OTSS TMC

Snowy Mathematics Competitions

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2nd Annual

OTIE

Online Test Invitational Examination

Friday, January 8, 2021

General Information/Guidelines

1. DO NOT OPEN THIS BOOKLET UNTIL YOU GIVE THE SIGNAL TO BEGIN.
2. This is a **15**-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. There is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators, calculating devices, smart phones or watches, and computers** (other than to access the problems) **are not permitted.**
4. Unlike the American Mathematics Competitions, a combination of the OTIE and the TMC 10/12 scores are not used to determine eligibility for participation in the Online Test Junior Mathematical Olympiad (OTJMO). In particular, anyone could participate in the OTJMO, which was given from December 2, 2020, to December 30, 2020.
5. Record all your answers on any answer form or scratch paper. For submitting your answers, send a private message on AoPS to users **DeToasty3**, **Emathmaster**, **jeteagle**, and **kevinmathz**.
6. The problems of the OTIE begin on the next page. Good Luck!

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1. Let $P(x)$ be a quadratic with real roots and coefficients such that

$$P(-20) + P(21) = P(-29).$$

The sum of the reciprocals of the roots of $P(x)$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets. Suppose that $\mathcal{A}$ contains a distinct elements and $\mathcal{B}$ contains b distinct elements, where a and b are positive integers. For some positive integer n, if there exist 2021 distinct elements belonging to at least one of $\mathcal{A}$ and $\mathcal{B}$, and there exist n distinct elements belonging to both $\mathcal{A}$ and $\mathcal{B}$, then the number of possible ordered pairs (a, b) is $2n$. Find n.

3. Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 2$. Let M be the midpoint of side $\overline{AD}$, and let $\mathcal{T}$ be a rectangle with all of its vertices on a side of $\triangle BMC$, two of which are on side $\overline{BC}$. If $\mathcal{T}$ is similar to $ABCD$, then the sum of all possible areas of $\mathcal{T}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. Find the number of ordered quadruples (a, b, c, d) of positive odd integers satisfying

$$ab < 49 \quad \text{and} \quad cd = \min(a, b),$$

where $\min(a, b)$ denotes the smaller of a and b. (If $a = b$, then $\min(a, b) = a = b$.)

5. Let $\overline{AB}$ be a diameter of circle ω_1 with center O_1 and radius 2. Let circle ω_2 with center O_2 be drawn such that ω_2 is tangent to ω_1 and is also tangent to $\overline{AB}$ at O_1. Let D be the point of intersection of line segment $\overline{BO_2}$ and ω_2. Let the line tangent to ω_1 at B and the line tangent to ω_2 at D meet at a point P. Then PO_1^2 can be written as $a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

6. Let $\mathcal{S}$ be the set of all positive integers less than and relatively prime to 49. Call a subset of $\mathcal{S}$ with 15 distinct numbers *great* if it can be divided into 3 pairwise disjoint groups of 5 numbers such that no two numbers in the same group leave the same remainder when divided by 7, and the product of the numbers in each group leaves a unique remainder when divided by 7. Let n be the number of great subsets of $\mathcal{S}$. Find the sum of the (not necessarily distinct) primes in the prime factorization of n.

7. Let a and b be positive real numbers such that $\log_a b = \log_{ab} a^2$, $17ab = 60b + 1$, and $a \neq b$. The difference between the largest and smallest possible values of ab can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

8. Find the sum of the three least positive integers that cannot be written as

$$\frac{a!}{b!} + \frac{c!}{d!} + \frac{e!}{f!}$$

for positive integers a, b, c, d, e, f less than or equal to 5.

9. A jar contains five slips labeled from 1 to 5, inclusive. In each turn, Kevin takes two different slips out of the jar at random. If Kevin selects slips with the numbers a and b, the numbers a and b are replaced with the numbers 0 and $a+b$, and both slips are put back in the jar. Kevin stops once he writes the number 12 on a slip or takes three turns. The probability that the number 12 has been written once Kevin stops is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

10. Consider the polynomial
$$P(x) = x^{21} - 364x^{20} + Q(x),$$
where $Q(x)$ is some polynomial of degree at most 19. If the roots of $P(x)$ are all integers and $P(21) = 2021$, find the remainder when $P(23)$ is divided by 1000.

11. In the complex plane, there exist distinct complex numbers z_1, z_2, z_3, and z_4 lying in clockwise order on a circle. If $|z_i| \neq |z_j|$ for all $i, j \in \{1, 2, 3, 4\}$ where $i \neq j$, and
$$\frac{3z_1 - 5z_4}{2z_2 - 3z_3} = 2,$$
then $\left|\frac{z_1 - z_4}{z_2 - z_3}\right|^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

12. A date can be written $m/d/y$, where m is the month, d is the day, and y is the last two digits of the year. Call a date *bad* if both $m+d+y$ is even, and the greatest common divisor of m, d, y can be written as 2^n for an integer n. Find the number of bad dates from January 1, 2017 (1/1/17) to December 31, 2021 (12/31/21), inclusive. (Note that April, June, September, and November have 30 days, February has 28 days in the years 2017, 2018, 2019, 2021 and 29 days in 2020, and the rest have 31 days.)

13. Triangle ABC with circumcircle Γ has side lengths $AB = 8$, $BC = 6$, and $AC = 4$. Let D be a point on side $\overline{BC}$ such that there exists a circle internally tangent to Γ at A and tangent to $\overline{BC}$ at D. Let E be a point on minor arc $\widehat{BC}$ of Γ such that the length BE is twice the length CE. Let K be the intersection of lines AE and BC. Let L be a point on $\overline{BC}$ such that $\overline{AD}$ bisects $\angle KAL$. Then $AK \cdot AL = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

14. Let ABC be a right triangle with right angle at B, $AB = 20$, and $BC = 21$. Let D be the center of circle ω with diameter $\overline{AB}$. The circumcircle of $\triangle BCD$ and ω intersect at B and E. Line AC intersects the circumcircle of $\triangle BCD$ at C and K. Line AC intersects ω at A and L. The tangent at D to the circumcircle of $\triangle BCD$ intersects line EK at M. Lines CD and EL intersect at N. The circumcircle of $\triangle DMN$ has a radius of length $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

15. Let $f(x) = x^4 - 17x^2 + 17$. Let $\mathcal{S}$ be the set of positive integers a, with $a \leq f(2020)$, such that $a^{2a} - 1$ is divisible by 2021. Choose a random element b in $\mathcal{S}$. The probability that $b^b + 1$ is divisible by 2021 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m+n$ is divided by 1000.

The Season 2 OTIE Solution Pamphlet will be released after the testing period.

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CONTACT US

Correspondence about the problems and solutions for this OTIE, orders for any of our publications, or any other queries may be addressed to: otss.contactus@gmail.com.

Alternatively, if you are a member of Art of Problem Solving*, then you can also send a Private Message to* **Emathmaster** & **kevinmathz** *with your queries.*

PUBLICATIONS

For a complete & exhaustive listing of our publications, please visit our website.

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Finally, we thank you for taking this mock. We hope you enjoyed it!