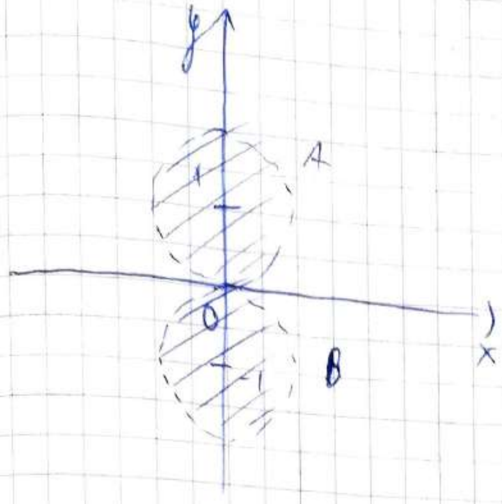


Exerciții suplimentare

1. Fie $x = (-1; 2; 3)$, $y(-2; 1; -3) \in \mathbb{R}^3$

a) $y \in B(x; r) = \{x_0 \in \mathbb{R}^3 / \|x - x_0\| < r\} \Rightarrow$

$\Rightarrow \|x - y\| < r \Leftrightarrow \sqrt{1+1+36} < r \Leftrightarrow r \in (0, \sqrt{38})$

b) $t \in \mathbb{R}$; $(1; -1; t) \in \overline{B}(x, 5) \Rightarrow$

$\Rightarrow \|x - (1, -1, t)\| \le 5 \Leftrightarrow \sqrt{4 + 9 + (3-t)^2} \le 5 \Leftrightarrow$

$\Leftrightarrow (3-t)^2 \le 12 \Leftrightarrow -2\sqrt{3} \le t-3 \le 2\sqrt{3} \Leftrightarrow$

$\Leftrightarrow t \in [3 - 2\sqrt{3}, 3 + 2\sqrt{3}]$

2. $x, y \in \mathbb{R}^n$ 3

a) $x \cdot y = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$

$\cancel{\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = \frac{1}{4}(x+y-x+y)}$

$\cancel{x \cdot y = \sqrt{(x \cdot y)(x \cdot y)}}$

$\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) =$

$= \frac{1}{4}((x+y)^2 - (x-y)^2) =$

$= \frac{1}{4}(x \cdot x + 2x \cdot y + y \cdot y - x \cdot x + 2x \cdot y - y \cdot y) =$

$= \frac{1}{4}(4 \cdot x \cdot y) = x \cdot y$, q.e.d.

l) $|\,\|x\| - \|y\|\,| \le \|x-y\| \iff -\|x-y\| \le \|x\| - \|y\| \le \|x-y\| \iff$

~~$|\sqrt{x \cdot x} - \sqrt{y \cdot y}| \le \sqrt{(x-y)\cdot(x-y)} \iff$~~

~~$\iff x \cdot x - 2\sqrt{x \cdot x \cdot y \cdot y} + y \cdot y \le x \cdot x - 2x \cdot y + y \cdot y \iff$~~

~~$\iff x \cdot y \le \sqrt{(x \cdot x)(y \cdot y)} \iff$~~

$$\iff \begin{cases} \|x-y\| \ge \|y\| - \|x\| \\ \|x\| - \|y\| \le \|x-y\| \end{cases} \iff \begin{cases} \|y\| - \|x\| \le \|y-x\| \\ \|x\| - \|y\| \le \|x-y\| \end{cases} \text{(A, cf. (*))} \Rightarrow$$

$\Rightarrow |\,\|x\| - \|y\|\,| \le \|x-y\|$

$\|y\| - \|x\| \le \|y-x\| \iff \|y\| \le \|y-x\| + \|x\| \iff$

$\iff \|(y-x)+x\| \le \|y-x\| + \|x\|$ (A), $\forall x,y \in \mathbb{R}^2 \Rightarrow$

$\Rightarrow \|y\| - \|x\| \le \|y-x\|, \forall x,y \in \mathbb{R}^2$ (*)

SAU

$|\,\|x\| - \|y\|\,| \le \|x-y\| \iff \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \le \|x-y\|^2 \iff$

$\iff x \cdot x - 2\sqrt{(x \cdot x)(y \cdot y)} + y \cdot y \le (x-y)(x-y) \iff$

$\iff -2\sqrt{(x \cdot x)(y \cdot y)} \le -2\,x \cdot y \iff x \cdot y \le \sqrt{(x \cdot x)(y \cdot y)} \iff$

$\iff x \cdot y \le \|x\|\|y\|$ (A, cf. (1)) $\Rightarrow |\,\|x\| - \|y\|\,| \le \|x-y\|$

$x \cdot y \le |x \cdot y| \le \|x\|\|y\|$ (1)

③ $x, y \in \mathbb{R}^m$

x, y s.n. ortogonali dacă $x \cdot y = 0$

$x \cdot y = 0 \iff \|x-y\|^2 = \|x\|^2 + \|y\|^2$

$\|x-y\|^2 = \sqrt{(x-y)\cdot(x-y)}^2 = (x-y)\cdot(x-y) = x\cdot x - 2(x\cdot y) +$

$+ y\cdot y = \sqrt{x\cdot x}^2 - 2(x\cdot y) + \sqrt{y\cdot y}^2 = \|x\|^2 - 2(x\cdot y) + \|y\|^2 \quad (*)$

„$\Rightarrow$"

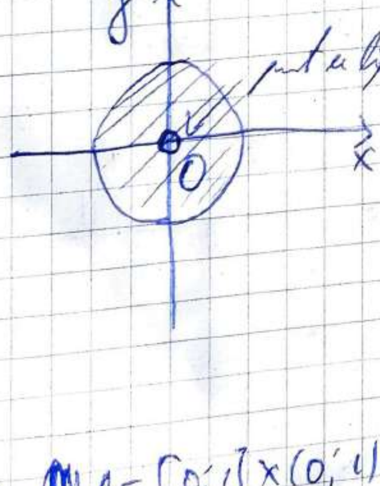
$x\cdot y = 0 \overset{(*)}{\Rightarrow} \|x-y\|^2 = \|x\|^2 - 2\cdot 0 + \|y\|^2 = \|x\|^2 + \|y\|^2$

„$\Leftarrow$"

$\|x-y\|^2 = \|x\|^2 + \|y\|^2 \overset{(*)}{\Rightarrow} -2(x\cdot y) = 0 \iff x\cdot y = 0.$

Q.E.D.

④ a) $A = \bar{B}(O_2, 1) \setminus \{O_2\} \subseteq \mathbb{R}^2$



$\text{int} A = B(O_2, 1) \setminus \{O_2\}$

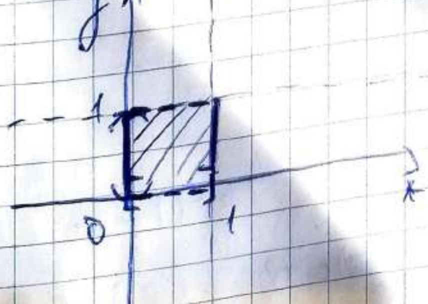
$\text{Fr} A = \{O_2\} \cup \{(x,y) \in \mathbb{R}^2 / x^2+y^2 = 1\}$

~~Fr A ∩ int A =~~

$(\text{Fr} A) \cap A = \{(x,y)\in\mathbb{R}^2 / x^2+y^2=1\} \neq \emptyset$
$\neq \text{Fr} A$ 3

A nu este nici închisă, nici deschisă

b) $A = [0;1] \times (0;1) \subseteq \mathbb{R}^2$



$\text{int} A = (0;1)\times(0;1)$

$\text{Fr} A = \{0;1\}\times[0;1] \cup [0;1]\times\{0;1\}$

A nu este nici deschisă, nici închisă

1) $A = \mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{R}^2$

$\text{int} A = \{ (x,y) \in \mathbb{R}^2 / \exists r_0 > 0 \ \ B((x,y), r_0) \subseteq A \}$

~~Ar trebui să arăt că dacă $x \in \text{int} A \Rightarrow x \in A$, dar nu ...~~

$x = (a; b) \in A$ a.î.

Fie $x \in \text{int} A \Rightarrow x \in A \Rightarrow x = (a; b)$, $a, b \in \mathbb{Q}$

$\cancel{B((x,y), r_0)}$ $B(x, r_0)$

$x \in \text{int} A \Rightarrow \exists r_0 > 0$ a.î. $B(x, r_0) \subseteq A$

$B(x, r_0) = \{ (a_0, b_0) \in \mathbb{R} \times \mathbb{R} / (a_0 - a)^2 + (b_0 - b)^2 < r_0^2 \}$

~~Fie $x_0 \in \text{int} A \Rightarrow x_0 \in A \Rightarrow x_0 \in \mathbb{Q} \times \mathbb{Q}$~~

~~Fie $x_0 = (a; b) \in \text{int} A \Rightarrow \exists r_0$ a.î. $B(x_0, r_0) \subseteq A$~~

Fie $(a; b) \in \text{int} A \Rightarrow \exists r_0 > 0$ a.î. $B((a; b), r_0) \cap (\mathbb{R}^2 \setminus A) = \emptyset$

$B((a; b), r_0) = \{ (x, y) \in \mathbb{R}^2 / \|(x,y) - (a,b)\| < r_0 \}$

$\|(x,y) - (a,b)\| < r_0 \Leftrightarrow (x-a)^2 + (y-b)^2 < r_0^2$

~~Aleg $z_0 = (x, b)$ din $B((a, b), r_0)$~~

Alegem $x = a$; $y = b + \frac{r_0}{2} \Rightarrow$

$\Rightarrow 0 + \frac{r_0^2}{4} < r_0^2 \Leftrightarrow$

~~Aleg $x = a$; $y = b$~~

Aleg $x = a \Rightarrow (y - b)^2 < r_0^2 \Leftrightarrow |y - b| < r_0 \Leftrightarrow$

$\Leftrightarrow -r_0 < y - b < r_0 \Leftrightarrow -r_0 + b < y < r_0 + b.$

Știu că în $(-r_0+b;\ b+r_0)$, există cel puțin un nr. irațional q. Aleg $y=q$

$\Rightarrow (a;q)\in B((a;b), r_0)$ și $q\notin \mathbb{Q} \Rightarrow (a;q)\notin \mathbb{Q}\times\mathbb{Q}$ $\Rightarrow B((a;b), r_0)\cap \mathbb{Q}^2\setminus A \neq \emptyset \Rightarrow$

$\Rightarrow B((a;b), r_0) \not\subset A,\ \forall r_0>0 \Rightarrow \text{int } A=\emptyset.$

$\text{fr } A = \mathbb{R}\times\mathbb{R}$

~~Fie $(a;b)\in \text{fr} A \Rightarrow \forall r>0,\ B((a;b), r)\cap A\neq\emptyset$ și $B((a;b), r)\cap(\mathbb{R}^2\setminus A)\neq\emptyset$~~

Fie $(a;b)\in \mathbb{R}^2$

Dem. să demonstrăm că $\forall r>0,\ B((a;b), r)\cap A\neq\emptyset.$

~~Fie $(x,y)\in\mathbb{R}^2$ a.î. $x\in(a;\ a+r)$~~

Fie $x\in(a-r;\ a+r)$

Fie $(x,y)\in\mathbb{R}^2$; Dem. să demonstrăm că $\forall r>0,\ B((x;y), r)\cap A\neq\emptyset.$

~~$B((x,y), r)=\{(a,b)\in\mathbb{R}^2/$~~

$B((x,y), r)\cap A=\{(a,b)\in\mathbb{Q}^2/\ (a-x)^2+(b-y)^2<r^2\}$

~~Fie $x\in(0;a)$~~

~~Iau $a_0\in(0;x)\cap\mathbb{Q}$ (știu că există unul)~~

~~$(b-y)^2<r^2-(a-a_0)^2 \Leftrightarrow |b-y|<\sqrt{r^2}$~~

~~(Iau $a_0\in(0;r)\cap A$)~~

~~Iau $x_0\in(x-r;\ a+r)\cap\mathbb{Q}$ (există)~~

~~$(a-a_0)^2<r^2 \Leftrightarrow 0<r^2-(a-a_0)^2$~~

~~$(b-y)^2<r^2-(a-a_0)^2$~~

$\overset{\Leftrightarrow}{\cancel{\ldots}}\ x_0 \in \mathbb{Q}$

Luăm $a = a_0 \in (x-r;\ x+r) \Rightarrow (a_0 - x)^2 < r^2$

$(a_0 - x)^2 + (b-y)^2 < r^2 \Leftrightarrow (b-y)^2 < r^2 - (a_0 - x)^2$ (aleg pătr. 2 parțial)

$$|b-y| < \sqrt{r^2 - (a_0 - x)^2} \Rightarrow b \in \overbrace{\left(y - \sqrt{r^2 - (a_0-x)^2},\ y + \sqrt{r^2 - (a_0-x)^2}\right)}^{V},$$

$(\cap \mathbb{Q}$, știu că mult.$)$

Luăm $b = b_0 \in V \cap \mathbb{Q}$ (există)

Deci $\exists (a_0, b_0) \in A$ a.î. $(a_0, b_0) \in B((x,y), r)$, $\forall r > 0$, $\forall (x,y) \in \mathbb{R}^2$

Analog $\Rightarrow \exists (a_0, b_0) \in \mathbb{R}^n \setminus A$ a.î. $(a_0, b_0) \in B((x,y), r)$, $\forall r > 0$, $\forall (x,y) \in \mathbb{R}^2$

$\Rightarrow \operatorname{Fr} A = \mathbb{R}^2$

A nu e nici închisă, nici deschisă

d) $A = \left\{ \left(1 + \frac{1}{n}\right)^2 / n \in \mathbb{N}^* \right\} \subseteq \mathbb{R}$

$\operatorname{int} A = \{ x \in \mathbb{R} / \exists r_0 > 0$ a.î. $B(x, r_0) \subseteq A \} =$

$B(x, r_0) = \{ y / |x-y| < r_0 \} = \{ y / x - r_0 < y < x + r_0 \} = (x - r_0, x + r_0)$

$\operatorname{int} A = \{ x \in \mathbb{R} / \exists r_0$ a.î. $(x - r_0;\ x + r_0) \subseteq A \}$

în $(x - r_0,\ x + r_0)$ sunt o infinit., iar A are doar elem. infinit $\Rightarrow \operatorname{int} A = \emptyset$

$\operatorname{int} A = \emptyset$

$\operatorname{Fr} A = \{ x \in \mathbb{R} / \forall r > 0,\ \overbrace{(x-r;\ x+r) \cap (\mathbb{R} \setminus A) \neq \emptyset}^{\text{mereu (A), știm că pe } (x-r;\ x+r) \text{ mereu există elemente care nu sunt din } A}$ și $(x-r;\ x+r) \cap A \neq \emptyset \} =$

$= \{ x \in \mathbb{R} / \forall r > 0,\ (x-r,\ x+r) \cap A \neq \emptyset \} =$

$= \{x \in \mathbb{R} \mid \forall r > 0, \exists a \in A \text{ a.î. } a \in (x-r; x+r)\} = A \cup \{e\}$

$\cancel{(\forall r > 0, x = (1 + \frac{1}{n})^n)}$

$\operatorname{Fr} A = A \cup \{e\}$

A nu are nici interior, nici punct aderent

(5) $\forall A \subseteq \mathbb{R}^n, A \neq \emptyset$

a) $A' \subseteq A \cup \operatorname{Fr} A \iff A' \cap (\mathbb{R}^n \setminus (A \cup \operatorname{Fr} A)) = \emptyset$

~~Fie $x \in A' \Rightarrow \forall r > 0, B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$~~

~~Fie $x \in A'$ și $x \in L$~~

Fie $x \in A' \cap (\mathbb{R}^n \setminus (A \cup \operatorname{Fr} A)) \iff x \in A'$ și $x \in \mathbb{R}^n \setminus (A \cup \operatorname{Fr} A) \iff$

$\iff x \in A'$ și $x \notin A$ și $x \notin \operatorname{Fr} A \iff$

$\iff (\forall r > 0, B(x, r) \cap (A \setminus \{x\}) \neq \emptyset)^{(1)}$ și $x \in \mathbb{R}^n \setminus A$ și $^{(2)}$

$(\exists r_0 > 0, (B(x, r) \cap A = \emptyset)^{(3)}$ sau $B(x, r_0) \cap (\mathbb{R}^n \setminus A) = \emptyset)^{(4)}$ (F)

~~$\iff \forall r > 0, B(x, r)$~~

Dacă $(3)(A) \Rightarrow (1)(F)$, deci $(1)(A)$ și (F)

~~Dacă $(2)(A) \Rightarrow (2)(F)$~~ Dacă $(2)(A) \Rightarrow (4)(F \to$ contradicție$)$ $\Big| \Rightarrow$

$\Rightarrow A' \subseteq A \cup \operatorname{Fr} A$

~~Fie $x \in A'$, I Dacă $x \in A$,~~ ✓

~~II Dacă $x \notin A \Rightarrow x \in$~~

b) $\mathrm{int} A \cap \mathrm{int}(\mathbb{R}^n \setminus A) = \emptyset$

Fie $x \in \mathrm{int} A \cap \mathrm{int}(\mathbb{R}^n \setminus A) \Rightarrow \cancel{(\forall r>0,\ B(x,r) \subseteq A}$

$\Rightarrow (\exists r_1 > 0$ a.î. $B(x, r_1) \subseteq A)$ și $(\exists r_2 > 0$ a.î. $B(x, r_2) \subseteq \mathbb{R}^n \setminus A)$;
$x \in B(x, r_1)$ $\quad$ $x \in B(x, r_2)$

$\Rightarrow x \in A$ și $x \in \mathbb{R}^n \setminus A$ (F) $\Rightarrow \mathrm{int} A \cap \mathrm{int}(\mathbb{R}^n \setminus A) = \emptyset$

c) $\mathrm{fr} A = \mathrm{fr}(\mathbb{R}^n \setminus A)$

Fie $x \in \mathrm{fr} A \Leftrightarrow \forall r > 0,\ B(x, r) \cap A \neq \emptyset$ și $B(x, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset \Leftrightarrow$
$\Leftrightarrow \forall r > 0,\ B(x, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$ și $B(x, r) \cap (\mathbb{R}^n \setminus (\mathbb{R}^n \setminus A))$
$\Leftrightarrow x \in \mathrm{fr}(\mathbb{R}^n \setminus A) \Rightarrow \mathrm{fr} A = \mathrm{fr}(\mathbb{R}^n \setminus A)$

d) $\mathrm{int} A = A \setminus \mathrm{fr} A$ ($\subseteq$)

Fie $x \in \mathrm{int} A \Rightarrow x \in A$.

Dacă $x \in \mathrm{fr} A \Rightarrow \forall r > 0,\ B(x, r) \cap A \neq \emptyset$ (A) și $B(x, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$

$\Rightarrow \forall r > 0\ B(x, r) \cap (\mathbb{R}^n \setminus A) \neq \emptyset$ (F) $\Rightarrow$
$B(x, r_0) \subseteq A \Rightarrow B(x, r_0) \cap (\mathbb{R}^n \setminus A) = \emptyset$

$\Rightarrow x \notin \mathrm{fr} A \Rightarrow x \in A \setminus \mathrm{fr}(A) \Rightarrow \mathrm{int} A \subseteq A \setminus \mathrm{fr} A$

Fie $x \in A \setminus \mathrm{fr} A \Rightarrow x \in A$ și $x \notin \mathrm{fr} A \Rightarrow x \in A$ și $\exists r_0$ a.î. $(B(x, r_0) \cap A = \emptyset$ (F)
sau $B(x, r_0) \cap (\mathbb{R}^n \setminus A) = \emptyset$ (V))

$\Leftrightarrow x \in A$ și $B(x, r_0) \cap (\mathbb{R}^n \setminus A) = \emptyset \Rightarrow x \in \mathrm{int} A \Rightarrow$
~~$\mathrm{fr} A \subseteq A \setminus \mathrm{fr} A) \Rightarrow \mathrm{int} A \subseteq A \setminus \mathrm{fr} A$~~ $A \setminus \mathrm{fr}(A) \subseteq \mathrm{int} A$

$\Rightarrow$ ~~$A = A \setminus \mathrm{fr}$~~ $\mathrm{int} A = A \setminus \mathrm{fr} A$

(6) $A \subseteq \mathbb{R}^n$, $A \neq \emptyset$

a) A este deschisă $\Leftrightarrow$ ~~$A \cap \partial A = \emptyset$~~ $A = \text{int} A$]

~~$A' \subseteq$~~ $A \cup \partial A$ $\Rightarrow \forall x \in A, \exists r_0 > 0$ a.î. $B(x, r_0) \subseteq A \Rightarrow$

$\Rightarrow \forall x \in A, \exists$

a) ~~$(A' \subseteq)$~~ A deschisă $\Rightarrow A \subseteq A' \Leftrightarrow A \cap (\mathbb{R}^n \setminus A') = \emptyset$.

A deschisă $\Leftrightarrow A = \text{int} A$

~~Fie $x \in A$~~ PRA) Fie $x \in A \cap (\mathbb{R}^n \setminus A') \Leftrightarrow x \in A$ și $x \in \mathbb{R}^n \setminus A' \Leftrightarrow$

$\Leftrightarrow x \in A$ și $x \notin A' \Leftrightarrow x \in \text{int} A$ și $x \notin A' \Leftrightarrow$

$\Leftrightarrow (\exists r_0 > 0$ a.î. $B(x, r_0) \subseteq A)$ și $(\exists r$ a.î. $B(x, r) \cap (A \setminus \{x\}) = \emptyset)$

~~(F),~~

$B(x, r_0) \cap B(x, r) \neq \emptyset$

Fie $a \in B(x, r_0) \cap B(x, r) \Rightarrow a \in A$ și $a \notin A \setminus \{x\} \Rightarrow a = x$ (F)

$\Rightarrow A \subseteq A'$

b) A închisă $\Rightarrow A' \subseteq A \Leftrightarrow A' \cap (\mathbb{R}^n \setminus A) = \emptyset$

A închisă $\Leftrightarrow \mathbb{R}^n \setminus A$ deschisă $\Leftrightarrow \mathbb{R}^n \setminus A = \text{int}(\mathbb{R}^n \setminus A)$

Fie $x \in A' \cap (\mathbb{R}^n \setminus A) \Leftrightarrow x \in A'$ și $x \in (\mathbb{R}^n \setminus A) \Leftrightarrow$

$\Leftrightarrow \forall r > 0, B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$ și $x \in \text{int}(\mathbb{R}^n \setminus A) \Leftrightarrow$

$\Leftrightarrow \forall r > 0, B(x, r) \cap (A \setminus \{x\}) \neq \emptyset$ și $\exists r_0 > 0$ a.î. $B(x, r_0) \subseteq \mathbb{R}^n \setminus A \Leftrightarrow$

$\Rightarrow B(x, r_0) \cap A \setminus \{x\} \neq \emptyset$ și $B(x, r_0) \subseteq \mathbb{R}^n \setminus A \Leftrightarrow$

$\Leftrightarrow B(x, r_0) \cap (A \setminus \{x\}) \neq \emptyset$ și $B(x, r_0) \cap A = \emptyset$

$\exists a \in B(x, r_0) \cap A \setminus \{x\} \Rightarrow a \in B(x, r_0) \cap A \neq \emptyset$ (F) $\Rightarrow x \in \emptyset$

$\Rightarrow A' \subseteq A$

(~~construct pentru a explica~~)

Fie A a.î. $A = A' \Rightarrow A \subseteq A'$ și $A' \subseteq A \Rightarrow$ A este deschisă, închisă și deci (?)

(A, deschisă $A = \mathbb{R}^n$ sau $A = \emptyset$)

$\Rightarrow$ Reciproca nu e adevărată

(7) Fie $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

Definim $\|x\|_1 \overset{not}{=} |x_1| + |x_2| + \ldots + |x_n|$ $\rightarrow$ (numită) NORMA MINKOWSKI a lui 1

~~Vom~~ <u>Vom verifica proprietățile normei euclidiene</u>:

$\forall x, y \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}$, au loc:

(1.) $\|x\| = 0 \Leftrightarrow x = 0_n$

„$\Rightarrow$"

$\|x\| = 0 \Leftrightarrow |x_1| + |x_2| + \ldots + |x_n| = 0 \Rightarrow |x_1| = |x_2| = \ldots = |x_n| = 0 \Rightarrow$

$\Rightarrow x_1 = x_2 = \ldots = x_n = 0 \Rightarrow x = 0_n$

pozitive

„$\Leftarrow$"

$x = 0_n \Rightarrow \|x\| = |0| + |0| + \ldots + |0| = 0$

(2.) $\|\alpha x\| = |\alpha| \, \|x\|$

$\|\alpha x\| = \|(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)\| = |\alpha x_1| + |\alpha x_2| + \ldots + |\alpha x_n| =$

$= |\alpha| (|x_1| + |x_2| + \ldots + |x_n|) = |\alpha| \, \|x\|$

(3.) $\|x + y\| \leq \|x\| + \|y\|$

$|x_i + y_i| \leq |x_i| + |y_i| \Rightarrow \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i| \Leftrightarrow$

$\Leftrightarrow \|(x + y)\| \leq \|x\| + \|y\|$

$\Rightarrow$ verifică proprietățile normei euclidiene.

Exerciții rezolvate

a) $\lim_{(x,y)\to(0,2)} \frac{\sin(xy)}{x} = \lim_{(x,y)\to(0,2)} \frac{\sin(xy)}{xy} \cdot \frac{xy}{x} = \lim_{(x,y)\to(0,2)} y = 2$

$\cancel{b)\ \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \cdot f \to \frac{x\cdot y}{\sqrt{1+\frac{y}{x}}}}$

$\cancel{\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{\sqrt{2}}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{2}n} = 0}$

$\cancel{\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^3}}{\sqrt{\frac{n^2+1}{n^4}}} = \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n^2}{\sqrt{n^2+1}}}$

$\cancel{\lim_{n\to\infty} f(}$

$\cancel{b)\ \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}; \lim_{n\to\infty} f\left(\frac{1}{n}, 0\right) = \lim_{n\to\infty} 0 = 0}$

$\cancel{\lim_{n\to\infty} f\left(\frac{1}{n}\right.}$

$\cancel{\ldots}$

b) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

$0 \le |f(x) - 0| = \sqrt{\frac{x^2y^2}{x^2+y^2}} \le \sqrt{\frac{x^2y^2}{y^2}} = \sqrt{x^2} \to 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x) = 0$

c) $\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2y^2} = \lim_{(x,y)\to(0,0)} \left(\frac{x}{y}\right)^2 - \left(\frac{y}{x}\right)^2$

$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{2}{n}\right) = \frac{1}{4} - 4 \ne \frac{1}{9} - 9 = \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{3}{n}\right) \Rightarrow \lim_{(x,y)\to(0,0)} f(x) \nexists$

d) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$

$0 \le |f(x) - 0| \le \frac{x^2|y|}{x^2} = |y| \to 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$

a) $l = \lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = \lim_{(x,y)\to(0,0)} e^{x^2y^2 \ln(x^2+y^2)}$ ~~$\lim_{(x,y)\to(0,0)} x^2y^2\ln(x^2+y^2) = e^0 = 1$~~

~~$0 \le |f(x,y) - 0| = x^2y^2 |\ln(x^2+y^2)|$~~

~~$\lim_{n\to\infty} f\left(\frac{1}{n}, 0\right) = \frac{1}{n^2}\cdot 0 \ln\left(\frac{1}{n^2}\right) = 0$~~

~~$\lim f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{1}{n^2}\cdot \ln($~~

~~$(0 \le |f(x,y) - 0| \le (x^2)^{x^2y^2} \le x^2)$~~

~~$\lim_{n\to\infty} f\left(0, \frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n^2}\right)^0 = 1 \; ; \; \lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = e^{\lim_{n\to\infty} \frac{1}{n^2}\cdot\ln\left(\frac{2}{n^2}\right)} = e^{-\infty} = 0$~~

$0 \le |x^2y^2\ln(x^2+y^2) - 0| \le |(x^2y^2 + x^4)\ln(x^2+y^2)| = |x^2|\,|(y^2+x^2)\ln(y^2+x^2)| \xrightarrow[(x,y)\to(0,0)]{} 0$

$\Rightarrow l = e^0 = 1$

f) $\lim_{(x,y)\to(\infty,\infty)} \frac{x+y}{x^2+xy+y^2}$; $0 \le |f(x,y) - 0| = \frac{x+y}{x^2+xy+y^2}$ ~~$\le \frac{x+y}{(x+y)^2 - xy}$~~ $= \frac{x+y}{(x+y)^2 - xy}$

~~$\frac{x+y}{x^2+xy+y^2} = \frac{x+y}{(x+y)^2-xy} \ge \frac{x+y}{y}$~~ $\le \frac{x+y}{y(x+y)} = \frac{1}{y} \to 0 \Rightarrow \lim_{(x,y)\to(\infty,\infty)} f(x,y) = 0$

g) $\lim_{(x,y)\to(1,1)} \frac{\ln(1+x^2) - \ln(1+y^2)}{x^2-y^2} =$ ~~$\lim_{(x,y)\to(0,0)} \frac{\ln(1+(x+1)^2) - \ln(1+(y+1)^2)}{(x+1)^2-(y+1)^2} =$~~

$= \lim_{(x,y)\to(1,1)} \frac{\ln\left(\frac{1+x^2}{1+y^2}\right)}{x^2-y^2} = \lim_{(x,y)\to(1,1)} \frac{\ln\left(1+\frac{x^2-y^2}{1+y^2}\right)}{\frac{x^2-y^2}{1+y^2}} \cdot \frac{\frac{x^2-y^2}{1+y^2}}{x^2-y^2} = \lim_{(x,y)\to(1,1)} \frac{1}{1+y^2} = \frac{1}{2}$

~~h) $\lim_{(x,y,z)\to 0_3} \frac{xyz}{x^2+y^2+z^2}$~~ ; ~~$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n^2}\right) = \frac{\frac{1}{n^4}}{\frac{1}{n^4}+\frac{2}{n^2}} = \frac{1}{1+2n^2} = 0$~~

~~$0 \le |f(x,y,z) - 0| \le \frac{\left(\frac{x+y+z}{3}\right)^3}{x^2+y^2+z^2}$~~ ~~$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right) = \frac{\frac{1}{n^3}}{\frac{3}{n^2}} = \frac{1}{3n} = 0$~~ ~~$\frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^4}+\frac{1}{n^8}} = \frac{2}{n^6}$~~

$0 \le |f(x,y,z) - 0| \le |xy| \xrightarrow{\lim} 0 \Rightarrow \lim_{(x,y,z)\to 0_3} f(x,y,z) = 0$

i) $\lim_{(x,y,z)\to 0_3} (xy+yz+zx)^2 \ln(x^2+y^2+z^2)$ | $0 \le |f(x,y,z) - 0| \le (x^2+y^2+z^2)^2 \ln(x^2+y^2+z^2) \xrightarrow{\lim} 0$

$\Rightarrow \lim_{(x,y,z)\to 0_3} f(x,y,z) = 0$

$p \le e^x$

$2p$

$= \xi + x$

$3p = p^3$

$\Rightarrow p = \sqrt{3}$

Luni 10 (ianuarie)

Exerciții rezolvate:

(2) a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases} \cancel{\frac{x^2+y^2}{x^2+y^2}, (x,y)\neq(0,0)} \ \frac{x^3+y^3}{x^2+y^2}, (x,y)\neq(0,0) \\ 0, (x,y)=(0,0)\end{cases}$

$0 \le \cancel{\lim_{(x,y)\to(0,0)}} |f(x,y)-0| = \frac{|x^3+y^3|}{x^2+y^2} \le \frac{|x|^3+|y|^3}{x^2+y^2} = \frac{|x|^3}{x^2+y^2} + \frac{|y|^3}{x^2+y^2} \le$

$\le \frac{|x|^3}{x^2} + \frac{|y|^3}{y^2} = \cancel{(x+y\to 0)} \ \left|\frac{x^3}{x^2}\right| + \left|\frac{y^3}{y^2}\right| =$

$= |x|+|y| \to 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) \Rightarrow f$ continuă în O_2

b) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=\begin{cases}(1+x^2y^2)^{\frac{1}{x^2+y^2}}, (x,y)\neq(0,0)\\ 1, (x,y)=(0,0)\end{cases}$

$\lim_{(x,y)\to(0,0)} (1+x^2y^2)^{\frac{1}{x^2+y^2}} = e^{\lim_{(x,y)\to(0,0)} \frac{\ln(1+x^2y^2)}{x^2y^2}\cdot\frac{x^2y^2}{x^2+y^2}} =$

$= e^{\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}}$ $\left(0 \le \left|\frac{x^2y^2}{x^2+y^2}-0\right| \le \frac{x^2(y^2+x^2)}{x^2+y^2} = x^2 \to 0\right)$ $= 1 = f(0,0) \Rightarrow$

$\Rightarrow f$ cont. în $\cancel{O}$ f și O_2

(1) $f:\mathbb{R}^3\to\mathbb{R}$, $\frac{xyz}{x^2+y^2+z^2} = f(x,y,z)$; $\lim_{(x,y,z)\to O_3} f(x,y,z)$

$0 \le |f(x,y,z)-0| = \frac{|xyz|}{x^2+y^2+z^2} = \frac{\sqrt{x^2y^2z^2}}{x^2+y^2+z^2} \le \frac{\sqrt{(x^2+y^2+z^2)^3}}{x^2+y^2+z^2} = \sqrt{x^2+y^2+z^2} \to 0 \Rightarrow$

$\Rightarrow \lim_{(x,y,z)\to O_3} f(x,y,z) = 0$

$\frac{|xy|}{x^2+y^2+1} \le \frac{1}{2}\left(\frac{x^2+y^2}{x^2+y^2+1}\right) = \frac{1}{2}\left(1-\frac{1}{x^2+y^2+1}\right)$

(3) a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = \frac{xy}{x^2+y^2+1}$

$m_G \le m_A \Leftrightarrow \sqrt{x^2 y^2} \le \frac{x^2+y^2}{2} \Leftrightarrow |xy| \le \frac{x^2+y^2}{2} \le \frac{x^2+y^2+1}{2} \Leftrightarrow \frac{|xy|}{x^2+y^2+1} \le \frac{1}{2} \Rightarrow$

$2xy \ge x^2+y^2+1 \Rightarrow$ $f(x,y) = \frac{1}{2} \Rightarrow$

$\Rightarrow f(x,y) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \to$ mărg.

$\cancel{\Rightarrow (x+y)^2 = -1}$

(3.) a) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \dfrac{xy}{x^2+y^2+1}$ $\left(\cancel{= \dfrac{1}{\frac{x}{y}+\frac{y}{x}+\frac{1}{xy}}}\right) = \lambda$, $\lambda \in \mathbb{R}$

$xy = \lambda x^2 + \lambda y^2 + \lambda \Leftrightarrow \lambda \cdot x^2 + (-y)x + (\lambda + \lambda y^2) = 0$

$\Delta = y^2 - 4\lambda(\lambda + \lambda y^2) \geq 0 \Leftrightarrow y^2 - 4\lambda^2 - 4\lambda^2 y^2 \geq 0 \Leftrightarrow$

$\Leftrightarrow (1-4\lambda^2)y^2 - 4\lambda^2 \geq 0$

I $1-4\lambda^2 > 0 \Leftrightarrow \frac{1}{4} > \lambda^2 \Rightarrow \lambda \in \left(-\frac{1}{2}; \frac{1}{2}\right)$, $\exists\, y$

II $1-4\lambda^2 = 0 \Leftrightarrow \lambda = \pm\frac{1}{2} \Rightarrow \Delta = -1 \leq 0$ (F)

III $1-4\lambda^2 < 0 \Leftrightarrow \lambda \in \left(-\infty; -\frac{1}{2}\right) \cup \left(\frac{1}{2}; +\infty\right)$

$y^2 \leq \dfrac{4\lambda^2}{1-4\lambda^2} < 0$ (F)

I, II, III $\Rightarrow \lambda \in \left(-\frac{1}{2}; \frac{1}{2}\right) \Rightarrow \operatorname{Im} f = \left(-\frac{1}{2}; \frac{1}{2}\right)$ $\pm\frac{1}{2}$

b) $f: \bar{B}(0_2, 1) \to \mathbb{R}$, $f(x,y) = x^2 + xy + y^2$, $x^2+y^2 \leq 1$

~~$0 \leq x^2+y^2 \leq 1$~~

~~$|x|, |y| < 1 \Rightarrow |xy| < 1 \Leftrightarrow -1 < xy < 1$~~

~~$\Rightarrow -1 \leq x^2+y^2 < 2$~~

$x^2 \leq 1-y^2 \Rightarrow |x| \leq \sqrt{1-y^2} \Leftrightarrow x \in \left[-\sqrt{1-y^2}; \sqrt{1-y^2}\right] \Rightarrow$

~~$\Rightarrow$~~

~~$x^2 \leq 1-y^2 \Rightarrow x \in \left[-\sqrt{1-y^2}; \sqrt{1-y^2}\right]$~~

$f(x,y)=0 \Rightarrow \cancel{x^2+y\cdot x+(y^2-a)=0}$

$\cancel{\Delta = y^2-4(y^2-a) = -3y^2+4a)}$

$f(x,y) = x^2+y^2+xy$

$f(x,y) \geq 0 \Leftrightarrow x^2+y\cdot x+y^2 \geq 0 \quad (R)$

$\Delta = y^2-4y^2 = -3y^2 \leq 0 \Rightarrow (R)\ (A) \Rightarrow$

$\Rightarrow f(x,y) \geq 0,\ \forall x,y \in \overline{B(0,1)}$

$f(0,0)=0$

Din T. Weierstrass $\Rightarrow$ f își atinge extremele (O funcție continuă pe domeniu compact)

$f(x,y) \leq 1+xy \leq 1+|xy| = 1+\sqrt{x^2\cdot y^2} \leq 1+\sqrt{x^2(1-x^2)} \rightarrow$

$\rightarrow 1+\sqrt{t(1-t)} \rightarrow t\in[0,1],\ g(t)=t-t^2 \rightarrow \max = \frac{-1}{-2} = \frac{1}{2}$

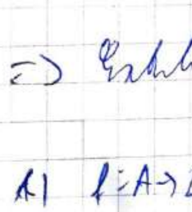
$\cancel{(g'(t) = -2t+1 = 0 \Rightarrow t=\frac{1}{2})}$

Deci, $f(x,y) \leq 1+\sqrt{\frac{1}{2}\cdot\frac{1}{2}} = 1+\frac{1}{2} = \frac{3}{2}$

$f\left(\frac{\sqrt{2}}{2};\frac{\sqrt{2}}{2}\right) = 1+\frac{2}{4} = \frac{3}{2}$

$\Rightarrow$ Extremele funcției sunt 0 și $\frac{3}{2}$, iar acestea sunt absolute.

4) $f: A\to\mathbb{R},\ f(x,y) = xy+\frac{1}{xy},\ A = \{(x,y)\in\mathbb{R}^2 / x>0, y>0, x+y\leq 2\}$



$xy, \frac{1}{xy} > 0 \Rightarrow \sqrt{xy\cdot\frac{1}{xy}} \leq \frac{f(x,y)}{2} \Rightarrow 2 \leq f(x,y)$

$f(x,y)=2 \Leftrightarrow t+\frac{1}{t} = 2 \Leftrightarrow t^2-2t+1=0 \Leftrightarrow t=1 \Leftrightarrow$

$\Leftrightarrow xy=1 \Rightarrow y=\frac{1}{x};\ x+\frac{1}{x} \leq 2\ (F)$.

1. 2 nu e valoare influențată prin ale ...

$f(x,y) = xy + \frac{1}{xy}$; $\lim_{(x,y)\to(0,0)} f(x,y) = 0 + \frac{1}{0_+ 0_+} = +\infty$

⇒ limita superioară a lui f este $+\infty$ pe care nu o atinge

2) $A = \overline{B}(O_3, 1) \setminus \{O_3\}$, $f: A \to \mathbb{R}$, $f(x,y,z) = \frac{x+y+z}{\sqrt{x^2+y^2+z^2}}$

a) $\lim_{n\to\infty} f\left(\frac{1}{n}, -\frac{1}{n}, 0\right) = 0$

$\lim_{n\to\infty} f\left(\frac{1}{n}, 0, 0\right) = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = 1$

⇒ f nu are limită în O_3 ⇒ ~~[illegible]~~

b) ~~A nu e compactă deoarece~~ $O_3 \in \text{Fr}A$, dar $O_3 \notin A$ ⇒ $\text{Fr}A \not\subset A$ ⇒

⇒ ~~[illegible]~~ A nu e compactă

c) $f(x,y,z) \le \frac{|x+y+z|}{\sqrt{x^2+y^2+z^2}} \le \frac{\sqrt{x^2}+\sqrt{y^2}+\sqrt{z^2}}{\sqrt{x^2+y^2+z^2}} \le \sqrt{x^2+y^2+z^2} \le 1$

~~$f(x,y,z) = 1 \Rightarrow (x+y+z)^2 = x^2+y^2+z^2 \Leftrightarrow 2(xy+xz+yz) = 0$~~

~~$\Leftrightarrow xy + yz + xz = 0$~~

$f(1,0,0) = \frac{1}{\sqrt{1}} = 1$ ⇒ f atinge sup

Analog ⇒ $f(x,y,z) \ge -\frac{|x+y+z|}{\sqrt{x^2+y^2+z^2}} \ge -1$; $f(-1,0,0) = -\frac{1}{1} = -1$

⇒ Există ... ± 1 ...

Exerciții suplimentare

① a) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\sin^2(x^3+y)$

$$\frac{\partial}{\partial x} f(x,y)=2\sin(x^3+y)\cos(x^3+y)\cdot 3x^2$$

$$\frac{\partial}{\partial y} f(x,y)=2\sin(x^3+y)\cos(x^3+y)$$

$$\nabla f(x,y)=\left(6x^2\sin(x^3+y)\cos(x^3+y),\ 2\sin(x^3+y)\cos(x^3+y)\right)$$

$$df(x,y)(u_1,u_2)=u_1\, 6x^2\sin(x^3+y)\cos(x^3+y)+u_2\cdot 2\sin(x^3+y)\cos(x^3+y)$$

b) $f:\mathbb{R}^3\to\mathbb{R},\ f(x,y,z)=(x+y+z)e^{x^2+y^2+z^2}$

$$\frac{\partial}{\partial x} f(x,y,z)=(1+2x^2+2xy+2xz)e^{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial y} f(x,y,z)=(1+2xy+2y^2+2yz)e^{x^2+y^2+z^2}$$

$$\frac{\partial}{\partial z} f(x,y,z)=(1+2xz+2yz+2z^2)e^{x^2+y^2+z^2}$$

$$\nabla f(x,y,z)\Big((1+2x^2+2xy+2xz)e^{x^2+y^2+z^2},\ (1+2xy+2y^2+2yz)e^{x^2+y^2+z^2},\ (1+2xz+2yz+2z^2)e^{x^2+y^2+z^2}\Big)$$

$$df(x,y,z)(u_1,u_2,u_3)=e^{x^2+y^2+z^2}\Big[u_1(1+2x^2+2xy+2xz)+u_2(1+2xy+2y^2+2yz)+u_3(1+2xz+2yz+2z^2)\Big]$$

② $J(f) \to (1,1)$

a) $J f(x,y) = \begin{pmatrix} 2x-y & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}$, $J(f)(1,1) = \begin{pmatrix} 1 & -1 \\ 3 & -2 \\ 2 & 4 \end{pmatrix}$

$f(x,y) = \left(\frac{1}{xy}, \operatorname{arctg} \frac{y}{x}\right)$

b) $J(f)(x,y) = \begin{pmatrix} \frac{1}{y}\cdot\frac{(-1)}{x^2} & \frac{1}{x}\cdot\frac{(-1)}{y^2} \\ \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\,\frac{(-1)}{x^2} & \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} \end{pmatrix}$

$J(f)(1,1) = \begin{pmatrix} -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Funcții omogene

Fie $p \in \mathbb{R}$. O funcție $f:(0,+\infty)^n \to \mathbb{R}$ se numește omogenă (de grad p)

dacă $f(tx) = t^p f(x)$, $\forall x \in (0,+\infty)^n$ și $\forall t > 0$

③ $f:(0,+\infty)^3$, $f(x,y,z) = \frac{1}{x+y+z}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

$f(t(x,y,z)) = f(tx,ty,tz) = \frac{1}{tx+ty+tz}\left(\frac{1}{tx}+\frac{1}{ty}+\frac{1}{tz}\right) =$

$= \frac{1}{t^2}\,\frac{1}{x+y+z}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = \frac{1}{t^2}\, f(x,y,z) = t^{-2} f(x,y,z) \Rightarrow$

$\Rightarrow f$ este omogenă de grad -2.

$$\cancel{\frac{\partial f}{\partial x}(x,y,z) = \left(\frac{-1}{(x+y+z)^2}\right)\cdot\left(\frac{-1}{x^2}\right) = \frac{1}{x^2(x+y+z)^2}}$$

$$\cancel{\frac{1}{x(x+y+z)^2} + \frac{1}{y(x+y+z)^2} + \frac{1}{z(x+y+z)^2} =}$$

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{-1}{(x+y+z)^2}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) + \frac{1}{x+y+z}\cdot\frac{(-1)}{x^2} =$$

$$\cancel{f\ \frac{1}{x+y+z}}\cdot\left(\frac{1}{x+y+z}\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$= \frac{-1}{x+y+z}\cdot f + \frac{(-1)}{x^2(x+y+z)}$$

Relația devine: $\frac{-x}{x+y+z}\cdot f + \frac{(-1)}{x(x+y+z)} + \frac{-y}{x+y+z} f + \frac{(-1)}{y(x+y+z)} +$

$$+\frac{(-z)}{x+y+z} f + \frac{(-1)}{z(x+y+z)} = -f + (-1)\frac{1}{x+y+z}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) =$$

$$= -2f$$

(4) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases}\frac{xy}{x+y}, & x+y\neq 0\\ 0, & x+y=0\end{cases}$

$$\lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\frac{1}{n^2}}{\frac{2}{n}} = \lim_{n\to\infty}\frac{1}{2n} = 0.$$

$$\lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n^2}-\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\frac{1}{n^3}-\frac{1}{n^2}}{\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{n^3}-\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1-n}{n} = -1$$

$\Rightarrow$ f nu este continuă în $(0,0)$

Ex: $v = (\alpha, \beta)$

$$\lim_{t\to 0} \frac{f(x+v\cdot t) - f(x)}{t} = \lim_{t\to 0} \frac{f(x+\alpha t, y+\beta t) - f(x,y)}{t} =$$

$$= \lim_{t\to 0} \frac{f(\alpha t, \beta t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{\alpha t \beta t}{\alpha t + \beta t}}{t} = \lim_{t\to 0} \frac{\alpha\beta}{\alpha+\beta} = \frac{\alpha\beta}{\alpha+\beta} \in \mathbb{R}$$

$\Rightarrow$ f derivabilă parțial după orice direcție în $(0,0)$

~~4) $g \circ f$; $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x)$~~

5) $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (x^2 - y,\ 3x - 2y,\ 2xy + y^2)$

$g = g(u,v,w): \mathbb{R}^3 \to \mathbb{R} \to$ clasă C^1

$g \circ f: \mathbb{R}^2 \to \mathbb{R}$

~~$\nabla_{g \circ f} =$~~

$$\nabla(g\circ f)(x,y) = \nabla g(f(x,y))\, J(f)(x,y)$$

$$\left(\frac{\partial g\circ f}{\partial x}(x,y),\ \frac{\partial g \circ f}{\partial y}(x,y)\right) = \left(\frac{\partial g}{\partial u}(f(x,y)),\ \frac{\partial g}{\partial v}(f(x,y)),\ \frac{\partial g}{\partial w}(f(x,y)\right.$$

$$\begin{pmatrix} 2x & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}$$

$$\frac{\partial g\circ f}{\partial x}(x,y) = \frac{\partial g}{\partial u}(f(x,y))\cdot 2x + \frac{\partial g}{\partial v}(f(x,y))\cdot 3 + \frac{\partial g}{\partial w}(f(x,y))\cdot 2y$$

$$\frac{\partial g\circ f}{\partial y}(x,y) = \frac{\partial g}{\partial u}(f(x,y))(-1) + \frac{\partial g}{\partial v}(f(x,y))(-2) + \frac{\partial g}{\partial w}(f(x,y))(2x+2y)$$

6. $v\frac{\partial g}{\partial u}(u,v) - u\frac{\partial g}{\partial v}(u,v) = 1, \ \forall u,v \in (0,+\infty)$

$(x,y)\in(0,+\infty)\times\left(0,\frac{\pi}{2}\right)$, $u = x\cos y$; $v = x\sin y$

Fie $f(x,y) = (x\cos y, x\sin y)$

$x\sin y\,\frac{\partial g}{\partial u}(f(x,y)) - x\cos y\,\frac{\partial g}{\partial v}(f(x,y)) = 1$

$$J(f)(x,y) = \begin{pmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{pmatrix}$$

~~$\frac{\partial g\circ f}{\partial u}(x,y) =$~~

~~$\frac{\partial g\circ f}{\partial x}(x,y) = \frac{\partial g}{\partial u}(f(x,y))\cdot\cos y +$~~

$\nabla(g\circ f)(x,y) = \nabla g(f(x,y))\cdot J(f)(x,y)$

$\frac{\partial g\circ f}{\partial y}(x,y) = \frac{\partial g}{\partial u}(f(x,y))\cdot(-x\sin y) + \frac{\partial g}{\partial v}(f(x,y))\cdot x\cos y =$

$\Rightarrow \boxed{\frac{\partial g\circ f}{\partial y}(x,y) = -1} \Rightarrow (g\circ f)(x,y) = -y \Rightarrow$

Exerciții rezolvate

$\Rightarrow g(f(x,y)) = -y \Rightarrow g(x\cos y, x\sin y) = -y$

$$\begin{cases} u = x\cos y \\ v = x\sin y \end{cases} \ominus$$

$\frac{v}{u} = \operatorname{tg} y \Rightarrow y = \operatorname{arctg}\frac{v}{u}$

$g(x\cos y, x\sin y) = -\operatorname{arctg}\frac{v}{u}$

---

(7) $f(x,y) = (x^2+y^2)\operatorname{arctg}\frac{y}{x} = (x^2+y^2)(-1)\operatorname{arctg}\frac{y}{x}$ ~~$= (-1)f(y,x)$~~

$\frac{\partial f}{\partial x}(x,y) = 2x\operatorname{arctg}\frac{y}{x} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\cdot\frac{(-1)}{x^2} =$

~~$\frac{\partial^2 f}{\partial x^2}(x,y) = 2\operatorname{arctg}\frac{y}{x} + 2x\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\cdot\frac{(-1)}{x^2}$~~

$= 2x\operatorname{arctg}\frac{y}{x} + \frac{(-1)y}{x^2+y^2}$

$\frac{\partial^2 f}{\partial x^2}(x,y) = 2\operatorname{arctg}\frac{y}{x} + 2x\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}$

2) $f(x,y) = (x^2+y^2)\,\mathrm{arctg}\,\frac{y}{x}$

$$\frac{\partial}{\partial x} f(x,y) = 2x\cdot \mathrm{arctg}\,\frac{y}{x} + (x^2+y^2)\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\cdot\frac{(-1)}{x^2} =$$

$$= 2x\,\mathrm{arctg}\,\frac{y}{x} + (x^2+y^2)\,\frac{-y}{x^2+y^2} = 2x\,\mathrm{arctg}\,\frac{y}{x} - y$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} f(x,y) = 2\,\mathrm{arctg}\,\frac{y}{x} + 2x\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\cdot\frac{(-1)}{x^2}$$

$$= \boxed{2\,\mathrm{arctg}\,\frac{y}{x} - \frac{2xy}{x^2+y^2}}$$

$$\frac{\partial}{\partial y} f(x,y) = 2y\,\mathrm{arctg}\,\frac{y}{x} + (x^2+y^2)\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} =$$

$$= 2y\,\mathrm{arctg}\,\frac{y}{x} + (x^2+y^2)\,\frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = 2y\,\mathrm{arctg}\,\frac{y}{x} + x$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = 2y\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot y\cdot\frac{(-1)}{x^2} + 1 = (-2y^2)\,\frac{1}{x^2+y^2} + 1 =$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2} = \boxed{\frac{x^2-y^2}{x^2+y^2}}$$

$$\frac{\partial^2}{\partial y^2} f(x,y) = 2\,\mathrm{arctg}\,\frac{y}{x} + 2y\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = 2\,\mathrm{arctg}\,\frac{y}{x} + 2\,\frac{y}{x}\cdot\frac{x^2}{x^2+y^2} =$$

$$= \boxed{2\,\mathrm{arctg}\,\frac{y}{x} + \frac{2xy}{x^2+y^2}}$$

$$x^2\left(2\,\mathrm{arctg}\,\frac{y}{x}-\frac{2xy}{x^2+y^2}\right)+2xy\cdot\frac{x^2-y^2}{x^2+y^2}+y^2\left(2\,\mathrm{arctg}\,\frac{y}{x}+\frac{2xy}{x^2+y^2}\right)=$$

$$=(x^2+y^2)\cdot 2\,\mathrm{arctg}\,\frac{y}{x}+\frac{x^2(-2xy)+2xy(x^2-y^2)+y^2\cdot 2xy}{x^2+y^2}=$$

$$=2f+0=2f,\ \text{q.e.d.}$$

$d_2 : (0;2) \to \mathbb{R},\ d_2(x) = f(x,4) = x^2 - 8x + 8$

$d_2'(x) = 2x - 8 = 0 \Rightarrow x \in \emptyset$

$d_3 : (0,4) \to \mathbb{R},\ d_3(y) = f(0,y) = 2y$

$d_3'(y) = 2 \neq 0 \Rightarrow y \in \emptyset$

$d_4 : (0;4) \to \mathbb{R},\ d_4(y) = f(2,y) = 4 - 4y + 2y = 4 - 2y$

~~$\Rightarrow y = 2 \Rightarrow (2,2) \Rightarrow f(2,2) = 1$~~

$d_4'(y) = -2 \neq 0 \Rightarrow y \in \emptyset$

$f(0;0) = 0;\ f(0;4) = 8;\ f(2;0) = 4;\ f(2;4) = 4 - 16 + 16 = 4$

$\Rightarrow (0;0) \to \min;\ (0;4) \to \max$

Extreme cu legături

(1) $a \in \mathbb{R};\ f : \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = 4x^2 + 4xy + ay^2$

$\nabla f = (8x + 4y;\ 4x + 2ay) = (0;0) \Leftrightarrow \begin{cases} y = -2x \\ 4x - 4xa = 0 \end{cases} \Leftrightarrow \begin{cases} y = -2x \\ 4x(1-a) = 0 \end{cases}$

I $1 - a = 0 \Rightarrow a = 1 \Rightarrow f(x,y) = (2x + y)^2 \geq 0,\ \forall (x,y) \in \mathbb{R}^2$

$f(x,y) = 0 \Leftrightarrow y = -2x$.

II $1 - a \neq 0 \Leftrightarrow a \in \mathbb{R} \setminus \{1\}$

$\begin{cases} y = 0 \\ x = 0 \end{cases} \to f(0;0) = 0$

~~$H(f)(x) = \begin{pmatrix} 8 & 4 \\ 4 & 2a \end{pmatrix};\ \Delta_1 = 8 > 0,\ \Delta_2 = 16a$~~

$f(x,y) = (2x + y)^2 + (a-1)y^2$ | Dacă $a - 1 > 0 \Rightarrow f(x,y) \geq 0\ \forall (x,y) \in \mathbb{R}^2$

Deci $(0;0)$ min

Deci $a-1<0 \Leftrightarrow a<1$

$f(0,y) = a\cdot y^2 \Rightarrow a<0$ [illegible] $\Rightarrow f(0,y)<0$

$f(x,0) = 4x^2>0$

$\rightarrow$ Deci nu [illegible]

~~$f(x,y) = 4x^2+4xy+ay^2.$~~

~~$f(x,-2y) = -2(a-1)y^2.$ Deci $a-1<0 \Rightarrow$~~

$f(-\frac{y}{2}, y) = (a-1)y^2<0 \rightarrow (0,0)$ nu [illegible]

R: $a\in[1,+\infty)$

② $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = (x^2-y)(x^2-3y) = x^4-4yx^2+3y^2$

$\nabla f = (4x^3-8yx, -4x^2+6y)$; $\nabla f(0,0) = (0,0) \Rightarrow$

$\Rightarrow a = (0,0)$ punct stationar.

~~$f(x,y)$~~ $f(x,0) = x^4 > 0 \rightarrow$ [illegible]

$f(x, \frac{2x^2}{3}) = \frac{2x^2}{3}\cdot(-x^2) = -\frac{4}{4}x^2<0.$

③ a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = x^3+y^3-3xy$

$\nabla f = (3x^2-3y, 3y^2-3x) = (0,0) \Rightarrow \begin{cases} y = x^2 \\ x = y^2 \end{cases} \Leftrightarrow \begin{cases} y(y^3-1) = 0 \\ x = y^2 \end{cases}$

$\Leftrightarrow \begin{cases} y(y-1)(y^2+y+1) = 0 \\ x = y^2 \end{cases}$

I $y = 0 \Rightarrow x = 0$, $y = 1 \Rightarrow x = \pm 1$

$(x,y)\in\{(0,0); (1;0); (-1;0)\}$

$$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}; \quad d^2 f(x,y) ?$$

$$d^2 f(x,y)u = 6x u_1^2 + 6y u_2^2 - 6u_1u_2$$

$d^2 f(0,0)(u) = 0$?

$d^2 f(-1,0)u = -6u_1^2 < 0, \forall u_1 \in \mathbb{R} \setminus \{0_2\} \Rightarrow (-1,0)$ punct de max

$d^2 f(1,0) = 6u_1^2 > 0 \Rightarrow (1,0)$ min

$$\left.\begin{array}{l} f(0,0) = 0 \\ f(x,0) > 0, \quad x > 0 \\ f(x,0) < 0, \quad x < 0 \end{array}\right| \Rightarrow (0,0) \text{ punct } şa$$

4) $f: \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = x^3 - x + y^2 + z^2$

$\nabla f = (3x^2 - 1, 2y, 2z) = 0_3 \Rightarrow a \in \left\{\left(\frac{1}{\sqrt{3}}; 0; 0\right); \left(-\frac{1}{\sqrt{3}}; 0; 0\right)\right\}$

$$H(f) = \begin{pmatrix} 6x & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}; \quad \text{I } a = \left(\frac{1}{\sqrt{3}}; 0; 0\right)$$

$\Delta_1 > 0; \Delta_2 > 0; \Delta_3 > 0 \Rightarrow$

$\Rightarrow a$ punct de minim

II $a = \left(-\frac{1}{\sqrt{3}}; 0; 0\right)$ $\Delta_1 < 0$; $\Delta_2 < 0$

$$d^2f\left(-\frac{1}{\sqrt{3}}; 0; 0\right)(u_1, u_2, u_3) = -2\sqrt{3}u_1^2 + 2u_2^2 + 2u_3^2 \quad ?$$

u_1	u_2	u_3
1	0	0
0	1	1

$\Rightarrow \left(-\frac{1}{\sqrt{3}}; 0; 0\right) \to$ punct şa

c) $f:(0;+\infty)\times\mathbb{R}\to\mathbb{R}$, $f(x,y)=x(y^2+\ln^2 x) = xy^2 + x\ln^2 x$

$\nabla f = \left(y^2+\ln^2 x + 2\ln x,\ \cancel{x(x\cdot 2y)}\ x2y\right)$

$\nabla f = 0 \Rightarrow \begin{cases} y^2+\ln^2 x+2\ln x=0 \\ x2y=0 \end{cases} \Leftrightarrow \begin{cases} \ln x(\ln x+2)=0 \\ y=0 \end{cases} \Rightarrow$

$\Rightarrow c \in \{(1;0); (e^{2};0)\}$

$H(f) = \begin{pmatrix} 2\frac{\ln x}{x} + \frac{2}{x} & 2y \\ 2y & 2x \end{pmatrix}$; $H(f)(1;0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\Delta_1 = 2>0$; $\Delta_2 = 4>0 \Rightarrow (1;0)$ minim

$H(f)(e^2;0) = \begin{pmatrix} \frac{4}{e^2} + \frac{2}{e^2} & 0 \\ 0 & 2e^2 \end{pmatrix}$ $\Delta_1>0$; $\Delta_2>0 \Rightarrow (e^2;0)$ - minim

d) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y,z) = z^2(1+xy)+xy = z^2+xyz^2+xy$

$\nabla f = \left(yz^2+y,\ xz^2+x,\ \cancel{z(1+xy)}\ 2z(1+xy)\right) = 0_3 \Rightarrow$

$= 2z+2xyz$

$\Rightarrow \begin{cases} y(z^2+1)=0 \\ x(z^2+1)=0 \\ 2z(1+xy)=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \Rightarrow c=(0;0;0)$

$$H(f) = \begin{pmatrix} 0 & z^2+1 & 2yz \\ z^2+1 & 0 & 2xz \\ 2yz & 2xz & 2+2xy \end{pmatrix}$$ ← Dozrale. [illegible]

$~~H(f)(0) = O_{3\times 3}~~$

$d^2f(x,y,z)u = (2+2xy)u_3^2 + 2(z^2+1)u_1u_2 + 4yz\,u_1u_3 + 4xz\,u_2u_3$

$d^2f(0,0,0)u = 2u_3^2 + 2u_1u_2 \rightarrow$ punkt [illegible] $(0,0,0)$

$u)\ f:(0,+\infty)^2 \to \mathbb{R},\ f(x,y) = xy + \frac{8}{x} + \frac{8}{y}$

$Df = \left(y - \frac{8}{x^2};\ x - \frac{8}{y^2}\right) = O_2 \Rightarrow \begin{cases} y = \frac{8}{x^2} \\ x = \frac{8}{\left(\frac{8}{x^2}\right)^2} = 8\cdot\left(\frac{x^2}{8}\right)^2 = \frac{x^4}{8} \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} y = \frac{8}{x} \\ x(x-1)=0 \end{cases} \Leftrightarrow \begin{cases} y=8 \\ x=1 \end{cases} \Leftrightarrow \begin{cases} y = \frac{8}{x^2} \\ x(x^3-8)=0 \end{cases} \Leftrightarrow \begin{cases} y=2 \\ x=2 \end{cases} \Rightarrow a=(2;2)$

$H(f) = \begin{pmatrix} \frac{16}{x^3} & 1 \\ 1 & \frac{16}{y^3} \end{pmatrix};\ H(f)(2;2) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix};\ \begin{matrix} \Delta_1 = 2 > 0 \\ \Delta_2 = 3 > 0 \end{matrix} \Rightarrow$

$\Rightarrow (2;2) \rightarrow$ minim

Fie $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (1+e^x)\cos y - xe^x = \cos y + e^x \cos y - xe^x$

$$Df = \left(e^x \cos y + (-e^x) + (-x)e^x,\ (1+e^x)(-\sin y)\right) \Rightarrow$$

$$\Rightarrow \begin{cases} \cos y - 1 - x = 0 \\ \sin y = 0 \end{cases} \Leftrightarrow \begin{cases} (-1)^k - 1 - x = 0 \\ y = k\pi \end{cases} \Leftrightarrow \begin{cases} x = (-1)^k - 1 \\ y = k\pi \end{cases}$$

$$H(f) = \begin{pmatrix} e^x \cos y + (-e^x) + (-e^x) + (-x)e^x & -e^x \sin y \\ e^x(-\sin y) & (1+e^x)(-\cos y) \end{pmatrix}$$

$$H(f)(c) = \begin{pmatrix} e^{(-1)^k - 1}\left((-1)^k - 2 - (-1)^k + 1\right) & 0 \\ 0 & (1+e^{(-1)^k - 1})(-1)^{k+1} \end{pmatrix} = \begin{pmatrix} -e^{(-1)^k - 1} & 0 \\ 0 & (1+e^{(-1)^k - 1})(-1)^{k+1} \end{pmatrix}$$

$\Delta_1 < 0$

$\cancel{\Delta_2 = (-e^{(-1)^k - 1})(1+e^{(-1)^k})(-1)^{k+1} \to \dots (-1)^k}$

$\cancel{d^2 f(c)h = }$

I $k \in 2\mathbb{Z}$

$$H(f)(c) = \begin{pmatrix} -1 & 0 \\ 0 & -1-e \end{pmatrix};\quad d^2 f(c)h = -h_1^2 + (-1)(1+e)h_2^2 < 0$$
$\to$ max

II $k \in 2\mathbb{Z}+1$

$$H(f)(c) = \begin{pmatrix} -e^{-2} & 0 \\ 0 & 2 \end{pmatrix};\quad d^2 f(c)h = -\frac{h_1^2}{e^2} + 2h_2^2 \ \substack{<0 \\ >0}$$

$\Rightarrow$ f are punct şa

a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x+y$, $S=\{(x,y)\in\mathbb{R}^2 \mid x^2+xy+y^2=1\}$.

Fie $F:\mathbb{R}^2\to\mathbb{R}$, $F(x,y)=x^2+xy+y^2-1$

$L:\mathbb{R}^3\to\mathbb{R}$, $L(x,y,\lambda)=f(x,y)+\lambda F(x,y)=$

$= x+y+\lambda(x^2+xy+y^2-1)=x+y+\lambda x^2+\lambda xy+\lambda y^2-\lambda$

$\nabla L=\left(1+2\lambda x+\lambda y \quad 1+2\lambda y+\lambda x,\ x^2+xy+y^2-1\right)=(0,0,0)$

$\Rightarrow \begin{cases} 1+2\lambda x+\lambda y=0 \\ 1+2\lambda y+\lambda x=0 \\ x^2+xy+y^2-1=0 \end{cases} \Leftrightarrow \begin{cases} \lambda y=-2\lambda x-1 \\ 1-4\lambda x-2+\lambda x=0 \\ x^2+xy+\cancel{\ }+y^2-1=0 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \lambda y=-\frac{1}{3} \\ \lambda x=-\frac{1}{3} \\ x^2+xy+y^2-1=0 \end{cases} \Leftrightarrow \begin{cases} x=-\frac{1}{3\lambda} \\ y=-\frac{1}{3\lambda} \\ \cancel{(x^2+xx)} \end{cases} \Leftrightarrow \begin{cases} \lambda=-\frac{1}{3x} \\ y=x \\ x^2=\frac{1}{3} \end{cases} \Rightarrow$

~~$(x,y)\in\left\{\left(-\frac{1}{\sqrt{3}};-\frac{1}{\sqrt{3}}\right);\left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right)\right\}$~~

~~$(x,y,\lambda)\in\left\{\left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}};-\frac{1}{\sqrt{3}}\right);\left(-\frac{1}{\sqrt{3}};-\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right)\right\}$~~

$\left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right)\to$ max; $\left(-\frac{1}{\sqrt{3}};-\frac{1}{\sqrt{3}}\right)\to$ min

2) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xyz$, $S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0, x^2+y^2+z^2=1\}$

$F_1(x,y,z) = x+y+z$

$F_2(x,y,z) = x^2+y^2+z^2-1$

$L: \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}$, $f(x,y,z,\lambda,\beta) = xyz + \lambda(x+y+z) + \beta(x^2+y^2+z^2-1)$

$\nabla L = (yz+\lambda+2\beta x,\ xz+\lambda+2\beta y,\ xy+\lambda+2\beta z,\ x+y+z,\ x^2+y^2+z^2-1)$

$$\nabla L = 0_5 \Rightarrow \begin{cases} yz+\lambda+2\beta x = 0 \\ xz+\lambda+2\beta y = 0 \\ xy+\lambda+2\beta z = 0 \\ x+y+z=0 \\ x^2+y^2+z^2-1=0 \end{cases} \Leftrightarrow \begin{cases} xy+yz+xz+3\lambda = 0 \\ \cancel{-x^2+\lambda-2\beta x=0} \end{cases}$$

$$\Leftrightarrow \begin{cases} -\frac{1}{2}+3\lambda = 0 \\ xz+\lambda+2\beta y=0 \\ xy+\lambda+2\beta z=0 \\ xy+yz+xz=\cancel{(0-1)}-\frac{1}{2} \\ x^2+y^2+z^2-1=0 \\ x+y+z=0 \end{cases} \Leftrightarrow \begin{cases} \lambda = \frac{1}{6} \end{cases}$$

$$\Leftrightarrow \begin{cases} 6xz+1+12\beta y=0 \\ 6xy+1+12\beta z=0 \\ x^2+y^2+z^2-1=0 \\ x+y+z=0 \end{cases} \Leftrightarrow \begin{cases} 6x(z-y)+12\beta(y-z)=0 \\ 6xy+1+12\beta z=0 \\ x^2+y^2+z^2-1=0 \\ x+y+z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (x+2\beta)(y-z)=0 \\ 6xy+1+2\beta z=0 \\ x^2+y^2+z^2-1=0 \\ x+y+z=0 \end{cases}$$

I $x+2\beta=0 \Rightarrow \cancel{\beta}\ 2\beta=-x \Leftrightarrow \beta=-\frac{x}{2}$

$$\begin{cases} 6xy-xz+1=0 \\ x^2+y^2+z^2-1=0 \\ x+y+z=0 \end{cases} \Leftrightarrow \begin{cases} (y+z)(z-6y)+1=0 \\ x^2+y^2+z^2-1=0 \\ x=-y-z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} yz-6y^2+z^2-6yz=-1 \\ x^2+y^2+z^2-1=0 \\ x=-y-z \end{cases} \Leftrightarrow \begin{cases} z^2-5yz-6y^2=-1 \end{cases}$$

II $y-z=0 \Rightarrow y=z$

$$\begin{cases} 6xy+1+2\beta y=0 \\ x^2+2y^2-1=0 \\ x+2y=0 \end{cases} \Leftrightarrow \begin{cases} -12y^2+1+2\beta y=0 \\ y^2=\frac{1}{6} \\ x=-2y \end{cases} \Leftrightarrow \begin{cases} -1+2\beta y=0 \\ y=\pm\frac{1}{\sqrt{6}} \\ x=\mp\frac{2}{\sqrt{6}} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \beta=\frac{1}{2y}=\pm\frac{\sqrt{6}}{2} \\ y=\pm\frac{1}{\sqrt{6}} \\ x=\mp\frac{2}{\sqrt{6}} \end{cases}$$

$\underline{II}$ $\lambda+2\beta=0 \Leftrightarrow \beta=-\frac{\lambda}{2}$

$$\begin{cases} 6xy+1-xz=0 \\ x^2+y^2+z^2-1=0 \\ x+y+z=0 \end{cases} \Leftrightarrow \begin{cases} (-y-z)(6y-z)+1=0 \\ y^2+z^2+2zy+y^2+z^2-1=0 \Leftrightarrow \\ x=-y-z \end{cases}$$

$$\Leftrightarrow \begin{cases} -6y^2+zy-6zy+z^2+1=0 \\ 2(y^2+zy+z^2)-1=0 \\ \cancel{x+y}\ x=-y-z \end{cases} \Leftrightarrow \begin{cases} z^2-5zy-6y^2+1=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} z^2-5zy-6y^2+1=0 \\ 2y^2+2zy+2z^2-1=0 \\ x=-y-z \end{cases} \Leftrightarrow \begin{cases} 7z^2+zy-1=0 \Leftrightarrow y=\frac{1-7z^2}{z} \end{cases}$$

$xy+\lambda(x^2+2y^2-2)$

; etc

(5) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=xy$, $S=\{(x,y)\in\mathbb{R}^2 \mid x^2+2y^2\le 2\}$

$S_1=\{(x,y)\in\mathbb{R}^2 \mid x^2+2y^2<2\}$ → mulțime deschisă

~~$f'(x)$~~

$\nabla f(x,y)=(y,x)=(0,0) \Rightarrow x=y=0$

$$J(1)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\quad d^2 L(x,y)\,h=2h_1h_2<0 \text{ și } >0$$

~~$(f:\mathbb{R}^2\to\mathbb{R})$~~ Fie $F:\mathbb{R}^2\to\mathbb{R}$, $F(x,y)=x^2+2y^2-2$

Fie $L:\mathbb{R}^3\to\mathbb{R}$, $L(x,y,\lambda)=f(x,y)+\lambda F(x,y)$

$\nabla L(x,y,\lambda)=(y+2\lambda x,\ x+2\lambda y,\ x^2+2y^2-2)=0 \Rightarrow$

$$\Rightarrow \begin{cases} y = -2t x \\ x = -2t y \\ x^2 + 2y^2 - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 4t^2 y \\ x = 4t^2 x \\ x^2 + 2y^2 - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} y(1-4t^2) = 0 \\ x(1-4t^2) = 0 \\ x^2 + 2y^2 - 2 = 0 \end{cases}$$

I $y = 0$ ⟵ Ind. $\bar{\text{II}}$; II $1 - 4t^2 = 0 \Leftrightarrow t = \pm\frac{1}{2}$

$$\begin{cases} y = 0 \\ t = \pm\frac{1}{\sqrt{2}} \\ x = \pm\sqrt{2} \end{cases}$$

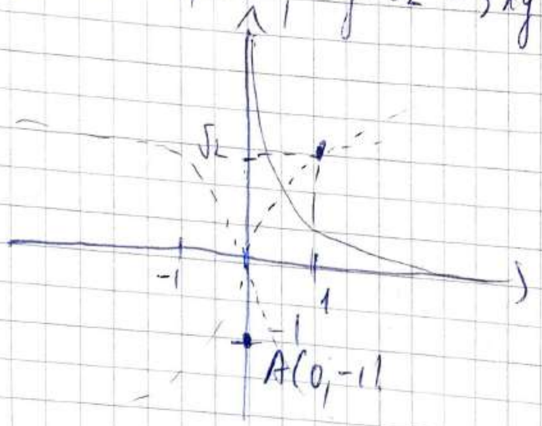
$$x^2 + 2y^2 - 2 = 0 \Rightarrow x^2 = 2(1-y^2);$$

$$\Rightarrow x = \pm\sqrt{2(1-y^2)},\ y \in [-1;1].$$

$$f(\pm\sqrt{2(1-y^2)}, y) = y\sqrt{2(1-y^2)} = \sqrt{2}\cdot y \cdot \sqrt{1-y^2}$$

$$\sqrt{2} \sin x \cos x = \sqrt{2} \sin 2x \cdot \frac{1}{2} = \frac{\sqrt{2}}{2} \sin 2x \rightarrow -\frac{\sqrt{2}}{2} \min;\ \frac{\sqrt{2}}{2} \max$$

⑥ $a = (0; -1)$; $xy = \sqrt{2} \Rightarrow xy - \sqrt{2} = 0,\ x>0, y>0$



$$d(A, B) = \sqrt{(0 - x_B)^2 + (-1 - y_B)^2} = \sqrt{x_B^2 + y_B^2 + 2y_B + 1}$$

$f: \mathbb{R}^2 \to \mathbb{R},\ f(x,y) = d(A, (x,y)) = \sqrt{x^2 + y^2 + 2y + 1} = \sqrt{x^2 + (y+1)^2}$
$(0, +\infty)^2$

$F: \mathbb{R}^2 \to \mathbb{R},\ F(x,y) = xy - \sqrt{2}$

$L: \mathbb{R}^3 \to \mathbb{R}\quad L(x, y, t) = \sqrt{x^2 + (y+1)^2} + t\,xy - \sqrt{2}\,t.$

$DL(x, y, \lambda) = \Big(\frac{1}{2\sqrt{x^2+(y+1)^2}} \cdot (2x) + \lambda y,\ \frac{1}{2\sqrt{x^2+(y+1)^2}} \cdot 2(y+1) + \lambda x,$

$, xy - \sqrt{2} \Big) = (0, 0, 0) \Rightarrow$

$$\Rightarrow \begin{cases} \frac{x}{\sqrt{x^2+(y+1)^2}} + \lambda y = 0 \\ \frac{y+1}{\sqrt{x^2+(y+1)^2}} + \lambda x = 0 \\ xy - \sqrt{2} = 0 \end{cases} \Leftrightarrow \begin{cases} x + \lambda y\sqrt{x^2+(y+1)^2} = 0 \\ (y+1) + \lambda x\sqrt{x^2+(y+1)^2} = 0 \\ x = \frac{\sqrt{2}}{y} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \lambda = -\frac{y}{x}\sqrt{x^2+(y+1)^2} \\ y+1 - y\sqrt{x^2+(y+1)^2} = 0 \\ x = \frac{\sqrt{2}}{y} \end{cases} \Leftrightarrow \begin{cases} \lambda = -\frac{y}{x}\sqrt{x^2+(y+1)^2} \\ (y+1)^2 = y^2(x^2+(y+1)^2) \\ x = \frac{\sqrt{2}}{y} \end{cases} \Leftrightarrow$$

$\Leftrightarrow \begin{cases} (y+1)^2 = \end{cases}$

$(y+1)^2 = y^2\left(\frac{2}{y^2} + (y+1)^2\right) \Leftrightarrow$

$\Leftrightarrow (y+1)^2 = 2 + y^2(y+1)^2 \Leftrightarrow$

$\Leftrightarrow$

$\Leftrightarrow y^2 + 2y + 1 = 2 + y^4 + 2y^3 + y^2 \Leftrightarrow$

$\Leftrightarrow y^4 + 2y^3 - 2y + 1 = 0$

$$\Leftrightarrow \begin{cases} 2 = f\left(\frac{y}{x}\sqrt{x^2+(y+1)^2}\right) - \frac{x}{y}\cdot\frac{1}{\sqrt{x^2+(y+1)^2}} \\ y+1\quad x - \frac{x^2}{y} = 0 \\ \cancel{(xy-\sqrt{2}=0)} \\ y = \frac{\sqrt{2}}{x} \end{cases} \Leftrightarrow$$

$$\frac{\sqrt{2}}{x} + 1 - \frac{x^2}{\frac{\sqrt{2}}{x}} = \frac{\sqrt{2}}{x} + - \frac{x^3}{\sqrt{2}} = 0 \Leftrightarrow \frac{2}{x} - x^3 = 0 \Leftrightarrow$$

$$\Leftrightarrow 2 - x^4 = 0 \Leftrightarrow x^4 = 2 \Rightarrow x = 2^{\frac{1}{4}} = \sqrt[4]{2}$$

$$y = \frac{\sqrt{2}}{\sqrt[4]{2}} = \frac{2^{\frac{1}{2}}}{2^{\frac{1}{4}}} = 2^{\frac{1}{2}-\frac{1}{4}} = 2^{\frac{1}{4}} = \sqrt[4]{2}$$

$$\cancel{d = \sqrt{\sqrt{2}}}$$

$$d = f(\sqrt[4]{2}, \sqrt[4]{2}) = \sqrt{\sqrt{2} + (\sqrt[4]{2}+1)^2} = \sqrt{\sqrt{2} + \sqrt{2} + 2\sqrt[4]{2} + 1} =$$

$$= \sqrt{2\sqrt{2} + 2\sqrt[4]{2} + 1}$$