On Calculating Pi

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Abstract

I studied the history as well as the methods of calculating pi. The first and foremost method explained was Archimedes method, which I, on my own, derived a function for that can be used to calculate pi using trigonometry. I also used Reiman sums to create a summation function that calculates pi.

1. What is Pi?

Pi is a mathematical constant that denotes the ratio between a circle's diameter and its circumference. Pi is also irrational as a result the "actual" value of pi can never be known. As a result, many mathematicians and researchers have created different methods of calculating pi. Overtime, these methods tended to become more precise. In this research paper, I will discuss some of these methods.

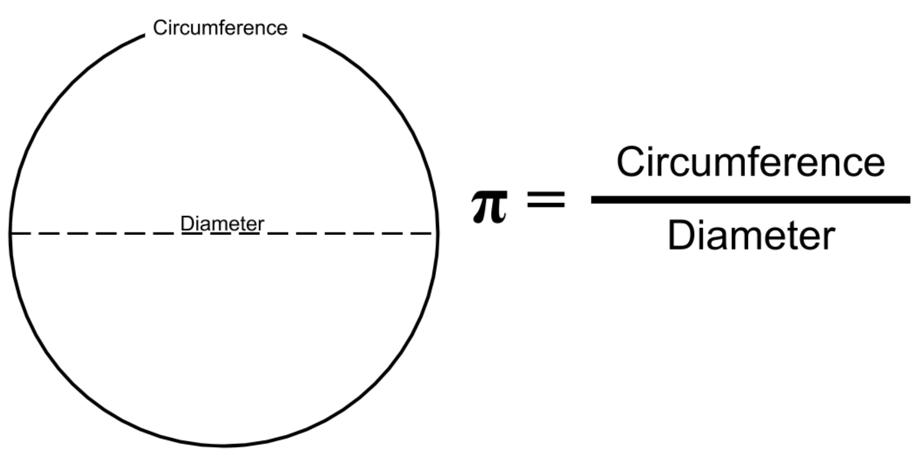

Figure 1

2. Calculating Pi through the History

One of the simplest approaches to understanding pi is through a hands on activity that many have partaken in at a younger age. One would use string to measure the circumference and diameter of a circular object. They would then divide the circumference by the diameter to an approximation of pi. This provides the understanding that there is a ratio, however it is very imprecise. To many early mathematicians, calculating pi was a challenge.

The ancient Babylonians commonly used 3 as pi in their calculations of the area of a circle. However, one tablet (ca. 1900–1680 BC) used 3.125 as pi. Another, ancient civilization that had approximations of pi is ancient Egypt. Based on the Rhind Papyrus (ca.1650 BC), the ancient Egyptians used 3.1605 as pi.

Figure 2

Rhind Papyrus (ca.1650 BC)

The first well known calculation of pi was done by Archimedes (287–212 BC). Archimedes used inscribed and circumscribed polygons to calculate pi. The ratio between the perimeter and the diameter would give the range that pi is within. This method is often also referred to as method of exhaustion. Archimedes used this to determine pi is within $3\frac{1}{7}$ and $3\frac{10}{71}$. This method will be thoroughly explained later. (For a visual representation using hexagons refer to figure 3 on the next page.)

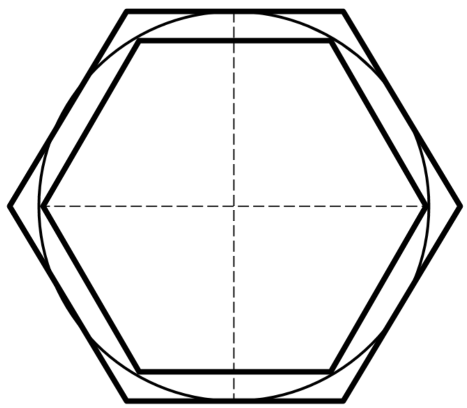

Figure 3

Representation of Archimedes method using hexagons.

Zu Chongzhi (429–501) used a similar method in approximating pi. He got 355/113 as his approximation which must have been tedious because he would have had to use a 24,576-gon. This included calculating hundreds of square roots to a precision of 9 decimal places.

Starting in the 1700s, the Greek letter π was designated as the ratio between a circle's circumference and area. The symbol was first used by Willian Jones in 1706. Leonard Euler started using this symbol in 1737 and made it conventional. Also, in the 1700s, Georges Buffon created a method of calculating pi using probability.

Relatively recently, mathematicians have used infinite series to calculate pi. These scientists include Ramanujan and Leibniz.

3. Equation for Archimedes Method

Archimedes method (also known as method of exhaustion) can be used to approximate pi using regular polygons. As the sides of the polygon increases the resemblance to a circle also increases. Refer to figure 4 for a representation. Similarly, the higher the number of sides, the more precise the calculation is.

The Archimedes method is where the area of a polygon inscribed or circumscribed hexagons in a circle is used to determine an approximation of pi. Inscribed polygons will be less than pi assuming it does not have infinite sides. Circumscribed polygons will be more than pi assuming it does not have infinite sides. Both of these can be used to find the range pi is within. As the sides of the polygons increases the accuracy of the estimation increases similar to how its similarity in appearance increases. Look below.

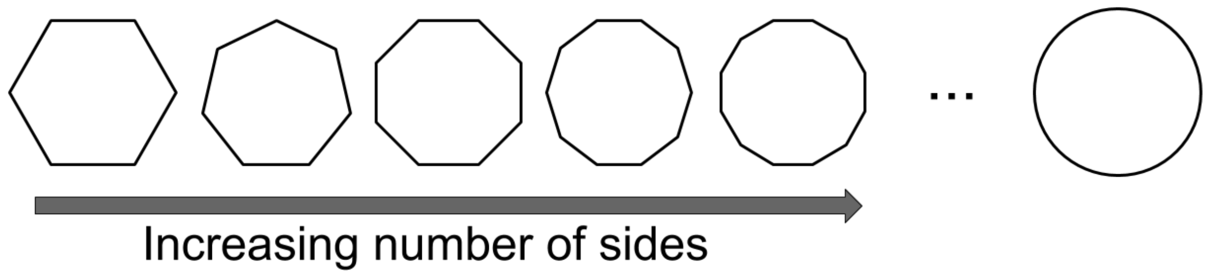

Figure 4

Note how the resemblance to that of a circle increases as the number of sides of a polygon increases.

Let us derive an equation for calculating pi through mode of exhaustion.

A regular polygon with n sides can be broken down into congruent isosceles triangles. Two of the sides of these triangles are the radii. The number of these triangles should correspond with the number of sides. Refer to figure 5 for a visual representation. Also the vertex angle of each of these triangles should be 360 divided by n where n is the number of sides.

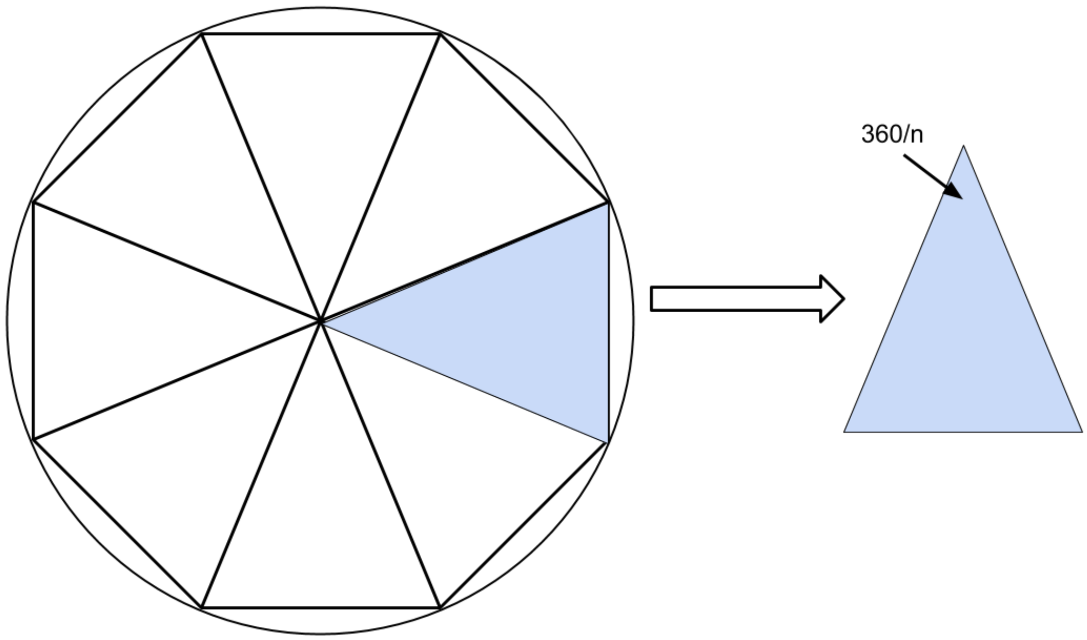

Figure 5

A polygon can be broken down into congruent isosceles triangles where two of the sides of these triangles are the radii. In this case, an octagon is used. An octagon has eight sides and can be divided into 8 congruent isosceles triangles.

First a median should be drawn from the vertex to the base. This median is also perpendicular to the base and bisects the vertex angle. This means 2 congruent right triangles should be formed in one of the aforementioned isosceles triangles. The vertex angle of the isosceles triangle is 360/n so the angles formed from the bisection of the vertex should be 180/n. Refer to figure 6 for a representation.

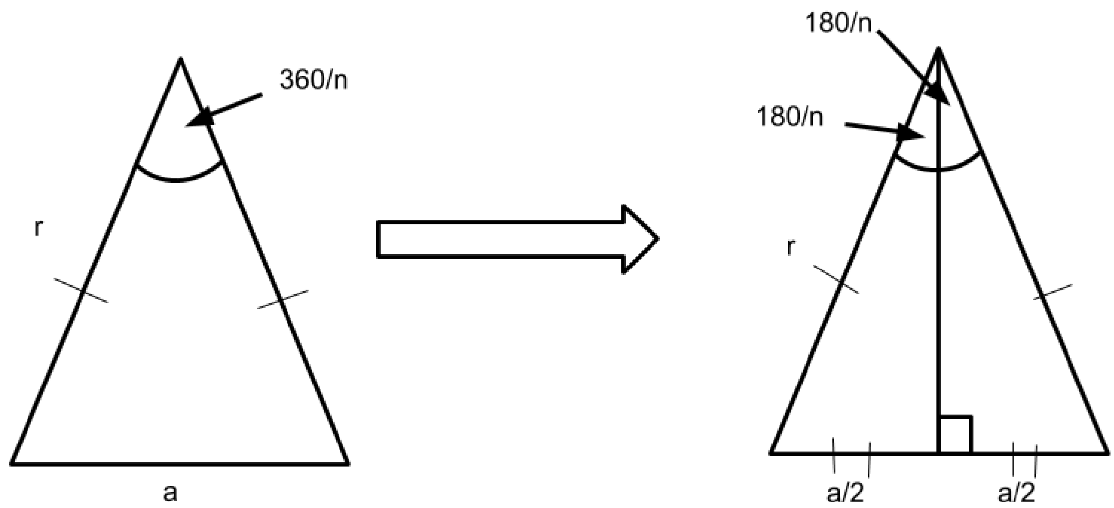

Figure 6

The vertex angle is bisected forming angles of 180/n. The angle bisector is also the perpendicular bisector. The two triangles formed are congruent right triangles. A represents the side length of the base of the congruent isosceles triangles.

The perimeter can be calculated by finding the length of the bases for all the congruent isosceles triangles and multiplying it by the number of sides because the number of these triangles should correspond with the number of sides.

So, we will find the value of each side length (a). In the previous steps we formed right angles. The side length opposite the 180/n degree angle is a/2 as the angle bisector is also the median. We will use trigonometry to find the value of a/2.

Since the angle opposite the 180/n degree angle is a/2 and the hypotenuse is r (radius),

$$sin(\frac{180}{n}) = \frac{\frac{a}{2}}{r}.$$

We will now isolate a.

$$r * sin(\frac{180}{n}) = \frac{a}{2}$$

$$2r * sin(\frac{180}{n}) = a$$

We have now found the side length of a. If we multiply this by the number of sides we will get the perimeter because the number of these congruent isosceles triangle corresponds with the number of sides. We will then substitute $2r * sin(\frac{180}{n})$ for a

$$p = na$$

$$p = 2nr * sin(\frac{180}{n})$$

Since pi is equal to the circumference over diameter. If we divide the perimeter by the diameter we will get a function for approximating pi. So we must substitute all the values and simplify to get our function.

f(n) is equal to the approximation. Let p equal perimeter. Let d be the diameter.

f(n) = p/d

$$f(n) = \frac{2nr*sin(\frac{180}{n})}{2r}$$

$$f(n) = n * sin(180n)$$

Our function is f(n) = n*sin(180n). Refer to table one to see the approximations for different numbers of sides.

Table 1

$$f(n) = n * sin(180n)$$

Number of Sides (n)	Value (f(n))	Number of precise decimal places
10	3.09016994	0
50	3.139525976	1
250	3.1415099708	4
1250	3.14158934625	4
6250	3.141592521296	7

Note how the accuracy increases as the number of sides increases.

4. Another Trigonometric Approach

A circle with a radius of 1 has an area of pi. So we will inscribe a polygon into a circle with a radius of pi. This would allow the area of the polygon to be the approximation for pi. We will also divide the polygon into multiple congruent isosceles triangles. A regular polygon with n sides can be broken down into congruent isosceles triangles. Two of the sides of these triangles are the radii. The number of these triangles should correspond with the number of sides. Refer to figure 5 for a visual representation. Also the vertex angle of each of these triangles should be 360 divided by n where n is the number of sides.

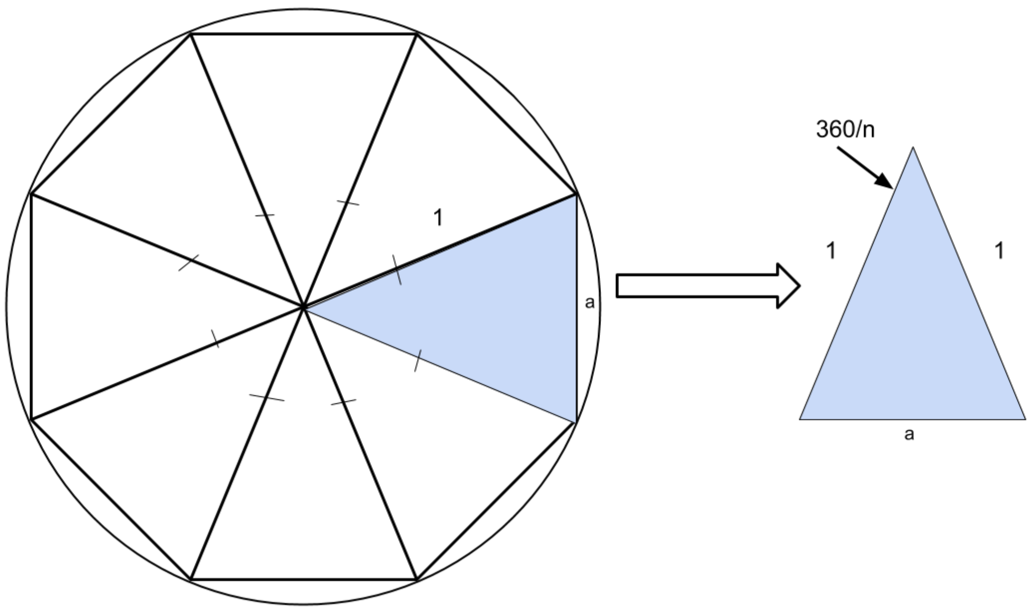

Figure 7

Note how this is very similar to the previous method. The only difference is the radius is one.

First a median should be drawn from the vertex to the base. This median is also perpendicular to the base and bisects the vertex angle. This means 2 congruent right triangles should be formed in one of the aforementioned isosceles triangles. The vertex angle of the isosceles triangle is 360/n so the angles formed from the bisection of the vertex should be 180/n. The base of the isosceles has also been bisected resulting in 2 side lengths of a/2. The perpendicular bisector is the height of the isosceles triangle as the median of an isosceles triangle is also perpendicular to the base and bisects the vertex angle. This height is represented by h. The two triangles formed are congruent right triangles.

Refer to figure 8 for a representation.

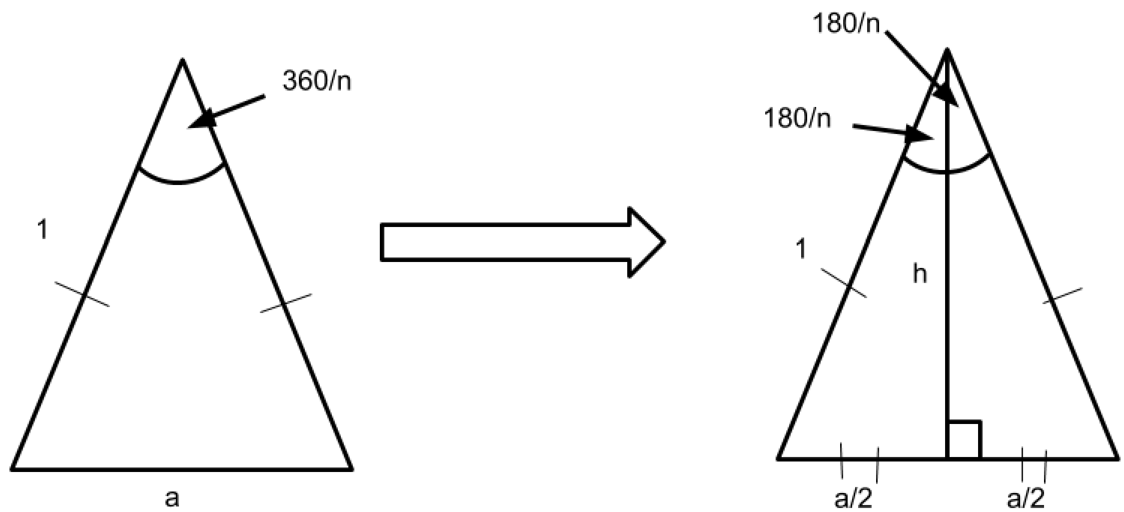

Figure 8

The vertex angle is bisected forming angles of 180/n. The variable a represents the side length of the base of the congruent isosceles triangles.

In order to calculate the area of one of these isosceles triangles we need to find the value of a and h. So, we will find the value of each side length (a).

In the previous steps we formed right angles. The side length opposite the 180/n degree angle is a/2 as the angle bisector is also the median. We will use trigonometry to find the value of a/2.

Since the angle opposite the 180/n degree angle is a/2 and the hypotenuse is 1 (radius),

$$sin(\frac{180}{n}) = \frac{a}{2}$$

We will know isolate a.

$$2 * sin(\tfrac{180}{n}) \ = a \ .$$

Now we also have to find the value of h (height). We can use trigonometry for that too.

Since the side opposite the 180/n angle is a/2, h is the “adjacent” side. So we must used cosine.

Remember the hypotenuse is 1.

$$cos(\tfrac{180}{n}) \ = \ h$$

We found the value of a and h. The area of a triangle is half of base (a) times height (h).

$$A_{isosceles\ triangle} = \tfrac{1}{2}ah$$

We will substitute the trigonometric expressions for a and h.

$$A_{isosceles\ triangle} = \tfrac{1}{2} * 2\ sin(\tfrac{180}{n})\ cos(\tfrac{180}{n})$$

Simplify the equation.

$$A_{isosceles\ triangle} = sin(\tfrac{180}{n})\ cos(\tfrac{180}{n})$$

Now we must find the total area of the polygon. As stated above, the number of congruent isosceles triangles is the same as the number of sides the regular polygon has. So, if we multiply the area of the isosceles triangle by the number of sides, we should get the area of the regular polygon.

$$A_{regular\ polygon} = A_{isosceles\ triangle} \ * \ n$$

Substitute the equation for the area of isosceles triangle.

$$A_{regular\ polygon} = \ n\ sin(\tfrac{180}{n})\ cos(\tfrac{180}{n})$$

Let f(n) be the area of the regular polygon.

$$f(n) = \ n\ sin(\tfrac{180}{n})\ cos(\tfrac{180}{n})$$

Below is a table which contains the value of the function given n.

Table 2

$$f(n) = n\ sin(\tfrac{180}{n})\ cos(\tfrac{180}{n})$$

Number of Sides (n)	Value (f(n))	Number of precise decimal places
10	2.938926261	0
50	3.1333308	1
250	3.14126193	3
1250	3.141579	4
6250	3.14159212	6

Note how the accuracy increases as the number of sides increases. Note also how the previous method was more accurate.

5. Riemann Sums

The area of a circle with a radius 2 is 4 pi. So a quarter of a circle with a radius of 2 is pi. The equation for a quarter of a circle with a radius of 2 with a center at the origin and where the x coordinates are greater than or equal to 0 is $y = \sqrt{4 - x^2}$ where $x\ \varepsilon\ [0,\ +\infty]$. Look at figure 9.

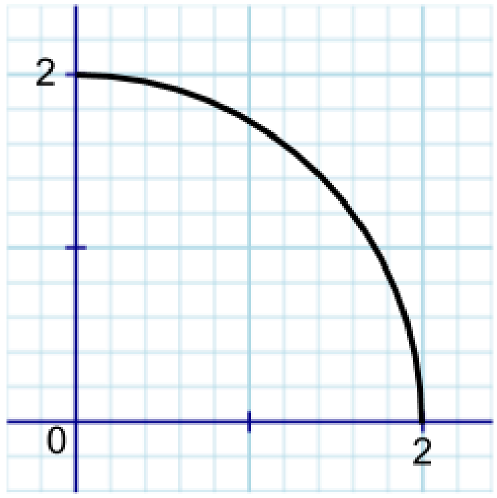

Figure 9

This is the graph for $y = \sqrt{4 - x^2}$ where $x \varepsilon [0, +\infty]$.

Now we will calculate the area under the curve using Reimann sums. Reimann sums is a method of evaluating the area under a graph. It uses rectangles to approximate the area. As the number of rectangles increases the higher accuracy. We will use left bound rectangles whose widths are the same. Look at figure 10 for an example.

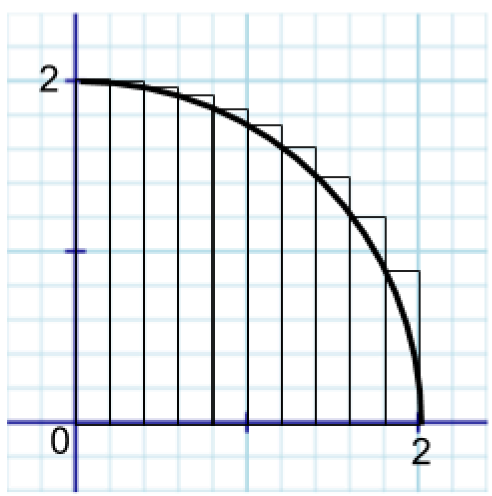

Figure 10

An example of how the rectangles would look.

Lets start off by coming up with a function that finds the area of a given rectangle. The first rectangle is f(1) and last one is f(n) where n is the number of rectangles. We must find the width and height in roder to find the area. Since all the widths are equal and the total length can be equal to the 2 which is the radius, the width is 2 divided by the number of rectangles (n). Look at figure 11.

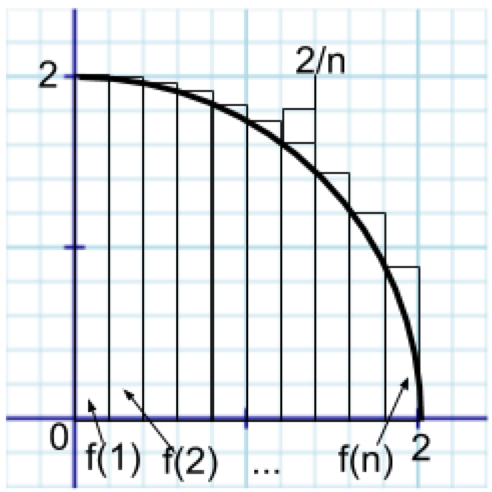

Figure 11

F(x) is the function that finds the area of a rectangle. The width is 2/n.

The challenge is to find the height. In order to find the height we will need to find the x coordinate. The widths are the same. Since the rectangles will be left justified we need to find the x coordinate of the lower left corner of the rectangles. The first x coordinate is 0 for the rectangle f(1). This can be shown as the number of the rectangle minus 1. So it starts at x-1. Every other point is a multiple of 2/n apart from the first point. The x coordinate for the second triangle should be 2/n. The number of rectangles between the x coordinate and the first coordinate times the width should give the x coordinate. For example, the x coordinate for the second triangle should be 2/n because there is only one rectangle between. The number of rectangles between can be represented as X - 1 where X is the number of a rectangle. So if we multiply X -1 by 2/n we will get the x coordinate.

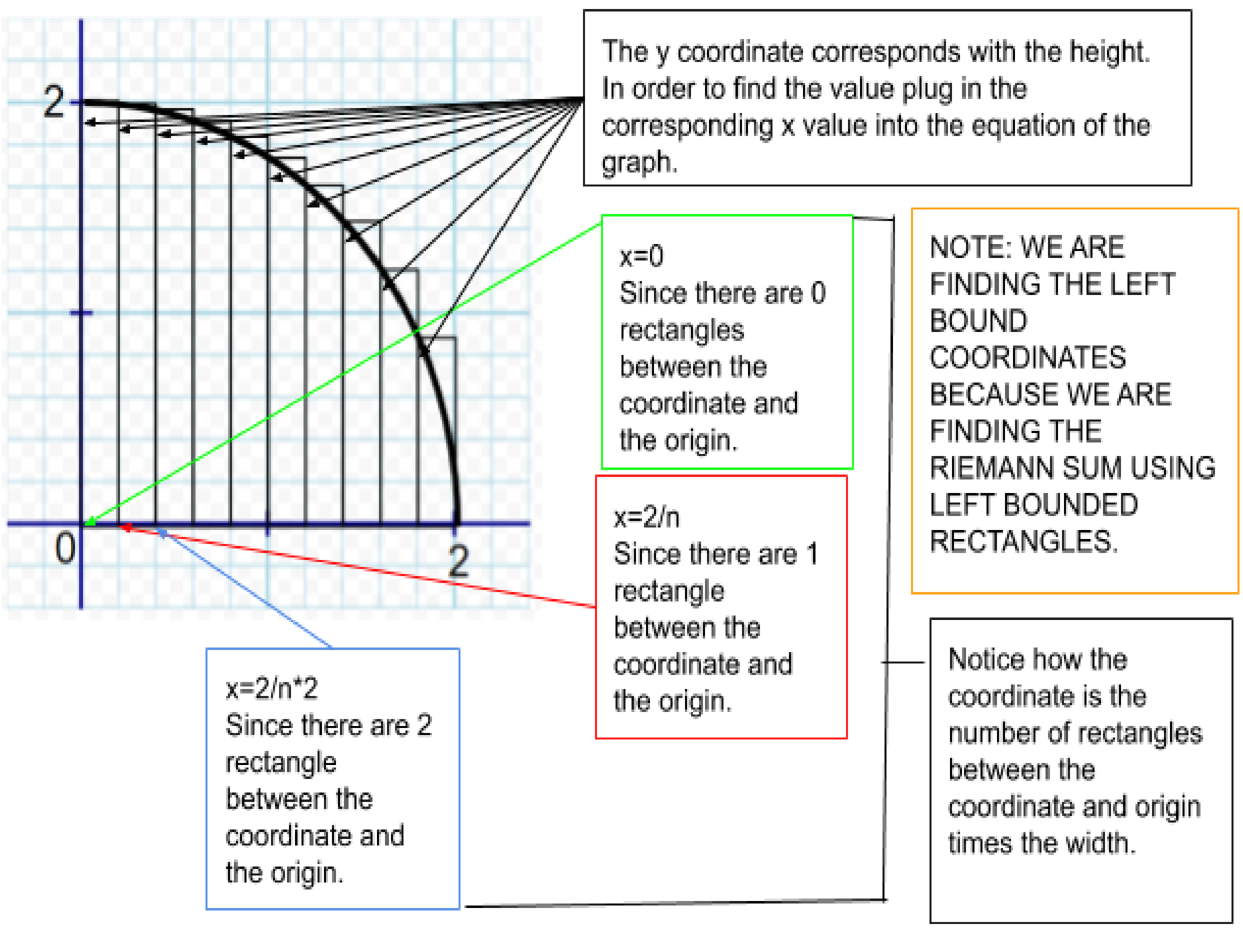

Figure 11

Now we need the height. We can plug the x coordinate into the equation for a quarter of a circle.

$$y = \sqrt{4 - x^2} \text{ where } x \, \varepsilon \, [0, \; +\infty]$$

$$\text{x} = (X - 1)(\tfrac{2}{n})$$

$$y = \sqrt{4 - ((X-1)(2/n))^2} \text{ where } x \, \varepsilon \, [0, \; +\infty] \; \textit{and } n \text{ is the total number of triangles.}$$

Now we can multiply the height (provided by the y coordinate) by the width (which is 2/n)

$$f(X) = \tfrac{2}{n}\sqrt{4 - ((X-1)(2/n))^2} \text{ where } x \, \varepsilon \, [0, \; +\infty] \; \textit{and } n \text{ is the total number of triangles.}$$

This will give us the area for a given triangle. We need to find the area of all the triangles. We can represent this as a sigma function. This function should give us an approximation for pi. As n increases, the precision increases. X represents the number of a rectangle. n represents the total number of rectangles.

$$\lim_{n\to\infty} \sum_{X=1}^{n} \tfrac{2}{n}\sqrt{4 - ((X-1)(2/n))^2}$$

Table 3

$$\lim_{n\to\infty} \sum_{X=1}^{n} \tfrac{2}{n}\sqrt{4 - ((X-1)(2/n))^2}$$

Number of Sides (n)	Value	Number of precise decimal places
10	3.3045183	0
50	3.17826851	1
250	3.1492951875	2
1250	3.1431660448	2
6250	3.14191027358	3

Note how the accuracy increases as the number of sides increases.

7. Conclusion

There are many methods of calculating pi. Each of these methods have different precision. However this precision might come at a cost of requiring higher levels of math and extreme complexity. In the future, I can research how other methods of calculating pi work such as Ramanujan's formula and Chudnovsky algorithm which is extremely precise.

References

"A Brief History of Pi (π)." *Exploratorium*, 14 Mar. 2019, https://www.exploratorium.edu/pi/history-of-pi#:~:targetText=The first calculation of π,mathematicians of the ancient world.&targetText=Mathematicians began using the Greek,who adopted it in 1737.