OTSS TMC

Spring Mathematics Competitions

Spring Mathematics Competitions

1st Annual

TMC 12B

Saturday, May 16, 2020

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU TELL YOURSELF.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the TMC 12 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded; however, this mock will be graded by people.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have taken the test during the period when make-ups are eligible.
9. When you finish the exam, don't sign your name in the space provided on the Answer Form.

The Committee on the Test Mathematics Competitions (CTMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CTMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this TMC 12 will not be invited, but rather encouraged, to take the 1st annual Olympiad Test Invitational Examination (OTIE) on Saturday, May 23, 2020 or (Date TBD). More details about the OTIE and other information are not on the back page of this test booklet.

1. In a bag of marbles, $\frac{3}{4}$ of the marbles are blue. If $\frac{1}{3}$ of the blue marbles are taken out of the bag, what fraction of the remaining marbles in the bag are blue?

   (A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

2. In equilateral triangle ABC with $AB = 1$, let M denote the midpoint of $\overline{BC}$ and N denote the midpoint of $\overline{AC}$. What is the area of AMN?

   (A) $\frac{\sqrt{3}}{32}$ **(B)** $\frac{\sqrt{3}}{16}$ **(C)** $\frac{3\sqrt{3}}{32}$ **(D)** $\frac{\sqrt{3}}{9}$ **(E)** $\frac{\sqrt{3}}{8}$

3. The sum of consecutive prime numbers p and q is a multiple of 20. What is the least possible value of the product $p \cdot q$?

   (A) 51 **(B)** 77 **(C)** 221 **(D)** 667 **(E)** 899

4. Let f be a function such that $f(f(x)) = \frac{x}{2} - 2$ for all real numbers x. If $f(f(f(f(y)))) = 100$ for some integer y, what is the sum of the digits of y?

   (A) 4 **(B)** 7 **(C)** 8 **(D)** 10 **(E)** 11

5. Shenlar has a sum S that is initially set equal to 0. For each integer n from 1 to 100 inclusive, Shenlar adds the value of $\frac{n}{2^n}$ to S if and only if 2^n is divisible by n. When S is written in binary (base-two), what is the sum of the digits of S after the point? (For example, if the number was 0.01101_2, the sum would be 3.)

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

6. Let rectangle $ABCD$ be such that $AB = CD = 8$ and $BC = AD = 5$. Construct four squares such that the sides of the rectangle are the diagonals of the squares. What is the sum of the areas of the regions that are in only one of the four squares?

   (A) 75.5 **(B)** 80 **(C)** 84.5 **(D)** 89 **(E)** 93.5

7. Given that x and y are positive real numbers, with x and y each less than $\frac{\pi}{2}$, that satisfy the equations $x + y = \frac{\pi}{2}$ and $\sin(x) + 2\cos(y) = \frac{3\sqrt{3}}{2}$, what is $|x - y|$?

   (A) 0 **(B)** $\frac{\pi}{12}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{4}$ **(E)** $\frac{\pi}{3}$

8. Let real numbers a, b, c, d satisfy the system of equations

$$\begin{aligned} \log_2 a + \log_3 b + \log_4 c &= 13 \\ \log_2 a + \log_3 b + \log_5 d &= 11 \\ \log_2 a + \log_4 c + \log_5 d &= 15 \\ \log_3 b + \log_4 c + \log_5 d &= 12. \end{aligned}$$

   Find the value of $a + b + c + d$.

   (A) 1722 **(B)** 4190 **(C)** 4762 **(D)** 7246 **(E)** 7256

9. Let O be the circumcenter of $\triangle ABC$ with circumradius 6. The internal angle bisectors of $\angle ABO$ and $\angle ACO$ meet $\overline{AO}$ at points D and E, respectively. If $AB = 9$ and $AC = 10$, then $DE = \frac{m}{n}$ for relatively prime positive integers m and n. What is $m + n$?

   (A) 23 **(B)** 29 **(C)** 33 **(D)** 41 **(E)** 57

10. Let (B, J) be an ordered pair of positive integers such that either $B + J = 60$ or $BJ = 60$. Bela is assigned the number B and Jenn is assigned the number J. Each of them only knows their own number and that either $B + J = 60$ or $BJ = 60$. Once they are assigned their numbers, Bela says, "I don't know your number." Then, Jenn replies, "I don't know your number." How many possible ordered pairs (B, J) exist, if Bela and Jenn always tell the truth and are infinitely intelligent?

    (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 or more

11. Arnold and Betty play a game. They each randomly write an integer between 1 and 6, inclusive, on a sheet of paper. Then, a fair six-sided dice is rolled. A person wins if the number they wrote down is closer to the number rolled on the dice than the other person's number is. The game results in a draw if the two numbers that were written are the same distance from the number rolled. What is the probability that Arnold wins?

    (A) $\frac{1}{18}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{7}{18}$ **(E)** $\frac{1}{2}$

12. For how many ordered pairs of positive integers (A, B), with $A \leq 3$ and $B \leq 2$, does the equation $x^3 - Ax - B = 0$ have a real solution x in the interval $[1.5, 2)$?

    (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

13. Edwin has two chess pieces that he places both on the center square of a 5×5 chessboard. He sets a border one square wide on the edges of the chessboard, leaving a 3×3 area in the middle. In one move, each piece moves as follows:

    - The white piece moves one square either vertically or horizontally and then two squares in a perpendicular direction.
    - The black piece moves one square either vertically or horizontally.

    Each piece moves repeatedly until it first lands on a square in the border, at which point it stops moving. If both pieces move randomly but always abide by their rules, what is the probability that the white and black pieces will end up on the same square after they each stop moving?

    (A) $\frac{1}{64}$ **(B)** $\frac{1}{16}$ **(C)** $\frac{1}{9}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

14. When the solutions to $y^4 + y^2(-1 + 2i\sqrt{3}) - 2 + 2i\sqrt{3} = 0$ are plotted in the complex plane, the area of the polygon with the roots as its vertices is A. Find A^2.

    (A) 12 **(B)** 16 **(C)** 21 **(D)** 24 **(E)** 36

15. There are two 2-digit prime numbers p less than 30 such that when the quantity $2^{2020} + 19^{1010}$ is divided by p, the result is an integer. What is the sum of these primes?

    (A) 30 **(B)** 36 **(C)** 42 **(D)** 46 **(E)** 52

16. In triangle ABC with $AB = 2$ and $AC = 4$, construct squares $YZAB$, $WXAC$, and $UVBC$ on the exteriors of sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$, respectively. Given that the sum of the squares of the side lengths of hexagon $XWUVYZ$ is 128, the length of $\overline{BC}$ can be expressed as $a\sqrt{b}$, where a and b are positive integers and b is not divisible by the square of any prime. Compute the value of $a + b$.

    (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

17. There are n integers x in the interval $2 \leq x \leq 2020$ such that when each of the values

$$\left\lfloor \frac{x^{10}}{x-1} \right\rfloor \text{ and } \left\lfloor \frac{x^9}{x-1} \right\rfloor$$

    are divided by 48, their remainders are equal. What is the sum of the digits of n? (Here, $\lfloor r \rfloor$ denotes the greatest integer less than or equal to r.)

    (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

18. A pyramid $ABCDE$ has square base $ABCD$ and apex E such that $AE = BE = CE = DE$. Suppose the planes determined by $\triangle ABE$ and $\triangle BCE$ form a $120°$ angle. If $AB = 2\sqrt{3}$, what is AE?

    (A) $\sqrt{3}$ **(B)** 3 **(C)** $2\sqrt{3}$ **(D)** 4 **(E)** $2\sqrt{5}$

19. Let $\overline{BC}$ be a line segment, and let F be a point on $\overline{BC}$. Construct points A and D, and let E be the intersection of $\overline{AC}$ and $\overline{BD}$ so that A, D, and E are all on the same side of $\overline{BC}$, and $\overline{AB} \parallel \overline{CD} \parallel \overline{EF}$. Given that lengths AB and CD are positive integers and $EF = 24$, find the number of possible ordered pairs of lengths (AB, CD). (Assume that all six points are distinct.)

    (A) 15 **(B)** 18 **(C)** 21 **(D)** 24 **(E)** 27

20. Richard writes the quadratic $f(x) = ax^2 + bx + c$ on a whiteboard, where a, b, and c are distinct nonzero complex numbers. Matthew sees Richard's quadratic, and rearranges the order of the coefficients (i.e. permutes the order of a, b, and c) to make his own six distinct quadratics: $g_1(x)$, $g_2(x)$, $g_3(x)$, $g_4(x)$, $g_5(x)$, and $g_6(x)$ (one of which is equal to $f(x)$). What is the minimum number of possible distinct roots of

$$\prod_{k=1}^{6} (f(x) + g_k(x))?$$

    (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 10

21. In a room with 10 people, each person knows exactly 4 different languages. A conversation is held between every pair of people with a language in common. If a total of 36 different languages are known throughout the room, and no two people have more than one language in common, what is the sum of all possible values of n such that a total of n conversations are held?

(A) 19 **(B)** 25 **(C)** 32 **(D)** 35 **(E)** 37

22. Let $(a \oplus b)$ denote the bitwise exclusive-or (XOR) of a and b. This is equivalent to adding a and b in binary (base-two), but discarding the "carry" to the next place value if it is applicable. For instance, $(1_2 \oplus 1_2) = 0_2$, $(1_2 \oplus 0_2) = 1_2$, and $(5 \oplus 3) = (101_2 \oplus 011_2) = 110_2$. How many ordered pairs of nonnegative integers (x, y) both less than 32 satisfy $(x \oplus y) > x \geq y$?

(A) 63 **(B)** 99 **(C)** 127 **(D)** 155 **(E)** 255

23. A real number x in the interval $0 \leq x \leq \frac{\pi}{2}$ satisfies the equation

$$\sin(x + \pi \sin(x)) = \cos(x + \pi \cos(x)).$$

Then, the value of $\sin(x)$ can be written as $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime. What is $a^2 + b^2 + c^2$?

(A) 26 **(B)** 42 **(C)** 52 **(D)** 66 **(E)** 86

24. Let O_1 be a circle with radius r. Let O_2 be a circle with radius between $\frac{r}{2}$ and r, exclusive, that goes through the center of circle O_1. Denote points X and Y as the intersections of the two circles. Let P be a point on the major arc $\widehat{XY}$ of O_1. Let $\overline{PX}$ intersect O_2 at A, strictly between P and X. Let $\overline{PY}$ intersect O_2 at B, strictly between P and Y. Let E be the midpoint of $\overline{PX}$ and F be the midpoint of $\overline{PY}$. If $AY = 100$, $AB = 65$, and $EF = 52$, what is BX?

(A) 104 **(B)** 105 **(C)** 106 **(D)** 107 **(E)** 108

25. There are N ordered pairs (x, y) of integers with $0 \leq x, y < 39375$ such that when each of the values

$$(x + y)^3 \text{ and } (x^3 + y^3)$$

are divided by 39375, their remainders are equal. How many positive integer divisors does N have?

(A) 72 **(B)** 96 **(C)** 108 **(D)** 144 **(E)** 192

2020 TMC 12

DO NOT OPEN UNTIL SATURDAY, May 16, 2020

Olympiad Tests Spring Series

Questions and comments about problems and solutions for this exam should be sent by PM to:

kevinmathz and
Emathmaster.

The 1st Annual OTIE will be held on Saturday, May 23, 2020, with the alternate on (Date TBD). It is a 15-question, 3-hour, integer-answer exam. You will not be invited, but rather encouraged, to participate based on your score on this competition. The top scoring students from both the TMC and the OTIE will not be invited, but rather encouraged, to take the 1st Olympiad Test Junior Math Olympiad (OTJMO) on (Date TBD) and (Date TBD).

A complete listing of our previous publications may be found at our web site:

`https://online-test-seasonal-series.github.io/`

****Administration On An Earlier Date Will Literally Be Impossible****

1. All the information needed to administer this exam is contained in the TMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL EVERY DAY BEFORE THE ACTUAL TMC.

2. YOU must not verify on the non-existent MOCK TMC 10/12 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.

3. All TMC 12 Answer Sheets must be returned to OTSS a week after the competition. Ship with inappropriate postage without using a tracking method. FedEx or AoPS is strongly recommended.

4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, Discord, Facebook, Hangouts or other digital media of any type during this period is a violation of the competition rules.

The **2020 Olympiad Spring Tests** *was made possible by the contributions of the following people:*

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Finally, we thank you for taking this mock. We hope you enjoyed it!