{R}^m \setminus A) = \mathbb{R}^m$

a) Fie $x \in \text{int } A \Rightarrow \exists r > 0 : B(x, r) \subseteq A$
$x \in B(x, r) \Leftrightarrow \underbrace{\|x - x\|}_{0_m} < r \Leftrightarrow 0 < r$ $\Big| \Rightarrow x \in A \Rightarrow \text{int } A \subseteq A$

b) pp că $\exists x \in \text{int } A \cap \text{fr } A \Rightarrow \begin{cases} x \in \text{int } A \Rightarrow \exists r_0 > 0 : B(x, r_0) \subseteq A \ (1) \\ \text{și} \\ x \in \text{fr } A \Rightarrow \forall r > 0 : B(x, r) \cap A \neq \emptyset \\ \quad \text{și } B(x, r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset \end{cases}$

alegem $r = r_0$ $\quad B(x, r_0) \nsubseteq A$, (2)

Din (1), (2) $\Rightarrow$ contradicție $\Rightarrow \text{int } A \cap \text{fr } A = \emptyset$

c) Fie $\underline{x \in A}$, putem presupune că $x \notin \text{int } A \Rightarrow \forall r > 0 : B(x, r) \nsubseteq A \Rightarrow$
$\Rightarrow \forall r > 0 : B(x, r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset$ $\Big\} \Rightarrow$

De arătat că $\underline{\forall r > 0 : B(x, r) \cap A \neq \emptyset}$ căci $x \in A$ și $x \in B(x, r)$.

$\Rightarrow x \in \text{fr } A$.

*): Dacă A este închisă, $\text{int } A \cup \text{fr } A \subseteq A$.
$\left.\begin{array}{l} \text{int } A \subseteq A \\ \text{fr } A \subseteq A \end{array}\right| \Rightarrow$

$\boxed{\text{Dacă } A \subseteq \mathbb{R}^m \text{ este închisă, } A = \text{int } A \cup \text{fr } A}$ (și invers e adevărat)

d) $\forall x \in \mathbb{R}^m$

$\operatorname{int} A \cup \operatorname{fr} A \cup \operatorname{int}(\mathbb{R}^m \setminus A) = \mathbb{R}^m$

pp că x nu e în două din ele

pp. $\underbrace{x \notin \operatorname{int} A}_{(1)}$ și $\underbrace{x \notin \operatorname{int}(\mathbb{R}^m \setminus A)}_{(2)}$

(1) $\forall r>0 : B(x,r) \nsubseteq A \Rightarrow \underline{B(x,r) \cap (\mathbb{R}^m \setminus A) \neq \emptyset}$

(2) $\forall r>0 : B(x,r) \nsubseteq \mathbb{R}^m \setminus A \Rightarrow \underline{B(x,r) \cap A \neq \emptyset}$

$\Rightarrow x \in \operatorname{fr} A$

Seminar 10

① Studiați existența limitelor de funcții:

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{1+xy}-1} \overset{xy=t\to 0}{=} \lim\limits_{t\to 0} \dfrac{t}{\sqrt{1+t}-1} = \lim\limits_{t\to 0} \dfrac{\cancel{t}(\sqrt{1+t}+1)}{\cancel{1+t-1}} = 2.$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

$\lim\limits_{x\to 0} \lim\limits_{y\to 0} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2} = 1$

$\lim\limits_{y\to 0} \lim\limits_{x\to 0} \dfrac{x^2-y^2}{x^2+y^2} = \lim\limits_{y\to 0} -\dfrac{y^2}{y^2} = -1$

limitele iterate nu sunt egale $\Rightarrow \nexists$ limita

c) $\lim\limits_{(x,y)\to(\infty,\infty)} \dfrac{x^2+y^2}{x^4+y^4}$ intuim că limita e 0

Notăm $f(x,y) = \dfrac{x^2+y^2}{x^4+y^4}$

$|f(x,y) - \underset{\substack{\uparrow \\ \text{limita} \\ \text{intuită}}}{0}| = \dfrac{x^2+y^2}{x^4+y^4} = \dfrac{x^2}{x^4+y^4} + \dfrac{y^2}{x^4+y^4} \le \dfrac{x^2}{x^4} + \dfrac{y^2}{y^4} = \dfrac{1}{x^2} + \dfrac{1}{y^2} \xrightarrow[(x,y)\to(\infty,\infty)]{} 0$

$\Rightarrow \lim\limits_{(x,y)\to(\infty,\infty)} f(x,y) = 0$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x \cdot \sin(x^2-y^2)}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \underbrace{\dfrac{\sin(x^2-y^2)}{x^2-y^2}}_{\downarrow 1} \cdot \dfrac{x(x^2-y^2)}{x^2+y^2} =$

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$$= \underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{=1} \cdot \lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{x^2+y^2} =$$

$$f(x,y) = \frac{x(x^2-y^2)}{x^2+y^2}$$

$$|f(x,y) - 0| = \left|\frac{x(x^2-y^2)}{x^2+y^2}\right| = |x| \cdot \underbrace{\left|\frac{x^2-y^2}{x^2+y^2}\right|}_{\leq 1} \leq |x| \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

e) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{xy}$

$a_n \to (0,0)$

$b_n \to (0,0)$ $\quad n \to +\infty$

a.î. $\lim\limits_{n\to\infty} f(a_n) \neq \lim\limits_{n\to\infty} f(b_n) \Rightarrow \nexists \lim\limits_{(x,y)\to 0_2} f(x,y)$

$f(x,y) = \dfrac{x^3+y^3}{xy}$ $\quad$ alegem $a_n = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$, $n \to +\infty$

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{\frac{1}{n^3}+\frac{1}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2\cancel{n^2}}{\cancel{n^3}\,n} = \lim_{n\to\infty} \frac{2}{n} = 0$$

$b_n = \left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$ $\quad n \to +\infty$

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \frac{\frac{1}{n^3}+\frac{1}{n^6}}{\frac{1}{n^3}} = \lim_{n\to\infty} 1 + \frac{1}{n^3} = 1.$$

$$\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n) \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

<u>Clase de șiruri din $\mathbb{R}^2$ cu limita $(0,0)$</u>

$$\boxed{\begin{array}{ll} a_n = \left(\frac{1}{n}, \frac{\alpha}{n}\right) & \alpha \in \mathbb{R} \\ b_n = \left(\frac{1}{n}, \frac{1}{n^\alpha}\right) & \alpha > 0. \end{array}}$$

f) $\lim\limits_{(x,y)\to(1,1)} \dfrac{(x-1)(y-1)}{xy-1}$

$a_n = \left(\frac{n+1}{n}, \frac{n+1}{n}\right) \to (1,1)$ $\quad n \to \infty$ (e mai greu așa, deci revenim la limite cu 0,0)

$u = x-1 \to 0$

$v = y-1 \to 0$

$$\lim_{(u,v)\to(0,0)} \frac{u\cdot v}{(u+1)(v+1)-1} = \lim_{(u,v)\to(0,0)} \underbrace{\frac{uv}{uv+u+v}}_{g(u,v)}$$

$$g(u,v) = \frac{uv}{uv+u+v}$$

Cu șiruri:

alegem $a_n = \left(\frac{1}{n}, 0\right) \to (0,0)$ $n\to\infty$

$$\lim_{n\to\infty} g(a_n) = \lim_{n\to\infty} \frac{0}{0+\frac{1}{n}} = 0$$

alegem $b_n = \left(\frac{1}{n}, -\frac{1}{n}\right) \to (0,0)$ $n\to\infty$

$$\lim_{n\to\infty} g(b_n) = \lim_{n\to\infty} \frac{-\frac{1}{n^2}}{-\frac{1}{n^2} + \underbrace{\cancel{\frac{1}{n}} - \cancel{\frac{1}{n}}}_{=0}} = 1$$

$$\lim_{n\to\infty} g(a_n) \neq \lim_{n\to\infty} g(b_n) \Rightarrow \nexists \lim_{(u,v)\to O_2} g(u,v)$$

g) $\displaystyle\lim_{(x,y,z)\to O_3} \frac{(x+y+z)^2}{x^2+y^2+z^2}$ $\qquad f(x,y,z) = \frac{(x+y+z)^2}{x^2+y^2+z^2}$

$\mathbb{R}^3$

$a_n = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right) \to O_3$ $n\to\infty$

$b_n = \left(\frac{1}{n}, \frac{1}{n}, -\frac{2}{n}\right) \to O_3$ $n\to\infty$

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} \frac{\left(\frac{3}{n}\right)^2}{3\cdot\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\cancel{9}\,3}{\cancel{n^2}}\cdot\frac{\cancel{n^2}}{\cancel{3}} = 3$$

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} \frac{0}{2\cdot\frac{1}{n^2}+\frac{4}{n^2}} = 0$$

$\Rightarrow \nexists \lim_{(x,y,z)\to O_3} f(x,y,z)$

② $f:\mathbb{R}^2\to\mathbb{R}$ $\quad f(x,y) = \begin{cases} x\cdot\cos\frac{1}{y^2} + y\cdot\cos\frac{1}{x^2}, & xy\neq 0 \\ 0, & xy = 0. \end{cases}$

Este funcția continuă în $(0,0)$? Dar în $(1,0)$?

f cont în $(0,0) \Leftrightarrow \exists \underbrace{\lim_{(x,y)\to(0,0)} f(x,y)}$ $= f(0,0) = 0$

$x \neq 0,\ y \neq 0$

$$f(x,y) = x \cdot \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2} \quad \text{Ar. că } l = 0$$

$$|f(x,y) - 0| = \left| x \cdot \cos\frac{1}{y^2} + y \cdot \cos\frac{1}{x^2} \right| \leq |x| \cdot \underbrace{\left|\cos\frac{1}{y^2}\right|}_{\leq 1} + |y| \cdot \underbrace{\left|\cos\frac{1}{x^2}\right|}_{\leq 1}$$

$$\leq |x| + |y| \to 0 \quad (x,y) \to (0,0)$$

$$\Rightarrow \exists \lim_{(x,y)\to 0_2} f(x,y) = 0 \Rightarrow f \text{ cont în } (0,0)$$

Arătăm că $\nexists \lim_{(x,y)\to(1,0)} f(x,y) \to f$ nu e cont în $(1,0)$

alegem $a_n = \left(1, \frac{1}{\sqrt{2\pi n}}\right) \to (1,0)\ n\to\infty$

$$f(a_n) = 1 \cdot \cos(2\pi n) + \frac{1}{\sqrt{2\pi n}} \cdot \cos 1 \to 1,\ n\to\infty$$

$$b_n = \left(1, \frac{1}{\sqrt{2\pi n + \frac{\pi}{3}}}\right) \to (1,0)\ n\to\infty$$

$$f(b_n) = 1 \cdot \cos\left(2\pi n + \frac{\pi}{3}\right) + \frac{1}{\sqrt{2\pi n + \frac{\pi}{3}}} \cdot \cos 1 \to \frac{1}{2}$$

limitele sunt diferite $\Rightarrow \nexists \lim_{(x,y)\to(1,0)} f(x,y) \Rightarrow f$ nu e cont în $(1,0)$

3. Val extreme

a) $f : (0,+\infty)^2 \to \mathbb{R}$ $f(x,y) = \frac{x}{y} + \frac{y}{x}$

$$A = (0,+\infty)^2 = (0,+\infty) \times (0,+\infty),\ x,y > 0$$

$$f(x,y) = \frac{x}{y} + \frac{y}{x} \geq 2 \cdot \sqrt{\frac{x}{y} \cdot \frac{y}{x}} = 2,\ \forall\, x,y > 0$$

$$\inf f(A) = 2 = f(1,1)$$

$f(n,1) = n + \frac{1}{n} \to \infty,\ n \to +\infty \Rightarrow \sup f(A) = +\infty$ - nu se atinge

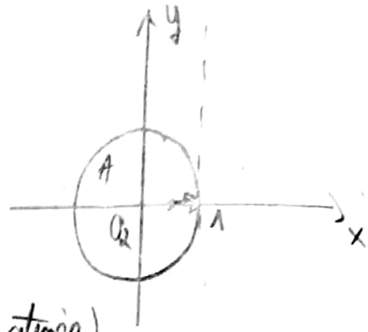
A nu e închisă pt. că nu-și conține frontiera | $\Rightarrow$ A nu e compactă $\Rightarrow$ nu se aplică Th lui Weierstrass

A nu e mărginită

b) $A = B(O_2, 1) = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ $f(x,y) = \dfrac{1}{1+x^2+y^2}$

$0 \le x^2+y^2 < 1 \Rightarrow 1 \le 1+x^2+y^2 < 2 \Rightarrow 1 \ge f(x,y) > \frac{1}{2}$

$\Rightarrow \inf(f(A)) = \frac{1}{2}$.



$a_n = (1-\frac{1}{n}, 0) \to (1,0)$

$f(1-\frac{1}{n}, 0) = \dfrac{1}{1+(1-\frac{1}{n})^2} \to \frac{1}{2}, \ n \to +\infty$

(asta justifică că $\inf(f(A)) = \frac{1}{2}$ și nu se atinge)

$\sup(f(A)) = 1 = f(0,0)$

A nu e închisă $\Rightarrow$ A nu e compactă $\Rightarrow$ nu se aplică T. lui Weierstrass

c) $f: A \to \mathbb{R}$ $f(x,y) = xy(1-x-y)$

$A = \{(x,y) \in \mathbb{R}^2 \mid x \ge 0, y \ge 0, x+y \le 1\}$

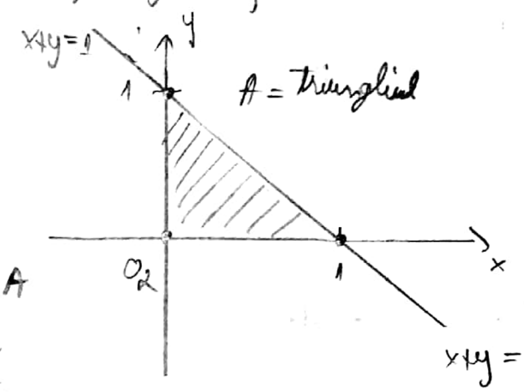
A compactă

$Fr A \subseteq A$ (A închisă)

A mărginită

f continuă (f elem.)

$\overset{T.W}{\Longrightarrow}$ f mărginită pe A și își atinge valorile extreme



$x \ge 0$

$y \ge 0$

$(1-x-y) \ge 0$

$(x+y=1)$

$\Rightarrow f(x,y) \ge 0 = f(0,0) = \inf(f(A))$

funcția este nulă pe laturile triunghiului.

Folosim inegalitatea mediilor

$a,b,c \ge 0$: $\dfrac{a+b+c}{3} \ge \sqrt[3]{abc}$ („=" când $a=b=c$)

$a = x$, $b = y$, $c = 1-x-y$ } sunt pozitive $\Rightarrow \frac{1}{3} \ge \sqrt[3]{x \cdot y(1-x-y)}$

$\Leftrightarrow \frac{1}{3} \ge \sqrt[3]{f(x,y)} \Leftrightarrow f(x,y) \le \frac{1}{27} = f(\frac{1}{3}, \frac{1}{3})$

$= \sup(f(A))$ și se atinge

$x=y \quad = 1-x-y$

$$\begin{cases} x=y \\ y=1-x-y \\ x=1-x-y \end{cases} \Rightarrow 2y=1-x \Rightarrow 3x=1 \Rightarrow x=y=\frac{1}{3}$$

Problemă tip examen

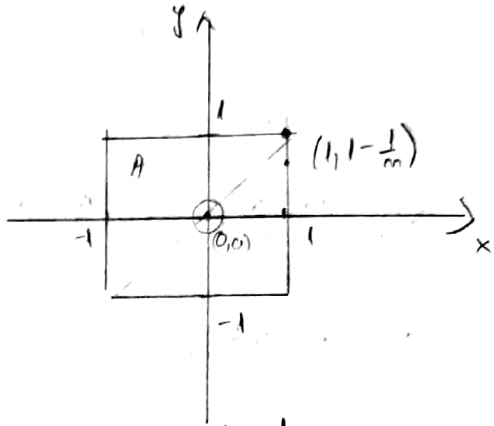
5) Se dă $A = \{ (x,y) \in [-1,1]^2 \mid x \neq y \}$ – Temă

$f: A \to \mathbb{R} \qquad f(x,y) = \dfrac{x^2+y^2}{(x-y)^2}$

a) Există limita lui f în O_2 (origine)?

b) A compactă?

c) val. extreme? + f atinge valorile?



a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{(x-y)^2}$

fără prima bisectoare.

$x \neq y$

$a_n = \left(\frac{1}{n}, 0\right) \to (0,0)$; $\lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n^2}+0}{\frac{1}{n^2}} = 1$

$b_n = \left(\frac{1}{n}, -\frac{1}{n}\right) \to (0,0)$; $\lim\limits_{n\to\infty} f(b_n) = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n^2}+\frac{1}{n^2}}{\frac{4}{n^2}} = \lim\limits_{n\to\infty} \frac{2}{n^2}\cdot\frac{n^2}{4} = \frac{1}{2}$

limitele sunt diferite $\Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

A compactă $\Leftrightarrow$ A mărginită + închisă

b) A mărginită

$(1,1) \in \mathrm{Fr}A$ și $(1,1) \notin A \Rightarrow \mathrm{Fr}A \not\subset A \Rightarrow A$ nu este închisă $\Rightarrow A$ nu e compactă

(A închisă $\Leftrightarrow \mathrm{Fr}A \subset A$.

frontiera: diagonala cu verde + laturile pătratului

c) $f(x,y) = \dfrac{x^2+y^2}{(x-y)^2} \qquad (x,y) \in [-1,1]^2 \quad x \neq y$

$a_n = \left(1, 1-\frac{1}{n}\right)$

$f\left(1, 1-\frac{1}{n}\right) = \dfrac{1+\left(1-\frac{1}{n}\right)^2}{\left(1-1+\frac{1}{n}\right)^2} = \dfrac{n^2}{1}\cdot\left[1+\left(1-\frac{1}{n}\right)^2\right] \xrightarrow[n\to\infty]{} \infty \Rightarrow$

$\sup f(A) = +\infty$ (nu se atinge)

Arătăm că $f(x,y) \geq \frac{1}{2}$, $\forall (x,y) \in A$

$$(=) \frac{x^2+y^2}{(x-y)^2} \geq \frac{1}{2} (=) 2x^2+2y^2 \geq x^2+y^2-2xy$$

$$(=) x^2+y^2+2xy \geq 0$$

$$(=) (x+y)^2 \geq 0. \checkmark$$

$f\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{2}$ $\Rightarrow \inf f(A) = \frac{1}{2}$ - se atinge (în punctele de pe cealaltă diagonală a pătratului)

Seminar 11:

① Calculați derivatele parțiale de ordinul 1, gradientul ∇f și diferențiala df pentru:

a) $f: \mathbb{R}^3 \to \mathbb{R}$

$$f(x,y,z) = x^2y^3 + y \cdot \sin x - 2z$$

$$\frac{\partial f}{\partial x} = 2xy^3 + y \cdot \cos x$$

$$\frac{\partial f}{\partial y} = 3y^2 \cdot x^2 + \sin x$$

$$\frac{\partial f}{\partial z} = -2$$

$$\nabla f = (2xy^3 + y\cos x, 3y^2x^2 + \sin x, -2)$$

vectorul din derivatele parțiale scris în ordinea (x,y,z)

↓ e o funcție

$df(x,y,z): \mathbb{R}^3 \to \mathbb{R}$

$$df(x,y,z)(u_1,u_2,u_3) = (2xy^3 + y\cos x) \cdot u_1 + (3y^2x^2 + \sin x) \cdot u_2 - 2u_3$$

b) $f : (0,+\infty)^2 \to \mathbb{R}$ $\quad f(x,y) = \operatorname{arctg} \frac{x-y}{x+y}$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2} \cdot \left(\frac{x-y}{x+y}\right)'_x = \frac{\cancel{(x+y)^2}}{(x+y)^2+(x-y)^2} \cdot \frac{x+y-(x-y)}{\cancel{(x+y)^2}}$$

$$= \frac{2y}{2x^2+2y^2} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x-y}{x+y}\right)^2} \cdot \left(\frac{x-y}{x+y}\right)'_y = \frac{\cancel{(x+y)^2}}{(x+y)^2+(x-y)^2} \cdot \frac{-(x+y)-(x-y)}{\cancel{(x+y)^2}} =$$

$$= \frac{-2x}{2x^2+2y^2} = -\frac{x}{x^2+y^2} \quad , \forall (x,y) \in (0,+\infty)^2$$

c) $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \sqrt{x^2+y^2} + x \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot (x^2+y^2)'_x$$

$$= \cancel{\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} + x \cdot \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}x$$

$$= \frac{2x^2+y^2}{\sqrt{x^2+y^2}} \quad , \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\frac{\partial f}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}}$$

$3 \cdot \sqrt{y^2+9}$ $\quad \frac{1}{2\sqrt{y^2+3^2}} \cdot 2y$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{x|x|-0}{x} = \lim_{x\to 0} |x| = 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

! în origine trebuiau calculate separat

② Ar. că $f(x,y) = y\cdot \ln(x^2-y^2)$ verifică relația:

$$\frac{1}{x}\cdot\frac{\partial f}{\partial x}(x,y) + \frac{1}{y}\cdot\frac{\partial f}{\partial y}(x,y) = \frac{1}{y^2}\cdot f(x,y),\ \forall\ x>y>0$$

$$\frac{\partial f}{\partial x}(x,y) = y\cdot\frac{1}{x^2-y^2}\cdot 2x = \frac{2xy}{x^2-y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \ln(x^2-y^2) + y\cdot\frac{1}{x^2-y^2}\cdot(-2y) = \ln(x^2-y^2) - \frac{2y^2}{x^2-y^2}$$

$$\Rightarrow \frac{1}{\cancel{x}}\cdot\frac{2\cancel{x}y}{x^2-y^2} + \frac{1}{y}\left(\ln(x^2-y^2) - \frac{2y^2}{x^2-y^2}\right) =$$

$$= \cancel{\frac{2y}{x^2-y^2}} + \frac{\ln(x^2-y^2)}{y} - \cancel{\frac{2y}{x^2-y^2}} = \frac{f(x,y)}{y^2}\ \checkmark$$

③ Studiați existența derivatelor parțiale în origine și a derivatelor după direcție în origine pentru:

$$f:\mathbb{R}^2\to\mathbb{R}\quad f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0)\end{cases}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to0}\frac{0}{x} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to0}\frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to0}\frac{0}{y^2} = 0$$

$\Rightarrow f$ este derivabilă parțial în $(0,0)$

Derivata după direcție:

$$f'_v(x^0) = \lim_{t\to0}\frac{1}{t}\left[f(x^0+t\cdot v) - f(x^0)\right]$$

$x^0=(0,0)$, $v = (v_1,v_2) \neq 0_2$

vector din $\mathbb{R}^2$ (nu ia diferit de $(0,0)$)

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$x^0 + t\cdot v = (0,0) + t(v_1, v_2) = (tv_1, tv_2)$

$$f'_v(0,0) = \lim_{t\to 0} \frac{1}{t}\left[f(tv_1, tv_2) - f(0,0)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[\frac{(tv_1)^2\cdot tv_2}{(tv_1)^4 + (tv_2)^2} - 0\right] =$$

$$= \lim_{t\to 0} \frac{\cancel{t^3}\, v_1^2 v_2}{\cancel{t^3}(t^2 v_1^4 + v_2^2)} = \begin{cases} \dfrac{v_1^2}{v_2}, & v_2 \neq 0 \\ 0, & v_2 = 0 \end{cases}$$

Cazuri particulare:

$v = e^1 = (1,0)$: $f_{e^1}(0,0) = 0 = \dfrac{\partial f}{\partial x}(0,0)$

$v = e^2 = (0,1)$: $f_{e^2}(0,0) = 0 = \dfrac{\partial f}{\partial y}(0,0)$

derivata după direcție = cât variază funcția pe direcția respectivă

(5) Calculați derivatele parțiale ale funcției compuse $g\circ f$ unde $f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (\underbrace{xe^y + x\cdot e^{-y}}_{f_1}, \underbrace{xe^y - xe^{-y}}_{f_2})$ și $g = g(u,v): \mathbb{R}^2 \to \mathbb{R}$ o funcție oarecare de clasă C^1 pe $\mathbb{R}^2$

f_1, f_2 sunt f de clasă $C_1 \to$ f e de clasă C_1

$$\nabla(g\circ f)(x,y) = \nabla g(f(x,y)) \cdot \underline{J(f)(x,y)}$$

$$J(f)(x,y) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} e^y + e^{-y} & x\cdot e^y - x\cdot e^{-y} \\ e^y - e^{-y} & x\cdot e^y + x\cdot e^{-y} \end{pmatrix}$$

$$\underbrace{\left(\frac{\partial}{\partial x}(g\circ f)(x,y), \frac{\partial}{\partial y}(g\circ f)(x,y)\right)}_{\nabla(g\circ f)(x,y)} = \left(\frac{\partial g}{\partial u}(f(x,y)), \frac{\partial g}{\partial v}(f(x,y))\right) \begin{pmatrix} e^y + e^{-y} & x\cdot e^y - x\cdot e^{-y} \\ e^y - e^{-y} & x\cdot e^y + x\cdot e^{-y} \end{pmatrix}$$

$$\frac{\partial}{\partial x}(g\circ f)(x,y) = \frac{\partial g}{\partial u}(f(x,y))\cdot e^{y}+e^{-y} + \frac{\partial g}{\partial v}(f(x,y))\cdot(e^{y}-e^{-y})$$

analog $\frac{\partial}{\partial y}(g\circ f)(x,y) = \dots$

⑤ Exprimați ecuația

$$u\cdot\frac{\partial g}{\partial u}(u,v) + v\cdot\frac{\partial g}{\partial v}(u,v) = \sqrt{u^2+v^2}, \ \forall (u,v)\in(0,+\infty)^2$$

în variabilele $(x,y)\in(0,+\infty)\times(0,\frac{\pi}{2})$, efectuând transformarea

$u = x\cdot\cos y$, $v = x\cdot\sin y$.

Determinați apoi o funcție g de clasă C^1 ce verifică relația respectivă.

Coord. polare: $g:\mathbb{R}^2\to\mathbb{R}$.

$$\begin{cases} u = x\cdot\cos y \\ v = x\cdot\sin y. \end{cases} \to$$

$$\to \cancel{x}\cos y\cdot\frac{\partial g}{\partial u}(x\cos y, x\sin y) + \cancel{x}\sin y\cdot\frac{\partial g}{\partial v}(x\cos y, x\sin y) = \cancel{x}\,1\ (*)$$

împărțim la $x(*)$

$$\sqrt{u^2+v^2} = \sqrt{x^2\cos^2 y + x^2\sin^2 y} = \sqrt{x^2} = |x| = x \quad (x\in(0,+\infty))$$

Fie $f:(0,+\infty)\times(0,\frac{\pi}{2})\to\mathbb{R}^2$, $f(x,y) = (\overset{f_1}{x\cos y}, \overset{f_2}{x\sin y})$

$g\circ f:(0,+\infty)\times(0,\frac{\pi}{2})\to\mathbb{R}$

$$\nabla(g\circ f)(x,y) = \nabla g(f(x,y))\cdot J(f)(x,y)$$

$$\left(\frac{\partial}{\partial x}(g\circ f)(x,y), \frac{\partial}{\partial y}(g\circ f)(x,y)\right) = \left(\frac{\partial g}{\partial u}(f(x,y)), \frac{\partial g}{\partial v}(f(x,y))\right)\cdot$$

$$\begin{pmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{pmatrix}$$

$$\frac{\partial}{\partial x}(g\circ f)(x,y) = \frac{\partial g}{\partial u}(x\cos y, x\sin y)\cdot \cos y + \frac{\partial g}{\partial v}(x\cos y, x\sin y)\cdot \sin y.$$

Relația asta este egală cu (*)

$$\Rightarrow \frac{\partial}{\partial x}\underbrace{(g\circ f)}_{h}(x,y) = 1, \quad \forall x,y \in (0,+\infty)\times(0,\tfrac{\pi}{2})$$

$$\Rightarrow \frac{\partial h}{\partial x}(x,y) = 1 \quad, \forall (x,y)\in A$$

alegem $h(x,y) = x$

$$g(f(x,y)) = x$$

$$g(x\cos y, x\sin y) = x$$

$$g(u,v) = \sqrt{u^2+v^2}$$

Verificare: calculăm deriv. parțiale ale lui g și verif. dacă e ok relația din enunț

⑥ Calculați derivatele parțiale de ordinul 2 ale funcțiilor:

a) $f:(1,+\infty)\times\mathbb{R}\to\mathbb{R}$

$$f(x,y) = \ln(x+y^2-1)$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x+y^2-1}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2y}{x+y^2-1}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = \left(\frac{1}{x+y^2-1}\right)'_x = \frac{-1}{(x+y^2-1)^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = \left(\frac{2y}{x+y^2-1}\right)' = \frac{2(x+y^2-1)-2y\cdot 2y}{(x+y^2-1)^2} =$$

$$= \frac{2x-2y^2-2}{(x+y^2-1)^2}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = \left(\frac{2y}{x+y^2-1}\right)'_x = \frac{-2y}{(x+y^2-1)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = \left(\frac{1}{x+y^2-1}\right)'_y = \frac{-2y}{(x+y^2-1)^2} \Rightarrow$$ din criteriul Schwarz.

5) $f: \mathbb{R} \times (0,+\infty) \to \mathbb{R} \quad f(x,y) = xy \cdot e^{\frac{x}{y}}$

$$\frac{\partial f}{\partial x}(x,y) = y \cdot e^{\frac{x}{y}} + xy \cdot e^{\frac{x}{y}} \left(\frac{x}{y}\right)'_x = y \cdot e^{\frac{x}{y}} + x\cancel{y} \cdot e^{\frac{x}{y}} \cdot \frac{1}{\cancel{y}}$$

$$= e^{\frac{x}{y}}(x+y)$$

$$\frac{\partial f}{\partial y}(x,y) = x \cdot e^{\frac{x}{y}} + xy \cdot e^{\frac{x}{y}} \cdot \left(\frac{x}{y}\right)'_y = x \cdot e^{\frac{x}{y}} + x\cancel{y} \cdot e^{\frac{x}{y}} \cdot \frac{-x}{y^{\cancel{2}}}$$

$$= xe^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left[e^{\frac{x}{y}}(x+y)\right]'_x =$$

$$= e^{\frac{x}{y}} \cdot \left(\frac{x}{y}\right)'_x \cdot (x+y) + e^{\frac{x}{y}} = e^{\frac{x}{y}} \cdot \frac{1}{y}(x+y) + e^{\frac{x}{y}}$$

$$= e^{\frac{x}{y}}\left(2 + \frac{x}{y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \left[x \cdot e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)\right]'_y = x \cdot e^{\frac{x}{y}} \cdot \frac{-x}{y^2} \cdot \left(1 - \frac{x}{y}\right) + x \cdot e^{\frac{x}{y}} \cdot \frac{x}{y^2} =$$

$$= \frac{x^2}{y^2} \cdot e^{\frac{x}{y}}\left(\frac{x}{y} - \cancel{1} + \cancel{1}\right) = \frac{x^3}{y^3} \cdot e^{\frac{x}{y}}$$

Temă:

Seminar 12:

① $f:\mathbb{R}^2 \to \mathbb{R}$ $f(x,y) = x^3 + 3xy^2 - 15x - 12y$

$a = (-2,-1)$

a) $\nabla f(a)$, $H(f)(a)$ și $d^2 f(a)$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) = \left(3x^2 + 3y^2 - 15,\ 6xy - 12\right)$$

$$\nabla f(-2,-1) = (12+3-15,\ 12-12) = (0,0)$$

$$H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial y \partial x}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix}$$

$$H(f)(x,y) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix}$$

$$d^2 f(-2,-1)(u_1,u_2) = -12u_1^2 - 12u_2^2 - 12u_1u_2$$

e o funcție (cu fel de polinom de gr 2)

minim maxim

b) natura punctului a

$\nabla f(a) = 0_2 \Rightarrow a =$ punct critic pt. funcția f → pct de extrem local / pct. șa

$$d^2 f(-2,-1)(u_1,u_2) = -12(u_1^2 + u_2^2 + u_1u_2)$$ → trb. să vedem semnul.

$< 0, \forall (u_1,u_2) \neq 0_2$

Criteriul Sylvester: (pe matricea hessiană)

$\Delta_1 = -12 < 0$

$\Delta_2 = 144 - 36 > 0$

$\Rightarrow d^2 f(a)$ negativ definită $\Rightarrow$ a - pct de maxim local

② Det. punctele critice și pct. de extrem local pt.:

a) $f:\mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$

Etapa I: Determinarea punctelor critice

(nabla) $$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(4x - y + 2z,\ -x - 1 + 3y^2,\ 2x + 2z\right)$$

$$\nabla f = 0_3 \Rightarrow \begin{cases} 4x - y + 2z = 0 & \Rightarrow -4z - y + 2z = 0 \\ -x - 1 + 3y^2 = 0 & \Rightarrow z - 1 + 3y^2 = 0 \\ 2x + 2z = 0 & \Rightarrow x = -z \end{cases}$$

$$\Rightarrow \begin{cases} -2z-y=0 \Rightarrow y=-2z \\ z-1+3y^2=0 \Rightarrow z-1+12z^2=0 \end{cases}$$

$$12z^2+z-1=0$$

$$\Delta = 1+48=49$$

$$z_{1,2} = \frac{-1 \pm 7}{24} \nearrow z_1 = \frac{1}{4},\ x_1 = -\frac{1}{4},\ y_1 = -\frac{1}{2}$$
$$\searrow z_2 = -\frac{1}{3},\ x_2 = \frac{1}{3},\ y_2 = \frac{2}{3}$$

$\Rightarrow \underbrace{\left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)}_{b}$, $\underbrace{\left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)}_{a}$ puncte critice

Etapa II. Natura punctelor critice (avem nevoie de matricea Hessiană)

$$H(f)(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 6y & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$H(f)(a) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} \qquad H(f)(b) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 4 > 0$

$\Delta_2 = 15 > 0$

$\Delta_3 = 32 - 16 - 2 = 14 > 0$

Crit Sylvester $\Rightarrow d^2 f(a)$ pozitiv definită

$\Rightarrow a$ - pct de minim local

$\Delta_1 = 4 > 0$

$\Delta_2 = -13 < 0$

$\Delta_3 = -24 + 12 - 2 = -14 < 0$

$\hookrightarrow$ nu mai are rost să calculăm

Crit Sylvester $\Rightarrow$ nu e nici poz. definită, nici negativ definită

$$d^2 f(b)(u_1,u_2,u_3) = 4u_1^2 - 3u_2^2 + 2u_3^2 - 2u_1u_2 + 4u_1u_3 + 0.$$

Semnul expresiei:

$d^2 f(b)(0,1,0) = -3 < 0$

$d^2 f(b)(1,0,0) = 4 > 0$

$\Rightarrow d^2 f(b)$ este indefinită

$\Rightarrow b$ - punct șa

5) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^4 + y^4 - 2x^2$

I. Det. pct. critice

$$\nabla f(x,y) = (4x^3 - 4x,\ 4y^3)$$

$$\nabla f(x,y) = O_2 \Rightarrow \begin{cases} 4x(x^2-1) = 0 \\ 4y^3 = 0 \Rightarrow y = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$$

$\Rightarrow$ soluții : $(0,0)$ $(-1,0)$ $(1,0)$ $\to$ punctele critice ale lui f

II. Det. natura punctelor critice

$$H(f)(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 12x^2 - 4 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

$$H(f)(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow \Delta_2 = 0$ (în ambele cazuri) $\Rightarrow$ nu ne încadrăm în niciuna

$$H(f)(\pm 1, 0) = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$$

$$d^2 f(0,0)(u_1,u_2) = -4u_1^2 \le 0$$

$$d^2 f(\pm 1,0)(u_1,u_2) = 8u_1^2 \ge 0$$

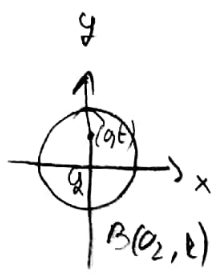
cele 2 diferențiale nu sunt pozitiv definite, nu sunt negativ definite și nu sunt indefinite

Teoria nu se aplică. !

- pt. $(0,0)$

$(0,0)$ pct. de minim local $\Leftrightarrow \exists\, r > 0 : \forall (x,y) \in B(O_2, r) \cap \mathbb{R}^2$

$$f(0,0) \le f(x,y)$$



$f(0,0) = \underline{\underline{0}}$

Ar. că nu e așa $(0 \le f(x,y))$

$\forall r > 0.$

$f(0,t) = t^4 > 0,\ t \ne 0.$

$f(t,0) = t^4 - 2t^2 = t^2(t - \sqrt{2})(t + \sqrt{2}) < 0.$

$\quad +\quad -\quad +$

$\forall t \in (-\sqrt{2}, \sqrt{2})$

$\Rightarrow (0,0)$ punct șa

- pt $(1,0), (-1,0)$

$f(1,0) = f(-1,0) = -1$

$f(x,y) = x^4 - 2x^2 + 1 + y^4 - 1 = \underbrace{(x^2-1)^2 + y^4}_{\geq 0} - 1 \geq -1 \Rightarrow f(x,y) \geq -1, \forall (x,y) \in \mathbb{R}^2$

$\Rightarrow (1,0), (-1,0)$ puncte de minim local

③ Det. punctele de extrem condiționat (+tipul) și valorile extreme ale funcției f relativ la mulțimea S indicată (e compactă)

$f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

$f(x,y) = (1-x)(1-y)$

Fie $\overset{\text{restricție}}{F}: \mathbb{R}^2 \to \mathbb{R}$ $\quad F(x,y) = x^2+y^2-1$

$S = \{(x,y) \in \mathbb{R}^2 \mid F(x,y) = 0\}$

Formăm: Funcție a lui Lagrange $L: \mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}$ $\leftarrow$ o singură restricție

$L(x,y,\lambda) = f(x,y) + \lambda \cdot F(x,y)$

$L(x,y,\lambda) = (1-x)(1-y) + \lambda(x^2+y^2-1)$

gradientul lui L:

$\frac{\partial L}{\partial x} = -(1-y) + 2\lambda x$

$\frac{\partial L}{\partial y} = -(1-x) + 2\lambda y$

$\frac{\partial L}{\partial \lambda} = x^2+y^2-1$

Determinăm pct. critice: $\nabla L = O_3$

$\Rightarrow \begin{cases} y - 1 + 2\lambda x = 0 \\ x - 1 + 2\lambda y = 0 \\ x^2+y^2-1=0 \end{cases} \Big| \ominus \Rightarrow y - x + 2\lambda(x-y) = 0$

$(y-x)(1-2\lambda) = 0$

caz i) $x = y \Rightarrow 2x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}} = y \quad \lambda = \frac{1 \pm \sqrt{2}}{2}$

caz ii) $\lambda = \frac{1}{2} \Rightarrow x + y - 1 = 0 \Rightarrow y = 1 - x$

$x^2 + 1 - 2x + x^2 - 1 = 0$

$2x^2 - 2x = 0$

$2x(x-1) = 0 \Rightarrow \begin{cases} x_1 = 0 \Rightarrow y_1 = 1 \\ x_2 = 1 \Rightarrow y_2 = 0 \end{cases}$

$\Rightarrow$ soluțiile $\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \frac{1+\sqrt{2}}{2}\right), \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \frac{1-\sqrt{2}}{2}\right), \left(0, 1, \frac{1}{2}\right), \left(1, 0, \frac{1}{2}\right)$

puncte critice ale lui L

Th. multiplicat. lui Lagrange $\Rightarrow$
$\Rightarrow$ punctele de extrem se află printre punctele astea (primele 2 coord)

S compactă

Folosim Teorema lui Weierstrass

$\left.\begin{matrix} f \text{ cont.} \\ S \text{ compactă} \end{matrix}\right| \xRightarrow{T.W} f$ mărginită și își atinge extremele pe S.

$\Rightarrow f$ are punct de minim și pct. de maxim condiționat relativ la S

Testăm valorile funcției în pct. găsite.

$f\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) = \left(1 - \sqrt{\frac{1}{2}}\right)^2$

$f\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right) = \left(1 + \sqrt{\frac{1}{2}}\right)^2 = \max f(S)$

$f(0,1) = f(1,0) = 0 = \min f(S)$

$\Rightarrow (0,1), (1,0)$ pct. de minim condiționat

$\Rightarrow \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$ pct de maxim condiționat

(5) Det val. extreme ale f., relativ la mulțimea S indicată

a) $f: \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z) = x + 2y + 3z$ $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$

$S = \bar{B}(O_3, 1)$ compactă

o bilă (sferă cu interior cu tot)
= bilă închisă de centru O_3 și rază 1

$\Rightarrow$ nu putem aplica teorema mult. lui Lagrange (nu avem =)

S închisă

$S = \text{int } S \cup \text{Fr } S$

Etapa I: Punctele de extrem în int S.

$\nabla f(x,y,z) = (1,2,3) \neq 0_3 \Rightarrow f$ nu are puncte critice

$\Rightarrow f$ nu are pct. de extrem în int S.

Ca să găsim punctele și valorile f(puncte) ca să le comparăm cu alea din fr S

Etapa II : Puncte de extrem pe fr $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$

Th. mult. lui Lagrange

Fie $F(x,y,z) = x^2+y^2+z^2-1$

$$L(x,y,z,\lambda) = x+2y+3z+\lambda(x^2+y^2+z^2-1)$$

$$\frac{\partial L}{\partial x} = 1+2\lambda x$$

$$\frac{\partial L}{\partial y} = 2+2\lambda y$$

$$\frac{\partial L}{\partial z} = 3+2\lambda z$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1$$

$$\nabla L = 0_4 \Rightarrow \begin{cases} 1+2\lambda x = 0 \Rightarrow x = -\frac{1}{2\lambda} \\ 2+2\lambda y = 0 \Rightarrow y = -\frac{2}{2\lambda} \\ 3+2\lambda z = 0 \Rightarrow z = -\frac{3}{2\lambda} \\ x^2+y^2+z^2-1=0 \Rightarrow \frac{1}{4\lambda^2}+\frac{4}{4\lambda^2}+\frac{9}{4\lambda^2} = 1 \end{cases}$$

$$\Rightarrow 4\lambda^2 = 14 \Rightarrow \lambda = \pm\sqrt{\frac{7}{2}}$$

$\Rightarrow$ soluțiile $(\underbrace{\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}}_{a}, -\sqrt{\frac{7}{2}})$, $(\underbrace{-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}}_{b}, \sqrt{\frac{7}{2}})$

puncte critice pt. L

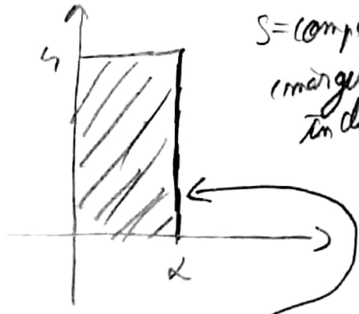
Aplicăm Th. Weierstrass

f continuă, S compactă $\Big| \Rightarrow f$ admite punct de minim și pct. de maxim condiționat relativ la S

$f(a) = \sqrt{14} = \max f(S)$ $\quad$ a – punct de minim cond

$f(b) = -\sqrt{14} = \min f(S)$ $\quad$ b – punct de maxim cond.

5) S: la fel cu int S și fr S



S = compactă (mărginită + închisă)

$(2,y)$, $y \in [0,5]$.

calculăm val. lui f în pct $(2,y)$ și la fel pt. celelalte și aflăm valorile extreme