Fie $s = \sup B \Rightarrow$ $s \in MAJ(B) \Rightarrow s \geq x, \forall x \in B$

Fie $y \in A$, $A \subseteq B$ $\Rightarrow y \in B$ $\Rightarrow s \geq y, \forall y \in A \Rightarrow$

$\Rightarrow s \in MAJ(A)$

Dar $M \leq M', \forall M' \in MAJ(A)$ $\Rightarrow M \leq s \Leftrightarrow \sup . A \leq \sup B$

SAU

Fie $M = \sup B \Rightarrow$ $\overset{M \in MAJ(B)}{x \leq M}, \forall x \in B.$

$A \subseteq B$ $\Rightarrow y \leq M, \forall y \in A. \Rightarrow$

$\Rightarrow M \in MAJ(A)$, $\sup A \leq M', \forall M' \in MAJ(A)$ $\Rightarrow \sup A \leq M. \Rightarrow \sup A \leq \sup B$

Exerciții suplimentare

1. a) $A = [-\pi; \pi] \cap \mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3\}$

~~$MIN(A) = (-\infty; -\pi]$; $MAJ(A) = [\pi; +\infty)$~~

~~$\inf(A) = -\pi$ $\sup(A) = \pi$~~

~~$\min(A) = \nexists$ $\max(A) = \nexists$~~

$MIN(A) = (-\infty; -3]$; $MAJ(A) = [3; +\infty)$

$\inf(A) = -3$ $\sup(A) = 3$

$\min(A) = -3$ $\max(A) = 3$

b) $A = \left\{ \frac{n}{1-n^2} \,/\, n \in \mathbb{N}, n \geq 2 \right\} = \left\{ -\frac{2}{3}; -\frac{3}{8}; -\frac{4}{15}; \dots \right\}$

$MIN(A) = \left(-\infty; -\frac{2}{3}\right]$ $MAJ(A) = [0; +\infty)$

$\inf(A) = -\frac{2}{3}$ $\sup(A) = 0$

$\min(A) = -\frac{2}{3}$ $\max(A) = \nexists$

$x_{n+1} - x_n = \frac{n+1}{1-n^2-2n-1} - \frac{n}{1-n^2} = \frac{n+1}{-n^2-2n} - \frac{n}{1-n^2} =$

$= \frac{n}{n^2-1} - \frac{n+1}{n^2+2n} = \frac{n^3+2n^2-(n+1)(n^2-1)}{n(n+2)(n-1)(n+1)} =$

$\forall n \geq 2$

$= \frac{n^3+2n^2-n^3+n-n^2+1}{n(n+1)(n-1)(n+2)} = \frac{n^2+n+1}{\dots} > 0 \Rightarrow (x_n)_{n\in\mathbb{N}^*}$ strict cresc. $\Rightarrow$

~~$\lim\limits_{n\to\infty} (x_n) = 0 \Rightarrow x_n \leq 0, \forall n, n \geq 2$~~

~~(x_n) cresc. $\Rightarrow x_n \geq x_2 = -\frac{2}{3}, \forall n \geq 2$.~~

SAU

~~Fie $\varepsilon > 0$ a.î. $\varepsilon < a, \forall a \in A \Rightarrow \varepsilon < \frac{n}{n^2-1}$, $\forall n \geq 2$~~

$\Rightarrow x_n \leq x_2, \forall n \in \mathbb{N}, n \geq 2 \Rightarrow x_n \leq -\frac{2}{3} = x_2, \forall n \in \mathbb{N}, n \geq 2 \Rightarrow$

$\Rightarrow \min(A) = x_2 = -\frac{2}{3} = \inf(A)$

~~$\Rightarrow -\frac{2}{3} \leq a \leq 0, \forall n \geq 2$~~

~~$\lim\limits_{n\to\infty} x_n = 0 \Rightarrow x_n \leq 0, \forall n \in \mathbb{N}^*, n \geq 2 \Rightarrow a \leq 0, \forall a \in A$~~

SAU

~~$\frac{n}{1-n^2} < 0 \Leftrightarrow n > 0$ (A)~~

$\frac{n}{1-n^2} \leq 0 \Leftrightarrow n \geq 0$ (A), $\forall n \geq 2$ $\left.\right\}$ $\Rightarrow a \leq 0, \forall a \in A \Rightarrow 0 \in MAJ(A)$

$1-n^2 < 0, \forall n \geq 2$

Fie $\varepsilon < 0$ a.î. $x_n < \varepsilon, \forall n \in \mathbb{N}, n \geq 2 \Rightarrow \frac{n}{1-n^2} < \varepsilon \Leftrightarrow$

$\Leftrightarrow n > \varepsilon - \varepsilon n^2 \Leftrightarrow \varepsilon n^2 + n - \varepsilon > 0, \forall n \geq 2$

$\Delta = 1 + 4\varepsilon^2 > 0, \forall \varepsilon < 0$

x	$-\infty$		α_1		α_2		$+\infty$
$\varepsilon x^2+x-\varepsilon$		$- - - - -$	0	$+ +$	0	$- - -$	

I Dacă $\alpha_2 < 2 \Rightarrow \exists\, n \in \mathbb{N},\ n \geq 2$ a.î. $x_n > \varepsilon$

II Dacă $\alpha_2 \geq 2 \Rightarrow \exists\, [\alpha_2]+1 \in \mathbb{N},\ [\alpha_2]+1 \geq 2$ a.î. $x_{[\alpha_2]+1} > \varepsilon$ $\Big| \Rightarrow$

$\Rightarrow$ Nu există $\varepsilon < 0$ a.î. $a < \varepsilon,\ \forall a \in A \Rightarrow$ ~~MAJ∩(~~

$\Rightarrow MAJ(A) \cap (-\infty; 0) = \emptyset \Rightarrow MAJ = [0; +\infty)$ ~~⟹ max A~~ $\Rightarrow$

$\Rightarrow \sup A = 0.$

Dacă există $\max A \Rightarrow \max A = 0 \Rightarrow$

$\max A \in A \Rightarrow \max A = \frac{n}{(1-n^2)},$

$n \geq 2$

$\Rightarrow \frac{n}{1-n^2} = 0 \Leftrightarrow n = 0,\ n \geq 2$ (F)

$\Rightarrow \max(A) = \nexists$

c) $A = \left\{ x + \frac{1}{x} \,/\, x > 0 \right\} = [2; +\infty)$

DACĂ UTILIZĂM LIMITE....: $\lim\limits_{x\to\infty}\left(x+\frac{1}{x}\right) = \infty;$

$\lim\limits_{\substack{x\to 0\\ x>0}}\left(x+\frac{1}{x}\right) = 0 + \frac{1}{0_+} = 0 + \infty = \infty$

$MIN(A) = (-\infty; 2]$ $\qquad$ $MAJ(A) = \emptyset$

$\inf(A) = 2$ $\qquad$ $\sup(A) = +\infty$

$\min(A) = 2$ $\qquad$ $\max(A) = \nexists$

$x > 0 \Rightarrow \frac{1}{x} > 0$

$m_A \geq m_G \Leftrightarrow \frac{x+\frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} \Leftrightarrow x + \frac{1}{x} \geq 2,\ \forall x \in (0; +\infty) \Rightarrow$

$\Rightarrow \inf(A) = \min(A) = 2$

~~Fie $\varepsilon \in (0;+\infty)$ a.î.~~

~~Fie $\varepsilon \in \mathbb{R}$ a.î. $a < \varepsilon, \forall a \in A$ $\Rightarrow$ Fie $a = x + \frac{1}{x}, x > 0$. $\in A$~~

~~$\Rightarrow x$~~

~~$\Rightarrow x + \frac{1}{x} < \varepsilon$~~

Fie $\varepsilon \in \mathbb{R}$ a.î. $a < \varepsilon, \forall a \in A, a = x + \frac{1}{x}, x \in (0;+\infty) \Rightarrow$

$\Rightarrow x + \frac{1}{x} < \varepsilon, \forall x \in (0;+\infty)$

I $\varepsilon \in (-\infty; 2)$

$\varepsilon < 2 \leq x + \frac{1}{x}, \forall x \in (0;+\infty)$ (F) $\Rightarrow \varepsilon \in \emptyset$

II $\varepsilon \in [2;+\infty)$

Fie $x = [\varepsilon]+1$

$\varepsilon \in [2;+\infty) \Rightarrow [\varepsilon] \in [2;+\infty) \Rightarrow [\varepsilon]+1 \in [3;+\infty) \Rightarrow$

$\Rightarrow x \in [3;+\infty) \subset (0;+\infty)$

$[\varepsilon]+1 + \frac{1}{[\varepsilon]+1} < \varepsilon < [\varepsilon]+1 \Rightarrow [\varepsilon]+1 + \frac{1}{[\varepsilon]+1} < [\varepsilon]+1 \Leftrightarrow$

$\Leftrightarrow \frac{1}{[\varepsilon]+1} < 0 \Leftrightarrow \frac{1}{x} < 0$ (F), deorece $x > 0$, deci $\frac{1}{x} > 0$

~~I, II $\Rightarrow \forall \varepsilon \in \mathbb{R}$ $\nexists \varepsilon \in \mathbb{R}$ a.î. $a < \varepsilon, \forall a \in A$.~~

I, II $\Rightarrow \forall \varepsilon \in \mathbb{R}$ $\exists a \in A$ a.î. $a > \varepsilon \Rightarrow MAJ(A) = \emptyset \Rightarrow$

$\Rightarrow \sup(A) = +\infty$, $\max(A) = \nexists$

a) $A = \{ x \in \mathbb{R} \,/\, |x^2 - x| \leq 1 \} = \left[\frac{1-\sqrt{5}}{2} ; \frac{1+\sqrt{5}}{2} \right]$

$$|x^2 - x| = |x|\,|x-1| = \begin{cases} x^2 - x, & x \in (-\infty; 0] \cup [1; +\infty) \\ x - x^2, & x \in (0; 1) \end{cases}$$

I $|x^2 - x| = x^2 - x \iff x \in (-\infty; 0] \cup [1; +\infty)$

$x^2 - x \leq 1 \iff x^2 - x - 1 \leq 0$

$a = 1 > 0$

$\Delta = 1 + 4 = 5$

$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow x \in \left[\frac{1-\sqrt{5}}{2} ; \frac{1+\sqrt{5}}{2} \right]$

$\frac{1-\sqrt{5}}{2} \leq 0 \iff 1 \leq \sqrt{5}$ (A) $\Rightarrow \frac{1-\sqrt{5}}{2} \leq 0$

$\frac{1+\sqrt{5}}{2} \geq 1 \iff 1 + \sqrt{5} \geq 2 \iff \sqrt{5} \geq 1$ (A) $\Rightarrow \frac{1+\sqrt{5}}{2} \geq 1$

$x \in \left[\frac{1-\sqrt{5}}{2} ; 0 \right] \cup \left[1 ; \frac{1+\sqrt{5}}{2} \right]$

~~II $x - x^2 \geq 1 \iff x^2 - x + 1 \geq 0$~~

~~$\Delta = 1 - 4 = -3 < 0$~~ ~~$\Rightarrow x \in \mathbb{R}$~~

~~$x \in [0; 1]$~~

II $|x - x^2| = x - x^2 \iff x \in [0; 1]$

$x - x^2 \leq 1 \iff x^2 - x + 1 \geq 0$

$a = 1 > 0$

$\Delta = 1 - 4 = -3 < 0$

$\Rightarrow x \in \mathbb{R}$

$x \in [0; 1]$

$\left.\begin{matrix} \text{I} \\ \text{II} \end{matrix}\right| \Rightarrow A = \left[\frac{1-\sqrt{5}}{2} ; \frac{1+\sqrt{5}}{2} \right]$

$MIN(A) = (-\infty; \frac{1-\sqrt{5}}{2}]$ $\qquad$ $MAJ(A) = [\frac{1+\sqrt{5}}{2}; +\infty)$

$\inf(A) = \frac{1-\sqrt{5}}{2}$ $\qquad$ $\inf(A) = \frac{1+\sqrt{5}}{2}$

$\min(A) = \frac{1-\sqrt{5}}{2}$ $\qquad$ $\max(A) = \frac{1+\sqrt{5}}{2}$

② $A, B \subseteq \mathbb{R}$, A, B nevide, A, B mărginite superior

~~A, B nevide~~
~~A, B mărginite~~

1) $A \cup B$ mărginită superior

2) $\sup(A \cup B) = \max\{\sup A, \sup B\}$

$\left.\begin{array}{l} A, B \text{ nevide} \\ A, B \text{ mărginite superior} \end{array}\right\} \Rightarrow A, B$ prezintă supremumuri; $\sup A, \sup B \in \mathbb{R}$ ~~$a = \sup A, b = \sup B, a, b \in \mathbb{R}$~~

Fie $m = \max\{\sup A, \sup B\} \Rightarrow \sup A \leq m, \sup B \leq m, m \in \mathbb{R}$ (*)

~~Fie a, b~~

Fie $a \in A, b \in B \Rightarrow \begin{array}{l} a \leq \sup A, \forall a \in A \\ b \leq \sup B, \forall b \in B \end{array} \overset{(*)}{\Rightarrow} \begin{array}{l} a \leq \sup A \leq m, \forall a \in A \\ b \leq \sup B \leq m, \forall b \in B \end{array} \Rightarrow$

$\Rightarrow \left.\begin{array}{l} a \leq m, \forall a \in A \\ b \leq m, \forall b \in B \end{array}\right\} \Rightarrow x \leq m, \forall x \in A \cup B \Rightarrow \underset{m \in MAJ(A \cup B)\ (**)}{} MAJ(A \cup B) \neq \emptyset \Rightarrow$

$\Rightarrow \boxed{A \cup B \text{ mărginită superior} \quad 1)}$

I $\sup A \leq \sup B \Rightarrow m = \sup B$

Presupunem prin absurd că m nu este $\sup(A \cup B) \Rightarrow$ ~~$\exists M \in MAJ(A \cup B)$ (...), $M \leq (A \cup B)$~~, ~~$M = \sup(A \cup B) \Rightarrow$~~ $M < m$

$\Rightarrow \exists M \in MAJ(A \cup B), M \neq m, M = \sup(A \cup B) \Rightarrow \begin{cases} M \geq x, \forall x \in A \cup B \\ M \leq M', \forall M' \in MAJ(A \cup B) \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x \leq M, \forall x \in A \cup B \\ M < m \end{cases} \Leftrightarrow \begin{cases} x \leq M, \forall x \in A \cup B \\ M < \sup B \end{cases} \Leftrightarrow \begin{cases} x \leq M, \forall x \in B \\ M < \sup B \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} M \in MAJ(B) \\ M < \sup B \end{cases} \Rightarrow \begin{cases} \sup B \le M \\ M < \sup B \end{cases}$ (F) $\Rightarrow \sup(A \cup B) = m$

II $\sup A > \sup B \Rightarrow m = \sup A$

Analog cu I

~~III $\sup A = \sup B = m$~~

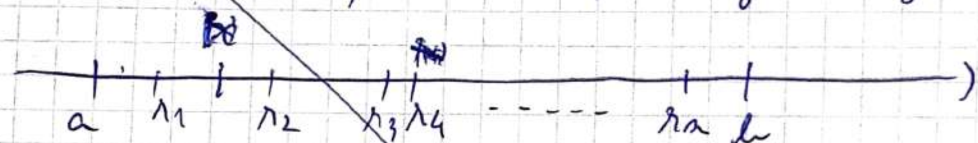
~~$m = \sup A \Rightarrow$~~

$\left.\begin{matrix} I \\ II \end{matrix}\right\} \Rightarrow m = \sup(A \cup B) \Rightarrow \sup(A \cup B) = \max\{\sup A, \sup B\}$

(3) Fie $a, b \in \mathbb{R}, a < b$

Presupunem prin absurd că între a și b există un nr. finit de nr. raționale r_i:

$A = \{r_1, r_2, r_3, \dots, r_n\}, n \in \mathbb{N}^*, (r_i < r_j \Leftrightarrow i < j), i, j = \overline{1,n}$



~~(ADICĂ $r_1 < r_2$)~~

(ADICĂ $a < r_1 < r_2 < r_3 < \dots < r_n < b$)

$r_1, r_2 \in \mathbb{Q} \Rightarrow r_1, r_2 \in \mathbb{R} \overset{(6)}{\Rightarrow} \exists r \in \mathbb{Q}, r \in (r_1; r_2), (r_1; r_2) \subseteq (a; b) \Rightarrow$

$\Rightarrow r \in A$

I $r = r_1 \Rightarrow r_1 \in (r_1; r_2)$ (F)

II $r = r_2 \Rightarrow r_2 \in (r_1; r_2)$ (F)

III $r = r_k, k \in \{3; 4; \dots; n\}$

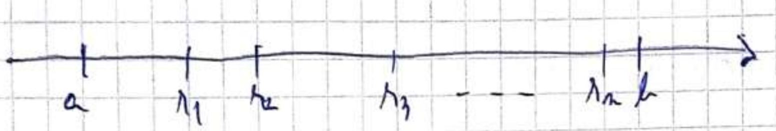
~~(Fie a,)~~

③ Fie $a, b \in \mathbb{R}$, $a < b$

Presupunem prin absurd că între a și b există un nr. finit de nr. raționale, n:

$A=\{r_1, r_2, r_3, \dots, r_n\}, n\in\mathbb{N}^*$, $(r_i<r_j \Leftrightarrow i<j)$, $i,j\in\{1;2;\dots;n\}$

(ADICĂ $a<r_1<r_2<\dots<r_n<b$), $r_k\in\mathbb{Q}, \forall k\in\{1;2;3;\dots;n\}$



$a, r_1 \in \mathbb{R} \Rightarrow \exists r\in\mathbb{Q}$ a.î. $r\in(a;r_1)\subset(a;b)\Rightarrow r\in A$.
$r\in(a;r_1)\Rightarrow r\neq r_1$ | $\Rightarrow r\in\{r_2;r_3;\dots;r_n\} \Rightarrow$

$r_1<r_k, \forall k\in\{2;3;\dots;n\}$

~~($\Rightarrow r>r_1$)~~

$\Rightarrow r_1<r$
Dar $r<r_1$ | → Contradicție ⇒ Există o infinitate de nr. raționale între a și b

~~(Analog pentru nr.)~~

Analog ⇒ Există o infinitate de nr. iraționale între a și b

~~Fie $x_n = 2^{-n}$,~~

~~Fie $(x_n)_{n\in\mathbb{N}^*}$, $x_n=2^{-n}$~~

~~$x_n\ge0, \forall n\in\mathbb{N}^*$~~

~~$2^{-n}\le1 \Leftrightarrow -n\le0, \forall n\in\mathbb{N}^*$ (A) | $\Rightarrow x_n\in[0;1], \forall$~~

~~$x\in[0;1) \Leftrightarrow x+\sqrt{2}\in(a\sqrt{2};1+\sqrt{2})$~~

Fie $(x_n)_{n\in\mathbb{N}^*}$, $x_n=2^{-\frac{1}{n+1}}$

$x_n\ge0, \forall n\in\mathbb{N}^*$

$2^{-\frac{1}{n+1}}\le1 \Leftrightarrow -\frac{1}{n+1}\le0 \Leftrightarrow \frac{1}{n+1}\ge0$ (A), $\forall n\in\mathbb{N}^*$ | $\Rightarrow x_n\in[0;1]$, $\forall n\in\mathbb{N}^*$ (*)

$x_n = 2^{-\frac{1}{n+1}} \Leftrightarrow x^{n+1} = 2^{-1} \Leftrightarrow 2x^{n+1} = 1 \Leftrightarrow 2x^{n+1} - 1 = 0$

$2, -1 \in \mathbb{Z}$

Presupunem prin absurd că $x \in \mathbb{Q} \Rightarrow$

$\Rightarrow x = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$ $\Rightarrow$

$\Rightarrow p \in D_1; q \in D_2 \Rightarrow x \in \{\pm 1; \pm \frac{1}{2}\}$

$x = 1 \Rightarrow 2 - 1 = 0 \Leftrightarrow 1 = 0$ (F)

$x = -1 \Rightarrow \pm 2 - 1 = 0 \Leftrightarrow \pm 2 = 1$ (F)

$x = \frac{1}{2} \Rightarrow 2^{-n} = 1 \Rightarrow n = 0$ (F)

$x = -\frac{1}{2} \Rightarrow \pm 2^{-n} = 1 \Rightarrow n = 0$ (F)

$\Rightarrow x \in \emptyset \Rightarrow x \notin \mathbb{Q} \Rightarrow$

$\Rightarrow x_n \in \mathbb{R} \setminus \mathbb{Q}, \forall n \in \mathbb{N}^* \overset{(*)}{\Rightarrow}$ | $A = \{x_n / n \in \mathbb{N}^*\}$ care e mulțime infinită de nr. iraționale, $A \subset [0; 1]$

4. Fie $x = \frac{\sqrt{2}}{1 + \sqrt[3]{2}}$

$x = \frac{\sqrt{2}}{1 + \sqrt[3]{2}} \Leftrightarrow x + \sqrt[3]{2}x = \sqrt{2} \Leftrightarrow$ ~~$x - \sqrt{2}\ \sqrt[3]{2}x = 0$~~

$\Leftrightarrow \sqrt[3]{2}x = \sqrt{2} - x \Leftrightarrow 2x^3 = 2\sqrt{2} - 3\sqrt{4} \cdot x + 3\sqrt{2}x^2 - x^3 \Leftrightarrow$

$\Leftrightarrow 3x^3 = 2\sqrt{2} - 6x + 3\sqrt{2}x^2 \Leftrightarrow 3x^3 + 6x = \sqrt{2}(2 + 3x^2) \Leftrightarrow$

$\Leftrightarrow 9x^6 + 36x^4 + 36x^2 = 2(4 + 12x^2 + 9x^4) \Leftrightarrow$

$\Leftrightarrow 9x^6 + 36x^4 + 36x^2 = 8 + 24x^2 + 18x^4 \Leftrightarrow$

$\Leftrightarrow 9x^6 + 18x^4 + 12x^2 - 8 = 0$. $\leftarrow$ coeficienți în $\mathbb{Z}$ $\Rightarrow$

Presupunem prin absurd că $x \in \mathbb{Q} \Rightarrow x = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0$

$\Rightarrow p \in D_8; q \in D_9 \Leftrightarrow p \in \{\pm 1; \pm 2; \pm 4; \pm 8\}; q \in \{\pm 1; \pm 3; \pm 9\}$

$\sqrt{2}>0$; $1+\sqrt[3]{2}>0 \Rightarrow x>0$

$\frac{\sqrt{2}}{1+\sqrt[3]{2}} < 1 \Leftrightarrow \sqrt{2} < 1+\sqrt[3]{2} \Leftrightarrow \sqrt{2}-1 < \sqrt[3]{2}$

$\left.\begin{array}{l} \sqrt{2}-1<1 \Leftrightarrow \sqrt{2}<2 \Leftrightarrow 2<4 \text{ (A)} \\ 1<\sqrt[3]{2} \Leftrightarrow 1<2 \text{ (A)} \end{array}\right| \Rightarrow \sqrt{2}-1<\sqrt[3]{2}$ $\Bigg| \Rightarrow$

$\Rightarrow \frac{\sqrt{2}}{1+\sqrt[3]{2}} < 1 \Rightarrow x<1 \Rightarrow p<q$

$x \in \{ \frac{1}{3}; \frac{1}{9}; \frac{2}{3}; \frac{2}{9}; \frac{4}{9}; \frac{8}{9} \}$

$\frac{\sqrt{2}}{1+\sqrt[3]{2}} > \frac{1}{2} \Leftrightarrow 2\sqrt{2} > 1+\sqrt[3]{2} \Leftrightarrow \sqrt{2}+\sqrt{2} > 1+\sqrt[3]{2}$

$\left.\begin{array}{l} \sqrt{2}>1 \Leftrightarrow 2>1 \text{ (A)} \Rightarrow \sqrt{2}>1 \\ \sqrt{2}>\sqrt[3]{2} \Leftrightarrow 2^3>2^2 \text{ (A)} \Rightarrow \sqrt{2}>\sqrt[3]{2} \end{array}\right| \Rightarrow 2\sqrt{2}>\sqrt[3]{2} \Rightarrow x>\frac{1}{2} \Rightarrow$

$\cancel{\frac{1}{3} > \frac{1}{2} \Leftrightarrow 2>3 \text{ (F)}}$

$\Rightarrow x \in \{ \frac{2}{3}; \frac{8}{9} \}$

$\frac{2}{3} > \frac{1}{2} \Leftrightarrow 4>3 \text{ (A)}$; $\frac{2}{9} > \frac{1}{2} \Leftrightarrow 4>9 \text{ (F)}$

$\frac{4}{9} > \frac{1}{2} \Leftrightarrow 8>9 \text{ (F)}$; $\frac{8}{9} > \frac{1}{2} \Leftrightarrow 16>9 \text{ (A)}$

$\cancel{x = \frac{2}{3} \Rightarrow}$

I $x=\frac{2}{3} \Rightarrow$ $\cancel{(2+\sqrt[3]{2})}$ $\frac{2}{3} = \frac{\sqrt{2}}{1+\sqrt[3]{2}} \Leftrightarrow 2+2\sqrt[3]{2} = 3\sqrt{2}$

$\left.\begin{array}{l} \sqrt[3]{2}>\sqrt{2} \Rightarrow 2\sqrt[3]{2}>2\sqrt{2} \\ 2>\sqrt{2} \end{array}\right| \Rightarrow$ $\Bigg| \Rightarrow x \neq \frac{2}{3}$

$\Rightarrow 2+2\sqrt[3]{2} > 3\sqrt{2}$.

$\underline{\text{II}}$ $x = \frac{8}{9} \Rightarrow \frac{\sqrt{2}}{1+\sqrt[3]{2}} = \frac{8}{9} \Rightarrow 9\sqrt{2} = 8+8\sqrt[3]{2} \Rightarrow$

$\cancel{\Rightarrow 162 = 64 + 128\sqrt[3]{2} + 64\sqrt[3]{4} \Rightarrow}$

$\cancel{\Rightarrow 98 = 64(2\sqrt[3]{2} + \sqrt[3]{4}) \Leftrightarrow 49 = 32(2\sqrt[3]{2}+\sqrt[3]{4})}$

$\cancel{\Rightarrow (8\sqrt[3]{2} = 8 - 9\sqrt{2}) \; 9\sqrt{2} - 8 = 8\sqrt[3]{2} \Rightarrow}$

$\cancel{\Rightarrow 9\sqrt{2} - 8 = 8\sqrt[3]{2} \Rightarrow (9\sqrt{2}-8)^3 = 8^3 \cdot 2 \Rightarrow}$

$\cancel{\Rightarrow 9?}$

$162 = 64(1+2\sqrt[3]{2}+\sqrt[3]{4}) = 64(1+\sqrt[3]{4}+\sqrt[3]{16}) \Big| \cdot (1-\sqrt[3]{4}) \Leftrightarrow$

$\Leftrightarrow 164(1-\sqrt[3]{4}) = 64(1-4) \Leftrightarrow 164 - 164\sqrt[3]{4} = -192 \Leftrightarrow$

$\Leftrightarrow 356 = 164\sqrt[3]{4} \Leftrightarrow 89 = 41\sqrt[3]{4} \Leftrightarrow 89^3 = 41^3 \cdot 4 \Rightarrow$

$\left(\Leftrightarrow \left(\frac{89}{41}\right)^3 = 4\right) \Rightarrow u(89^3) = u(41^3 \cdot 4) \Rightarrow u(9^3) = 4 \Rightarrow$

$\Leftrightarrow 7 = 4 \text{ (F)} \Rightarrow x \neq \frac{8}{9}$

$\cancel{(\Leftrightarrow x \notin \mathbb{Q} \Rightarrow x \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow \frac{\sqrt{2}}{1+\sqrt[3]{2}} \in \mathbb{R}\setminus\mathbb{Q})}$

$\cancel{\Rightarrow x \in \emptyset \text{ (F)} \Rightarrow x \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow}$

$\cancel{\Rightarrow x \in \emptyset \text{ (F)} \Rightarrow x \notin \mathbb{Q} \underset{x\in\mathbb{R}}{} \Big| \Rightarrow x \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow \frac{\sqrt{2}}{1+\sqrt[3]{2}} \in \mathbb{R}\setminus\mathbb{Q}}$

$\cancel{\Rightarrow x \in \emptyset \text{ (F)} \Leftrightarrow}$

$\Rightarrow x \in \emptyset$ (F) $\rightarrow$ Contradicție $\Rightarrow x \notin \mathbb{Q} \Rightarrow \dfrac{\sqrt{2}}{1+\sqrt[3]{2}} \notin \mathbb{Q}$

12.10.2020

Un șir $(x_n)_{n\in\mathbb{N}^*}$, $x_n \in \mathbb{R}$, s.n.:

$\rightarrow$ mărginit inferior dacă mulțimea $\{x_n \mid n \in \mathbb{N}^*\}$ e mărginită inferior

$\rightarrow$ mărginit superior dacă mulțimea $\{x_n \mid n \in \mathbb{N}^*\}$ e mărginită superior

$\rightarrow$ mărginit dacă e mărginit atât inferior cât și superior

SAU

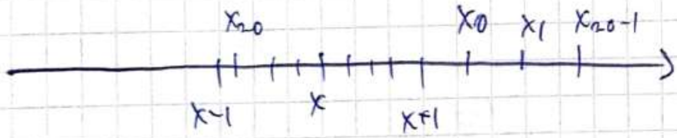
• dacă $\exists\, m, M \in \mathbb{R}$ a.î. $m \leq x_n \leq M, \forall n \in \mathbb{N}$

Obs. ① Orice șir convergent e mărginit; ② Reciproca nu e adevărată

① Dem: $(x_n)_{n\in\mathbb{N}^*}$ convergent, $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$

$\varepsilon = 1 \Rightarrow |x_n - x| < 1, \forall n \geq n_0$

$x_n \in (x-1; x+1)$



$m = \min(x-1, x_0, x_1, \ldots; x_{n_0-1})$

$M = \max(x+1, x_0, x_1, \ldots, x_{n_0-1})$

② Ex: $(x_n)_{n\in\mathbb{N}^*} = (-1)^n, \forall n \in \mathbb{N}$

e) $x_n = \left(1+\frac{\cos(n\pi)}{n}\right)^n$

Din Crit. Cleşte $\Rightarrow \lim_{n\to\infty} \frac{\cos(n\pi)}{n} = 0$

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} e^{\cos(n\pi)} \dots$

$\lim_{n\to\infty} x_{2k} = \lim_{n\to\infty} e^{\cos(2k\pi)} = e$

$\lim_{n\to\infty} x_{2k+1} = \lim_{n\to\infty} e^{\cos(\pi)} = \frac{1}{e}$

$Lim(x_n) = \{e; e^{-1}\}$

$\lim_{n\to\infty} \inf x_n = e^{-1}$, $\lim_{n\to\infty} \sup x_n = e$

<u>Exerciţii suplimentare</u>

① $\cancel{\forall\varepsilon}$ $\lim_{n\to\infty} \frac{n^3-n}{n^3+n} = 1$, $n\in\mathbb{N}^*$ $\quad x_n = \frac{n^3-n}{n^3+n} = \frac{n^2-1}{n^2+1}$

$\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}^*$ a.î. $\forall n \geq n_0$ $|x_n - 1| < \varepsilon \Leftrightarrow$

$\Leftrightarrow \left|\frac{n^2-1-n^2-1}{n^2+1}\right| = \frac{2}{n^2+1} < \frac{2}{n^2} < \varepsilon \Leftrightarrow \frac{2}{\varepsilon} < n^2 \Rightarrow n_0 = \left[\sqrt{\frac{2}{\varepsilon}}\right]+1 \Rightarrow$

$\Rightarrow \forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}^*$ a.î. $\forall n \geq n_0$ $|x_n - 1| < \varepsilon \Rightarrow \lim_{n\to\infty} \frac{n^3-n}{n^3+n} = 1.$

② a) $x_n = 1,\underbrace{99\ldots9}_{\text{de n ori}} = 1+\frac{9}{10}+\frac{9}{10^2}+\dots+\frac{9}{10^n} =$

$$= 1+9\sum_{k=1}^{n}\frac{1}{10^k} = 1+9\cdot\frac{1}{10}\cdot\frac{\left(\frac{1}{10}\right)^n-1}{\frac{1}{10}-1} = 1+\frac{9}{10}\cdot\frac{1-\frac{1}{10^n}}{\frac{9}{10}} =$$

$= 1+1-\frac{1}{10^n} = 2-\frac{1}{10^n} \Rightarrow \lim_{n\to\infty} x_n = 2\in\mathbb{R} \Rightarrow (x_n)_{n\in\mathbb{N}}$ convergent

b) $x_n = 1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}$

$x_{n+1}-x_n = \frac{1}{(n+1)^2} > 0,\ \forall n\geq 1 \Rightarrow (x_n)_{n\in\mathbb{N}^*}$ strict crescător $\overset{(*)}{\Rightarrow} x_n \geq x_1 > 0,\ \forall n\in\mathbb{N}$

$$x_n = \sum_{k=1}^{n}\frac{1}{k\cdot k} \leq 1+\sum_{k=2}^{n}\frac{1}{(k-1)\cdot k} = 1+\left(1-\frac{1}{2}+\frac{1}{2}-\dots+\frac{1}{n-1}-\frac{1}{n}\right)$$

$= 2-\frac{1}{n} < 2,\ \forall n\in\mathbb{N}^* \Rightarrow x_n\in[0;2],\ \forall n\in\mathbb{N}^* \overset{(*)}{\underset{\text{T. Weierstrass}}{\Rightarrow}} x_n$ convergent.

Euler demonstrează (dar printr-o neriguroasă) că $\lim_{n\to\infty} x_n = \frac{\pi^2}{6}$.

c) $x_n = \frac{\sin(1)}{5}+\frac{\sin(2)}{5^2}+\dots+\frac{\sin(n)}{5^n} = \sum_{k=1}^{n}\frac{\sin(k)}{5^k}$

~~$(x_n)_{n\in\mathbb{N}^*}$ fundamental $\Leftrightarrow \forall\varepsilon>0, \exists n_\varepsilon\in\mathbb{N}$ a.î. $\forall n\geq n_0$ $|x_m-x_n|<\varepsilon$.~~

$(x_n)_{n\in\mathbb{N}^*}$ fundamental $\Leftrightarrow \forall\varepsilon>0, \exists n_0\in\mathbb{N}$ a.î. $\forall n\geq n_0, \forall p\in\mathbb{N}$ $|x_{n+p}-x_n|<\varepsilon$

$$|x_{n+p}-x_n| = \left|\sum_{k=1}^{p}\frac{\sin(n+k)}{5^{n+k}}\right| \leq \sum_{k=1}^{p}\frac{|\sin(n+k)|}{5^{n+k}} \leq \sum_{k=1}^{p}\frac{1}{5^{n+k}} = \frac{1}{5^{n+1}}\cdot\frac{\left(\frac{1}{5}\right)^p-1}{\frac{1}{5}-1} =$$

$$= \frac{5}{4}\cdot\frac{1}{5^{n+1}}\cdot\frac{1-\frac{1}{5^p}}{1} = \frac{1}{4}\cdot\frac{1}{5^n}\left(1-\frac{1}{5^p}\right) < \frac{1}{4}\cdot\frac{1}{5^n} < \varepsilon \Leftrightarrow \frac{1}{5^n} < 4\varepsilon$$

$\Leftrightarrow n = -\log_5(4\varepsilon) \Rightarrow n_0 = [-\log_5(4\varepsilon)]+1 \Rightarrow x_n$ fundamental $\Rightarrow$

$\Rightarrow (x_n)_{n\in\mathbb{N}^*}$ convergent (*)

d) $x_{n+1} = \sqrt{2x_n+3}$, $x_1=\sqrt{3}$

$P(n)$: " $x_{n+1} > x_n$ ", $\forall n \in \mathbb{N}$

I Verificare

$P(1)$: " $x_2 > x_1$ " (A)

$\sqrt{2\sqrt{3}+3} > \sqrt{3} \Leftrightarrow 2\sqrt{3}>0$ (A)

II Demonstrație

$P(k)$: " $x_{k+1} > x_k$ " (A)

$P(k+1)$: " $x_{k+2} > x_{k+1}$ "

$x_{k+2} > x_{k+1} \Leftrightarrow 2x_{k+1}+3 > 2x_k+3 \Leftrightarrow x_{k+1} > x_k$ (A) din $P(k) \Rightarrow$

$\Rightarrow P(k+1)$ (A)

I, II $\Rightarrow P(n)$ (A), $\forall n\in\mathbb{N} \Rightarrow x_{n+1} > x_n$, $\forall n\in\mathbb{N} \Rightarrow (x_n)_{n\in\mathbb{N}}$ strict crescător $\overset{(*)}{\Rightarrow}$

$\Rightarrow x_n \ge x_1=\sqrt{3}$, $\forall n \in \mathbb{N}^*$

$P(n)$: " $x_n \le 3$ ", $\forall n\in\mathbb{N}^*$

I Verificare

$P(1)$: " $x_1 \le 3$ " (A)

II Demonstrație

$P(k)$: " $x_k \le 3$ " (A)

$P(k+1)$: " $x_{k+1} \le 3$ "

$x_{k+1} \le 3 \Leftrightarrow \sqrt{2x_k+3} \le 3 \Leftrightarrow 2x_k \le 6 \Leftrightarrow x_k \le 3$ (A) $\Rightarrow P(k+1)$ (A)

I, II $\Rightarrow P(n)$ (A), $\forall n\in\mathbb{N}^* \Rightarrow x_n \le 3$, $\forall n\in\mathbb{N}^* \Rightarrow x_n \in [\sqrt{3}; 3]$, $\forall n\in\mathbb{N}^* \overset{(**)}{\Rightarrow}$

$\overset{(*)}{\Rightarrow} (x_n)_{n\in\mathbb{N}^*}$ convergent, Fie $l=\lim_{n\to\infty} x_n \in \mathbb{R}$

$x_{n+1}=\sqrt{2x_n+3} \Rightarrow l=\sqrt{2l+3} \Rightarrow l^2=2l+3 \Leftrightarrow l^2-2l-3=0$

$\Delta = 4+12 = 16 \Rightarrow l_{1,2} = \frac{2\pm 4}{2}$

$\Rightarrow l\in\{3;-1\}$

$l\in[0;3]$ $\Rightarrow$

$\Rightarrow l = 3 \Rightarrow \lim_{n\to\infty} x_n = 3$

2) $x_{n+1} = 1 + \frac{1}{x_n}$, $x_1 = 1$.

$x_1 = \frac{1}{1}$; $x_2 = \frac{2}{1}$; $x_3 = \frac{3}{2}$; $x_4 = \frac{5}{3}$; $x_5 = \frac{8}{5}$; $x_6 = \frac{13}{5}$.

$$x_{n+2} = 1 + \frac{1}{x_{n+1}} = 1 + \frac{1}{1+\frac{1}{x_n}} = 1 + \frac{x_n}{x_n+1} = 2 - \cancel{\frac{1}{x_n+1}} \frac{1}{x_n+1}$$

$P(n)$: „ $x_{2k-1} < x_{2k+1}$ și $x_{2k} > x_{2k+2}$ " A, $\forall n \geq 1$.

<u>I Verificare:</u>

$P(1)$: „ $x_1 < x_3$ și $x_2 > x_4$ " (A)

<u>II Demonstrare:</u>

$P(k)$: „ $x_{2k-1} < x_{2k+1}$ și $x_{2k} > x_{2k+2}$ " (A)

$P(k+1)$: „ $x_{2k+1} < x_{2k+3}$ și $x_{2k+2} > x_{2k+4}$ "

$$x_{2k+1} < x_{2k+3} \wedge x_{2k+2} > x_{2k+4} \Leftrightarrow \cancel{2} - \frac{1}{x_{2k-1}+1} < \cancel{2} - \frac{1}{x_{2k+1}+1} \wedge$$

$$\wedge\ \cancel{2} - \frac{1}{x_{2k}+1} > \cancel{2} - \frac{1}{x_{2k+2}+1} \Leftrightarrow x_{2k+1} > x_{2k-1} \wedge x_{2k+2} < x_{2k} \text{ (A)}$$

$\Rightarrow P(k+1)$ (A)

$\left.\begin{matrix} I \\ II \end{matrix}\right| \Rightarrow P(n)$ (A), $\forall n \in \mathbb{N}^* \Rightarrow (x_{2n-1})_{n\in\mathbb{N}^*}$ strict. crescător $\overset{a_1}{\Rightarrow} x_{2n-1} \geq 1$, $\forall n \in \mathbb{N}^*$

$(x_{2n})_{n\in\mathbb{N}^*}$ strict descrescător $\overset{a_2}{\Rightarrow} x_{2n} \leq 2$, $\forall n \in \mathbb{N}^*$

$P(n)$: „ $1 \le x_n \le 2$ ", $\forall n \in \mathbb{N}^*$

I Verificare:

$P(1)$: „ $1 \le x_1 \le 2$ " (A)

II Demonstrație:

$P(k)$: „ $1 \le x_k \le 2$ " (A)

$P(k+1)$: „ $1 \le x_{k+1} \le 2$ "

$1 \le x_{k+1} \le 2 \Leftrightarrow 1 \le 1 + \frac{1}{x_k} \le 2 \Leftrightarrow 0 \le \frac{1}{x_k} \le 2 \Leftrightarrow \begin{cases} 0 \le \frac{1}{x_k} \text{ (A)} \Leftrightarrow \frac{1}{2} \le x_k \text{ (A)} \\ \frac{1}{x_k} \le 1 \end{cases}$

$\Rightarrow P(k+1)$ (A)

$\left.\begin{matrix} \text{I} \\ \text{II} \end{matrix}\right| \Rightarrow P(n)$ (A), $\forall n \in \mathbb{N}^* \Rightarrow x_n \in [1;2], \forall n \in \mathbb{N}^* \xRightarrow[\text{T. Weierstrass}]{(A)\,(1)(2)}$ $(x_{2n-1})_{n \in \mathbb{N}^*}$ și $(x_{2n})_{n \in \mathbb{N}^*}$ convergente

Fie $l = \lim_{n \to \infty} x_{2n-1} \Rightarrow l = 2 - \frac{1}{l+1}$ ~~$\frac{2l+1}{l+1} \Leftrightarrow 2l$~~ $\Big| \Leftrightarrow$

$\Leftrightarrow l^2 + l = 2l + 2 - 1 \Leftrightarrow l^2 - l - 1 = 0.$

$\Delta = 1 + 4 = 5$

$\left.\begin{matrix} l_{1,2} = \frac{1 \pm \sqrt{5}}{2} \\ l \ge 0 \end{matrix}\right| \Rightarrow l = \frac{1+\sqrt{5}}{2} \Rightarrow \lim_{n \to \infty} x_{2n-1} = \frac{1+\sqrt{5}}{2}$

Analog $\Rightarrow \lim_{n \to \infty} x_{2n} = \frac{1+\sqrt{5}}{2}$

$\Rightarrow \lim_{n \to \infty} x_n = \frac{1+\sqrt{5}}{2} \Rightarrow x_n \to$ convergent

③ a) $x_n = \frac{1}{2+\sqrt{n}\cos(n\pi)}$

$x_{2k} = \frac{1}{2+\sqrt{2k}}$; $x_{2k+1} = \frac{1}{2-\sqrt{2k}}$

$\lim\limits_{k\to\infty} x_{2k} = \frac{1}{\infty} = 0$; $\lim\limits_{k\to\infty} x_{2k+1} = \frac{1}{2-\infty} = \frac{1}{-\infty} = 0 \Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}\sup x_n = \lim\limits_{n\to\infty}\inf x_n = 0 \Rightarrow LIM(x_n) = \{0\}$

b) $x_n = \left(\frac{n+3}{n+1}\right)^{n\sin\frac{n\pi}{3}} = \left(1+\frac{2}{n+1}\right)^{n\sin\frac{n\pi}{3}}$

$\lim\limits_{n\to\infty} x_n = e^{2\lim\limits_{n\to\infty}\sin\frac{n\pi}{3}}$

$\lim\limits_{k\to\infty} x_{6k} = e^{2\cdot 0} = 1$; $\lim\limits_{k\to\infty} x_{6k+1} = e^{2\cdot\frac{\sqrt{3}}{2}} = e^{\sqrt{3}}$

$\lim\limits_{k\to\infty} x_{6k+2} = e^{2\cdot\frac{\sqrt{3}}{2}} = e^{\sqrt{3}}$; $\lim\limits_{k\to\infty} x_{6k+3} = e^{2\cdot 0} = 1$

$\lim\limits_{k\to\infty} x_{6k+4} = e^{2\cdot\left(-\frac{\sqrt{3}}{2}\right)} = e^{-\sqrt{3}}$ $\lim\limits_{k\to\infty} x_{6k+5} = e^{-2\left(\frac{\sqrt{3}}{2}\right)} = e^{-\sqrt{3}}$

$LIM(x_n) = \{e^{-\sqrt{3}}; e^{0}; e^{\sqrt{3}}\}$; $\lim\limits_{n\to\infty}\inf x_n = e^{-\sqrt{3}}$

$\lim\limits_{n\to\infty}\sup x_n = e^{\sqrt{3}}$

$$S_n = 1 - \frac{2^{n+1}}{(n+2)!} = 1 - x_n$$

$$x_n = \frac{2^{n+1}}{(n+2)!}$$

$$\frac{x_{n+1}}{x_n} = \frac{2^{n+2}}{(n+3)!} \cdot \frac{(n+2)!}{2^{n+1}} = \frac{2}{n+3} \overset{<1}{\cancel{\ge 1}} \Rightarrow x_n \text{ strict descrescător}$$

$x_n > 0, \forall n \in \mathbb{N}^*$ $\Big| \Rightarrow$

$\Rightarrow x_n$ monoton și mărginit $\overset{T.\ Weierstrass}{\Rightarrow}$

$$\Rightarrow \exists\, l \in \mathbb{R},\ l = \lim_{n\to\infty} x_n$$

$$x_{n+1} = \frac{2 x_n}{n+3} \Rightarrow l = \frac{2}{\infty} \cdot l = 0 \Rightarrow \lim_{n\to\infty} x_n = 0$$

$$\lim_{n\to\infty} S_n = 1 - 0 = 1 \Rightarrow \left(\sum_{k=1}^{\infty}\right) \sum_{n=1}^{\infty} \frac{n \cdot 2^n}{(n+2)!}$$

~~Temă Seminar 4~~

① $y_n = \dfrac{1+\sqrt{2}+\dots+\sqrt{n}}{n\sqrt{n}}$ $\left(\cancel{\sum_{k=1}^{n} \frac{\sqrt{k}}{n\sqrt{n}}}\right)$

Fie $a_n = 1+\sqrt{2}+\dots+\sqrt{n}$; $b_n = n\sqrt{n} = n^{\frac{3}{2}} \to$ strict ~~cresc~~ monoton și divergent

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \left(\lim_{n\to\infty} \frac{\sqrt{n+1}}{n\sqrt{n+1}}\right) \lim_{n\to\infty} \frac{\sqrt{n+1}}{(n+1)\sqrt{n+1} - n\sqrt{n}} =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}\left((n+1)\sqrt{n+1} + n\sqrt{n}\right)}{(n+1)^3 - n^3} = \cancel{\lim_{n\to\infty} \sqrt{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^2\sqrt{1+n^{-1}}\left((1+n^{-1})\sqrt{1+n^{-1}}+1\right)}{3n^2+3n+1} = \lim_{n\to\infty} \frac{\sqrt{1+n^{-1}}\left((1+n^{-1})\sqrt{1+n^{-1}}+1\right)}{3+3n^{-1}+n^{-2}}$$

$$= \frac{\sqrt{1}\,(1\cdot 1+1)}{3} = \frac{2}{3} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3} \Rightarrow \lim_{n\to\infty} y_n = \frac{2}{3}$$

b) $y_n = \dfrac{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}{\ln n}$

Fie $a_n = 1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}$; $b_n = \ln n \to$ șir strict crescător și divergent

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n+1}}}{\ln\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}}\cdot\frac{1}{\underbrace{\ln\left(1+\frac{1}{n}\right)}_{\to 1}\cdot\frac{1}{n}} =$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}}\cdot n = \lim_{n\to\infty}\frac{n^{\frac{1}{2}}}{\sqrt{1+\frac{1}{n}}} = \frac{\infty}{\sqrt{1}} = \infty \Rightarrow \lim_{n\to\infty}\frac{a_n}{b_n} = +\infty$$

$$\Rightarrow \lim_{n\to\infty} y_n = +\infty$$

c) $y_n = \dfrac{\sqrt[n]{(n+1)(n+2)\dots(n+n)}}{n}$ $= \sqrt[n]{\dfrac{(n+1)(n+2)\dots(n+n)}{n^n}} =$

$$= \sqrt[n]{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{n}{n}\right)}$$

~~Fie $a_n = \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{n}{n}\right) \to$ șir cu termeni strict pozitivi~~

Fie $a_n = \dfrac{(n+1)(n+2)\dots(n+n)}{n^n} \to$ șir cu termeni strict pozitivi

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{\cancel{(n+2)(n+3)\dots(n+n)}(n+n+1)(2n+2)}{(n+1)^{n+1}}\cdot\frac{n^n}{(n+1)\cancel{(n+2)\dots(n+n)}} =$$

$$= \lim_{n\to\infty}\frac{(2n+1)(2n+2)\cdot n^n}{(n+1)^{n+2}} = \lim_{n\to\infty}\frac{(2+n^{-1})(2+2n^{-1})\cancel{n^{n+2}}}{\left(1+\frac{1}{n}\right)^{n+2}\cdot\cancel{n^{n+2}}} =$$

$$= 4\cdot\frac{1}{e^{\lim_{n\to\infty}\frac{n+2}{n}}} = 4\cdot\frac{1}{e^1} = \frac{4}{e}$$

d) $y_n = \frac{\ln n!}{n \ln n}$; $y_n = \frac{a_n}{b_n}$

Fie $a_n = \ln n!$, $b_n = n \ln n$ → pr ~~[illegible]~~ strict crescător și divergent

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \cancel{\lim_{n\to\infty} \frac{\ln\frac{(n+1)!}{n!}}{n\ln n}} \lim_{n\to\infty} \frac{\ln\left(\frac{(n+1)!}{n!}\right)}{(n+1)\ln(n+1)-n\ln n} =$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{(n+1)\ln(n+1)-n\ln(n)} = \lim_{n\to\infty} \frac{\ln(n+1)}{n\ln(n+1)-n\ln n+\ln(n+1)} =$$

$$= \lim_{n\to\infty} \frac{\ln(n+1)}{n\ln\left(1+\frac{1}{n}\right)+\ln(n+1)} = \lim_{n\to\infty} \frac{1}{\frac{n}{\ln(n+1)}\ln\left(1+\frac{1}{n}\right)+1} =$$

$$= \lim_{n\to\infty} \frac{1}{\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \cdot \frac{1}{\ln(n+1)} + 1} = \frac{1}{1 \cdot \frac{1}{\infty} + 1} = \frac{1}{0+1} = \frac{1}{1} = 1$$

$\Rightarrow \lim_{n\to\infty} y_n = 1$

(2) Fie $(x_n)_{n\in\mathbb{N}}$, ~~$x_n > 0$, $x_n \to l$~~ → termeni strict pozitivi, c.î. $\exists \lim_{n\to\infty} x_n = l$

Fie ~~y_n~~ $\Rightarrow y_n = \frac{x_1+x_2+\dots+x_n}{n}$; Fie $a_n = x_1+x_2+\dots+x_n$, $b_n = n$
→ strict crescător și divergent

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{x_{n+1}}{1} = \lim_{n\to\infty} x_{n+1} = \underset{\in\overline{\mathbb{R}}}{l} \Rightarrow \lim_{n\to\infty} y_n = l$$

Fie $z_n = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$; Fie $c_n = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ → termeni strict pozitivi

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} x_{n+1} = l \Rightarrow \lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} z_n = l.$$

$\Rightarrow \lim_{n\to\infty} z_n = l$

③ a) $\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n = -1+\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n = -1+\frac{1}{1+\frac{2}{3}} = -1+\frac{3}{5} = -\frac{2}{5}$

$S_n = \sum_{k=1}^{n}\left(-\frac{2}{3}\right)^n = \sum_{k=0}^{n}\left(-\frac{2}{3}\right)^n - 1 = -1+\sum_{k=0}^{n}\left(-\frac{2}{3}\right)^n \Rightarrow$

$\Rightarrow \lim_{n\to\infty} S_n = -1+\frac{1}{1+\frac{2}{3}} = -1+\frac{3}{5} = -\frac{2}{5} \Rightarrow S = -\frac{2}{5}$

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③ a) $\sum_{n=1}^{\infty}\left(-\frac{2}{3}\right)^n = -1+\sum_{n=0}^{\infty}\left(-\frac{2}{3}\right)^n = -1+\frac{1}{1+\frac{2}{3}} = -1+\frac{3}{5} = -\frac{2}{5}$

b) $\sum_{n=2}^{\infty}\frac{1}{C_n^2}$

$S_n = \sum_{k=2}^{n}\frac{1}{C_k^2} = \sum_{k=2}^{n}\frac{1}{\frac{(k-1)k}{2}} = \sum_{k=2}^{n}\frac{2}{(k-1)k} = 2\sum_{k=2}^{n}\frac{k-(k-1)}{k(k-1)} =$

$= 2\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right) = 2\left(\frac{1}{1}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\dots+\cancel{\frac{1}{n-1}}-\frac{1}{n}\right) =$

$= 2\left(1-\frac{1}{n}\right) \Rightarrow \lim_{n\to\infty} S_n = 2(1-0) = 2 \Rightarrow S = \frac{1}{2} \Rightarrow \cancel{\lim_{n\to\infty}}$

$\Rightarrow \sum_{n=2}^{\infty}\frac{1}{C_n^2} = \frac{1}{2}$

c) $\sum_{n=1}^{\infty}\operatorname{arctg}\frac{1}{n^2+n+1}$

$\left(S_n = \sum_{k=1}^{n}\operatorname{arctg}\frac{n^3+1-n^3}{n^2+n+1} = \sum_{k=1}^{n}\operatorname{arctg}\left(-\frac{n^3-1+n^3}{n^2+n+1}\right) = \right.$

$\left(= \sum_{k=1}^{n}\operatorname{arctg}\left(\frac{n^3}{n^2+n+1}-n+1\right)\right)$

$\cancel{\left(-\frac{\pi}{2};\frac{\pi}{2}\right)}$

Fix $x,y\in\mathbb{R}$; Fix $\alpha = \operatorname{arctg} x - \operatorname{arctg} y \Rightarrow \operatorname{tg}\alpha = \frac{x-y}{1+xy} \Rightarrow \alpha = \operatorname{arctg}\frac{x-y}{1+xy} \Rightarrow$

$\Rightarrow \operatorname{arctg} x - \operatorname{arctg} y = \operatorname{arctg}\frac{x-y}{1+x\cdot y}$ | $\Rightarrow \operatorname{arctg}(n+1) - \operatorname{arctg} n = \operatorname{arctg}\frac{n+1-n}{1+n(n+1)} =$

Fix $x = n+1$; $y = n$

$= \operatorname{arctg}\frac{1}{n^2+n+1}$

$\cancel{S_n = \sum_{k=1}^{n} \frac{1}{k^2+k+1} = \sum_{k=1}^{n} \operatorname{arctg}}$

$S_n = \sum_{k=1}^{n} \frac{1}{}$

$\cancel{S_n = \sum_{k=1}^{n} \operatorname{arctg} \frac{1}{n^2+n}}$

$S_n = \sum_{k=1}^{n} \operatorname{arctg} \frac{1}{k^2+k+1} = \sum_{k=1}^{n} (\operatorname{arctg}(k+1) - \operatorname{arctg} k) = \sum_{k=1}^{n} (-\operatorname{arctg} k + \operatorname{arctg}(k+1)) =$

$\cancel{(= \operatorname{arctg} 2 - \operatorname{arctg} 1 + \operatorname{arctg} 3 - \operatorname{arctg})}$

$= -\operatorname{arctg} 1 + \cancel{\operatorname{arctg} 2} - \cancel{\operatorname{arctg} 2} + \cancel{\operatorname{arctg} 3} - \ldots - \cancel{\operatorname{arctg} n} + \operatorname{arctg}(n+1) =$

$= \operatorname{arctg}(n+1) - \frac{\pi}{4} \Rightarrow \lim_{n\to\infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \Rightarrow S = \frac{\pi}{4} \Rightarrow \sum_{n=1}^{\infty} \operatorname{tg} \frac{1}{n^2+n+1}$

d) $\sum_{n=0}^{\infty} \frac{1+a^n}{(1+a)^n}, \; a>0$

$\cancel{S_n = \sum_{n=0}^{\infty} \frac{1+a^n}{(1+a)^n}} \; \cancel{\sum_{k=0}^{n} \frac{1+a^k}{(1+a)^k} - \sum_{k=0}^{n} \frac{1}{(1+a)^k}}$

$S_n = \sum_{k=0}^{n} \frac{1+a^k}{(1+a)^k} = \sum_{k=0}^{n} \underbrace{\left(\frac{1}{1+a}\right)}_{\in(0;1)}^k + \sum_{k=0}^{n} \underbrace{\left(\frac{a}{1+a}\right)}_{\in(0;1)}^k \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-\frac{1}{1+a}} + \frac{1}{1-\frac{a}{a+1}} =$

$\cancel{\sum_{k=0}^{n} \frac{1}{1}}$ $= \frac{1}{\frac{a}{a+1}} + \frac{1}{\frac{1}{a+1}} = \frac{a+1}{a} + a+1 = \frac{a+1+(a+1)a}{a} =$

$= \frac{a^2+2a+1}{a} = \frac{(a+1)^2}{a} \Rightarrow S = \frac{(a+1)^2}{a}; \; \sum_{n=0}^{\infty} \frac{1+a^n}{(1+a)^n} = \frac{(a+1)^2}{a}$

Criteriul STOLZ - CESARO (0/0)

Fie $(a_n)_{n\in\mathbb{N}}$ → pe dar să fie $(b_n)_{n\in\mathbb{N}}$ strict monotonă; a.î. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$

(f. zi) $\exists \lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = \ell \in \overline{\mathbb{R}}$

~~$\mathrm{I}\ \ell \in \mathbb{R} \Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 \left| \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} - \ell \right| < \varepsilon$~~

~~$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{a_n - a_{n-1} + a}{b_n - b_{n-1}}$~~

~~$\lim\limits_{n\to\infty}$~~

I $\ell \in \overline{\mathbb{R}}$ $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \ell \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0, \left| \dfrac{a_n}{b_n} - \ell \right| < \varepsilon \Leftrightarrow$

$\Leftrightarrow -\varepsilon + \ell < \dfrac{a_n}{b_n} < \varepsilon + \ell$

LUĂM $n_0 = \max(n_1, n_2, n_3)$, n_1, n_2, n_3 indici unde condiție pt 3 puncte

$\lim\limits_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = \ell \Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 \left| \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} \right| < \varepsilon \Leftrightarrow$

~~$\Leftrightarrow < \varepsilon + \ell < \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} < \varepsilon + \ell,$~~

~~$-\varepsilon < a_n < \varepsilon \Rightarrow -2\varepsilon < a_{n+1} - a_n < 2\varepsilon \Leftrightarrow |a_{n+1} - a_n| < 2\varepsilon$~~

~~$-\varepsilon < b_n < \varepsilon \Rightarrow -2\varepsilon < b_{n+1} - b_n < 2\varepsilon \Leftrightarrow |b_{n+1} - b_n| < 2\varepsilon \Leftrightarrow$~~

~~$\Leftrightarrow \dfrac{1}{2\varepsilon} < \dfrac{1}{|b_{n+1}-b_n|}$~~

$\dfrac{|a_{n+1}-a_n|}{|b_{n+1}-b_n|} < \dfrac{2\varepsilon}{|b_{n+1}-b_n|}$

II Fie $\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ convergentă $\Rightarrow \lim_{n\to\infty} \frac{x_n}{1+x_n} = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{1+\frac{1}{x_n}} = 0 \Rightarrow$

$\Rightarrow \lim_{n\to\infty} \frac{1}{x_n} = +\infty \Rightarrow \lim_{n\to\infty} x_n = 0$

Folosim criteriul comparației cu limită $\left(\lim_{n\to\infty} \frac{x_n}{1} \right) \lim_{n\to\infty} \frac{\frac{x_n}{1+x_n}}{x_n} =$

$= \lim_{n\to\infty} \frac{1}{1+x_n} = 1 \in (0, +\infty) \Rightarrow \sum_{n=1}^{\infty} x_n \sim \sum_{n=1}^{\infty} \frac{x_n}{1+x_n} \rightarrow$ convergentă

(1) a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = \sum_{n=1}^{\infty} e^{-\frac{1}{n}\ln(n)} = \sum_{n=1}^{\infty} e^{-\frac{\ln(n)}{n}}$

$\lim_{n\to\infty} e^{-\frac{\ln(n)}{n}} = e^{-0} = 1 \neq 0 \Rightarrow S = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} \rightarrow$ divergentă

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$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1 \neq 0 \Rightarrow S \rightarrow$ divergentă

b) $\sum_{n=0}^{\infty} \frac{2^n}{n+3^n} = \sum_{n=0}^{\infty} \frac{\left(\frac{2}{3}\right)^n}{\frac{n}{3^n}+1} = \sum_{n=0}^{\infty} x_n$

~~$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n}{\frac{n}{3^n}+1} \cdot \frac{\frac{n+1}{3^{n+1}}+1}{\left(\frac{2}{3}\right)^{n+1}} = \frac{3}{2} \lim_{n\to\infty} \frac{\frac{n+1+3^{n+1}}{3^{n+1}}}{\frac{n+3^n}{3^n}} =$~~

~~$= \frac{3}{2} \lim_{n\to\infty} \frac{n+1+3^{n+1}}{n\cdot 3+3^{n+1}} = \frac{3}{2} \lim_{n\to\infty} \frac{n}{3^{n+1}}+1$~~

$$x = \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \frac{\frac{2}{3}}{\left(\frac{n}{3^n}+1\right)^{\frac{1}{n}}} = \frac{2}{3}\lim_{n\to\infty} \frac{1}{\left(\frac{n}{3^n}+1\right)^{\frac{1}{n}}}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n}{3^n}\cdot\frac{3^{n+1}}{n+1} = 3\lim_{n\to\infty}\frac{n}{n+1} = 3>1 \Rightarrow \lim_{n\to\infty} a_n = 0$$

$$C = \frac{2}{3}\cdot\frac{1}{(0+1)^{0}} = \frac{2}{3}\cdot\frac{1}{1^0} = \frac{2}{3}\cdot 1 = \frac{2}{3} < 1 \Rightarrow \sum_{n=0}^{\infty} x_n \to \text{convergentă}$$

$x_n \in C_I,\ \forall n \in \mathbb{N}^*$

c) $\sum_{n=1}^{\infty} \sin^3\frac{1}{n} = \sum_{n\geq 1} x_n$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^3 : \left(\frac{\sin\frac{1}{n+1}}{\frac{1}{n+1}}\right)^{-3}\cdot\frac{1}{n^3}\cdot\frac{(n+1)^3}{1} =$$

$= 1\cdot 1^3\cdot 1 = 1 \to$ Nu putem trage concluzia

$$\lim_{n\to\infty} \frac{x_n}{\frac{1}{n^3}} = \lim_{n\to\infty}\left(\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right)^3 = 1^3 = 1 \in (0,+\infty)$$

$$\Rightarrow \sum_{n\geq 1}^{\infty} x_n \sim \sum_{n\geq 1}\frac{1}{n^3} \Rightarrow S \to \text{convergentă}$$

d) $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1} = \sum_{n\geq 1}^{\infty} x_n$

$$D = \lim_{n\to\infty}\frac{x_n}{x_{n+1}} = \lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}\cdot\frac{(2n+2)!!}{(2n+1)!!}\cdot(2n+3) =$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+3)}{(2n+1)^2} = 1$$

$$R = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+10n+6-4n^2-4n-1}{(2n+1)^2}\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{6n-1}{(2n+1)^2}\right) = \frac{6}{4} = \frac{3}{2} > 1 \Rightarrow S \to \text{convergentă}$$

11) $\sum_{n=1}^{\infty} a^{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}},\ a>0 = \sum_{n=1}^{\infty} x_n$

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} a^{-\frac{1}{n+1}} \to a^{-0} = 1$

$R = \lim_{n\to\infty} n\left(a^{-\frac{1}{n+1}} - 1\right) = \lim_{n\to\infty} \frac{a^{-\frac{1}{n+1}}-1}{-\frac{1}{n+1}} \cdot \frac{-1}{n+1} \cdot n = \ln a\,(-1)$

I $R<1 \iff \ln(a) > -1 \iff a > e^{-1} \iff a \in \left(0; \frac{1}{e}\right) \to S$ diverg

II $R>1 \iff a \in \left(\frac{1}{e}; +\infty\right) \to S$ converg

III $R=1 \iff a = \frac{1}{e}$

$\cancel{B = \lim_{n\to\infty} \ln(n)\left(n\left(e^{\frac{1}{n+1}}-1\right)-1\right) = \lim_{n\to\infty} \ln(n)\left(\frac{e^{\frac{1}{n+1}}-1}{\frac{1}{n}}-1\right)}$

$\cancel{= \lim_{n\to\infty} \ln(n)\left(n \cdot e^{\frac{1}{n+1}} - n - 1\right) = \lim_{n\to\infty} n\ln(n)\left(e^{\frac{1}{n+1}} - 1 - \frac{1}{n}\right) =}$

$\cancel{= \lim \quad y+1 \quad e^{\ln(n)+e^{\frac{1}{n+1}}} - n - 1}$

$\cancel{B = \lim_{n\to\infty} \frac{n\left(e^{\frac{1}{n+1}}-1\right)-1}{(\ln(n))^{-1}} = \lim_{n\to\infty} \frac{e^{\frac{1}{n}}-1}{\frac{1}{n}} - 1}$

$\cancel{B = \lim_{n\to\infty}}$

$B = \lim_{n\to\infty} \ln(n) \cdot \left[n\left(e^{\frac{1}{n+1}}-1\right)-1\right]$

Fie $f:[0,+\infty)\to\mathbb{R}$, $f(x) = \ln(x)\cdot\left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right]$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{\frac{1}{x\left(e^{\frac{1}{x+1}}-1\right)-1}} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{l'H}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{-1}{\left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right]^2}\cdot\left[e^{\frac{1}{x+1}}-1+x\cdot e^{\frac{1}{x+1}}\cdot\frac{-1}{(x+1)^2}\right]}$

$=$

$$= \lim_{x\to\infty} \frac{\left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right]^2}{x\left[e^{\frac{1}{x+1}}-1+x\,\frac{e^{\frac{1}{x+1}}(-1)}{(x+1)^2}\right]} =$$

$$= \lim_{x\to\infty} \frac{(x+1)^2\left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right]^2}{x\left[(x+1)^2\left(e^{\frac{1}{x+1}}-1\right) - x e^{\frac{1}{x+1}}\right]} = \lim_{x\to\infty} \frac{\left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right]^2}{\left[(x+1)^2\left(e^{\frac{1}{x+1}}-1\right)-e^{\frac{1}{x+1}}\right]}$$

$$= \lim_{x\to\infty} \frac{\left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right]^2}{x e^{\frac{1}{x+1}} - x - 1} = (\ominus) \lim_{x\to\infty} \frac{\left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right]^2}{x\left(e^{\frac{1}{x+1}}-1\right)-1} =$$

Lưu ý $x_2 = 2 \Rightarrow B = 0 < 1 \Rightarrow$

$$= \lim_{x\to\infty} \left[x\left(e^{\frac{1}{x+1}}-1\right)-1\right] = 1-1=0.\ \cancel{L<1 \Rightarrow S \to \text{dâng}} \Rightarrow S \to \text{chặn}$$

SAI!

~~Đặt $y = e^{\frac{1}{x+1}}-1 \Leftrightarrow y+1 = e^{\frac{1}{x+1}} \Leftrightarrow \ln(y+1) = \frac{1}{x+1} \Rightarrow$~~

~~$\Rightarrow x+1 = \frac{1}{\ln(y+1)} \Leftrightarrow x = \frac{1}{\ln(y+1)} - 1 \quad (x\to\infty \Rightarrow y\to 0)$~~

~~$B = \lim_{\substack{y\to 0\\ y>0}} \ln\left(\frac{1}{\ln(y+1)}-1\right)\left[\left(\frac{1}{\ln(y+1)}-1\right)\cdot y - 1\right] =$~~

~~$= \lim_{\substack{y\to 0\\ y>0}} \ln\left(\frac{1}{\ln(y+1)}-1\right)\Big[$~~

$$\sum_{n=1}^{\infty} \frac{(an)^n}{n!}, \ a>0 \quad = \sum_{n=1}^{\infty} x_n$$

$$\frac{x_n}{x_{n+1}} = \frac{(an)^n}{n!} \cdot \frac{(n+1)!}{[a(n+1)]^{n+1}} = (n+1) \cdot \frac{1}{a} \cdot \frac{n^n}{(n+1)^{n+1}} =$$

$$= \frac{1}{a} \cdot \frac{n^n}{(n+1)^n} = \frac{1}{a} \cdot \left(1 - \frac{1}{n+1}\right)^n$$

$$D = \lim_{n \to \infty} \frac{x_n}{x_{n+1}} = \frac{1}{a} \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right)^{-\frac{n}{n+1} \cdot (-(n+1))} = \frac{1}{a} \cdot e^{\lim_{n \to \infty} -\frac{n}{n+1}} = \frac{1}{ea}$$

I $D < 1 \Rightarrow e^{-1} < a \Rightarrow a \in (e^{-1}, +\infty) \to S$ divergentă

II $D > 1 \Rightarrow a \in (0, e^{-1}) \to S$ convergentă

III $D = 1 \Rightarrow a = e^{-1} \Rightarrow \frac{x_n}{x_{n+1}} = e\left(1 - \frac{1}{n+1}\right)^n$

$$R = \lim_{n \to \infty} n\left[e\left(1 - \frac{1}{n+1}\right)^n - 1\right] =$$

$$\cancel{= \frac{1}{e} \lim_{n \to \infty} n\left[\left(1 - \frac{1}{n+1}\right)^n - e\right]}$$

$$= \lim_{n \to \infty} n\left[e^{-1} \cdot e^{n \ln\left(1 - \frac{1}{n+1}\right)} - 1\right] = \lim_{n \to \infty} n \cdot \left(e^{n \ln\left(1 - \frac{1}{n+1}\right) - 1} - 1\right) =$$

$$= \lim_{n \to \infty} n \cdot \frac{e^{n \ln\left(1 - \frac{1}{n+1}\right) - 1} - 1}{n \ln\left(1 - \frac{1}{n+1}\right) - 1} \cdot \left(n \ln\left(1 - \frac{1}{n+1}\right) - 1\right) =$$

$$= \lim_{n \to \infty} n\left[n \ln\left(1 - \frac{1}{n+1}\right) - 1\right]$$

Fie $f : (0, +\infty) \to \mathbb{R}$, $f(x) = x\left[e\left(1 - \frac{1}{x+1}\right)^x - 1\right]$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{e\left(1-\frac{1}{k+1}\right)^k - 1}{\frac{1}{k}} \overset{[\frac{0}{0}]}{\underset{l'H}{=}} \lim_{k\to\infty} =$$

$$= \lim_{k\to\infty} \frac{e^{k\ln\left(1-\frac{1}{k+1}\right)+1}-1}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k\ln\left(1-\frac{1}{k+1}\right)+1}{\frac{1}{k}} \overset{[\frac{0}{0}]}{\underset{l'H}{=}}$$

$$= \lim_{k\to\infty} \frac{\ln\left(1-\frac{1}{k+1}\right) + k\left(\frac{1}{k}-\frac{1}{k+1}\right)}{\frac{1}{k}} = 1 + \lim_{k\to\infty} \ln\left(1-\frac{1}{k+1}\right)\cdot k =$$

$\lim x_n = n$

$$= 1 + \lim_{k\to\infty} -\frac{k}{k+1} = 0 \Rightarrow R = 0 \Rightarrow S \to \text{divergent}$$

g) $\sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^{n^2} = \sum_{n=0}^{\infty}\left(1-\frac{1}{n+2}\right)^{n^2} = \sum_{n=0}^{\infty} x_n$ ~~f~~

$$C = \lim \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty}\left(1-\frac{1}{n+2}\right)^n = e^{\lim_{n\to\infty} -\frac{n}{n+2}} = \frac{1}{e} < 1$$

$\Rightarrow S \to$ convergent

h) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2} = \sum_{n=2}^{\infty} x_n$

~~$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\ln n}{n^2}\cdot\frac{(n+1)^2}{\ln(n+1)} = \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1+\ln(1+n^{-1})}$~~

~~$= 1 \to$ the ratio test inconclusive~~

~~$R = \lim_{n\to\infty} n\left(\frac{\ln n}{\ln(n+1)}\cdot\frac{(n+1)^2}{n^2} - 1\right) = \lim_{n\to\infty} \frac{\ln(n)\cdot(n+1)^2 - \ln(n+1)\cdot n^2}{n\ln(n+1)}$~~

~~$= \lim_{n\to\infty} \left(\frac{\ln(n)}{\ln(n+1)}\cdot\right)$~~

$$\frac{n}{2^n}\cdot\frac{2^{n+1}}{n+1} = \frac{2}{n+1} \Rightarrow 0$$

$$\sum_{n=2}^{\infty} \frac{\ln 2^n}{(2^n)^2}\cdot 2^n = \sum_{n=2}^{\infty} \frac{n\ln 2}{2^n}$$

$\frac{\ln n}{n^2} \le \frac{1}{n^{\frac{3}{2}}} \iff \ln n \cdot n^{\frac{3}{2}} \le n^{\frac{4}{2}} \iff$

$\iff 0 \le n^{\frac{3}{2}}(\sqrt{n} - \ln n) \Big|_{n \ge 2}$

$\iff \cancel{\sqrt{n} \le \ln n}$ $\ln n \le \sqrt{n} \iff \ln x - \sqrt{x} < 0$

Fie $f:(0,+\infty)$, $f(x) = \ln x - \sqrt{x}$

$f'(x) = \frac{1}{x} - \frac{1}{2}\cdot x^{-\frac{1}{2}} = \frac{1}{x} - \frac{1}{2\sqrt{x}} = \frac{2-\sqrt{x}}{2x}$

$f'(x) = 0 \Rightarrow x = 4.$

$\cancel{f'(9) = -\frac{1}{18} < 0 \Rightarrow f \text{ descresc}}$ $f'(9) = -\frac{1}{18} < 0$

$f'(9) = -\frac{1}{18} < 0 \Rightarrow f(x$ $f'(1) = \frac{1}{2} > 0$

x	0		4		
$f'(x)$	+	+ +	0	− − −	− −
$f(x)$		↗		↘	

$f(4) = 2\ln 2 - 2 = 2(\ln 2 - 1)(<0) < 0$

$\Rightarrow f(x) < 0, \forall x \in (0,+\infty)$

$\Rightarrow \ln n \le \sqrt{n}, \forall n \in \mathbb{N}, n \ge 2.$

$\Rightarrow \frac{\ln n}{n^2} \le \frac{1}{n^{\frac{3}{2}}}$

$\frac{1}{n^{\frac{3}{2}}}, \frac{\ln n}{n^2} \ge 0$; $\sum_{n=2}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converg

$\Rightarrow S \rightarrow$ converg

SAU? $\lim_{n\to\infty} \frac{x_n}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} (\ln n)\cdot \frac{1}{\sqrt{n}} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converg

$\Rightarrow \sum_{n=1}^{\infty} x_n$ converg

(2) a) $\sum_{n=1}^{\infty} (-1)^n \cdot n = \sum_{n=1}^{\infty} x_n$

$\left.\begin{array}{l} \lim_{n\to\infty} x_{2n} = \infty \\ \lim_{n\to\infty} x_{2n+1} = -\infty \end{array}\right| \Rightarrow \nexists \lim_{n\to\infty} x_n \Rightarrow$ Seria nu e convergentă $\Rightarrow$ Seria nu e absolut convergentă

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+\sqrt{2}} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n$

$a_{n+1} - a_n = \frac{\sqrt{n+1}}{n+1+\sqrt{2}} - \frac{\sqrt{n}}{n+\sqrt{2}} \le 0 \iff \sqrt{n+1}\,(n+\sqrt{2}) \le \sqrt{n}\,(n+1+\sqrt{2})$

$\iff (n+1)(n^2+2\sqrt{2}n+2) \le n(n^2+2n+1+2\sqrt{2}(n+1)+2) \iff$

$\iff \cancel{n^3} + n^2 + \cancel{2\sqrt{2}n^2} + \cancel{2\sqrt{2}n} + 2n + 2 \le \cancel{n^3} + 2n^2 + n + \cancel{2\sqrt{2}n^2} + \cancel{2\sqrt{2}n} + 2n \iff n^2 + 2 \le 2n^2 + n \iff$

$\iff 0 \le n^2 + n - 2$. (A, $\forall n \ge 1$)

$\Delta = 1 + 8 = 9$

$n_{1,2} = \frac{-1 \pm 3}{2} \Rightarrow n \in \{-2; 1\}$

$(a_n)_{n\in\mathbb{N}} \to$ descrescător $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+\sqrt{2}} \to$ convergentă.

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+\sqrt{2}} = \sum_{n=1}^{\infty} x_n$

$\cancel{x_n \ge \frac{1}{n} \iff n\sqrt{n} \ge n+\sqrt{2} \iff n^2 \cdot n \ge n^2 + 2\sqrt{2}n + 2 \iff}$

$x_n = \frac{\sqrt{n}}{n+\sqrt{2}} = \frac{1}{\sqrt{n} + \sqrt{\frac{2}{n}}}$

$\lim_{n\to\infty} \frac{x_n}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{2}} = 1 \in (0, +\infty) \Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+\sqrt{2}} \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \Rightarrow$

~~S este absolut convergentă~~ $\Rightarrow$ S nu e absolut convergentă

2) $\sum_{n=1}^{\infty} \sin\left(\pi\sqrt{n^2+1}\right) = \sum_{n=1}^{\infty} \sin\left(\pi n\sqrt{1+\frac{1}{n^2}}\right) = \sum_{n=1}^{\infty} x_n$

$$\cancel{\text{Fie } S_n = \sum_{n=1}^{\infty}\left(\sin\left(\pi n\sqrt{1+\frac{1}{n^2}}\right)\right)}$$

$$\cancel{= \sum_{n=1}^{\infty} \sin\left(\pi n\sqrt{1+\frac{1}{n^2}} - 2\pi n\right) = \sum_{n=1}^{\infty} \sin\left(\pi n\left(\sqrt{1+\frac{1}{n}}-1\right)\right)}$$

$$\cancel{\text{Fie } S_n = \sum_{n=1}^{\infty}\left(\sin\left(\pi\sqrt{n^2+1}\right)\right)}$$

$$\cancel{= \sum_{n=1}^{\infty} \sin\left[n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right)\right]}$$

$$x_n = \sin\left(\pi n\sqrt{1+\frac{1}{n^2}}\right) = \sin\left(\pi n\sqrt{1+\frac{1}{n^2}} - n\pi + n\pi\right) =$$

$$= \sin\left(\pi n\sqrt{1+\frac{1}{n^2}} - n\pi\right)\cdot\cos(n\pi) - \cos\left(\pi n\sqrt{1+\frac{1}{n^2}} - n\pi\right)\sin(n\pi) =$$

$$= (-1)^n \sin\left(n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right)\right)$$

$$\cancel{\lim_{n\to\infty} \frac{\sin\left(n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right)\right)}{n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right)} \cdot n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right) =}$$

$$\cancel{= \lim_{n\to\infty} n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right)}$$

$$l = \lim_{n\to\infty} \sin\left(n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right)\right) = 0$$

$$\lim_{n\to\infty} n\pi\cdot\left(\sqrt{1+\frac{1}{n^2}}-1\right) = \lim_{n\to\infty} n\pi\cdot\frac{1}{n^2}\cdot\frac{1}{2} = \lim_{n\to\infty}\frac{\pi}{2n} = 0.$$

$$\left(l = \lim_{n\to\infty} n\pi\left(\sqrt{1+\frac{1}{n^2}}-1\right) = 0.\right)$$

$x_n = (-1)^n \sin\left(\pi\left(\sqrt{n^2+1} - n\right)\right) = (-1)^n \cdot \sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right) = (-1)^n \cdot a_n$

$\frac{\pi}{\sqrt{n^2+1}+n}$ → descrescător și ... C_1 ⇒ a_n descrescător, a_n pozitiv

$$0 \le \frac{\pi}{\sqrt{n^2+1}+n} \le \frac{\pi}{2} \iff 0 \le \frac{1}{\sqrt{n^2+1}+n} \le \frac{1}{2} \iff$$ (A)

$\iff 2 \le \sqrt{n^2+1}+n$ (A), $\forall n \ge 1$.

Cum $\lim_{n\to\infty} a_n = 0$ și a_n descrescător ⇒ $\sum_{n=1}^{\infty} x_n$ → convergentă.

$$\sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} \sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right) = \sum_{n=1}^{\infty} y_n$$

$$\lim_{n\to\infty} \frac{y_n}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{\sqrt{n^2+1}+n}\right)}{\frac{\pi}{\sqrt{n^2+1}+n}} \cdot \frac{\pi}{\sqrt{n^2+1}+n} \cdot n = \frac{\pi}{1+1} = \frac{\pi}{2} \in (0, \infty)$$

⇒ $\sum_{n=1}^{\infty} |x_n| \sim \sum_{n=1}^{\infty} \frac{1}{n}$ ⇒ x_n nu este absolut convergentă

(3.) $x_n = \frac{3^n \cdot n!}{n^n} > 0, \forall n \in \mathbb{N}$, rezultă:

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{3^n \cdot n!}{n^n} \cdot \frac{(n+1)^{n+1}}{3^{n+1} \cdot (n+1)!} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} :$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1 \Rightarrow \lim_{n\to\infty} x_n = 0.$$

$$y_n = \ln(x_n) = \frac{n \ln 3 (\ln 1 + \ln 2 + \dots + \ln n)}{n \ln n} = \frac{\ln 3(\ln 1 + \ln 2 + \dots}{\ln n}$$

(8.)

(4) $S = \sum_{n=1}^{\infty} x_n \to$ convergentă, S.T.P.

i) $S_1 = \sum_{n=1}^{\infty} x_n^2$ nu convergentă

S convergentă $\Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 \mid x_n < \varepsilon$.

Luăm $\varepsilon = 1$ ~~...~~ $\Rightarrow$

$S_1 \sim \sum_{n=n_0}^{\infty} x_n^2$.

$x_n^2 \leq x_n \Leftrightarrow 0 \leq x_n(1-x_n)$.

$x_n \geq 0, \forall n \in \mathbb{N}^*$.

$x_n < 1, \forall n \geq n_0 \Rightarrow 0 < 1 - x_n, \forall n \geq n_0$ $\Big| \Rightarrow$

$\Rightarrow x_n^2 \leq x_n, \forall n \geq n_0$ $\Big|$ Crit. comparației $\Rightarrow \sum_{n=n_0}^{\infty} x_n^2$ convergentă $\Rightarrow$

$\sum_{n=n_0}^{\infty} x_n$ convergentă

$\Rightarrow$ S_1 convergentă

ii) Fie $\sum_{n=1}^{\infty} \sqrt{x_n}$ convergentă

$\sum_{n=1}^{\infty} \frac{1}{n^2} \to$ convergentă

$\sum_{n=1}^{\infty} \sqrt{\frac{1}{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n} \to$ divergentă

$\Big| \Rightarrow$ ii) (F)

(5) $(x_n)_{n\in\mathbb{N}} \to$ [illegible]

i) $\sum_{n=1}^{\infty} x_n$ convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ convergent

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent

$\sum_{n=1}^{\infty} \frac{1}{n^2 \cdot n}$

$\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ divergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ divergent

$F \to T \wedge T \to F$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.i. $\forall n \geq n_0 \; \frac{\sqrt{x_n}}{n} \geq \varepsilon \Rightarrow$

$\Rightarrow \frac{x_n}{n^2} \geq \varepsilon^2 \Leftrightarrow x_n \geq \varepsilon^2 n^2 \geq \varepsilon^2 = \varepsilon' \Rightarrow$

$\Rightarrow \forall \varepsilon' > 0, \exists n_0 \in \mathbb{N}$ a.i. $\forall n \geq n_0 \; x_n$

$\sum_{n=1}^{\infty} x_n$ convergent $\Rightarrow$

$\frac{\sqrt{x_n}}{n} \leq x_n \Leftrightarrow 0 \leq \sqrt{x_n}\left(\sqrt{x_n} - \frac{1}{n}\right), \forall n \geq n_0.$

$\sqrt{x_n} \geq 0, \forall n \in \mathbb{N}^*$

I $\sqrt{x_n} - \frac{1}{n} \overset{\text{De la } n_0}{\geq} 0 \Rightarrow \frac{\sqrt{x_n}}{n} \geq x_n,$

$\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n} \leq \left(\sum_{n=1}^{\infty} x_n \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \sqrt{\frac{x_n}{n^2}} \leq \sqrt{\sum_{n=1}^{\infty} x_n \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}}$ [illegible]

conv. conv.

ii) $\sum_{n=1}^{\infty} x_n$ divergent $\Rightarrow \sum_{n=1}^{\infty} x_n$

ii) $\sum_{n=1}^{\infty} x_n$ divergent $\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ divergent

$\sum_{n=1}^{\infty} \frac{1}{n}$ divergent

$\sum_{n=1}^{\infty} \frac{\frac{1}{\sqrt{n}}}{n} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} \to$ convergent

$\Rightarrow (F)$

a) $S=\sum_{n=1}^{\infty} x_n$ convergent $\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ convergent. ~~(⇐)~~

~~$(\Leftarrow)\ \sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ divergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ divergent~~

~~PRIN INDUCȚIE~~

~~DESTUL DE SATISFĂCĂT~~

$S_n = \sum_{k=1}^{n} \frac{\sqrt{x_n}}{n} = \sum_{k=1}^{n} \sqrt{\frac{x_n}{n^2}} = \sum_{k=1}^{n} \sqrt{x_n \cdot \frac{1}{n^2}} \le \sqrt{\sum_{k=1}^{n} x_n \cdot \sum_{k=1}^{n} \frac{1}{n^2}}$ (DIN *)

~~$a_k = \sqrt{x_k}$, $b_k = \sqrt{\frac{1}{n^2}}$~~

Șirurile x_n, $\frac{1}{n^2}$ sunt convergente $\Rightarrow \sum_{k=1}^{n} x_n$, și $\sum_{k=1}^{n} \frac{1}{n^2}$ mărginite $\Rightarrow$

$\Rightarrow \exists M_1, M_2 \in \mathbb{R}$ a.î. $0 < \sum_{k=1}^{n} x_n \le M_1$, $0 < \sum_{k=1}^{n} \frac{1}{n^2} \le M_2$, $\forall n \in \mathbb{N}^* \Rightarrow$

$\Rightarrow 0 < \sqrt{\sum_{k=1}^{n} x_n \cdot \sum_{k=1}^{n} \frac{1}{n^2}} \le \sqrt{M_1 \cdot M_2} \Rightarrow S_n \le \sqrt{M_1 \cdot M_2}$, $\forall n \in \mathbb{N}^* \Rightarrow S_n$ mărginit $\Rightarrow$

$\Rightarrow S$ convergent

$P(n)$: „~~$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \le \sum_{k=1}^{n} a_k$~~ $\sum_{k=1}^{n} a_k b_k \cdot \sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$”, $n \ge 1$

I Verificare: $P(1)$: „$a_1 b_1 \cdot a_1 b_1 \le a_1^2 b_1^2$” (A)

II Demonstrație: $P(n)$: „$\sum_{k=1}^{n} a_k b_k \cdot \sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$” (A)

$P(n+1)$: „$\sum_{k=1}^{n+1} a_k b_k \cdot \sum_{k=1}^{n+1} a_k b_k \le \sum_{k=1}^{n+1} a_k^2 \cdot \sum_{k=1}^{n+1} b_k^2$”

Din $P(n) \Rightarrow \sum_{\substack{1\le i\le n\\ 1\le j\le n}} a_i b_i a_j b_j \le \sum_{\substack{1\le i\le n\\ 1\le j\le n}} a_i^2 b_j^2 \Leftrightarrow$

$B(n+1) =$

$\Leftrightarrow \sum_{\substack{1\le i\le n\\ 1\le j\le n}} a_i b_i a_j b_j + \cancel{a_{n+1} b_{n+1}}) a_{n+1}^2 \sum_{k=1}^{n} b_k^2 + b_{n+1}^2 \sum_{k=1}^{n} a_k^2 + b_{n+1}^2 a_{n+1}^2 \le$

$\le \sum_{\substack{1\le i\le n+1\\ 1\le j\le n+1}} a_i^2 b_j^2 = A(n+1)$ | $\sum_{\substack{1\le i\le n+1\\ 1\le j\le n+1}} a_i b_i a_j b_j \le B(n+1) \Leftrightarrow$

$\Leftrightarrow a_{n+1} b_{n+1} \sum_{k=1}^{n} a_k b_k + a_{n+1} b_{n+1} \sum_{k=1}^{n} a_k b_k + \cancel{(a_{n+1} b_{n+1})^2} \le a_{n+1}^2 \sum_{k=1}^{n} b_k^2 + b_{n+1}^2 \sum_{k=1}^{n} a_k^2$

$+ \cancel{a_{n+1}^2 b_{n+1}^2} \Leftrightarrow \sum_{k=1}^{n} (a_{n+1}^2 b_k^2 - 2 a_{n+1} b_{n+1} a_k b_k + b_{n+1}^2 a_k^2) =$

$= \sum_{k=1}^{n} (a_{n+1} b_k - b_{n+1} a_k)^2 \ge 0$ (A) ~~(A)~~ $\Rightarrow$ ~~$P(n+1)$~~ (A) $\sum_{\substack{1\le i\le n+1\\ 1\le j\le n+1}} a_i b_i a_j b_j \le \sum_{\substack{1\le i\le n\\ 1\le j\le n}} a_i^2 b_j^2$

$\Rightarrow P(n+1)$ (A) | I, II $\Rightarrow P(n)$ (A), $\forall n \in \mathbb{N}^* \Rightarrow \sum_{k=1}^{n} a_k b_k \le \sqrt{\sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2}$

Aleg $a_k = \sqrt{x_k}$ și $b_k = \sqrt{\frac{1}{n^2}} \Rightarrow \sum_{k=1}^{n} \sqrt{\frac{x_k}{n^2}} \le \sqrt{\sum_{k=1}^{n} x_k \cdot \sum_{k=1}^{n} \frac{1}{n^2}}$ (*)