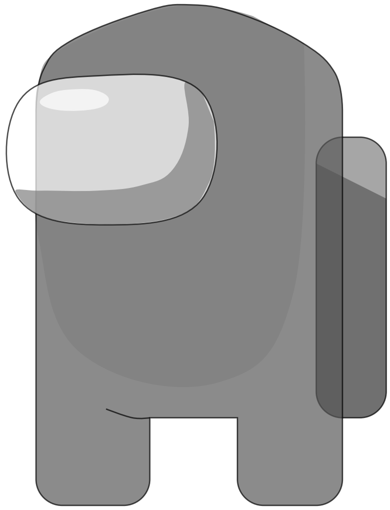
OTSS TMC

Test Math Competitions

OTSS'

3rd Season of

TIME



Timed Invite to aiME

General Guidelines — Timed Invite to aiME (TIME)

1. DO NOT OPEN THIS BOOKLET UNTIL YOU GIVE THE SIGNAL TO BEGIN.
2. This is a **15**-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers. Each correct answer will yield you **1 point**. There is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators, calculating devices, smart phones or watches, *and* computers** (other than to access the problems) **are not permitted.**
4. Unlike the American Mathematics Competitions, a combination of the TIME and the TMC 10/12 scores are not used to determine eligibility for participation in the OTJMO. In particular, anyone can participate in the OTJMO, which will be given from August 1 to August 29, 2021.
5. Record all your answers on any answer form or scratch paper. For submitting your answers, send a private message on AoPS to users **DeToasty3**, **Emathmaster**, **jeteagle**, and **kevinmathz.**
6. The problems of the TIME begin on the next page. Good Luck!

---

1. A fair coin is flipped 2021 times. Consider the conditions below:

   - If the 20th flip comes up heads, then so must both the 21st and 22nd flips.
   - If the 201st flip comes up heads, then so must the 202nd flip.

   The probability that both conditions are satisfied can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

2. Find the smallest positive integer n such that the sum
$$1^2 + 2^2 + 3^2 + \cdots + n^2$$
is divisible by 36.

3. In trapezoid $ABCD$ with parallel sides $\overline{AB}$ and $\overline{CD}$, the lengths AB and CD as well as the area of $ABCD$ all have integer values. Let E and F be the midpoints of sides $\overline{AD}$ and $\overline{BC}$, respectively. Given that the ratio of the area of $ABFE$ to the area of $EFCD$ is $19 : 49$, find the smallest possible area of trapezoid $ABCD$.

4. Find the number of positive integers $n \leq 1000$ such that
$$n(n+1)(n+\tfrac{1}{2})(n+\tfrac{1}{3})(n+\tfrac{1}{4})$$
is an integer.

5. Let $P(x) = x^2 + ax + b$ be a quadratic with not necessarily distinct real roots r and s, where a and b are positive integers. If the quadratic $Q(x) = x^2 + 2ax + 3b$ has real roots r and $t \neq s$, find the maximum value of $P(1) + Q(1)$ less than 1000.

6. Find the number of ordered triples of not necessarily distinct positive integers (p, q, r) with $1 \leq p, q, r \leq 16$ such that both the products pq and qr are perfect squares.

7. During fencing practice, six people split into three pairs to spar. Two more people join afterwards. The eight people then randomly rearrange themselves into four pairs to spar. The probability that no one spars with someone that they previously sparred with is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

8. In triangle ABC, $AB = 15$, $AC = 13$, and $BC = 14$. The angle bisector of $\angle BAC$ intersects $\overline{BC}$ at D, and the angle bisector of $\angle ABC$ intersects $\overline{AC}$ at E. Let P be the projection from A to $\overline{BE}$, and let Q be the projection from B to $\overline{AD}$. The length PQ can be expressed as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

9. For positive integers m and n with $mn \geq 3$, draw a $m \times n$ grid and randomly place the numbers $1, 4, 7, \ldots, 3mn-2$ into the grid, with each square containing exactly one number and each number contained within exactly one square, such that every pair of numbers differing by three are in adjacent squares. If the expected value of the number in any one of the corner squares is 202, find the sum of all values of n.

10. In triangle ABC with $AB = 26$, $BC = 28$, and $AC = 30$, let O and $\overline{AD}$ be the center and a diameter of the circumcircle of $\triangle ABC$, respectively. Two distinct lines pass through O, are parallel to $\overline{AB}$ and $\overline{AC}$, respectively, and meet side $\overline{BC}$ at points M and N, respectively. Let lines DM and DN meet the circumcircle of $\triangle ABC$ at points P and Q, respectively, both distinct from D. Find the area of $APDQ$.

11. A group of 48 people are in a gathering, where it is known that any two people are either colleagues or rivals. Also, for any set of three people $\{A, B, C\}$, if A and B are colleagues, and A and C are rivals, then B and C are also rivals. Finally, there does not exist any set of four people such that all 6 combinations of two people from this set are rivals. Find the maximum possible number of pairs of people that are rivals.

12. Non-negative real numbers r_1, r_2, and r_3 satisfy
$$r_1^2 + r_2^2 + r_3^2 + 70 = 35r_1r_2 + 7r_1r_3 + 5r_2r_3.$$
The least possible value of $r_1^2 + 2r_2^2 + 3r_3^2$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. Six pebbles with weights $\{1, 2, 3, 4, 5, 6\}$ form the vertices of a regular hexagon in that order. The pebbles are colored either red or blue. A pebble with weight i has a probability $\frac{i}{7}$ of being colored red and a probability $\frac{7-i}{7}$ of being colored blue. The expected number of triangles formed by three red pebbles can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

14. Two circles ω_1 and ω_2 with radii 32 and 12, respectively, have centers 16 units apart. The length of the curve determined by the set of the centroid (intersection of the medians) of all triangles with ω_1 as their circumcircle and ω_2 as their inscribed circle is equal to $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. Find the remainder when the sum of all positive integers $n \le 1000$ that satisfy
$$\nu_p(1^{16} + 2^{16} + \cdots + n^{16}) = \nu_p(1 + 2 + \cdots + n) - 1$$
is divided by 1000, where p is a prime, and $n = p^k$ for some positive integer k.

Note. For a positive integer m and prime t, $\nu_t(m)$ denotes the largest integer such that m is divisible by $t^{\nu_t(m)}$.

The Season 3 TIME Solution Pamphlet will be released after the testing period.

OTSS Tests a Sarcastic Series

CONTACT US — *Correspondence about the problems and solutions for this TIME (or our publications), or any other queries may be addressed to:* otss.contactus@gmail.com.

Alternatively, if you are a member of Art of Problem Solving, *then you can also send a Private Message to* **Emathmaster** & **jeteagle** *with your queries.*

PUBLICATIONS — For a complete listing of our publications, please visit our website.

The Timed Invite to aiME

A program of the Online Test Sarcastic Series

Supported by major contributions from

Aathreyakadambi, ARMLlegend,
Awesome_guy, AwesomeYRY, azduncan,
DeToasty3, Emathmaster, Ish_Sahh, ivyzheng,
jeteagle, kevinmathz, MathCounts145, NJOY,
Orestis_Lignos, PCChess, reaganchoi,
rqhu, twinbrian, & zhao_andrew

Finally, we thank you for taking this mock. We hope you enjoyed it!