Image inpainting

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From functional to Laplace equation

We have the functional

$$\arg\min_{u\in W^{1,2}(\Omega)} \int_D |\nabla u(x)|^2 dx$$

According to the Necessary condition for the extremum[1], the minimum of the functional is the solution of

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\frac{\partial \mathcal{F}}{\partial p_i} + \frac{\partial \mathcal{F}}{\partial u} = 0$$

where $\nabla u(u) = (p_1, p_2)$ and $\mathcal{F}$ is the functional.

[1] We saw this result at the first class of the module

From functional to Laplace equation

Replacing the given functional, we get

$$0 = -\frac{\partial}{\partial x}\frac{\partial(u_x^2 + u_y^2)}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial(u_x^2 + u_y^2)}{\partial u_y} + \frac{\partial(u_x^2 + u_y^2)}{\partial u} =$$

$$= -\frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) + 0 = 2u_{xx} + 2u_{yy}$$

In conclusion,

$$2(u_{xx} + u_{yy}) = 0 \Longrightarrow \Delta u = 0$$

The problem of inpainting

The problem of inpainting can be modeled as

$$\begin{cases} \arg\min_{u \in W^{1,2}(\Omega)} \displaystyle\int_D |\nabla u(x)|^2 dx, \\ u|_{\partial D} = f \end{cases}$$

where f is the image to inpaint.
With the result obtained, this problem is equivalent to find the solution of

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{in } \partial D \end{cases}$$

The equation is completed with homogeneous Neumann boundary conditions at the boundary of the image.

The laplacian can be computed as

$$\Delta u = div(\nabla u).$$

Discretization of the gradient

Using forward differences, we obtain

$$\nabla u_{ij} = \left(\frac{u_{i+1,j} - u_{i,j}}{h_i}, \frac{u_{i,j+1} - u_{i,j}}{h_j} \right)$$

Discretization of the divergence

Taking $v = (v_1, v_2)$ and using backward differences, we obtain

$$div(v)_{i,j} = \frac{v_{1_{i,j}} - v_{1_{i-1,j}}}{h_i} + \frac{v_{2_{i,j}} - v_{2_{i,j-1}}}{h_j}$$

Joining the previous discretizations and doing some calculus we obtain the following equation for each pixel in D

$$\frac{1}{h_j^2}u_{i,j-1} + \frac{1}{h_i^2}u_{i-1,j} - \left(\frac{2}{h_i^2} + \frac{2}{h_j^2}\right)u_{i,j} + \frac{1}{h_i^2}u_{i+1,j} + \frac{1}{h_j^2}u_{i,j+1} = 0$$