Gasdynamics task 1

2024/2025

Chapter 1

1.1. Prove that sound propagation is isentropic
1.2. Jump equations in thermodynamic variables
1.3. The t,x diagrams
1.4. One Strong Shock

Chapter 2

2.1. Linear Wave Interaction
2.2. Moving a compressible fluid
2.3. Waves interacting with a contact discontinuity
2.4. Linear theory: Driving a free piston in a tube

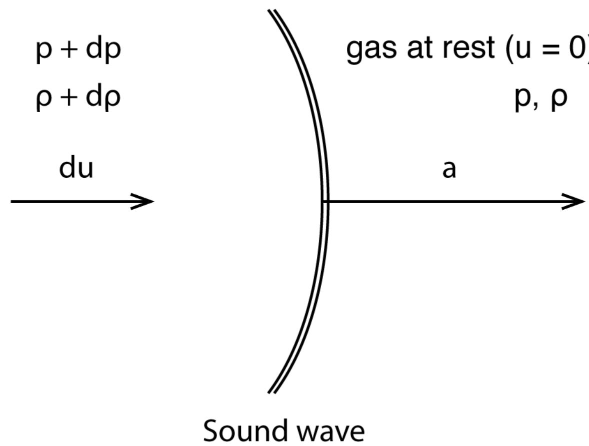
Chapter 3

3.1. Various Riemann problems
3.2. Compressions
3.3. Wave interactions
3.4. Fully non-linear: Moving a gas

Chapter 1

Problem 1.1. ***Prove that sound propagation is isentropic***

Consider a sound wave travelling with sound velocity a into a gas at rest, having pressure p and density ρ. When the sound wave has passed, the gas attains a (infinitesimally small) velocity du following the travelling sound wave. Due to the passage of the sound wave, the pressure and the density in the gas increases to ρ + dρ and to p + dp respectively, see figure below.



a. Take a small part of the sound wave and view it in laboratory frame as a one-dimensional unsteady process. Draw the picture of this process in a reference frame that moves with the sound wave. In this reference frame identify the velocities and thermodynamic variables on both sides of the sound wave.

b. Apply the one-dimensional conservation laws for mass and momentum for an inviscid compressible flow in integral form to derive that: $a^2 = {}^{dp}/_{d\rho}$.

c. Use energy conservation (and the result from the previous question) to show that sound propagation is an isentropic process: $ds = 0$ such that $a^2 = \left({}^{\partial p}/_{\partial \rho}\right)_s$.

Problem 1.2. ***Jump equations in thermodynamic variables***

Start from the jump equations for a steady shock and use the perfect-gas law

$$p = (\gamma - 1)\rho e,$$

to derive the following relations:

a. $[e] + \langle p \rangle [v] = 0,$

b. $\frac{[p]}{\langle p \rangle} + \gamma \frac{[v]}{\langle v \rangle} = 0.$

Remember that: $v = \frac{1}{\rho}$ = specific volume

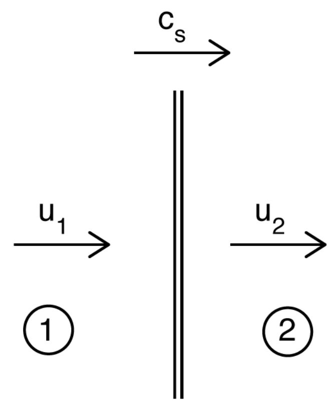
$$[\cdot] = (\cdot)_2 - (\cdot)_1,$$

$$\langle \cdot \rangle = \frac{1}{2}\{(\cdot)_1 + (\cdot)_2\}$$

c. What differential relations do these jump relations resemble? Comment on the distinction, if you can.

Problem 1.3. ***t,x diagrams***

In laboratory frame, we observe a shock moving with velocity c_s separating the pre-shock state 1 and the post-shock state 2. The velocities in these zones (states) are u_1 and u_2 respectively, see figure below.



Moving shock wave

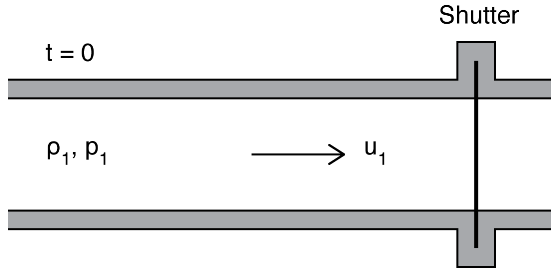
Draw the (t,x) diagram (including shocks and particle paths) for the following cases, give and explain the conditions for c_s, u_1 and u_2 with respect to each other to have a <u>valid</u> shock.

a)	$c_s > 0$		
b)	$c_s = 0$		
c1)	$c_s < 0$	$u_1 > 0$	$u_2 > 0$
c2)	$c_s < 0$	$u_1 > 0$	$u_2 = 0$
c3)	$c_s < 0$	$u_1 > 0$	$u_2 < 0$
c4)	$c_s < 0$	$u_1 = 0$	$u_2 < 0$
c5)	$c_s < 0$	$u_1 < 0$	$u_2 < 0$

Give typical values for the velocities in the limiting case of a strong shock in air ($\rho_2/\rho_1 \rightarrow 6$).

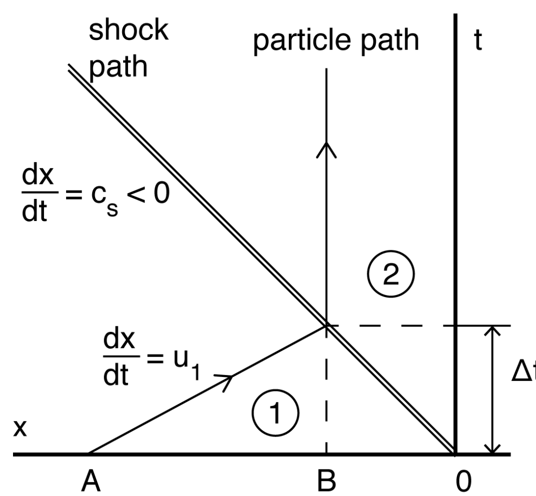
Problem 1.4. ***One Strong Shock***

A <u>uniform</u> hypersonic flow of <u>Helium</u> through a tube (state quantities labelled by subscript 1) is disturbed at t = 0, when a shutter at the end of the tube is suddenly closed. A shock recedes (moves to the left) from the closed end with speed c_s. Behind the shock the quantities are labelled by subscript 2.



a. <u>Conservation of mass</u>

Consider the (t,x) diagram for this flow. The gas, originally contained in the space interval AO, is shocked during a time interval Δt, and is then contained in the space interval BO. Using only this fact, (no credit for other approaches!) and the formula for the density ratio in the <u>very-strong-shock limit</u>, show that $c_s = -1/3\ u_1$.



b. <u>Conservation of momentum</u>
From the momentum jump equation for a moving shock and the formulas derived in (a), show that:

$$p_2 \approx 4/3\ \rho_1 u_1^2.$$

Hint: Remember that $u_1 \gg a_1$!

c. <u>Conservation of energy</u>
From the previous formulas, show that:

$$e_2 \approx 1/2\ u_1^2.$$

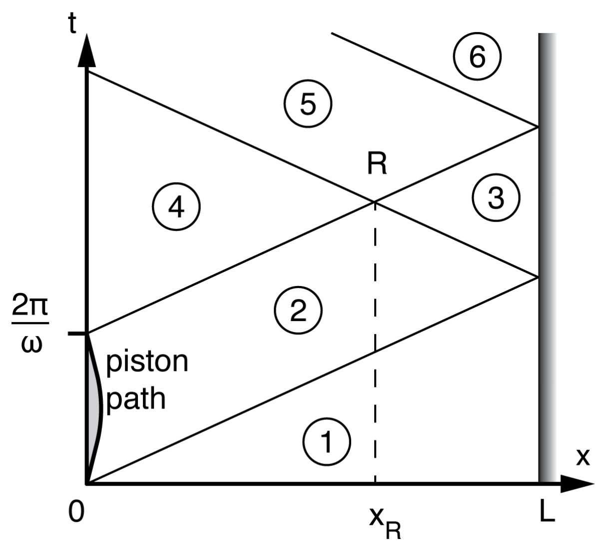
NB: the energy jump equation is not needed (information replaced by the density-ratio formula).

d. You could have produced the last equation without even looking at the jump equation. How?

Chapter 2.

Problem 2.1. ***Linear Wave Interaction***

A constant area tube of uniform gas at rest (labelled 1) is closed at one end (x = L). At the other end (x = 0) there is a movable piston. At t = 0, the piston starts moving with a velocity $u_p = \varepsilon\, a_0 \sin(\omega t),\ \varepsilon > 0$ and small. At $t = \frac{2\pi}{\omega}$ the piston is returned at its starting position (x = 0) and remains there for $t > 2\pi/\omega$. During the movement of the piston it produces an expansion/compression wave that lasts until it reflects from the far end, see the (t, x) diagram. In the diagram, six regions are distinguished.



a. Identify the regions that are unperturbed, simple waves and non-simple waves and justify your answer.

b. Take a point in region ② and show that
$$\tilde{s}(x_2, t_2) = M(x_2, t_2) = \varepsilon \sin \omega \left(t_2 - \frac{x_2}{a_0}\right)$$

c. Take a point in region ③ and show
$$M(x_3, t_3) = \varepsilon \sin \omega \left(t_3 - \frac{x_3}{a_0}\right) - \varepsilon \sin \omega \left(t_3 - \frac{2L}{a_0} + \frac{x_3}{a_0}\right)$$

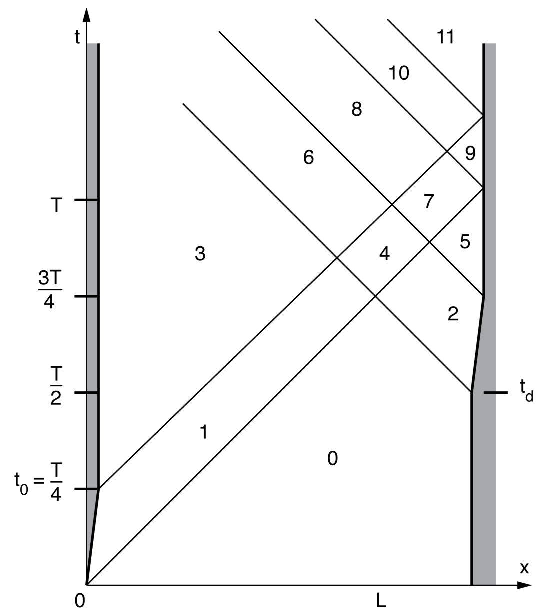
d. Determine the pressure as a function of time on the piston (x = 0) and at the end of the tube (x = L). Compare the pressure signals and comment on the differences.

e. What does an observer experience at $x < x_R$ if time proceeds.

f. Let the cross section of the tube be A_t; find the increase in kinetic energy of the gas in the tube and the work done by the piston. Are these values the same, why not?

Problem 2.2. ***Moving a compressible fluid***

A uniform gas ($M = \tilde{s} = 0$, speed of sound = a_0) is held in a tube by a piston on the left ($x = 0$) and one on the right ($x = L$). At $t = 0$ the left piston suddenly starts to move with a small positive speed εa_0; it stops again at $t = t_0$. The right piston does the same but with some delay; it starts to move with speed εa_0 at $t = t_d$ and stops at $t = t_d + t_0$.



In the figure, t_0 and t_d have the value T/4 and T/2 respectively where $T = L/a_0$ is the time it takes a sound wave to move from one piston to the other. Each change of motion creates a wave; the wave paths divide the (t,x)-plane into lots of regions. Twelve of these regions are labelled from 0 to 11.

a. Graphically determine M and $\tilde{s}$ in the regions 1-11, using five $(M, \tilde{s})$-diagrams:

   Diagram 1 connects regions 0, 1, 2, 4.
   Diagram 2 connects regions 1, 3, 4, 6.
   Diagram 3 connects regions 2, 4, 5, 7.
   Diagram 4 connects regions 6, 7, 8, 9, 10.
   Diagram 5 connects regions 10, 11.

   Indicate wave type (compression/expansion) in these diagrams.

b. Besides region 0 there are 4 other numbered regions where the gas is unperturbed; List these.

c. In which region is the velocity highest? Why?

d. In which region is the density highest? Why?

e. In which region is the velocity negative? Why?

f. Now assume $t_d = T$, all else remaining the same. Draw the wave paths and one particle path in an (x,t)-diagram. Discuss how the above system can be modified to move a gas while making (almost) no waves.

Problem 2.3. ***Waves interacting with a contact discontinuity***

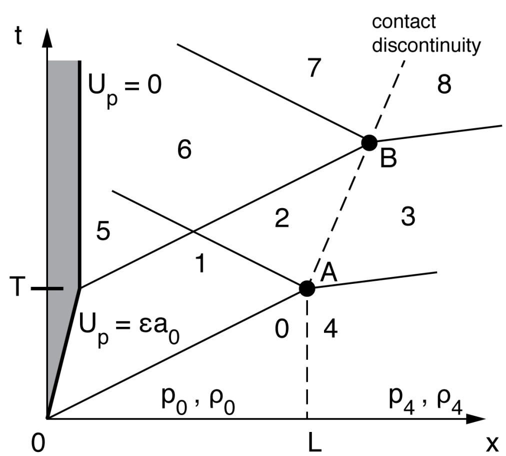
Consider a semi-infinite tube having a piston at its left end (x = 0). At t<0 the tube contains two uniform states, 0 and 4 of the same gas at rest separated by a contact discontinuity (c.d.) at x = L.

Conditions:

left state 0, $0 < x < L$: p_0, ρ_0,
right state 4, $x > L$: $p_4 = p_0$ but $\rho_4 \neq \rho_0$.

At t = 0 the piston accelerates instantaneously and moves with speed U_p into the gas; at t = T the piston stops and keeps its position for t > T.
The piston creates a wave that moves into the gas (with state 0) and interacts with the c.d. Due to interaction a transmitted wave (state 3) moves into the gas (state 4) and a reflected wave (state 6) moves to the left.

Assume for $0 < t < T$, $U_p = \varepsilon a_0$ (where ε is small) linear theory is applicable and waves can be treated as acoustic waves. We distinguish the domains 0-8 in the (t,x)-diagram below.



a. Express the acoustic speed a in terms of p and ρ and determine the slopes of the characteristics left and right from the c.d. (distinguish the cases $\rho_4 > \rho_0$ and $\rho_4 < \rho_0$).

b. Show that the c.d. separating 0 and 4 is not moving. Determine the time at which this c.d. starts moving. Determine the values of M and $\tilde{s}$ in state 1 and also the values of u_1 and p_1.

c. Now concentrate on the c.d. separating states 2 and 3. Using the conditions $p_2 = p_3$ and $u_2 = u_3$ show (in linear theory) that the c.d. moves with the velocity.

$$u_{c.d.} = \frac{\rho_0 a_0}{\frac{1}{2}\left(\rho_0 a_0 + \rho_4 a_4\right)} \quad U_p$$

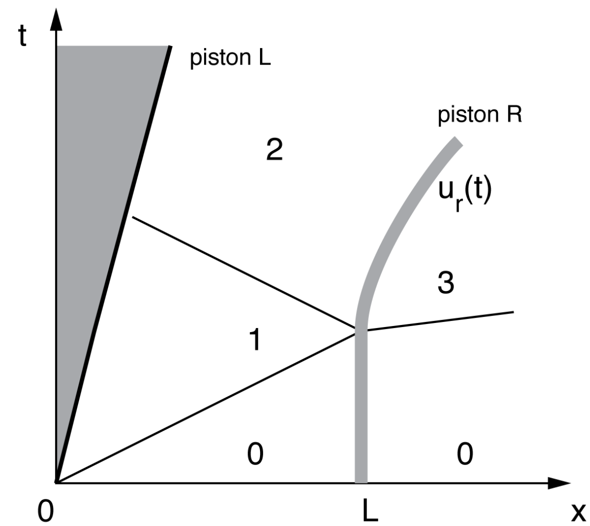
Using this value, now the values of M and $\tilde{s}$ in the domains 2, 3 and 6 have to be determined. (Hint: Be aware that the reference values of states left and right from the c.d. differ!)

d. Comment on the behaviour of the reflected wave (state 6) and the transmitted wave (state 3) with respect to the incoming wave (state 1), for the two limiting cases:
$\rho_4 = \rho_0$ (actually no c.d.) $\rho_4 >> \rho_0$ (c.d. is a rigid wall)

Problem 2.4. ***Linear theory: Driving a free piston in a tube***

A semi-infinite tube with constant area (A) filled with gas at (M = 0, $\tilde{s} = 0$, sound speed a_0, pressure p_0) has a movable piston L at its left and (x = 0) and a second movable piston R (mass m) at $x = \ell > 0$ in the tube.

At time t = 0 the left piston is forced to move with constant speed $u_\ell = \varepsilon a_0$ (ε > 0 and small) causing an acoustic (compression) wave running into the gas. The right piston R at $x = \ell$ stays there until $t = \ell / a_0$ when it is hit by the compression wave, then the piston R starts moving too, causing a wave running into the gas to the right of this piston, see (t,x)-diagram below:



a. Let the velocity of the piston R be denoted by $u_r(t)$. Using linear theory, express the value of $\tilde{s}$ on each face of piston R in terms of the velocity $u_r(t)$ and shows that the results are:

$$\tilde{s}_2 = 2\varepsilon - \frac{u_r(t)}{a_0},$$

$$\tilde{s}_3 = \frac{u_r(t)}{a_0},$$

b. Apply Newton's Second Law on the piston R to find its motion and show that $u_r(t)$ follows from the O.D.E.

$$\frac{du_r}{dt} + \lambda u_r = \lambda \varepsilon a_0, \qquad \text{with} \qquad \lambda = \frac{2 A a_0 \rho_0}{m}$$

c. The general solution of this O.D.E. can be written in the form

$$u_r = Ce^{-\lambda t} + a_0 \varepsilon$$

where C is a constant determined by the initial conditions.

Formulate the appropriate initial conditions for piston R and determine the constant C

d. Draw the graph $u_r(t)$ for t > 0 and comment on the motion of piston R if $t \to \infty$.

Chapter 3

Problem 3.1. ***Various Riemann problems***

Consider the following initial conditions. Indicate them in a *p,u* diagram and draw the Poisson and/or Hugoniot curves connecting the initial states to the post states (solution). Using the *p,u* diagram, draw the x,t diagram containing the waves and particle paths. Indicate the type of the wave (compression or expansion). For cases d) and e) also comment on the relative strength of the waves (which of the two is stronger and why?).

Initial conditions:

a) $u_1 < u_4, p_1 = p_4$

b) $u_1 > u_4, p_1 = p_4$

c) $u_1 = u_4, p_1 \neq p_4$

d) $0 < u_1 - u_4 = \frac{p_1 - p_4}{\rho_4 a_4}$

e) $0 \geq u_1 - u_4 = -\frac{p_1 - p_4}{\rho_1 a_1}$

Problem 3.2. ***Compressions***

a. Starting from rest a piston is accelerated into a tube filled with uniform He at rest, sound speed a_0, creating a centered compression wave. Once the piston has reached a speed $u_1 = 3a_0$, it continues its motion at that speed. Compute the density ratio ρ_1/ρ_0 across the wave, valid before the wave has focussed into a shock.

b. If the piston were accelerated instantaneously to speed u_1, creating a shock, would the density ratio across the shock be greater than, equal to, or less than the ratio found under (a), justify your answer?

Problem 3.3. ***Wave interactions***

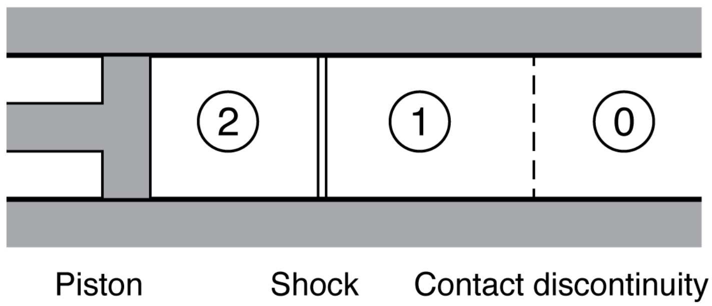
A shock running through a uniform gas encounters a contact discontinuity. The interaction creates a second non-linear wave.

Determine the nature of this wave (compression or expansion) from a *p,u* diagram.

You must distinguish between two cases: $\rho_1 > \rho_0$ and $\rho_1 < \rho_0$. Use the notation of the figure below.

(Hint: points 0 and 1 are indistinguishable in a *p,u* diagram; what matters is the value of ρa since this is equal to the slope of the Poisson or Hugoniot curves.)

Do the answers obtained make sense, explain?



Problem 3.4. ***Fully non-linear: Moving a gas***

A tube filled wit a uniform gas at rest is closed by pistons at both ends; they are a distance ℓ apart. At time t = 0 both pistons are forced to move at the same speed $u_p>0$. Shock- and expansion waves are generated. They run from one piston to the other, interact and reflect. Part of this process is sketched (not to scale) in the (t,x)-diagram below; various domains are distinguished. (Note that shocks get curved when they interact).

a. In the labelled area of the (t,x)-diagram there are three regions of constant entropy; the entropy levels are different. Indicate these regions (each may extend over several labelled domains) and number them I, II and III in the order of increasing entropy. Discuss your answer.

b. Among the labelled domains (A-H) there are domains having uniform conditions (u = constant, a = constant). Indicate those which have a velocity $u = u_p$.

c. Among the labelled domains there are three simple waves. Indicate them and explain which of the Riemann invariant J^+ or J^- is constant in which region $\left(dJ^{\pm} = du \pm \frac{dp}{\rho a}\right)$.

d. In homentropic flow, the Riemann invariants can be integrated leading to $J^{\pm} = u \pm \frac{2a}{\gamma-1}$. Which domains do not satisfy the homentropic conditions?