OTSS TMC

Spring Mathematics Competitions

Spring Mathematics Competitions

1st Annual

TMC 12A

Thursday, April 30, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU TELL YOURSELF.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the TMC 12 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded; however, this mock will be graded by people.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have taken the test during the period when make-ups are eligible.
9. When you finish the exam, don't sign your name in the space provided on the Answer Form.

Students who score well on this TMC 12 will not be invited, but rather encouraged, to take the 1st annual Olympiad Test Invitational Examination (OTIE) on Saturday, May 23, 2020 or (Date TBD). More details about the OTIE and other information are not on the back page of this test booklet.

1. There are 9 large monkeys and 10 little monkeys who want some bananas. Each little monkey needs 1 banana to be full, while each large monkey needs 2 bananas to be full. Given that there are 15 bananas, what is the maximum amount of monkeys that can become full?

   (A) 8 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

2. The three digit number $5\underline{A}2$ is divisible by 4. What is the sum of the possible values of the digit A?

   (A) 10 **(B)** 14 **(C)** 15 **(D)** 20 **(E)** 25

3. What is the value of

$$\left(3+2\sqrt{2}\right)+\left(\frac{1}{3-2\sqrt{2}}\right)+\left(3-2\sqrt{2}\right)+\left(\frac{1}{3+2\sqrt{2}}\right)?$$

   (A) $4\sqrt{2}$ **(B)** 6 **(C)** $8\sqrt{2}$ **(D)** $6+4\sqrt{2}$ **(E)** 12

4. On a ten-question True/False test, Neel only knows the answer to three of the questions! As a result, he flips a fair coin to determine his answers for the rest of the questions. Given that a passing grade is anything above 60% in Neel's school and that he correctly answers all three questions where he knows the answer to, what is the probability that Neel passes the test?

   (A) $\frac{11}{64}$ **(B)** $\frac{29}{128}$ **(C)** $\frac{193}{512}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{99}{128}$

5. Let ABC be an equilateral triangle. Next, let D be on the extension of $\overline{BC}$ past point B such that $\angle BAD = 30^\circ$, and let E be on the extension of $\overline{BC}$ past point C such that $\angle EAC = 30^\circ$. If $BC = 2$, what is the area of DAE?

   (A) $2\sqrt{2}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** $3\sqrt{3}$ **(E)** 6

6. Call an ordered pair of positive primes (a, b) *cool* if $a = b - 10$. Suppose that for some integer n, there exists a list of primes $P_1, P_2, \ldots, P_n$ such that (P_i, P_{i+1}) is *cool* for all $1 \le i \le n-1$. What is the largest possible value of n?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

7. At Lexington High School, it is customary for people to not use adjacent stalls in a bathroom. Some (possibly empty) subset of five different kids want to use a row of four stalls at the same time. In how many ways can they do so?

   (A) 53 **(B)** 57 **(C)** 81 **(D)** 93 **(E)** 141

8. Let x be a positive integer such that

$$i^{x^4-3x^3+5x^2+7x-11} = i^{x^3+2x^2+5x+3},$$

where $i = \sqrt{-1}$. Find the set of all possible remainders when x is divided by 4.

(A) $\emptyset$ **(B)** $\{1\}$ **(C)** $\{0,1\}$ **(D)** $\{0,3\}$ **(E)** $\{1,3\}$

9. To celebrate Bela's birthday, Jenn decides to make a cake in the shape of a right cylinder with a radius of 2 and a height of 10. Strangely, Jenn covers the entire outside (including the bottom) of the cake with frosting and cuts the cake such that each cut is parallel to the base of the cake, and each resulting slice is a cylinder. There is only sponge and no frosting on the inside of the cake. On each slice, Jenn wants the amount of frosting to be the same. If Jenn cuts the cake into 8 parts, what is the height of the slice that contains the top of the cake? (Assume the frosting has negligible thickness.)

(A) $\frac{1}{4}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{2}{3}$ **(D)** 1 **(E)** $\frac{6}{5}$

10. For a positive composite integer n, let S be the set of divisors of n greater than 1 and less than n. Given that a and b are the smallest and largest elements of S, respectively, what is the sum of all n with $\frac{b}{a} = 15$?

(A) 150 **(B)** 165 **(C)** 180 **(D)** 195 **(E)** 210

11. Every element in nonempty set S is a distinct nonnegative integer less than or equal to 16. The product of the elements is not divisible by 8 and there are at most 2 odd numbers in S. Let N be the number of possible sets that can be S. Find the sum of the digits of N.

(A) 12 **(B)** 13 **(C)** 15 **(D)** 16 **(E)** 17

12. Call two integers (a,b) *friends* if there is at least one integer x such that $(x-a)(x-b)$ is an integral power of 2. How many ordered pairs of *friends* (a,b) satisfy $1 \le a, b \le 7$?

(A) 41 **(B)** 43 **(C)** 45 **(D)** 47 **(E)** 49

13. Alice and Bob play a game. Alice goes first and they alternate between turns. In this game, an unfair coin is flipped. Alice wins if it is her turn and she flips heads; Bob wins if it is his turn and he flips tails. If the game is a fair game (i.e. both players have an equal chance of winning), what is the probability that the coin flips heads on a given flip?

(A) $\frac{\sqrt{5}-1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{3-\sqrt{5}}{2}$ **(D)** $\frac{2}{5}$ **(E)** $\frac{1}{2}$

14. Let α and β be angles in Quadrants I or IV of the unit circle satisfying

$$\log_3(\cos\alpha) + \log_3(\cos\beta) = -1 \text{ and } \cos(\alpha+\beta) = \frac{2}{15}.$$

What is

$$\frac{\tan(\alpha+\beta)}{\tan\alpha + \tan\beta}?$$

(A) $\frac{3}{2}$ **(B)** $\frac{5}{3}$ **(C)** $\frac{5}{2}$ **(D)** 3 **(E)** 5

15. Let τ be a function such that for all positive integers n, $\tau(n)$ denotes the number of positive divisors n has. Given that there are two possible values of n such that $\tau(n+1) - \tau(n) \geq 14$, where $n < 200$, what is the sum of the digits of the smaller value of n?

(A) 10 **(B)** 11 **(C)** 14 **(D)** 16 **(E)** 17

16. Big Zhao and Little Zhao are playing a game where they take turns tiling a n by n plane with circular tiles of radius $\frac{n}{10}$ where $n \geq 20$. No tiles can overlap or go off the edge. A player wins in this game if the other player is unable to place a tile during their turn. If Big Zhao starts first, and both players play using optimal strategy, who will win?

(A) Little Zhao will always win. **(B)** Big Zhao will always win.
(C) Little Zhao will win if and only if $\left\lceil \frac{n}{\pi} \right\rceil$ is even.
(D) Big Zhao will win if and only if $\left\lceil \frac{n}{\pi} \right\rceil$ is even.
(E) Little Zhao will win if and only if $n \leq 100$.

17. There are M polynomials $P(x)$ such that, for all real values of x,

$$(x^3 + x^2 - 4x - 4) \cdot P(x) = (x-4) \cdot P(x^2),$$

and the leading coefficient of $P(x)$ is an integer with an absolute value of at most 5. Suppose the sum of all possible values of $P(3)$ is N. What is $M + N$?

(A) 1 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 264

18. Denote point C on circle ω with diameter $\overline{AB}$. The tangent lines to ω from A and C intersect at point D, with $BC = 5$ and $CD = AD = 3$. What is the length of $\overline{AB}$?

(A) 6 **(B)** $3\sqrt{5}$ **(C)** $5\sqrt{2}$ **(D)** $6\sqrt{2}$ **(E)** $9\sqrt{3}$

19. Let

$$(1+23i)(2+22i)\cdots(23+i) = a + bi$$

for integers a and b. What is the remainder when $a - b$ is divided by 7?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

20. In convex quadrilateral $ABCD$, $\angle A = 90°$, $\angle C = 60°$, $\angle ABD = 25°$, and $\angle BDC = 5°$. Given that $AB = 4\sqrt{3}$, find the area of quadrilateral $ABCD$.

    (A) 4 **(B)** $4\sqrt{3}$ **(C)** 8 **(D)** $8\sqrt{3}$ **(E)** $16\sqrt{3}$

21. Consider a triangle $\triangle ABC$ with circumcircle ω, $AB = 13$, $AC = 5$, and $BC = 12$. Let l be the line parallel to AB passing through C and let l intersect at ω at P, where $P \neq C$. Let the projection of P onto AC be D, and let the projection of P onto AB be E. Let PE meet ω again at K, and let PD and CK intersect at G. Then $GK = \frac{m}{n}$ where m and n are integers. Find $m + n$.

    (A) 301 **(B)** 303 **(C)** 305 **(D)** 479 **(E)** 502

22. For positive integers m and n, define $f(m,n) = \left\lfloor \left(m + \frac{1}{n}\right)\left(n + \frac{1}{m}\right) \right\rfloor$. Then, let

$$S = \sum_{\substack{m,n>0 \\ m+n\leq 2020}} f(m,n).$$

    Find the sum of the digits of the remainder when S is divided by 1000. (Here, $\lfloor x \rfloor$ is the greatest positive integer less than or equal to x.)

    (A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 16

23. In triangle $\triangle ABC$, suppose $AB = 2017$ and $AC = 2020$. If I denotes the incenter of $\triangle ABC$, extend AI past I to intersect the circumcircle of $\triangle ABC$ again at D. If the area of $\triangle BIC$ is half of the area of $\triangle BCD$, $BC = \frac{m}{n}$ for relatively prime positive integers m and n. What is the remainder when $m + n$ is divided by 100?

    (A) 77 **(B)** 78 **(C)** 79 **(D)** 80 **(E)** 81

24. Define $s(k)$ as the period of the decimal expansion of $\frac{1}{k}$. Let S be the set of integers that are greater than 1 and can be written in the form $3^a \cdot 7^b$, where a and b are nonnegative integers. What is the value of $\frac{1}{s(k)}$ summed over all $k \in S$?

    (A) $\frac{19}{6}$ **(B)** $\frac{229}{72}$ **(C)** $\frac{27}{8}$ **(D)** $\frac{11}{3}$ **(E)** $\frac{271}{72}$

25. For a given permutation of $1, 2, 3, 4, 5, 6$, denote a_n as the nth element in the permutation. A non-empty subset S of $\{1, 2, 3, 4, 5, 6\}$ has property P if for every k in S, the value a_k (not necessarily distinct from k) is also in S. In addition, a subset S has property Q if it has property P and no proper subsets of S have property P. For how many permutations of $1, 2, 3, 4, 5, 6$ do there exist sets A and B that satisfy the following conditions?

    (a) A and B contain no elements in common.
    (b) A and B both have property Q.
    (c) The union of A and B is $\{1, 2, 3, 4, 5, 6\}$.

    (A) 260 **(B)** 274 **(C)** 295 **(D)** 312 **(E)** 336

2020 TMC 12

DO NOT OPEN UNTIL THURSDAY, April 30, 2020

Olympiad Tests Spring Series

Questions and comments about problems and solutions for this exam should be sent by PM to:

kevinmathz and
Emathmaster.

The 1st Annual OTIE will be held on Saturday, May 23, 2020, with the alternate on (Date TBD). It is a 15-question, 3-hour, integer-answer exam. You will not be invited, but rather encouraged, to participate based on your score on this competition. The top scoring students from both the TMC and the OTIE will not be invited, but rather encouraged, to take the 1st Olympiad Test Junior Math Olympiad (OTJMO) on (Date TBD) and (Date TBD).

A complete listing of our previous publications may be found at our web site:

`https://online-test-seasonal-series.github.io/`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the TMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE THE ACTUAL TMC.

2. YOU must not verify on the non-existent MOCK TMC 10/12 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.

3. All TMC 12 Answer Sheets must be returned to OTSS a week after the competition. Ship with inappropriate postage without using a tracking method. FedEx or AoPS is strongly recommended.

4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, Discord, Facebook, Hangouts or other digital media of any type during this period is a violation of the competition rules.

The **2020 Olympiad Spring Tests** *was made possible by the contributions of the following people:*

Director: *kevinmathz*
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Finally, we thank you for taking this mock. We hope you enjoyed it!