

Image inpainting

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From functional to Laplace equation

We have the functional

$$\arg \min_{u \in W^{1,2}(\Omega)} \int_D |\nabla u(x)|^2 dx$$

According to the Necessary condition for the extremum¹, the minimum of the functional is the solution of

$$-\sum_{i=1}^2 \frac{\partial}{\partial x_i} \frac{\partial \mathcal{F}}{\partial p_i} + \frac{\partial \mathcal{F}}{\partial u} = 0$$

where $\nabla u(u) = (p_1, p_2)$ and \mathcal{F} is the functional.

¹We saw this result at the first class of the module

From functional to Laplace equation

Replacing the given functional, we get

$$\begin{aligned} 0 &= -\frac{\partial}{\partial x} \frac{\partial(u_x^2 + u_y^2)}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial(u_x^2 + u_y^2)}{\partial u_y} + \frac{\partial(u_x^2 + u_y^2)}{\partial u} = \\ &= -\frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) + 0 = 2u_{xx} + 2u_{yy} \end{aligned}$$

In conclusion,

$$2(u_{xx} + u_{yy}) = 0 \implies \Delta u = 0$$

The problem of inpainting

The problem of inpainting can be modeled as

$$\begin{cases} \arg \min_{u \in W^{1,2}(\Omega)} \int_D |\nabla u(x)|^2 dx, \\ u|_{\partial D} = f \end{cases}$$

where f is the image to inpaint.

With the result obtained, this problem is equivalent to find the solution of

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{in } \partial D \end{cases}$$

The equation is completed with homogeneous Neumann boundary conditions at the boundary of the image.

The laplacian can be computed as

$$\Delta u = \operatorname{div}(\nabla u).$$

Discretization of the gradient

Using forward differences, we obtain

$$\nabla u_{ij} = \left(\frac{u_{i+1,j} - u_{i,j}}{h_i}, \frac{u_{i,j+1} - u_{i,j}}{h_j} \right)$$

Discretization of the divergence

Taking $v = (v_1, v_2)$ and using backward differences, we obtain

$$\operatorname{div}(v)_{i,j} = \frac{v_{1,i,j} - v_{1,i-1,j}}{h_i} + \frac{v_{2,i,j} - v_{2,i,j-1}}{h_j}$$

Joining the previous discretizations and doing some calculus we obtain the following equation for each pixel in D

$$\frac{1}{h_j^2} u_{i,j-1} + \frac{1}{h_i^2} u_{i-1,j} - \left(\frac{2}{h_i^2} + \frac{2}{h_j^2} \right) u_{i,j} + \frac{1}{h_i^2} u_{i+1,j} + \frac{1}{h_j^2} u_{i,j+1} = 0$$

Boundary conditions

Equations for all the points of the image

Results(images)