Image inpainting

Onofre Martorell, Lidia Talavera

October 12, 2016

From functional to Laplace equation

We have the functional

$$\arg\min_{u\in W^{1,2}(\Omega)} \int_D |\nabla u(x)|^2 dx$$

According to the Necessary condition for the extremum[1], the minimum of the functional is the solution of

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\frac{\partial \mathcal{F}}{\partial p_i} + \frac{\partial \mathcal{F}}{\partial u} = 0$$

where $\nabla u(u) = (p_1, p_2)$ and $\mathcal{F}$ is the function inside the functional.

[1]We saw this result at the first class of the module

From functional to Laplace equation

Replacing the given functional, we get

$$0 = -\frac{\partial}{\partial x}\frac{\partial(u_x^2+u_y^2)}{\partial u_x} - \frac{\partial}{\partial y}\frac{\partial(u_x^2+u_y^2)}{\partial u_y} + \frac{\partial(u_x^2+u_y^2)}{\partial u} =$$

$$= -\frac{\partial}{\partial x}(2u_x) - \frac{\partial}{\partial y}(2u_y) + 0 = 2u_{xx} + 2u_{yy}$$

In conclusion,

$$2(u_{xx}+u_{yy}) = 0 \Longrightarrow \Delta u = 0$$

The problem of inpainting

The problem of inpainting can be modelled as

$$\begin{cases} \arg\min_{u \in W^{1,2}(\Omega)} \displaystyle\int_D |\nabla u(x)|^2 dx, \\ u|_{\partial D} = f \end{cases}$$

where f is the image to inpaint.
With the result obtained, this problem is equivalent to find the solution of

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = f & \text{in } \partial D \end{cases}$$

The equation is completed with homogeneous Neumann boundary conditions at the boundary of the image.

Discretization of the laplacian

The laplacian can be computed as

$$\Delta u = div(\nabla u).$$

Discretization of the gradient

Using forward differences, we obtain

$$\nabla u_{ij} = \left(\frac{u_{i+1,j} - u_{i,j}}{h_i}, \frac{u_{i,j+1} - u_{i,j}}{h_j} \right)$$

Discretization of the divergence

Taking $v = (v_1, v_2)$ and using backward differences, we obtain

$$div(v)_{i,j} = \frac{v_{1_{i,j}} - v_{1_{i-1,j}}}{h_i} + \frac{v_{2_{i,j}} - v_{2_{i,j-1}}}{h_j}$$

Joining the previous discretizations and doing some calculus we obtain the following equation for each pixel in D

$$\frac{1}{h_j^2}u_{i,j-1} + \frac{1}{h_i^2}u_{i-1,j} - \left(\frac{2}{h_i^2} + \frac{2}{h_j^2}\right)u_{i,j} + \frac{1}{h_i^2}u_{i+1,j} + \frac{1}{h_j^2}u_{i,j+1} = 0$$

Boundary conditions

For the boundary of the image we have added one row or column to each side of the image and we have computed the Neumann boundary conditions. For example, at the east side, we get

$$\frac{u_{i,1} - u_{i,2}}{h_j} = 0 \Longrightarrow u_{i,1} = u_{i,2}$$

System of equations

To sum up, we have an equation for each type of pixel in the image

- Pixels belonging to D:

$$\frac{1}{h_j^2}u_{i,j-1}+\frac{1}{h_i^2}u_{i-1,j}-\left(\frac{2}{h_i^2}+\frac{2}{h_j^2}\right)u_{i,j}+\frac{1}{h_i^2}u_{i+1,j}+\frac{1}{h_j^2}u_{i,j+1}=0$$

- Pixels not belonging to D:

$$u_{i,j}=f_{i,j}$$

- Pixels on the boundary:

$$u_{i,1}=u_{i,2},\ u_{i,n_j}=u_{i,n_j-1},\ u_{1,j}=u_{2,j} \text{ or } u_{n_i,j}=u_{n_i-1,j},$$

  depending if the pixel belongs to east, west, north or south boundary.

System of equations

Ordering the pixels of the image as

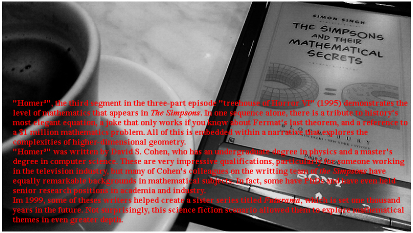
$$x = (u_{1,1}, u_{2,1}, \cdots , u_{i,j}, u_{i+1,j}, \cdots , u_{n_i+2,n_j+2})^T$$

the previous equations can be written as a linear system of equations

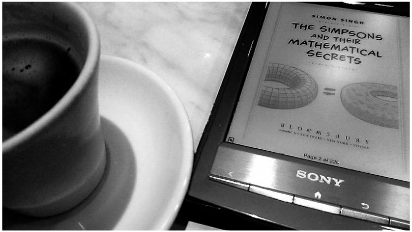
$$Ax = b,$$

which can be solved using Matlab.

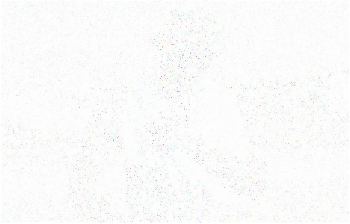
Results



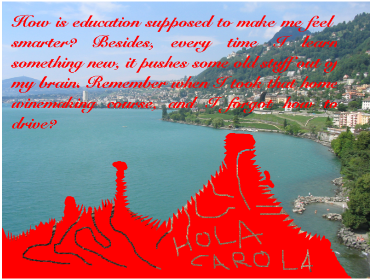
After



Results



Results


How is education supposed to make me feel smarter? Besides, every time I learn something new, it pushes some old stuff out of my brain. Remember when I took that home winemaking course, and I forgot how to drive?
HOLA
CAROLA

Conclusions

Conclusions

- The inpainted image has a clear definition for the images which we know most of the information.
- In the image with 99% information lost, the restored image shows a whole view of the image, but it is not clearly defined.
- In the goal image, it restores quite well the zone with letters, but not the zone fully painted of red.