# Season 2 OTIE Solutions

Online Test Snowy Series

January 8, 2021, to January 22, 2021

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## Answer Key: *Click on the Problem Number to go to the solution of that problem*

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|---|---|---|---|---|
| **1.** (031) | **2.** (674) | **3.** (127) | **4.** (083) | **5.** (049) |
| **6.** (075) | **7.** (067) | **8.** (170) | **9.** (533) | **10.** (535) |
| **11.** (013) | **12.** (864) | **13.** (129) | **14.** (033) | **15.** (601) |

## Solutions:

1. *(ARMLlegend)* Let $P(x)$ be a quadratic with real, nonzero roots and coefficients such that
$$P(-20) + P(21) = P(-29).$$
The sum of the reciprocals of the roots of $P(x)$ can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Answer:* ***031***

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Let $P(x) = ax^2 - bx + c$, and let the roots of $P(x)$ be $p$ and $q$. Then, $\frac{1}{p} + \frac{1}{q} = \frac{b}{c}$, by Vieta's Formulas. Now,

$$\begin{aligned} P(-29) &= P(-20) + P(21) \\ \implies (29^2a + 29b + c) &= (20^2a + 20b + c) + (21^2a - 21b + c) \\ &= 29^2a - b + 2c \\ \implies 30b = c &\implies \frac{b}{c} = \frac{1}{30}. \end{aligned}$$

Thus, $m + n = \mathbf{031}$. ■

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2. *(DeToasty3)* Let $\mathcal{A}$ and $\mathcal{B}$ be two sets. Suppose that $\mathcal{A}$ contains $a$ distinct elements and $\mathcal{B}$ contains $b$ distinct elements, where $a$ and $b$ are positive integers. For some positive integer $n$, if there exist $2021$ distinct elements belonging to at least one of $\mathcal{A}$ and $\mathcal{B}$, and there exist $n$ distinct elements belonging to both $\mathcal{A}$ and $\mathcal{B}$, then the number of possible ordered pairs $(a, b)$ is $2n$. Find $n$.

*Answer:* ***674***

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Note that the number of distinct elements belonging to $\mathcal{A}$ but not $\mathcal{B}$ is $a - n$, the number of distinct elements belonging to both $\mathcal{A}$ and $\mathcal{B}$ is $n$, and the number of distinct elements belonging to $\mathcal{B}$ but not $\mathcal{A}$ is $b-n$. Then, we have that $a-n+n+b-n = 2021$, or $a+b = 2021+n$. However, we also need $a-n \geq 0$ and $b-n \geq 0$. The ordered pair with the smallest $a$ is $(n, 2021)$, and the ordered pair with the largest $a$ is $(2021, n)$. We see that there exists exactly one ordered pair for each integer $a$ from $n$ to $2021$, inclusive, so the number of possible ordered pairs $(a, b)$ is $2021 - n + 1 = 2022 - n$. We let $2022 - n = 2n$, which gives $3n = 2022 \implies n = \mathbf{674}$. ■

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3. *(Emathmaster)* Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 2$. Let $M$ be the midpoint of side $\overline{AD}$, and let $\mathcal{T}$ be a rectangle with all of its vertices on a side of $\triangle BMC$, two of which are on side $\overline{BC}$. If $\mathcal{T}$ is similar to $ABCD$, then the sum of all possible areas of $\mathcal{T}$ can be written as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

*Answer:* ***127***

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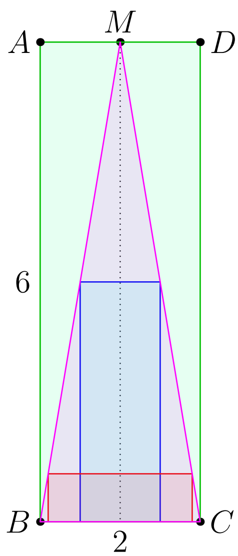


As $\mathcal{T}$ is similar to $ABCD$, the height must either be 3 times, or one third of the width, giving us two possible configurations of the rectangle. We use similar triangles, the one cut out by the right edge of $\mathcal{T}$ and half the large triangle, which has height 6 and width 1. For the larger one (the blue one in the above figure), where the height is three times the width $w$,

$$\frac{1 - \frac{3w}{2}}{1} = \frac{3w}{6} \implies w = 1,$$

so the area is 3. For the second (the red one in the above figure), we have that the width is three times the height $h$,

$$\frac{1 - \frac{3h}{2}}{1} = \frac{h}{6} \implies h = \frac{3}{5},$$

thus the area of the second configuration is $\frac{27}{25}$.

The sum of possible areas of $\mathcal{T}$ is $3 + \frac{27}{25} = \frac{102}{25}$, so our answer is $102 + 25 = \mathbf{127}$. ■

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4. *(Emathmaster)* Find the number of ordered quadruples $(a, b, c, d)$ of positive odd integers satisfying

$$ab < 49 \quad \text{and} \quad cd = \min(a, b),$$

where $\min(a, b)$ denotes the smaller of $a$ and $b$. (If $a = b$, then $\min(a, b) = a = b$.)

*Answer:* ***083***

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We do casework on $\min(a, b)$. It is obvious to note that $\min(a, b) < 7$, since, otherwise, if $\min(a, b) \geq 7$, then $ab \geq 7 \cdot 7 = 49$, which is a contrast to the problem statement.

**Case 1.** $\min(a, b) = 1$. Then $c$ and $d$ must both be 1. One of $a$ and $b$ must be 1. If $a$ is 1, then $b$ can be any odd integer from 1 to 47, giving 24 cases. Similarly, for $b = 1$ we also have 24 cases. However, we have over-counted the case $(a, b) = (1, 1)$, so we subtract 1 to get **47** cases in this case.

**Case 2.** $\min(a, b) = 3$. Then $c$ and $d$ must be 1 and 3 in some order, so there are 2 choices for $c$ and $d$. Note that if $a = 3$, then $b$ can only be 3, 5, 7, 9, 11, 13, or 15 for 7 cases. We also get 7 cases for $b = 3$, and we need to subtract 1 again for over-counting $(3, 3)$. This gives $13 \cdot 2 = \mathbf{26}$ cases.

**Case 3.** $\min(a, b) = 5$. Then $c$ and $d$ must be 1 and 5 in some order, so there are 2 choices for $c$ and $d$. Note that if $a = 5$, then $b$ can only be 5, 7, or 9 for 3 cases. We also get 3 cases for $b = 5$, and we need to subtract 1 again for over-counting $(5, 5)$. This gives $5 \cdot 2 = \mathbf{10}$ cases.

The answer is $47 + 26 + 10 = \mathbf{083}$. ■

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5. *(PCChess)* Let $\overline{AB}$ be a diameter of circle $\omega_1$ with center $O_1$ and radius 2. Let circle $\omega_2$ with center $O_2$ be drawn such that $\omega_2$ is tangent to $\omega_1$ and is also tangent to $\overline{AB}$ at $O_1$.

Let $D$ be the point of intersection of line segment $\overline{BO_2}$ and $\omega_2$. Let the line tangent to $\omega_1$ at $B$ and the line tangent to $\omega_2$ at $D$ meet at a point $P$. Then $PO_1^2$ can be written as $a - b\sqrt{c}$, where $a$, $b$, and $c$ are positive integers, and $c$ is not divisible by the square of any prime. Find $a + b + c$.

*Answer:* ***049***

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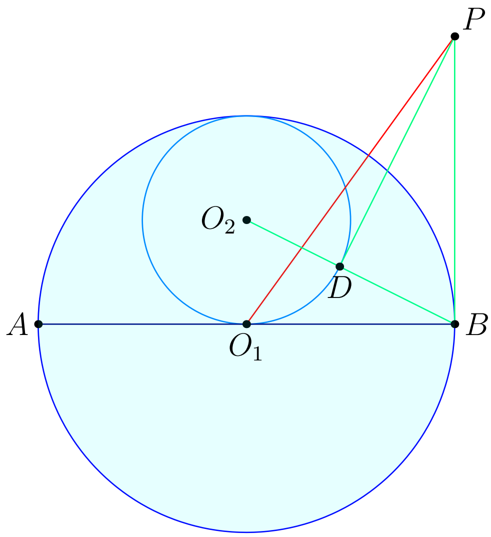


By the tangent conditions, we have that $\angle PBO_1 = \angle PDB = 90^\circ$. By angle chasing, we get that $\angle O_2BO_1 = \angle BPD$ and $\angle O_2O_1B = \angle PDB = 90^\circ$. By $AA$ similarity, we have that $\triangle O_2BO_1 \sim \triangle BPD$. Since $BO_1 = 2$ and $O_1O_2 = 1$, we have that $BO_2 = \sqrt{BO_1^2 + O_1O_2^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$ by the Pythagorean Theorem. Also, since $DO_2 = 1$, we get that $BD = BO_2 - DO_2 = \sqrt{5} - 1$. By similarity, we have that

$$\frac{O_1O_2}{BO_2} = \frac{BD}{BP} \implies \frac{1}{\sqrt{5}} = \frac{\sqrt{5}-1}{BP} \implies BP = 5 - \sqrt{5}.$$

By using the Pythagorean Theorem on $\triangle PBO_1$, we get that

$$PO_1^2 = BP^2 + BO_1^2 = (5 - \sqrt{5})^2 + 2^2 = 25 - 10\sqrt{5} + 5 + 4 = 34 - 10\sqrt{5},$$

so our answer is $34 + 10 + 5 =$ **049**. ■

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6. *(DeToasty3)* Let $\mathcal{S}$ be the set of all positive integers less than and relatively prime to 49. Call a subset of $\mathcal{S}$ with 15 distinct numbers *great* if it can be divided into 3 pairwise disjoint groups of 5 numbers such that no two numbers in the same group leave the same remainder when divided by 7, and the product of the numbers in each group leaves a unique remainder when divided by 7. Let $n$ be the number of great subsets of $\mathcal{S}$. Find the sum of the (not necessarily distinct) primes in the prime factorization of $n$.

*Answer:* ***075***

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First, we note that $6! \equiv 6 \pmod 7$. For each product of a group, there exists a unique number which must be removed from the set $\{1, 2, 3, 4, 5, 6\}$ to obtain a unique remainder.

Call the number of times a residue modulo 7 appears in the subset the number of its *mentions*. Within the subset, there must exist 3 residues with 3 mentions, and 3 residues with 2 mentions. Within $\mathcal{S}$, there exist seven numbers which are 1 modulo 7, seven numbers which are 2 modulo 7, and so on. Therefore, for each of the residues with 3 mentions, there exist $\binom{7}{3} = 35$ ways to choose the three numbers, and $\binom{7}{2} = 21$ ways to choose the two numbers for the 3 residues with 2 mentions. But we also have $\binom{6}{3} = 20$ ways to choose which of the 3 residues will have 3 mentions.

Thus, we have that

$$n = 35^3 \cdot 21^3 \cdot 20 = 2^2 \cdot 3^3 \cdot 5^4 \cdot 7^6,$$

so the sum of the primes is $2(2) + 3(3) + 5(4) + 7(6) = \mathbf{075}$. ■

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7. *(DeToasty3)* Let $a$ and $b$ be positive real numbers such that $\log_a b = \log_{ab} a^2$, $17ab = 60b + 1$, and $a \neq b$. The difference between the largest and smallest possible values of $ab$ can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Answer:* ***067***

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Add $\log_a a = 1$ to both sides of the first equation to obtain $\log_a ab = 2\log_{ab} a + 1$. Note that $\log_a ab$ and $\log_{ab} a$ are reciprocals of each other. Let $x = \log_a ab$. Then, we have that $x = \frac{2}{x} + 1$. Multiplying both sides by $x$, we get $x^2 = 2 + x$, so $x^2 - x - 2 = 0$, from which we get that $x = -1, 2$. Therefore, $\log_a ab = 1 + \log_a b = -1, 2 \implies \log_a b = -2, 1$. Therefore, we have that either $\frac{1}{a^2} = b$ or $a = b$. Since $a$ and $b$ are distinct, we can discard the latter case to get $\frac{1}{a^2} = b$ as our only solution.

We have that $\frac{1}{a^2} = b$ and $17ab - 60b - 1 = 0$. Rearrange the second equation to get $b = \frac{1}{17a - 60}$. Then, we must have that $\frac{1}{a^2} = \frac{1}{17a-60} \implies a^2 - 17a + 60 = 0$. From here, we get that $a = 5, 12$. Plugging $a = 5$ and $a = 12$ into the equation $b = \frac{1}{a^2} \implies ab = \frac{1}{a}$, we get $ab = \frac{1}{5}$ and $ab = \frac{1}{12}$, respectively. We see that $\frac{1}{5} > \frac{1}{12}$, so our difference is $\frac{1}{5} - \frac{1}{12} = \frac{7}{60}$. Thus, we have that $m + n = 7 + 60 = \mathbf{067}$. ■

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8. *(DeToasty3)* Find the sum of the three least positive integers that cannot be written as

$$\frac{a!}{b!} + \frac{c!}{d!} + \frac{e!}{f!}$$

for positive integers $a, b, c, d, e, f$ less than or equal to 5.

*Answer:* ***170***

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Notice that the only numbers that we can create with one of these fractions are 1, 2, 3, 4, 5, 6, 12, 20, 24, 60, 120, and their reciprocals. We can create 1 with $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}$ and 2 with $\frac{1}{2}+\frac{1}{2}+1$.

Now consider the possible sums of two of the fractions. We can create 1 with $\frac{1}{2}+\frac{1}{2}$, 2 with $1+1$, and we can make every number until 12 by only using the numbers 1, 2, 3, 4, 5, 6. We can create 13 through 18 by using 12 and 1 through 6. Although we cannot create 19 or 20 with two fractions, we can create these with three fractions by using a sum with two fractions and adding a number from 1 to 6 as the third fraction. This means that we must find the first time when there is a gap of more than 6 numbers that we can create by summing two fractions.

Continuing, we see that 21 through 30, 32, 36, 40, 44, and 48 can be created by summing two fractions. However, the next number is 61, which is more than 6 away from 48. We can create 49 through 54 by summing three fractions, but we cannot create 55 through 61 with 3 fractions, if the third fraction is one of 1, 2, 3, 4, 5, 6. However, we can create 56 with $12+20+24$.

Thus, the three least positive integers are 55, 57, and 58, and their sum is **170**. ■

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9. *(DeToasty3)* A jar contains five slips labeled from 1 to 5, inclusive. In each turn, Kevin takes two different slips out of the jar at random. If Kevin selects slips with the numbers $a$ and $b$, the numbers $a$ and $b$ are replaced with the numbers 0 and $a+b$, and both slips are put back in the jar. Kevin stops once he writes the number 12 on a slip or takes three turns. The probability that the number 12 has been written once Kevin stops is $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

*Answer:* ***533***

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Notice that in order to have 12 be written, we have to combine either 3, 4, and 5, or 1, 2, 4, and 5. We may treat the numbers we want to combine as $X$s, the numbers we don't want to combine as $Y$s, and the zeroes as $O$s. Observe that if we select an $O$ at any move, the highest number will not change. This means that we have to combine two $X$s in order to progress to 12. We also don't want to combine an $X$ and a $Y$ at any point.

**Case 1.** 3, 4, and 5 are chosen. Notice that we start with $XXXYY$. We have two possibilities: we select two $X$s with a $\frac{3}{10}$ chance, or we select two $Y$s with a $\frac{1}{10}$ chance.

- **Subcase 1.** Two $X$s. Then, we have $XXYYO$. From here, there are three possibilities: we select two $X$s with a $\frac{1}{10}$ chance, we select two $Y$s with a $\frac{1}{10}$ chance, and we select one $O$ and one $X$ or one $Y$ with a $\frac{2}{5}$ chance. If we select two $X$s, then we have $XYYOO$, and we are done. If we select two $Y$s, then we have to select two $X$s for our third move, which has a $\frac{1}{10}$ chance. Finally, if we select one $O$ and one $X$ or one $Y$, then we have to select two $X$s on our third move, which has a $\frac{1}{10}$ chance.

- **Subcase 2.** Two $Y$s. Then, we have $XXXYO$. From here, we must select only two $X$s for our next two moves in order to arrive at 12 by three total moves. Picking out two $X$s on the second move has a $\frac{3}{10}$ chance, and then picking out two $X$s on the third move has a $\frac{1}{10}$ chance.

Our total probability for Case 1 is $(\frac{3}{10} \cdot \frac{1}{10})+(\frac{3}{10} \cdot \frac{1}{10} \cdot \frac{1}{10})+(\frac{3}{10} \cdot \frac{2}{5} \cdot \frac{1}{10})+(\frac{1}{10} \cdot \frac{3}{10} \cdot \frac{1}{10}) = \frac{6}{125}$.

**Case 2.** 1, 2, 4, and 5 are chosen. Notice that we start with $XXXXY$. We see that we must choose only two $X$s in each move in order to arrive at 12 in three moves. There is a $\frac{3}{5}$ chance for the first move, a $\frac{3}{10}$ chance for the second move, and a $\frac{1}{10}$ chance for the third move, for a total probability of $\frac{3}{5} \cdot \frac{3}{10} \cdot \frac{1}{10} = \frac{9}{500}$.

Adding, we arrive at the total probability of $\frac{6}{125} + \frac{9}{500} = \frac{33}{500}$, so $m + n = \mathbf{533}$. ■

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10. *(DeToasty3)* Consider the polynomial

$$P(x) = x^{21} - 364x^{20} + Q(x),$$

where $Q(x)$ is some polynomial of degree at most 19. If the roots of $P(x)$ are all integers and $P(21) = 2021$, find the remainder when $P(23)$ is divided by 1000.

*Answer:* ***535***

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Write $P(x)$ as

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_{21}),$$

for roots $r_1, r_2, \ldots, r_{21}$. Define a new sequence $a_1, a_2, \ldots, a_{21}$ such that $a_i = 21 - r_i$ for all $1 \le i \le 21$. Now, we have

$$(x - (21 - a_1))(x - (21 - a_2)) \cdots (x - (21 - a_{21})).$$

By Vieta's Formulas, we have that

$$\begin{aligned} 364 &= r_1 + r_2 + \cdots + r_{21} \\ \implies 364 &= (21 - a_1) + (21 - a_2) + \cdots + (21 - a_{21}) \\ \implies 77 &= a_1 + a_2 + \cdots + a_{21}. \end{aligned}$$

Note that from the condition $P(21) = 2021$, we have that $a_1 \cdot a_2 \cdots a_{21} = 2021$ as well.

Our goal is to find the exact integer values of $a_1, a_2, \ldots, a_{21}$. Notice that $2021 = 43 \cdot 47$, and $43 + 47 = 90$, and since 90 is close to 77, and all other non 1 or $-1$ values seem too far off to be true (e.g. 2021), we have that 43 and 47 should be two of the values.

Now, for the other values, we must have either 1 or $-1$. We want the sum to be equal to 77, so we must have 3 1s and 16 $-1$s.

From this, we arrive at the polynomial

$$\begin{aligned} & P(x) = (x-(21-(-1)))^{16}(x-(21-1))^3(x-(21-43))(x-(21-47)) \\ \implies & P(x) = (x-22)^{16}(x-20)^3(x+22)(x+26). \end{aligned}$$

Finally, plugging in 23 yields $P(23) = 27 \cdot 45 \cdot 49 = 59,535$, and the remainder when $59,535$ is divided by 1000 is **535**. ■

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11. *(NJOY)* In the complex plane, there exist distinct complex numbers $z_1$, $z_2$, $z_3$, and $z_4$ lying in clockwise order on a circle. If $|z_i| \neq |z_j|$ for all $i, j \in \{1,2,3,4\}$ where $i \neq j$, and

$$\frac{3z_1 - 5z_4}{2z_2 - 3z_3} = 2,$$

then $\left|\frac{z_1 - z_4}{z_2 - z_3}\right|^2 = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

*Answer:* ***013***

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First, draw the complex plane and let $A$, $B$, $C$, $D$ respectively represent the points $z_1$, $z_2$, $z_3$, and $z_4$. Let $P$ be the intersection of the diagonals $AC$ and $BD$ of the cyclic quadrilateral $ABCD$. Note that we are asked to find

$$\left|\frac{z_1 - z_4}{z_2 - z_3}\right|^2 = \frac{AD^2}{BC^2}.$$

Now, rewrite the given expression as

$$\frac{3z_1 + 6z_3}{9} = \frac{4z_2 + 5z_4}{9}.$$

This implies (by the Section Formula) that $P$ divides $AC$ in the ratio $6:3 = 2:1$ and $BD$ in the ratio $5:4$. Now, let $AP = 2x$, $PC = x$, $BP = 5y$, $PD = 4y$. Then, using Power of a Point with $P$,

$$2x \cdot x = 5y \cdot 4y \implies x^2 = 10y^2 \tag{1}$$

Now, suppose that the angle $\angle DAP = \theta = \angle CBP$. Then, by the **Law of Cosines**, and using (1), we obtain $AD^2 = 4x^2 + 16y^2 - 16xy\cos\theta = 8(7y^2 - 2xy\cos\theta)$, and $BC^2 = x^2 + 25y^2 - 10xy\cos\theta = 5(7y^2 - 2xy\cos\theta)$. Therefore, we have

$$\frac{AD^2}{BC^2} = \frac{8}{5},$$

so $m + n = $ **013**. ■

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12. *(kevinmathz)* A date can be written $m/d/y$, where $m$ is the month, $d$ is the day, and $y$ is the last two digits of the year. Call a date *bad* if both $m+d+y$ is even, and the greatest common divisor of $m$, $d$, $y$ can be written as $2^n$ for an integer $n$. Find the number of bad dates from January 1, 2017 (1/1/17) to December 31, 2021 (12/31/21), inclusive. (Note that April, June, September, and November have 30 days, February has 28 days in the years 2017, 2018, 2019, 2021 and 29 days in 2020, and the rest have 31 days.)

*Answer:* ***864***

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Clearly, the values of $y$ will be 17, 18, 19, 20, and 21. First, we will find how many times $m+d+y$ is even. For each year, there are 179 dates where $m+d+y$ is even and $1 \leq d \leq 30$ (The Leap day will not make a difference in our count of bad dates as $2+29+20$ is even). Then if $d = 31$, then $m+y$ must be odd. Going through the values of $m$ where $d$ can be 31 and the given values of $y$, there are 17 different occasions where $m+y$ is odd. In total, there are $5 \cdot 179 + 17 = 912$ different dates where $m+d+y$ is even.

Next, we have to subtract the dates where $\gcd(m,d,y)$ is not a power of 2. If $\gcd(m,d,y)$ is not a power of 2, then they must all share an odd prime factor that is not a multiple of 2. Since $1 \leq m \leq 12$, this prime number could be 3, 5, 7, or 11, depending on the year, month, and day. If the year is 2018 or 2021, then the prime number could be 3. For this case, 5 days would be subtracted from each month the number of those days would be divisible by 3, and there are 8 different months that satisfy this condition. If the year is 2020, the prime number could be 5. For this case, 3 days would be subtracted from each month the number of those days would be divisible by 5, and there are 2 different months that satisfy this condition. If the year is 2021, the prime number could be 7. For this case, 2 days would be subtracted from each month the number of those days would be divisible by 7, and there is 1 month that satisfies this condition.

In total, there are $912 - 8(5) - 2(3) - 1(2) = \mathbf{864}$ bad dates. ■

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13. *(Awesome_guy)* Triangle $ABC$ with circumcircle $\Gamma$ has side lengths $AB = 8$, $BC = 6$, and $AC = 4$. Let $D$ be a point on side $\overline{BC}$ such that there exists a circle internally tangent to $\Gamma$ at $A$ and tangent to $\overline{BC}$ at $D$. Let $E$ be a point on minor arc $\widehat{BC}$ of $\Gamma$ such that the length $BE$ is twice the length $CE$. Let $K$ be the intersection of lines $AE$ and $BC$. Let $L$ be a point on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle KAL$. Then $AK \cdot AL = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m+n$.

*Answer:* ***129***

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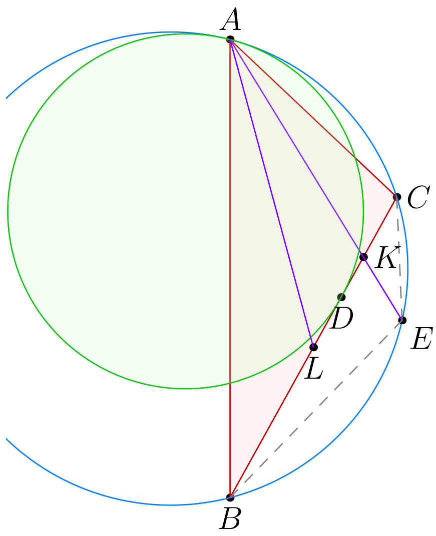


**Solution 1 (kevinmathz)** Note that since $BE = 2CE$, then $AB \cdot CE = AC \cdot BE$ and thus $ABEC$ is harmonic. Now, we see that $\overline{AD}$ is the angle bisector of $\angle BAC$ because a homothety maps $D$ to the midpoint of minor arc $\widehat{BC}$. Now, since $ABEC$ is harmonic, then $\overline{AK}$ is the $A$-symmedian of $\triangle ABC$, and $\overline{AL}$ is the $A$-median of $\triangle ABC$. Due to that, $AB^2 \cdot CK = AC^2 \cdot BK$ and thus, with the given sides, we see that $BK = \frac{6}{5}$ and $CK = \frac{24}{5}$.

Now, using Stewart's Theorem,

$$\frac{864}{25} + 6AK^2 = \frac{768}{5}$$

so $AK^2 = \frac{496}{25}$ meaning $AK = \frac{4}{5} \cdot \sqrt{31}$. Similarly, $2AL^2 = AC^2 + AB^2 - 2BL^2 = 16 + 64 - 18 = 62$ so $AL = \sqrt{31}$. Hence, $AK \cdot AL = \frac{124}{5}$, meaning our answer is $124 + 5 = \mathbf{129}$. ■

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**Solution 2** We first claim that $\overline{AD}$ is the angle bisector of $\angle BAC$. Let $\omega$ be the circle internally tangent to $\Gamma$ at $A$ and $\overline{BC}$ at $D$ with center $O_1$, and let $O_2$ be the center of $\Gamma$. If we extend line $AD$ to intersect $\Gamma$ again at a point $D' \neq A$, we see that $\triangle AO_1D \sim \triangle AO_2D'$ by using common angle $DAO_1$ and the fact that both triangles are isosceles. Since $\overline{DO_1}$ is perpendicular to $\overline{BC}$, we have that $\overline{D'O_2}$ is also perpendicular to $\overline{BC}$. Since $O_2$ is the circumcenter of $\triangle ABC$, the point of intersection of $\overline{D'O_2}$ and $\overline{BC}$ is the midpoint of $\overline{BC}$, meaning that $D'$ is the midpoint of arc $\widehat{BC}$, and $\angle BAD = \angle CAD$, as desired.

We now claim that $L$ is the midpoint of $\overline{BC}$. Let line $AL$ intersect $\Gamma$ at another point $F \neq A$. Since line $AD$ bisects both $\angle BAC$ and $\angle KAL$, we have the lengths $BE = 2CE$ are preserved as $BF = 2CF$. Let $BF = x$. By angle chasing, we get that $\triangle BLA \sim \triangle FLC$ and $\triangle BLF \sim \triangle ALC$. We have the ratios

$$\frac{FL}{CL} = \frac{x}{4} \quad \text{and} \quad \frac{FL}{BL} = \frac{x}{4},$$

so we get that

$$\frac{BL}{CL} = 1 \implies BL = CL,$$

as desired. The rest is a length chase. Proceed with Solution 1.

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14. *(Awesome_guy)* Let $ABC$ be a right triangle with right angle at $B$, $AB = 20$, and $BC = 21$. Let $D$ be the center of circle $\omega$ with diameter $\overline{AB}$. The circumcircle of $\triangle BCD$ and $\omega$ intersect at $B$ and $E$. Line $AC$ intersects the circumcircle of $\triangle BCD$ at $C$ and $K$. Line $AC$ intersects $\omega$ at $A$ and $L$. The tangent at $D$ to the circumcircle of $\triangle BCD$ intersects line $EK$ at $M$. Lines $CD$ and $EL$ intersect at $N$. The circumcircle of $\triangle DMN$ has a radius of length $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Answer:* ***033***

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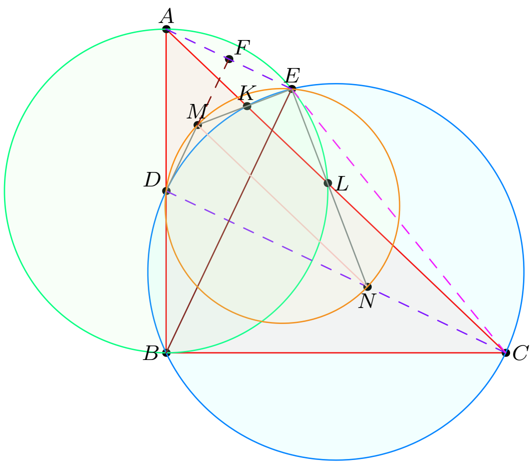


First, note that since $BCDE$ is cyclic, $\angle CED = 180^\circ - \angle CBD = 90^\circ$. Thus $\overline{CE}$ is tangent to $\omega$ and $\overline{CD}$ bisects $\overline{BE}$. Note that since $\widehat{CB} = \widehat{CE}$, $\angle CDB = \angle CKE = \angle LKE$. Furthermore, $\angle KLE = \angle ALE = \frac{\widehat{AE}}{2} = \angle ABE = \angle DBE = \angle DCE = \angle DCB$. Thus $\triangle EKL \sim \triangle BDC$, and $\angle NEM = \angle LEK = 90^\circ$.

Note that $CD$ is the diameter of the circumcircle of $\triangle BCD$, thus $90^\circ = \angle CDM = \angle NDM$. Thus $DEMN$ is cyclic, and $\angle DNM = \angle DEM = \angle DEK = \frac{\widehat{DK}}{2} = \angle DCK$. Therefore, we have $\overline{MN} \parallel \overline{CK}$, and $\triangle EMN \sim \triangle EKL \sim \triangle BDC$.

Let $F$ be the intersection between lines $AE$ and $DM$. Note that $F$ is the midpoint of $\overline{AE}$. Next, note that since $\overline{BC}$ and $\overline{CE}$ are tangent to $\omega$, line $AC$ is the $A$-symmedian of $\triangle ABE$. Since $\overline{DF} \parallel \overline{BE}$, there exists a homothety with center $A$ that maps $\triangle ABE$ to $\triangle ADF$, and thus $AC$ is the $A$-symmedian $\triangle ADF$.

Note that $\angle MAE = \angle MEA$. Since $\angle AEB = \angle MEN = 90^\circ$, $\angle MEA = \angle NEB$. Note that $\angle NEB = \angle LEB = \angle LAB$. Thus $\angle MAE = \angle LAB$, and $\angle MAB =$

$\angle LAE$. This implies line $AM$ is the reflection of $AC$ over the $A$-angle bisector of $\triangle ADF$. Since $AC$ is the $A$-symmedian of $\triangle ADF$, $\overline{AM}$ is a median, and thus $M$ is the midpoint of $\overline{DF}$.

Note that since $DEMN$ is cyclic, the circumradius of $\triangle DMN$ is equivalent to $\frac{MN}{2}$. Note that $\triangle BCD \sim \triangle EBA \sim \triangle FDA$. Since $CD = \sqrt{10^2 + 21^2} = \sqrt{541}$ and $DA = 10$, we know $DM = MF = \frac{21}{2} \cdot \frac{10}{\sqrt{541}} = \frac{105}{\sqrt{541}}$. Similarly, $AH = 10 \cdot \frac{10}{\sqrt{541}} = \frac{100}{\sqrt{541}}$, thus by the Pythagorean Theorem, $ME = AM = \frac{145}{\sqrt{541}}$. This yields $MN = DC \cdot \frac{ME}{DB} = \frac{29}{2}$. Thus $\frac{m}{n} = \frac{29}{4}$, and $m + n = \mathbf{033}$. ■

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15. *(reaganchoi)* Let $f(x) = x^4 - 17x^2 + 17$. Let $\mathcal{S}$ be the set of positive integers $a$, with $a \leq f(2020)$, such that $a^{2a} - 1$ is divisible by $2021$. Choose a random element $b$ in $\mathcal{S}$. The probability that $b^b + 1$ is divisible by $2021$ can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find the remainder when $m + n$ is divided by $1000$.

*Answer:* ***201***

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**Solution (P_Groudon)** The prime factorization of 2021 is $43 \cdot 47$. It is clear that no element $a$ in $\mathcal{S}$ can be divisible by 43 or 47, since then, it would be impossible for $a^{2a} - 1$ to be divisible by 2021.

To start, by Fermat's Little Theorem, $X^{42} \equiv 1 \pmod{43}$ for all integers $X$ not divisible by 43. Thus, $(X + 43 \cdot 42)^{X + 43 \cdot 42} \equiv X^X \pmod{43}$. Hence, under modulo 43, the value $X^X$ is periodic with length $43 \cdot 42$, where the base is taken modulo 43 and the exponent is taken modulo 42.

Similarly, $X^{46} \equiv 1 \pmod{47}$, implying that $(X + 47 \cdot 46)^{X + 47 \cdot 46} \equiv X^X \pmod{47}$. Hence, under modulo 47, the value $X^X$ is periodic with length $47 \cdot 46$, where the base is taken modulo 47 and the exponent is taken modulo 46.

Merging these two observations, the value of $X^X \pmod{2021}$ is periodic with length $\operatorname{lcm}(43 \cdot 42, 47 \cdot 46) = 2 \cdot 43 \cdot 47 \cdot 21 \cdot 23$. Call this period length $p$.

Note that

$$f(x) = (x^2 - 1)(x^2 - 16) + 1 = (x + 4)(x + 1)(x - 1)(x - 4) + 1,$$

so plugging in $x = 2020$ gives

$$f(2020) = 2024 \cdot 2021 \cdot 2019 \cdot 2016 + 1.$$

It can be seen that $f(2020) - 1$ is divisible by $p$. Let $c = \frac{f(2020) - 1}{p}$. Computing $c$,

$$c = \frac{2024 \cdot 2021 \cdot 2019 \cdot 2016}{2 \cdot 43 \cdot 47 \cdot 21 \cdot 23} = 88 \cdot 2019 \cdot 48.$$

Thus, as $a$ ranges from 1 to $f(2020)$, inclusive, it will make $c$ complete cycles of length $p$ and then achieve 1 last integer that is $1 \pmod{p}$. Since $p$ is divisible by 2021, the last integer that is $1 \pmod p$ will be in $\mathcal{S}$, but if $b = f(2020)$, then $b^b + 1$ will not be divisible by 2021.

Let $\mathcal{B}$ be the number of integers $b$ such that $1 \le b \le p$ and $b^b \equiv -1 \pmod{2021}$. Similarly, let $\mathcal{A}$ be the number of integers $a$ such that $1 \le a \le p$ and $a^{2a} \equiv 1 \pmod{2021}$. Since $X^X \pmod{2021}$ repeats with period $p$, and there are $c$ complete cycles of $p$ as $a$ varies (with one last integer), the desired probability is given by

$$\frac{c \cdot \mathcal{B}}{c \cdot \mathcal{A} + 1}.$$

Now, it remains to compute $\mathcal{B}$ and $\mathcal{A}$.

Let $u$ and $v$ be primitive roots modulo 43 and 47, respectively. These exist, since 43 and 47 are both prime. By a property of a primitive root, $u^{42} \equiv 1 \pmod{43}$ and $\{u, u^2, u^3, \ldots, u^{42}\} \equiv \{1, 2, 3, \ldots, 42\} \pmod{43}$. Essentially, as an integer $k$ ranges from 1 to 42, the value $u^k \pmod{43}$ uniquely achieves all nonzero residues modulo 43, so the value is uniquely determined by $k \pmod{42}$. Similarly, the value $v^k \pmod{47}$ is determined by $k \pmod{46}$.

To compute $\mathcal{B}$, let $b \equiv u^i \pmod{43}$ and $b \equiv v^j \pmod{47}$ for some integers $i$ and $j$. Then,

$$b^b \equiv -1 \pmod{2021} \iff u^{ib} \equiv -1 \pmod{43} \text{ and } v^{jb} \equiv -1 \pmod{47}$$

This implies that $ib \equiv 21 \pmod{42}$ and $jb \equiv 23 \pmod{46}$. Clearly, $b$ cannot be even and must be odd, which determines $b \pmod 2$. Since $i \pmod{42}$ determines $b \pmod{43}$, $j \pmod{46}$ determines $b \pmod{47}$, it suffices to examine $i$ and $b \pmod{42}$ and $j$ and $b \pmod{46}$ to determine $b \pmod p$, which determines $b^b \pmod{2021}$.

Decomposing the congruences to have a prime power modulus (which is valid by Chinese Remainder Theorem),

$$\begin{aligned} ib &\equiv 1 \pmod 2 \\ ib &\equiv 0 \pmod 3 \\ ib &\equiv 0 \pmod 7 \\ jb &\equiv 1 \pmod 2 \\ jb &\equiv 0 \pmod{23} \end{aligned}$$

From the first and fourth congruences, $i \equiv j \equiv b \equiv 1 \pmod 2$.

There are $3^2 - 2^2 = 5$ pairs $(i, b)$ in modulo 3 that satisfy the second congruence, since there are $3^2$ pairs in general, but $2^2$ of these pairs have neither variable congruent to 0. Similarly, there are $7^2 - 6^2 = 13$ pairs $(i, b)$ in modulo 7 that satisfy the third congruence, and $23^2 - 22^2 = 45$ pairs $(i, b)$ in modulo 23 that satisfy the last congruence.

By Chinese Remainder Theorem, the values of $i$ and $b$ in modulo 2, 3, and 7, as well as $j$ and $b$ in modulo 2 and 23 can be independently chosen. Hence, the total number of combinations of $(i, b) \pmod{42}$ in tandem with $(j, b) \pmod{46}$ is $5 \cdot 13 \cdot 45 = 2925$. Thus, $\mathcal{B} = 2925$.

To compute $\mathcal{A}$, let $a \equiv u^i \pmod{43}$ and $a \equiv v^j \pmod{47}$ as before. Then,

$$a^{2a} \equiv 1 \pmod{2021} \iff u^{2ia} \equiv 1 \pmod{43} \quad \text{and} \quad v^{2ja} \equiv 1 \pmod{47}.$$

This implies that $ia \equiv 0 \pmod{21}$ and $ja \equiv 0 \pmod{23}$. These are almost exactly the same congruences found when computing $\mathcal{B}$; the difference is that here, the parity of the three variables can be freely chosen. Hence, $\mathcal{A} = 2^3 \cdot \mathcal{B}$.

Returning to the requested probability, it is

$$\frac{c \cdot \mathcal{B}}{c \cdot 8\mathcal{B} + 1}.$$

It is clear the numerator and denominator are relatively prime, so $m + n = 9c \cdot \mathcal{B} + 1$, where $c = 88 \cdot 2019 \cdot 48$ from earlier and $\mathcal{B} = 2925$.

Obviously, $c$ is divisible by 8, so $m + n \equiv 1 \pmod{8}$. To compute $m + n \pmod{125}$, note that 2925 is divisible by 25. Hence,

$$9c \cdot \mathcal{B} + 1 = 25(117 \cdot 88 \cdot 2019 \cdot 48 \cdot 9) + 1.$$

It remains to compute the stuff inside the parentheses modulo 5, which is $3 \pmod{5}$. Hence, $m + n \equiv 3 \cdot 25 + 1 \equiv 76 \pmod{125}$. Merging the two congruences implies $m + n \equiv \mathbf{201} \pmod{1000}$. ■

**Remark.** By the way, $m + n = 224,506,339,201$, in case you're wondering.